Proof Theory of Modal Logic

Lecture : Nested Sequents

Tiziano Dalmonte, Marianna Girlando

Free University of Bozen-Bolzano, University of Amsterdam

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Recap

Nested sequents for the S5-cube: Soundness

Independently introduced in:

- [Bull, 1992]; [Kashima, 1994] $\rightsquigarrow$ *nested seuquents*
- [Brünnler, 2006], [Brünnler, 2009] $\rightsquigarrow$ *deep sequents*
- [Poggiolesi, 2008], [Poggiolesi, 2010] $\rightsquigarrow$ *tree-hypersequents*

Main references for this lecture:

- [Lellmann & Poggiolesi, 2022 (arXiv)]
- [Brünnler, 2009], [Brünnler, 2010 (arXiv)]
- [Marin & Straßburger, 2014]

One-sided sequents

Sequent $\quad \Gamma \Rightarrow \Delta \quad \Gamma, \Delta$ multisets of formulas

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Rules of G3cp^{one}

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Exercise. $\vdash_{\mathrm{G3cp}} \Gamma \Rightarrow \Delta$ iff $\vdash_{\mathrm{G3cp}^{one}} \overline{\Gamma}, \Delta$, where $\overline{\Gamma} = \{\overline{A} \mid A \in \Gamma\}$.

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  We call $[\Gamma]$ a boxed sequent.

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Nested sequents are multisets of formulas and boxed sequents:

$$A_1, \ldots, A_m, [\Delta_1], \ldots, [\Delta_n]$$

$$\Gamma = A_1, \dots, A_m, [\Delta_1], \dots, [\Delta_n]$$

To a nested sequent Γ there corresponds the following tree $tr(\Gamma)$, whose nodes $\gamma, \delta, \dots$ are multisets of formulas:

The formula interpretation $i(\Gamma)$ of a nested sequent Γ is defined as:

- If $m = n = 0$, then $i(\Gamma) := \bot$
- Otherwise, $i(\Gamma) := A_1 \vee \cdots \vee A_m \vee \Box(i(\Delta_1)) \vee \cdots \vee \Box(i(\Delta_n))$

Examples

A context is a nested sequent with one or multiple holes, denoted by $\{\,\}$, each taking the place of a formula in the nested sequent.

- Unary context $\Gamma\{\,\}$
- Binary context $\Gamma\{\,\}\{\,\}$

The depth $\mathit{depth}(\Gamma\{\,\})$ of a unary context $\Gamma\{\,\}$ is defined as:

- $\mathit{depth}(\{\,\}) := 0$;
- $\mathit{depth}(\Gamma\{\,\}, \Delta) := \mathit{depth}(\Gamma\{\,\})$;
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- Unary context $\Gamma\{\,\}$ $\rightsquigarrow$ $\Gamma\{\Delta\}$: filling $\Gamma\{\,\}$ with a nested sequent Δ
- Binary context $\Gamma\{\,\}\{\,\}$ $\rightsquigarrow$ $\Gamma\{\Delta_1\}\{\Delta_2\}$: filling $\Gamma\{\,\}\{\,\}$ with Δ_1, Δ_2

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Rules of NK

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Example. Proof of $(\Diamond p \to \Box q) \to \Box(p \to q)$ in NK

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Roadmap

Validity of nested sequents [Kuznets & Straßburger, 2018]

For a nested sequent Γ and a model $\mathcal{M} = \langle W, R, v \rangle$, an $\mathcal{M}$-map for Γ is a map $f : tr(\Gamma) \to W$ such that whenever δ is a child of γ in $tr(\Gamma)$, then $f(\gamma) R f(\delta)$.

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$$\mathcal{M}, f(\delta) \models B, \ \text{ for some } \delta \in tr(\Gamma), \ \text{ for some } B \in \delta$$

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For $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$, a nested sequent is X-valid iff it is satisfied by all $\mathcal{M}$-map for Γ, for all models $\mathcal{M}$ satisfying the frame conditions in X.

Lemma. If Γ is derivable in NK then $\bigvee \Gamma$ is valid in all Kripke frames.

Rules for extensions: $\mathsf{NK} \cup \mathsf{X}^\diamond$

$$\mathsf{d}^\diamond \frac{\Gamma\{\diamond A, [A]\}}{\Gamma\{\diamond A\}} \qquad \mathsf{t}^\diamond \frac{\Gamma\{\diamond A, A\}}{\Gamma\{\diamond A\}} \qquad \mathsf{b}^\diamond \frac{\Gamma\{[\Delta, \diamond A], A\}}{\Gamma\{[\Delta, \diamond A]\}}$$

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Example. Proof of $\square p \to \square\square p$ in $\mathrm{NK} \cup \{\mathrm{t}, 4\}$

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$$\text{wk}\ \frac{\Gamma\{\emptyset\}}{\Gamma\{\Delta\}} \qquad \text{ctr}\ \frac{\Gamma\{\Delta, \Delta\}}{\Gamma\{\Delta\}} \qquad \text{cut}\ \frac{\Gamma\{A\} \quad \Gamma\{\overline{A}\}}{\Gamma\{\emptyset\}}$$

Structural rules [Brünnler, 2009]

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For $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$:

Lemma. The rules wk and ctr are hp-admissible in $\mathsf{NK} \cup \mathsf{X}^{\Diamond}$.

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Proposition. Rule $5^{\Diamond}$ is derivable in $\mathsf{NK} \cup \{5_1^{\Diamond}, 5_2^{\Diamond}, 5_3^{\Diamond}\} \cup \{\mathsf{wk}\}$.

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Lemma. If Γ is derivable in $\mathsf{NK} \cup \mathsf{X}^{\Diamond}$ then $\bigvee \Gamma$ is valid in all X-frames.

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Nested sequents with explicit contraction [Marin & Straßburger, 2014]

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Lemma. The rule wk is hp-admissible in $\mathsf{NK} \cup \mathsf{X}^{\Diamond}$.

Proposition. Γ is derivable in $\mathsf{NK} \cup \mathsf{X}^{\Diamond}$ iff Γ is derivable in $\mathsf{NK}_{\mathsf{ctr}} \cup \mathsf{X}^{\Diamond}_{\mathsf{ctr}}$.

Roadmap

Nested sequents for the S5-cube: Completeness

Three problems for completeness

- Axiom 5, that is, $\Diamond A \to \Box\Diamond A$, is valid in all $\{\mathrm{b}, 4\}$-frames, but it is not derivable in $\mathrm{NK} \cup \{\mathrm{b}^{\Diamond}, 4^{\Diamond}\}$.

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Failed proof of $\Diamond A \to \Box\Diamond A$ in $NK \cup \{b^\Diamond, 4^\Diamond\}$

$$
\dfrac{\dfrac{\dfrac{\dfrac{[\bar{p}], p, [\Diamond p]}{[\bar{p}], [\Diamond p]}\, b^\Diamond}{\Box\bar{p}, [\Diamond p]}\, \Box}{\Box\bar{p}, \Box\Diamond p}\, \Box}{\Box\bar{p} \vee \Box\Diamond p}\, \vee
$$

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- Axiom 4, that is, $A \to \Box\Box A$, is valid in all $\{t, 5\}$-frames, but it is not derivable in $\mathsf{NK} \cup \{t^\Diamond, 5^\Diamond\}$.

Failed proof of $\Diamond A \to \Box\Diamond A$ in $\mathsf{NK} \cup \{b^\Diamond, 4^\Diamond\}$

$$\dfrac{\dfrac{\dfrac{\dfrac{[\bar{p}], p, [\Diamond p]}{[\bar{p}], [\Diamond p]}\,b^\Diamond}{\Box\bar{p}, [\Diamond p]}\,\Box}{\Box\bar{p}, \Box\Diamond p}\,\Box}{\Box\bar{p} \vee \Box\Diamond p}\,\vee$$

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- Axiom 4, that is, $A \to \Box\Box A$, is valid in all $\{\mathrm{b}, 5\}$-frames, but it is not derivable in $\mathrm{NK} \cup \{\mathrm{b}^\Diamond, 5^\Diamond\}$.

Failed proof of $\Diamond A \to \Box\Diamond A$ in $\mathrm{NK} \cup \{\mathrm{b}^\Diamond, 4^\Diamond\}$

$$\dfrac{\dfrac{\dfrac{\dfrac{[\bar{p}], p, [\Diamond p]}{[\bar{p}], [\Diamond p]}\,\mathrm{b}^\Diamond}{\Box\bar{p}, [\Diamond p]}\,\Box}{\Box\bar{p}, \Box\Diamond p}\,\Box}{\Box\bar{p} \vee \Box\Diamond p}\,\vee$$

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For $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$, the 45-closure of X is defined as:

$$\hat{\mathsf{X}} = \begin{cases} \mathsf{X} \cup \{4\} & \text{if } \{\mathsf{b}, 5\} \subseteq \mathsf{X} \text{ or } \{\mathsf{t}, 5\} \subseteq \mathsf{X} \\ \mathsf{X} \cup \{5\} & \text{if } \{\mathsf{b}, 4\} \subseteq \mathsf{X} \\ \mathsf{X} & \text{otherwise} \end{cases}$$

We say that X is 45-closed if $\mathsf{X} = \hat{\mathsf{X}}$.

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Proposition. For $X \subseteq \{d, t, b, 4, 5\}$ X is 45-closed iff, for $\rho \in \{4, 5\}$, it holds that if ρ is valid in all X-frames, then $\rho \in X$.

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Proposition. For $X \subseteq \{d, t, b, 4, 5\}$ X is 45-closed iff, for $\rho \in \{4, 5\}$, it holds that if ρ is valid in all X-frames, then $\rho \in X$.

To prove:

Theorem (Completeness). For $X \subseteq \{d, t, b, 4, 5\}$, if Γ is X-valid, then Γ is derivable in $\mathsf{NK} \cup \hat{X}^{\diamond}$.

Theorem (Completeness). For $X \subseteq \{d, t, b, 4, 5\}$, if Γ is X-valid, then Γ is derivable in $NK \cup \hat{X}^{\Diamond}$.

Theorem (Cut-elimination). For $X \subseteq \{d, t, b, 4, 5\}$ 45-closed, if Γ is derivable in $NK \cup X^{\diamond} \cup \{cut\}$, then it is derivable in $NK \cup X^{\diamond}$.

Theorem (Cut-elimination). For $X \subseteq \{d, t, b, 4, 5\}$ 45-closed, if Γ is derivable in $NK \cup X^{\Diamond} \cup \{cut\}$, then it is derivable in $NK \cup X^{\Diamond}$.

The proof uses:

- A generalised version of cut (eliminable)

$$\dfrac{\square\,\dfrac{\Gamma\{[A],[\Delta]\}}{\Gamma\{\square A,[\Delta]\}} \qquad \mathrm{tr}^{\Diamond}\,\dfrac{\Gamma\{\Diamond\overline{A},[\Diamond\overline{A},\Delta]\}}{\Gamma\{\Diamond\overline{A},[\Delta]\}}}{\Gamma\{[\Delta]\}}\,\mathrm{cut}$$

- Additional structural modal rules (admissible)

$$\mathsf{cut}\ \frac{\Gamma\{A\} \quad \Gamma\{\overline{A}\}}{\Gamma\{\emptyset\}} \qquad\qquad \mathsf{Y\text{-}cut}\ \frac{\Gamma\{\Box A\}\{\emptyset\}^n \quad \Gamma\{\Diamond\overline{A}\}\{\Diamond\overline{A}\}^n}{\Gamma\{\emptyset\}\{\emptyset\}^n}$$

In the Y-cut:

- $\{\Delta\}^n$ denotes $\overbrace{\{\Delta\}\ldots\{\Delta\}}^{n \text{ times}}$;
- $n \geq 0$;
- $\mathsf{Y} \subseteq \{4, 5\}$;
- there is a derivation of $\Gamma\{\Diamond\overline{A}\}\{\Diamond\overline{A}\}^n$ to $\Gamma\{\Diamond\overline{A}\}\{\emptyset\}^n$ in system $\mathsf{Y}^\Diamond$.

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The rank of the cut formula A is defined as the complexity of A, plus one. The cut rank of a derivation is the maximum of the ranks of its cuts.

$$\text{cut}\ \frac{\Gamma\{A\} \quad \Gamma\{\overline{A}\}}{\Gamma\{\emptyset\}} \qquad\qquad \text{Y-cut}\ \frac{\Gamma\{\Box A\}\{\emptyset\}^n \quad \Gamma\{\Diamond\overline{A}\}\{\Diamond\overline{A}\}^n}{\Gamma\{\emptyset\}\{\emptyset\}^n}$$

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The notions of cut rank-preserving admissible rule and cut rank-preserving invertible rule are defined analogously to the notions of hp admissible rule and hp invertible rule.

Example: 4-cut

$$\text{cut}\,\frac{\Gamma\{A\} \quad \Gamma\{\overline{A}\}}{\Gamma\{\emptyset\}} \qquad\qquad \text{Y-cut}\,\frac{\Gamma\{\Box A\}\{\emptyset\}^n \quad \Gamma\{\Diamond\overline{A}\}\{\Diamond\overline{A}\}^n}{\Gamma\{\emptyset\}\{\emptyset\}^n}$$

If $\mathsf{Y} = \{4\}$, then $\Gamma\{\ \}\{\ \}^n$ is of the form $\Gamma_1\{\{\ \}, \Gamma_2\{\ \}^n\}$:

$$\text{4-cut}\,\frac{\Gamma_1\{\{\Box A\}, \Gamma_2\{\emptyset\}^n\} \quad \Gamma_1\{\{\Diamond A\}, \Gamma_2\{\Diamond A\}^n\}}{\Gamma_1\{\{\emptyset\}, \Gamma_2\{\emptyset\}^n\}}$$

$$\text{cut}\,\frac{\Box\,\dfrac{\Gamma\{[A], [\Delta]\}}{\Gamma\{\Box A, [\Delta]\}} \quad 4^{\Diamond}\,\dfrac{\Gamma\{\Diamond\overline{A}, [\Diamond\overline{A}, \Delta]\}}{\Gamma\{\Diamond\overline{A}, [\Delta]\}}}{\Gamma\{[\Delta]\}} \quad \rightsquigarrow \quad \text{4-cut}\,\frac{\Box\,\dfrac{\Gamma\{[A], [\Delta]\}}{\Gamma\{\Box A, [\Delta]\}} \quad \Gamma\{\Diamond\overline{A}, [\Diamond\overline{A}, \Delta]\}}{\Gamma\{[\Delta]\}}$$

$$\mathsf{d}^{[\,]}\ \frac{\Gamma\{[\emptyset]\}}{\Gamma\{\emptyset\}} \qquad \mathsf{t}^{[\,]}\ \frac{\Gamma\{[\Delta]\}}{\Gamma\{\Delta\}} \qquad \mathsf{b}^{[\,]}\ \frac{\Gamma\{[\Sigma,[\Delta]]\}}{\Gamma\{\Delta,[\Sigma]\}}$$

$$4^{[\,]}\ \frac{\Gamma\{[\Delta],[\Sigma]\}}{\Gamma\{[[\Delta],\Sigma]\}} \qquad 5^{[\,]}\ \frac{\Gamma\{\Delta\}\{\emptyset\}}{\Gamma\{\emptyset\}\{\Delta\}}\ \mathit{depth}(\Gamma\{\,\}\{\emptyset\}) > 0$$

For $\mathsf{X} \subseteq \{\mathsf{d},\mathsf{t},\mathsf{b},4,5\}$, we write $\mathsf{X}^{[\,]}$ for the corresponding subset of $\{\mathsf{d}^{[\,]},\mathsf{t}^{[\,]},\mathsf{b}^{[\,]},4^{[\,]},5^{[\,]}\}$.

Structural modal rules

$$\mathsf{d}^{[\,]}\ \frac{\Gamma\{[\emptyset]\}}{\Gamma\{\emptyset\}} \qquad \mathsf{t}^{[\,]}\ \frac{\Gamma\{[\Delta]\}}{\Gamma\{\Delta\}} \qquad \mathsf{b}^{[\,]}\ \frac{\Gamma\{[\Sigma,[\Delta]]\}}{\Gamma\{\Delta,[\Sigma]\}}$$

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Example. Proof of $\Diamond A \to \Box\Diamond A$ in $\mathsf{NK} \cup \{\mathsf{b}^{[\,]},4^{[\,]}\}$

$$\vee\ \cfrac{\Box\ \cfrac{\Box\ \cfrac{\mathsf{b}^{[\,]}\ \cfrac{4^{[\,]}\ \cfrac{\Diamond\ \cfrac{\mathsf{init}\ \cfrac{}{[[[\bar{p},p],\Diamond p]]}}{[[[\bar{p}],\Diamond p]]}}{[[[\bar{p}]],\Diamond p]}}{[\bar{p}],[\Diamond p]}}{\Box\bar{p},[\Diamond p]}}{\Box\bar{p},\Box\Diamond p}}{\Box\bar{p}\vee\Box\Diamond p}$$

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Problem: Rule $\mathsf{d}^{[\,]}$ is not admissible in the presence of cut.

Solution:

- Show how derivations in $\mathsf{NK} \cup \{\mathsf{t}^{\Diamond}, \mathsf{b}^{\Diamond}, 4^{\Diamond}, 5^{\Diamond}\} \cup \{\mathsf{d}^{[\,]}\} \cup \{\mathsf{cut}\}$ can be transformed into derivations in $\mathsf{NK} \cup \{\mathsf{t}^{\Diamond}, \mathsf{b}^{\Diamond}, 4^{\Diamond}, 5^{\Diamond}\} \cup \{\mathsf{d}^{[\,]}\}$;
- Show that $\mathsf{d}^{[\,]}$ is admissible in $\mathsf{NK} \cup \mathsf{X}^{\Diamond}$.

Problem: Rule $\mathsf{d}^{[\,]}$ is not admissible in the presence of cut.

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- Show that $\mathsf{d}^{[\,]}$ is admissible in $\mathsf{NK} \cup \mathsf{X}^\Diamond$.

For $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$:

Lemma (Weakening, Contraction). The rules wk and ctr are height- and cut-rank preserving admissible in $\mathsf{NK} \cup \mathsf{X}^\Diamond \cup \{\mathsf{d}^{[\,]}\} \cup \{\mathsf{cut}\}$.

Problem: Rule $\mathsf{d}^{[]}$ is not admissible in the presence of cut.

Solution:

- Show how derivations in $\mathsf{NK} \cup \{\mathsf{t}^{\Diamond}, \mathsf{b}^{\Diamond}, 4^{\Diamond}, 5^{\Diamond}\} \cup \{\mathsf{d}^{[]}\} \cup \{\mathsf{cut}\}$ can be transformed into derivations in $\mathsf{NK} \cup \{\mathsf{t}^{\Diamond}, \mathsf{b}^{\Diamond}, 4^{\Diamond}, 5^{\Diamond}\} \cup \{\mathsf{d}^{[]}\}$;
- Show that $\mathsf{d}^{[]}$ is admissible in $\mathsf{NK} \cup \mathsf{X}^{\Diamond}$.

For $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$:

Lemma (Weakening, Contraction). The rules wk and ctr are height- and cut-rank preserving admissible in $\mathsf{NK} \cup \mathsf{X}^{\Diamond} \cup \{\mathsf{d}^{[]}\} \cup \{\mathsf{cut}\}$.

Lemma (Invertibility). All the rules of $\mathsf{NK} \cup \mathsf{X}^{\Diamond} \cup \{\mathsf{d}^{[]}\} \cup \{\mathsf{cut}\}$ are height- and cut-rank preserving invertible.

Lemma (Admissibility of structural modal rules).

(*i*) Let $X \subseteq \{t, b, 4, 5\}$ be 45-closed, and let $\rho \in X$. Then rule $\rho^{[\,]}$ is cut-rank preserving admissible in $NK \cup X^{\diamond} \cup \{cut\}$ and in $NK \cup X^{\diamond} \cup \{cut\} \cup \{d^{[\,]}\}$.

(*ii*) Let $X \subseteq \{d, t, b, 4, 5\}$ be 45-closed, and let $d \in X$. Then rule $d^{[\,]}$ is admissible in $NK \cup X^{\diamond}$.

Proof. Case $b^{[\,]}$ is admissible in $NK \cup \{b^{\diamond}, 4^{\diamond}, 5^{\diamond}\} \cup \{cut\} \cup \{d^{[\,]}\}$.

$$b^{[\,]}\,\frac{\Gamma\{[\Sigma, [\Delta]]\}}{\Gamma\{\Delta, [\Sigma]\}}$$

$$4^{\diamond}\,\cfrac{\Gamma\{[\diamond A, \Sigma, [, \diamond A, \Delta]]\}}{b^{[\,]}\,\cfrac{\Gamma\{[\diamond A, \Sigma, [\Delta]]\}}{\Gamma\{\Delta, [\diamond A, \Sigma]\}}} \quad \rightsquigarrow \quad b^{[\,]}\,\cfrac{[\diamond A, \Sigma, [, \diamond A, \Delta]]}{5^{\diamond}\,\cfrac{\Gamma\{\Delta, \diamond A, [\diamond A, \Sigma]\}}{\Gamma\{\Delta, [\diamond A, \Sigma]\}}}$$

Let $\mathsf{X} \subseteq \{\mathsf{t}, \mathsf{b}, 4, 5\}$, and let Y be a subset of $\{4, 5\} \cap \mathsf{X}$. Then:

- Let $\mathcal{D}$ be a proof in $\mathsf{NK} \cup \mathsf{X}^{\diamond} \cup \{\mathsf{cut}\}$ (or in $\mathsf{NK} \cup \mathsf{X}^{\diamond} \cup \{\mathsf{cut}\} \cup \{\mathsf{ser}^{[\,]}\}$) as displayed below, with $cr(\mathcal{D}_1) = cr(\mathcal{D}_2) = p = c(A)$. Then, we can construct the proof $\mathcal{D}^*$ below in the same system, with $cr(\mathcal{D}^*) = p$.

$$\mathcal{D} = \quad \text{cut}\,\dfrac{\overset{\mathcal{D}_1}{\Gamma\{A\}} \qquad \overset{\mathcal{D}_2}{\Gamma\{\overline{A}\}}}{\Gamma\{\emptyset\}} \quad \rightsquigarrow \quad \overset{\mathcal{D}^*}{\Gamma\{\emptyset\}}$$

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$$\mathcal{D} = \quad \mathsf{Y}\text{-cut}\,\dfrac{\overset{\mathcal{D}_1}{\Gamma\{\Box A\}\{\emptyset\}^n} \qquad \overset{\mathcal{D}_2}{\Gamma\{\Diamond\overline{A}\}\{\Diamond\overline{A}\}^n}}{\Gamma\{\emptyset\}\{\emptyset\}^n} \quad \rightsquigarrow \quad \overset{\mathcal{D}^*}{\Gamma\{\emptyset\}}$$

Proof: By induction on the sum of heights of $\mathcal{D}_1$ and $\mathcal{D}_2$.

Two cases of the Reduction Lemma

$$
\text{cut}\,\dfrac{\Box\,\dfrac{\Gamma\{[A],[\Delta]\}}{\Gamma\{\Box A,[\Delta]\}} \qquad 4^{\Diamond}\,\dfrac{\Gamma\{\Diamond\overline{A},[\Diamond\overline{A},\Delta]\}}{\Gamma\{\Diamond\overline{A},[\Delta]\}}}{\Gamma\{[\Delta]\}}
\quad\rightsquigarrow\quad
\text{4-cut}\,\dfrac{\Box\,\dfrac{\Gamma\{[A],[\Delta]\}}{\Gamma\{\Box A,[\Delta]\}} \qquad \Gamma\{\Diamond\overline{A},[\Diamond\overline{A},\Delta]\}}{\Gamma\{[\Delta]\}}
$$

$$
\text{4-cut}\,\dfrac{\Box\,\dfrac{\Gamma\{[A],[[\Sigma]]\}}{\Gamma\{\Box A,[[\Sigma]]\}} \qquad \Diamond\,\dfrac{\Gamma\{\Diamond\overline{A},[\Diamond\overline{A},[\overline{A},\Sigma]]\}}{\Gamma\{\Diamond\overline{A},[\Diamond\overline{A},[\Sigma]]\}}}{\Gamma\{[[\Sigma]]\}}
\quad\rightsquigarrow
$$

$$
\rightsquigarrow\quad
\text{cut}\,\dfrac{4^{[\,]}\,\dfrac{4^{[\,]}\,\dfrac{\Gamma\{[A],[[\Sigma]]\}}{\Gamma\{[[A],[\Sigma]]\}}}{\Gamma\{[[A,\Sigma]]\}} \qquad \text{4-cut}\,\dfrac{\text{wk}\,\dfrac{\Box\,\dfrac{\Gamma\{[A],[[\Sigma]]\}}{\Gamma\{\Box A,[[\Sigma]]\}}}{\Gamma\{\Box A,[[\overline{A},\Sigma]]\}} \qquad \Gamma\{\Diamond\overline{A},[\Diamond\overline{A},[\overline{A},\Sigma]]\}}{\Gamma\{[[\overline{A},\Sigma]]\}}}{\Gamma\{[[\Sigma]]\}}
$$

Theorem (Cut-elimination). For $X \subseteq \{d, t, b, 4, 5\}$ 45-closed, if Γ is derivable in $NK \cup X^{\diamond} \cup \{cut\}$, then it is derivable in $NK \cup X^{\diamond}$.

Solution # 2 [Marin & Straßburger, 2014]

Can we get rid of the 45-closure condition?

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YES: by adding to NK both the propagation rules $\mathsf{X}^{\Diamond}$ and the structural rules $\mathsf{X}^{[\,]}$. The price to pay is that contraction is no longer admissible.

Theorem. For $\mathsf{X} = \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$, and Γ a set of formulas, it holds that Γ is derivable in $\mathsf{NK}_{\mathsf{ctr}} \cup \mathsf{X}^{\Diamond}_{\mathsf{ctr}} \cup \mathsf{X}^{[\,]}$ iff Γ is X-valid.

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Can we get rid of the propagation rules, and use $NK_{ctr} \cup X^{[\,]}$?

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Can we get rid of the propagation rules, and use $\mathrm{NK}_{\mathrm{ctr}} \cup X^{[\,]}$?

NO, some combinations are incomplete, and one example is given in [Marin & Straßburger, 2014].

Roadmap

Conclusions

Comparison

Within the S5-cube ($X \subseteq \{d, t, b, 4, 5\}$):

	fml. interpr.	invertible rules	analyticity	termination proof search	counterm. constr.	modularity
$\mathsf{labK} \cup X$	no	yes	yes	yes, for most	yes, easy!	yes
HS5	yes	yes	yes	yes	yes, easy!	no
$\mathsf{NK} \cup X^{\Diamond}$	yes	yes	yes	yes	yes	45-clause

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HS5	yes	yes	yes	yes	yes, easy!	no
$\mathsf{NK} \cup \mathsf{X}^{\diamond}$	yes	yes	yes	yes	yes	45-clause

And beyond the S5-cube?

A few words in conclusion

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- We have presented the most standard (and possibly simplest) extensions of the sequent calculus. However, once established that one can extend the language or the structure, there is no limit to imagination: 2-sequents, display calculus, sequents with histories, linear nested sequents, grafted hypersequents, etc.

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- We have presented labelled and structured calculi for the S5 cube of normal modal logics because it is a well-known family of modal logics, and it is the context where this solutions have been initially developed. However, the same or similar solutions have been applied to many other kinds of logics: non-normal modal logics, intuitionistic modal logics, conditional logics, temporal logics, intermediate logics, etc.

Questions?

	fml. interpr.	invertible rules	analyticity	termination proof search	counterm. constr.	modularity
labK $\cup$ X	no	yes	yes	yes, for most	yes, easy!	yes
HS5	yes	yes	yes	yes	yes, easy!	no
NK $\cup$ $X^{\diamond}$	yes	yes	yes	yes	yes	45-clause

A few words in conclusion

- Questions, suggestions, discussion etc. are very welcome
  "m.girlando at uva dot nl"   "tiziano.dalmonte at unibz dot it"
- Thank you for attending, we hope you enjoyed the course