Proof Theory of Modal Logic

Lecture 3, part 1: Labelled Proof Systems

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The plan

- Labelled sequent calculus for K
- Frame conditions: a general recipe
- Semantic completeness

Recap

Geometric implications can be expressed as conjunctions of geometric axioms, i.e., closed formulas of $\mathcal{L}(\sigma)$ having the form:

$$\forall \vec{x}\Big(P \to \big(\exists \vec{y}_1(Q_1) \vee \cdots \vee \exists \vec{y}_m(Q_m)\big)\Big)$$

- $\vec{x}$, $\vec{y}_1, \ldots, \vec{y}_m$ are (possibly empty) vectors of variables;
- $m \geq 0$;
- P, $Q_1, \ldots, Q_m$ are (possibly empty) conjunctions of atomic formulas of $\mathcal{L}(\sigma)$;
- $\vec{y}_1, \ldots, \vec{y}_m$ do not occur in P.

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Geometric axioms can be turned into sequent calculus rules:

$$\text{GA}\,\frac{\Xi_1[\vec{z}_1/\vec{y}_1], \Pi, \Gamma \Rightarrow \Delta \quad \cdots \quad \Xi_m[\vec{z}_m/\vec{y}_m], \Pi, \Gamma \Rightarrow \Delta}{\Pi, \Gamma \Rightarrow \Delta}$$

- Π is the multiset of atomic formulas in P;
- Ξ_i is the multiset of atomic formulas in Q_i, for each $i \leq m$;
- $\vec{z}_1, \ldots, \vec{z}_m$ do not occur in $\Gamma \cup \Delta$.

From geometric axioms to labelled rules [Negri, 2003]

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Labelled calculi for the S5-cube [Negri, 2005]

$$\text{ser}\ \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta}\ y \text{ fresh} \qquad \text{ref}\ \frac{xRx, \mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta} \qquad \text{sym}\ \frac{yRx, xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

$$\text{tr}\ \frac{xRz, xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta} \qquad \text{euc}\ \frac{yRz, xRy, xRz, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, xRz, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

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For $\mathrm{X} \subseteq \{\mathrm{d}, \mathrm{t}, \mathrm{b}, 4, 5\}$, $\mathrm{labK} \cup \mathrm{X}$ is defined by adding to labK the rules for frame conditions corresponding to elements of X, plus the rules obtained by to satisfy the closure condition (contracted instances of the rules):

$$\text{euc}\ \frac{yRy, xRy, xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, xRy, \mathcal{R}, \Gamma \Rightarrow \Delta} \quad \rightsquigarrow \quad \text{euc}'\ \frac{yRy, xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

Example: $\mathrm{labK} \cup \{5\}$ denotes the proof system $\mathrm{labK} \cup \{\mathrm{euc}, \mathrm{euc}'\}$.

We denote by $\vdash_{\mathrm{labK} \cup \mathrm{X}} \mathcal{S}$ derivability of labelled sequent $\mathcal{S}$ in $\mathrm{labK} \cup \mathrm{X}$.

For X ⊆ {d, t, b, 4, 5}:

Theorem (Soundness). If $\vdash_{\mathsf{labK}\cup\mathsf{X}}\ \mathcal{R}, \Gamma \Rightarrow \Delta$ then $\models_{\mathcal{X}}\ \mathcal{R}, \Gamma \Rightarrow \Delta$.

Example. If the premiss of rule ser is valid in all serial models, then its conclusion is valid in all serial models.

$$\text{ser}\ \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta}\ y \text{ fresh}$$

Lemma (Cut). The cut rule is admissible in labK ∪ X:

$$\text{cut}\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \qquad x{:}A, \mathcal{R}', \Gamma' \Rightarrow \Delta'}{\mathcal{R}, \mathcal{R}', \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

For Γ set of formulas and $x{:}\Gamma = \{x{:}G \mid \text{for each } G \in \Gamma\}$:

Theorem (Syntactic Completeness). If $\Gamma \vdash_{\mathsf{K}\cup\mathsf{X}} A$ then $\vdash_{\mathsf{labK}\cup\mathsf{X}}\ x{:}\Gamma \Rightarrow x{:}A$.

Beyond geometric axioms

- Systems of rules [Negri, 2016], to capture theories / logics characterized by generalized geometric implications:

$$
\begin{aligned}
GA_0 &= \forall\vec{x}\Big(P \to \big(\exists\vec{y}_1(Q_1) \vee \cdots \vee \exists\vec{y}_m(Q_m)\big)\Big)\\
GA_1 &= \forall\vec{x}\Big(P \to \big(\exists\vec{y}_1(\bigwedge GA_0) \vee \cdots \vee \exists\vec{y}_m(\bigwedge GA_0)\big)\Big)\\
GA_{n+1} &= \forall\vec{x}\Big(P \to \big(\exists\vec{y}_1(\bigwedge GA_{k_1}) \vee \cdots \vee \exists\vec{y}_m(\bigwedge GA_{k_m})\big)\Big)
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for $k_1, \ldots, k_m \geq n$

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  Converse well-foundedness: there are no infinite R-chains

  [Negri, 2005]: labelled proof system for GL!

Semantic completeness

For $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$:

Theorem (Proof or Countermodel). For $\mathcal{S}$ labelled sequent, either $\vdash_{\mathsf{labK} \cup \mathsf{X}} \mathcal{S}$ or $\mathcal{S}$ has a countermodel satisfying the frame conditions in X.

For $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$,
Γ set of formulas and $x{:}\Gamma = \{x{:}G \mid \text{for each } G \in \Gamma\}$:

Theorem (Semantic completeness). If $\Gamma \models_{\mathcal{X}} A$ then $\vdash_{\mathsf{labK} \cup \mathsf{X}} x{:}\Gamma \Rightarrow x{:}A$.

Proof search algorithm for labK ∪ X [Negri, 2009]

0. Given a sequent $\mathcal{S}_0$, place $\mathcal{S}_0$ at the root of $\mathcal{T}$.
1. For every rule $\mathsf{R} \in \{\wedge_\mathsf{L}, \wedge_\mathsf{R}, \vee_\mathsf{L}, \vee_\mathsf{R}, \rightarrow_\mathsf{L}, \rightarrow_\mathsf{R}, \Box_\mathsf{L}, \Box_\mathsf{R}, \Diamond_\mathsf{L}, \Diamond_\mathsf{R}\}$, apply the following:
   a) If every topmost sequent of $\mathcal{T}$ is initial, terminate.
   $\rightsquigarrow$ $\mathcal{S}_0$ is provable in labK ∪ X, and $\mathcal{T}$ defines a labK ∪ X proof for it.
   b) Otherwise, write above each non-initial sequent $\mathcal{S}_i$ of $\mathcal{T}$ the sequent(s) obtained by exhaustively apply rule R to $\mathcal{S}_i$.
2. For every rule $\mathsf{R} \in \{\mathsf{ref}, \mathsf{tr}, \mathsf{sym}, \mathsf{ser}, \mathsf{euc}\}$ in labK ∪ X (if any), apply the following:
   a) If every topmost sequent of $\mathcal{T}$ is initial, terminate.
   $\rightsquigarrow$ $\mathcal{S}_0$ is provable in labK ∪ X, and $\mathcal{T}$ defines a labK ∪ X proof for it.
   b) Otherwise, write above each non-initial sequent $\mathcal{S}_i$ of $\mathcal{T}$ the sequent(s) obtained by exhaustively apply rule R to $\mathcal{S}_i$.
3. If there is a topmost sequent $\mathcal{S}_i$ of $\mathcal{T}$ which is non-initial and to which none of the steps in 1 and 2 applied, then terminate.
   $\rightsquigarrow$ $\mathcal{S}_0$ is not provable in labK ∪ X, and the branch $\mathcal{B}^\times$ of $\mathcal{T}$ to which $\mathcal{S}_i$ belongs defines a countermodel for $\mathcal{S}_0$.
   Otherwise, go to step 1.

Back to the theorem [Negri, 2009]

Theorem (Proof or Countermodel). For $\mathcal{S}$ labelled sequent, either $\vdash_{\mathsf{labK} \cup \mathsf{X}} \mathcal{S}$ or $\mathcal{S}$ has a countermodel satisfying the frame conditions in $\mathcal{X}$.

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Proof. Run the proof search algorithm for labK $\cup$ X taking $\mathcal{S}_0 = \mathcal{S}$. Then:

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Proof. Run the proof search algorithm for labK ∪ X taking $\mathcal{S}_0 = \mathcal{S}$. Then:

- If the algorithm terminates in Step 1 or Step 2, then $\vdash_{\mathsf{labK} \cup \mathsf{X}} \mathcal{S}$.

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Proof. Run the proof search algorithm for labK ∪ X taking $\mathcal{S}_0 = \mathcal{S}$. Then:

- If the algorithm terminates in Step 1 or Step 2, then $\vdash_{\mathsf{labK}\cup\mathsf{X}} \mathcal{S}$.
- If the algorithm terminates in Step 3: We construct a countermodel for $\mathcal{S}$ from the finite branch $\mathscr{B}^{\times}$ produced by the algorithm.

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- If the algorithm terminates in Step 1 or Step 2, then $\vdash_{\mathsf{labK}\cup\mathsf{X}} \mathcal{S}$.
- If the algorithm terminates in Step 3: We construct a countermodel for $\mathcal{S}$ from the finite branch $\mathcal{B}^{\times}$ produced by the algorithm.
- If the algorithm does not terminate, then all branches of $\mathcal{T}$ are infinite. We construct a countermodel for $\mathcal{S}$ from any infinite branch $\mathcal{B}^{\times}$ of $\mathcal{T}$.

Let $\mathcal{B}^{\times} = (\mathcal{R}_i, \Gamma_i \Rightarrow \Delta_i)_{i<k}$ be a finite branch in $\mathcal{T}$ produced by the algorithm ($k \in \mathbb{N}$), or an infinite branch in $\mathcal{T}$ ($k = \omega$).
In both cases, $\mathcal{S} = \mathcal{R}_0, \Gamma_0 \Rightarrow \Delta_0$.

We construct a countermodel $\mathcal{M}^{\times}$ from $\mathcal{B}^{\times}$ as follows:

- $W^{\times} = \{x \mid x \text{ occurs in } \mathcal{B}^{\times}\}$;
- $xR^{\times}y$ iff xRy occurs in $(\mathcal{R}_i)_{i<k}$;
- $v^{\times}(p) = \{x \mid x{:}p \text{ occurs in } (\Gamma_i)_{i<k}\}$.

Constructing the countermodel for $\mathcal{S}$ [Negri, 2009]

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Truth Lemma. Take $\rho^{\times}(x) = x$, for each label x occurring in $\mathcal{B}^{\times}$. Then:

- If $x{:}A \in (\Gamma_i)_{i<k}$, then $\mathcal{M}^{\times}, \rho^{\times} \models x{:}A$
- If $x{:}A \in (\Delta_i)_{i<k}$, then $\mathcal{M}^{\times}, \rho^{\times} \not\models x{:}A$

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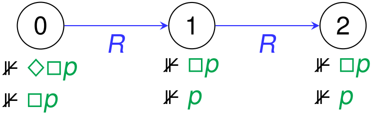
- If $x{:}A \in (\Delta_i)_{i<k}$, then $\mathcal{M}^\times, \rho^\times \not\models x{:}A$

Therefore, $\mathcal{M}^\times, \rho^\times \not\models \mathcal{S}$.

Example

Proof search for $\Rightarrow 0{:}\Diamond\Box p$ in $\mathsf{labK} \cup \{\mathsf{t}, 4\}$

$$
\begin{array}{rc}
 & 0R2, 2R2, 1R2, 1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p, 1{:}\Box p, 2{:}p, 2{:}\Box p \\ \cline{2-2}
\Diamond_R & 2R2, 0R2, 1R2, 1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p, 1{:}\Box p, 2{:}p \\ \cline{2-2}
\mathrm{tr} & 2R2, 1R2, 1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p, 1{:}\Box p, 2{:}p \\ \cline{2-2}
\mathrm{ref} & 1R2, 1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p, 1{:}\Box p, 2{:}p \\ \cline{2-2}
\Box_R & 1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p, 1{:}\Box p \\ \cline{2-2}
\Diamond_R & 1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p \\ \cline{2-2}
\mathrm{ref} & 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p \\ \cline{2-2}
\Box_R & 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p \\ \cline{2-2}
\Diamond_R & 0R0 \Rightarrow 0{:}\Diamond\Box p
\end{array}
$$



Bounding proof search

Sources of non-termination, 1

$$
\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{1{:}2,\ 1{:}\Box q,\ 2{:}q,\ 2{:}q,\ 2{:}q \Rightarrow}\,\Box_L}{1{:}2,\ 1{:}\Box q,\ 2{:}q,\ 2{:}q \Rightarrow}\,\Box_L}{1{:}2,\ 1{:}\Box q,\ 2{:}q \Rightarrow}\,\Box_L}{1{:}2,\ 1{:}\Box q \Rightarrow}\,\Box_L
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{2R3,\ 1R2,\ 0R1 \Rightarrow 0{:}p}\,\mathrm{ser}}{1R2,\ 0R1 \Rightarrow 0{:}p}\,\mathrm{ser}}{0R1 \Rightarrow 0{:}p}\,\mathrm{ser}}{\Rightarrow 0{:}p}\,\mathrm{ser}
$$

Sources of non-termination, 2

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{0R2, 1R2, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p, 1{:}\Box p, 2{:}p, 2{:}\Box p}\;\Box_R}{0R2, 1R2, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p, 1{:}\Box p, 2{:}p}\;\Diamond_R}{1R2, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p, 1{:}\Box p, 2{:}p}\;\mathrm{tr}}{0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p, 1{:}\Box p}\;\Box_R}{0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p}\;\Diamond_R}{0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p}\;\Box_R}{0R0 \Rightarrow 0{:}\Diamond\Box p}\;\Diamond_R}{\Rightarrow 0{:}\Diamond\Box p}\;\mathrm{ref}
$$

In the literature:

- [Negri, 2005]: Minimality argument for some logics in the S5-cube (K, T, S4, S5);
- [Negri, 2014]: Termination for intermediate logics;
- [Garg, Genovese and Negri, 2012]: Termination for multi-modal logics (without symmetry).

As a case study, we shall consider labK $\cup$ {t, 4}, shortened in labS4.

Theorem (Proof or Finite Countermodel). For $\mathcal{S} = x{:}\Gamma \Rightarrow x{:}A$ labelled sequent, either $\vdash_{\mathsf{labS4}} \mathcal{S}$ or $\mathcal{S}$ has a finite countermodel satisfying ref, tr.

A cumulative version of labS4: labS4^{c}

$$\text{init}\ \frac{}{\mathcal{R}, x{:}p, \Gamma \Rightarrow \Delta, x{:}p} \qquad \bot_{\mathrm{L}}\ \frac{}{\mathcal{R}, x{:}\bot, \Gamma \Rightarrow \Delta}$$

$$\wedge_{\mathrm{R}}\ \frac{\mathcal{R}, x{:}A \wedge B, x{:}A, x{:}B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x{:}A \wedge B, \Gamma \Rightarrow \Delta} \qquad \vee_{\mathrm{R}}\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \vee B, x{:}A, x{:}B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \vee B}$$

$$\wedge_{\mathrm{L}}\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \wedge B, x{:}A \quad \mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \wedge B, x{:}B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \wedge B}$$

$$\vee_{\mathrm{L}}\ \frac{\mathcal{R}, x{:}A \vee B, x{:}A, \Gamma \Rightarrow \Delta \quad \mathcal{R}, x{:}A \vee B, x{:}B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x{:}A \vee B, \Gamma \Rightarrow \Delta}$$

$$\rightarrow_{\mathrm{L}}\ \frac{\mathcal{R}, x{:}A \rightarrow B, \Gamma \Rightarrow \Delta, x{:}A \quad \mathcal{R}, x{:}A \rightarrow B, x{:}B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x{:}A \rightarrow B, \Gamma \Rightarrow \Delta}$$

$$\rightarrow_{\mathrm{R}}\ \frac{\mathcal{R}, x{:}A, \Gamma \Rightarrow \Delta, x{:}A \rightarrow B, x{:}B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}A \rightarrow B}$$

$$\Box_{\mathrm{L}}\ \frac{xRy, \mathcal{R}, y{:}A, x{:}\Box A, \Gamma \Rightarrow \Delta}{xRy, \mathcal{R}, x{:}\Box A, \Gamma \Rightarrow \Delta} \qquad \Box_{\mathrm{R}}\ \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, x{:}\Box A, y{:}A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x{:}\Box A}\ y \text{ fresh}$$

$$\Diamond_{\mathrm{L}}\ \frac{xRy, \mathcal{R}, y{:}A, x{:}\Diamond A, \Gamma \Rightarrow \Delta}{\mathcal{R}, x{:}\Diamond A, \Gamma \Rightarrow \Delta}\ y \text{ fresh} \qquad \Diamond_{\mathrm{R}}\ \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, x{:}\Diamond A, y{:}A}{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, x{:}\Diamond A}$$

Main ideas

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Formally: A rule application R to formulas in $\mathcal{S} = \mathcal{R}, \Gamma \Rightarrow \Delta$ is redundant if condition (R) is satisfied:

(ref) If x occurs in $\mathcal{S}$, then xRx occurs in $\mathcal{R}$;

(tr) If xRy and yRz occur in $\mathcal{R}$, then xRz occurs in $\mathcal{R}$;

($\wedge_L$) If $x{:}A \wedge B$ occurs in Γ, then both $x{:}A$ and $x{:}B$ occur in Γ;

($\wedge_R$) If $x{:}A \wedge B$ occurs in Δ, then $x{:}A$ occurs in Δ or $x{:}B$ occur in Δ;

($\Box_L$) If xRy occurs in $\mathcal{R}$ and $x{:}\Box A$ occurs in Γ, then $y{:}A$ occurs in Γ;

($\Box_R$) If $x{:}\Box A$ occurs in Δ, then there is a y such that xRy occurs in $\mathcal{R}$ and $y{:}A$ occurs in Δ.

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($\Box_R$) If $x{:}\Box A$ occurs in Δ, then either

a) there is a k such that kRx occurs in $\mathcal{R}$ and $k \sim x$; otherwise

b) there is a y such that xRy occurs in $\mathcal{R}$ and $y{:}A$ occurs in Δ.

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If *a*) holds, we say that x is a $\Box$-copy of k at $\mathcal{S}$.

Does $\Gamma \models_{\{\mathrm{ref,tr}\}} A$ hold?

0. Place $\mathcal{S}_0 = x{:}\Gamma \Rightarrow x{:}A$ at the root of $\mathcal{T}$.
1. For every topmost sequent $\mathcal{S}_i$ of $\mathcal{T}$, apply as much as possible non-redundant instances of the rules: ref, tr, $\wedge_L, \wedge_R, \vee_L, \vee_R, \rightarrow_L, \rightarrow_R, \Box_L, \Diamond_R$.
2. If every topmost sequent of $\mathcal{T}$ is initial, terminate. $\rightsquigarrow$ $x{:}\Gamma \Rightarrow x{:}A$ is provable in labS4.
3. Otherwise, pick a non-initial topmost sequent $\mathcal{S}_k$ of $\mathcal{T}$.
   a) If there are non-redundant $\Box_R$- or $\Diamond_L$- rule instances that can be applied, apply one such instance. Go to Step 1.
   b) Otherwise terminate. $\rightsquigarrow$ $x{:}\Gamma \Rightarrow x{:}A$ is not provable in labS4.

A countermodel $\mathcal{M}^{\times}$ for a sequent $\mathcal{S} = \mathcal{R}, \Gamma \Rightarrow \Delta$ which is non-initial and to which only redundant rules can be applied is defined as follows:

- $W^{\times} = \{x \mid x \text{ occurs in } \mathcal{S}\}$;
- To define $R^{\times}$, first define:
  - $xR_1^{\times}y$ iff xRy occurs in $\mathcal{R}$;
  - $kR_2^{\times}x$ iff x is a $\Box$-copy (or $\Diamond$-copy) of k.

  $\mathcal{R}^{\times}$ is the reflexive and transitive closure of $R_1^{\times} \cup R_2^{\times}$.
- $v^{\times}(p) = \{x \mid x{:}p \text{ occurs in } \Gamma\}$.

It is easy to verify that $\mathcal{M}^{\times}$ satisfies the frame conditions ref, tr.

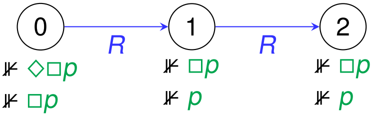
Truth Lemma. Take $\rho^{\times}(x) = x$, for each label x occurring in $\mathcal{S}$. Then:

- If $x{:}A$ occurs in Γ, then $\mathcal{M}^{\times}, \rho^{\times} \models x{:}A$
- If $x{:}A$ occurs in Δ, then $\mathcal{M}^{\times}, \rho^{\times} \not\models x{:}A$

Example

Does $\models_{\{\text{ref},\text{tr}\}} \Diamond\Box p$ hold?

$$
\begin{array}{rc}
 & 0R2, 2R2, 1R2, 1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p, 1{:}\Box p, 2{:}p, 2{:}\Box p \\
\hline
\Diamond_R & 2R2, 0R2, 1R2, 1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p, 1{:}\Box p, 2{:}p \\
\hline
\text{tr} & 2R2, 1R2, 1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p, 1{:}\Box p, 2{:}p \\
\hline
\text{ref} & 1R2, 1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p, 1{:}\Box p, 2{:}p \\
\hline
\Box_R & 1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p, 1{:}\Box p \\
\hline
\Diamond_R & 1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p \\
\hline
\text{ref} & 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p \\
\hline
\Box_R & 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p \\
\hline
\Diamond_R & 0R0 \Rightarrow 0{:}\Diamond\Box p \\
\hline
\text{ref} & \Rightarrow 0{:}\Diamond\Box p
\end{array}
$$



Example

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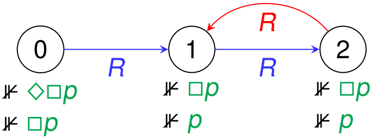
$$
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{
        \cfrac{
          \cfrac{
            \cfrac{
              \cfrac{
                \cfrac{
                  0R2, 2R2, 1R2, 1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p, 1{:}\Box p, 2{:}p, 2{:}\Box p
                }{
                  2R2, 0R2, 1R2, 1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p, 1{:}\Box p, 2{:}p
                }\Diamond_R
              }{
                2R2, 1R2, 1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p, 1{:}\Box p, 2{:}p
              }\text{tr}
            }{
              1R2, 1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p, 1{:}\Box p, 2{:}p
            }\text{ref}
          }{
            1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p, 1{:}\Box p
          }\Box_R
        }{
          1R1, 0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p
        }\Diamond_R
      }{
        0R1, 0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p, 1{:}p
      }\text{ref}
    }{
      0R0 \Rightarrow 0{:}\Diamond\Box p, 0{:}\Box p
    }\Box_R
  }{
    0R0 \Rightarrow 0{:}\Diamond\Box p
  }\Diamond_R
}{
  \Rightarrow 0{:}\Diamond\Box p
}\text{ref}
$$



Termination. The algorithm terminates in a finite number of steps, yielding either a proof or a sequent from which a countermodel can be extracted.

Theorem (Proof or Finite Countermodel). For $\mathcal{S} = x{:}\Gamma \Rightarrow x{:}A$ labelled sequent, either $\vdash_{\mathsf{labS4}} \mathcal{S}$ or $\mathcal{S}$ has a finite countermodel satisfying ref, tr.

Theorem (Semantic completeness). If $\Gamma \models_{\{ref,tr\}} A$ then $\vdash_{\mathsf{labS4}} x{:}\Gamma \Rightarrow x{:}A$.

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Corollary. The validity problem of S4 is decidable.

Properties of labK $\cup$ X

	fml. interpr.	invertible rules	analyticity	termination proof search	counterm. constr.	modularity
labK $\cup$ X	no	yes	yes	yes, for most	yes, easy!	yes

1. Check whether $\models_{\{\text{ref},\text{tr}\}} \Diamond\Box(p \vee \Box(p \to \bot))$ using the terminating algorithm for S4. If the formula is not valid, produce a countermodel.

2. Let $\mathcal{M}^{\times}$ be the countermodel for a sequent $\mathcal{S}$ as defined in Slide 20. Verify that $\mathcal{M}^{\times}$ satisfies the frame conditions ref, tr.
   Then, for $\rho^{\times}(x) = x$, for each label x occurring in $\mathcal{S}$, verify that the Truth Lemma holds, for the cases:
   - If $x{:}\Box A$ occurs in Γ, then $\mathcal{M}^{\times}, \rho^{\times} \models x{:}\Box A$
   - If $x{:}\Box A$ occurs in Δ, then $\mathcal{M}^{\times}, \rho^{\times} \not\models x{:}\Box A$