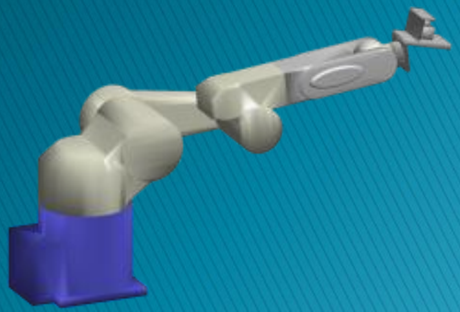
Redundant Manipulators



A redundant manipulator has more joints than required by the task

The differential kinematics describes a system of m equations with n unknowns:

$$\dot{\mathbf{x}} = \mathbf{J}(\mathbf{q})\dot{\mathbf{q}}$$

$\dot{\mathbf{x}} \in \mathbb{R}^{\mathrm{m}}$ – a vector of end-effector velocities with m rows
$\dot{\mathbf{q}} \in \mathbb{R}^{\mathrm{n}}$ – a vector of joint velocities with n rows
$\mathbf{J}(\mathbf{q}) \in \mathbb{R}^{\mathrm{m}\times\mathrm{n}}$ – a matrix with m rows and n columns

For a redundant robot, $\mathrm{m} < \mathrm{n}$, and hence there are **infinite choices** for the joint velocities $\dot{\mathbf{q}}$.

What can we do with all these choices?

- Avoid joint limits
- Minimize joint velocities
- Avoid singularities
- Avoid obstacles
- Minimize joint torque

Achieving this is quite hard, and won't be considered in this course.

A prudent choice is to minimize the (weighted) sum of joint velocities

$$\min_{\dot{\mathbf{q}}} f(\dot{\mathbf{q}}) = \tfrac{1}{2}\dot{\mathbf{q}}^T\mathbf{W}\dot{\mathbf{q}}$$

$$\text{subject to: } \mathbf{g}(\dot{\mathbf{q}}) = \dot{\mathbf{x}} - \mathbf{J}(\mathbf{q})\dot{\mathbf{q}} = \mathbf{0}$$

$\boldsymbol{g}(\dot{\boldsymbol{q}})$ is a set of equality constraints that ensures the desired end-effector velocity will be achieved.

$$\mathbf{W} = \mathrm{diag}([\mathrm{w}_1 \quad \cdots \quad \mathrm{w}_n]), \mathrm{w}_i = (0, \infty)\ \forall i$$

Combine as a single equation:

$$\begin{aligned} L(\dot{\mathbf{q}}, \boldsymbol{\lambda}) &= f(\dot{\mathbf{q}}) + \mathbf{g}(\dot{\mathbf{q}})^T\boldsymbol{\lambda} \\ &= \tfrac{1}{2}\dot{\mathbf{q}}^T\mathbf{W}\dot{\mathbf{q}} + (\dot{\mathbf{x}} - \mathbf{J}(\mathbf{q})\dot{\mathbf{q}})^T\boldsymbol{\lambda} \end{aligned}$$

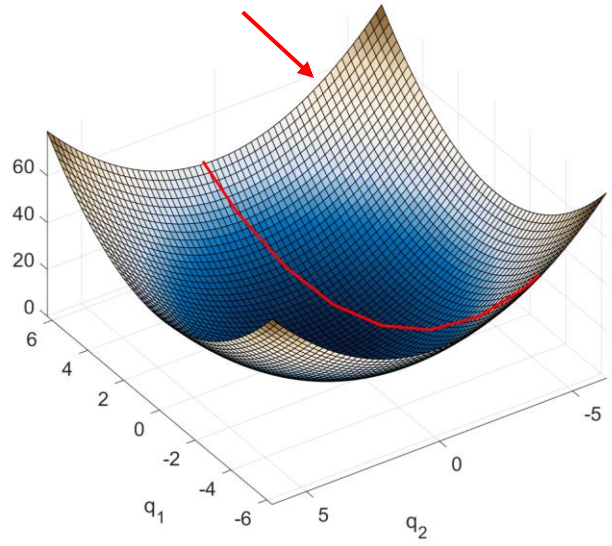
$L(\dot{\boldsymbol{q}}, \boldsymbol{\lambda}) = f(\dot{\boldsymbol{q}}) + \boldsymbol{g}(\dot{\boldsymbol{q}})^T\boldsymbol{\lambda}$ is known as a Lagrangian Multiplier.

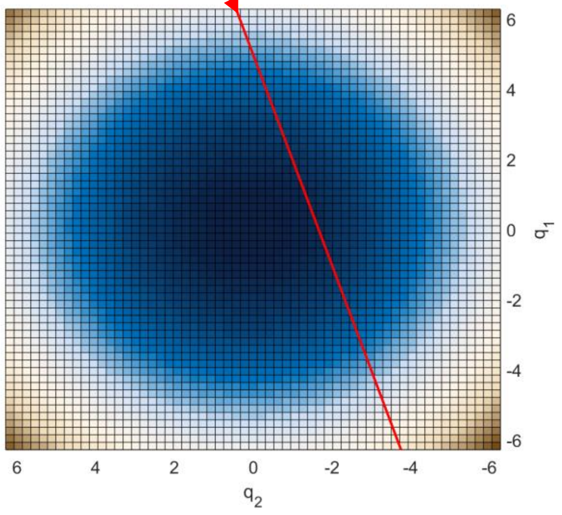
For example, a 2-Link planar robot, ignoring $\dot{y}$:

$$\dot{x} = [J_{11} \quad J_{12}]\begin{bmatrix}\dot{q}_1 \\ \dot{q}_2\end{bmatrix}$$

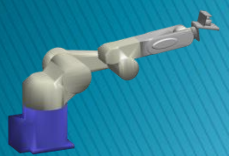
$$\dot{x} = J_{11}\dot{q}_1 + J_{12}\dot{q}_2$$

$$\tfrac{1}{2}\dot{\mathbf{q}}^T\mathbf{W}\dot{\mathbf{q}} = \tfrac{1}{2}w_1\dot{q}_1 + \tfrac{1}{2}w_2\dot{q}_2$$





"Find the point on the red line where the height is lowest"



The optimal solution is where the derivatives are both zero

$$\mathrm{L}(\dot{\mathbf{q}}, \boldsymbol{\lambda}) = \mathrm{f}(\dot{\mathbf{q}}) + \mathbf{g}(\dot{\mathbf{q}})^{\mathrm{T}}\boldsymbol{\lambda}$$
$$= \tfrac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{W}\dot{\mathbf{q}} + (\dot{\mathbf{x}} - \mathbf{J}(\mathbf{q})\dot{\mathbf{q}})^{\mathrm{T}}\boldsymbol{\lambda} \quad ①$$

$$\frac{\partial \mathrm{L}}{\partial \boldsymbol{\lambda}} = \dot{\mathbf{x}} - \mathbf{J}(\mathbf{q})\dot{\mathbf{q}} = \mathbf{0}$$
$$\mathbf{J}(\mathbf{q})\dot{\mathbf{q}} = \dot{\mathbf{x}}$$

$$\frac{\partial \mathrm{L}}{\partial \dot{\mathrm{q}}} = \mathbf{W}\dot{\mathbf{q}} - \mathbf{J}(\mathbf{q})^{\mathrm{T}}\boldsymbol{\lambda} = \mathbf{0}$$
$$\mathbf{W}\dot{\mathbf{q}} = \mathbf{J}(\mathbf{q})^{\mathrm{T}}\boldsymbol{\lambda}$$
$$\dot{\mathbf{q}} = \mathbf{W}^{-1}\mathbf{J}(\mathbf{q})^{\mathrm{T}}\boldsymbol{\lambda} \quad ②$$

Substituting ② in to ① :

$$\mathbf{J}(\mathbf{q})\mathbf{W}^{-1}\mathbf{J}(\mathbf{q})^{\mathrm{T}}\boldsymbol{\lambda} = \dot{\mathbf{x}}$$
$$\boldsymbol{\lambda} = \left(\mathbf{J}(\mathbf{q})\mathbf{W}^{-1}\mathbf{J}(\mathbf{q})^{\mathrm{T}}\right)^{-1}\dot{\mathbf{x}} \quad ③$$

Substitute ③ in to ② :

$$\boxed{\begin{aligned}\dot{\mathbf{q}} &= \mathbf{W}^{-1}\mathbf{J}(\mathbf{q})^{\mathrm{T}}\left(\mathbf{J}(\mathbf{q})\mathbf{W}^{-1}\mathbf{J}(\mathbf{q})^{\mathrm{T}}\right)^{-1}\dot{\mathbf{x}} \\ &= \mathbf{J}_{\mathrm{W}}^{\dagger}(\mathbf{q})\dot{\mathbf{x}}\end{aligned}}$$

We can achieve our desired motion even with the velocity optimization and joint weights

$$\begin{aligned}\dot{\mathbf{q}} &= \mathbf{W}^{-1}\mathbf{J}(\mathbf{q})^{\mathrm{T}}\left(\mathbf{J}(\mathbf{q})\mathbf{W}^{-1}\mathbf{J}(\mathbf{q})^{\mathrm{T}}\right)^{-1}\dot{\mathbf{x}} \\ &= \mathbf{J}_{\mathrm{W}}^{\dagger}(\mathbf{q})\dot{\mathbf{x}}\end{aligned}$$

Multiplying through by the Jacobian:

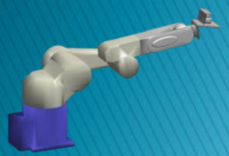
$$\mathbf{J}(\mathbf{q})\dot{\mathbf{q}} = \mathbf{J}(\mathbf{q})\mathbf{W}^{-1}\mathbf{J}(\mathbf{q})^{\mathrm{T}}\left(\mathbf{J}(\mathbf{q})\mathbf{W}^{-1}\mathbf{J}(\mathbf{q})^{\mathrm{T}}\right)^{-1}\dot{\mathbf{x}}$$
$$\mathbf{J}(\mathbf{q})\dot{\mathbf{q}} = \dot{\mathbf{x}}$$

The desired end-effector velocity is achieved!

What if $\mathbf{J}(\mathbf{q})$ is square??

$$\begin{aligned}\dot{\mathbf{q}} &= \mathbf{W}^{-1}\mathbf{J}(\mathbf{q})^{\mathrm{T}}\left(\mathbf{J}(\mathbf{q})\mathbf{W}^{-1}\mathbf{J}(\mathbf{q})^{\mathrm{T}}\right)^{-1}\dot{\mathbf{x}} \\ &= \mathbf{W}^{-1}\mathbf{J}(\mathbf{q})^{\mathrm{T}}\left(\mathbf{J}(\mathbf{q})^{\mathrm{T}}\right)^{-1}\mathbf{W}\mathbf{J}(\mathbf{q})^{-1}\dot{\mathbf{x}} \\ &= \mathbf{W}^{-1}\mathbf{W}\mathbf{J}(\mathbf{q})^{-1}\dot{\mathbf{x}} \\ &= \mathbf{J}(\mathbf{q})^{-1}\dot{\mathbf{x}}\end{aligned}$$

The weighting matrix $\mathbf{W}$ and the velocity minimization have no effect if we don't have redundancy!



What if the Jacobian is square?

$$\min_{\dot{\mathbf{q}}} f(\dot{\mathbf{q}}) = \tfrac{1}{2}\dot{\mathbf{q}}^T \mathbf{W} \dot{\mathbf{q}}$$

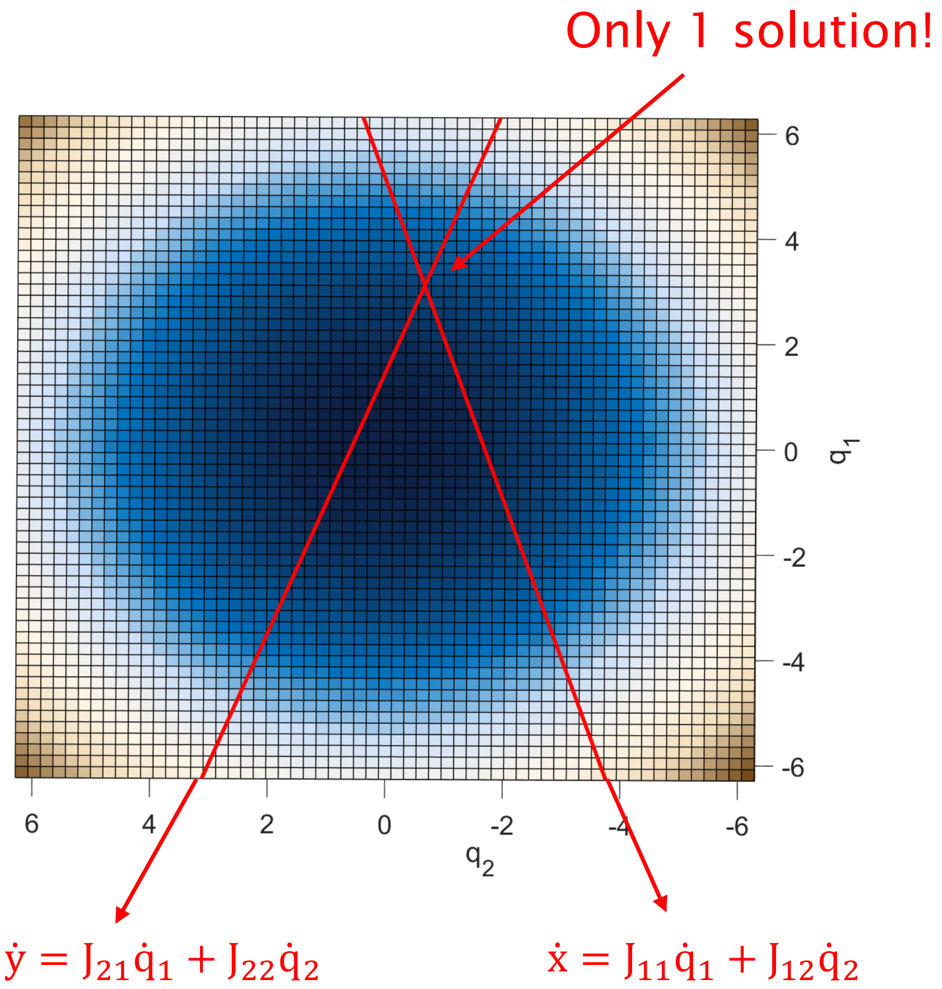
subject to: $\mathbf{g}(\dot{\mathbf{q}}) = \dot{\mathbf{x}} - \mathbf{J}(\mathbf{q})\dot{\mathbf{q}} = \mathbf{0}$

Returning to the 2-Link planar example…

$$\mathbf{g}(\dot{\mathbf{q}}) = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} - \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

$$\dot{x} = J_{11}\dot{q}_1 + J_{12}\dot{q}_2$$
$$\dot{y} = J_{21}\dot{q}_1 + J_{22}\dot{q}_2$$

We must satisfy both constraint equations



The Weighting matrix can be chosen to avoid joint limits

$$h(q_i) = \frac{(q_{i,max} - q_{i,min})(2q_i - q_{i,max} - q_{i,min})}{c_i\left(q_{i,max} - q_i\right)^2\left(q_i - q_{i,min}\right)^2}$$

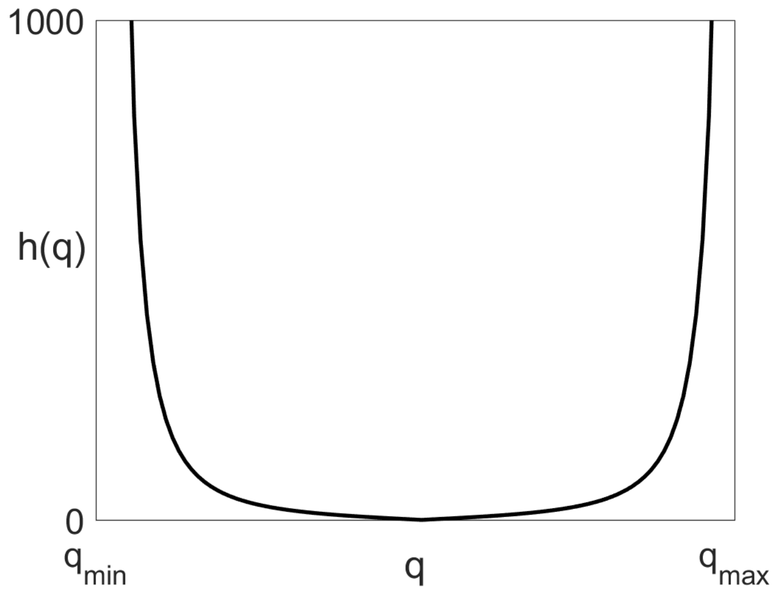
where c_i is a scalar that can be tuned to suit the individual joint.

Then assign the elements of $\mathbf{W}$ as:

$$w_i = \begin{cases} 1 + |h(q_i)| & \text{for } \Delta h(q_i) > 0 \\ 1 & \text{otherwise} \end{cases}$$

Note that:

$$\lim_{q \to q_{lim}} |h(q)| = \infty$$



The joint velocity $\dot{q} \to 0$ *as the joint approaches its limits.*

Redundant manipulators can rearrange themselves to perform other complex tasks

$$\min_{\dot{\mathbf{q}}} (\dot{\mathbf{q}} + \mathbf{y}_2)^{\mathrm{T}} (\dot{\mathbf{q}} + \mathbf{y}_2)$$

subject to: $\dot{\mathbf{x}} - \mathbf{J}(\mathbf{q})\dot{\mathbf{q}} = \mathbf{0}$

$$\dot{\mathbf{q}} = \mathbf{J}^{\dagger}(\mathbf{q})\dot{\mathbf{x}} + \left(\mathbf{I} - \mathbf{J}^{\dagger}(\mathbf{q})\mathbf{J}(\mathbf{q})\right)\mathbf{y}_2$$

Multiplying through by the Jacobian…

$$\begin{aligned}
\mathbf{J}(\mathbf{q})\dot{\mathbf{q}} &= \mathbf{J}(\mathbf{q})\mathbf{J}^{\dagger}(\mathbf{q})\dot{\mathbf{x}} + \mathbf{J}(\mathbf{q})\left(\mathbf{I} - \mathbf{J}^{\dagger}(\mathbf{q})\mathbf{J}(\mathbf{q})\right)\mathbf{y}_2 \\
&= \dot{\mathbf{x}} - \left(\mathbf{J}(\mathbf{q}) - \mathbf{J}(\mathbf{q})\mathbf{J}^{\dagger}(\mathbf{q})\mathbf{J}(\mathbf{q})\right)\mathbf{y}_2 \\
&= \dot{\mathbf{x}} - \left(\mathbf{J}(\mathbf{q}) - \mathbf{J}(\mathbf{q})\right)\mathbf{y}_2 \\
&= \dot{\mathbf{x}}
\end{aligned}$$

This secondary, "redundant" task $\mathbf{y}_2$ has no effect on the end-effector motion $\dot{\mathbf{x}}$!

$\mathbf{N} = \mathbf{I} - \mathbf{J}^{\dagger}(\mathbf{q})\mathbf{J}(\mathbf{q})$ is known as the **null space** of the manipulator:

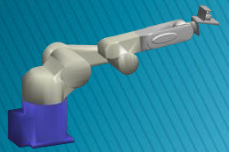
$$\mathbf{J}(\mathbf{q})\mathbf{N} = \mathbf{0}$$

The redundant task is usually chosen to be proportional to the gradient vector of a scalar cost function:

$$\mathbf{y}_2 = \alpha \nabla f(\mathbf{q})$$

Typical choices for the redundant task are:

- Distance from a singularity (gradient of the measure of manipulability)
- Distance from obstacles, or joint velocity needed to avoid an obstacle



Summary

- A redundant robot has more joints than task dimensions
  - The Jacobian is not square, and cannot be directly inverted

$$\mathbf{J}(\mathbf{q}) \in \mathbb{R}^{m \times n},\ m < n$$

- There are infinite choices for the joint velocities to perform a given end-effector task using a redundant manipulator

- The *weighted pseudoinverse Jacobian* gives the smallest possible combination of joint velocities to achieve the desired task

$$\begin{aligned}\dot{\mathbf{q}} &= \mathbf{W}^{-1}\mathbf{J}(\mathbf{q})^{\mathrm{T}}\left(\mathbf{J}(\mathbf{q})\mathbf{W}^{-1}\mathbf{J}(\mathbf{q})^{\mathrm{T}}\right)^{-1}\dot{\mathbf{x}} \\ &= \mathbf{J}_{\mathrm{W}}^{\dagger}(\mathbf{q})\dot{\mathbf{x}}\end{aligned}$$

- The weighted, minimum-velocity solution does not work for a non-redundant robot!

$$\mathbf{J}(\mathbf{q})\mathbf{J}_{\mathrm{W}}^{\dagger}(\mathbf{q})\dot{\mathbf{x}} = \mathbf{J}(\mathbf{q})^{-1}\dot{\mathbf{x}}, \qquad \text{if } \mathbf{J}(\mathbf{q}) \in \mathbb{R}^{m \times m}$$

- The weighting matrix can be chosen to avoid joint limits

- Redundant manipulators can perform complex manoeuvres through *null space projection*

$$\dot{\mathbf{q}} = \mathbf{J}^{\dagger}(\mathbf{q})\dot{\mathbf{x}} + \left(\mathbf{I} - \mathbf{J}^{\dagger}(\mathbf{q})\mathbf{J}(\mathbf{q})\right)\mathbf{y}_2$$