2.2 Rotation Matrices

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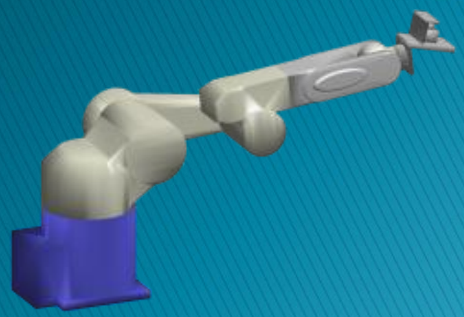


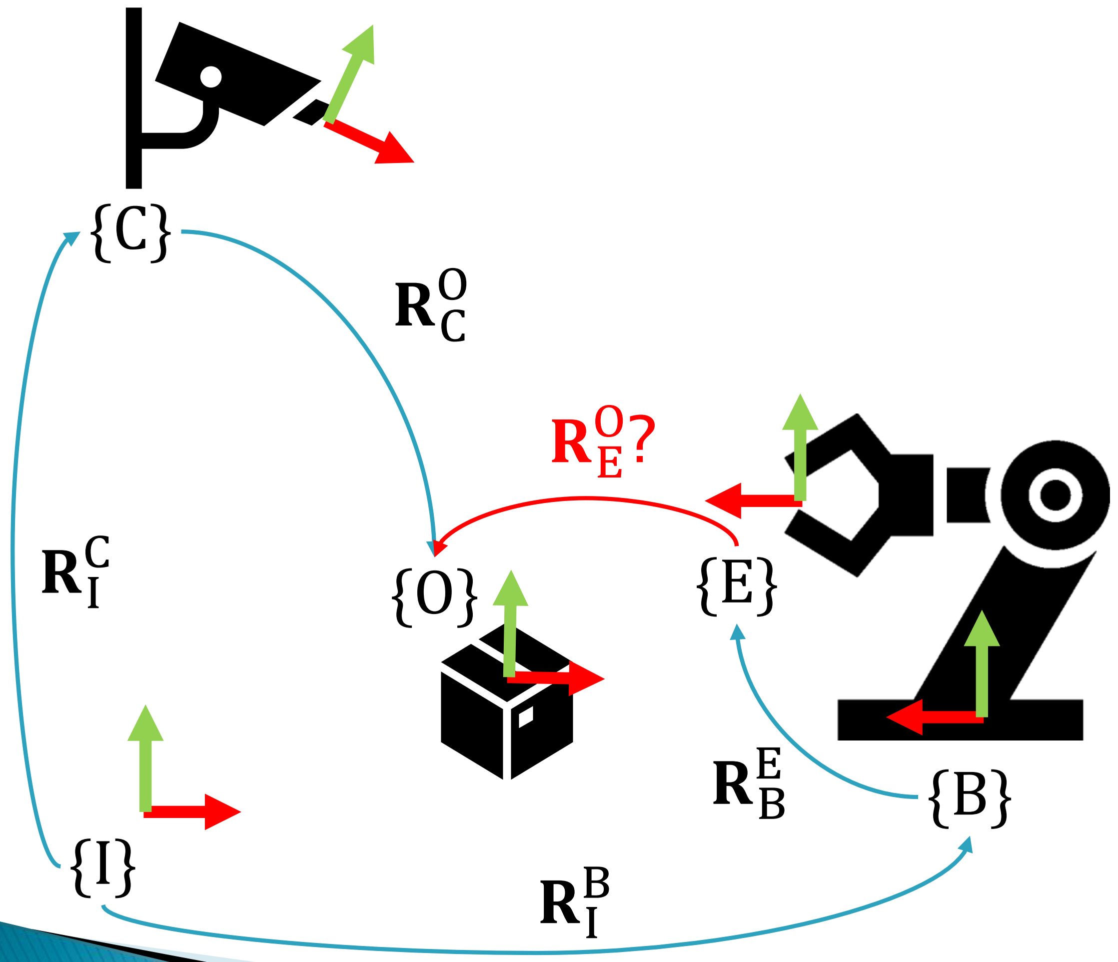
Roadmap

Rotation Matrices

$$\mathbf{R} = [\hat{x} \quad \hat{y} \quad \hat{z}]$$

How can we describe the relative orientation between reference frames?

How Should the Robot Orient its End-Effector?

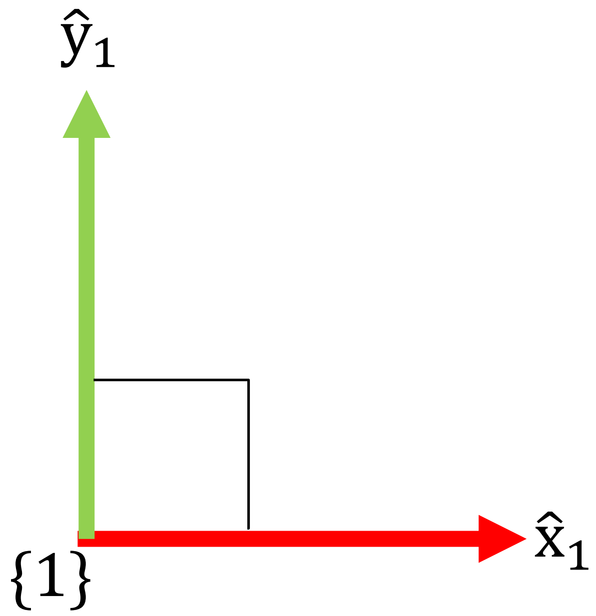


Given the following rotations:

- $\mathbf{R}_I^C \rightarrow$ Inertial to Camera
- $\mathbf{R}_C^O \rightarrow$ Camera to Object
- $\mathbf{R}_I^B \rightarrow$ Inertial to Base
- $\mathbf{R}_B^E \rightarrow$ Base to End-Effector

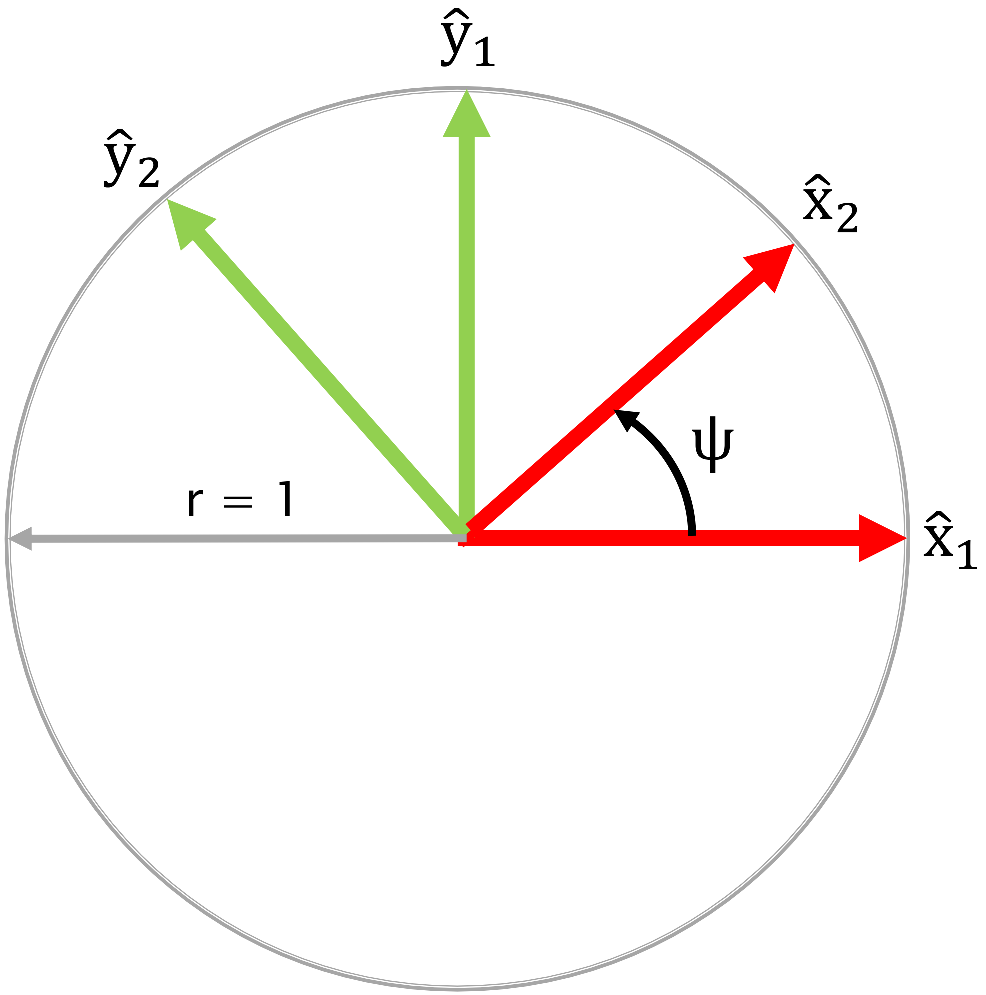
What is $\mathbf{R}_E^O$?

Constructing a Reference Frame



- $\hat{x}$ and $\hat{y}$ are unit vectors:
  - $\|\hat{x}\| = 1$
  - $\|\hat{y}\| = 1$
- $\hat{x}$ is orthogonal to $\hat{y}$:
  - $\hat{x}^T\hat{y} = 0$
- $\hat{x}$ and $\hat{y}$ are **orthonormal**
  - Orthogonal + normalized

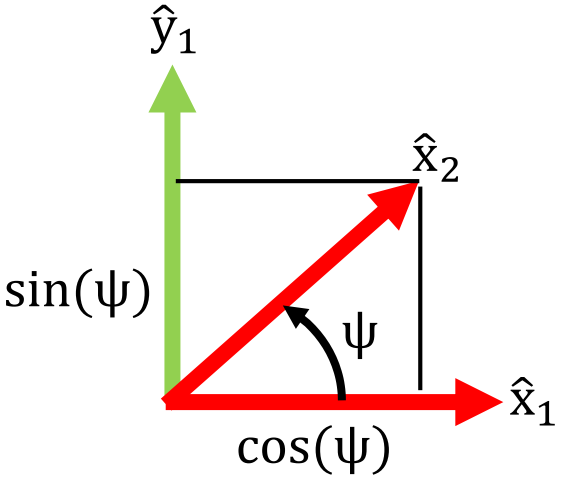
Rotation Between 2 Reference Frames

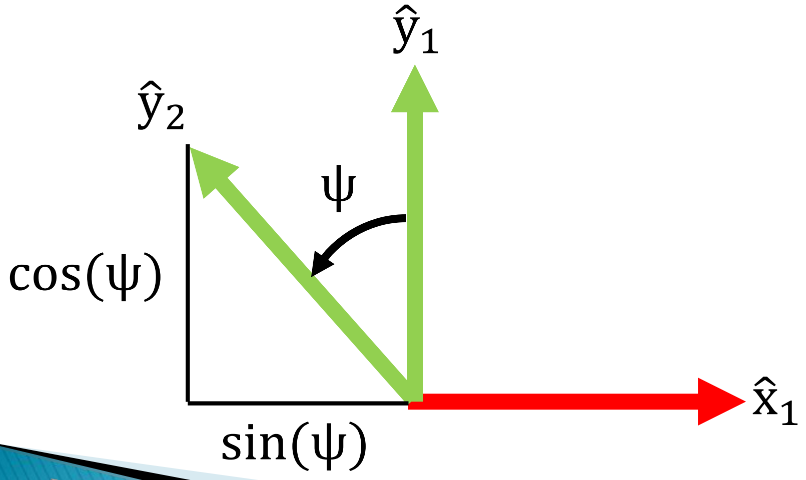


Suppose we rotate a second reference frame {2} about unit circle by ψ.

How can we describe {2} with respect to {1} mathematically?

Rotation Between 2 Reference Frames





Express axes of frame {1} as functions of axes of frame {2}:

$$\begin{bmatrix}\hat{x}_1\\ \hat{y}_1\end{bmatrix} = \begin{bmatrix}\hat{x}_2\cos(\psi) - \hat{y}_2\sin(\psi)\\ \hat{x}_2\sin(\psi) + \hat{y}_2\cos(\psi)\end{bmatrix}$$

$$= \begin{bmatrix}\cos(\psi) & -\sin(\psi)\\ \sin(\psi) & \cos(\psi)\end{bmatrix}\begin{bmatrix}\hat{x}_2\\ \hat{y}_2\end{bmatrix}$$

Define the **Rotation Matrix** from {1} to {2} as:

$$\boxed{\mathbf{R}_1^2(\psi) = \begin{bmatrix}\cos(\psi) & -\sin(\psi)\\ \sin(\psi) & \cos(\psi)\end{bmatrix}}$$

Components of $\hat{x}_2$

Components of $\hat{y}_2$

The Transpose of a Rotation Matrix is Equivalent to its Inverse

Multiply the rotation matrix by its transpose:

$$\mathbf{R}\mathbf{R}^{\mathrm{T}} = \begin{bmatrix} \cos(\psi) & -\sin(\psi) \\ \sin(\psi) & \cos(\psi) \end{bmatrix} \begin{bmatrix} \cos(\psi) & \sin(\psi) \\ -\sin(\psi) & \cos(\psi) \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2(\psi) + \sin^2(\psi) & \cos(\psi)\sin(\psi) - \cos(\psi)\sin(\psi) \\ \sin(\psi)\cos(\psi) - \sin(\psi)\cos(\psi) & \sin^2(\psi) + \cos^2(\psi) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

We also know that:

$$\mathbf{R}\mathbf{R}^{-1} = \mathbf{I}$$
$$\therefore \mathbf{R}^{-1} = \mathbf{R}^{\mathrm{T}}$$

The Rotation Matrix is **orthogonal.**

Make sense since column vectors $\hat{\mathrm{x}}^{\mathrm{T}}\hat{\mathrm{y}} = 0$ (orthogonal) by definition!

The Reverse of a Rotation is its Inverse

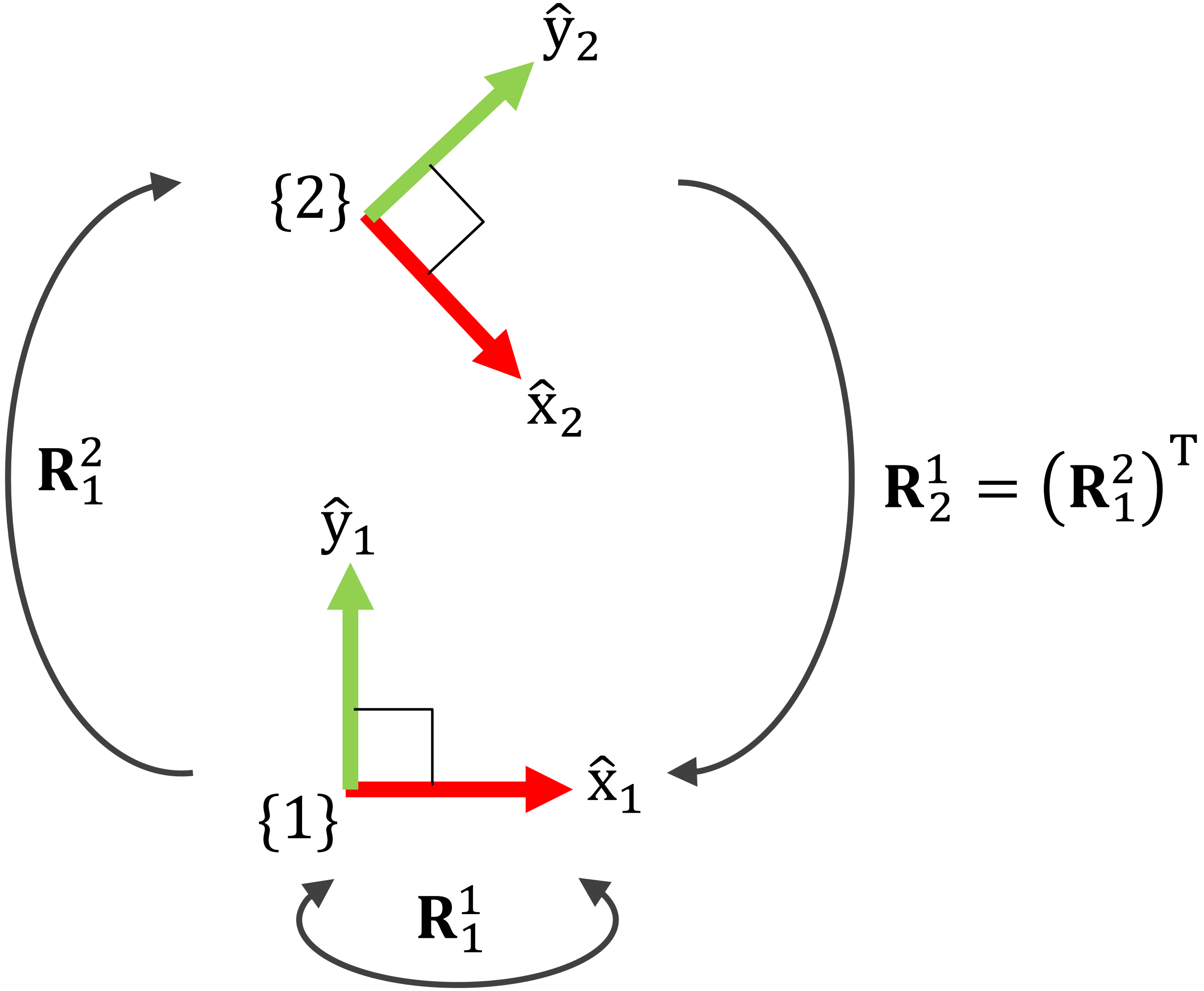
Rotation from {1} to {2}:

$\mathbf{R}_1^2$

Rotation from {2} to {1}:

$\mathbf{R}_2^1 = \left(\mathbf{R}_1^2\right)^{\mathrm{T}}$

Rotation from {1} to {1}:



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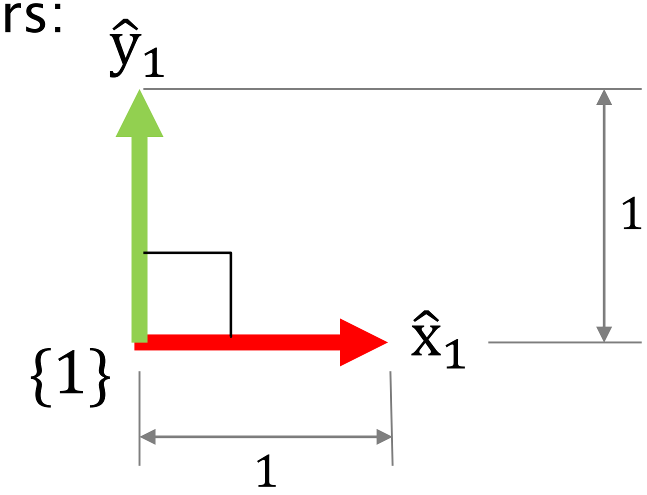
The Euclidean Norm of a Rotation Matrix is 1

Recall that $\mathbf{R}$ is a concatenation of unit vectors:

$$\mathbf{R} = [\hat{\mathrm{x}} \quad \hat{\mathrm{y}}]$$

$$= \begin{bmatrix} \cos(\psi) & -\sin(\psi) \\ \sin(\psi) & \cos(\psi) \end{bmatrix}$$

The Euclidean Norm of $\mathbf{R}$ is the largest norm of its column vectors:



$$\|\mathbf{R}\| = \max[\|\hat{\mathrm{x}}\|, \|\hat{\mathrm{y}}\|]$$

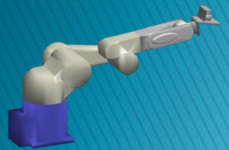
$$= \max\left[\sqrt{\cos^2(\psi) + \sin^2(\psi)}, \sqrt{\sin^2(\psi) + \cos^2(\psi)}\right]$$

$$= \max[1,1]$$

$$= 1$$

This is self evident as, by definition:

$$\|\hat{\mathrm{x}}\| = \|\hat{\mathrm{y}}\| = 1$$



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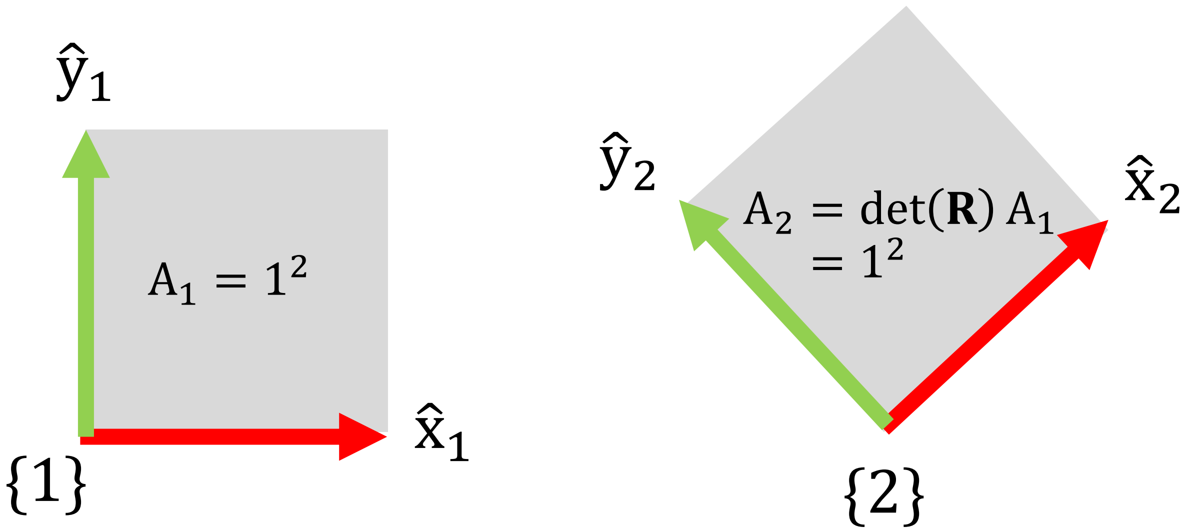
The Determinant of a Rotation Matrix is 1

$$\det(\mathbf{R}) = \begin{vmatrix} \cos(\psi) & -\sin(\psi) \\ \sin(\psi) & \cos(\psi) \end{vmatrix}$$
$$= \cos(\psi) \times \cos(\psi) - \sin(\psi) \times - \sin(\psi)$$
$$= \cos^2(\psi) + \sin^2(\psi)$$
$$= 1$$

This means the area (or volume) bounded by the axes remains constant.

i.e. scaled by 1.



A Rotation Matrix is in the Special Orthogonal Group

Special Orthogonal $\mathbb{SO}$ group:

- $\mathbf{R}\mathbf{R}^{\mathrm{T}} = \mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}$
- $\|\mathbf{R}\| = 1$
- $\det(\mathbf{R}) = 1$

If $\mathbf{R}$ is an $\mathrm{n} \times \mathrm{n}$ matrix with the above properties, then $\mathbf{R} \in \mathbb{SO}(\mathrm{n})$

Invariance Under Rotation

If we rotate a vector $\mathbf{b}$ such that:

$$\mathbf{a} = \mathbf{R}\mathbf{b}$$

Then its magnitude will remain the same:

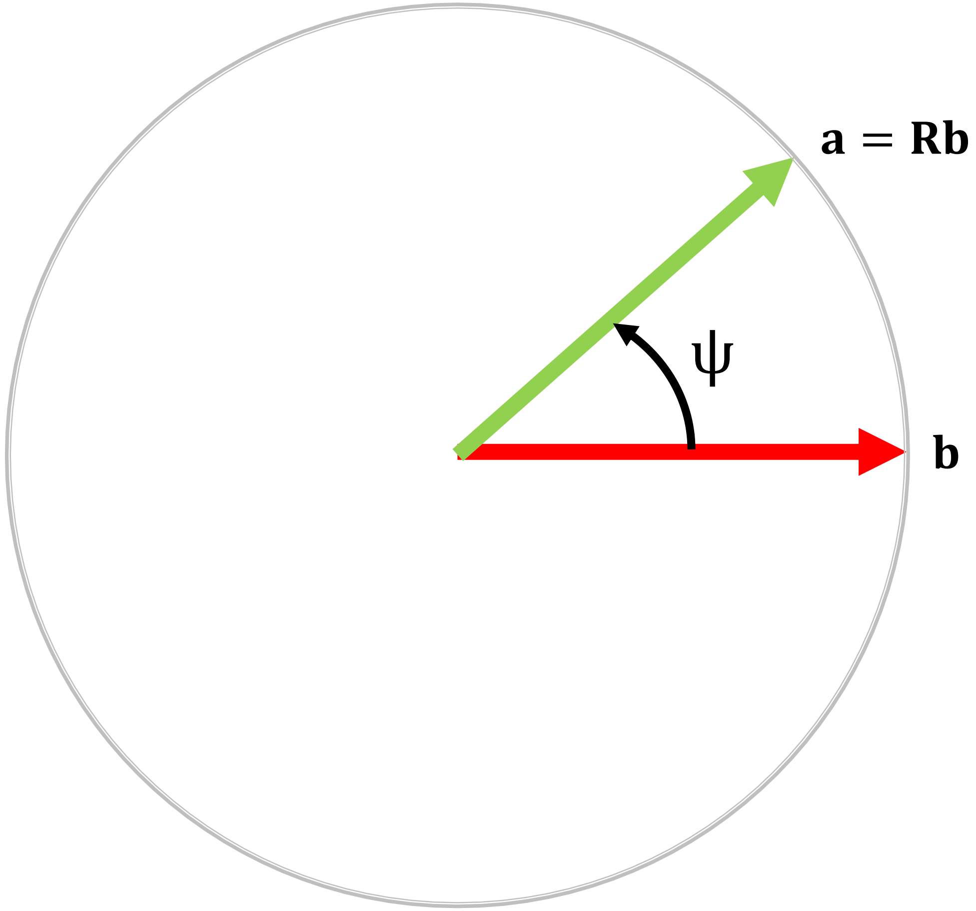
$$\|\mathbf{a}\| = \|\mathbf{b}\|$$

Proof:

$$\|\mathbf{a}\|^2 = \|\mathbf{R}\mathbf{b}\|^2$$
$$\mathbf{a}^{\mathrm{T}}\mathbf{a} = \mathbf{b}^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}\mathbf{R}\mathbf{b}$$
$$\mathbf{a}^{\mathrm{T}}\mathbf{a} = \mathbf{b}^{\mathrm{T}}\mathbf{b}$$

Hence, $\|\mathbf{a}\| = \|\mathbf{R}\mathbf{b}\| = \|\mathbf{b}\|$.

Obvious by rotating a vector around a circle.



Implications to Invariance Under Rotation

Velocity for observer $\{A\}$:

$$^{A}\mathbf{v}$$

Velocity for observer $\{B\}$:

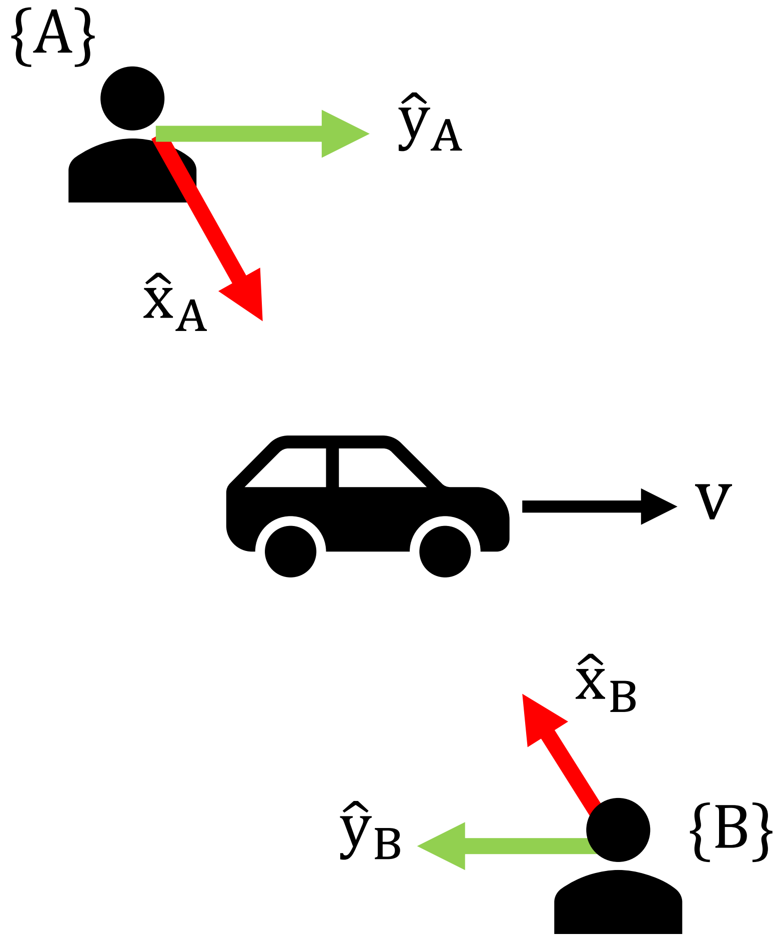
$$^{B}\mathbf{v} = \mathbf{R}_{B}^{A} \cdot {}^{A}\mathbf{v}$$

Direction of the vectors is different:

$$^{A}\mathbf{v} \neq {}^{B}\mathbf{v}$$

But the magnitude will be the same:

$$\left\| {}^{A}\mathbf{v} \right\| = \left\| {}^{B}\mathbf{v} \right\|$$



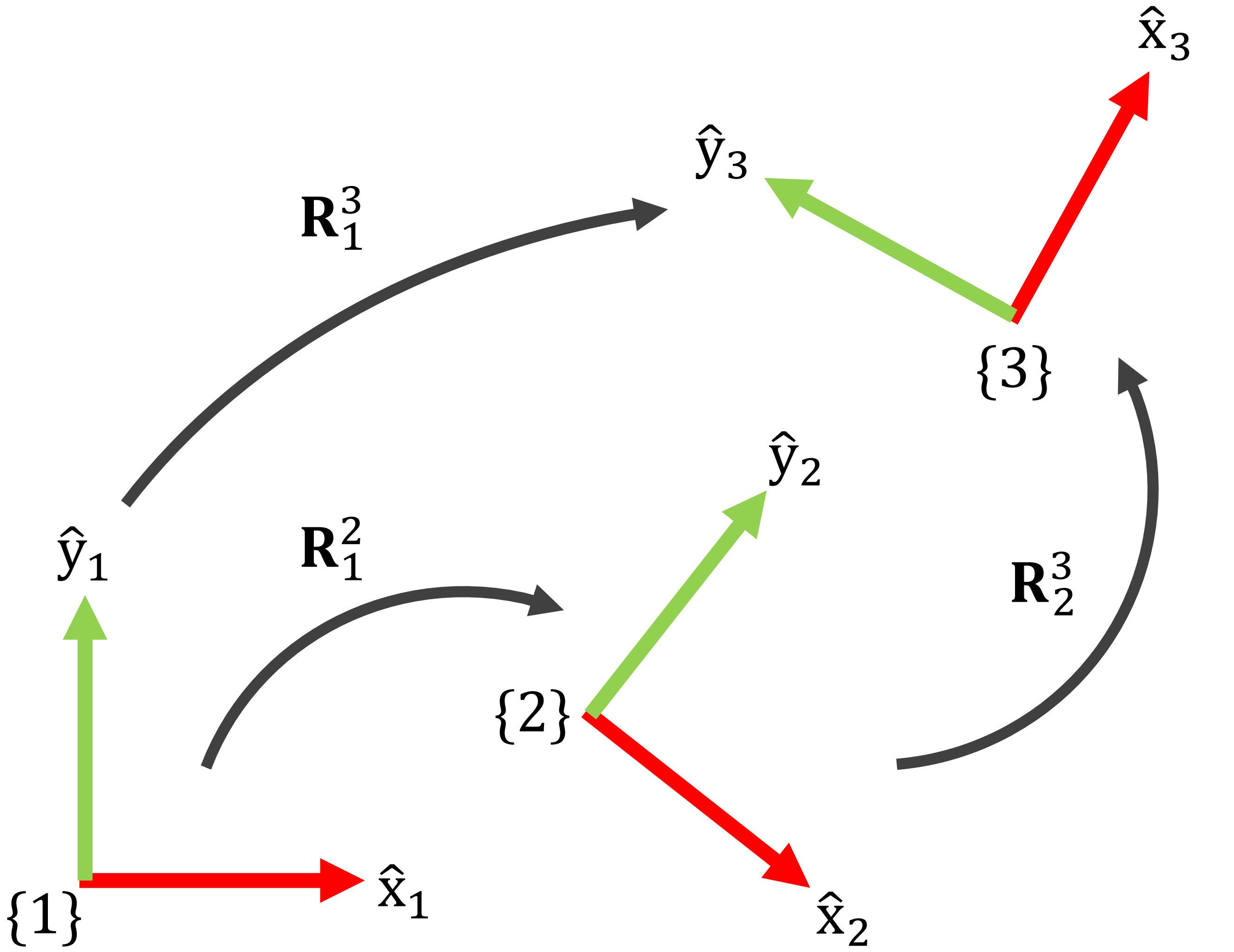
Multiplying Rotation Matrices makes another Rotation Matrix

$$\mathbf{R}_1^3 = \mathbf{R}_1^2\mathbf{R}_2^3$$

$$\begin{aligned}\mathbf{R}_1^3\left(\mathbf{R}_1^3\right)^{\mathrm{T}} &= \left(\mathbf{R}_1^2\mathbf{R}_2^3\right)^{\mathrm{T}}\left(\mathbf{R}_1^2\mathbf{R}_2^3\right)\\ &= \left(\mathbf{R}_2^3\right)^{\mathrm{T}}\left(\mathbf{R}_1^2\right)^{\mathrm{T}}\mathbf{R}_1^2\mathbf{R}_2^3\\ &= \left(\mathbf{R}_2^3\right)^{\mathrm{T}}\mathbf{R}_2^3\\ &= \mathbf{I}\end{aligned}$$

$$\begin{aligned}\det\left(\mathbf{R}_1^3\right) &= \det\left(\mathbf{R}_1^2\mathbf{R}_2^3\right)\\ &= \det\left(\mathbf{R}_1^2\right)\det\left(\mathbf{R}_2^3\right)\\ &= 1 \times 1\\ &= 1\end{aligned}$$

$\mathbf{R}_1^3 \in \mathbb{SO}$ is another rotation matrix!



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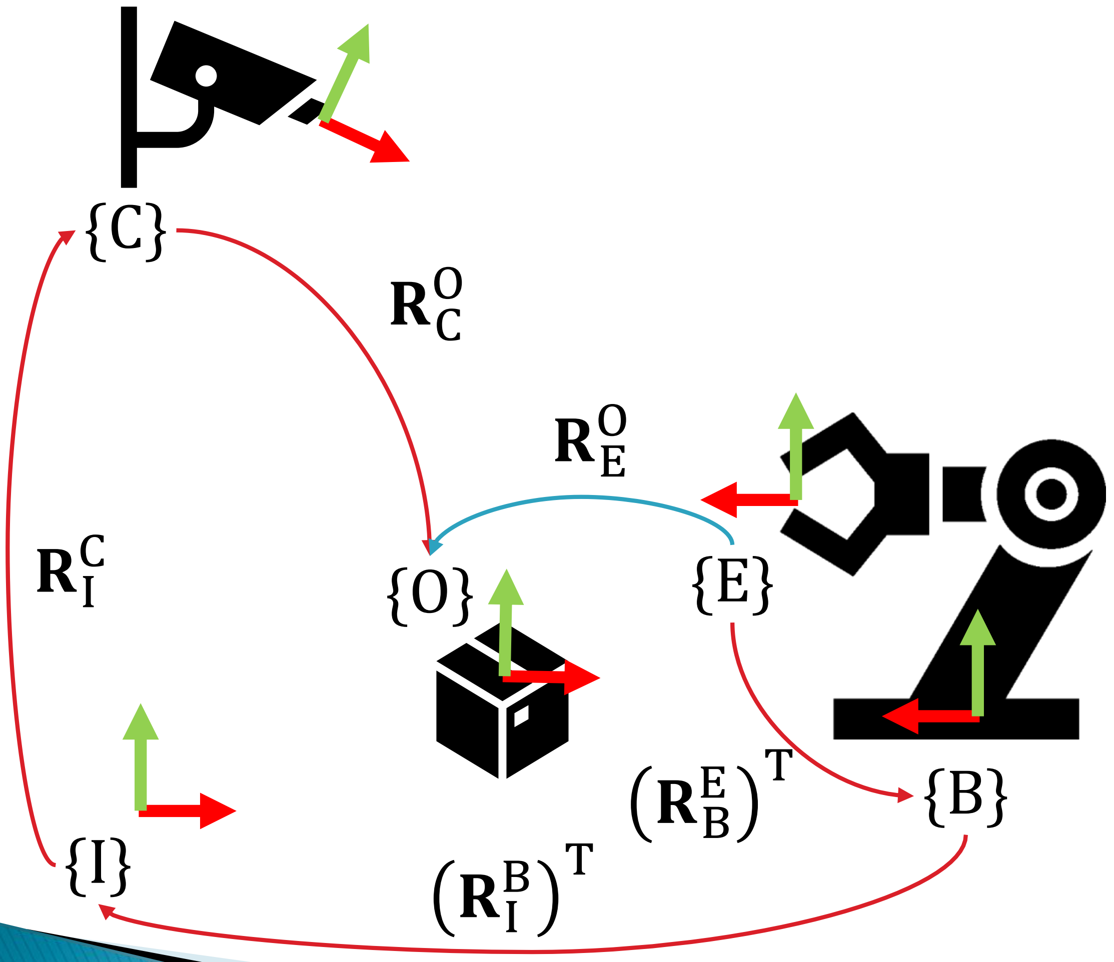
How Should the Robot Orient its End-Effector?



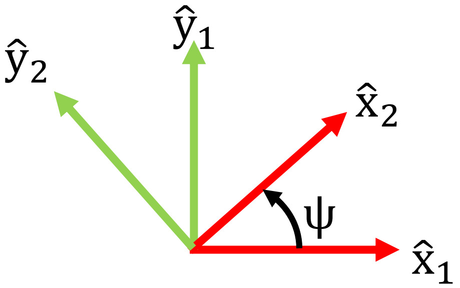
First, chain the rotation matrices:

$$\mathbf{R}_{\mathrm{E}}^{\mathrm{O}} = \mathbf{R}_{\mathrm{E}}^{\mathrm{B}}\mathbf{R}_{\mathrm{B}}^{\mathrm{I}}\mathbf{R}_{\mathrm{I}}^{\mathrm{C}}\mathbf{R}_{\mathrm{C}}^{\mathrm{O}}$$

Then, transpose (or invert) the relevant rotations to get opposite direction:

$$\mathbf{R}_{\mathrm{E}}^{\mathrm{O}} = \left(\mathbf{R}_{\mathrm{B}}^{\mathrm{E}}\right)^{\mathrm{T}}\left(\mathbf{R}_{\mathrm{I}}^{\mathrm{B}}\right)^{\mathrm{T}}\mathbf{R}_{\mathrm{I}}^{\mathrm{C}}\mathbf{R}_{\mathrm{C}}^{\mathrm{O}}$$

Summary of Rotation Matrices



Rotations describe relative orientation between reference frames

$\mathbf{R} = [\hat{x} \quad \hat{y}]$	The rotation matrix is a concatenation of unit vectors
$\mathbf{R} \in \mathbb{SO}(n)$	The rotation matrix is in the Special Orthogonal group
$\mathbf{R}\mathbf{R}^{\mathrm{T}} = \mathbf{I} \implies \mathbf{R}^{\mathrm{T}} = \mathbf{R}^{-1}$	The transpose of a rotation matrix is its inverse
$\lVert\mathbf{R}\rVert = 1$	The Euclidean norm of a rotation matrix is 1
$\det(\mathbf{R}) = 1$	The determinant of a rotation matrix is 1
$\lVert\mathbf{v}\rVert = \lVert\mathbf{R}\mathbf{v}\rVert$	The magnitude of a vector is invariant under rotation
$\mathbf{R}_2^3 = \mathbf{R}_1^2\mathbf{R}_2^3 \in \mathbb{SO}$	Multiplying 2 rotations makes another rotation
$\mathbf{R}_2^1 = \left(\mathbf{R}_1^2\right)^{\mathrm{T}}$	The reverse of a rotation is its transpose/inverse

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