Proof Theory of Modal Logic

Lecture 4 ^{15}: Nested Sequents

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Nested sequents for the S5-cube: Soundness

Nested sequents in the literature

Independently introduced in:

- ▹ [Bull, 1992]; [Kashima, 1994] $\rightsquigarrow$ *nested se~~u~~quents*
- ▹ [Brünnler, 2006], [Brünnler, 2009] $\rightsquigarrow$ *deep sequents*
- ▹ [Poggiolesi, 2008], [Poggiolesi, 2010] $\rightsquigarrow$ *tree-hypersequents*

Main references for this lecture:

- ▹ [Lellmann & Poggiolesi, 2022 (arXiv)]
- ▹ [Brünnler, 2009], [Brünnler, 2010 (arXiv)]
- ▹ [Marin & Straßburger, 2014]

Sequent	$\Gamma \Rightarrow \Delta$	Γ, Δ multisets of formulas
One-sided sequent	Γ	Γ multiset of formulas

$$A, B ::= p \mid \overline{p} \mid A \wedge B \mid A \vee B$$

$$\overline{A \wedge B} := \overline{A} \vee \overline{B} \qquad \overline{A \vee B} := \overline{A} \wedge \overline{B}$$

$$A \to B := \overline{A} \vee B \qquad \bot := p \wedge \overline{p}$$

Rules of G3cp^{one}

$$\text{init}\ \frac{}{\Gamma, p, \overline{p}} \qquad \wedge\ \frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \qquad \vee\ \frac{\Gamma, A, B}{\Gamma, A \vee B}$$

Exercise. $\vdash_{\mathrm{G3cp}} \Gamma \Rightarrow \Delta$ iff $\vdash_{\mathrm{G3cp}^{one}} \overline{\Gamma}, \Delta$, where $\overline{\Gamma} = \{\overline{A} \mid A \in \Gamma\}$.

Nested sequents for modal logic

$$A, B ::= p \mid \overline{p} \mid A \wedge B \mid A \vee B \mid \Box A \mid \Diamond A$$

$$\overline{A \wedge B} := \overline{A} \vee \overline{B} \qquad \overline{A \vee B} := \overline{A} \wedge \overline{B} \qquad \overline{\Box A} := \Diamond \overline{A} \qquad \overline{\Diamond A} := \Box \overline{A}$$

$$A \rightarrow B := \overline{A} \vee B \qquad \bot := p \wedge \overline{p}$$

Nested sequents (denoted $\Gamma, \Delta, \ldots$) are inductively generated as follows:

- A multiset of formulas is a nested sequent;
- If Γ and Δ are nested sequents, then Γ, Δ is a nested sequent;
- If Γ is a nested sequent, then $[\Gamma]$ is a nested sequent.
  We call $[\Gamma]$ a boxed sequent.

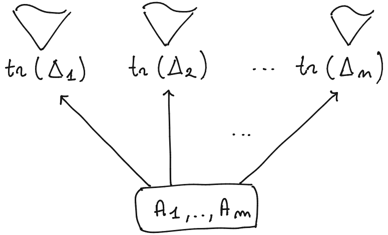
Nested sequents are multisets of formulas and boxed sequents:

$$\underbrace{A_1, \ldots, A_m}_{}, [\Delta_1], \ldots, [\Delta_n]$$

$$\Gamma = A_1, \ldots, A_m, [\Delta_1], \ldots, [\Delta_n]$$

To a nested sequent Γ there corresponds the following tree $tr(\Gamma)$, whose nodes $\gamma, \delta, \ldots$ are multisets of formulas:

$tr(\Gamma) :=$



The formula interpretation $i(\Gamma)$ of a nested sequent Γ is defined as:

- If $m = n = 0$, then $i(\Gamma) := \bot$
- Otherwise, $i(\Gamma) := A_1 \vee \cdots \vee A_m \vee \Box(i(\Delta_1)) \vee \cdots \vee \Box(i(\Delta_n))$

Examples

Examples

1) $\Gamma = A, [B_1, B_2]$

what is $tr(\Gamma)$?

a)
$$\begin{tikzcd} B_1, B_2 \\ A \arrow[u] \end{tikzcd}$$

b)
$$\begin{tikzcd} B_1 & & B_2 \\ & A \arrow[ul] \arrow[ur] & \end{tikzcd}$$

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$A, [B_1], [B_2]$

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2) $\Gamma = A, [B_1, B_2, [C], D]$

what is $tr(\Gamma)$?

a)
$$\begin{tikzcd} C \\ B_1, B_2, D \arrow[u] \\ A \arrow[u] \end{tikzcd}$$

b)
$$\begin{tikzcd} C & & D \\ & B_1, B_2 \arrow[ul] \arrow[ur] & \\ & A \arrow[u] & \end{tikzcd}$$

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3) $\Gamma = A, [B_1, B_2], [C, [D]]$

what is $tr(\Gamma)$?

a)
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b)
$$\begin{tikzcd} & & D \\ B_1, B_2 & & C \arrow[u] \\ & A \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Examples

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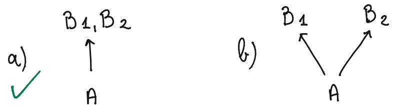

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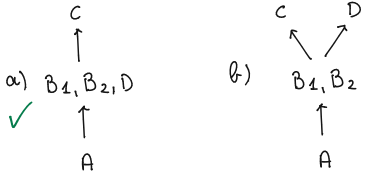

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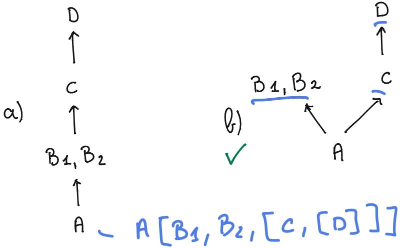

A context is a nested sequent with one or multiple holes, denoted by $\{\,\}$, each taking the place of a formula in the nested sequent.

- Unary context $\Gamma\{\,\}$
- Binary context $\Gamma\{\,\}\{\,\}$

The depth $\mathit{depth}(\Gamma\{\,\})$ of a unary context $\Gamma\{\,\}$ is defined as:

- $\mathit{depth}(\{\,\}) := 0$;
- $\mathit{depth}(\Gamma\{\,\}, \Delta) := \mathit{depth}(\Gamma\{\,\})$;
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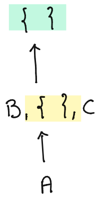

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Δ_1, Δ_2 nested sequents

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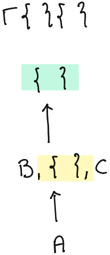

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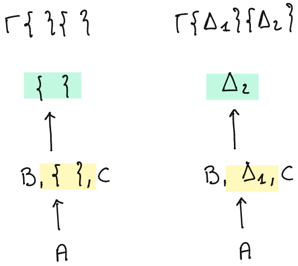

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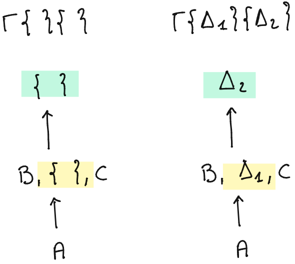
$\Gamma\{\Delta_1\}\{\emptyset\} = A, [B, \Delta_1, [\,], C]$



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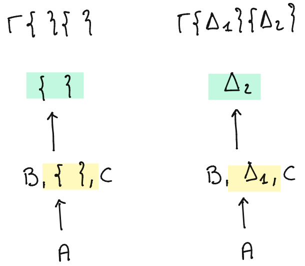

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$depth(\Gamma\{\,\}\{\Delta_1\}) = 1$

$depth(\Gamma\{\Delta_1\}\{\,\}) = 2$

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$$\Box\,\frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \Diamond\,\frac{\Gamma\{\Diamond A, [A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}$$

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Example. Proof of $(\Diamond p \to \Box q) \to \Box(p \to q)$ in NK

$$
\vee\,\cfrac{\Box\,\cfrac{\vee\,\cfrac{\wedge\,\cfrac{\Diamond\,\cfrac{\text{init}\,\cfrac{}{\Diamond p, [p, \bar{p}, q]}}{\Diamond p, [\bar{p}, q]} \qquad \Diamond\,\cfrac{\text{init}\,\cfrac{}{\Diamond \bar{q}, [\bar{q}, \bar{p}, q]}}{\Diamond \bar{q}, [\bar{p}, q]}}{\Diamond p \wedge \Diamond \bar{q}, [\bar{p}, q]}}{\Diamond p \wedge \Diamond \bar{q}, [\bar{p} \vee q]}}{\Diamond p \wedge \Diamond \bar{q}, \Box(\bar{p} \vee q)}}{(\Diamond p \wedge \Diamond \bar{q}) \vee \Box(\bar{p} \vee q)}
$$

Roadmap

Γ set of formulas, A formula

HILBERT-STYLE
AXIOM SYSTEM

$\Gamma \vdash_K A$ $\Longleftrightarrow$ $\Gamma \models A$

LOGICAL
CONSEQUENCE

$\vdash \bar{\Gamma}, A$

NK

NESTED
SEQUENTS

For a nested sequent Γ and a model $\mathcal{M} = \langle W, R, v \rangle$, an $\mathcal{M}$-map for Γ is a map $f : tr(\Gamma) \to W$ such that whenever δ is a child of γ in $tr(\Gamma)$, then $f(\gamma) R f(\delta)$.

Validity of nested sequents [Kuznets & Straßburger, 2018]

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$$\mathcal{M}, f(\delta) \models B, \text{ for some } \delta \in tr(\Gamma), \text{ for some } B \in \delta$$

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For $X \subseteq \{d, t, b, 4, 5\}$, a nested sequent is X-valid iff it is satisfied by all $\mathcal{M}$-map for Γ, for all models $\mathcal{M}$ satisfying the frame conditions in X.

Lemma. If Γ is derivable in NK then $\cancel{\bigvee}\,\Gamma$ is valid in all Kripke frames.

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Proof. By induction on the height of the derivation of Γ. We need to show that initial sequents are valid, and inference rules preserve validity.

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To prove: If $\Gamma\{\Box A\}$ is <u>not</u> valid, then $\Gamma\{[A]\}$ is <u>not</u> valid.

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In particular: $\mathcal{M}, f(\gamma) \not\models \Box A$.

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Let $\delta = \{A\} \in tr(\Gamma\{[A]\})$. Define $g(\delta) = w$, and $g(\eta) = f(\eta)$, for all $\eta \neq \delta$ belonging to $tr(\Gamma\{[A]\})$. It holds that g is an $\mathcal{M}$-map for $\Gamma\{[A]\}$, and it refutes the sequent. $\Box$

Rules for extensions: $\mathsf{NK} \cup \mathsf{X}^{\diamond}$

$$\mathsf{d}^{\diamond}\ \frac{\Gamma\{\diamond A, [A]\}}{\Gamma\{\diamond A\}} \qquad \mathsf{t}^{\diamond}\ \frac{\Gamma\{\diamond A, A\}}{\Gamma\{\diamond A\}} \qquad \mathsf{b}^{\diamond}\ \frac{\Gamma\{[\Delta, \diamond A], A\}}{\Gamma\{[\Delta, \diamond A]\}}$$

$$\mathsf{4}^{\diamond}\ \frac{\Gamma\{\diamond A, [\diamond A, \Delta]\}}{\Gamma\{\diamond A, [\Delta]\}} \qquad \mathsf{5}^{\diamond}\ \frac{\Gamma\{\diamond A\}\{\diamond A\}}{\Gamma\{\diamond A\}\{\emptyset\}}\ \mathit{depth}(\Gamma\{\ \}\{\emptyset\}) > 0$$

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$$4^{\diamond}\ \frac{\Gamma\{\diamond A, [\diamond A, \Delta]\}}{\Gamma\{\diamond A, [\Delta]\}} \qquad 5^{\diamond}\ \frac{\Gamma\{\diamond A\}\{\diamond A\}}{\Gamma\{\diamond A\}\{\emptyset\}}\ \mathit{depth}(\Gamma\{\ \}\{\emptyset\}) > 0$$

For $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$, we write $\mathsf{X}^{\diamond}$ for the corresponding subset of $\{\mathsf{d}^{\diamond}, \mathsf{t}^{\diamond}, \mathsf{b}^{\diamond}, 4^{\diamond}, 5^{\diamond}\}$. We shall consider the calculi $\mathsf{NK} \cup \mathsf{X}^{\diamond}$.

$$
\mathsf{d}^{\Diamond}\ \dfrac{\Gamma\{\Diamond A, [A]\}}{\Gamma\{\Diamond A\}}
\qquad
\mathsf{t}^{\Diamond}\ \dfrac{\Gamma\{\Diamond A, A\}}{\Gamma\{\Diamond A\}}
\qquad
\mathsf{b}^{\Diamond}\ \dfrac{\Gamma\{[\Delta, \Diamond A], A\}}{\Gamma\{[\Delta, \Diamond A]\}}
$$

$$
\mathsf{4}^{\Diamond}\ \dfrac{\Gamma\{\Diamond A, [\Diamond A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}
\qquad
\mathsf{5}^{\Diamond}\ \dfrac{\Gamma\{\Diamond A\}\{\Diamond A\}}{\Gamma\{\Diamond A\}\{\emptyset\}}\ \mathit{depth}(\Gamma\{\ \}\{\emptyset\}) > 0
$$

For $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$, we write $\mathsf{X}^{\Diamond}$ for the corresponding subset of $\{\mathsf{d}^{\Diamond}, \mathsf{t}^{\Diamond}, \mathsf{b}^{\Diamond}, \mathsf{4}^{\Diamond}, \mathsf{5}^{\Diamond}\}$. We shall consider the calculi $\mathsf{NK} \cup \mathsf{X}^{\Diamond}$.

Example. Proof of $\Box p \rightarrow \Box\Box p$ in $\mathsf{NK} \cup \{\mathsf{t}, \mathsf{4}\}$

$$
\vee\ \dfrac{\Box\ \dfrac{\Box\ \dfrac{\mathsf{4}^{\Diamond}\ \dfrac{\mathsf{4}^{\Diamond}\ \dfrac{\mathsf{t}^{\Diamond}\ \dfrac{\mathsf{init}\ \dfrac{}{\Diamond\bar{p}, [\Diamond\bar{p}, [\Diamond\bar{p}, \bar{p}, p]]}}{\Diamond\bar{p}, [\Diamond\bar{p}, [\Diamond\bar{p}, p]]}}{\Diamond\bar{p}, [\Diamond\bar{p}, [p]]}}{\Diamond\bar{p}, [[p]]}}{\Diamond\bar{p}, [\Box p]}}{\Diamond\bar{p}, \Box\Box p}}{\Diamond\bar{p} \vee \Box\Box p}
$$

Structural rules [Brünnler, 2009]

$$\mathsf{wk}\,\frac{\Gamma\{\emptyset\}}{\Gamma\{\Delta\}} \qquad \mathsf{ctr}\,\frac{\Gamma\{\Delta,\Delta\}}{\Gamma\{\Delta\}} \qquad \mathsf{cut}\,\frac{\Gamma\{A\} \quad \Gamma\{\overline{A}\}}{\Gamma\{\emptyset\}}$$

Structural rules [Brünnler, 2009]

$$\mathsf{wk}\,\frac{\Gamma\{\emptyset\}}{\Gamma\{\Delta\}} \qquad \mathsf{ctr}\,\frac{\Gamma\{\Delta,\Delta\}}{\Gamma\{\Delta\}} \qquad \mathsf{cut}\,\frac{\Gamma\{A\} \quad \Gamma\{\overline{A}\}}{\Gamma\{\emptyset\}}$$

For $\mathsf{X} \subseteq \{\mathsf{d},\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}\}$:

Lemma. The rules wk and ctr are hp-admissible in $\mathsf{NK} \cup \mathsf{X}^{\diamond}$.

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For $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$:

Lemma. The rules wk and ctr are hp-admissible in $\mathsf{NK} \cup \mathsf{X}^{\Diamond}$.

Lemma. All the rules of $\mathsf{NK} \cup \mathsf{X}^{\Diamond}$ are hp-invertible.

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Proposition. Rule $5^{\diamond}$ is derivable in $\mathsf{NK} \cup \{5_1^{\diamond}, 5_2^{\diamond}, 5_3^{\diamond}\} \cup \{\mathsf{wk}\}$.

$$5^{\diamond}\,\frac{\Gamma\{\diamond A\}\{\diamond A\}}{\Gamma\{\diamond A\}\{\emptyset\}}\ \mathit{depth}(\Gamma\{\ \}\{\emptyset\}) > 0$$

$$5_1^{\diamond}\,\frac{\Gamma\{[\Delta,\diamond A],\diamond A\}}{\Gamma\{[\Delta,\diamond A]\}} \qquad 5_2^{\diamond}\,\frac{\Gamma\{[\Delta,\diamond A],[\Lambda,\diamond A]\}}{\Gamma\{[\Delta,\diamond A],[\Lambda]\}} \qquad 5_3^{\diamond}\,\frac{[\Delta,\diamond A,[\Lambda,\diamond A]]}{\Gamma\{[\Delta,\diamond A,[\Lambda]]\}}$$

$$\mathsf{wk}\ \frac{\Gamma\{\emptyset\}}{\Gamma\{\Delta\}} \qquad \mathsf{ctr}\ \frac{\Gamma\{\Delta,\Delta\}}{\Gamma\{\Delta\}} \qquad \mathsf{cut}\ \frac{\Gamma\{A\} \quad \Gamma\{\overline{A}\}}{\Gamma\{\emptyset\}}$$

For $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$:

Lemma. The rules wk and ctr are hp-admissible in $\mathsf{NK} \cup \mathsf{X}^{\Diamond}$.

Lemma. All the rules of $\mathsf{NK} \cup \mathsf{X}^{\Diamond}$ are hp-invertible.

Proposition. Rule $5^{\Diamond}$ is derivable in $\mathsf{NK} \cup \{5^{\Diamond}_1, 5^{\Diamond}_2, 5^{\Diamond}_3\} \cup \{\mathsf{wk}\}$.

$$5^{\Diamond}\ \frac{\Gamma\{\Diamond A\}\{\Diamond A\}}{\Gamma\{\Diamond A\}\{\emptyset\}}\ \mathit{depth}(\Gamma\{\ \}\{\emptyset\}) > 0$$

$$5^{\Diamond}_1\ \frac{\Gamma\{[\Delta, \Diamond A], \Diamond A\}}{\Gamma\{[\Delta, \Diamond A]\}} \qquad 5^{\Diamond}_2\ \frac{\Gamma\{[\Delta, \Diamond A], [\Lambda, \Diamond A]\}}{\Gamma\{[\Delta, \Diamond A], [\Lambda]\}} \qquad 5^{\Diamond}_3\ \frac{[\Delta, \Diamond A, [\Lambda, \Diamond A]]}{\Gamma\{[\Delta, \Diamond A, [\Lambda]]\}}$$

Lemma. If Γ is derivable in $\mathsf{NK} \cup \mathsf{X}^{\Diamond}$ then Γ is valid in all X-frames.

Nested sequents with explicit contraction [Marin & Straßburger, 2014]

Rules of $\mathsf{NK}_{\mathsf{ctr}}$

$$\mathsf{init}\,\frac{}{\Gamma\{p,\overline{p}\}} \qquad \mathsf{ctr}\,\frac{\Gamma\{\Delta,\Delta\}}{\Gamma\{\Delta\}} \qquad \wedge\,\frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}} \qquad \vee\,\frac{\Gamma\{A,B\}}{\Gamma\{A \vee B\}}$$

$$\Box\,\frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \Diamond_{\mathsf{ctr}}\,\frac{\Gamma\{\cancel{\Diamond A,}\,[A,\Delta]\}}{\Gamma\{\Diamond A,[\Delta]\}}$$

Rules for extensions

$$\mathsf{d}^{\Diamond}_{\mathsf{ctr}}\,\frac{\Gamma\{\cancel{\Diamond A,}\,[A]\}}{\Gamma\{\Diamond A\}} \qquad \mathsf{t}^{\Diamond}_{\mathsf{ctr}}\,\frac{\Gamma\{\cancel{\Diamond A,}\,A\}}{\Gamma\{\Diamond A\}} \qquad \mathsf{b}^{\Diamond}_{\mathsf{ctr}}\,\frac{\Gamma\{[\Delta,\cancel{\Diamond A}],A\}}{\Gamma\{[\Delta,\Diamond A]\}}$$

$$4^{\Diamond}_{\mathsf{ctr}}\,\frac{\Gamma\{\Diamond A,[\cancel{\Diamond A,}\,\Delta]\}}{\Gamma\{\Diamond A,[\Delta]\}} \qquad 5^{\Diamond}_{\mathsf{ctr}}\,\frac{\Gamma\{\cancel{\Diamond A}\}\{\Diamond A\}}{\Gamma\{\Diamond A\}\{\emptyset\}}\ \mathit{depth}(\Gamma\{\ \}\{\emptyset\}) > 0$$

For $\mathsf{X} \subseteq \{\mathsf{d},\mathsf{t},\mathsf{b},4,5\}$, we write $\mathsf{X}^{\Diamond}_{\mathsf{ctr}}$ for the corresponding subset of $\{\mathsf{d}^{\Diamond}_{\mathsf{ctr}},\mathsf{t}^{\Diamond}_{\mathsf{ctr}},\mathsf{b}^{\Diamond}_{\mathsf{ctr}},4^{\Diamond}_{\mathsf{ctr}},5^{\Diamond}_{\mathsf{ctr}}\}$.

Rules of $\mathsf{NK}_{\mathsf{ctr}}$

$$\text{init}\ \frac{}{\Gamma\{p, \bar{p}\}} \qquad \text{ctr}\ \frac{\Gamma\{\Delta, \Delta\}}{\Gamma\{\Delta\}} \qquad \wedge\ \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}} \qquad \vee\ \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}}$$

$$\Box\ \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \Diamond_{\mathsf{ctr}}\ \frac{\Gamma\{\cancel{\Diamond A,} [A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}$$

Rules for extensions

$$\mathsf{d}^{\Diamond}_{\mathsf{ctr}}\ \frac{\Gamma\{\cancel{\Diamond A,} [A]\}}{\Gamma\{\Diamond A\}} \qquad \mathsf{t}^{\Diamond}_{\mathsf{ctr}}\ \frac{\Gamma\{\cancel{\Diamond A,} A\}}{\Gamma\{\Diamond A\}} \qquad \mathsf{b}^{\Diamond}_{\mathsf{ctr}}\ \frac{\Gamma\{[\Delta, \cancel{\Diamond A}], A\}}{\Gamma\{[\Delta, \Diamond A]\}}$$

$$4^{\Diamond}_{\mathsf{ctr}}\ \frac{\Gamma\{\Diamond A, [\cancel{\Diamond A,} \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}} \qquad 5^{\Diamond}_{\mathsf{ctr}}\ \frac{\Gamma\{\cancel{\Diamond A}\}\{\Diamond A\}}{\Gamma\{\Diamond A\}\{\emptyset\}}\ \mathit{depth}(\Gamma\{\ \}\{\emptyset\}) > 0$$

For $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$, we write $\mathsf{X}^{\Diamond}_{\mathsf{ctr}}$ for the corresponding subset of $\{\mathsf{d}^{\Diamond}_{\mathsf{ctr}}, \mathsf{t}^{\Diamond}_{\mathsf{ctr}}, \mathsf{b}^{\Diamond}_{\mathsf{ctr}}, 4^{\Diamond}_{\mathsf{ctr}}, 5^{\Diamond}_{\mathsf{ctr}}\}$.

Lemma. The rule wk is hp-admissible in $\mathsf{NK} \cup \mathsf{X}^{\Diamond}$.

Rules of $\mathsf{NK_{ctr}}$

$$\mathsf{init}\ \frac{}{\Gamma\{p, \bar{p}\}} \qquad \mathsf{ctr}\ \frac{\Gamma\{\Delta, \Delta\}}{\Gamma\{\Delta\}} \qquad \wedge\ \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}} \qquad \vee\ \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}}$$

$$\Box\ \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \Diamond_{\mathsf{ctr}}\ \frac{\Gamma\{\cancel{\Diamond A,}\, [A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}$$

Rules for extensions

$$\mathsf{d}^{\Diamond}_{\mathsf{ctr}}\ \frac{\Gamma\{\cancel{\Diamond A,}\, [A]\}}{\Gamma\{\Diamond A\}} \qquad \mathsf{t}^{\Diamond}_{\mathsf{ctr}}\ \frac{\Gamma\{\cancel{\Diamond A,}\, A\}}{\Gamma\{\Diamond A\}} \qquad \mathsf{b}^{\Diamond}_{\mathsf{ctr}}\ \frac{\Gamma\{[\Delta, \cancel{\Diamond A}], A\}}{\Gamma\{[\Delta, \Diamond A]\}}$$

$$4^{\Diamond}_{\mathsf{ctr}}\ \frac{\Gamma\{\Diamond A, [\cancel{\Diamond A,}\, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}} \qquad 5^{\Diamond}_{\mathsf{ctr}}\ \frac{\Gamma\{\cancel{\Diamond A}\}\{\Diamond A\}}{\Gamma\{\Diamond A\}\{\emptyset\}}\ \mathit{depth}(\Gamma\{\ \}\{\emptyset\}) > 0$$

For $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$, we write $\mathsf{X}^{\Diamond}_{\mathsf{ctr}}$ for the corresponding subset of $\{\mathsf{d}^{\Diamond}_{\mathsf{ctr}}, \mathsf{t}^{\Diamond}_{\mathsf{ctr}}, \mathsf{b}^{\Diamond}_{\mathsf{ctr}}, 4^{\Diamond}_{\mathsf{ctr}}, 5^{\Diamond}_{\mathsf{ctr}}\}$.

Lemma. The rule wk is hp-admissible in $\mathsf{NK} \cup \mathsf{X}^{\Diamond}$.

Proposition. Γ is derivable in $\mathsf{NK} \cup \mathsf{X}^{\Diamond}$ iff Γ is derivable in $\mathsf{NK_{ctr}} \cup \mathsf{X}^{\Diamond}_{\mathsf{ctr}}$.

Roadmap

$X \subseteq \{d, t, b, 4, 5\}$

Γ set of formulas, A formula

HILBERT-STYLE
AXIOM SYSTEM

LOGICAL
CONSEQUENCE

$\Gamma \vdash_{K \cup X} A \iff \Gamma \models_X A$

SOUNDNESS

$\vdash \overline{\Gamma}, A$

$NK \cup X_\Diamond$

NESTED
SEQUENTS