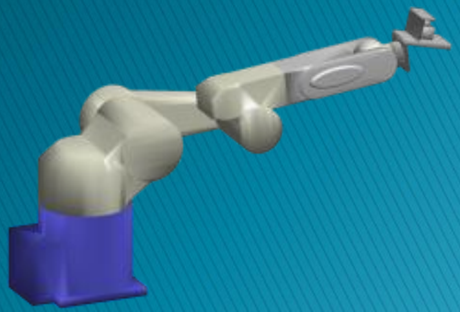
Manipulator Dynamics



Torque Equation

$$\boldsymbol{\tau} = \mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{c}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q})$$

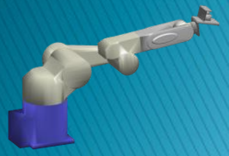
Where:

- $\boldsymbol{\tau} \in \mathbb{R}^n$ is a vector of joint torques

- $\mathbf{M}(\mathbf{q}) \in \mathbb{R}^{n \times n}$ is the inertia matrix
    - $\mathbf{M} = \mathbf{M}^T$
    - m_{ij} denotes the inertial coupling between link i and link j

- $\mathbf{c}(\mathbf{q}, \dot{\mathbf{q}}) \in \mathbb{R}^n$ is a vector of the centripetal and Coriolis forces
    - Sometimes written as $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}}$ (matrix and velocity vector)

- $\mathbf{g}(\mathbf{q})$ is the gravity vector, or the weight of each link



Lagrangian Mechanics is used to solve the dynamic equations

The Newton-Euler equation is:

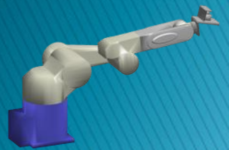
$$L = K - P$$

where:

- K is the sum of all kinetic energy in the system
- P is the sum of all potential energy in the system

Then:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} = \boldsymbol{\tau}$$



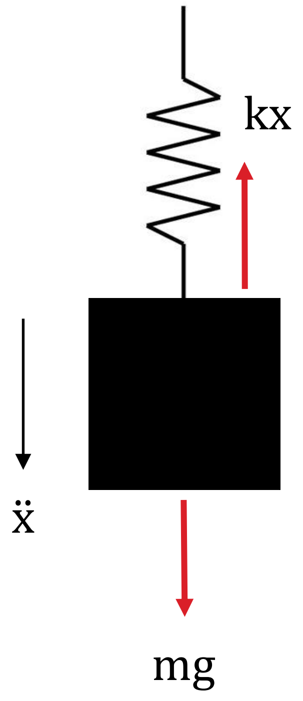
Mass-Spring System

Using Lagrangian Mechanics:

$$K = \tfrac{1}{2} m\dot{x}^2$$
$$P = \tfrac{1}{2} kx^2 - mgx$$

$$L = \tfrac{1}{2} m\dot{x}^2 - \tfrac{1}{2} kx^2 + mgx$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$
$$\frac{d}{dt}(m\dot{x}) + kx - mg = 0$$
$$m\ddot{x} + kx - mg = 0$$
$$\ddot{x} = g - m^{-1}kx$$



Using Newtonian Mechanics:

$$\downarrow \sum F = m\ddot{x}$$
$$= mg - kx$$
$$\ddot{x} = g - m^{-1}kx$$

- The same answer using both methods!
- Lagrangian mechanics is really useful for constrained motion
  - Such as serial-link manipulators

1–Link Planar Example

$$\begin{aligned} K &= \tfrac{1}{2}mv^2 \\ &= \tfrac{1}{2}m(r\dot{q})^2 \\ &= \tfrac{1}{2}mr^2\dot{q}^2 \end{aligned}$$

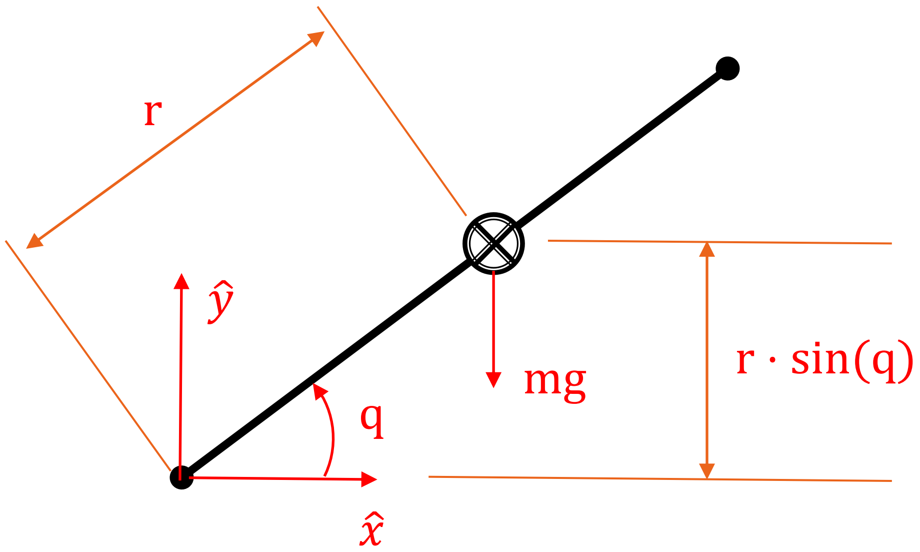
$$P = mgr \cdot \sin(q)$$

$$\begin{aligned} L &= K - P \\ &= \tfrac{1}{2}mr^2\dot{q}^2 - mg\sin(q) \end{aligned}$$

$$\begin{aligned} \boldsymbol{\tau} &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} \\ &= \frac{d}{dt}(mr^2\dot{q}) + mgr \cdot \cos(q) \\ &= mr^2\ddot{q} + mgr \cdot \cos(q) \end{aligned}$$

mr^2 is the rotational inertia of the link

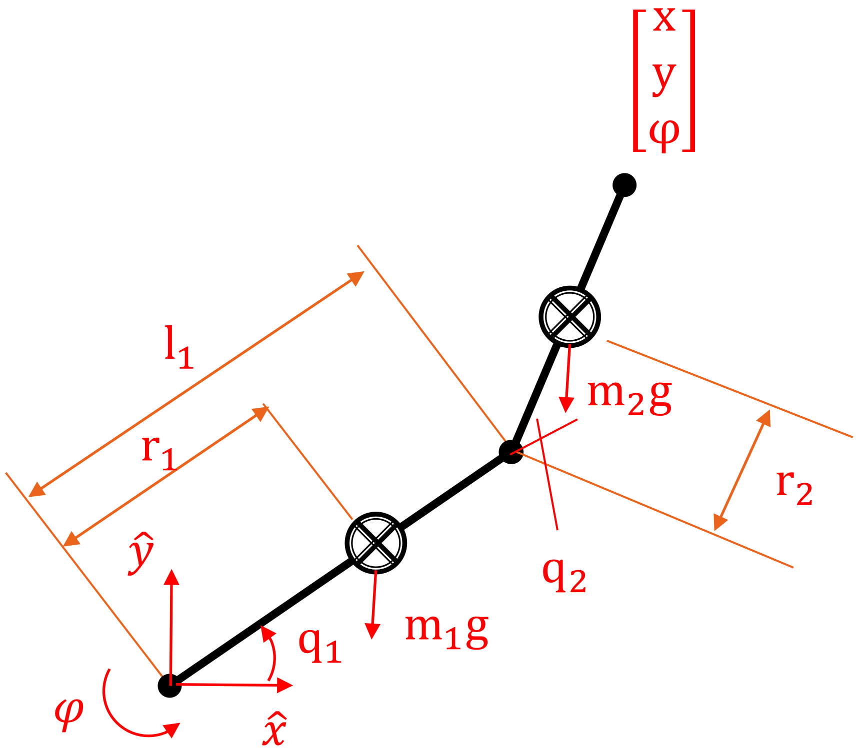
$mgr \cdot \cos(q)$ is the contribution of the link's own mass



2-Link Planar Robot

$$K = \tfrac{1}{2}m_1 r_1^2 \dot{q}_1^2 + \tfrac{1}{2}m_2(l_1^2 + r_2^2 + 2l_1 r_2 \cos(q_2))\dot{q}_1^2 + m_2(r_2^2 + l_1 r_2 \cos(q_2))\dot{q}_1 \dot{q}_2 + \tfrac{1}{2}m_2 r_2^2 \dot{q}_2^2$$

$$P = m_1 g r_1 \sin(q_1) + m_2 g(l_1 \sin(q_1) + r_2 \sin(q_1 + q_2))$$

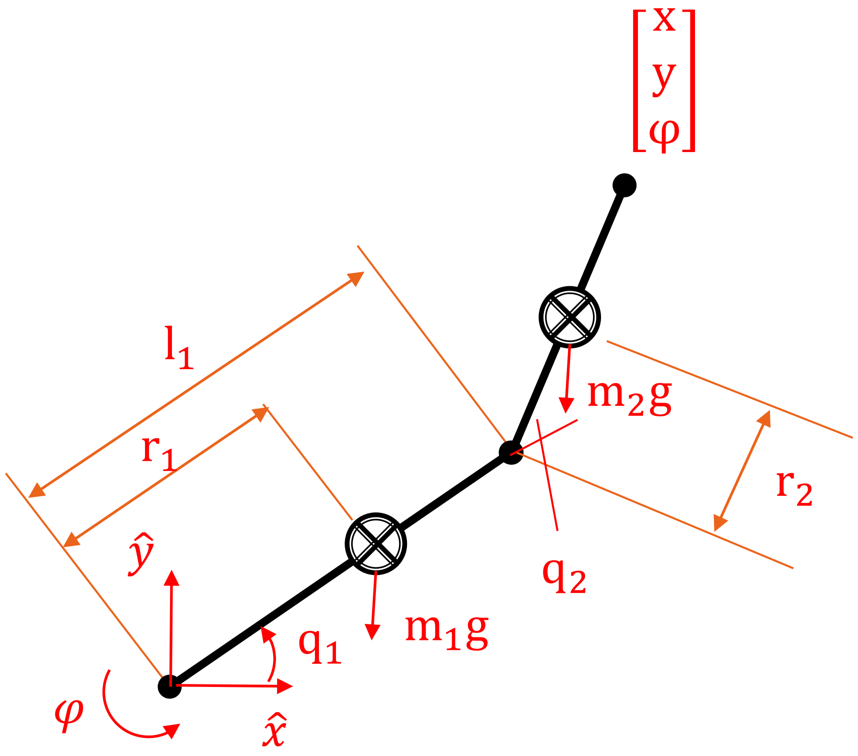


2-Link Planar Robot (cont.)

$$\mathbf{M}(\mathbf{q}) = \begin{bmatrix} (m_1 r_1^2 + m_2 l_1^2 + m_2 r_2^2 + 2 m_2 l_1 r_2 \cos(q_2)) & (m_2 r_2^2 + m_2 l_1 r_2 \cos(q_2)) \\ (m_2 r_2^2 + m_2 l_1 r_2 \cos(q_2)) & 0 \end{bmatrix}$$

$$\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} = \begin{bmatrix} -2 m_2 l_1 r_2 \sin(q_2)\dot{q}_2 & -m_2 l_1 r_2 \sin(q_2)\dot{q}_2 \\ m_2 l_1 r_2 \sin(q_2)\,\dot{q}_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

$$\mathbf{g}(\mathbf{q}) = \begin{bmatrix} r_1(m_1 g)\cos(q_1) + (m_2 g)(l_1 r_2 \cos(q_1)\cos(q_1 + q_2)) \\ r_2(m_2 g)\cos(q_1 + q_2) \end{bmatrix}$$



- Solving the dynamics for a manipulator can get very complex very fast
- It's best to let the a computer solve it for you!

To control a manipulator with motor torques, we need to know the required joint acceleration

Given a joint trajectory, solve for the joint accelerations needed to move to a desired joint angle:

$$\mathbf{q}(\mathrm{t}+1) = \mathbf{q}(\mathrm{t}) + \Delta\mathrm{t}\dot{\mathbf{q}}(\mathrm{t}) + \Delta\mathrm{t}^2\ddot{\mathbf{q}}(\mathrm{t})$$

$$\ddot{\mathbf{q}}(\mathrm{t}) = \Delta\mathrm{t}^{-2}\left(\mathbf{q}(\mathrm{t}+1) - \mathbf{q}(\mathrm{t}) - \Delta\mathrm{t}\dot{\mathbf{q}}(\mathrm{t})\right)$$

Then solve for the torque needed to accelerate the joint, as well as overcome gravitation, inertial, and Coriolis forces:

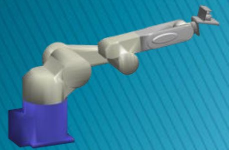
$$\boldsymbol{\tau} = \mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{c}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q})$$

Given an applied torque, we can find the resultant acceleration:

$$\ddot{\mathbf{q}} = \mathbf{M}^{-1}(\boldsymbol{\tau} - \mathbf{c} - \mathbf{g})$$

The resultant acceleration might not be the same for an unknown control disturbance $\boldsymbol{\tau}_{\mathrm{D}}$:

$$\ddot{\mathbf{q}} = \mathbf{M}^{-1}(\boldsymbol{\tau} - \mathbf{c} - \mathbf{g} - \boldsymbol{\tau}_{\mathrm{D}})$$



Summary

- Lagrangian mechanics can be used to solve for the dynamic equations

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial \mathrm{L}}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial \mathrm{L}}{\partial \mathbf{q}} = \boldsymbol{\tau}$$

- $\mathbf{M}(\mathbf{q})$ is the inertia matrix
  - It is symmetric and positive definite
- $\mathbf{c}(\mathbf{q}, \dot{\mathbf{q}})$ is a vector of Coriolis forces
- $\mathbf{g}(\mathbf{q})$ is the gravity vector
- To move between joint angles, solve for the joint accelerations

$$\ddot{\mathbf{q}}(\mathrm{t}) = \Delta\mathrm{t}^{-2}\big(\mathbf{q}(\mathrm{t}+1) - \mathbf{q}(\mathrm{t}) - \Delta\mathrm{t}\dot{\mathbf{q}}(\mathrm{t})\big)$$

- To accelerate the joint, solve the dynamics equation

$$\boldsymbol{\tau} = \mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{c}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q})$$