Lecture 10: Continuous Random Variables

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Random variables

Recall that a *random variable* is a numerical outcome of an experiment.

Last time we looked at *discrete random variables*. X is *discrete* if its support is a discrete set.

Discrete sets:

- $\{0, 1, 2, \ldots, n\}$
- $\mathbb{N} = \{1, 2, 3, \ldots\}$
- $\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$
- $\{\text{dog}, \text{cat}, \text{fish}\}$
- The set of all animals in existence.

Continuous sets:

- $[0, 1] = \{x : 0 \leq x \leq 1\}$
- $(-12, 67] = \{x : -12 < x \leq 67\}$
- $\mathbb{R} = (-\infty, \infty)$
- The waiting time for a bus at North Hall.

Continuous Random Variables

Suppose a bus arrives every 10 minutes. You arrival at the bus stop at a random time. How long will you have to wait?

Let X be the time you must wait for the next bus.

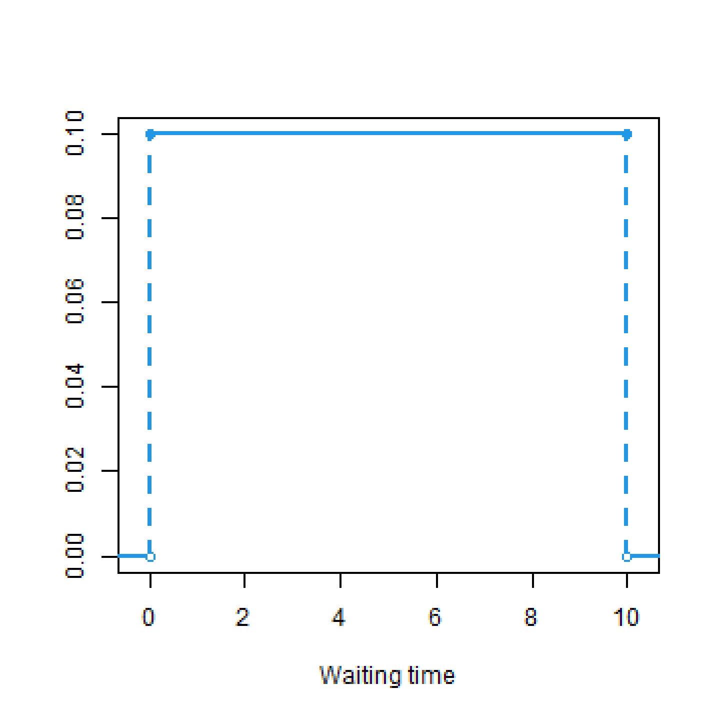
X is a continuous r.v. with support $(0, 10)$.

What is the **distribution** of X? This depends on some assumptions.

Uniform distribution

Suppose you are equally likely to arrive at any time. Draw pic

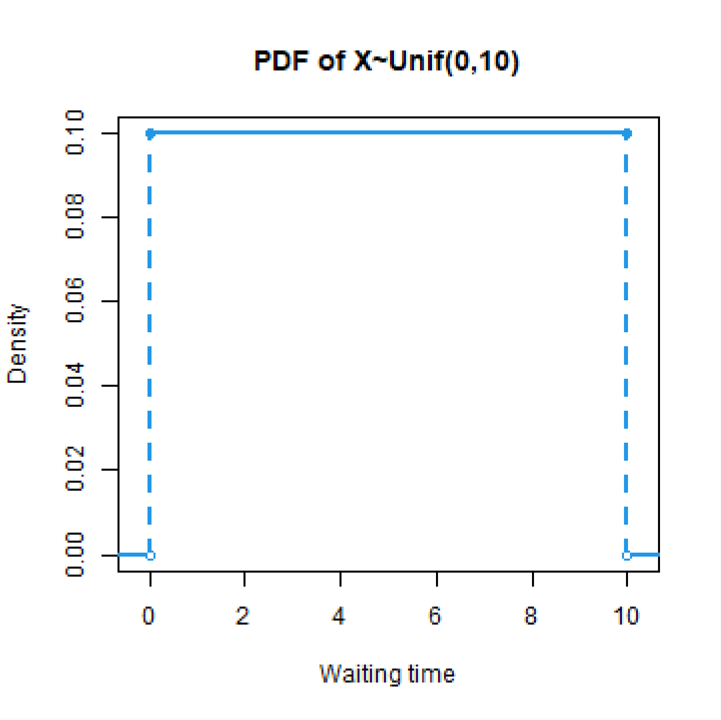
A plot of the "likelihood" might look like this:



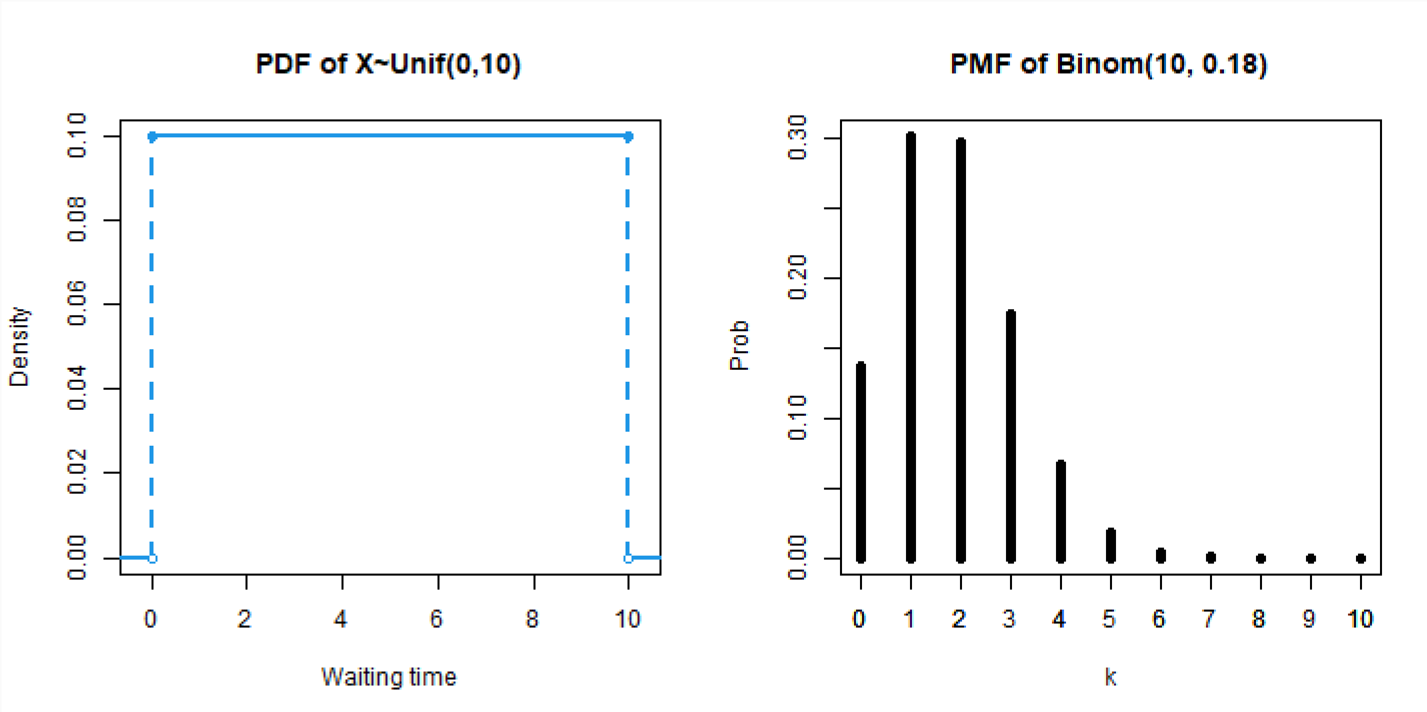
Uniform distribution

What we have plotted is the *probability density function (pdf)* of the **uniform distribution**.

Write $X \sim \text{Unif}(0, 10)$. The parameters specify the lower and upper bound of the support.



Uniform distribution



Unlike discrete r.v.s, **the y-axis does not give probabilities**.

The y-axis is the **density**, not the *mass* (p**d**f vs p**m**f).

Uniform distribution

The builtin R functions are `dunif`, `punif`, and `runif`. By default, `runif` generates observations from $\mathrm{Unif}(0, 1)$:

```r
runif(5)
```

```
## [1] 0.37219838 0.04382482 0.70968402 0.65769040 0.24985572
```

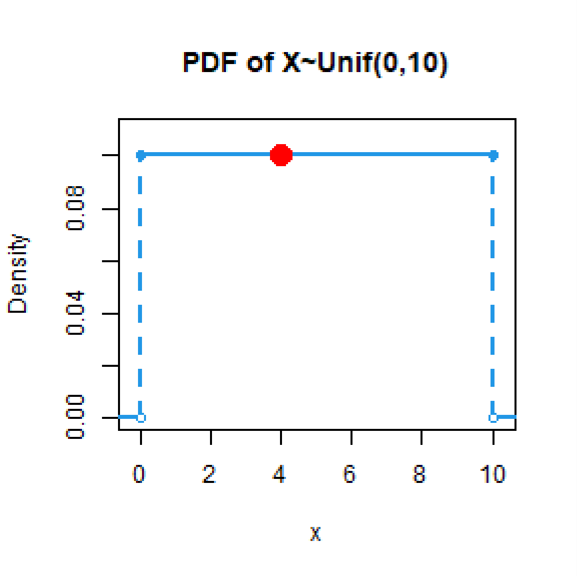
Generate waiting times for $X \sim \mathrm{Binom}(0, 10)$:

```r
runif(5, min = 0, max = 10)
```

```
## [1] 3.000548 5.848666 3.334671 6.220120 5.458286
```

Uniform distribution

`dunif` gives the **values of the density function**.




```
dunif(4, min = 0, max = 10)
```

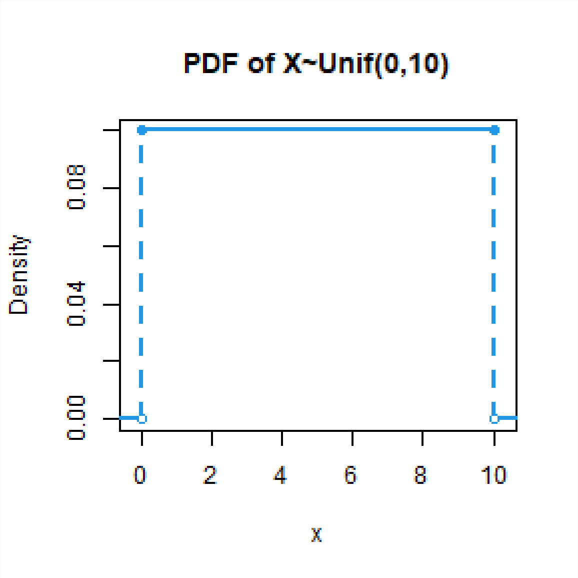
```
## [1] 0.1
```

ACHTUNG!

This **does not** say $\mathbb{P}(X = 4) = 0.1$.

Uniform distribution

Computing probabilities

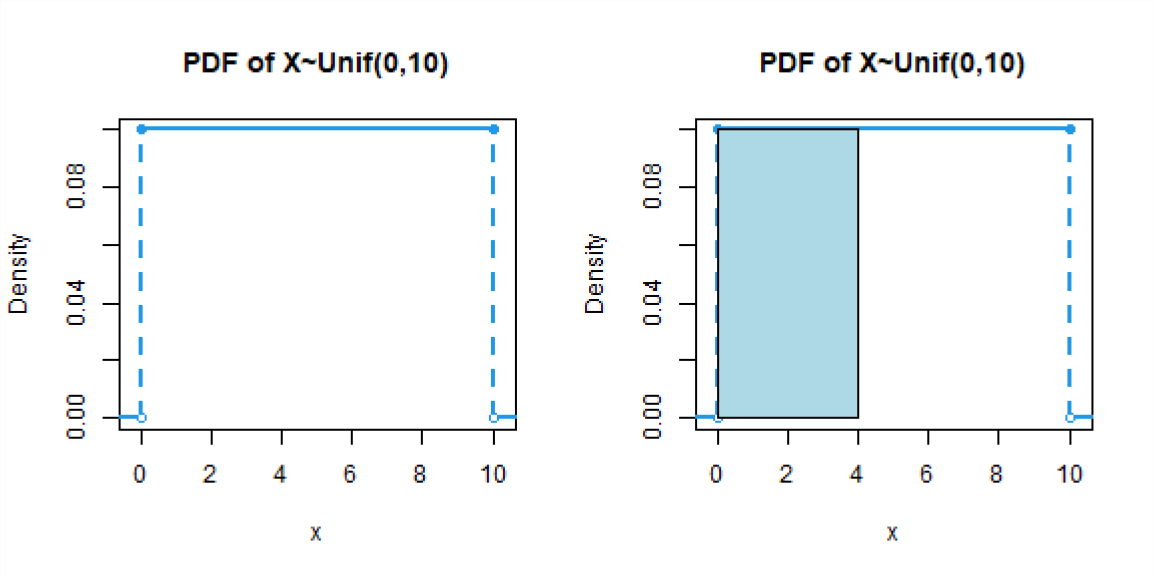


Note that the **area under the curve** equals 1.

Unlike discrete r.v.s, for continuous r.v.s we must compute the area under curve to find the probabilities.

What is $\mathbb{P}(X \leq 4)$?

Uniform distribution

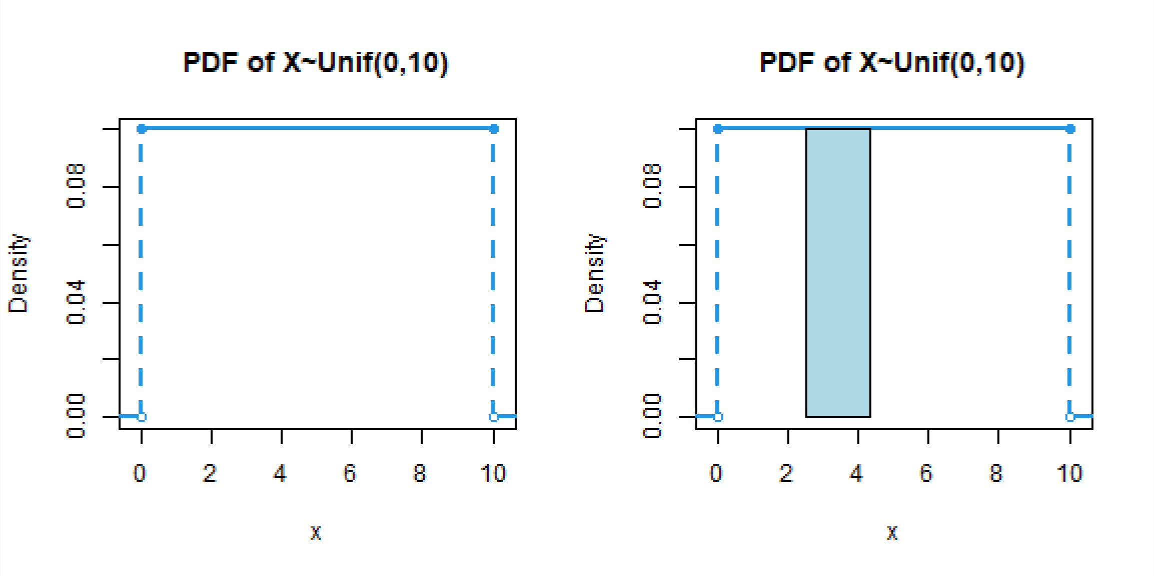


area of shaded region $= 0.10 \times 4 = 0.4$

Hence: $\mathbb{P}(X \leq 4) = 0.4$

Uniform distribution

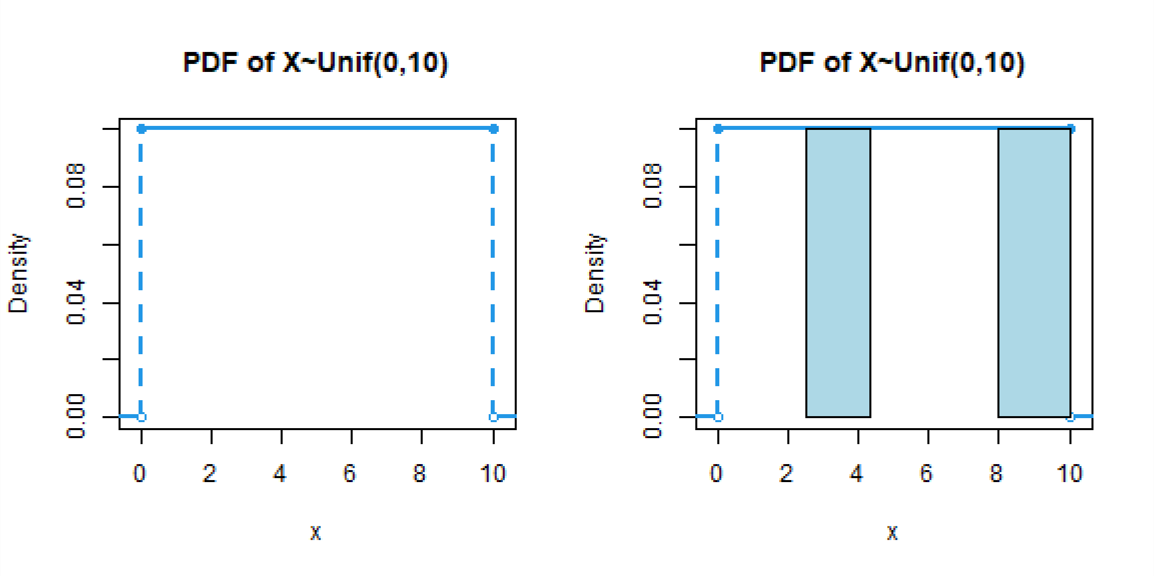
Compute $\mathbb{P}(2.5 \leq X \leq 4.3)$.



$$\mathbb{P}(2.5 \leq X \leq 4.3) = \text{area of shaded region} = 0.10 \times (4.3 - 2.5) = 0.18$$

Uniform distribution

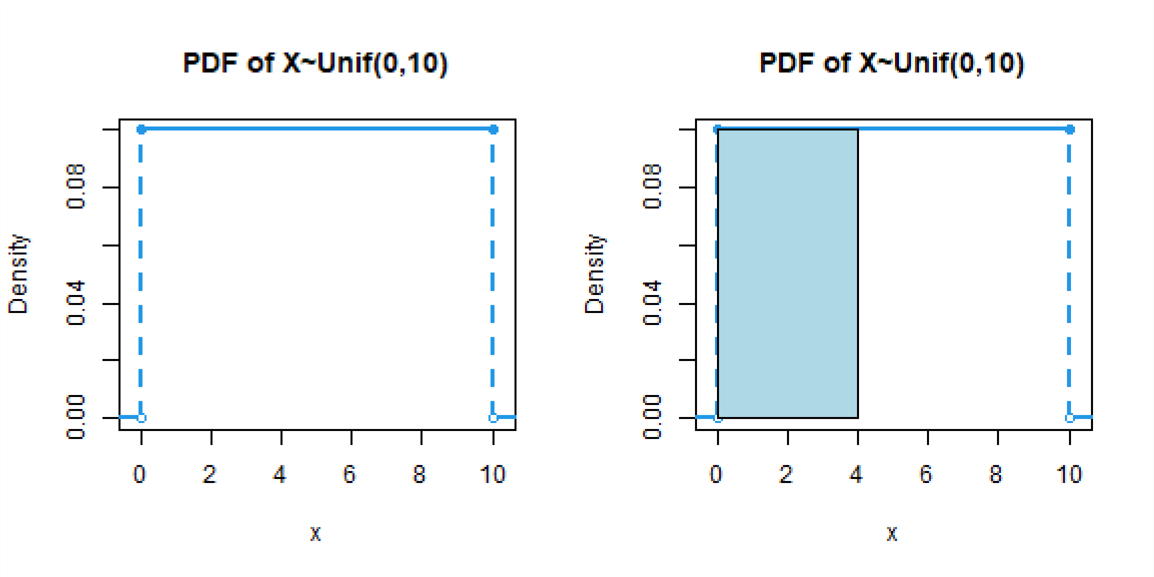
Compute $\mathbb{P}(2.5 \leq X \leq 4.3 \textbf{ OR } X > 8)$.



area of shaded region $= 0.10 \times (4.3 - 2.5) + 0.10 \times (10 - 8) = 0.38$

Since the total area under the curve equals 1, all probabilities are between 0 and 1, which is good.

Uniform distribution



Recall that *cumulative probabilities* are of the form $\mathbb{P}(X \leq k)$.

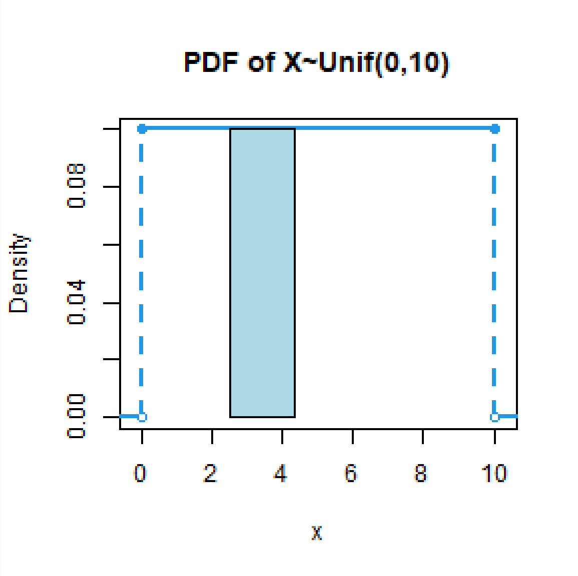
`punif` gives the cumulative probabilities.

```
punif(4, min = 0, max = 10)
```

```
## [1] 0.4
```

Uniform distribution

Compute $\mathbb{P}(2.5 \leq X \leq 4.3)$ using `punif`.

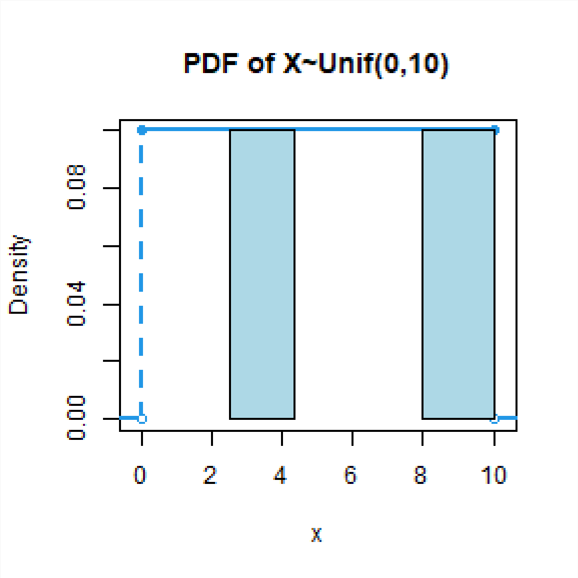



```
punif(4.3, min = 0, max = 10) - punif(2.5, min = 0, max = 10)
```

```
## [1] 0.18
```

Uniform distribution

Compute $\mathbb{P}(2.5 \leq X \leq 4.3 \textbf{ OR } X > 8)$ with `punif`.



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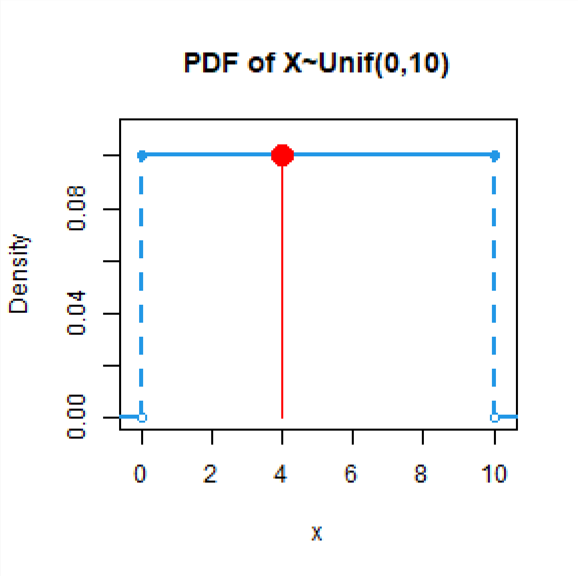
Uniform distribution

So what is $\mathbb{P}(X = 4)$?



Probabilities are areas under the curve. But the "area under" the curve at the point 4 is a one-dimensional line with no area.

The probability a continuous r.v. takes on a single point is ZERO.

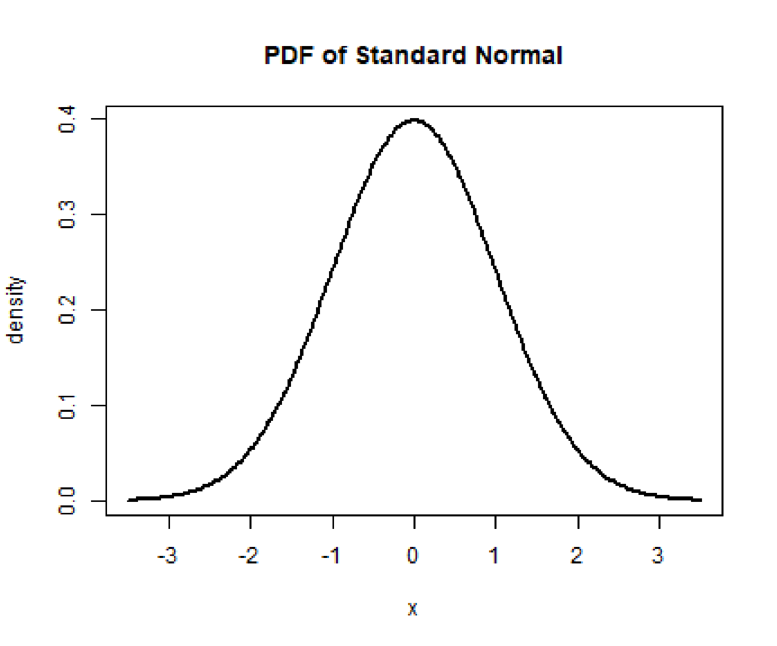
What does this say about $\mathbb{P}(X \leq 4)$ vs. $\mathbb{P}(X < 4)$?

Normal distribution

The *most important* distribution.

Many continuous r.v.s are (approximately) normally distributed:

- heights of people
- weights of similar animals
- measurements of items produced in a factory



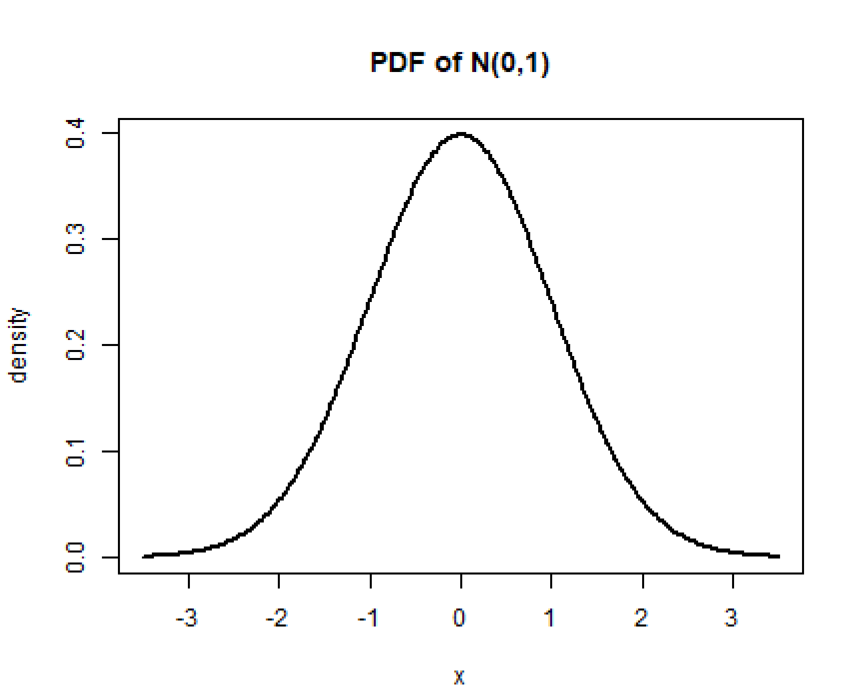
Normal distribution

The normal distribution has two parameters:

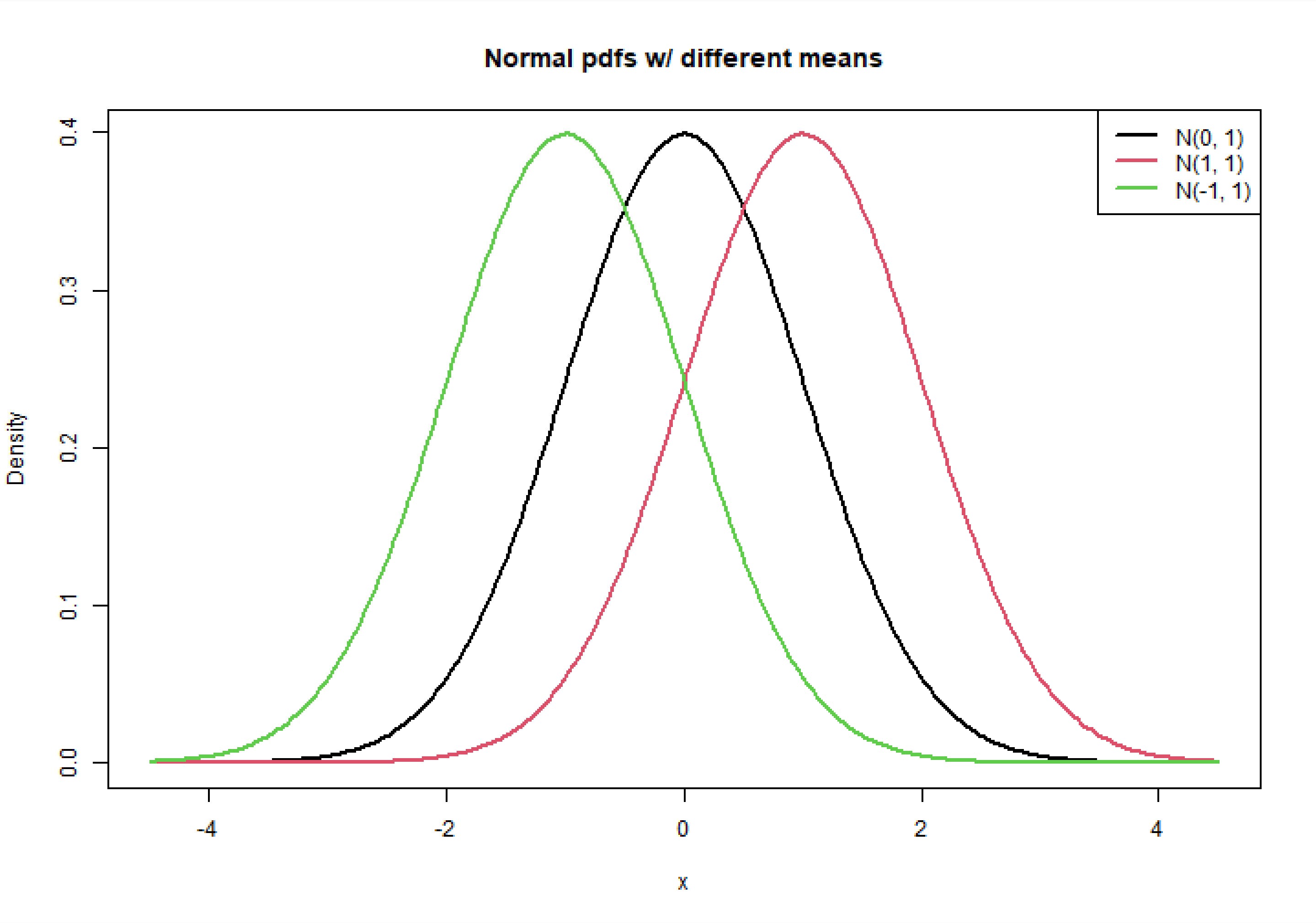
- the **mean** μ
- the **standard deviation** σ

Write $X \sim N(\mu, \sigma)$.

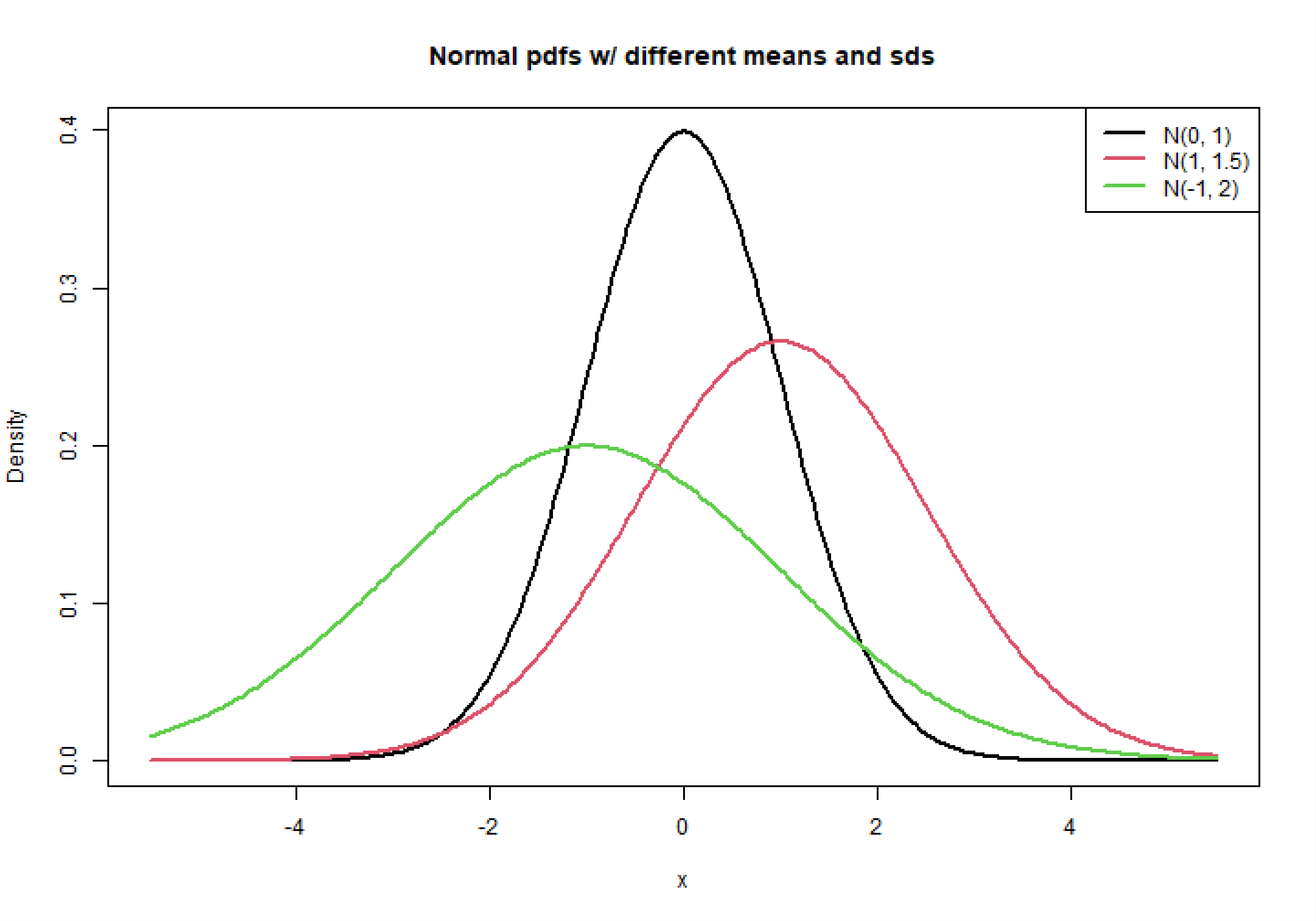
$N(0, 1)$ is called the **standard normal**.



Normal distribution



Normal distribution



Normal distribution

In R

binomial distribution $\text{Binom}(\text{size}, \text{prob})$

- `dbinom(x, size, prob)`
- `pbinom(q, size, prob)`
- `rbinom(n, size, prob)`

uniform distribution $\text{Unif}(\text{min}, \text{max})$

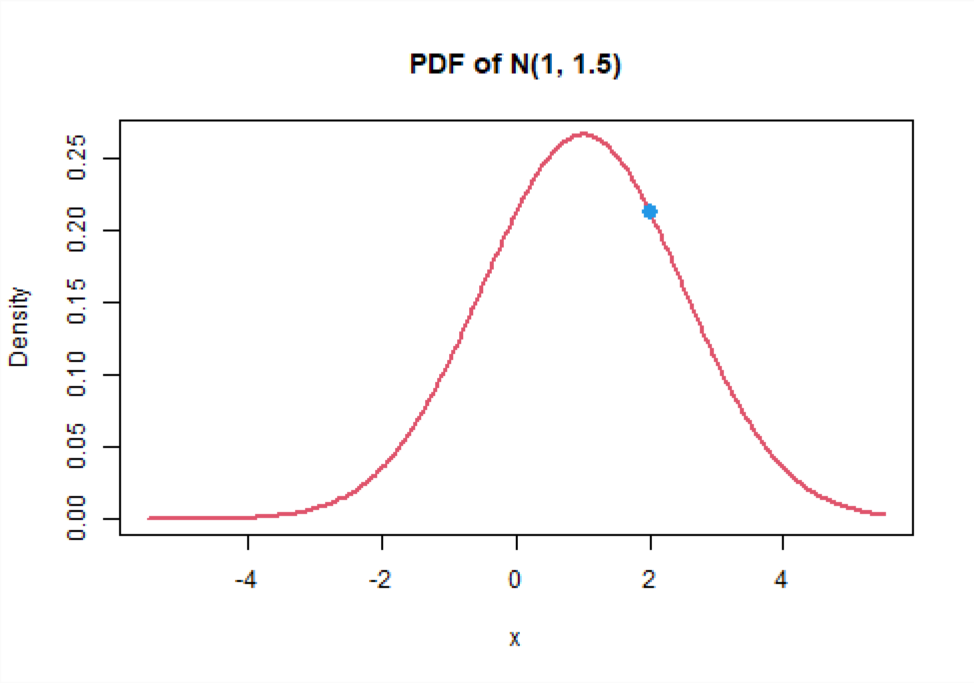
- `dunif(x, min, max)`
- `punif(q, min, max)`
- `runif(n, min, max)`

Can you guess the functions for the normal distribution?

normal distribution $N(\text{mean}, \text{sd})$

- `dnorm(x, mean, sd)`
- `pnorm(q, mean, sd)`
- `rnorm(n, mean, sd)`

Normal distribution

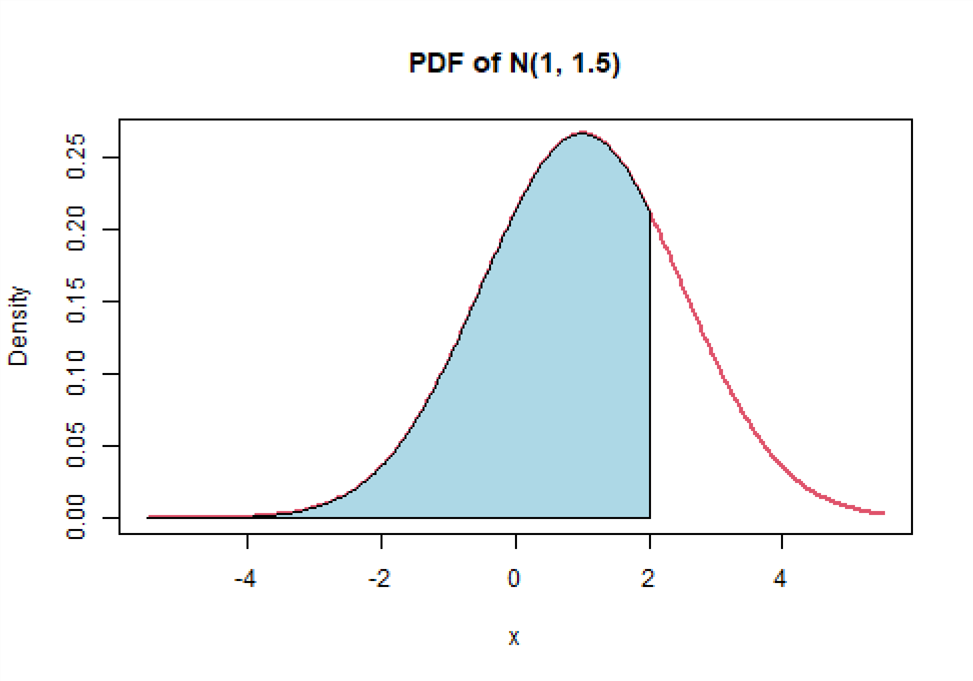



```r
dnorm(2, mean = 1, sd = 1.5)
```

```
## [1] 0.2129653
```

`dnorm` gives the *values of the normal pdf*

Normal distribution

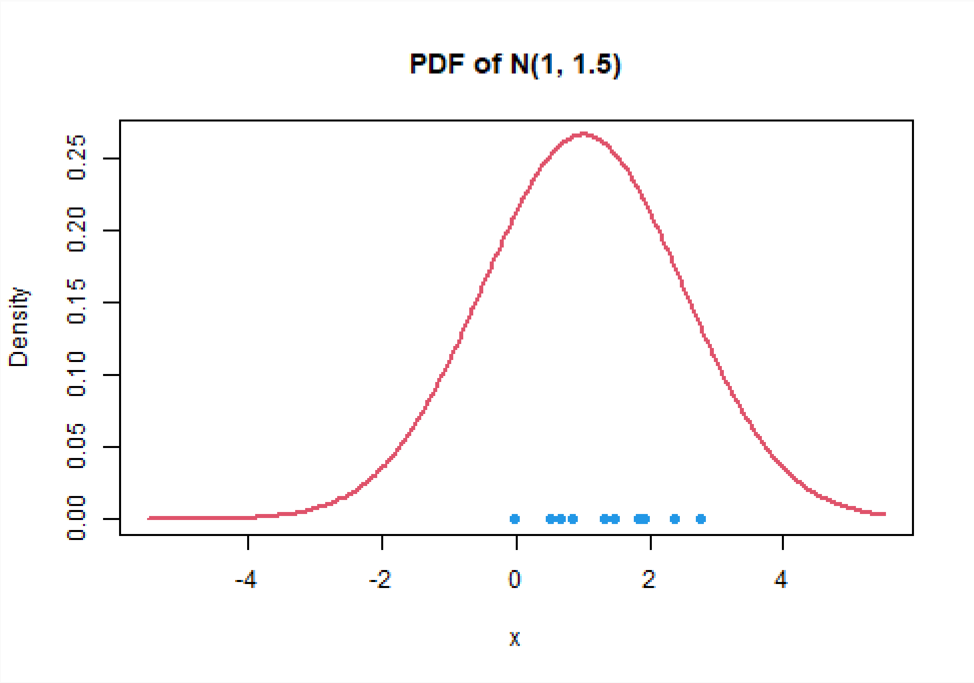



```r
pnorm(2, mean = 1, sd = 1.5)
```

```
## [1] 0.7475075
```

`pnorm` gives the *cdf* $\mathbb{P}(X \leq 2)$

Normal distribution



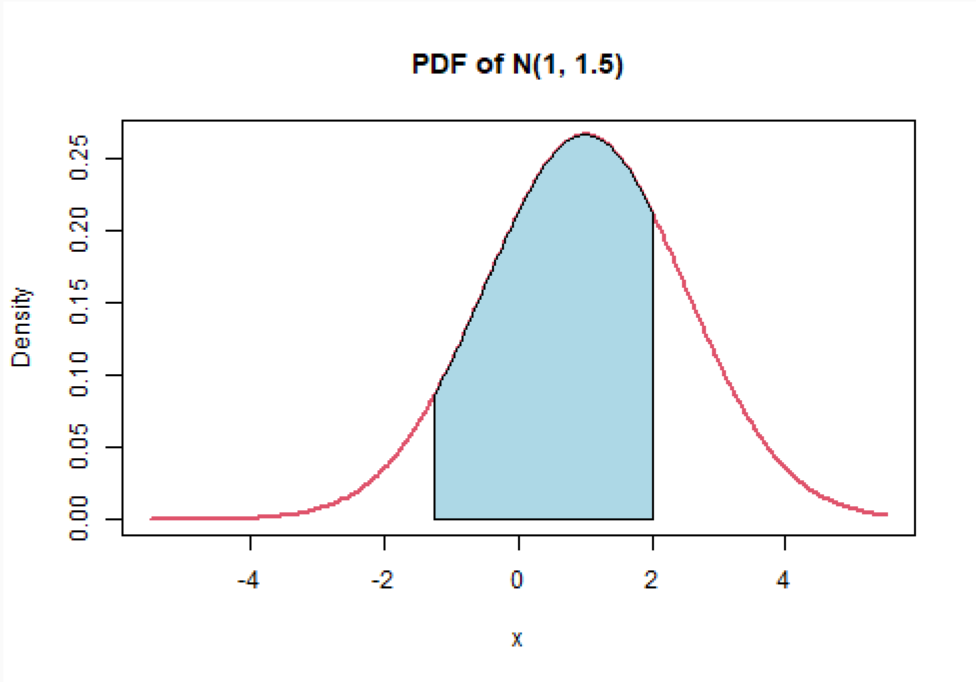

```r
set.seed(101)
rnorm(10, mean = 1, sd = 1.5)
```

```
##  [1]  0.51094526  1.82869278 -0.01241577  1.32153919  1.46615383  2.76094943
##  [7]  1.92818478  0.83089853  2.37554243  0.66511095
```

`rnorm` generates normal random variates (or observations)

Normal distribution

Let $X \sim N(1, 1.5)$. Compute $\mathbb{P}(-1.25 < X \leq 2)$



Method 1: Solve

$$\int_{-1.25}^{2} \frac{1}{1.5\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-1}{1.5}\right)^2} \mathrm{d}x$$

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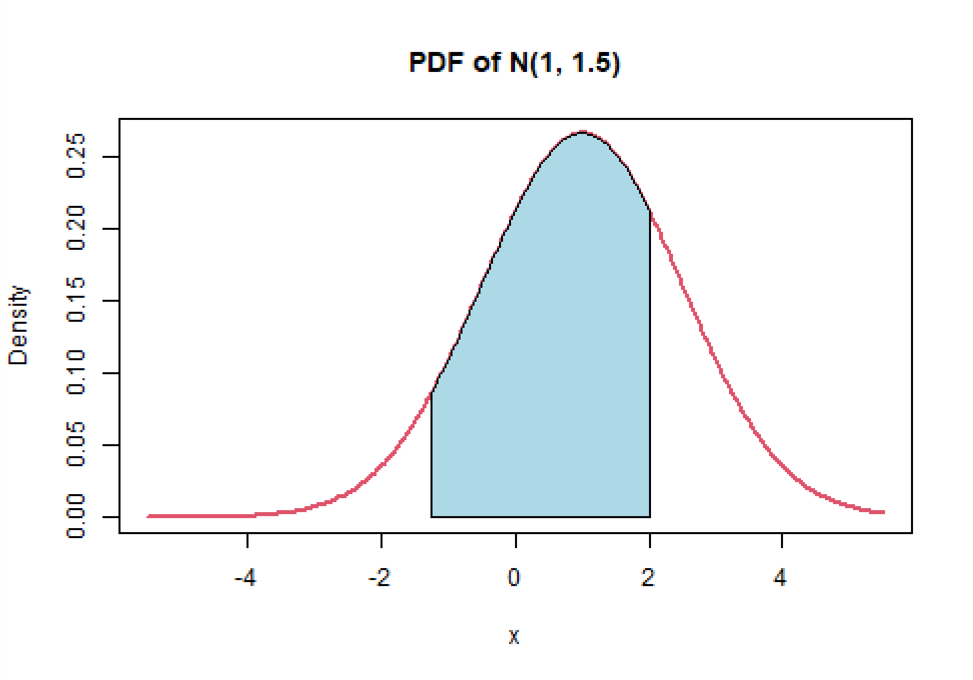
Normal distribution

Let $X \sim N(1, 1.5)$. Compute $\mathbb{P}(-1.25 < X \leq 2)$



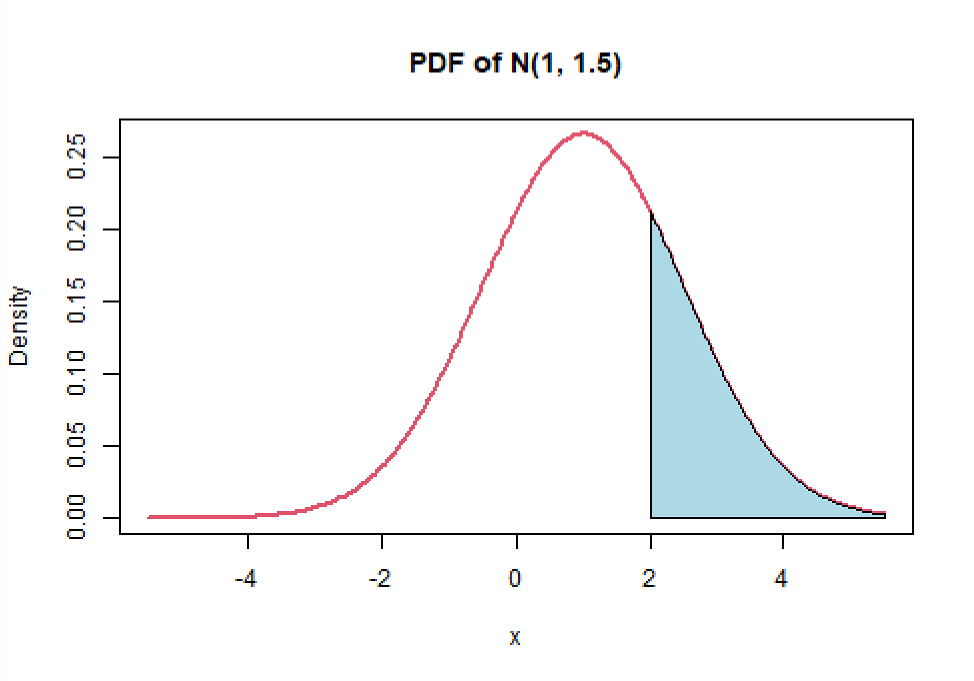
Method 2: `pnorm`

```
pnorm(2, mean = 1, sd = 1.5) - pnorm(-1.25, mean = 1, sd = 1.5)
```

```
## [1] 0.6807003
```

Normal distribution

Let $X \sim N(1, 1.5)$. Compute $\mathbb{P}(X \geq 2)$



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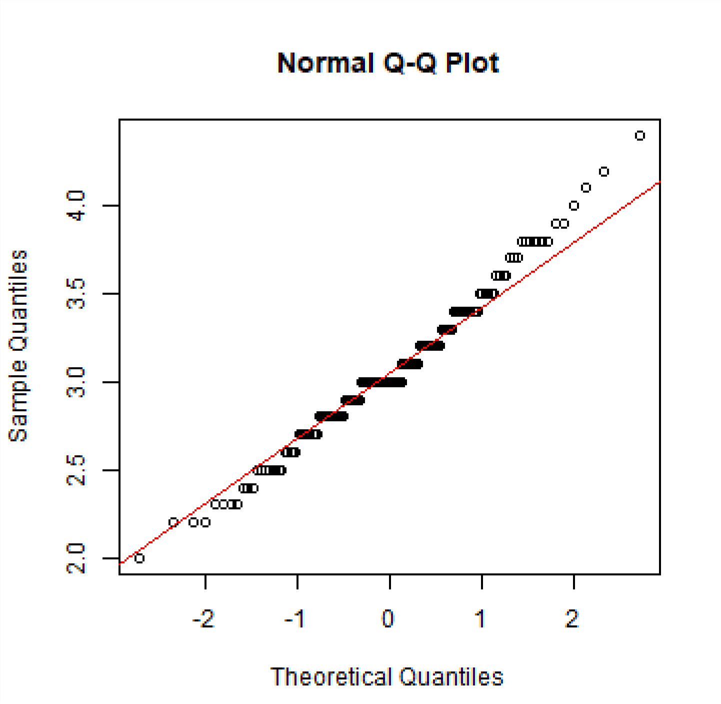
Assessing normality

Is our data normally distributed?

A visual way to check this is by using a *Q-Q plot* (quantile-quantile plot). The points should lie along the line.

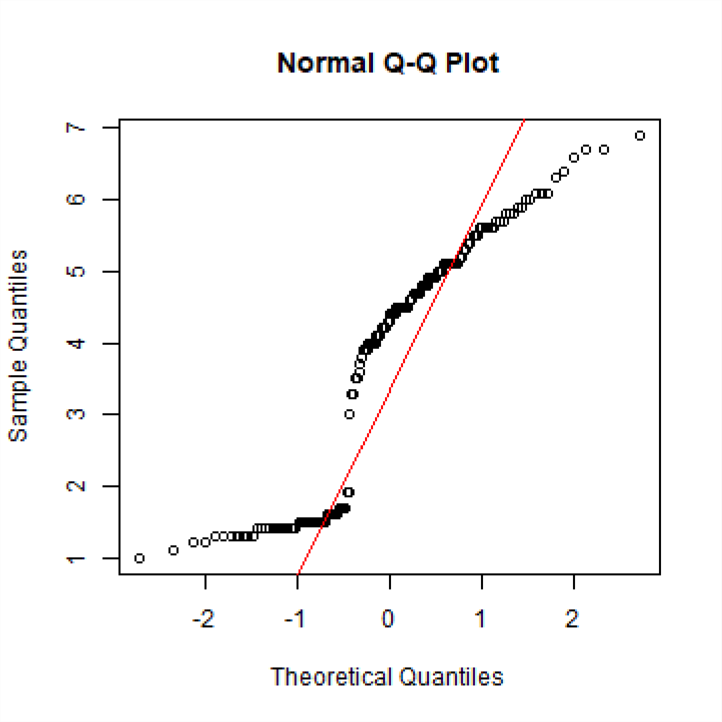
```
qqnorm(iris$Sepal.Width)
qqline(iris$Sepal.Width, col = "red")
```



Assessing normality

Is our data normally distributed?

```
qqnorm(iris$Petal.Length)
qqline(iris$Petal.Length, col = "red")
```



This is a visual, heuristic way to check normality, but sometimes its the best we've got.