Math 338: Exam 2

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$Y \sim \text{Exponential}(\tfrac{1}{2}) \qquad f(y) = 2e^{-2y},\ y > 0$

a) $P(Y > y) = \int_y^{\infty} 2e^{-2x}\,dx,\ x > 0$

$= \frac{2e^{-2x}}{-2}\Big|_y^{\infty} = -e^{-2\cdot\infty} + e^{-2y} = \frac{-1}{e^{\infty}} + e^{-2y} = \boxed{e^{-2y}} = S(y)$

b) $\underbrace{P(Y > \tfrac{3}{4})}_{\text{survival function set-up}} = \int_{3/4}^{\infty} 2e^{-2y}\,dy = S(\tfrac{3}{4}) = e^{-2(\frac{3}{4})} = \boxed{e^{-3/2}}$

c) $\underbrace{P(Y > \tfrac{6}{4} \mid Y > \tfrac{3}{4}) = P(Y > \tfrac{6}{4} - \tfrac{3}{4})}_{\text{true by the memoryless property } P(Y>a+b \mid Y>a) = P(Y>b) \text{ where } a=b=\frac{3}{4}} = P(Y > \tfrac{3}{4}) = \boxed{e^{-3/2}}$

d) $U \sim \text{Erlang}(n, \beta) \Rightarrow f(u) = \frac{1}{(n-1)!\,\beta^n} u^{n-1} e^{-u/\beta},\ u > 0,\ n \text{ is integer}$

$E(U) = \int_0^{\infty} u f(u)\,du = \int_0^{\infty} \frac{1}{(n-1)!\,\beta^n} u^n e^{-u/\beta}\,du = \frac{1}{(n-1)!\,\beta^n} \int_0^{\infty} \underbrace{u^n e^{-u/\beta}}_{\text{kernel of Erlang}(n+1,\beta)}\,du$

$\Rightarrow \int_0^{\infty} u^n e^{-u/\beta} = n!\,\beta^{n+1}$

$= \frac{1}{(n-1)!\,\beta^n}\left(n!\,\beta^{n+1}\right) = \frac{n!}{(n-1)!} \cdot \frac{\beta^{n+1}}{\beta^n} = \boxed{n\beta}$

e) $X \sim \text{Exponential}(1)$, X and Y are independent

$\Rightarrow f(x) = \frac{1}{1} e^{-x/1} = e^{-x},\ x > 0 \qquad \Rightarrow f(x,y) = f(x) f(y)$

$f(y) = 2e^{-2y},\ y > 0$

$\underbrace{f(x,y)}_{\text{joint PDF is product of the independent marginals}} = e^{-x} \cdot 2e^{-2y} = \boxed{2e^{-(x+2y)},\ \begin{matrix} x>0 \\ y>0 \end{matrix}}$

2 Model A : $Y \sim \text{Uniform}(0, \frac{3}{2}) \implies f(y) = \frac{1}{\frac{3}{2} - 0} = \frac{2}{3}, \quad 0 \le y \le \frac{3}{2}$

Model B: $f(y) = \frac{8}{9}\left(\frac{3}{2} - y\right) = \frac{4}{3} - \frac{8}{9}y, \quad 0 \le y \le \frac{3}{2}$

a) $$E(Y) = \int_0^{3/2} y \cdot \frac{2}{3}\, dy = \frac{y^2}{3}\Big|_0^{3/2} = \frac{\left(\frac{9}{4}\right)}{3} - 0 = \frac{9}{12} = \frac{3}{4}$$

$$E(Y^2) = \int_0^{3/2} y^2 \cdot \frac{2}{3}\, dy = \frac{2y^3}{9}\Big|_0^{3/2} = \frac{2}{9}\left(\frac{27}{8}\right) - 0 = \frac{3}{4}$$

$$\text{Var}(Y) = E(Y^2) - (E(Y))^2 = \frac{3}{4} - \left(\frac{3}{4}\right)^2 = \frac{3}{4} - \frac{9}{16} = \frac{12}{16} - \frac{9}{16} = \frac{3}{16}$$

b) $$E(Y) = \int_0^{3/2} y\left(\frac{4}{3} - \frac{8}{9}y\right) dy = \int_0^{3/2} \frac{4}{3}y - \frac{8}{9}y^2\, dy = \frac{2}{3}y^2 - \frac{8}{27}y^3\Big|_0^{3/2} = \frac{2}{3}\left(\frac{9}{4}\right) - \frac{8}{27}\left(\frac{27}{8}\right) - 0$$

$$= \frac{3}{2} - 1 = \frac{1}{2}$$

$$E(Y^2) = \int_0^{3/2} y^2\left(\frac{4}{3} - \frac{8}{9}y\right) dy = \int_0^{3/2} \frac{4}{3}y^2 - \frac{8}{9}y^3\, dy = \frac{4}{9}y^3 - \frac{2}{9}y^4\Big|_0^{3/2} = \frac{4}{9}\left(\frac{27}{8}\right) - \frac{2}{9}\left(\frac{81}{16}\right) - 0$$

$$= \frac{3}{2} - \frac{9}{8} = \frac{12-9}{8} = \frac{3}{8}$$

c) The sample mean is .854 and sample variance is .163.

Model A has a mean of .75 and a variance of .1875.

Model B has a mean of .5 and a variance of .375.

Model A has a closer mean <u>and</u> variance to the sample, so Model A is a better candidate model for the distribution of Y (since the sample comes from Y).

3 $f(x,y) = 2k - kx, \quad 0 \le y \le x < 2$

a) $k = \frac{3}{4} \to f(x,y) = \frac{3}{2} - \frac{3}{4}x$

$$\iint f(x,y)\, dx\, dy = \int_0^2 \int_y^2 \frac{3}{2} - \frac{3}{4}x\, dx\, dy = \int_0^2 \left[\frac{3}{2}x - \frac{3}{8}x^2 \Big|_y^2\right] dy = \int_0^2 \left[\left(3 - \frac{3}{2}\right) - \left(\frac{3}{2}y - \frac{3}{8}y^2\right)\right] dy$$

$$= \int_0^2 \frac{3}{2} + \frac{3}{8}y^2 - \frac{3}{2}y\, dy = \frac{3}{2}y + \frac{1}{8}y^3 - \frac{3}{4}y^2\Big|_0^2 = 3 + 1 - 3 = 1 \checkmark$$

integrates to 1 $\Rightarrow$ valid PDF

b) $f_x(x) = \int_0^x \frac{3}{2} - \frac{3}{4}x\,dy = \frac{3}{2}y - \frac{3}{4}xy \Big|_0^x = \frac{3}{2}x - \frac{3}{4}x^2 \quad 0 < x < 2$

$f_y(y) = \int_y^2 \frac{3}{2} - \frac{3}{4}x\,dx = \frac{3}{2}x - \frac{3}{8}x^2 \Big|_y^2 = \left(3 - \frac{3}{2}\right) - \left(\frac{3}{2}y - \frac{3}{8}y^2\right) = \frac{3}{2} - \frac{3}{2}y + \frac{3}{8}y^2 \quad 0 < y < 2$

c) $f_x(x) \cdot f_y(y) = \left(\frac{3}{2}x - \frac{3}{4}x^2\right)\left(\frac{3}{2} - \frac{3}{2}y + \frac{3}{8}y^2\right) \neq \frac{3}{2} - \frac{3}{4}x = f(x,y)$

X and Y are not independent because the product of their respective marginal distributions does not equal their joint distribution.

d) $f(y|x) = \dfrac{f(x,y)}{f_x(x)} = \dfrac{\frac{3}{2} - \frac{3}{4}x}{\frac{3}{2}x - \frac{3}{4}x^2} = \dfrac{\left(\frac{3}{2} - \frac{3}{4}x\right)}{x\left(\frac{3}{2} - \frac{3}{4}x\right)} = \dfrac{1}{x}$

$\boxed{4}$ $f(x,y) = 3x^2y \quad -1 \le x \le 1,\ 0 \le y \le 1$

a) $P(x > 0, y > .5) = \int_{1/2}^1 \int_0^1 3x^2y\,dx\,dy = \int_{.5}^1 \left[x^3y \Big|_0^1\right] dy = \int_{1/2}^1 y\,dy = \frac{y^2}{2} \Big|_{1/2}^1 = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}$

b) $P(x > y) = \int_0^1 \int_y^1 3x^2y\,dx\,dy = \int_0^1 x^3y \Big|_y^1 dy = \int_0^1 y - y^4\,dy = \frac{y^2}{2} - \frac{y^5}{5} \Big|_0^1 = \frac{1}{2} - \frac{1}{5} = \frac{3}{10}$

c) $f(x,y) = 3x^2y \qquad g(x) = 3x^2 \qquad h(y) = y \qquad f(x,y) = g(x)\,h(y) \qquad \begin{array}{l} -1 \le x \le 1 \\ 0 \le y \le 1 \end{array}$

Because the limits of the support of x and y are all constants and their joint distribution can be written as two separate functions (one of just x, the other of just y), X and Y are independent.

d) $E\left(\frac{(1-y)^3}{3x^2}\right) = \underbrace{E((1-y)^3) \cdot E\left(\frac{1}{3x^2}\right)}_{\text{possible because x and y are independent}} = \int_0^1 (1-y^3) f(y)\,dy \cdot \int_{-1}^1 \frac{1}{3x^2} f(x)\,dx$

$f(y) = \int_{-1}^1 3x^2y\,dx = x^3y \Big|_{-1}^1 = y - (-y) = 2y \qquad f(x) = \int_0^1 3x^2y\,dy = \frac{3}{2}x^2y^2 \Big|_0^1 = \frac{3}{2}x^2$

$$E\left((1-y)^3\right) = \int_0^1 2\overbrace{(1-y)^3 \cdot y}^{\text{kernel of Beta}(2,4)}\,dy \qquad \text{Beta}(2,4) = \frac{\Gamma(6)}{\Gamma(2)\Gamma(4)}\,y\,(1-y)^3 = \frac{5!}{3!}\,y(1-y)^3 = 20\,y(1-y)^3$$

$$= \frac{1}{10} \quad \text{because} \quad \int_0^1 20(1-y)^3 y\,dy = 1 \implies \int_0^1 2(1-y)^3 y\,dy = \frac{1}{10}$$

$$E\left(\frac{1}{3x^2}\right) = \int_{-1}^{1} \frac{1}{3x^2} \cdot \frac{3}{2}x^2\,dx = \int_{-1}^{1} \frac{1}{2}\,dx = \left.\frac{x}{2}\right|_{-1}^{1} = \frac{1}{2} - \left(\frac{-1}{2}\right) = 1$$

$$E\left((1-y)^3\right) \cdot E\left(\frac{1}{3x^2}\right) = \frac{1}{10} \cdot 1 = E\left(\frac{(1-y)^3}{3x^2}\right) = \frac{1}{10}$$