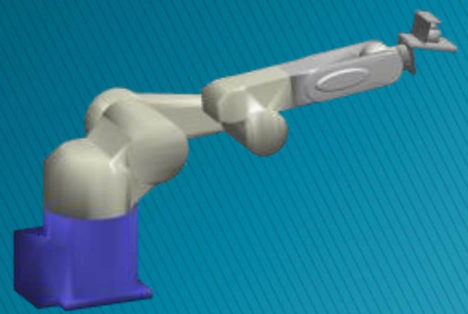
2.4 Derivatives of the Rotation Matrix

Centre for Autonomous Systems
University of Technology Sydney



Time Derivative of the Rotation Matrix

Recall that for the Special Orthogonal $\mathbb{SO}$ group:

$$\mathbf{R}\mathbf{R}^{\mathrm{T}} = \mathbf{I}$$

Taking the time derivative, and using the Product Rule:

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\mathbf{R}\mathbf{R}^{\mathrm{T}}\right) = \frac{\mathrm{d}}{\mathrm{dt}}(\mathbf{I})$$

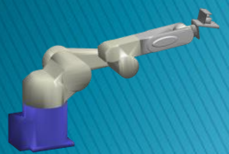
$$\dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} + \mathbf{R}\dot{\mathbf{R}}^{\mathrm{T}} = \mathbf{0}$$

$$\dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} = -\mathbf{R}\dot{\mathbf{R}}^{\mathrm{T}}$$

For the case $\mathbf{R} = \mathbf{I}$ we get:

$$\dot{\mathbf{R}} = -\dot{\mathbf{R}}^{\mathrm{T}}$$

Hence $\dot{\mathbf{R}}$ is skew symmetric.



Skew Symmetric Matrix

Suppose we have a 3D vector of angular velocities:

$$\boldsymbol{\omega} = [\omega_x \quad \omega_y \quad \omega_z]^T \in \mathbb{R}^3$$

Then the skew-symmetric matrix from this vector is:

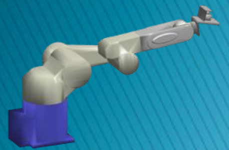
$$S(\boldsymbol{\omega}) = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

Where it can be seen that:

$$S(\boldsymbol{\omega})^T = -S(\boldsymbol{\omega})$$

$$S(\boldsymbol{\omega}) = -S(\boldsymbol{\omega})^T$$

We showed: $\dot{\mathbf{R}} = -\dot{\mathbf{R}}^T$



Skew Symmetric Matrices and Cross Products

Recall the calculation of instantaneous linear velocity of a rotating body:

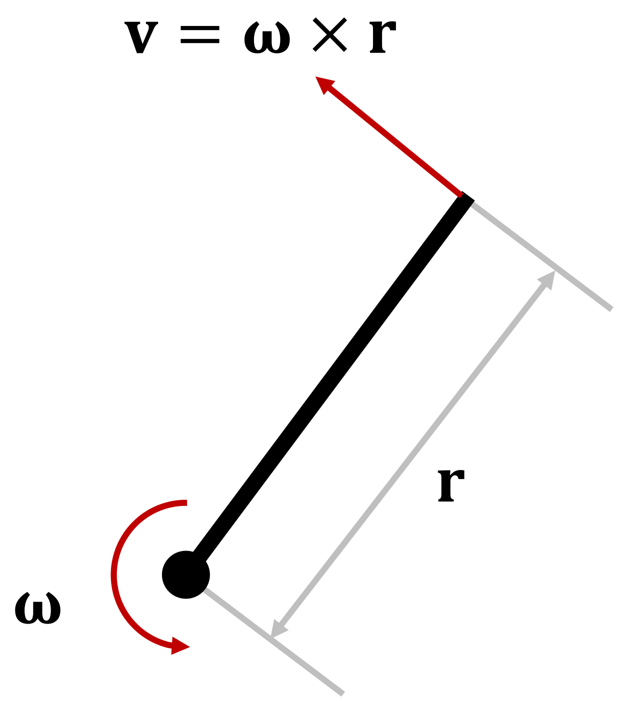
$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \text{ (cross product)}$$

This can be re-written as:

$$\mathbf{v} = \mathrm{S}(\boldsymbol{\omega})\mathbf{r}$$

And by exploiting properties of the skew-symmetric matrix:

$$\mathbf{v} = -\mathrm{S}(\mathbf{r})\boldsymbol{\omega}$$



Returning to the Time Derivative...

The time-derivative of the rotation matrix is skew-symmetric such that:

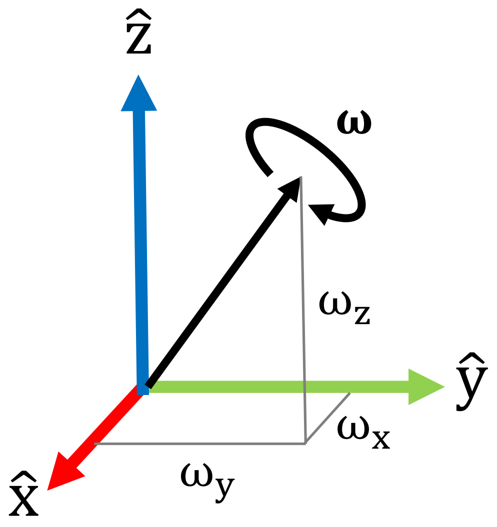
$$\dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} = -\mathbf{R}\dot{\mathbf{R}}^{\mathrm{T}}$$

Which can be defined as:

$$\boxed{\dot{\mathbf{R}} \triangleq \mathrm{S}(\boldsymbol{\omega})\mathbf{R}}$$

where $\boldsymbol{\omega} \in \mathbb{R}^3$ is the angular velocity. By substitution:

$$\begin{aligned} \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} &= -\mathbf{R}\dot{\mathbf{R}}^{\mathrm{T}} \\ \mathrm{S}(\boldsymbol{\omega})\mathbf{R}\mathbf{R}^{\mathrm{T}} &= -\mathbf{R}(\mathrm{S}(\boldsymbol{\omega})\mathbf{R})^{\mathrm{T}} \\ \mathrm{S}(\boldsymbol{\omega}) &= \mathbf{R}\mathbf{R}^{\mathrm{T}}\mathrm{S}(\boldsymbol{\omega}) \\ \mathrm{S}(\boldsymbol{\omega}) &= \mathrm{S}(\boldsymbol{\omega}) \checkmark \end{aligned}$$



Propagation of Rotation

The rotation matrix can be propagated for small time steps Δt:

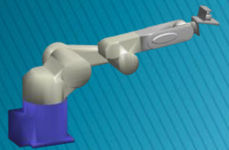
$$\mathbf{R}(t + \Delta t) \approx \mathbf{R}(t) + \Delta t\dot{\mathbf{R}}(t)$$

But, adding two rotations does not make another rotation:

$$\begin{aligned}\left\|\mathbf{R}(t) + \Delta t\dot{\mathbf{R}}(t)\right\|^2 &= \mathbf{R}(t)^{\mathrm{T}}\mathbf{R}(t) + \Delta t\mathbf{R}^{\mathrm{T}}\mathrm{S}(\boldsymbol{\omega})\mathbf{R} - \Delta t\mathbf{R}^{\mathrm{T}}\mathrm{S}(\boldsymbol{\omega})\mathbf{R} - \Delta t^2\mathbf{R}^{\mathrm{T}}\mathrm{S}(\boldsymbol{\omega})^2\mathbf{R} \\ &= \mathbf{I} - \Delta t^2\mathbf{R}^{\mathrm{T}}\mathrm{S}(\boldsymbol{\omega})^2\mathbf{R} \notin \mathbb{SO}(3)\end{aligned}$$

$$\lim_{\Delta t, \boldsymbol{\omega} \to 0} \left\|\mathbf{R}(t) + \Delta t\dot{\mathbf{R}}(t)\right\|^2 = \mathbf{I}$$

Only valid for small values of Δt, $\boldsymbol{\omega}$.
Errors accrue with subsequent propagations.
Need to re-normalize the column vectors.



Partial Derivative of the Rotation Matrix

The time-derivative of the rotation matrix is skew-symmetric:

$$\dot{\mathbf{R}} = S(\boldsymbol{\omega})\mathbf{R}$$

But, using the Chain Rule:

$$\dot{\mathbf{R}} = \left({}^{\partial \mathbf{R}}/_{\partial q}\right) \times \frac{dq}{dt}$$

Angular velocity about a joint is the axis of rotation multiplied by joint speed:

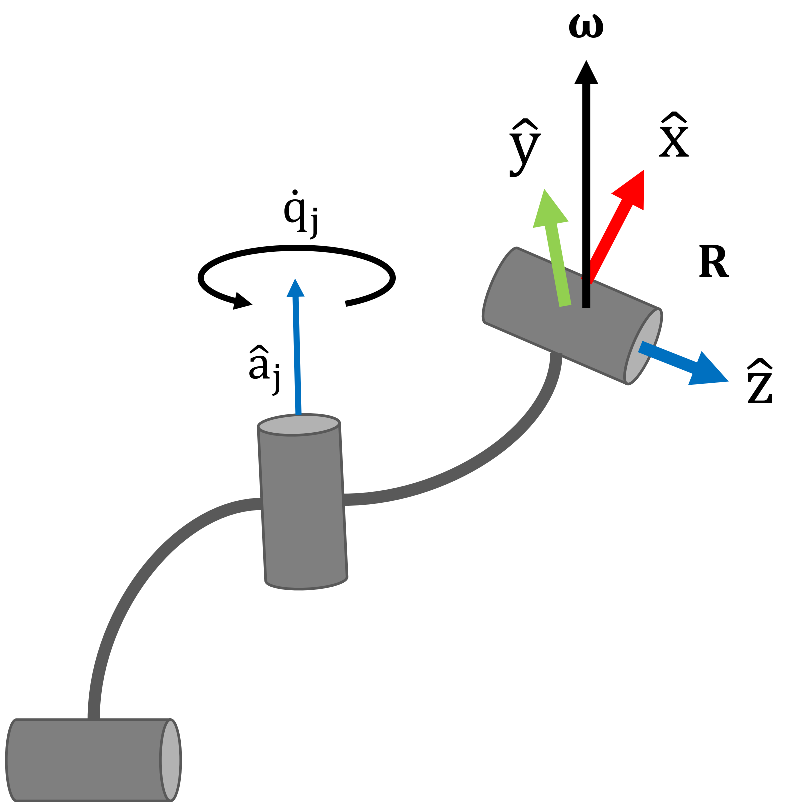
$$\boldsymbol{\omega} = \hat{a} \cdot \dot{q}$$

Then substituting this back in to the first equation:

$$\dot{\mathbf{R}} = S(\hat{a})\mathbf{R} \times \dot{q}$$

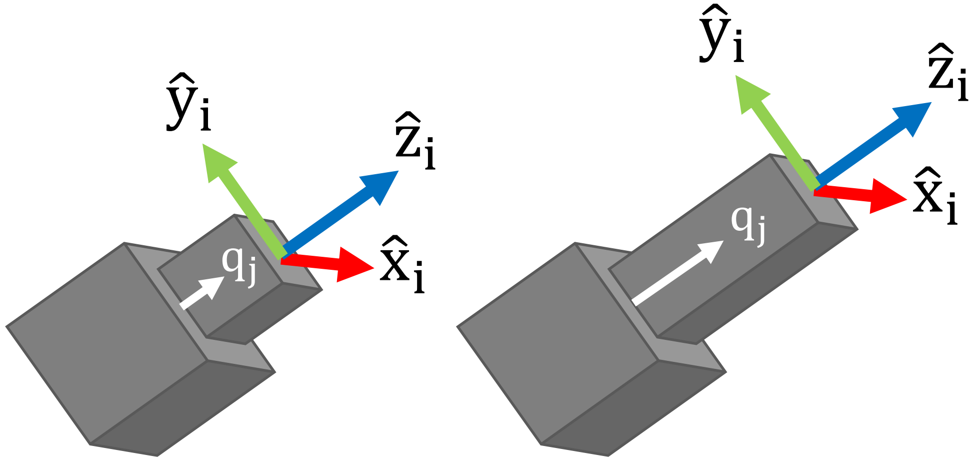
Thus:

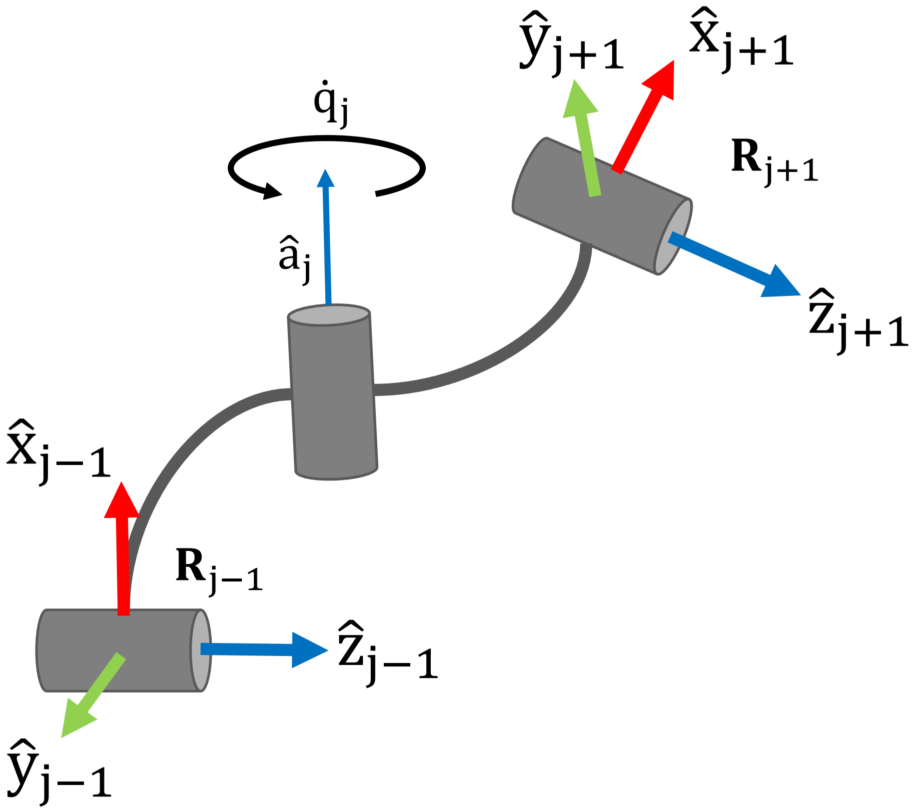
$$\boxed{\frac{\partial \mathbf{R}}{\partial q} = S(\hat{a})\mathbf{R}}$$



Partial Derivative of the Rotation Matrix

$$\frac{\partial \mathbf{R}_i}{\partial q_j} = \begin{cases} S(\hat{a}_j)\mathbf{R}_i & \text{if } j \le i \\ \mathbf{0}_{3\times3} & \text{if } j > i \\ \mathbf{0}_{3\times3} & \text{if prismatic} \end{cases}$$





Summary of Derivatives of the Rotation Matrix

$\dot{\mathbf{R}} = \mathrm{S}(\boldsymbol{\omega})\mathbf{R}$

The time derivative is skew-symmetric

$$\mathrm{S}(\cdot) = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

The skew-symmetric matrix has this form

$\mathbf{R}(t + \Delta t) \approx \mathbf{R}(t) + \Delta t \dot{\mathbf{R}}(t)$

Rotations can be propagated for small steps

$$\frac{\partial \mathbf{R}_i}{\partial q_j} = \begin{cases} \mathrm{S}(\hat{a}_j)\mathbf{R}_i & \text{if } j \le i \\ \mathbf{0}_{3\times 3} & \text{if } j > i \\ \mathbf{0}_{3\times 3} & \text{if } j \text{ prismatic} \end{cases}$$

Partial derivative with respect to a joint