Proof Theory of Modal Logic

Lecture : Nested Sequents

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Recap

Nested sequents for the S5-cube:
Soundness

Independently introduced in:

- [Bull, 1992]; [Kashima, 1994] $\rightsquigarrow$ *nested seuquents*
- [Brünnler, 2006], [Brünnler, 2009] $\rightsquigarrow$ *deep sequents*
- [Poggiolesi, 2008], [Poggiolesi, 2010] $\rightsquigarrow$ *tree-hypersequents*

Main references for this lecture:

- [Lellmann & Poggiolesi, 2022 (arXiv)]
- [Brünnler, 2009], [Brünnler, 2010 (arXiv)]
- [Marin & Straßburger, 2014]

One-sided sequents

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Rules of G3cp^{one}

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Exercise. $\vdash_{\mathrm{G3cp}} \Gamma \Rightarrow \Delta$ iff $\vdash_{\mathrm{G3cp}^{one}} \overline{\Gamma}, \Delta$, where $\overline{\Gamma} = \{\overline{A} \mid A \in \Gamma\}$.

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  We call $[\Gamma]$ a boxed sequent.

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Nested sequents are multisets of formulas and boxed sequents:

$$A_1, \ldots, A_m, [\Delta_1], \ldots, [\Delta_n]$$

$$\Gamma = A_1, \ldots, A_m, [\Delta_1], \ldots, [\Delta_n]$$

To a nested sequent Γ there corresponds the following tree $tr(\Gamma)$, whose nodes $\gamma, \delta, \ldots$ are multisets of formulas:

The formula interpretation $i(\Gamma)$ of a nested sequent Γ is defined as:

- If $m = n = 0$, then $i(\Gamma) := \bot$
- Otherwise, $i(\Gamma) := A_1 \vee \cdots \vee A_m \vee \Box(i(\Delta_1)) \vee \cdots \vee \Box(i(\Delta_n))$

Examples

A context is a nested sequent with one or multiple holes, denoted by $\{\,\}$, each taking the place of a formula in the nested sequent.

- Unary context $\Gamma\{\,\}$
- Binary context $\Gamma\{\,\}\{\,\}$

The depth $\mathit{depth}(\Gamma\{\,\})$ of a unary context $\Gamma\{\,\}$ is defined as:

- $\mathit{depth}(\{\,\}) := 0$;
- $\mathit{depth}(\Gamma\{\,\}, \Delta) := \mathit{depth}(\Gamma\{\,\})$;
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- Unary context $\Gamma\{\,\}$ $\rightsquigarrow$ $\Gamma\{\Delta\}$: filling $\Gamma\{\,\}$ with a nested sequent Δ
- Binary context $\Gamma\{\,\}\{\,\}$ $\rightsquigarrow$ $\Gamma\{\Delta_1\}\{\Delta_2\}$: filling $\Gamma\{\,\}\{\,\}$ with Δ_1, Δ_2

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Rules of NK

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Example. Proof of $(\Diamond p \to \Box q) \to \Box(p \to q)$ in NK

$$
\vee\ \cfrac{\Box\ \cfrac{\vee\ \cfrac{\wedge\ \cfrac{\Diamond\ \cfrac{\text{init}\ \cfrac{}{\Diamond p, [p, \bar{p}, q]}}{\Diamond p, [\bar{p}, q]} \qquad \Diamond\ \cfrac{\text{init}\ \cfrac{}{\Diamond \bar{q}, [\bar{q}, \bar{p}, q]}}{\Diamond \bar{q}, [\bar{p}, q]}}{\Diamond p \wedge \Diamond \bar{q}, [\bar{p}, q]}}{\Diamond p \wedge \Diamond \bar{q}, [\bar{p} \vee q]}}{\Diamond p \wedge \Diamond \bar{q}, \Box(\bar{p} \vee q)}}{(\Diamond p \wedge \Diamond \bar{q}) \vee \Box(\bar{p} \vee q)}
$$

Roadmap

For a nested sequent Γ and a model $\mathcal{M} = \langle W, R, v \rangle$, an $\mathcal{M}$-map for Γ is a map $f : tr(\Gamma) \to W$ such that whenever δ is a child of γ in $tr(\Gamma)$, then $f(\gamma) R f(\delta)$.

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$$\mathcal{M}, f(\delta) \models B, \text{ for some } \delta \in tr(\Gamma), \text{ for some } B \in \delta$$

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For $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$, a nested sequent is X-valid iff it is satisfied by all $\mathcal{M}$-map for Γ, for all models $\mathcal{M}$ satisfying the frame conditions in X.

Lemma. If Γ is derivable in NK then $\bigvee \Gamma$ is valid in all Kripke frames.

Rules for extensions: $\mathsf{NK} \cup \mathsf{X}^{\Diamond}$

$$\mathsf{d}^{\Diamond}\ \frac{\Gamma\{\Diamond A, [A]\}}{\Gamma\{\Diamond A\}} \qquad \mathsf{t}^{\Diamond}\ \frac{\Gamma\{\Diamond A, A\}}{\Gamma\{\Diamond A\}} \qquad \mathsf{b}^{\Diamond}\ \frac{\Gamma\{[\Delta, \Diamond A], A\}}{\Gamma\{[\Delta, \Diamond A]\}}$$

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Example. Proof of $\Box p \to \Box\Box p$ in $\mathsf{NK} \cup \{\mathsf{t}, 4\}$

$$
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$$

Structural rules [Brünner, 2009]

$$\mathsf{wk}\,\frac{\Gamma\{\emptyset\}}{\Gamma\{\Delta\}} \qquad \mathsf{ctr}\,\frac{\Gamma\{\Delta,\Delta\}}{\Gamma\{\Delta\}} \qquad \mathsf{cut}\,\frac{\Gamma\{A\} \quad \Gamma\{\overline{A}\}}{\Gamma\{\emptyset\}}$$

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Lemma. The rules wk and ctr are hp-admissible in $\mathsf{NK} \cup \mathsf{X}^{\diamond}$.

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Nested sequents with explicit contraction [Marin & Straßburger, 2014]

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For $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$, we write $\mathsf{X}^{\Diamond}_{\mathsf{ctr}}$ for the corresponding subset of $\{\mathsf{d}^{\Diamond}_{\mathsf{ctr}}, \mathsf{t}^{\Diamond}_{\mathsf{ctr}}, \mathsf{b}^{\Diamond}_{\mathsf{ctr}}, 4^{\Diamond}_{\mathsf{ctr}}, 5^{\Diamond}_{\mathsf{ctr}}\}$.

Lemma. The rule wk is hp-admissible in $\mathsf{NK} \cup \mathsf{X}^{\Diamond}$.

Proposition. Γ is derivable in $\mathsf{NK} \cup \mathsf{X}^{\Diamond}$ iff Γ is derivable in $\mathsf{NK}_{\mathsf{ctr}} \cup \mathsf{X}^{\Diamond}_{\mathsf{ctr}}$.