MT104-Linear Algebra

Matrices and Matrix Operations

DEFINITION 1 A *matrix* is a rectangular array of numbers. The numbers in the array are called the *entries* in the matrix.

Examples of Matrices

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}, \quad [2 \quad 1 \quad 0 \quad -3], \quad \begin{bmatrix} e & \pi & -\sqrt{2} \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad [4]$$

A matrix with m rows and n columns is referred to as an $m \times n$ **matrix** or as having **size** $m \times n$.

A general $m \times n$ matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad (1)$$

When a compact notation is desired, the preceding matrix can be written as

$$[a_{ij}]_{m \times n} \quad \text{or} \quad [a_{ij}]$$

The entry in row i and column j of a matrix A is also commonly denoted by the symbol $(A)_{ij}$. Thus, for matrix (1) above, we have

$$(A)_{ij} = a_{ij}$$

and for the matrix

$$A = \begin{bmatrix} 2 & -3 \\ 7 & 0 \end{bmatrix}$$

we have $(A)_{11} = 2$, $(A)_{12} = -3$, $(A)_{21} = 7$, and $(A)_{22} = 0$.

Row Vector and Column Vector

A matrix with only one row is called a ***row vector*** (or a ***row matrix***) , and a matrix with only one column is called a ***column vector*** (or a ***column matrix***).

Row and column vectors are of special importance, and it is common practice to denote them by boldface lowercase letters rather than capital letters. For such matrices, double subscripting of the entries is unnecessary. Thus a general $1 \times n$ row vector $\mathbf{a}$ and a general $m \times 1$ column vector $\mathbf{b}$ would be written as

$$\mathbf{a} = [a_1 \quad a_2 \quad \cdots \quad a_n] \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Square Matrix and Main Diagonal

A matrix A with n rows and n columns is called a ***square matrix of order n***, and the shaded entries $a_{11}, a_{22}, \ldots, a_{nn}$ in (2) are said to be on the ***main diagonal*** of A.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \tag{2}$$

Operations on Matrices

DEFINITION 2 Two matrices are defined to be ***equal*** if they have the same size and their corresponding entries are equal.

Problem

Given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$, discuss the possibility that $A = B$, $B = C$, $A = C$.

Solution ▶ $A = B$ is impossible because A and B are of different sizes: A is 2×2 whereas B is 2×3. Similarly, $B = C$ is impossible. But $A = C$ is possible provided that corresponding entries are equal: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ means $a = 1$, $b = 0$, $c = -1$, and $d = 2$.

DEFINITION 3 If A and B are matrices of the same size, then the ***sum*** $A + B$ is the matrix obtained by adding the entries of B to the corresponding entries of A, and the ***difference*** $A - B$ is the matrix obtained by subtracting the entries of B from the corresponding entries of A. Matrices of different sizes cannot be added or subtracted.

Consider the matrices

$$A = \begin{bmatrix} 2 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 4 & -2 & 7 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -4 & 3 & 5 & 1 \\ 2 & 2 & 0 & -1 \\ 3 & 2 & -4 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

Then

$$A + B = \begin{bmatrix} -2 & 4 & 5 & 4 \\ 1 & 2 & 2 & 3 \\ 7 & 0 & 3 & 5 \end{bmatrix} \quad \text{and} \quad A - B = \begin{bmatrix} 6 & -2 & -5 & 2 \\ -3 & -2 & 2 & 5 \\ 1 & -4 & 11 & -5 \end{bmatrix}$$

The expressions $A + C$, $B + C$, $A - C$, and $B - C$ are undefined. ◄

Scalar Multiples of a Matrix

DEFINITION 4 If A is any matrix and c is any scalar, then the ***product*** cA is the matrix obtained by multiplying each entry of the matrix A by c. The matrix cA is said to be a ***scalar multiple*** of A.

For the matrices

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 & 7 \\ -1 & 3 & -5 \end{bmatrix}, \quad C = \begin{bmatrix} 9 & -6 & 3 \\ 3 & 0 & 12 \end{bmatrix}$$

we have

$$2A = \begin{bmatrix} 4 & 6 & 8 \\ 2 & 6 & 2 \end{bmatrix}, \quad (-1)B = \begin{bmatrix} 0 & -2 & -7 \\ 1 & -3 & 5 \end{bmatrix}, \quad \tfrac{1}{3}C = \begin{bmatrix} 3 & -2 & 1 \\ 1 & 0 & 4 \end{bmatrix}$$

It is common practice to denote $(-1)B$ by $-B$. ◀

Product of Two Matrices

DEFINITION 5 If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then the ***product*** AB is the $m \times n$ matrix whose entries are determined as follows: To find the entry in row i and column j of AB, single out row i from the matrix A and column j from the matrix B. Multiply the corresponding entries from the row and column together, and then add up the resulting products.

Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

Since A is a 2×3 matrix and B is a 3×4 matrix, the product AB is a 2×4 matrix.

To determine, for example, the entry in row 2 and column 3 of AB, we single out row 2 from A and column 3 from B. Then, as illustrated below, we multiply corresponding entries together and add up these products.

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \square & \square & \square & \square \\ \square & \square & \boxed{26} & \square \end{bmatrix}$$

$$(2 \cdot 4) + (6 \cdot 3) + (0 \cdot 5) = 26$$

The entry in row 1 and column 4 of AB is computed as follows:

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \square & \square & \square & \boxed{13} \\ \square & \square & \square & \square \end{bmatrix}$$

$$(1 \cdot 3) + (2 \cdot 1) + (4 \cdot 2) = 13$$

The computations for the remaining entries are

$$
\begin{aligned}
(1 \cdot 4) + (2 \cdot 0) + (4 \cdot 2) &= 12 \\
(1 \cdot 1) - (2 \cdot 1) + (4 \cdot 7) &= 27 \\
(1 \cdot 4) + (2 \cdot 3) + (4 \cdot 5) &= 30 \\
(2 \cdot 4) + (6 \cdot 0) + (0 \cdot 2) &= 8 \\
(2 \cdot 1) - (6 \cdot 1) + (0 \cdot 7) &= -4 \\
(2 \cdot 3) + (6 \cdot 1) + (0 \cdot 2) &= 12
\end{aligned}
$$

$$
AB = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}
$$

In general, if $A = [a_{ij}]$ is an $m \times r$ matrix and $B = [b_{ij}]$ is an $r \times n$ matrix, then, as illustrated by the shading in the following display,

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ir} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rj} & \cdots & b_{rn} \end{bmatrix} \tag{4}$$

the entry $(AB)_{ij}$ in row i and column j of AB is given by

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{ir}b_{rj} \tag{5}$$

Formula (5) is called the ***row-column rule*** for matrix multiplication.

Partitioned Matrices

A matrix can be subdivided or ***partitioned*** into smaller matrices by inserting horizontal and vertical rules between selected rows and columns. **e.g**

$$A = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$A = \left[\begin{array}{cccc} a_{11} & a_{12} & a_{13} & a_{14} \\ \hline a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right] = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{bmatrix}$$

$$A = \left[\begin{array}{c|c|c|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right] = [\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{c}_3 \quad \mathbf{c}_4]$$

- Partitioning has many uses, one of which is for finding particular rows or columns of a matrix product AB without computing the entire product.

Matrix Multiplication by Columns and by Rows

$$AB = A[\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n] = [A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \cdots \quad A\mathbf{b}_n]$$

(AB computed column by column)

$$AB = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_m \end{bmatrix} B = \begin{bmatrix} \mathbf{a}_1 B \\ \mathbf{a}_2 B \\ \vdots \\ \mathbf{a}_m B \end{bmatrix}$$

(AB computed row by row)

In words, these formulas state that

$$j\text{th column vector of } AB = A[\, j\text{th column vector of } B]$$

$$i\text{th row vector of } AB = [i\text{th row vector of } A]B$$

Problem

Compute AB if $A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 4 & 7 \\ 0 & 1 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 8 & 9 \\ 7 & 2 \\ 6 & 1 \end{bmatrix}$

Solution ▶ The columns of B are $\mathbf{b}_1 = \begin{bmatrix} 8 \\ 7 \\ 6 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 9 \\ 2 \\ 1 \end{bmatrix}$

$$A\mathbf{b}_1 = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 4 & 7 \\ 0 & 1 & 8 \end{bmatrix} \begin{bmatrix} 8 \\ 7 \\ 6 \end{bmatrix} = \begin{bmatrix} 67 \\ 78 \\ 55 \end{bmatrix} \text{ and } A\mathbf{b}_2 = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 4 & 7 \\ 0 & 1 & 8 \end{bmatrix} \begin{bmatrix} 9 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 29 \\ 24 \\ 10 \end{bmatrix}.$$

$$AB = [A\mathbf{b}_1 \;\; A\mathbf{b}_2] = \begin{bmatrix} 67 & 29 \\ 78 & 24 \\ 55 & 10 \end{bmatrix}.$$

Powers of a Matrix

If A is an $n \times n$ matrix and if k is a positive integer, then A^k denotes the product of k copies of A:

$$A^k = \underbrace{A \cdots A}_{k}$$

For instance , in order to find A^2 we will multiply A with A.

Transpose of a Matrix

DEFINITION 7 If A is any $m \times n$ matrix, then the ***transpose of A***, denoted by A^T, is defined to be the $n \times m$ matrix that results by interchanging the rows and columns of A; that is, the first column of A^T is the first row of A, the second column of A^T is the second row of A, and so forth.

The following are some examples of matrices and their transposes.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{bmatrix}, \quad C = [1 \quad 3 \quad 5], \quad D = [4]$$

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \\ a_{14} & a_{24} & a_{34} \end{bmatrix}, \quad B^T = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \quad D^T = [4]$$

Properties of Transpose

THEOREM 1.4.8 *If the sizes of the matrices are such that the stated operations can be performed, then:*

(a) $(A^T)^T = A$

(b) $(A + B)^T = A^T + B^T$

(c) $(A - B)^T = A^T - B^T$

(d) $(kA)^T = kA^T$

(e) $(AB)^T = B^T A^T$

Properties of Matrix Addition and Scalar Multiplication

THEOREM 1.4.1 Properties of Matrix Arithmetic

Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid.

(a) $A + B = B + A$ **[Commutative law for matrix addition]**

(b) $A + (B + C) = (A + B) + C$ **[Associative law for matrix addition]**

(c) $A(BC) = (AB)C$ **[Associative law for matrix multiplication]**

(d) $A(B + C) = AB + AC$ **[Left distributive law]**

(e) $(B + C)A = BA + CA$ **[Right distributive law]**

(f) $A(B - C) = AB - AC$

(g) $(B - C)A = BA - CA$

(h) $a(B + C) = aB + aC$

(i) $a(B - C) = aB - aC$

(j) $(a + b)C = aC + bC$

(k) $(a - b)C = aC - bC$

(l) $a(bC) = (ab)C$

(m) $a(BC) = (aB)C = B(aC)$

WARNINGS:

1. In general, $AB \neq BA$.
2. The cancellation laws do *not* hold for matrix multiplication. That is, if $AB = AC$, then it is *not* true in general that $B = C$.
3. If a product AB is the zero matrix, you *cannot* conclude in general that either $A = 0$ or $B = 0$.

Example on Point 1

Let $A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$

$$AB = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 14 & 3 \\ -2 & -6 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 10 & 2 \\ 29 & -2 \end{bmatrix}$$

$\Rightarrow$ $AB \neq BA$

Example on Point 2

Consider the matrices

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$$

Here $$AB = AC = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$$

But $$B \neq C$$

Example on Point 3

Here are two matrices for which $AB = 0$, but $A \neq 0$ and $B \neq 0$:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix}$$

Trace of a Matrix

DEFINITION 8 If A is a square matrix, then the ***trace of A***, denoted by $\operatorname{tr}(A)$, is defined to be the sum of the entries on the main diagonal of A. The trace of A is undefined if A is not a square matrix.

The following are examples of matrices and their traces.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 3 & 5 & -8 & 4 \\ 1 & 2 & 7 & -3 \\ 4 & -2 & 1 & 0 \end{bmatrix}$$

$$\operatorname{tr}(A) = a_{11} + a_{22} + a_{33} \qquad \operatorname{tr}(B) = -1 + 5 + 7 + 0 = 11$$

Properties of Trace

1. $tr(A+B) = tr(A) + tr(B)$
2. $tr(AB) = tr(BA)$
3. $tr(cA) = c\,tr(A)$
4. $tr(A^t) = tr(A)$

Zero Matrices

A matrix whose entries are all zero is called a ***zero matrix***. Some examples are

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [0]$$

Identity Matrices

A square matrix with 1's on the main diagonal and zeros elsewhere is called an ***identity matrix***. Some examples are

$$[1], \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

An identity matrix is denoted by the letter I. If it is important to emphasize the size, we will write I_n for the $n \times n$ identity matrix.

Triangular Matrices

A square matrix in which all the entries above the main diagonal are zero is called ***lower triangular***, and a square matrix in which all the entries below the main diagonal are zero is called ***upper triangular***. A matrix that is either upper triangular or lower triangular is called ***triangular***.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

A general 4×4 upper triangular matrix

A general 4×4 lower triangular matrix

Symmetric Matrices

A square matrix A is said to be ***symmetric*** if $A = A^T$.

The following matrices are symmetric, since each is equal to its own transpose

$$\begin{bmatrix} 7 & -3 \\ -3 & 5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 4 & 5 \\ 4 & -3 & 0 \\ 5 & 0 & 7 \end{bmatrix}, \quad \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}$$

THEOREM 1.7.2 *If A and B are symmetric matrices with the same size, and if k is any scalar, then:*

(*a*) A^T *is symmetric.*

(*b*) $A + B$ *and* $A - B$ *are symmetric.*

(*c*) kA *is symmetric.*

THEOREM 1.7.3 *The product of two symmetric matrices is symmetric if and only if the matrices commute.*

Equivalent Matrices

DEFINITION 1 Matrices A and B are said to be ***row equivalent*** if either (hence each) can be obtained from the other by a sequence of elementary row operations.

e.g I_2 and the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$ **are equivalent because we can obtain A from identity matrix by multiplying row 2 by -3**

Elementary Matrix

DEFINITION 2 A matrix E is called an ***elementary matrix*** if it can be obtained from an identity matrix by performing a *single* elementary row operation.

Listed below are four elementary matrices and the operations that produce them

$$\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiply the second row of I_2 by -3.

Interchange the second and fourth rows of I_4.

Add 3 times the third row of I_3 to the first row.

Multiply the first row of I_3 by 1.

Determinants

The determinant of a 1×1 matrix $[a_{11}]$ is

$$\det[a_{11}] = a_{11}$$

For 2×2 Matrices the formula is

$$: \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Let's talk about minor and cofactor of an element with the help of which we will calculate determinants of higher order matrices.

Minor & Cofactor of an element
(In a Square Matrix)

DEFINITION 1 If A is a square matrix, then the ***minor of entry*** $\boldsymbol{a_{ij}}$ is denoted by M_{ij} and is defined to be the determinant of the submatrix that remains after the ith row and jth column are deleted from A. The number $(-1)^{i+j}M_{ij}$ is denoted by C_{ij} and is called the ***cofactor of entry*** $\boldsymbol{a_{ij}}$.

Let

$$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$

The minor of entry a_{11} is

$$M_{11} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 16$$

The cofactor of a_{11} is

$$C_{11} = (-1)^{1+1}M_{11} = M_{11} = 16$$

Similarly, the minor of entry a_{32} is

$$M_{32} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = 26$$

The cofactor of a_{32} is

$$C_{32} = (-1)^{3+2} M_{32} = -M_{32} = -26 \blacktriangleleft$$

Definition of a General Determinant

DEFINITION 2 If A is an $n \times n$ matrix, then the number obtained by multiplying the entries in any row or column of A by the corresponding cofactors and adding the resulting products is called the ***determinant of A***, and the sums themselves are called ***cofactor expansions of A***. That is,

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \tag{7}$$

[cofactor expansion along the *j*th column]

and

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \tag{8}$$

[cofactor expansion along the *i*th row]

Problem

Find the determinant of the matrix

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$

Solution By cofactor expansion along the first Row

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - 1\begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + 0\begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix}$$

$$= 3(-4) - (1)(-11) + 0 = -1$$

By cofactor expansion along the first coulmn

Solution

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix} + 5\begin{vmatrix} 1 & 0 \\ -4 & 3 \end{vmatrix}$$

$$= 3(-4) - (-2)(-2) + 5(3) = -1$$

Tthe best strategy for cofactor expansion is to expand along a row or column with the most zeros.

A Useful Technique for Evaluating 2×2 and 3×3 Determinants

Determinants of 2×2 and 3×3 matrices can be evaluated very efficiently using the pattern suggested in Figure 2.1.1.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

▶ Figure 2.1.1

e. g

$$\begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = (3)(-2) - (1)(4) = -10$$

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix} = \left|\begin{array}{ccc|cc} 1 & 2 & 3 & 1 & 2 \\ -4 & 5 & 6 & -4 & 5 \\ 7 & -8 & 9 & 7 & -8 \end{array}\right.$$

$$= [45 + 84 + 96] - [105 - 48 - 72] = 240$$

Properties of Determinants

Theorem 8.5.1 Determinant of a Transpose

If $\mathbf{A}^T$ is the transpose of the $n \times n$ matrix $\mathbf{A}$, then $\det \mathbf{A}^T = \det \mathbf{A}$.

For example, for the matrix $\mathbf{A} = \begin{pmatrix} 5 & 7 \\ 3 & -4 \end{pmatrix}$, we have $\mathbf{A}^T = \begin{pmatrix} 5 & 3 \\ 7 & -4 \end{pmatrix}$. Observe that

$$\det \mathbf{A} = \begin{vmatrix} 5 & 7 \\ 3 & -4 \end{vmatrix} = -41 \quad \text{and} \quad \det \mathbf{A}^T = \begin{vmatrix} 5 & 3 \\ 7 & -4 \end{vmatrix} = -41.$$

Theorem 8.5.2 Two Identical Rows

If any two rows (columns) of an $n \times n$ matrix $\mathbf{A}$ are the same, then $\det \mathbf{A} = 0$.

EXAMPLE 1 Matrix with Two Identical Rows

Since the second and third columns in the matrix $\mathbf{A} = \begin{pmatrix} 6 & 2 & 2 \\ 4 & 2 & 2 \\ 9 & 2 & 2 \end{pmatrix}$ are the same, it follows from Theorem 8.5.2 that

$$\det \mathbf{A} = \begin{vmatrix} 6 & 2 & 2 \\ 4 & 2 & 2 \\ 9 & 2 & 2 \end{vmatrix} = 0.$$

Theorem 8.5.3 Zero Row or Column

If all the entries in a row (column) of an $n \times n$ matrix $\mathbf{A}$ are zero, then $\det \mathbf{A} = 0$.

$$\text{zero row} \rightarrow \begin{vmatrix} 0 & 0 \\ 7 & -6 \end{vmatrix} = 0 \quad \text{and} \quad \begin{array}{c} \text{zero column} \downarrow \\ \begin{vmatrix} 4 & 6 & 0 \\ 1 & 5 & 0 \\ 8 & -1 & 0 \end{vmatrix} = 0. \end{array}$$

Theorem 8.5.4 Interchanging Rows

If $\mathbf{B}$ is the matrix obtained by interchanging any two rows (columns) of an $n \times n$ matrix $\mathbf{A}$, then $\det \mathbf{B} = -\det \mathbf{A}$.

For example, if $\mathbf{B}$ is the matrix obtained by interchanging the first and third rows of

$$\mathbf{A} = \begin{pmatrix} 4 & -1 & 9 \\ 6 & 0 & 7 \\ 2 & 1 & 3 \end{pmatrix}$$

then $$\det \mathbf{B} = \begin{vmatrix} 2 & 1 & 3 \\ 6 & 0 & 7 \\ 4 & -1 & 9 \end{vmatrix} = -\begin{vmatrix} 4 & -1 & 9 \\ 6 & 0 & 7 \\ 2 & 1 & 3 \end{vmatrix} = -\det \mathbf{A}.$$

Theorem 8.5.5 Constant Multiple of a Row

If $\mathbf{B}$ is the matrix obtained from an $n \times n$ matrix $\mathbf{A}$ by multiplying a row (column) by a nonzero real number k, then $\det \mathbf{B} = k \det \mathbf{A}$.

$$\begin{vmatrix} 5 & 8 \\ 20 & 16 \end{vmatrix} = \underset{\substack{\text{from} \\ \text{first column}}}{\overset{}{5}}\begin{vmatrix} 1 & 8 \\ 4 & 16 \end{vmatrix} = 5 \cdot \underset{\substack{\text{from} \\ \text{second column}}}{8}\begin{vmatrix} 1 & 1 \\ 4 & 2 \end{vmatrix} = 5 \cdot 8 \cdot \underset{\substack{\text{from} \\ \text{second row}}}{2}\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 80(1 - 2) = -80$$

$$\begin{vmatrix} 4 & 2 & -1 \\ 5 & -2 & 1 \\ 7 & 4 & -2 \end{vmatrix} = (-2)\begin{vmatrix} 4 & -1 & -1 \\ 5 & 1 & 1 \\ 7 & -2 & -2 \end{vmatrix} = (-2) \cdot 0 = 0$$

Theorem 8.5.6 Determinant of a Matrix Product

If $\mathbf{A}$ and $\mathbf{B}$ are both $n \times n$ matrices, then $\det \mathbf{AB} = \det \mathbf{A} \cdot \det \mathbf{B}$.

In other words, the determinant of a product of two $n \times n$ matrices is the same as the product of the determinants.

Suppose $\mathbf{A} = \begin{pmatrix} 2 & 6 \\ 1 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & -4 \\ -3 & 5 \end{pmatrix}$. Then $\mathbf{AB} = \begin{pmatrix} -12 & 22 \\ 6 & -9 \end{pmatrix}$. Now $\det \mathbf{AB} = -24$, $\det \mathbf{A} = -8$, $\det \mathbf{B} = 3$, and so we see that

$$\det \mathbf{A} \cdot \det \mathbf{B} = (-8)(3) = -24 = \det \mathbf{AB}.$$

≡

Theorem 8.5.7 Determinant Is Unchanged

Suppose **B** is the matrix obtained from an $n \times n$ matrix **A** by multiplying the entries in a row (column) by a nonzero real number k and adding the result to the corresponding entries in another row (column). Then $\det \mathbf{B} = \det \mathbf{A}$.

A Multiple of a Row Added to Another

Suppose $\mathbf{A} = \begin{pmatrix} 5 & 1 & 2 \\ 3 & 0 & 7 \\ 4 & -1 & 4 \end{pmatrix}$ and suppose the matrix **B** is defined as that matrix obtained from **A** by the elementary row operation

$$\mathbf{A} = \begin{pmatrix} 5 & 1 & 2 \\ 3 & 0 & 7 \\ 4 & -1 & 4 \end{pmatrix} \overset{-3R_1 + R_3}{\Longrightarrow} \begin{pmatrix} 5 & 1 & 2 \\ 3 & 0 & 7 \\ -11 & -4 & -2 \end{pmatrix} = \mathbf{B}.$$

Expanding by cofactors along, say, the second column, we find $\det \mathbf{A} = 45$ and $\det \mathbf{B} = 45$.

Theorem 8.5.8 Determinant of a Triangular Matrix

Suppose $\mathbf{A}$ is an $n \times n$ triangular matrix (upper or lower). Then

$$\det \mathbf{A} = a_{11}a_{22}\cdots a_{nn},$$

where $a_{11}, a_{22}, \ldots, a_{nn}$ are the entries on the main diagonal of $\mathbf{A}$.

$$\begin{vmatrix} 3 & 0 & 0 & 0 \\ 2 & 6 & 0 & 0 \\ 5 & 9 & -4 & 0 \\ 7 & 2 & 4 & -2 \end{vmatrix} = 3 \cdot 6 \cdot (-4) \cdot (-2) = 144.$$

$$\begin{vmatrix} -3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 4 \end{vmatrix} = (-3) \cdot 6 \cdot 4 = -72.$$

THEOREM 2.3.1 *Let A, B, and C be $n \times n$ matrices that differ only in a single row, say the rth, and assume that the rth row of C can be obtained by adding corresponding entries in the rth rows of A and B. Then*

$$\det(C) = \det(A) + \det(B)$$

The same result holds for columns.

$$\det\begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1+0 & 4+1 & 7+(-1) \end{bmatrix} = \det\begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1 & 4 & 7 \end{bmatrix} + \det\begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\det(kA) = k^n \det(A)$$

In general

$$\det(A + B) \neq \det(A) + \det(B)$$

Consider

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \quad A + B = \begin{bmatrix} 4 & 3 \\ 3 & 8 \end{bmatrix}$$

We have $\det(A) = 1$, $\det(B) = 8$, and $\det(A + B) = 23$; thus

$$\det(A + B) \neq \det(A) + \det(B)$$

Evaluating Determinants by Row Reduction

Evaluating the determinant of an $n \times n$ matrix by the method of cofactor expansion requires a Herculean effort when the order of the matrix is large. To expand the determinant of, say, a 5×5 matrix with nonzero entries requires evaluating five cofactors that are determinants of 4×4 submatrices; each of these in turn requires four additional cofactors that are determinants of 3×3 submatrices, and so on.

★ The idea of the method is to reduce the given matrix to upper triangular form by elementary row operations, then compute the determinant of the upper triangular matrix (an easy computation), and then relate that determinant to that of the original matrix.

Evaluate the determinant of $\mathbf{A} = \begin{pmatrix} 6 & 2 & 7 \\ -4 & -3 & 2 \\ 2 & 4 & 8 \end{pmatrix}$.

SOLUTION

$$\det \mathbf{A} = \begin{vmatrix} 6 & 2 & 7 \\ -4 & -3 & 2 \\ 2 & 4 & 8 \end{vmatrix}$$

$$= 2\begin{vmatrix} 6 & 2 & 7 \\ -4 & -3 & 2 \\ 1 & 2 & 4 \end{vmatrix}$$

2 is a common factor in third row

$$= -2\begin{vmatrix} 1 & 2 & 4 \\ -4 & -3 & 2 \\ 6 & 2 & 7 \end{vmatrix}$$

interchange first and third rows

$$= -2\begin{vmatrix} 1 & 2 & 4 \\ 0 & 5 & 18 \\ 6 & 2 & 7 \end{vmatrix}$$

4 times first row added to second row

$$= -2\begin{vmatrix} 1 & 2 & 4 \\ 0 & 5 & 18 \\ 0 & -10 & -17 \end{vmatrix}$$

-6 times first row added to third row

$$= -2\begin{vmatrix} 1 & 2 & 4 \\ 0 & 5 & 18 \\ 0 & 0 & 19 \end{vmatrix}$$

2 times second row added to third row

$$= (-2)(1)(5)(19) = -190$$

Compute the determinant of $A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 2 & 7 & 0 & 6 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -5 \end{bmatrix}$

Adjoint Matrix

DEFINITION 1 If A is any $n \times n$ matrix and C_{ij} is the cofactor of a_{ij}, then the matrix

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

is called the ***matrix of cofactors from A***. The transpose of this matrix is called the ***adjoint of A*** and is denoted by $\mathrm{adj}(A)$.

Let

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

$$\begin{array}{lll} C_{11} = 12 & C_{12} = 6 & C_{13} = -16 \\ C_{21} = 4 & C_{22} = 2 & C_{23} = 16 \\ C_{31} = 12 & C_{32} = -10 & C_{33} = 16 \end{array}$$

so the matrix of cofactors is

$$\begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{bmatrix}$$

and the adjoint of A is

$$\text{adj}(A) = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

Inverse of a Matrix

Definition 8.6.1 Inverse of a Matrix

Let $\mathbf{A}$ be an $n \times n$ matrix. If there exists an $n \times n$ matrix $\mathbf{B}$ such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}, \tag{1}$$

where $\mathbf{I}$ is the $n \times n$ identity, then the matrix $\mathbf{A}$ is said to be **nonsingular** or **invertible**. The matrix $\mathbf{B}$ is said to be the **inverse** of $\mathbf{A}$.

For example, the matrix $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ is nonsingular or invertible since the matrix $\mathbf{B} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ is its inverse. To verify this, observe that

$$\mathbf{AB} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$$

and

$$\mathbf{BA} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}.$$

Inverse of a 2 × 2 Matrix

THEOREM 1.4.5 *The matrix*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$, *in which case the inverse is given by the formula*

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{2}$$

Determine whether the matrix is invertible. If so, find its inverse. $A = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix}$

Solution The determinant of A is $\det(A) = (6)(2) - (1)(5) = 7$, which is nonzero. Thus, A is invertible, and its inverse is

$$A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix}$$

THEOREM 2.3.6 Inverse of a Matrix Using Its Adjoint

If A is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) \tag{6}$$

Problem

Find the inverse of the matrix $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$

Sol

$$\begin{array}{lll} C_{11} = 12 & C_{12} = 6 & C_{13} = -16 \\ C_{21} = 4 & C_{22} = 2 & C_{23} = 16 \\ C_{31} = 12 & C_{32} = -10 & C_{33} = 16 \end{array} \qquad \operatorname{adj}(A) = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix} \qquad \det(A) = 64$$

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{64}\begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix} = \begin{bmatrix} \frac{12}{64} & \frac{4}{64} & \frac{12}{64} \\ \frac{6}{64} & \frac{2}{64} & -\frac{10}{64} \\ -\frac{16}{64} & \frac{16}{64} & \frac{16}{64} \end{bmatrix}$$

Another method for finding inverse of a matrix is illustrated in the coming slides.

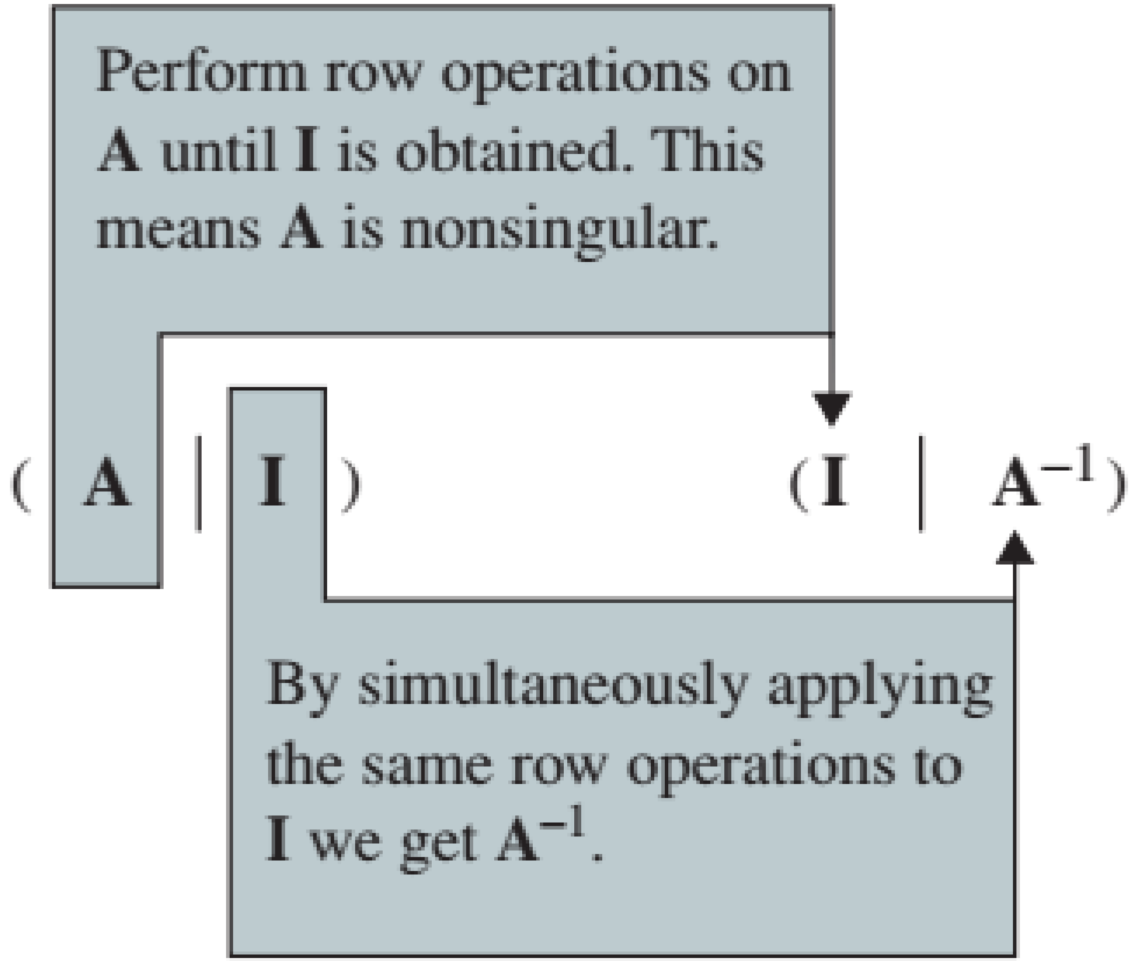
Theorem 8.6.4 Finding the Inverse

If an $n \times n$ matrix $\mathbf{A}$ can be transformed into the $n \times n$ identity $\mathbf{I}$ by a sequence of elementary row operations, then $\mathbf{A}$ is nonsingular. The same sequence of operations that transforms $\mathbf{A}$ into the identity $\mathbf{I}$ will also transform $\mathbf{I}$ into $\mathbf{A}^{-1}$.

It is convenient to carry out these row operations on $\mathbf{A}$ and $\mathbf{I}$ simultaneously by means of an $n \times 2n$ matrix obtained by augmenting $\mathbf{A}$ with the identity $\mathbf{I}$ as shown here:

$$(\mathbf{A}|\mathbf{I}) = \left(\begin{array}{cccc|cccc} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & \cdots & 1 \end{array}\right).$$

The procedure for finding $\mathbf{A}^{-1}$ is outlined in the following diagram:



Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

The computations are as follows:

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right] \longleftarrow$$

We added -2 times the first row to the second and -1 times the first row to the third.

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

We added 2 times the second row to the third.

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

We multiplied the third row by -1.

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

We added 3 times the third row to the second and -3 times the third row to the first.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

We added -2 times the second row to the first.

Thus,

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

Note

Often it will not be known in advance if a given $n \times n$ matrix A is invertible.

However, if it is not, then it will be impossible to reduce A to I_n by elementary row operations. This will be signaled by a row of zeros appearing on the *left side* of the partition at some stage of the inversion algorithm. If this occurs, then you can stop the computations and conclude that A is not invertible.

EXAMPLE 5 Showing That a Matrix Is Not Invertible

Consider the matrix

$$A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$$

$$\left[\begin{array}{rrr|rrr} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{rrr|rrr} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array}\right]$$

We added -2 times the first row to the second and added the first row to the third.

$$\left[\begin{array}{rrr|rrr} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right]$$

We added the second row to the third.

Since we have obtained a row of zeros on the left side, A is not invertible.

Properties

Let **A** and **B** be nonsingular matrices. Then

(*i*) $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$

(*ii*) $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

(*iii*) $(\mathbf{A}^{T})^{-1} = (\mathbf{A}^{-1})^{T}$

THEOREM 2.3.5 *If A is invertible, then*

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

THEOREM 1.4.9 *If A is an invertible matrix, then A^T is also invertible and*

$$(A^T)^{-1} = (A^{-1})^T$$

THEOREM 1.4.7 *If A is invertible and n is a nonnegative integer, then:*

(a) A^{-1} *is invertible and* $(A^{-1})^{-1} = A$.

(b) A^n *is invertible and* $(A^n)^{-1} = A^{-n} = (A^{-1})^n$.

(c) kA *is invertible for any nonzero scalar k, and* $(kA)^{-1} = k^{-1}A^{-1}$.

Diagonal Matrices

A square matrix in which all the entries off the main diagonal are zero is called a ***diagonal matrix***. Here are some examples:

$$\begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Remark Observe that diagonal matrices are both upper triangular and lower triangular since they have zeros below and above the main diagonal. Observe also that a *square* matrix in row echelon form is upper triangular since it has zeros below the main diagonal.

General Form of a Diagonal Matrix

A general $n \times n$ diagonal matrix D can be written as

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

Inverse of a Diagonal Matrix

A diagonal matrix is invertible if and only if all of its diagonal entries are nonzero; in this case the inverse of (1) is

$$D^{-1} = \begin{bmatrix} 1/d_1 & 0 & \cdots & 0 \\ 0 & 1/d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1/d_n \end{bmatrix}$$

Power of a Diagonal Matrix

Powers of diagonal matrices are easy to compute; we leave it for you to verify that if D is the diagonal matrix (1) and k is a positive integer, then

$$D^{k} = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

If

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

then

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \quad A^5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -243 & 0 \\ 0 & 0 & 32 \end{bmatrix}, \quad A^{-5} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{243} & 0 \\ 0 & 0 & \frac{1}{32} \end{bmatrix}$$

- *nonzero* row or column in a matrix means a row or column that contains at least one nonzero entry;
- a **leading entry** of a row refers to the leftmost nonzero entry (in a nonzero row).

Echelon Forms

A matrix is said to be in Echelon form if it has the following properties

1. **Every nonzero row precedes every zero row**.

2. **In each successive nonzero row, the number of zeros before the leading entry of a row increases row by row.**

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

- The leading entry in each nonzero row is 1.
- Each leading 1 is the only nonzero entry in its column.

The matrices

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix} \quad \text{are in echelon form.}$$

In fact, the second matrix is in reduced echelon form.

THEOREM

Uniqueness of the Reduced Echelon Form

Each matrix is row equivalent to one and only one reduced echelon matrix.

Rank of a Matrix

The number of nonzero rows in the echelon form of a matrix is called rank of the matrix.

THEOREM

An $n \times n$ square matrix $\mathbf{A}$ has rank n if and only if

$$\det \mathbf{A} \neq 0.$$

Linear System

A general linear system of m equations in the n unknowns $x_1, x_2, \ldots, x_n$ can be written as

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

A ***solution*** of a linear system in n unknowns $x_1, x_2, \ldots, x_n$ is a sequence of n numbers $s_1, s_2, \ldots, s_n$ for which the substitution

$$x_1 = s_1, \quad x_2 = s_2, \ldots, \quad x_n = s_n$$

makes each equation a true statement.

For example $x = 1, \quad y = -2$ is a solution of $\begin{array}{l} 5x + y = 3 \\ 2x - y = 4 \end{array}$

In matrix form, we can write (★) as

$$\mathbf{Ax} = \mathbf{b}$$

where,

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & \cdots & . \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

A is called the **coefficient matrix**

$$\tilde{\mathbf{A}} = \left[\begin{array}{cccc} a_{11} & \cdots & a_{1n} & b_1 \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$$

$\tilde{\mathbf{A}}$ is called the **augmented matrix** of the system

Homogeneous Linear Systems

A system of linear equations is said to be ***homogeneous*** if the constant terms are all zero; that is, the system has the form

$$
\begin{array}{ccccccc}
a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n & = & 0 \\
a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n & = & 0 \\
\vdots & & \vdots & & \vdots & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + \cdots + & a_{mn}x_n & = & 0
\end{array}
$$

Every homogeneous system of linear equations is consistent because all such systems have $x_1 = 0, x_2 = 0, \ldots, x_n = 0$ as a solution. This solution is called the ***trivial solution***; if there are other solutions, they are called ***nontrivial solutions***.

Because a homogeneous linear system always has the trivial solution, there are only two possibilities for its solutions:

- The system has only the trivial solution.
- The system has infinitely many solutions in addition to the trivial solution.

Important Points about Linear Systems

1. There are a lot of methods for solving linear systems, some methods work in special cases while some works for every type of linear system.

2. If in a linear system the number of equations is equal to the number of unknowns (i.e. if the coefficient matrix is a square matrix i.e. if $m = n$) then from determinant we can comment about the number of solution(s).

3. Any linear system in which $m = n$, and $|A| \neq 0$, has a unique solution. We can obtain this unique solution by any of the following methods:

   (i) Matrix Inversion Method (ii) Cramer's Rule

   (iii) Gauss Elimination (iv) Gauss-Jordan Elimination

   Furthermore, if the system is homogeneous then this unique solution is the trivial solution (zero solution).

4. A homogeneous system has always a zero solution called trivial solution. Thus a homogeneous system is always consistent.

5. A homogeneous linear system in which $m = n$, and $|A| = 0$, has infinite solutions (i.e. nontrivial solutions also exist).

5. In case of nonhomogeneous linear system in which $m = n$ and $|A| = 0$, there is possibility that the system has

(i) No Solution (ii) Infinite Solutions

6. If in a linear system the number of equations is not equal to the number of unknowns (i.e. if the coefficient matrix is not a square matrix i.e. if $m \neq n$) then we can comment about the number of solution(s) by looking into the rank of coefficient matrix, rank of augmented matrix and number of unknowns in the system. In this case we can find the solution(s) (if exist) by either Gauss Elimination or by Gauss-Jordan Elimination.

THEOREM 1

Fundamental Theorem for Linear Systems

(a) Existence. *A linear system of* m *equations in* n *unknowns* $x_1, \cdots, x_n$

(1)
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

is **consistent**, *that is, has solutions, if and only if the coefficient matrix* **A** *and the augmented matrix* $\tilde{\mathbf{A}}$ *have the same rank. Here,*

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \quad \textit{and} \quad \tilde{\mathbf{A}} = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$$

(b) Uniqueness. *The system* (1) *has precisely one solution if and only if this common rank* r *of* **A** *and* $\tilde{\mathbf{A}}$ *equals* n.

(c) Infinitely many solutions. *If this common rank r is less than n, the system (1) has infinitely many solutions. All of these solutions are obtained by determining r suitable unknowns in terms of the remaining $n - r$ unknowns, to which arbitrary values can be assigned.*

(d) Gauss elimination *If solutions exist, they can all be obtained by the Gauss elimination.* (This method will automatically reveal whether or not solutions exist.

THEOREM 1.2.2 *A homogeneous linear system with more unknowns than equations has infinitely many solutions.*

(Because for homogeneous linear system Rank of the coefficient matrix is always equal to Rank of the augmented Matrix and if number of equations is less than unknowns this mean that common rank < n, so by third part of fundamental theorem, system will have infinite solutions.

Methods for solving a linear System

Gaussian Elimination (or Gauss Elimination)

Gaussian Elimination with Back-Substitution

1. Write the augmented matrix of the system of linear equations.
2. Use elementary row operations to rewrite the matrix in row-echelon form.
3. Write the system of linear equations corresponding to the matrix in row-echelon form, and use back-substitution to find the solution.

Problem

Solve the system.

$$\begin{aligned} x_2 + x_3 - 2x_4 &= -3 \\ x_1 + 2x_2 - x_3 &= 2 \\ 2x_1 + 4x_2 + x_3 - 3x_4 &= -2 \\ x_1 - 4x_2 - 7x_3 - x_4 &= -19 \end{aligned}$$

SOLUTION The augmented matrix for this system is

$$\left[\begin{array}{rrrr|r} 0 & 1 & 1 & -2 & -3 \\ 1 & 2 & -1 & 0 & 2 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{array}\right].$$

$$\left[\begin{array}{rrrr|r} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{array}\right] \qquad R_1 \leftrightarrow R_2$$

$$\left[\begin{array}{rrrr|r} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 1 & -4 & -7 & -1 & -19 \end{array}\right] \qquad R_3 + (-2)R_1$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 0 & -6 & -6 & -1 & -21 \end{array}\right] \qquad R_4 + (-1)R_1$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 0 & 0 & 0 & -13 & -39 \end{array}\right] \qquad R_4 + (6)R_2$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & -13 & -39 \end{array}\right] \qquad \left(\tfrac{1}{3}\right)R_3$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \qquad \left(-\tfrac{1}{13}\right)R_4$$

The matrix is now in row-echelon form, and the corresponding system of linear equations is as shown below.

$$\begin{aligned} x_1 + 2x_2 - x_3 \phantom{{}-2x_4} &= 2 \\ x_2 + x_3 - 2x_4 &= -3 \\ x_3 - x_4 &= -2 \\ x_4 &= 3 \end{aligned}$$

Using back-substitution, you can determine that the solution is

$$x_1 = -1, \qquad x_2 = 2, \qquad x_3 = 1, \qquad x_4 = 3.$$

Gauss-Jordan Elimination

GAUSS-JORDAN ELIMINATION

With Gaussian elimination, you apply elementary row operations to a matrix to obtain a (row-equivalent) row-echelon form. A second method of elimination, called **Gauss-Jordan elimination** after Carl Friedrich Gauss and Wilhelm Jordan (1842–1899), continues the reduction process until a *reduced* row-echelon form is obtained.

Use Gauss-Jordan elimination to solve the system.

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y \phantom{{}+3z} &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

Associated Augmented Matrix

$$\left[\begin{array}{rrr|r} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right]$$

Add the first row to the second row to produce a new second row.

$$\left[\begin{array}{rrr|r} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 2 & -5 & 5 & 17 \end{array}\right] \quad R_2 + R_1 \rightarrow R_2$$

Add -2 times the first row to the third row to produce a new third row.

$$\left[\begin{array}{rrr|r} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{array}\right] \quad R_3 + (-2)R_1 \rightarrow R_3$$

Add the second row to the third row to produce a new third row.

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{array}\right] \quad R_3 + R_2 \rightarrow R_3$$

Multiply the third row by $\frac{1}{2}$ to produce a new third row.

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad \left(\tfrac{1}{2}\right)R_3 \rightarrow R_3$$

Now, apply elementary row operations until you obtain zeros above each of the leading 1's, as shown below.

$$\left[\begin{array}{ccc|c} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right] \qquad R_1 + (2)R_2 \to R_1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 9 & 19 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \qquad R_2 + (-3)R_3 \to R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \qquad R_1 + (-9)R_3 \to R_1$$

The matrix is now in reduced row-echelon form. Converting back to a system of linear equations, you have

$$\begin{aligned} x &= 1 \\ y &= -1 \\ z &= 2. \end{aligned}$$

Problem

Solve by Gauss–Jordan elimination.

$$\begin{aligned}
x_1 + 3x_2 - 2x_3 \qquad\quad + 2x_5 \qquad\quad &= 0\\
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1\\
5x_3 + 10x_4 \qquad\quad + 15x_6 &= 5\\
2x_1 + 6x_2 \qquad\quad + 8x_4 + 4x_5 + 18x_6 &= 6
\end{aligned}$$

Solution The augmented matrix for the system is

$$\left[\begin{array}{cccccc|c}
1 & 3 & -2 & 0 & 2 & 0 & 0\\
2 & 6 & -5 & -2 & 4 & -3 & -1\\
0 & 0 & 5 & 10 & 0 & 15 & 5\\
2 & 6 & 0 & 8 & 4 & 18 & 6
\end{array}\right]$$

If we reduced the augmented matrix to reduce echelon form we have

$$\left[\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} x_1 + 3x_2 \qquad + 4x_4 + 2x_5 \qquad &= 0 \\ x_3 + 2x_4 \qquad &= 0 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

Solving for the leading variables, we obtain

$$\begin{aligned} x_1 &= -3x_2 - 4x_4 - 2x_5 \\ x_3 &= -2x_4 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

THEOREM 3.11 Cramer's Rule

If a system of n linear equations in n variables has a coefficient matrix A with a nonzero determinant $|A|$, then the solution of the system is

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad . \; . \; ., \quad x_n = \frac{\det(A_n)}{\det(A)}$$

where the ith column of A_i is the column of constants in the system of equations.

Problem

Use Cramer's Rule to solve the system of linear equations for x.

$$\begin{aligned} -x + 2y - 3z &= 1 \\ 2x \qquad\quad + z &= 0 \\ 3x - 4y + 4z &= 2 \end{aligned}$$

SOLUTION

The determinant of the coefficient matrix is $|A| = \begin{vmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{vmatrix} = 10.$

The determinant is nonzero, so you know that the solution is unique. Apply Cramer's Rule to solve for x, as shown below.

$$x = \frac{\begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 2 & -4 & 4 \end{vmatrix}}{10} = \frac{(1)(-1)^5\begin{vmatrix} 1 & 2 \\ 2 & -4 \end{vmatrix}}{10} = \frac{(1)(-1)(-8)}{10} = \frac{4}{5}$$

Solving Linear Systems by Matrix Inversion

THEOREM 1.6.2 *If A is an invertible $n \times n$ matrix, then for each $n \times 1$ matrix $\mathbf{b}$, the system of equations $A\mathbf{x} = \mathbf{b}$ has exactly one solution, namely, $\mathbf{x} = A^{-1}\mathbf{b}$.*

e.g.

Consider the system of linear equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 5 \\ 2x_1 + 5x_2 + 3x_3 &= 3 \\ x_1 + 8x_3 &= 17 \end{aligned}$$

In matrix form this system can be written as $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

or $x_1 = 1, x_2 = -1, x_3 = 2$.