Proof Theory of Modal Logic

Lecture 2: Labelled Proof Systems

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ESSLLI 2024

Leuven, 5-9 August 2024

Partial references:

- [Kanger, 1957] Spotted formulas for S5
- [Fitting, 1983], [Goré 1998] Tableaux + labels
- [Simpson, 1994], [Viganò, 1998] Natural deduction + labels
- [Mints, 1997], [Viganò, 2000], [Negri,2005] Sequent calculus + labels

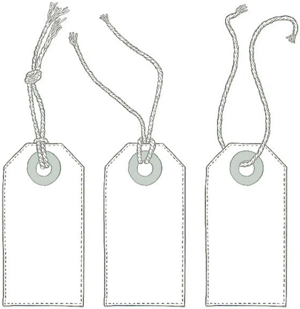
We follow the approach of Negri:

- *Proof analysis in modal logics* [Negri, 2005]
- *Contraction-free sequent calculi for geometric theories with an application to Barr's theorem* [Negri, 2003]

The plan

- Labelled calculi for K
- Frame conditions: a general recipe
- Semantic completeness

Labelled sequent calculus for K



Enriching the language

$$A, B ::= p \mid \bot \mid A \wedge B \mid A \vee B \mid A \to B \mid \Box A \mid \Diamond A$$

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Take countably many variables $x, y, z, \ldots$ (the lables)

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Labelled formulas

- xRy meaning 'x has access to y' (relational atoms)
- $x{:}A$ meaning 'x satisfies A'

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Labelled sequent

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where

- $\mathcal{R}$ is a multiset of relational atoms;
- Γ, Δ are multisets of labelled formulas *without* relational atoms.

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Labelled sequents lack a formula interpretation

Rules of labK

$$\text{init}\ \frac{}{\mathcal{R}, x{:}p, \Gamma \Rightarrow \Delta, x{:}p} \qquad \bot_L\ \frac{}{\mathcal{R}, x{:}\bot, \Gamma \Rightarrow \Delta}$$

Rules of labK

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$$\wedge_{\mathrm{R}}\ \frac{\mathcal{R}, x{:}A, x{:}B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x{:}A \wedge B, \Gamma \Rightarrow \Delta} \qquad \wedge_{\mathrm{L}}\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \quad \mathcal{R}, \Gamma \Rightarrow \Delta, x{:}B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \wedge B}$$

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$$\Box_{\mathrm{L}}\ \frac{xRy, \mathcal{R}, y{:}A, x{:}\Box A, \Gamma \Rightarrow \Delta}{xRy, \mathcal{R}, x{:}\Box A, \Gamma \Rightarrow \Delta} \qquad \Box_{\mathrm{R}}\ \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, y{:}A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}\Box A}\ y \text{ fresh}$$

y fresh means y does not occur in $\mathcal{R} \cup \Gamma \cup \Delta$

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We write $\vdash_{\mathsf{labK}} \mathcal{R}, \Gamma \Rightarrow \Delta$ if there is a derivation of $\mathcal{R}, \Gamma \Rightarrow \Delta$ in labK.

Example: $\vdash_{\mathsf{labK}} \Rightarrow x{:}(\Diamond p \to \Box q) \to \Box(p \to q)$

$$
\text{\scriptsize $\to_R$}\,\dfrac{
  \text{\scriptsize $\Box_R$}\,\dfrac{
    \text{\scriptsize $\to_R$}\,\dfrac{
      \text{\scriptsize $\to_L$}\,\dfrac{
        \text{\scriptsize $\Diamond_R$}\,\dfrac{
          \text{\scriptsize init}\,\dfrac{}{xRy, y{:}p \Rightarrow y{:}q, x{:}\Diamond p, y{:}p}
        }{xRy, y{:}A \Rightarrow y{:}q, x{:}\Diamond p}
        \qquad
        \text{\scriptsize $\Box_L$}\,\dfrac{
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  }{x{:}\Diamond p \to \Box q \Rightarrow x{:}\Box(p \to q)}
}{\Rightarrow x{:}(\Diamond p \to \Box q) \to \Box(p \to q)}
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labK: main results

Given a sequent $\mathcal{S} = \mathcal{R}, \Gamma \Rightarrow \Delta$, and a model $\mathcal{M} = \langle W, R, v \rangle$, let $\mathsf{Lb}(\mathcal{S}) = \{x \mid x \in \mathcal{R} \cup \Gamma \cup \Delta\}$, and $\rho : \mathsf{Lb}(\mathcal{S}) \rightarrow W$ (interpretation).

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Satisfiability of labelled formulas at $\mathcal{M}$ under ρ :

$$\mathcal{M}, \rho \Vdash xRy \quad \textit{iff} \quad \mathcal{M} \Vdash \rho(x) R \rho(y)$$
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Satisfiability of sequents at $\mathcal{M}$ under ρ (φ is xRy or $x{:}A$):

$$\mathcal{M}, \rho \Vdash \mathcal{R}, \Gamma \Rightarrow \Delta \quad \textit{iff}$$

if for all $\varphi \in \mathcal{R} \cup \Gamma$ it holds that $\mathcal{M}, \rho \Vdash \varphi$,

then for some $x{:}D \in \Delta$ it holds that $\mathcal{M}, \rho \Vdash x{:}D$.

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A sequent $\mathcal{R}, \Gamma \Rightarrow \Delta$ has a countermodel iff there are $\mathcal{M}$, ρ such that:

- $\mathcal{M}, \rho \models \varphi$, for all $\varphi \in \mathcal{R} \cup \Gamma$, and
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Validity of sequents in a class of frames $\mathcal{X}$:

$$\models_{\mathcal{X}} \mathcal{R}, \Gamma \Rightarrow \Delta \quad \textit{iff} \quad \text{for any } \rho \text{ and any } \mathcal{M} \in \mathcal{X},\ \mathcal{M}, \rho \Vdash \mathcal{R}, \Gamma \Rightarrow \Delta$$

Soundness of labK [Negri, 2009]

Theorem (Soundness). If $\vdash_{\mathsf{labK}} \mathcal{R}, \Gamma \Rightarrow \Delta$ then $\models \mathcal{R}, \Gamma \Rightarrow \Delta$

labK: main results

Towards cut-admissibility of labK 1/2 [Negri, 2005]

Substitution on labelled formulas:

$$xRy[z/y] \quad := \quad xRz$$
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Substitution on multisets of labelled formulas $\Gamma[z/y]$

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Lemma (Substitution). Rule subst is hp-admissible.

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Lemma (Weakening). Rules $\mathrm{wk_L}$, $\mathrm{wk_R}$ are hp-admissible (φ is xRy or $x{:}A$).

$$\mathrm{wk_L}\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta}{\varphi, \mathcal{R}, \Gamma \Rightarrow \Delta} \qquad \mathrm{wk_R}\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta, \varphi}$$

Lemma (Invertibility).
For every rule r, if the conclusion of r is derivable with a derivation of height h, then each of its premisses is derivable, with at most the same h.

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Lemma (Contraction). Rules $\mathsf{ctr_L}$, $\mathsf{ctr_R}$ are hp-admissible (φ is xRy or $x{:}A$).

$$\mathsf{ctr_L}\ \frac{\varphi, \varphi, \mathcal{R}, \Gamma \Rightarrow \Delta}{\varphi, \mathcal{R}, \Gamma \Rightarrow \Delta} \qquad \mathsf{ctr_R}\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, \varphi, \varphi}{\mathcal{R}, \Gamma \Rightarrow \Delta, \varphi}$$

Lemma (Cut). The cut rule is admissible.

$$\text{cut}\,\frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \quad x{:}A, \mathcal{R}', \Gamma' \Rightarrow \Delta'}{\mathcal{R}, \mathcal{R}', \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Proof. By induction on $(c(A), h_1 + h_2)$.

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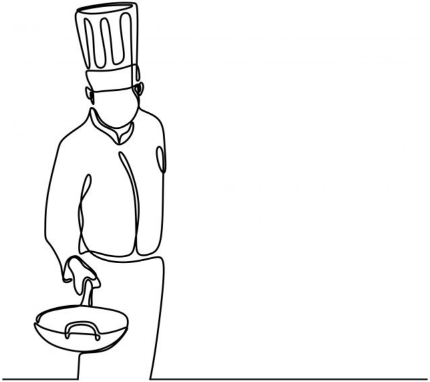
For Γ set of formulas and $x{:}\Gamma = \{x{:}G \mid \text{for each } G \in \Gamma\}$:

Theorem (Syntactic Completeness). If $\Gamma \vdash_K A$ then $\vdash_{labK} x{:}\Gamma \Rightarrow x{:}A$.

labK: main results

Frame conditions: a general recipe



What do we mean by modularity?

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d	$\Box A \to \Diamond A$	Seriality	$\forall x \exists y (xRy)$
t	$\Box A \to A$	Reflexivity	$\forall x (xRx)$
b	$A \to \Box\Diamond A$	Symmetry	$\forall x \forall y (xRy \to yRx)$
4	$\Box A \to \Box\Box A$	Transitivity	$\forall x \forall y \forall z ((xRy \wedge yRz) \to xRz)$
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We write $\Gamma \vdash_{\mathsf{K} \cup \mathsf{X}} A$ iff A is derivable from Γ in the axiom system $\mathsf{K} \cup \mathsf{X}$.

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We denote by $\mathcal{X}$ the class of frames satisfying properties in X.
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Let $\mathsf{K} = \mathsf{CPL} \cup \{\mathsf{k}, \mathsf{nec}\}$. Logic K is characterised by the class of all Kripke frames.

Name	Axiom	Frame condition	
d	$\Box A \to \Diamond A$	Seriality	$\forall x \exists y (xRy)$
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Theorem. For $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$, $\Gamma \vdash_{\mathsf{K} \cup \mathsf{X}} A$ iff $\Gamma \models_{\mathcal{X}} A$.

Main ingredients

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Frame conditions can be characterised by first-order logic formulas, in the language consisting of a single predicate symbol, $R(x, y)$.

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The first-order logic formulas corresponding to the frame conditions above (and many more!) are geometric implications

A first-order signature is a tuple $\sigma = \langle c, d, \ldots, f, g, \ldots p, q, \ldots \rangle$

- Constant symbols $c, d, \ldots$
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A first-order language over a signature σ, denoted $\mathcal{L}(\sigma)$, consists of:

- The terms generated from a countably many variables $x, y, \ldots$ using the constants and function symbols of σ;
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Example.
$\mathcal{L}^{=}(0, suc^1, +^2, \times^2)$ is the language of arithmetic
$\mathcal{L}(R^2)$ is the language we use to express frame conditions

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Example. Robinson arithmetic is a geometric theory over the language $\mathcal{L}^{=}(0, suc, +, \times)$.

Geometric implications can be expressed as conjunctions of geometric axioms, i.e., closed formulas of $\mathcal{L}(\sigma)$ having the form:

$$\forall \vec{x}\Big(P \to \big(\exists \vec{y}_1(Q_1) \vee \cdots \vee \exists \vec{y}_m(Q_m)\big)\Big)$$

- $\vec{x}$, $\vec{y}_1, \ldots, \vec{y}_m$ are (possibly empty) vectors of variables;
- $m \geq 0$;
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Geometric axioms can be turned into sequent calculus rules:

$$\text{GA}\,\frac{\Xi_1[\vec{z}_1/\vec{y}_1], \Pi, \Gamma \Rightarrow \Delta \quad \cdots \quad \Xi_m[\vec{z}_m/\vec{y}_m], \Pi, \Gamma \Rightarrow \Delta}{\Pi, \Gamma \Rightarrow \Delta}$$

- Π is the multiset of atomic formulas in P;
- Ξ_i is the multiset of atomic formulas in Q_i, for each $i \leq m$;
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For $X \subseteq \{d, t, b, 4, 5\}$, $\mathsf{labK} \cup X$ is defined by adding to labK the rules for frame conditions corresponding to elements of X, plus the rules obtained by to satisfy the closure condition (contracted instances of the rules):

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We denote by $\vdash_{\mathsf{labK} \cup \mathsf{X}} \mathcal{S}$ derivability of labelled sequent $\mathcal{S}$ in $\mathsf{labK} \cup \mathsf{X}$.

For $X \subseteq \{d, t, b, 4, 5\}$:

Theorem (Soundness). If $\vdash_{\mathsf{labK} \cup \mathsf{X}} \mathcal{R}, \Gamma \Rightarrow \Delta$ then $\models_{\mathcal{X}} \mathcal{R}, \Gamma \Rightarrow \Delta$.

Example. If the premiss of rule ser is valid in all serial models, then its conclusion is valid in all serial models.

$$\text{ser}\ \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta}\ y \text{ fresh}$$

Lemma (Cut). The cut rule is admissible in labK ∪ X:

$$\text{cut}\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \qquad x{:}A, \mathcal{R}', \Gamma' \Rightarrow \Delta'}{\mathcal{R}, \mathcal{R}', \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

For Γ set of formulas and $x{:}\Gamma = \{x{:}G \mid \text{for each } G \in \Gamma\}$:

Theorem (Syntactic Completeness). If $\Gamma \vdash_{\mathsf{K} \cup \mathsf{X}} A$ then $\vdash_{\mathsf{labK} \cup \mathsf{X}} x{:}\Gamma \Rightarrow x{:}A$.

labK ∪ X: main results

Beyond geometric axioms

- Systems of rules [Negri, 2016], to capture theories / logics characterized by generalized geometric implications:

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GA_0 &= \forall \vec{x}\Big(P \to \big(\exists \vec{y}_1(Q_1) \vee \cdots \vee \exists \vec{y}_m(Q_m)\big)\Big) \\
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  [Negri, 2005]: labelled proof system for GL!

- Derive axiom 4, that is, $\Box A \to \Box\Box A$, in $\mathsf{labK} \cup \{\mathsf{t}, 5\}$. Then, show that rule tr is derivable in $\mathsf{labK} \cup \{\mathsf{t}, 5\} \cup \{\mathsf{wk_L}, \mathsf{wk_R}\}$.
- Derive axiom 5, that is, $\Diamond A \to \Box\Diamond A$, in $\mathsf{labK} \cup \{\mathsf{b}, 4\}$. Then, show that rule euc is derivable in $\mathsf{labK} \cup \{\mathsf{b}, 4\} \cup \{\mathsf{wk_L}, \mathsf{wk_R}\}$.
- Write down the labelled rule corresponding to the frame condition of confluence:

$$\forall x, y, z \big((xRy \wedge xRz) \to \exists k(yRk \wedge zRk)\big)$$

- Write down the sequent calculus rules corresponding to the axioms of Robinson Arithmetic. Can we use the results from [Negri, 2003] to prove consistency of Robinson Arithmetic? If yes, how?