Proof Theory of Modal Logic

Lecture 4: Hypersequent calculi

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- Lecture 1: Sequent calculi
- Lecture 2: Labelled sequent calculi
- In this lecture we start looking at structured calculi, that extend sequent calculi with additional structural connectives

In particular, we now look at hypersequent calculi

- Simple generalisation of sequent calculi
- Introduced by [Mints, 1968] [Pottinger, 1983], [Avron, 1987] to provide cut-free calculi for modal and relevant logics

☞ In this lecture we focus only on modal logic S5

Axiomatisation of S5

$$\mathsf{K} + \begin{array}{ll} \mathsf{t} & \Box A \to A \\ \mathsf{4} & \Box A \to \Box\Box A \\ \mathsf{b} & A \vee \Box\neg\Box A \end{array} \qquad \text{or} \qquad \mathsf{K} + \begin{array}{ll} \mathsf{t} & \Box A \to A \\ \mathsf{5} & \Box A \vee \Box\neg\Box A \end{array}$$

Semantics of S5

Kripke models with equivalence relation

Complexity of S5

The validity/derivability problem for S5 is coNP-complete

Recap

- No cut-free, Gentzen-style sequent calculus for S5 (Lecture 1)
- Cut-free labelled calculus for S5 (Lecture 2)
- What about an internal, structured calculus for S5?

A hypersequent calculus for S5

Main reference for this calculus

- *A cut-free simple sequent calculus for modal logic S5* [Poggiolesi, 2008]: Definition of the calculus and structural analysis

Further references

- [Lellmann, 2016]: Optimal proof-search procedure in the calculus
- [Restall, 2007]: A version of the calculus with explicit structural rules

Hypersequent   Finite multiset of sequents, written

$$\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$$

where $\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n$ are the components of the hypersequent

Formula interpretation

$$i(\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n)$$
$$=$$
$$\Box(\bigwedge \Gamma_1 \to \bigvee \Delta_1) \vee \ldots \vee \Box(\bigwedge \Gamma_n \to \bigvee \Delta_n)$$

☞ Differently from labelled sequents, hypersequents can be interpreted as formulas

Initial hypersequents and propositional hypersequent rules

$$\text{init} \quad p, \Gamma \Rightarrow \Delta, p \qquad \rightsquigarrow \qquad \text{init} \quad \mathcal{H} \mid p, \Gamma \Rightarrow \Delta, p$$

$$\vee_R \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B} \qquad \rightsquigarrow \qquad \vee_R \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A, B}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \vee B}$$

Modal rules for S5

$$\Box_L \frac{\mathcal{H} \mid \Box A, \Gamma \Rightarrow \Delta \mid A, \Sigma \Rightarrow \Pi}{\mathcal{H} \mid \Box A, \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi} \qquad \Box_L^t \frac{\mathcal{H} \mid A, \Box A, \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Box A, \Gamma \Rightarrow \Delta}$$

$$\Box_R \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Box A}$$

Example. Derivation of axiom B

$$
\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \mid \Box A \Rightarrow}{\Rightarrow A \mid \Box A \Rightarrow}\,\Box_L}{\Rightarrow A \mid \Rightarrow \neg\Box A}\,\neg_R}{\Rightarrow A, \Box\neg\Box A}\,\Box_R}{\Rightarrow A \vee \Box\neg\Box A}\,\vee_R
$$

Exercise. Derive axioms k, t, 4, 5

Soundness

Theorem. If $\vdash_{\mathbf{HS5}} \mathcal{H}$, then $\vdash_{\mathrm{S5}} i(\mathcal{H})$

Proof sketch (i). We consider simple instances of the rules

$$\Box_L \; \frac{\Box A \Rightarrow \mid A \Rightarrow B}{\Box A \Rightarrow \mid \Rightarrow B}$$

i.	$\vdash \Box\neg\Box A \vee \Box(A \to B)$	$(i(P))$
ii.	$\vdash \Box\neg\Box A \vee \neg\Box\neg\Box A$	(CPL)
iii.	$\vdash \neg\Box\neg\Box A \to \Box A$	(axiom 5)
iv.	$\vdash \Box A \wedge \Box(A \to B) \to \Box B$	(axiom k)
v.	$\vdash \Box\neg\Box A \vee \Box B = i(C)$	(by classical reasoning)

Soundness

Theorem. If $\vdash_{\mathbf{HS5}} \mathcal{H}$, then $\vdash_{\mathrm{S5}} i(\mathcal{H})$

Proof sketch (ii). We consider simple instances of the rules

$$\Box_R \ \frac{B \Rightarrow C \mid \Rightarrow A}{B \Rightarrow C, \Box A}$$

i.	$\vdash \Box(B \to C) \lor \Box A$	$(i(P))$
ii.	$\vdash (B \to C) \to (B \to C \lor \Box A)$	(CPL)
iii.	$\vdash \Box(B \to C) \to \Box(B \to C \lor \Box A)$	(ii, by K valid rule)
iv.	$\vdash \Box A \to (B \to C \lor \Box A)$	(CPL)
v.	$\vdash \Box\Box A \to \Box(B \to C \lor \Box A)$	(iv, by K valid rule)
vi.	$\vdash \Box A \to \Box\Box A$	(axiom 4)
vii.	$\vdash \Box A \to \Box(B \to C \lor \Box A)$	(from iv, vi)
viii.	$\vdash \Box(B \to C \lor \Box A) = i(C)$	(from i, iii, vii)

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Proof sketch (ii). We consider simple instances of the rules

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ii.	$\vdash (B \to C) \to (B \to C \vee \Box A)$	(CPL)
iii.	$\vdash \Box(B \to C) \to \Box(B \to C \vee \Box A)$	(ii, by K valid rule)
iv.	$\vdash \Box A \to (B \to C \vee \Box A)$	(CPL)
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vi.	$\vdash \Box A \to \Box\Box A$	(axiom 4)
vii.	$\vdash \Box A \to \Box(B \to C \vee \Box A)$	(from iv, vi)
viii.	$\vdash \Box(B \to C \vee \Box A) = i(C)$	(from i, iii, vii)

Exercise. Prove soundness of all the rules of **HS5**

In order to syntactically prove the completeness of **HS5**, we need to analyse its structural properties

Relevant structural properties:

1. Hp-invertibility of all rules
2. Hp-admissibility of weakening and contraction
3. Admissibility of cut

☞ Interesting properties on their own

☞ Some dependeces

- Hp-admissibility of contraction depends on 1.
- Admissibility of cut depends on 1. and 2.

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$$\Box_R \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Box A}$$

(base case) If $h = 0$, then the conclusion $\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Box A$ is an initial hypersequent. There are three possibilities:

1. $\mathcal{H}$ is an intial hypersequent
2. $p \in \Gamma \cap \Delta$ for some p
3. $\bot \in \Gamma \cap \Delta$

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3. $\bot \in \Gamma \cap \Delta$

In each of these cases, the premiss $\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A$ is an initial hypersequent, hence it is derivable with height 0.

(inductive step) If $h > 0$, we need to consider the last rule application in the derivation of the conclusion $\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Box A$.

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1. $\Box A$ is principal in the last rule application
2. $\Box A$ is not principal in the last rule application

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1. $\Box A$ is principal in the last rule application
2. $\Box A$ is not principal in the last rule application

(case 1.) If $\Box A$ is principal in the last rule application, then the last rule application is precisely

$$\frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Box A}\ \Box_{\mathsf{R}}$$

which means that the premiss $\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A$ has a derivation of height $h - 1$.

(case 2.) If $\Box A$ is not principal in the last rule application, then, since $\mathcal{H}, \Gamma, \Delta$ can be any hypersequent and multisets, the last rule applied can be any rule of the calculus, hence one needs to consider all of them...

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$$\frac{\mathcal{H}' \mid \Sigma \Rightarrow \Pi \mid \Gamma \Rightarrow \Delta, \Box A \mid \Rightarrow B}{\mathcal{H}' \mid \Sigma \Rightarrow \Pi, \Box B \mid \Gamma \Rightarrow \Delta, \Box A} \ \Box_{\mathrm{R}}$$

where $(\mathcal{H}' \mid \Sigma \Rightarrow \Pi, \Box B) = \mathcal{H}$

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where $(\mathcal{H}' \mid \Sigma \Rightarrow \Pi, \Box B) = \mathcal{H}$ and the premiss $\mathcal{H}' \mid \Sigma \Rightarrow \Pi \mid \Gamma \Rightarrow \Delta, \Box A \mid \Rightarrow B$ has a derivation of height $h-1$.

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(Alternatively, one can have $\Box B \in \Delta$, the proof is analogous in this case.)

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(Alternatively, one can have $\Box B \in \Delta$, the proof is analogous in this case.)

Then, by the hp-invertibility of $\Box_R$, that holds at height $h - 1$ by inductive hypothesis, $\mathcal{H}' \mid \Sigma \Rightarrow \Pi \mid \Gamma \Rightarrow \Delta \mid \Rightarrow B \mid \Rightarrow A$ has a derivation $\mathcal{D}$ of height $h' \leq h - 1$.

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$$\frac{\mathcal{H}' \mid \Sigma \Rightarrow \Pi \mid \Gamma \Rightarrow \Delta, \Box A \mid \Rightarrow B}{\mathcal{H}' \mid \Sigma \Rightarrow \Pi, \Box B \mid \Gamma \Rightarrow \Delta, \Box A} \; \Box_R$$

where $(\mathcal{H}' \mid \Sigma \Rightarrow \Pi, \Box B) = \mathcal{H}$ and the premiss $\mathcal{H}' \mid \Sigma \Rightarrow \Pi \mid \Gamma \Rightarrow \Delta, \Box A \mid \Rightarrow B$ has a derivation of height $h - 1$.

(Alternatively, one can have $\Box B \in \Delta$, the proof is analogous in this case.)

Then, by the hp-invertibility of $\Box_R$, that holds at height $h - 1$ by inductive hypothesis, $\mathcal{H}' \mid \Sigma \Rightarrow \Pi \mid \Gamma \Rightarrow \Delta \mid \Rightarrow B \mid \Rightarrow A$ has a derivation $\mathcal{D}$ of height $h' \leq h - 1$. Therefore, by extending $\mathcal{D}$ with an application of $\Box_R$ to this hypersequent, we obtain a derivation of height $h' + 1 \leq h$ of $\mathcal{H}' \mid \Sigma \Rightarrow \Pi, \Box B \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A$, that is, $\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A$.

qed

Structural rules

$$\mathrm{wk_L}\ \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta}{\mathcal{H} \mid A, \Gamma \Rightarrow \Delta} \qquad \mathrm{wk_R}\ \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A}$$

$$\mathrm{ctr_L}\ \frac{\mathcal{H} \mid A, A, \Gamma \Rightarrow \Delta}{\mathcal{H} \mid A, \Gamma \Rightarrow \Delta} \qquad \mathrm{ctr_R}\ \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A, A}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A}$$

$$\mathrm{wk_{ext}}\ \frac{\mathcal{H}}{\mathcal{H} \mid \Gamma \Rightarrow \Delta} \qquad \mathrm{ctr_{ext}}\ \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Gamma \Rightarrow \Delta}$$

☞ Note: external forms of weakening and contraction

The cut rule

$$\mathrm{cut}\ \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \qquad \mathcal{H}' \mid A, \Gamma' \Rightarrow \Delta'}{\mathcal{H} \mid \mathcal{H}' \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Theorem. Left, right and external weakening and contraction are hp-admissible in **HS5**

Sketch of proof. By induction on the height of the derivation of the premiss (*exercise*)

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Hint. In order to prove the hp-admissibility of some structural rules you may need the following (nice) rule

$$\text{merge}\ \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi}{\mathcal{H} \mid \Gamma, \Sigma \Rightarrow \Delta, \Pi}$$

Theorem. The rule merge is hp-admissible in **HS5**

Sketch of proof. By induction on the height of the derivation of the premiss (*exercise*)

Theorem. Cut is admissible in **HS5**

Proof sketch. By induction on the complexity of the cut formula and subinduction on the cut height.

As an example, consider the following derivation, with the cut formula $\Box A$ principal in the last rule application in both premisses of cut

$$\dfrac{\dfrac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Box A}\,\Box_R \qquad \dfrac{\mathcal{H}' \mid A, \Box A, \Gamma' \Rightarrow \Delta'}{\mathcal{H}' \mid \Box A, \Gamma' \Rightarrow \Delta'}\,\Box^t_L}{\mathcal{H} \mid \mathcal{H}' \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\text{cut}$$

Converted into the following, with one application of cut at a lower height, and one application of cut with a cut formula of lower complexity

$$\dfrac{\dfrac{\dfrac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A \qquad \dfrac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Box A \qquad \mathcal{H}' \mid A, \Box A, \Gamma' \Rightarrow \Delta'}{\mathcal{H} \mid \mathcal{H}' \mid \Gamma, \Gamma', A \Rightarrow \Delta, \Delta'}\,\text{cut}}{\mathcal{H} \mid \mathcal{H} \mid \mathcal{H}' \mid \Gamma \Rightarrow \Delta \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\text{cut}}{\mathcal{H} \mid \mathcal{H} \mid \mathcal{H}' \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\text{wk}^*}{\mathcal{H} \mid \mathcal{H}' \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\text{ctr}_{\text{ext}} \times 2$$

(where wk* denotes multiple applications of (left and right) weakening

Completeness

Theorem. If $\vdash_{\text{S5}} A$, then $\vdash_{\mathbf{HS5}} \Rightarrow A$

Proof sketch.

- All axioms of S5 are derivable in **HS5** (*exercise*)
- The necessitation rule is admissible in **HS5** (*exercise*)
- Modus ponens is simulated by cut

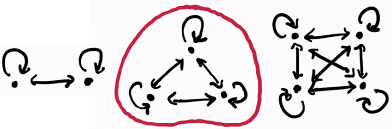
So far, purely syntactical analysis.
What about a semantics for the calculus?

Two semantics for S5

1. Kripke models with equivalence relation, or
2. Universal semantics $\mathcal{M} = \langle W, v \rangle$
   - No binary relation
   - $\mathcal{M}, w \Vdash \Box A$ *iff* for all $u \in W$, $\mathcal{M}, u \Vdash A$

☞ $\Box A$ true somewere iff A true everywhere

☞ Corresponds to choosing one cluster of a model with equivalence relation

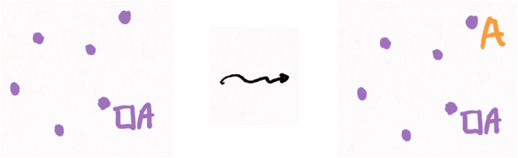


Notation. We denote $\mathcal{U}$ the class of all universal models

A semantic intuition of the rules

- Different components $\rightsquigarrow$ different worlds
- For each component, formulas on the left true, formulas on the right false in the corresponding world

$$\Box_L \ \frac{\mathcal{H} \mid \Box A, \Gamma \Rightarrow \Delta \mid A, \Sigma \Rightarrow \Pi}{\mathcal{H} \mid \Box A, \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi} \uparrow$$



- Different components $\rightsquigarrow$ different worlds
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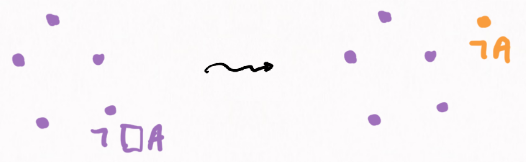
$$\Box_R \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \;\Rightarrow A}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Box A} \uparrow$$



Valid hypersequent $\mathcal{M} \models \mathcal{H}$ *iff* $\exists\, \Gamma \Rightarrow \Delta \in \mathcal{H} : \mathcal{M} \models \Gamma \Rightarrow \Delta$

- ☞ Semantically, a hypersequent is a disjunction of validities
- ☞ That is, $\mathcal{H}$ valid iff one component is valid

Soundness

Theorem. If $\vdash_{\mathbf{HS5}} \mathcal{H}$, then $\models_{\mathcal{U}} \mathcal{H}$

Proof sketch. One needs to show that the inital hypersequent are valid in $\mathcal{U}$ (trivial) and that all rules of **HS5** preserve validity in universal models (*exercise*).

We now prove the opposite direction (completeness of **HS5**), namely that

$$\text{if } \models_{\mathcal{U}} \mathcal{H}\text{, then } \vdash_{\textbf{HS5}} \mathcal{H}$$

1. First, we define a terminating (optimal) proof-search procedure in **HS5**
2. Then, we show that every failed proof constructed according to this procedure provides a countermodel of the root hypersequent: that is, if $\nvdash_{\textbf{HS5}} \mathcal{H}$, then $\not\models_{\mathcal{U}} \mathcal{H}$

A proof-search procedure in **HS5**

Main reference [Lellmann, 2016]

As a first step, we consider a cumulative formulation of **HS5**

Comulative formulation of a rule

The principal formula is copied to the premiss(es)

e.g.

$$\vee_R \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \vee B, A, B}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \vee B}$$

$$\vee_L \frac{\mathcal{H} \mid A, A \vee B, \Gamma \Rightarrow \Delta \qquad \mathcal{H} \mid B, A \vee B, \Gamma \Rightarrow \Delta}{\mathcal{H} \mid A \vee B, \Gamma \Rightarrow \Delta}$$

$$\Box_R \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Box A \mid \Rightarrow A}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Box A}$$

☞ Remark. The rules $\Box_L$ and $\Box^t_L$ are already in cumulative form

Notation. We call **HS5$_{\mathbf{cum}}$** the calculus defined by the cumulative formulation of the rules of **HS5**

Theorem (Soundness). If $\vdash_{\mathbf{HS5_{cum}}} \mathcal{H}$, then $\models_{\mathcal{U}} \mathcal{H}$

Proof. The cumulative rules are admissible in **HS5**.

Example: Admissibility of the cumulative version of $\Box_\mathrm{R}$ in **HS5**

$$\dfrac{\dfrac{\dfrac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Box A \mid \Rightarrow A}{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A \mid \Rightarrow A}\ \text{by invertibility of } \Box_\mathrm{R}}{\mathcal{H} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A}\ \mathrm{ctr_{ext}}}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Box A}\ \Box_\mathrm{R}$$

Therefore: if $\vdash_{\mathbf{HS5_{cum}}} \mathcal{H}$, then $\vdash_{\mathbf{HS5}} \mathcal{H}$, hence $\models_{\mathcal{U}} \mathcal{H}$

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Local loop-checking condition (LLCC)
An application of a hypersequent rule with premisses $\mathcal{G}_1, \ldots, \mathcal{G}_n$ and conclusion $\mathcal{H}$ satisfies the local loop checking condition if for each premiss $\mathcal{G}_i$, there exists a component $\Gamma \Rightarrow \Delta$ in $\mathcal{G}_i$ such that for no component $\Sigma \Rightarrow \Pi$ of the conclusion $\mathcal{H}$ we have $set(\Gamma) \subseteq set(\Sigma)$ and $set(\Delta) \subseteq set(\Pi)$

Example: the following rule applications violate the LLCC

$$\dfrac{\Rightarrow p \wedge q, q, p \qquad \Rightarrow p \wedge q, q, q}{\Rightarrow p \wedge q, q} \wedge_R \qquad \dfrac{p \Rightarrow q \mid r \Rightarrow \Box q \mid \Rightarrow q}{p \Rightarrow q \mid r \Rightarrow \Box q} \Box_R$$

☞ The LLCC prevents the applications of rules that do not add additional information to the hypersequents

Saturated hypersequent

A hypersequent which is not initial and such that no rule is backward applicable to it without violating the LLCC

Backward proof-search with LLCC for $\mathcal{H}$

The construction of a derivation tree from the root to the leaves such that the root is labelled with the hypersequent $\mathcal{H}$, and the branches are expanded by applying at each step a backwards applicable rule that satisfies the LLCC. The construction terminates when all leaves are labelled with hypersequents that are either initial or saturated

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☞ In particular, we show that from every failed proof for $\mathcal{H}$ we can extract a countermodel of $\mathcal{H}$

The LLCC restricts the backward applicability of the rules

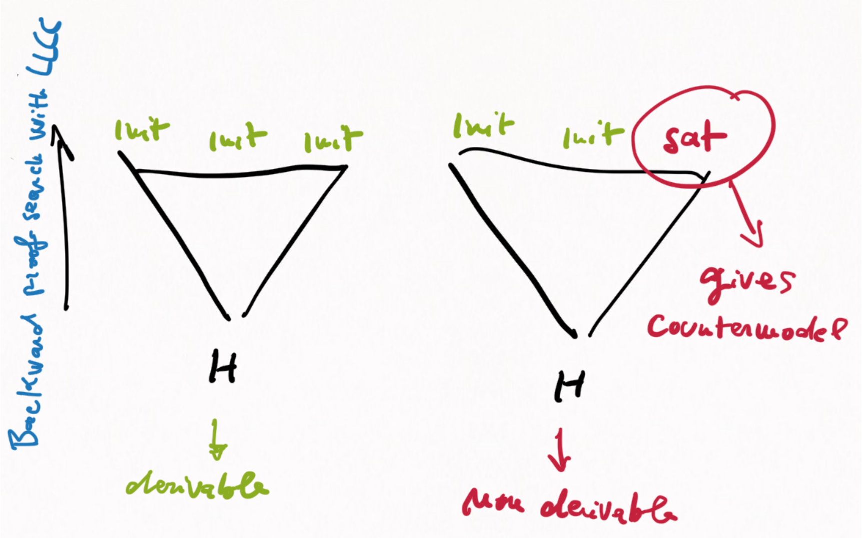
☞ Is proof-search with LLCC still complete?

☞ We now prove that proof-search with LLCC is complete by showing that every hypersequent $\mathcal{H}$ on which it fails is not valid in the universal semantics

☞ In particular, we show that from every failed proof for $\mathcal{H}$ we can extract a countermodel of $\mathcal{H}$

☞ More precisely, we show that each saturated hypersequent occurring in a failed proof of $\mathcal{H}$ provides the information needed to build such countermodel

The procedure in a picture



Let $\mathcal{H} = \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$ be a saturated hypersequent occurring in a failed proof for $\mathcal{G}$

Countermodel extracted from a saturated hypersequent

We define $\mathcal{M} = \langle W, v \rangle$ on the basis of $\mathcal{H}$ as follows

- $W = \{k \mid \Gamma_k \Rightarrow \Delta_k \in \mathcal{H}\}$
- For all $p \in Atm$, $v(p) = \{k \in W \mid p \in \Gamma_k\}$

Countermodel lemma

For all formulas A, for all components $\Gamma_k \Rightarrow \Delta_k$,

- if $A \in \Gamma_k$, then $k \Vdash A$
- if $A \in \Delta_k$, then $k \nVdash A$

Proved by induction on the construction of A (*exercise*)

Let $\mathcal{H} = \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$ be a saturated hypersequent occurring in a failed proof for $\mathcal{G}$, and $\mathcal{M}$ be the model defined on the basis of $\mathcal{H}$ as in the previous slide

The countermodel lemma implies that

- for all $\Gamma_k \Rightarrow \Delta_k \in \mathcal{H}, \; k \nVdash \bigwedge \Gamma_k \to \bigvee \Delta_k$
- hence, $\mathcal{M} \not\models \mathcal{H}$

Moreover, since all rules are cumulative, we have

$$\text{for all } \Sigma \Rightarrow \Pi \in \mathcal{G}, \; \text{there is } \Gamma_k \Rightarrow \Delta_k \in \mathcal{H} \text{ s.t. } \Sigma \subseteq \Gamma_k \text{ and } \Pi \subseteq \Delta_k$$

Therefore

- for all $\Sigma \Rightarrow \Pi \in \mathcal{G}, \;$ there is $k \in W$ s.t. $k \nVdash \bigwedge \Sigma \to \bigvee \Pi$
- hence, $\mathcal{M} \not\models \mathcal{G}$

☞ $\mathcal{M}$ is a countermodel of the root hypersequent $\mathcal{G}$

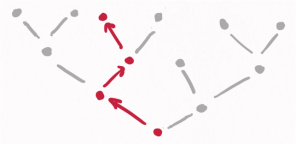
Example of failed proof

$$
\begin{array}{lcl}
 & \qquad w_1 \qquad\qquad w_2 \qquad\qquad w_3 & \\
 & p, p \vee q, \ldots \Rightarrow \ldots \mid q, \ldots \Rightarrow p \mid p, \ldots \Rightarrow q & \chi \\ \cline{2-2}
\checkmark & p \vee q, \ldots \Rightarrow \ldots \mid q, p \vee q \Rightarrow p \mid p, p \vee q \Rightarrow q & \vee_L \\ \cline{2-2}
\checkmark & p \vee q, \Box(p \vee q) \Rightarrow \ldots \mid q, p \vee q \Rightarrow p \mid p \vee q \Rightarrow q & \vee_L \\ \cline{2-2}
 & p \vee q, \Box(p \vee q) \Rightarrow \Box p \vee \Box q, \Box p, \Box q \mid p \vee q \Rightarrow p \mid p \vee q \Rightarrow q & \vee_L \\ \cline{2-2}
 & p \vee q, \Box(p \vee q) \Rightarrow \Box p \vee \Box q, \Box p, \Box q \mid p \vee q \Rightarrow p \mid \Rightarrow q & \Box_L \\ \cline{2-2}
 & p \vee q, \Box(p \vee q) \Rightarrow \Box p \vee \Box q, \Box p, \Box q \mid \Rightarrow p \mid \Rightarrow q & \Box_L \\ \cline{2-2}
 & \Box(p \vee q) \Rightarrow \Box p \vee \Box q, \Box p, \Box q \mid \Rightarrow p \mid \Rightarrow q & \Box_L^t \\ \cline{2-2}
 & \Box(p \vee q) \Rightarrow \Box p \vee \Box q, \Box p, \Box q \mid \Rightarrow p & \Box_R \\ \cline{2-2}
 & \Box(p \vee q) \Rightarrow \Box p \vee \Box q, \Box p, \Box q & \Box_R \\ \cline{2-2}
 & \Box(p \vee q) \Rightarrow \Box p \vee \Box q & \vee_R
\end{array}
$$

$$w_1, w_2, w_3 \Vdash \Box(p \vee q)$$
$$w_1, w_2, w_3 \nVdash \Box p \vee \Box q$$

Theorem. Backward proof-search with LLCC in **HS5** provides a NP decision procedure for non derivability in S5

☞ At each step, non deterministically chose an applicable rule satisfying the LLCC and a correct premiss



This result relies on two key remarks:

1. The length of branches in a proof built by backward proof-search with LLCC is polynomially bounded by the length of the root hypersequent (see next slide)
2. Verifying the LLCC takes polynomial time

Lemma. The length of branches in a proof for a hypersequent $\mathcal{H}$ built by backward proof-search with LLCC is polynomially bounded by the length n of $\mathcal{H}$

Sketch of proof.

Lemma. The length of branches in a proof for a hypersequent $\mathcal{H}$ built by backward proof-search with LLCC is polynomially bounded by the length n of $\mathcal{H}$

Sketch of proof. The length of a branch depends on the number of rule applications in that branch.

Lemma. The length of branches in a proof for a hypersequent $\mathcal{H}$ built by backward proof-search with LLCC is polynomially bounded by the length n of $\mathcal{H}$

Sketch of proof. The length of a branch depends on the number of rule applications in that branch. Each backward rule application of a cumulative rule introduces a new formula.

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Therefore, the length of hypersequents in the proof, hence the length of branches, is in $\mathcal{O}(n^2)$ qed

Mono- vs. Multi-modal logics

How many modalities can sequent calculi support?

Sequent and labelled sequent calculi can be extended to multimodal logics without essential modifications.

Example. Let K_n be the logic with n K-modalities $\Box_1, \dots, \Box_n$. The calculus $\mathbf{G3K_n}$ can be defined considering, for each $i \leq n$, the rule

$$\mathrm{k_i}\ \frac{\Gamma \Rightarrow A}{\Gamma', \Box_i\Gamma \Rightarrow \Box_i A, \Delta}$$

Similary, a labelled calculus for K_n can be defined considering relational symbols $R_1, \dots, R_n$ and, for each $i \leq n$, the rules

$$\Box_{\mathrm{L}}\ \frac{xR_iy, x : \Box_i A, y : A, \Gamma \Rightarrow \Delta}{xR_iy, x : \Box_i A, \Gamma \Rightarrow \Delta} \qquad \Box_{\mathrm{R}}\ \frac{xR_iy, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \Box_i A}\ (y!)$$

☞ The properties of the sequent and the labelled calculus for K hold also for the sequent and the labelled calculus for K_n

The same is not possible in **HS5**

- The hypersequent construct | can represent only one S5 modality
- After all, a model can have only one universal modality

However, the universal modality can be combined with other kinds of modalities

Example.
Let $K_{\mathcal{U}}$ be the logic with a K modality $\Box$ and a universal modality $\blacksquare$

Semantics $\mathcal{M} = \langle W, R, v \rangle$, with

- $\mathcal{M}, w \Vdash \Box A$ iff for all u s.t. wRu, $\mathcal{M}, u \Vdash A$
- $\mathcal{M}, w \Vdash \blacksquare A$ iff for all u, $\mathcal{M}, u \Vdash A$

(redundant but complete) (cf. [Goranko, Passy, 1992] for a more detailed analysis)

Axiomatisation

- K axiomatisation for $\Box$
- S5 axiomatisation for $\blacksquare$
- $\blacksquare A \to \Box A$

Hypersequent calculus S5 hypersequent calculus for $\blacksquare$, extended with the hypersequent formulation of the rule k for $\Box$:

$$\mathsf{k}\ \frac{\mathcal{H} \mid \Sigma \Rightarrow A}{\mathcal{H} \mid \Gamma, \Box\Sigma \Rightarrow \Box A, \Delta}$$

Exercise. Derive the axiom $\blacksquare A \to \Box A$

As we have seen, a sequent $\Gamma \Rightarrow \Delta$ represents a consequence relation between the antecedent Γ (the assumptions) and the consequent Δ

But... which kind of assumptions?

Global vs. local modal consequence relation

Syntactically $\Gamma \vdash A$ (Hilbert systems)

- Global Both propositional and modal rules (necessitation) can be applied to the assumptions
- Local Only propositional rules be applied to the assumptions

Semantically $\Gamma \models A$

- Global For all $\mathcal{M}$, $\mathcal{M} \models \bigwedge \Gamma$ implies $\mathcal{M} \models A$
- Local For all $\mathcal{M}$, for all w, $\mathcal{M}, w \Vdash \bigwedge \Gamma$ implies $\mathcal{M}, w \Vdash A$

Remark.

- The sequent rule

$$\rightarrow_R \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B}$$

  expresses the deduction theorem, that holds (in this form) for local consequence only

$$A \models_{local} B \;\rightsquigarrow\; \models_{local} A \rightarrow B$$
$$A \models_{global} B \;\not\rightsquigarrow\; \models_{global} A \rightarrow B$$

  e.g. $A \models_{global} \Box A$ but $\not\models_{global} A \rightarrow \Box A$

- Indeed, validity of modal sequents is defined exactly as the local consequence

☞ Modal sequents represent local consequence relations

The hypersequent calculus can be used to reasoning under global assumptions

Indeed, reasoning under global assumptions in K:

$$B_1, ..., B_n \vdash_{global} A$$

can be reduced to

$$\vdash_{K_{\mathcal{U}}} \blacksquare B_1 \wedge ... \wedge \blacksquare B_n \rightarrow A$$

which is expressed in $\mathbf{HK_U}$ with the sequent

$$\blacksquare B_1, \ldots, \blacksquare B_n \Rightarrow A$$

☞ We now show that $\mathbf{HK_U}$ provides a decision procedure for reasoning under global assumptions in K

Decision procedure analogous to **HS5**:

- Cumulative formulation of all rules of the calculus

  example: $$\text{k}\ \frac{\mathcal{H} \mid \Gamma, \Box\Sigma \Rightarrow \Box A, \Delta \mid \Sigma \Rightarrow A}{\mathcal{H} \mid \Gamma, \Box\Sigma \Rightarrow \Box A, \Delta}$$

- Loop checking and proof-search strategy defined as for **HS5**
- Termination of proof search: by measuring the size of maximal hypersequents in proof-search. Remark: exponential size!
- Completeness of proof-search: countermodel from every saturated hypersequent (next slide)

Let $\mathcal{H} = \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$ be a saturated hypersequent occurring in a failed proof for $\mathcal{G}$ in $\mathbf{HK_U}$

Countermodel extracted from a saturated hypersequent

We define $\mathcal{M} = \langle W, R, v \rangle$ on the basis of $\mathcal{H}$ as follows

- $W = \{k \mid \Gamma_k \Rightarrow \Delta_k \in \mathcal{H}\}$
- For all $k, \ell \in W$, $kR\ell$ iff $\Box A \in \Gamma_k$ or $\blacksquare A \in \Gamma_k$, then $A \in \Gamma_\ell$
- For all $p \in Atm$, $v(p) = \{k \in W \mid p \in \Gamma_k\}$

Countermodel lemma

For all formulas A, for all components $\Gamma_k \Rightarrow \Delta_k$,

- if $A \in \Gamma_k$, then $k \Vdash A$
- if $A \in \Delta_k$, then $k \nVdash A$

Proved by induction on the construction of A (*exercise*)

Exercise. Prove that $\Box q$ is derivable under assumptions p and $p \to q$ if and only if both assumptions are global

Exercise. Prove that $\Box q$ is derivable under assumptions p and $p \to q$ if and only if both assumptions are global

Possible solution

- $\Box q$ is derivable under global assumptions p and $p \to q$

$$
\cfrac{\cfrac{\cfrac{\cfrac{\overset{\checkmark}{\ldots \mid p, p \to q \Rightarrow q, p} \qquad \overset{\checkmark}{\ldots \mid q, p, p \to q \Rightarrow q}}{\blacksquare p, \blacksquare(p \to q) \Rightarrow \Box q \mid p, p \to q \Rightarrow q}\ \to_{\mathrm{L}}}{\blacksquare p, \blacksquare(p \to q) \Rightarrow \Box q \mid p \Rightarrow q}\ \blacksquare_{\mathrm{L}}}{\blacksquare p, \blacksquare(p \to q) \Rightarrow \Box q \mid\ \Rightarrow q}\ \blacksquare_{\mathrm{L}}}{\blacksquare p, \blacksquare(p \to q) \Rightarrow \Box q}\ \Box_{\mathrm{R}}
$$

Possible solution

- $\square q$ is not derivable under global assumption p and local assumption $p \to q$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\checkmark}{p, \blacksquare p, p \to q, p \Rightarrow \square q \mid p \Rightarrow q} \qquad \overset{\times}{q, p, \blacksquare p, p \to q \Rightarrow \square q \mid p \Rightarrow q}}{p, \blacksquare p, p \to q \Rightarrow \square q \mid p \Rightarrow q} \to_R}{\blacksquare p, p \to q \Rightarrow \square q \mid p \Rightarrow q} \blacksquare^t_L}{\blacksquare p, p \to q \Rightarrow \square q \mid \ \Rightarrow q} \blacksquare_L}{\blacksquare p, p \to q \Rightarrow \square q} \square_R$$

$$\overset{\mathbf{1}}{q, p, \blacksquare p, p \to q \Rightarrow \square q} \mid \overset{\mathbf{2}}{p \Rightarrow q}$$

$W = \{1, 2\} \quad 1R1, 1R2 \quad v(p) = \{1, 2\}, v(q) = \{1\}$

☞ $1 \Vdash \blacksquare p, \ 1 \Vdash p \to q, \ 1 \nVdash \square q$

Several alternative hypersequent calculi for S5: [Mints, 1971], [Pottinger, 1983], [Avron, 1993], [Restall, 2007], [Poggiolesi, 2008], [Kurokawa, 2013], [Lahav, 2013]

A nice and influential calculus: [Avron, 1993]

$$\mathsf{T}\ \frac{\mathcal{H} \mid A, \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Box A, \Gamma \Rightarrow \Delta} \qquad 4\ \frac{\mathcal{H} \mid \Box\Gamma \Rightarrow A}{\mathcal{H} \mid \Box\Gamma \Rightarrow \Box A}$$

$$\mathsf{MS}\ \frac{\mathcal{H} \mid \Box\Gamma, \Sigma \Rightarrow \Box\Delta, \Pi}{\mathcal{H} \mid \Box\Gamma \Rightarrow \Box\Delta \mid \Sigma \Rightarrow \Pi}$$

- "Modular" extension of a sequent calculus for S4
- S5 obtained with the addition of a hypersequential structural rule: the Modal Splitting (MS)

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Appendix:
Labelled vs. structured calculi.
Two separate worlds?

Labelled calculus (informal definition)
Any calculus which includes linguistic components that do not belong to the language of the logic

A simple labelled calculus: **LabS5**

- Labels $x, y, z, \ldots$
- Labelled formulas $x : A$
- No relational atoms
- Rules of **G3cp** enriched with labels $\vee_R \dfrac{\Gamma \Rightarrow \Delta, x : A, x : B}{\Gamma \Rightarrow \Delta, x : A \vee B}$
- Modal rules

$$\Box_L \dfrac{y : A, x : \Box A, \Gamma \Rightarrow \Delta}{x : \Box A, \Gamma \Rightarrow \Delta} (y \in \Gamma, \Delta) \qquad \Box_R \dfrac{\Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \Box A} (y!)$$

Remark. **LabS5** notational variant of predicate calculus

$$\Box_L \frac{y : A, x : \Box A, \Gamma \Rightarrow \Delta}{x : \Box A, \Gamma \Rightarrow \Delta} (y \in \Gamma, \Delta) \qquad \Box_R \frac{\Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \Box A} (y!)$$

$$\rightsquigarrow \qquad\qquad \rightsquigarrow$$

$$\forall_L \frac{A(y/x), \forall x A, \Gamma \Rightarrow \Delta}{\forall x A, \Gamma \Rightarrow \Delta} (y \in \Gamma, \Delta) \qquad \forall_R \frac{\Gamma \Rightarrow \Delta, A(y/x)}{\Gamma \Rightarrow \Delta, \forall x A} (y!)$$

☞ S5 corresponds to the “uniform monadic first-order predicate calculus” [Prior, Fine, 1977]

- Relational symbols with only one argument
- Formulas with at most one free variable

*Prior, Fine, Worlds, Times and Selves, Univ. Mass. Press, 1977.

- Mutual translations between semantically equivalent hyper- and labelled sequents

$$\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$$
$$\updownarrow$$
$$x_1 : \Gamma_1, \ldots, x_n : \Gamma_n \Rightarrow x_1 : \Delta_1, \ldots, x_n : \Delta_n$$

- Direct correspondence between hypersequent and labelled sequent rule applications

$$\begin{array}{ccc} \mathrm{hyp}(\Box_\mathrm{L}, \Box^\mathrm{t}_\mathrm{L}) & \qquad & \mathrm{hyp}(\Box_\mathrm{R}) \\ (x \neq y) \updownarrow (x = y) & & \updownarrow \\ \mathrm{lab}(\Box_\mathrm{L}) & & \mathrm{lab}(\Box_\mathrm{R}) \end{array}$$

- Direct correspondence between hypersequent and labelled sequent derivations