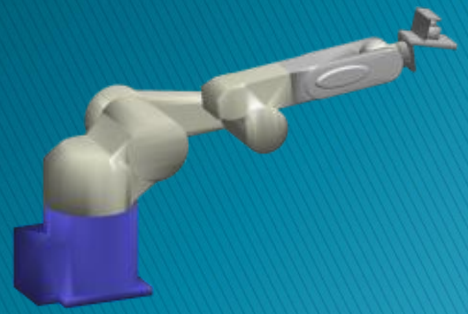
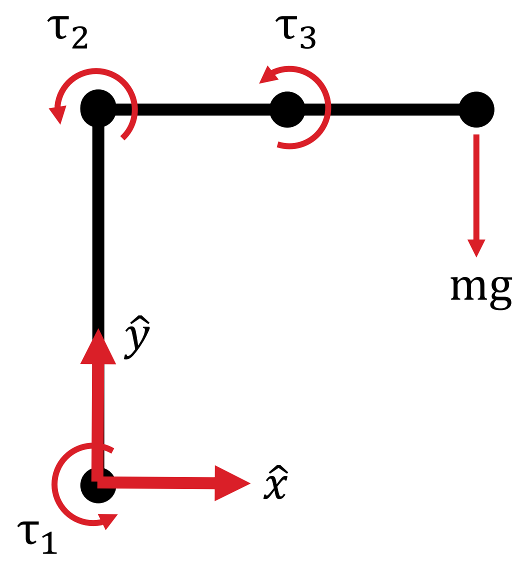
Manipulator Statics



Motivation

- How much torque do we need to apply at each joint to hold an object of known mass in the air?



Calculating the static torque

From the differential kinematics:

$$\dot{\mathbf{x}} = \mathbf{J}(\mathbf{q})\dot{\mathbf{q}}$$

Taking a linear approximation:

$$\frac{\Delta\mathbf{x}}{\Delta \mathrm{t}} = \mathbf{J}(\mathbf{q})\frac{\Delta\mathbf{q}}{\Delta \mathrm{t}}$$

For infinitesimally small displacements, it holds that:

$$\Delta\mathbf{x} = \mathbf{J}(\mathbf{q})\Delta\mathbf{q}$$

When the manipulator is stationary, virtual work/energy done by the system must be zero:

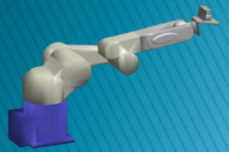
$$\mathrm{E} = \mathbf{w}^{\mathrm{T}}\Delta\mathbf{x} - \boldsymbol{\tau}^{\mathrm{T}}\Delta\mathbf{q} = 0$$

Where $\mathbf{w}$ is a *wrench* of forces and torques applied to the end-effector:

$$\mathbf{w}^{\mathrm{T}} = \begin{bmatrix} \mathrm{F_x} & \mathrm{F_y} & \mathrm{F_z} & \tau_\mathrm{x} & \tau_\mathrm{y} & \tau_\mathrm{z} \end{bmatrix}$$

And $\boldsymbol{\tau} \in \mathbb{R}^{\mathrm{n}}$ is a vector of joint torques:

$$\boldsymbol{\tau}^{\mathrm{T}} = \begin{bmatrix} \tau_1 & \cdots & \tau_\mathrm{n} \end{bmatrix}$$



The Jacobian maps a wrench of forces, torques at the end-effector to the joint torques

Rearranging the energy/work equation:

$$
\begin{aligned}
\mathrm{E} = \mathbf{w}^{\mathrm{T}}\Delta\mathbf{x} - \boldsymbol{\tau}^{\mathrm{T}}\Delta\mathbf{q} &= 0 \\
\mathbf{w}^{\mathrm{T}}\Delta\mathbf{x} &= \tau^{\mathrm{T}}\Delta\mathbf{q} \\
\mathbf{w}^{\mathrm{T}}\mathbf{J}(\mathbf{q})\Delta\mathbf{q} &= \boldsymbol{\tau}^{\mathrm{T}}\Delta\mathbf{q} \qquad \text{Since } \Delta\mathbf{x} = \mathbf{J}(\mathbf{q})\Delta\mathbf{q} \\
\mathbf{w}^{\mathrm{T}}\mathbf{J}(\mathbf{q}) &= \boldsymbol{\tau}^{\mathrm{T}} \\
&\boxed{\boldsymbol{\tau} = \mathbf{J}(\mathbf{q})^{\mathrm{T}}\mathbf{w}}
\end{aligned}
$$

The reaction torque for a wrench applied at the end-effector is related to the Jacobian.

How much torque do we need to hold 1kg of weight, at 1m distance away?

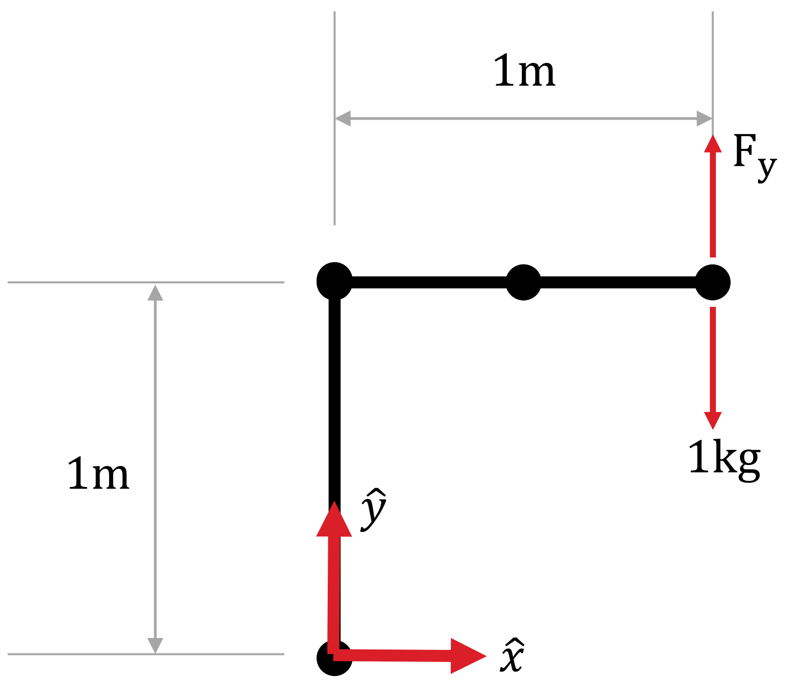
DH Parameters:

θ_j	d_j	a_j	α_j
q_1	0	1	0
q_2	0	0.5	0
q_3	0	0.5	0

$$\mathbf{w} = [0 \quad 9.81 \quad 0 \quad 0 \quad 0 \quad 0]^T$$

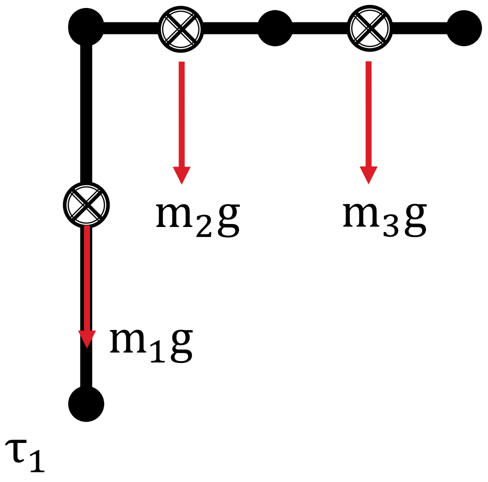
$$\boldsymbol{\tau} = \mathbf{J}(\mathbf{q})^T \mathbf{w}$$

$$= \begin{bmatrix} 9.810 \\ 9.810 \\ 4.905 \end{bmatrix} \text{Nm}$$



Note that the wrench is specified in the **base** frame!

Wait a minute! We still need to factor in the weight of the arm itself



$$\boldsymbol{\tau} = \mathbf{g}(\mathbf{q}) + \mathbf{J}(\mathbf{q})^T\mathbf{w}$$

- The static torque equation does not consider the torque needed to support the mass of each link

What if we want to move the object?

We need to consider the dynamic forces!

This is much more complicated.

$$\boldsymbol{\tau} = \mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{c}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q})$$