Proof Theory of Modal Logic

Lecture 3 (and a half): Hypersequent calculi (part I)

Tiziano Dalmonte & Marianna Girlando

Free University of Bozen-Bolzano
ILLC, Universtiy of Amsterdam

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- Lecture 1: Sequent calculi
- Lecture 2: Labelled sequent calculi
- In this lecture we start looking at structured calculi, that extend sequent calculi with additional structural connectives

In particular, we now look at hypersequent calculi

- Simple generalisation of sequent calculi
- Introduced by [Mints, 1968] [Pottinger, 1983], [Avron, 1987] to provide cut-free calculi for modal and relevant logics

☞ In this lecture we mostly focus on modal logic S5

Axiomatisation of S5

$$\mathsf{K} + \begin{array}{ll} \mathsf{t} & \Box A \to A \\ \mathsf{4} & \Box A \to \Box\Box A \\ \mathsf{b} & A \lor \Box\neg\Box A \end{array} \quad \text{or} \quad \mathsf{K} + \begin{array}{ll} \mathsf{t} & \Box A \to A \\ \mathsf{5} & \Box A \lor \Box\neg\Box A \end{array}$$

Semantics of S5

Kripke models with equivalence relation

Complexity of S5

The validity/derivability problem for S5 is coNP-complete

Recap

- No cut-free, Gentzen-style sequent calculus for S5 (Lecture 1)
- Cut-free labelled calculus for S5 (Lecture 2)
- What about an internal, structured calculus for S5?

A hypersequent calculus for S5

Main reference for this calculus

- *A cut-free simple sequent calculus for modal logic S5* [Poggiolesi, 2008]: Definition of the calculus and structural analysis

Further references

- [Lellmann, 2016]: Optimal proof-search procedure in the calculus
- [Restall, 2007]: A version of the calculus with explicit structural rules

Hypersequent Finite multiset of sequents, written

$$\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$$

where $\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n$ are the components of the hypersequent

Formula interpretation

$$i(\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n)$$
$$=$$
$$\Box(\bigwedge \Gamma_1 \to \bigvee \Delta_1) \vee \ldots \vee \Box(\bigwedge \Gamma_n \to \bigvee \Delta_n)$$

☞ Differently from labelled sequents, hypersequents can be interpreted as formulas

Initial hypersequents and propositional hypersequent rules

$$\text{init}\quad p, \Gamma \Rightarrow \Delta, p \qquad \rightsquigarrow \qquad \text{init}\quad \mathcal{H} \mid p, \Gamma \Rightarrow \Delta, p$$

$$\vee_R \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B} \qquad \rightsquigarrow \qquad \vee_R \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A, B}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \vee B}$$

Modal rules for S5

$$\Box_L \frac{\mathcal{H} \mid \Box A, \Gamma \Rightarrow \Delta \mid A, \Sigma \Rightarrow \Pi}{\mathcal{H} \mid \Box A, \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi} \qquad \Box_L^t \frac{\mathcal{H} \mid A, \Box A, \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Box A, \Gamma \Rightarrow \Delta}$$

$$\Box_R \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Box A}$$

Example. Derivation of axiom B

$$\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \mid \Box A \Rightarrow}{\Rightarrow A \mid \Box A \Rightarrow}\,\Box_L}{\Rightarrow A \mid \Rightarrow \neg\Box A}\,\neg_R}{\Rightarrow A, \Box\neg\Box A}\,\Box_R}{\Rightarrow A \vee \Box\neg\Box A}\,\vee_R$$

Exercise. Derive axioms k, t, 4, 5

Soundness

Theorem. If $\vdash_{\mathbf{HS5}} \mathcal{H}$, then $\vdash_{\mathrm{S5}} i(\mathcal{H})$

Proof sketch (i). We consider simple instances of the rules

$$\Box_L \; \frac{\Box A \Rightarrow \mid A \Rightarrow B}{\Box A \Rightarrow \mid \Rightarrow B}$$

i.	$\vdash \Box\neg\Box A \vee \Box(A \to B)$	$(i(P))$
ii.	$\vdash \Box\neg\Box A \vee \neg\Box\neg\Box A$	(CPL)
iii.	$\vdash \neg\Box\neg\Box A \to \Box A$	(axiom 5)
iv.	$\vdash \Box A \wedge \Box(A \to B) \to \Box B$	(axiom k)
v.	$\vdash \Box\neg\Box A \vee \Box B = i(C)$	(by classical reasoning)

Soundness

Theorem. If $\vdash_{\mathbf{HS5}} \mathcal{H}$, then $\vdash_{\mathrm{S5}} i(\mathcal{H})$

Proof sketch (ii). We consider simple instances of the rules

$$\Box_R \ \frac{B \Rightarrow C \mid \Rightarrow A}{B \Rightarrow C, \Box A}$$

i.	$\vdash \Box(B \to C) \vee \Box A$	$(i(P))$
ii.	$\vdash (B \to C) \to (B \to C \vee \Box A)$	(CPL)
iii.	$\vdash \Box(B \to C) \to \Box(B \to C \vee \Box A)$	(ii, by K valid rule)
iv.	$\vdash \Box A \to (B \to C \vee \Box A)$	(CPL)
v.	$\vdash \Box\Box A \to \Box(B \to C \vee \Box A)$	(iv, by K valid rule)
vi.	$\vdash \Box A \to \Box\Box A$	(axiom 4)
vii.	$\vdash \Box A \to \Box(B \to C \vee \Box A)$	(from iv, vi)
viii.	$\vdash \Box(B \to C \vee \Box A) = i(C)$	(from i, iii, vii)

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vii.	$\vdash \Box A \to \Box(B \to C \vee \Box A)$	(from iv, vi)
viii.	$\vdash \Box(B \to C \vee \Box A) = i(C)$	(from i, iii, vii)

Exercise. Prove soundness of all the rules of **HS5**

Theorem. All rules of **HS5** are hp-invertible

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(base case) If $h = 0$, then the conclusion $\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Box A$ of $\Box_R$ is an initial hypersequent. There are three possibilities:

1. $\mathcal{H}$ is an intial hypersequent
2. $p \in \Gamma \cap \Delta$ for some p
3. $\bot \in \Gamma \cap \Delta$

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In each of these cases, the premiss $\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A$ of $\Box_R$ is an initial hypersequent, hence it is derivable with height 0.

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1. $\Box A$ is principal in the last rule application
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There are two possibilities

1. $\Box A$ is principal in the last rule application
2. $\Box A$ is not principal in the last rule application

(case 1.) If $\Box A$ is principal in the last rule application, then the last rule application is precisely

$$\frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Box A} \; \Box_\mathsf{R}$$

which means that the premiss $\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A$ has a derivation of height $h - 1$.

(case 2.) If $\Box A$ is not principal in the last rule application, then the last rule application has the form

$$\frac{\mathcal{H}' \mid \Sigma \Rightarrow \Pi \mid \Gamma \Rightarrow \Delta, \Box A \mid \Rightarrow B}{\mathcal{H}' \mid \Sigma \Rightarrow \Pi, \Box B \mid \Gamma \Rightarrow \Delta, \Box A}\ \Box_R$$

where $\mathcal{H}' \mid \Sigma \Rightarrow \Pi, \Box B \ = \mathcal{H}$

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where $\mathcal{H}' \mid \Sigma \Rightarrow \Pi, \Box B \; = \mathcal{H}$ and the premiss $\mathcal{H}' \mid \Sigma \Rightarrow \Pi \mid \Gamma \Rightarrow \Delta, \Box A \mid \Rightarrow B$ has a derivation of height $h - 1$.

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(Alternatively, we may also have $\Box B \in \Delta$, the proof is analogous in this case.)

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(Alternatively, we may also have $\Box B \in \Delta$, the proof is analogous in this case.)

Then, by inductive hypothesis, $\mathcal{H}' \mid \Sigma \Rightarrow \Pi \mid \Gamma \Rightarrow \Delta \mid \Rightarrow B \mid \Rightarrow A$ has a derivation of height $\leq h-1$

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where $\mathcal{H}' \mid \Sigma \Rightarrow \Pi, \Box B = \mathcal{H}$ and the premiss $\mathcal{H}' \mid \Sigma \Rightarrow \Pi \mid \Gamma \Rightarrow \Delta, \Box A \mid \Rightarrow B$ has a derivation of height $h - 1$.

(Alternatively, we may also have $\Box B \in \Delta$, the proof is analogous in this case.)

Then, by inductive hypothesis, $\mathcal{H}' \mid \Sigma \Rightarrow \Pi \mid \Gamma \Rightarrow \Delta \mid \Rightarrow B \mid \Rightarrow A$ has a derivation of height $\leq h - 1$ and by applying $\Box_R$ to this hypersequent, we obtain a derivation of height $\leq h$ of $\mathcal{H}' \mid \Sigma \Rightarrow \Pi, \Box B \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A$ = $\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A$.

qed

Structural rules

$$\mathsf{wk_L}\ \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta}{\mathcal{H} \mid A, \Gamma \Rightarrow \Delta} \qquad \mathsf{wk_R}\ \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A}$$

$$\mathsf{ctr_L}\ \frac{\mathcal{H} \mid A, A, \Gamma \Rightarrow \Delta}{\mathcal{H} \mid A, \Gamma \Rightarrow \Delta} \qquad \mathsf{ctr_R}\ \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A, A}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A}$$

$$\mathsf{wk_{ext}}\ \frac{\mathcal{H}}{\mathcal{H} \mid \Gamma \Rightarrow \Delta} \qquad \mathsf{ctr_{ext}}\ \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Gamma \Rightarrow \Delta}$$

☞ Note: external forms of weakening and contraction

The cut rule

$$\mathsf{cut}\ \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \qquad \mathcal{H}' \mid A, \Gamma' \Rightarrow \Delta'}{\mathcal{H} \mid \mathcal{H}' \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Theorem. Left, right and external weakening and contraction are hp-admissible in **HS5**

Sketch of proof. By induction on the height of the derivation of the premiss (*exercise*)

Theorem. Left, right and external weakening and contraction are hp-admissible in **HS5**

Sketch of proof. By induction on the height of the derivation of the premiss (*exercise*)

Hint. In order to prove the hp-admissibility of some structural rules you may need the following (nice) rule

$$\text{merge}\ \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi}{\mathcal{H} \mid \Gamma, \Sigma \Rightarrow \Delta, \Pi}$$

Theorem. The rule merge is hp-admissible in **HS5**

Sketch of proof. By induction on the height of the derivation of the premiss (*exercise*)

Theorem. Cut is admissible in **HS5**

Proof sketch. By induction on the complexity of the cut formula and subinduction on the cut height.

As an example, consider the following derivation, with the cut formula $\Box A$ principal in the last rule application in both premisses of cut

$$\dfrac{\Box_R\dfrac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Box A} \qquad \dfrac{\mathcal{H}' \mid A, \Box A, \Gamma' \Rightarrow \Delta'}{\mathcal{H}' \mid \Box A, \Gamma' \Rightarrow \Delta'}\Box^t_L}{\mathcal{H} \mid \mathcal{H}' \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\text{cut}$$

Converted into the following, with one application of cut at a lower height, and one application of cut with a cut formula of lower complexity

$$\dfrac{\dfrac{\dfrac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A \qquad \dfrac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Box A \qquad \mathcal{H}' \mid A, \Box A, \Gamma' \Rightarrow \Delta'}{\mathcal{H} \mid \mathcal{H}' \mid \Gamma, \Gamma', A \Rightarrow \Delta, \Delta'}\text{cut}}{\mathcal{H} \mid \mathcal{H} \mid \mathcal{H}' \mid \Gamma \Rightarrow \Delta \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\text{cut}}{\mathcal{H} \mid \mathcal{H} \mid \mathcal{H}' \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\text{wk}^*}{\mathcal{H} \mid \mathcal{H}' \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\text{ctr}_{\text{ext}} \times 2$$

(where wk* denotes multiple applications of (left and right) weakening

Completeness

Theorem. If $\vdash_{S5} A$, then $\vdash_{\mathbf{HS5}} \Rightarrow A$

Proof sketch.

- All axioms of S5 are derivable in **HS5** (*exercise*)
- The necessitation rule is admissible in **HS5** (*exercise*)
- Modus ponens is simulated by cut

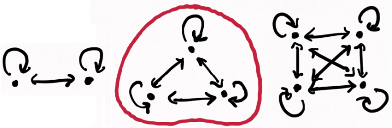
So far, purely syntactical analysis.
What about a semantics for the calculus?

Two semantics for S5

1. Kripke models with equivalence relation, or
2. Universal semantics $\mathcal{M} = \langle W, v \rangle$
   - No binary relation
   - $\mathcal{M}, w \Vdash \Box A$ *iff* for all $u \in W$, $\mathcal{M}, u \Vdash A$

☞ $\Box A$ true somewere iff A true everywhere

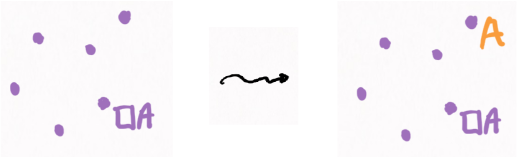
☞ Corresponds to choosing one cluster of a model with equivalence relation



Notation. We denote $\mathcal{U}$ the class of all universal models

- Different components $\rightsquigarrow$ different worlds
- For each component, formulas on the left true, formulas on the right false in the corresponding world

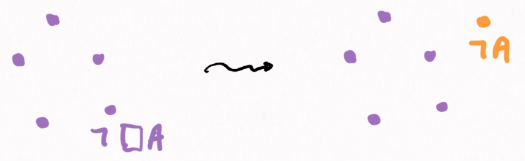
$$\Box_L \ \frac{\mathcal{H} \mid \Box A, \Gamma \Rightarrow \Delta \mid A, \Sigma \Rightarrow \Pi}{\mathcal{H} \mid \Box A, \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi} \uparrow$$



A semantic intuition of the rules

- Different components $\rightsquigarrow$ different worlds
- For each component, formulas on the left true, formulas on the right false in the corresponding world

$$\Box_R \; \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \; \Rightarrow A}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Box A} \uparrow$$



Valid hypersequent $\mathcal{M} \models \mathcal{H}$ *iff* $\exists\, \Gamma \Rightarrow \Delta \in \mathcal{H} : \mathcal{M} \models \Gamma \Rightarrow \Delta$

- ☞ Semantically, a hypersequent is a disjunction of validities
- ☞ That is, $\mathcal{H}$ valid iff one component is valid

Soundness

Theorem. If $\vdash_{\mathbf{HS5}} \mathcal{H}$, then $\models_{\mathcal{U}} \mathcal{H}$

Proof sketch. One needs to show that the inital hypersequent are valid in $\mathcal{U}$ (trivial) and that all rules of **HS5** preserve validity in universal models (*exercise*).

We now prove the opposite direction (completeness of **HS5**), namely that

$$\text{if } \models_{\mathcal{U}} \mathcal{H}, \text{ then } \vdash_{\textbf{HS5}} \mathcal{H}$$

1. First, we define a terminating (optimal) proof-search procedure in **HS5**
2. Then, we show that every failed proof constructed according to this procedure provides a countermodel of the root hypersequent: that is, if $\nvdash_{\textbf{HS5}} \mathcal{H}$, then $\not\models_{\mathcal{U}} \mathcal{H}$

A proof-search procedure in **HS5**

Main reference [Lellmann, 2016]

As a first step, we consider a cumulative formulation of **HS5**

Comulative formulation of a rule

The principal formula is copied to the premiss(es)

e.g.

$$\vee_R \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \vee B, A, B}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \vee B}$$

$$\vee_L \frac{\mathcal{H} \mid A, A \vee B, \Gamma \Rightarrow \Delta \qquad \mathcal{H} \mid B, A \vee B, \Gamma \Rightarrow \Delta}{\mathcal{H} \mid A \vee B, \Gamma \Rightarrow \Delta}$$

$$\Box_R \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Box A \mid \Rightarrow A}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Box A}$$

☞ Remark. The rules $\Box_L$ and $\Box_L^t$ are already in cumulative form

Notation. We call $\mathbf{HS5_{cum}}$ the calculus defined by the cumulative formulation of the rules of **HS5**

Theorem (Soundness). If $\vdash_{\mathbf{HS5_{cum}}} \mathcal{H}$, then $\models_{\mathcal{U}} \mathcal{H}$

Proof. The cumulative rules are admissible in **HS5**.

Example: Admissibility of the cumulative version of $\Box_{\mathrm{R}}$ in **HS5**

$$\dfrac{\dfrac{\dfrac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Box A \mid \Rightarrow A}{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A \mid \Rightarrow A}\text{ by invertibility of } \Box_{\mathrm{R}}}{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A}\,\mathrm{ctr_{ext}}}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Box A}\,\Box_{\mathrm{R}}$$

Therefore: if $\vdash_{\mathbf{HS5_{cum}}} \mathcal{H}$, then $\vdash_{\mathbf{HS5}} \mathcal{H}$, hence $\models_{\mathcal{U}} \mathcal{H}$

Clearly, the complexity of hypersequents is not reduced by backward applications of cumulative rules

☞ In order to ensure termination of backward proof-search, one needs to avoid redundant rule applications

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☞ In order to ensure termination of backward proof-search, one needs to avoid redundant rule applications

Local loop-checking condition (LLCC)

An application of a hypersequent rule with premisses $\mathcal{G}_1, \ldots, \mathcal{G}_n$ and conclusion $\mathcal{H}$ satisfies the local loop checking condition if for each premiss $\mathcal{G}_i$, there exists a component $\Gamma \Rightarrow \Delta$ in $\mathcal{G}_i$ such that for no component $\Sigma \Rightarrow \Pi$ of the conclusion $\mathcal{H}$ we have $set(\Gamma) \subseteq set(\Sigma)$ and $set(\Delta) \subseteq set(\Pi)$

Example: the following rule applications violate the LLCC

$$\dfrac{\Rightarrow p \wedge q, q, p \qquad \Rightarrow p \wedge q, q, q}{\Rightarrow p \wedge q, q} \wedge_R \qquad \dfrac{p \Rightarrow q \mid r \Rightarrow \Box q \mid \Rightarrow q}{p \Rightarrow q \mid r \Rightarrow \Box q} \Box_R$$

☞ The LLCC prevents the applications of rules that do not add additional information to the hypersequents

Saturated hypersequent

A hypersequent which is not initial and such that no rule is backward applicable to it without violating the LLCC

Backward proof-search with LLCC for $\mathcal{H}$

The construction of a derivation tree from the root to the leaves such that the root is labelled with the hypersequent $\mathcal{H}$, and the branches are expanded by applying at each step a backwards applicable rule that satisfies the LLCC. The construction terminates when all leaves are labelled with hypersequents that are either initial or saturated

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☞ In particular, we show that from every failed proof for $\mathcal{H}$ we can extract a countermodel of $\mathcal{H}$

The LLCC restricts the backward applicability of the rules

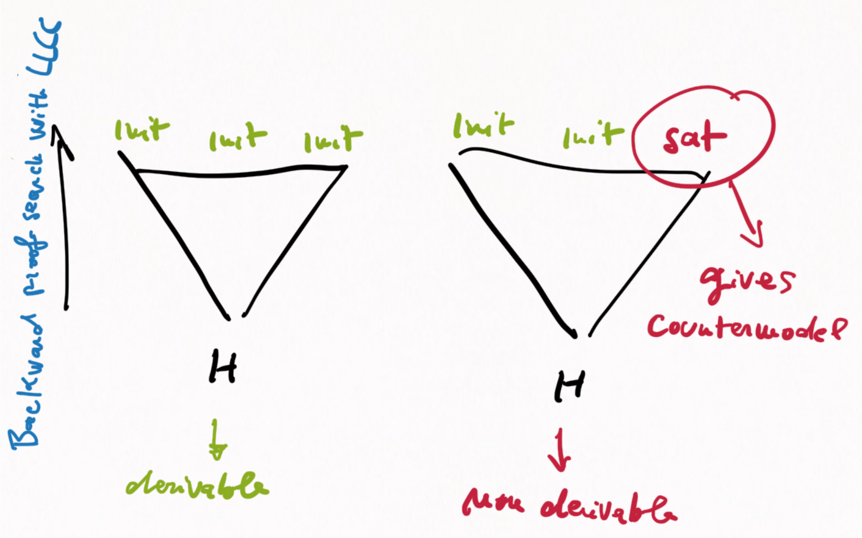
☞ Is proof-search with LLCC still complete?

☞ We now prove that proof-search with LLCC is complete by showing that every hypersequent $\mathcal{H}$ on which it fails is not valid in the universal semantics

☞ In particular, we show that from every failed proof for $\mathcal{H}$ we can extract a countermodel of $\mathcal{H}$

☞ More precisely, we show that each saturated hypersequent occurring in a failed proof of $\mathcal{H}$ provides the information needed to build such countermodel

The procedure in a picture



Let $\mathcal{H} = \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$ be a saturated hypersequent occurring in a failed proof for $\mathcal{G}$

Countermodel extracted from a saturated hypersequent

We define $\mathcal{M} = \langle W, v \rangle$ on the basis of $\mathcal{H}$ as follows

- $W = \{k \mid \Gamma_k \Rightarrow \Delta_k \in \mathcal{H}\}$
- For all $p \in Atm$, $v(p) = \{k \in W \mid p \in \Gamma_k\}$

Countermodel lemma

For all formulas A, for all components $\Gamma_k \Rightarrow \Delta_k$,

- if $A \in \Gamma_k$, then $k \Vdash A$
- if $A \in \Delta_k$, then $k \nVdash A$

Proved by induction on the construction of A (*exercise*)

Let $\mathcal{H} = \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$ be a saturated hypersequent occurring in a failed proof for $\mathcal{G}$, and $\mathcal{M}$ be the model defined on the basis of $\mathcal{H}$ as in the previous slide

The countermodel lemma implies that

- for all $\Gamma_k \Rightarrow \Delta_k \in \mathcal{H},\ k \nVdash \bigwedge \Gamma_k \to \bigvee \Delta_k$
- hence, $\mathcal{M} \not\models \mathcal{H}$

Moreover, since all rules are cumulative, we have

$$\text{for all } \Sigma \Rightarrow \Pi \in \mathcal{G},\ \text{there is } \Gamma_k \Rightarrow \Delta_k \in \mathcal{H} \text{ s.t. } \Sigma \subseteq \Gamma_k \text{ and } \Pi \subseteq \Delta_k$$

Therefore

- for all $\Sigma \Rightarrow \Pi \in \mathcal{G},\ $ there is $k \in W$ s.t. $k \nVdash \bigwedge \Sigma \to \bigvee \Pi$
- hence, $\mathcal{M} \not\models \mathcal{G}$

☞ $\mathcal{M}$ is a countermodel of the root hypersequent $\mathcal{G}$

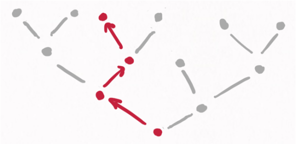
Example of failed proof

$$
\begin{array}{lcl}
 & w_1 \qquad\qquad\qquad w_2 \qquad\qquad w_3 & \\
 & p, p \vee q, \ldots \Rightarrow \ldots \mid q, \ldots \Rightarrow p \mid p, \ldots \Rightarrow q \qquad \chi & \\
\hline
\checkmark & p \vee q, \ldots \Rightarrow \ldots \mid q, p \vee q \Rightarrow p \mid p, p \vee q \Rightarrow q & \vee_L \\
\hline
\checkmark & p \vee q, \Box(p \vee q) \Rightarrow \ldots \mid q, p \vee q \Rightarrow p \mid p \vee q \Rightarrow q & \vee_L \\
\hline
 & p \vee q, \Box(p \vee q) \Rightarrow \Box p \vee \Box q, \Box p, \Box q \mid p \vee q \Rightarrow p \mid p \vee q \Rightarrow q & \vee_L \\
\hline
 & p \vee q, \Box(p \vee q) \Rightarrow \Box p \vee \Box q, \Box p, \Box q \mid p \vee q \Rightarrow p \mid \Rightarrow q & \Box_L \\
\hline
 & p \vee q, \Box(p \vee q) \Rightarrow \Box p \vee \Box q, \Box p, \Box q \mid \Rightarrow p \mid \Rightarrow q & \Box_L \\
\hline
 & \Box(p \vee q) \Rightarrow \Box p \vee \Box q, \Box p, \Box q \mid \Rightarrow p \mid \Rightarrow q & \Box_L^t \\
\hline
 & \Box(p \vee q) \Rightarrow \Box p \vee \Box q, \Box p, \Box q \mid \Rightarrow p & \Box_R \\
\hline
 & \Box(p \vee q) \Rightarrow \Box p \vee \Box q, \Box p, \Box q & \Box_R \\
\hline
 & \Box(p \vee q) \Rightarrow \Box p \vee \Box q & \vee_R
\end{array}
$$

$$w_1, w_2, w_3 \Vdash \Box(p \vee q)$$
$$w_1, w_2, w_3 \nVdash \Box p \vee \Box q$$

Theorem. Backward proof-search with LLCC in **HS5** provides a NP decision procedure for non derivability in S5

☞ At each step, non deterministically chose an applicable rule satisfying the LLCC and a correct premiss



This result relies on two key remarks:

1. The length of branches in a proof built by backward proof-search with LLCC is polynomially bounded by the length of the root hypersequent (see next slide)
2. Verifying the LLCC takes polynomial time

Lemma. The length of branches in a proof for a hypersequent $\mathcal{H}$ built by backward proof-search with LLCC is polynomially bounded by the length n of $\mathcal{H}$

Sketch of proof.

Lemma. The length of branches in a proof for a hypersequent $\mathcal{H}$ built by backward proof-search with LLCC is polynomially bounded by the length n of $\mathcal{H}$

Sketch of proof. The length of a branch depends on the number of rule applications in that branch.

Lemma. The length of branches in a proof for a hypersequent $\mathcal{H}$ built by backward proof-search with LLCC is polynomially bounded by the length n of $\mathcal{H}$

Sketch of proof. The length of a branch depends on the number of rule applications in that branch. Each backward rule application of a cumulative rule introduces a new formula.

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Sketch of proof. The length of a branch depends on the number of rule applications in that branch. Each backward rule application of a cumulative rule introduces a new formula. Hence, the number of possible rule applications is bounded by the maximal length of a hypersequent that can occur in a proof for $\mathcal{H}$.

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Number of components: New components are introduced only by the rule $\Box_R$.

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Number of components: New components are introduced only by the rule $\Box_R$. Moreover, the creation of new identical components is prevented by the LLCC.

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Sketch of proof. The length of a branch depends on the number of rule applications in that branch. Each backward rule application of a cumulative rule introduces a new formula. Hence, the number of possible rule applications is bounded by the maximal length of a hypersequent that can occur in a proof for $\mathcal{H}$. In turn, the length of a hypersequent depends on the number of components and the length of each component.

Number of components: New components are introduced only by the rule $\Box_R$. Moreover, the creation of new identical components is prevented by the LLCC. Hence, the number of components is bounded by the number of $\Box$-subformulas of $\mathcal{H}$.

Lemma. The length of branches in a proof for a hypersequent $\mathcal{H}$ built by backward proof-search with LLCC is polynomially bounded by the length n of $\mathcal{H}$

Sketch of proof. The length of a branch depends on the number of rule applications in that branch. Each backward rule application of a cumulative rule introduces a new formula. Hence, the number of possible rule applications is bounded by the maximal length of a hypersequent that can occur in a proof for $\mathcal{H}$. In turn, the length of a hypersequent depends on the number of components and the length of each component.

Number of components: New components are introduced only by the rule $\Box_{\mathrm{R}}$. Moreover, the creation of new identical components is prevented by the LLCC. Hence, the number of components is bounded by the number of $\Box$-subformulas of $\mathcal{H}$.

Length of each component: Because of analyticity, each component can only contain subformulas of $\mathcal{H}$.

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Sketch of proof. The length of a branch depends on the number of rule applications in that branch. Each backward rule application of a cumulative rule introduces a new formula. Hence, the number of possible rule applications is bounded by the maximal length of a hypersequent that can occur in a proof for $\mathcal{H}$. In turn, the length of a hypersequent depends on the number of components and the length of each component.

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Therefore, the length of hypersequents in the proof, hence the length of branches, is in $\mathcal{O}(n^2)$ qed

Mono- vs. Multi-modal logics

How many modalities can sequent calculi support?

Sequent and labelled sequent calculi can be extended to multimodal logics without essential modifications.

Example. Let K_n be the logic with n K-modalities $\Box_1, \dots, \Box_n$. The calculus $\mathbf{G3K_n}$ can be defined considering, for each $i \leq n$, the rule

$$\mathrm{k_i}\ \frac{\Gamma \Rightarrow A}{\Gamma', \Box_i \Gamma \Rightarrow \Box_i A, \Delta}$$

Similary, a labelled calculus for K_n can be defined considering relational symbols $R_1, \dots, R_n$ and, for each $i \leq n$, the rules

$$\Box_{\mathrm{L}}\ \frac{xR_i y, x : \Box_i A, y : A, \Gamma \Rightarrow \Delta}{xR_i y, x : \Box_i A, \Gamma \Rightarrow \Delta} \qquad \Box_{\mathrm{R}}\ \frac{xR_i y, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \Box_i A}\ (y!)$$

☞ The properties of the sequent and the labelled calculus for K hold also for the sequent and the labelled calculus for K_n

The same is not possible in **HS5**

- The hypersequent construct | can represent only one S5 modality
- After all, a model can have only one universal modality

However, the universal modality can be combined with other kinds of modalities

Example.
Let $\mathrm{K}_{\mathcal{U}}$ be the logic with a K modality $\Box$ and a universal modality $\blacksquare$

Semantics $\mathcal{M} = \langle W, R, v \rangle$, with

- $\mathcal{M}, w \Vdash \Box A$ iff for all u s.t. wRu, $\mathcal{M}, u \Vdash A$
- $\mathcal{M}, w \Vdash \blacksquare A$ iff for all u, $\mathcal{M}, u \Vdash A$

(Redundant but complete) axiomatisation (cf. [Goranko, Passy, 1992])

- K axiomatisation for $\Box$
- S5 axiomatisation for $\blacksquare$
- $\blacksquare A \to \Box A$

Hypersequent calculus S5 hypersequent calculus for $\blacksquare$,
extended with the hypersequent formulation of the rule k for $\Box$:

$$\mathrm{k}\ \frac{\mathcal{H} \mid \Sigma \Rightarrow A}{\mathcal{H} \mid \Gamma, \Box\Sigma \Rightarrow \Box A, \Delta}$$

Exercise. Derive the axiom $\blacksquare A \to \Box A$

As we have seen, a sequent $\Gamma \Rightarrow \Delta$ represents a consequence relation between the antecedent Γ (the assumptions) and the consequent Δ

But... which kind of assumptions?

Global vs. local modal consequence relation

Syntactically $\Gamma \vdash A$ (Hilbert systems)

- Global Both propositional and modal rules (necessitation) can be applied to the assumptions
- Local Only propositional rules be applied to the assumptions

Semantically $\Gamma \models A$

- Global For all $\mathcal{M}$, $\mathcal{M} \models \bigwedge \Gamma$ implies $\mathcal{M} \models A$
- Local For all $\mathcal{M}$, for all w, $\mathcal{M}, w \Vdash \bigwedge \Gamma$ implies $\mathcal{M}, w \Vdash A$

Remark.

- The sequent rule

$$\to_R \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}$$

expresses the deduction theorem, that holds (in this form) for local consequence only

$$A \models_{local} B \;\leadsto\; \models_{local} A \to B$$
$$A \models_{global} B \;\not\leadsto\; \models_{global} A \to B$$

e.g. $\quad A \models_{global} \Box A \quad$ but $\quad \not\models_{global} A \to \Box A$

- Indeed, validity of modal sequents is defined exactly as the local consequence

☞ Modal sequents represent local consequence relations

The hypersequent calculus can be used to reasoning under global assumptions

Indeed, reasoning under global assumptions in K:

$$B_1, ..., B_n \vdash_{global} A$$

can be reduced to

$$\vdash_{K_{\mathcal{U}}} \blacksquare B_1 \wedge ... \wedge \blacksquare B_n \rightarrow A$$

which is expressed in $\mathbf{HK_U}$ with the sequent

$$\blacksquare B_1, \ldots, \blacksquare B_n \Rightarrow A$$

☞ We now show that $\mathbf{HK_U}$ provides a decision procedure for reasoning under global assumptions in K

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