5.1 Inverse Kinematics

Centre for Autonomous Systems
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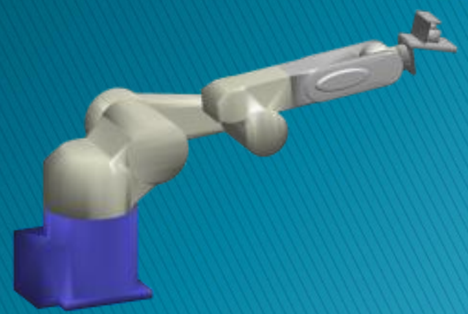


Roadmap

Rotation Matrices

$\mathbf{R} = [\hat{x} \quad \hat{y} \quad \hat{z}]$

→

Transformation Matrices

$\mathbf{T} = \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix}$

→

Forward Kinematics

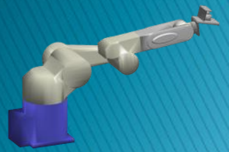
$\mathbf{x} = \mathbf{f}(\mathbf{q})$

↓

Inverse Kinematics

$\mathbf{q} = \mathbf{f}^{-1}(\mathbf{x})$

Given a pose of the robot tool-tip $\mathbf{x}$, what joint configurations $\mathbf{q}$ can achieve it?

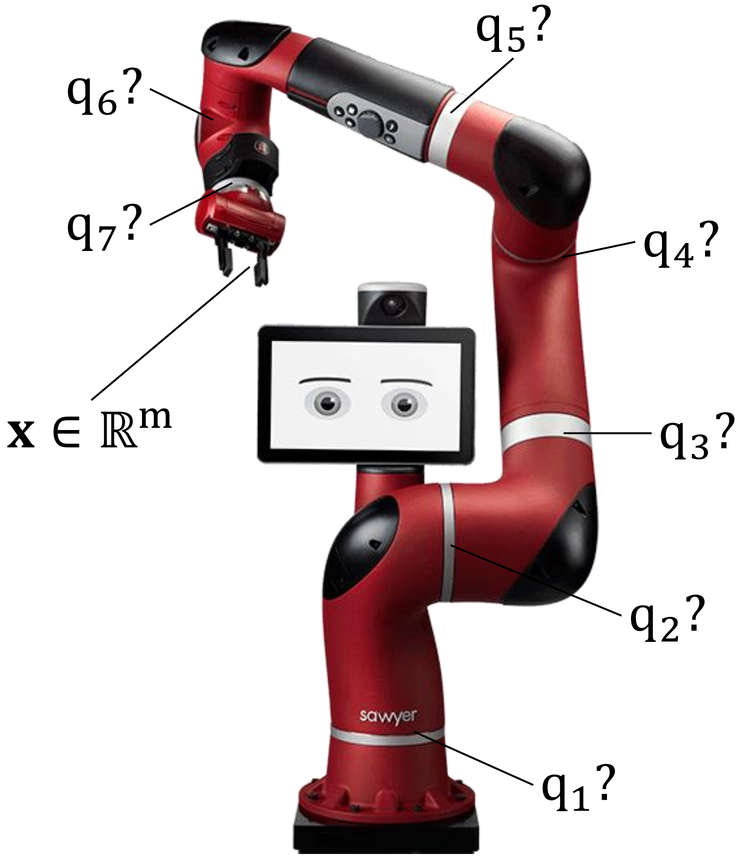


The Inverse Kinematics Problem

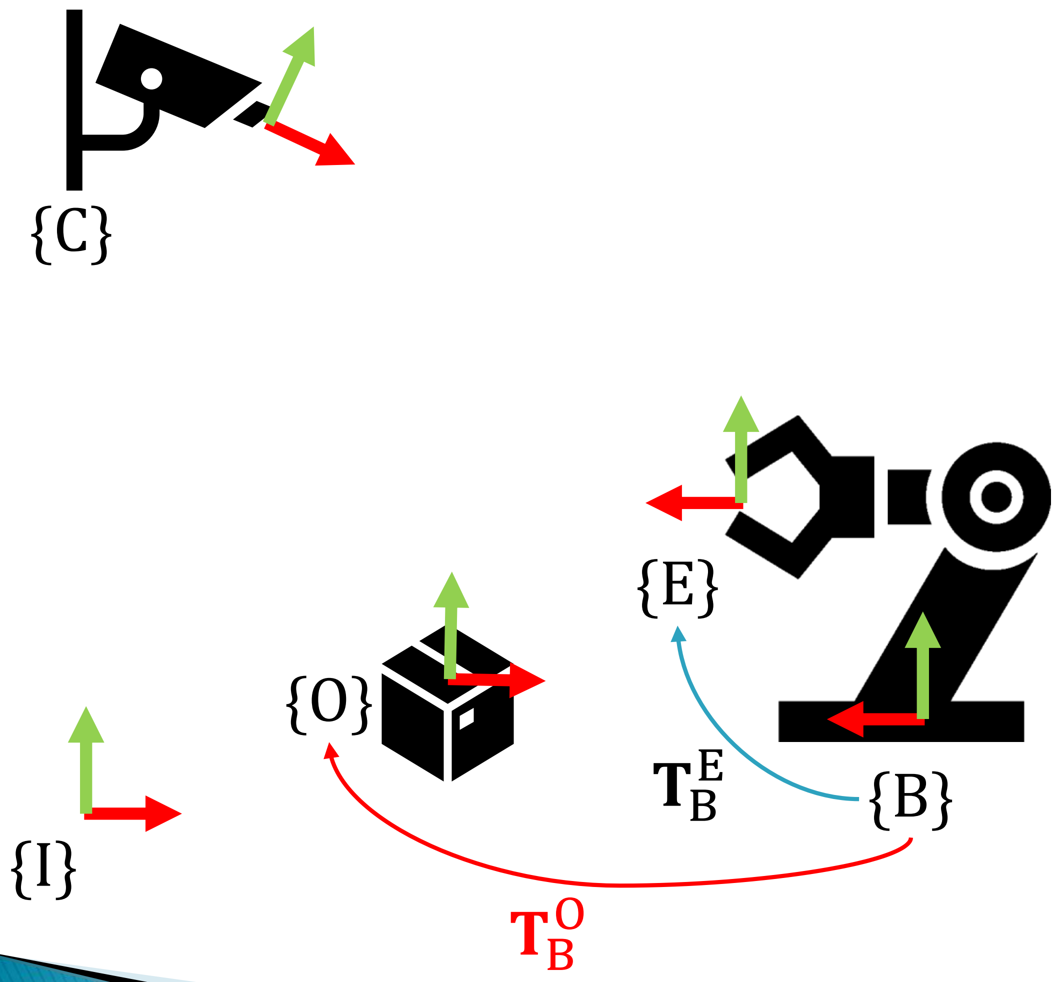
Given a *desired* end-effector pose $\mathbf{x} \in \mathbb{R}^m$…

… find a set of joint positions $\mathbf{q} \in \mathbb{R}^n$ that satisfies the forward kinematics $\mathbf{x} = \mathbf{f}(\mathbf{q})$ (or some equivalent).

In other words, solve $\mathbf{q} = \mathbf{f}^{-1}(\mathbf{x})$.



Motivation for Forward Kinematics



Desired end-effector transform to grasp object:

$$\mathbf{T}_{\text{desired}} = \mathbf{T}_{\mathrm{B}}^{\mathrm{O}}$$

Need to find a joint configuration $\mathbf{q}$ to reach the object:

$$\mathbf{T}_{\mathrm{B}}^{\mathrm{E}}(\mathbf{q}) = \mathbf{T}_{\text{desired}}$$

Two Approaches to Inverse Kinematics

Inverse Kinematics (IK)

- **Closed-Form Solutions:**

  $$\mathbf{q} = \mathbf{f}^{-1}(\mathbf{x})$$

- **Optimisation:**

  $$\min_{\mathbf{q}} \|\mathbf{x} - \mathbf{f}(\mathbf{q})\|$$

Inverse Kinematics of a 2-Link Planar Robot

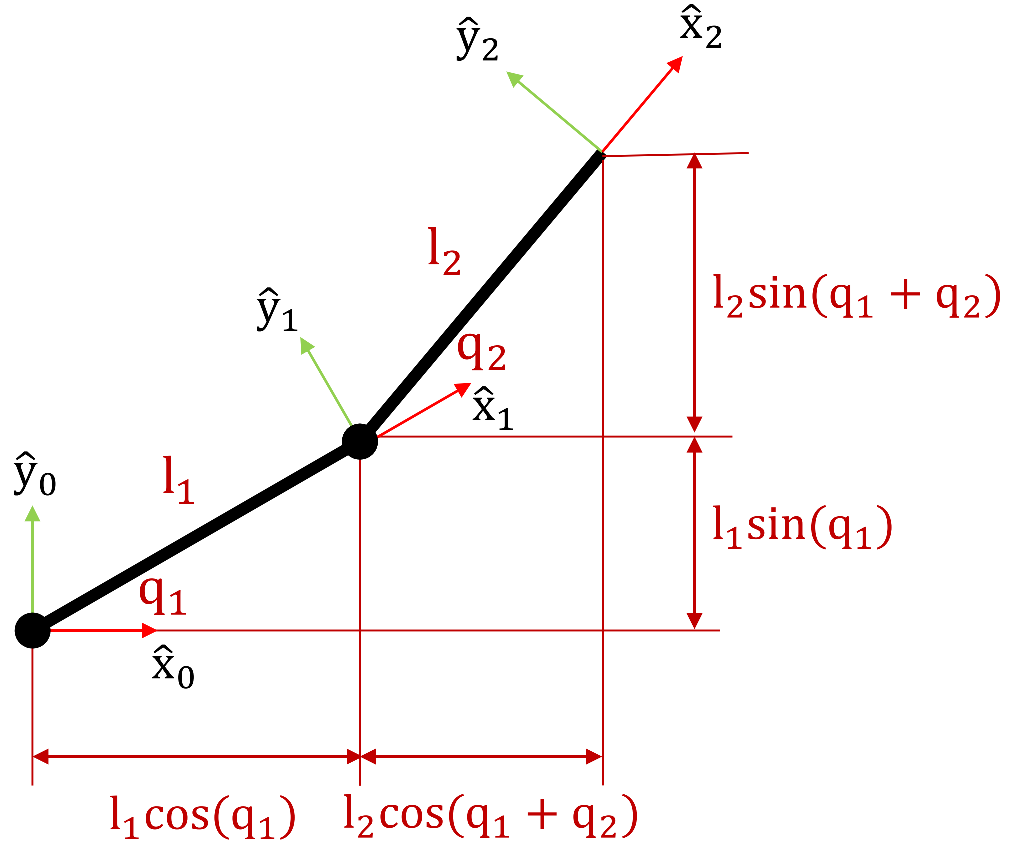
Forward kinematics:

$$\mathbf{x} = \mathbf{f}(\mathbf{q})$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l_1 \cos(q_1) + l_2 \cos(q_1 + q_2) \\ l_1 \sin(q_1) + l_2 \sin(q_1 + q_2) \end{bmatrix}$$

Inverse Kinematics:

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \mathbf{f}^{-1}(\mathbf{x})?$$



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Inverse Kinematics of a 2-Link Planar Robot

Pythagoras's Theorem:

$$d^2 = x^2 + y^2$$

Law of Cosines:

$$d^2 = l_1^2 + l_2^2 - 2l_1l_2\cos(\alpha)$$

$$\cos(\alpha) = \frac{l_1^2 + l_2^2 - d^2}{2l_1l_2}$$

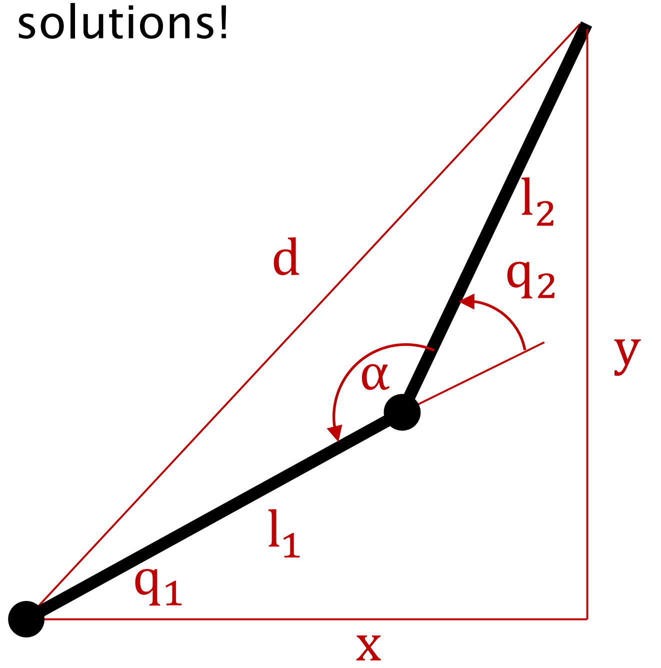
But also,

$$q_2 = \pi - \alpha$$

$$\begin{aligned}\cos(q_2) &= \cos(\pi - \alpha) \\ &= \cos(\pi)\cos(\alpha) + \sin(\pi)\sin(\alpha) \\ &= -\cos(\alpha) \\ &= \frac{d_2 - l_1^2 - l_2^2}{2l_1l_2}\end{aligned}$$

$$q_2 = \begin{cases}\pi - \cos^{-1}\left(\dfrac{l_1^2 + l_2^2 - x^2 - y^2}{2l_1l_2}\right) \\ \cos^{-1}\left(\dfrac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1l_2}\right)\end{cases}$$

Two solutions!



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Inverse Kinematics of a 2-Link Planar Robot

Using tangent rule:

$$\tan(q_1 + \beta) = \frac{y}{x}$$

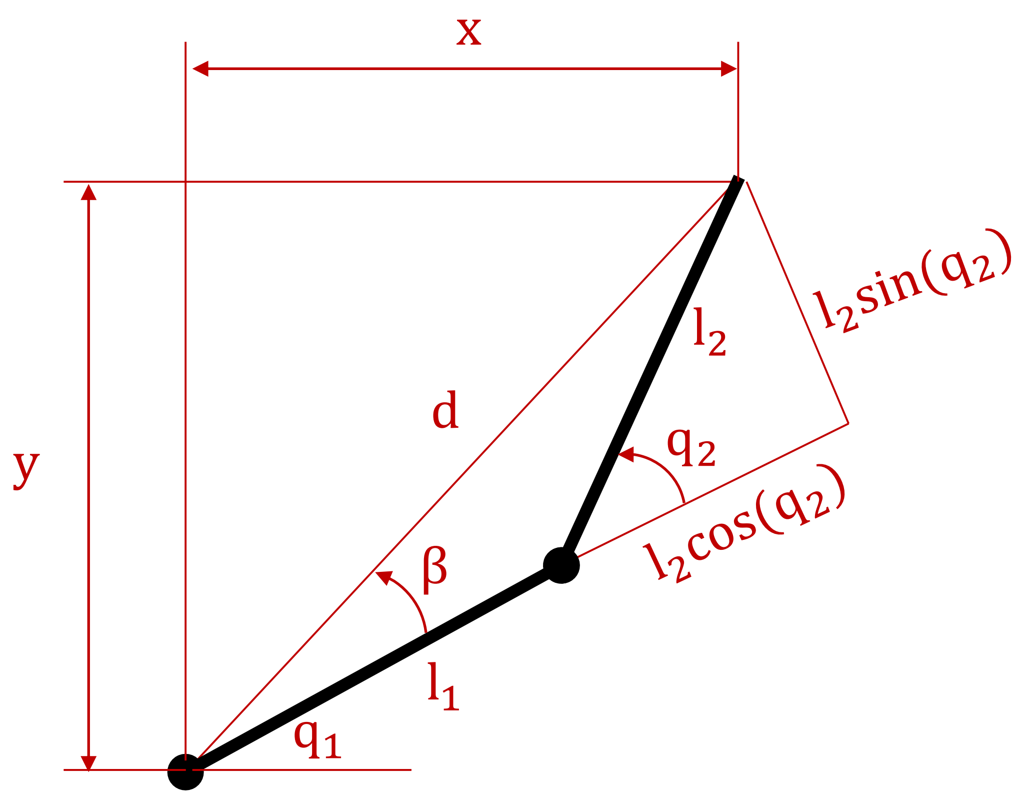
$$q_1 = \tan^{-1}\left(\frac{y}{x}\right) - \beta$$

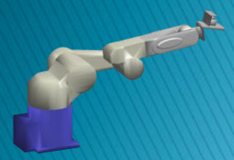
Using Law of Cosines:

$$\beta = \tan^{-1}\left(\frac{l_2 \sin(q_2)}{l_1 + l_2 \cos(q_2)}\right)$$

Hence:

$$q_1 = \tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{l_2 \sin(q_2)}{l_1 + l_2 \cos(q_2)}\right)$$





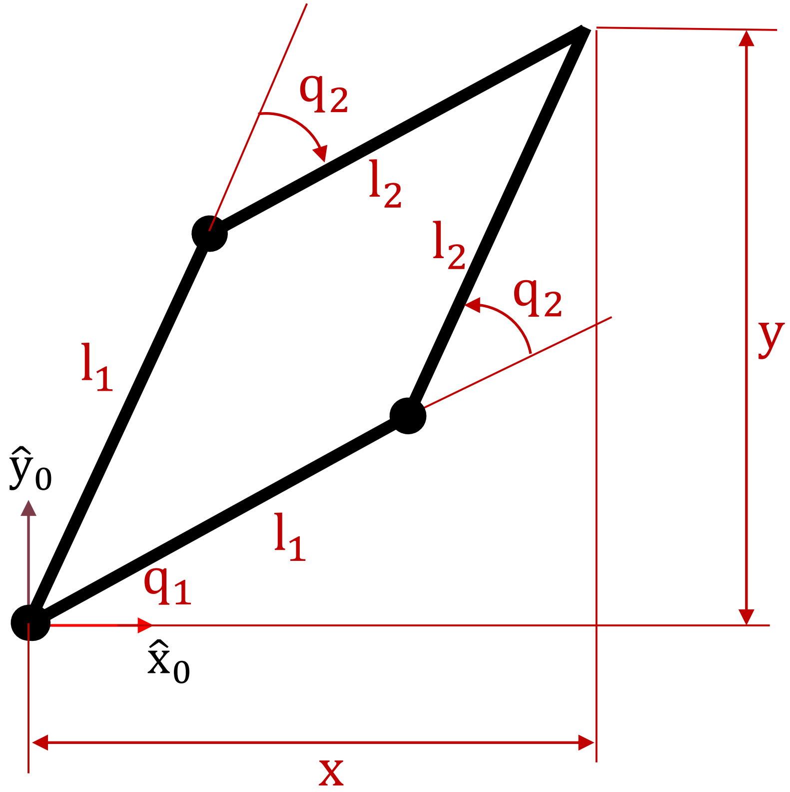
Inverse Kinematics of a 2-Link Planar Robot

There are **2 distinct joint configurations** $\mathbf{q} \in \mathbb{R}^2$ for a given end-effector pose $\mathbf{x} \in \mathbb{R}^2$.

$$\mathbf{q} = \begin{bmatrix} \tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{l_2 \sin(q_2)}{l_1 + l_2\cos(q_2)}\right) \\ \pi - \cos^{-1}\left(\frac{l_1^2 + l_2^2 - x^2 - y^2}{2l_1 l_2}\right) \end{bmatrix}$$

Or

$$\mathbf{q} = \begin{bmatrix} \tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{l_2 \sin(q_2)}{l_1 + l_2\cos(q_2)}\right) \\ \cos^{-1}\left(\frac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1 l_2}\right) \end{bmatrix}$$



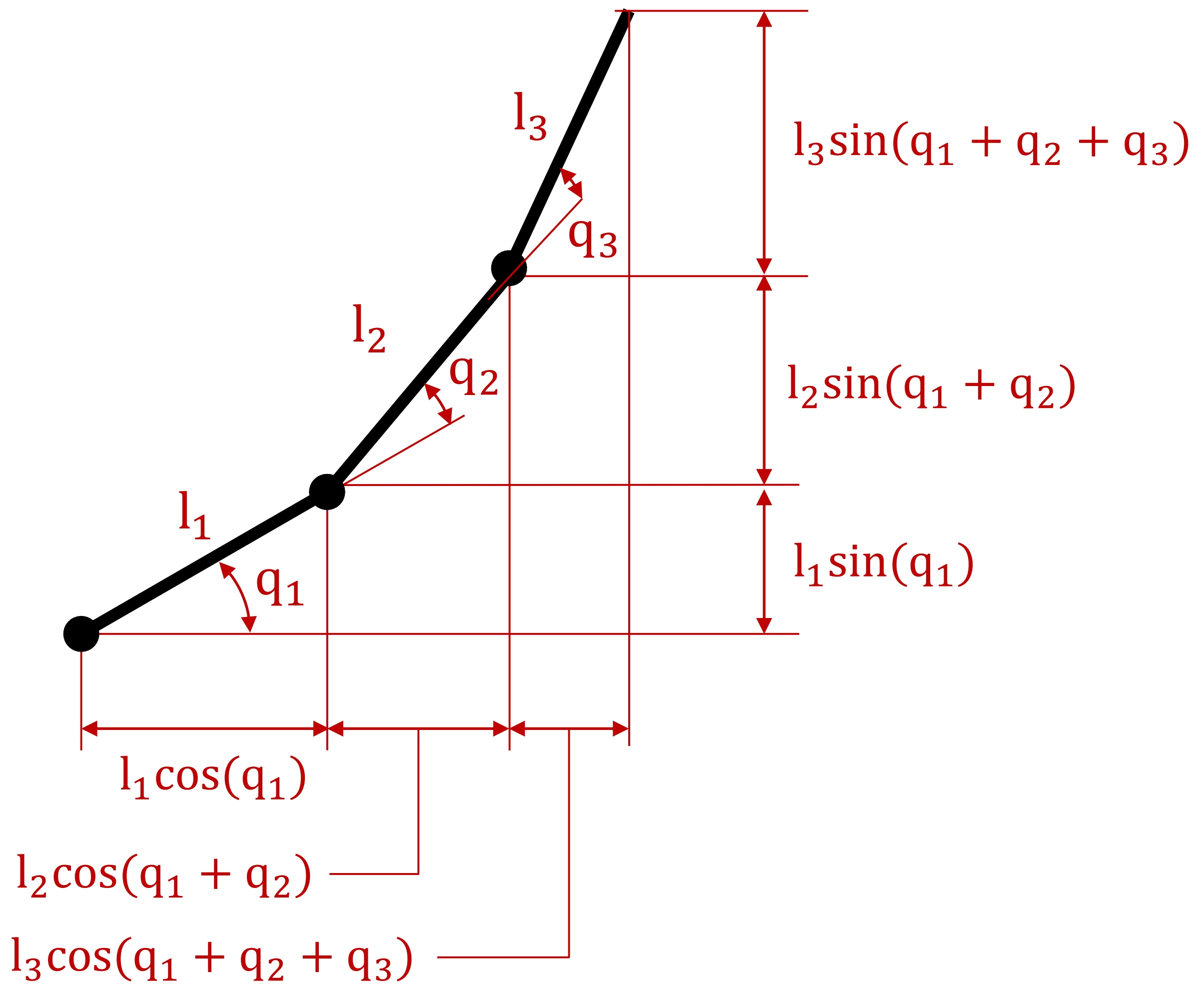
Inverse Kinematics for a 3-Link Manipulator

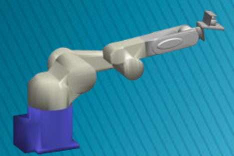
Forward kinematics:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l_1 c_1 + l_2 c_{12} + l_3 c_{123} \\ l_1 s_1 + l_2 s_{12} + l_3 s_{123} \end{bmatrix}$$

Where:

- $c_1 = \cos(q_1)$
- $c_{12} = \cos(q_1 + q_2)$
- $c_{123} = \cos(q_1 + q_2 + q_3)$
- $s_1 = \sin(q_1)$
- $s_{12} = \sin(q_1 + q_2)$
- $s_{123} = \sin(q_1 + q_2 + q_3)$





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Inverse Kinematics for a 3-Link Manipulator

Task space:

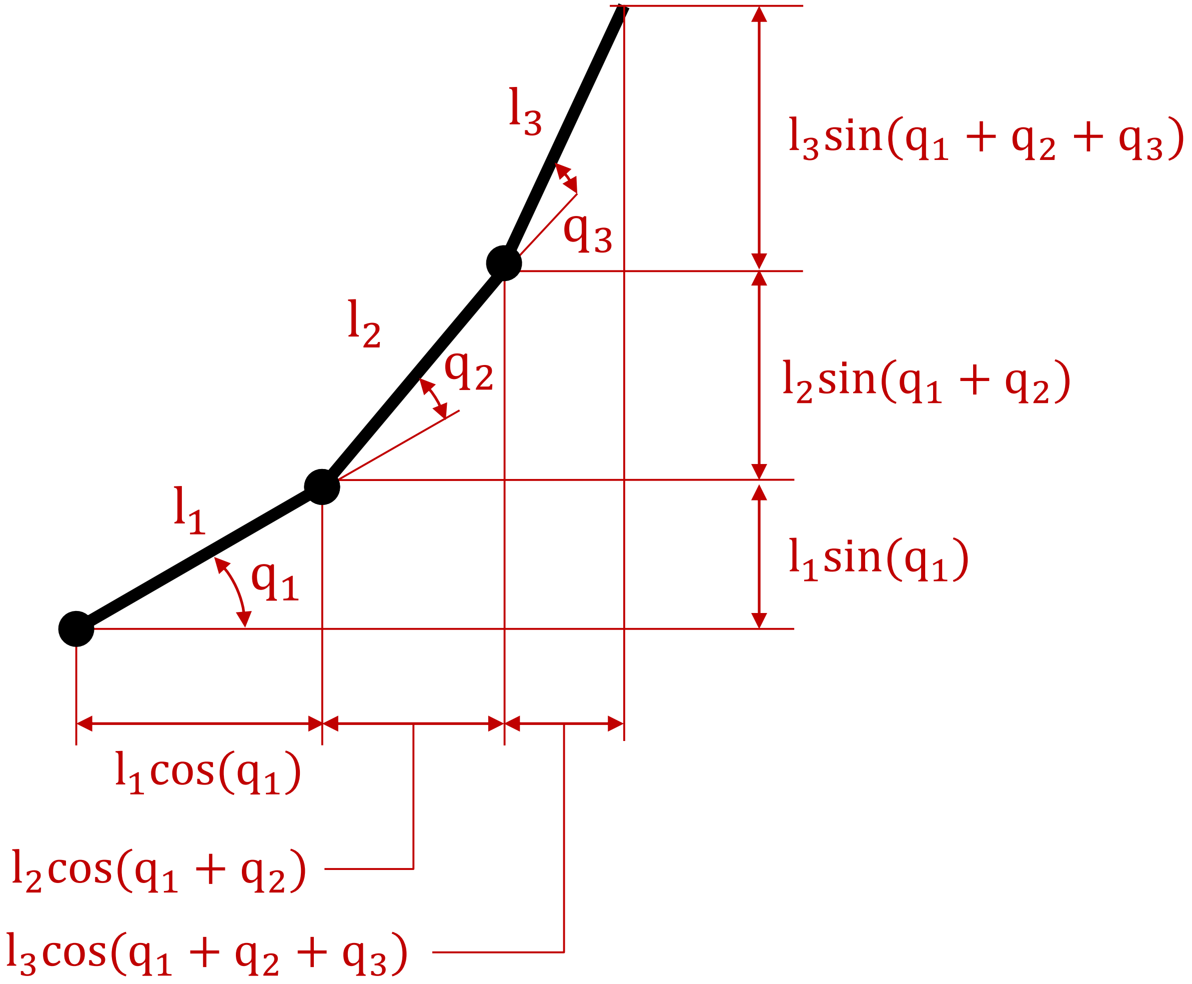
$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$$

Joint/control space:

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \in \mathbb{R}^3$$

$m = 2 < n = 3.$

This system is underdetermined and thus has **infinitely many solutions.**



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Inverse Kinematics for a 3-Link Manipulator

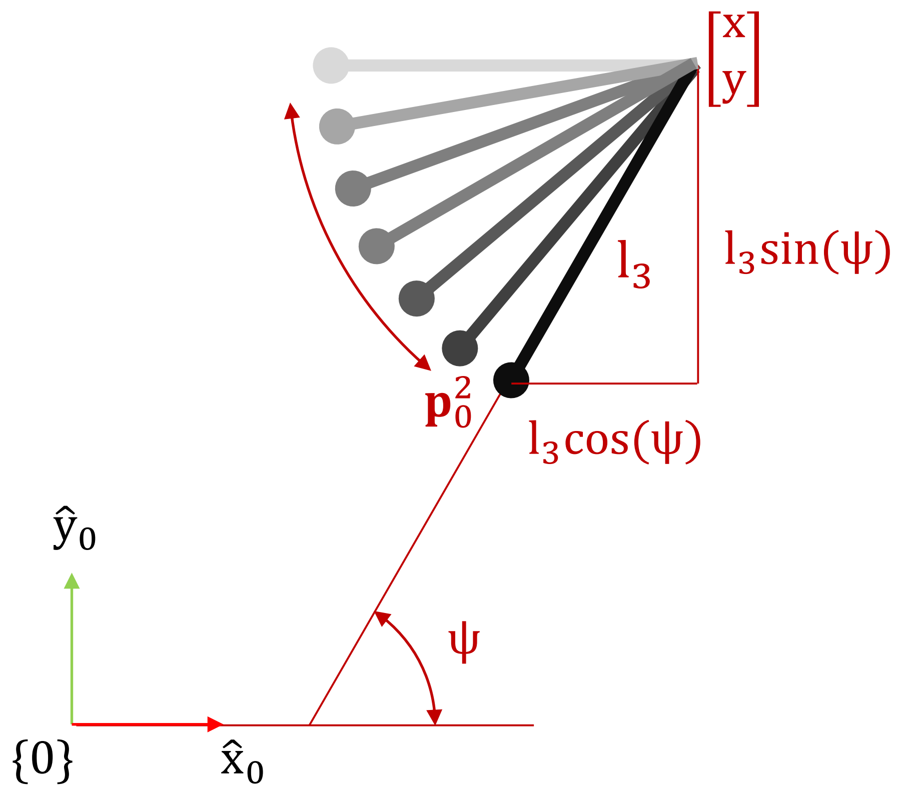
Since we have infinite joint solutions, the end-effector orientation ψ can be set (almost) arbitrarily.

Then the position of the 2nd link frame is:

$$\mathbf{p}_0^2 = \begin{bmatrix} x - l_3\cos(\psi) \\ y - l_3\sin(\psi) \end{bmatrix}$$

Then we can use inverse kinematics to solve the first 2 joints:

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \mathbf{f}^{-1}(\mathbf{p}_0^2)$$



Inverse Kinematics for a 3-Link Manipulator

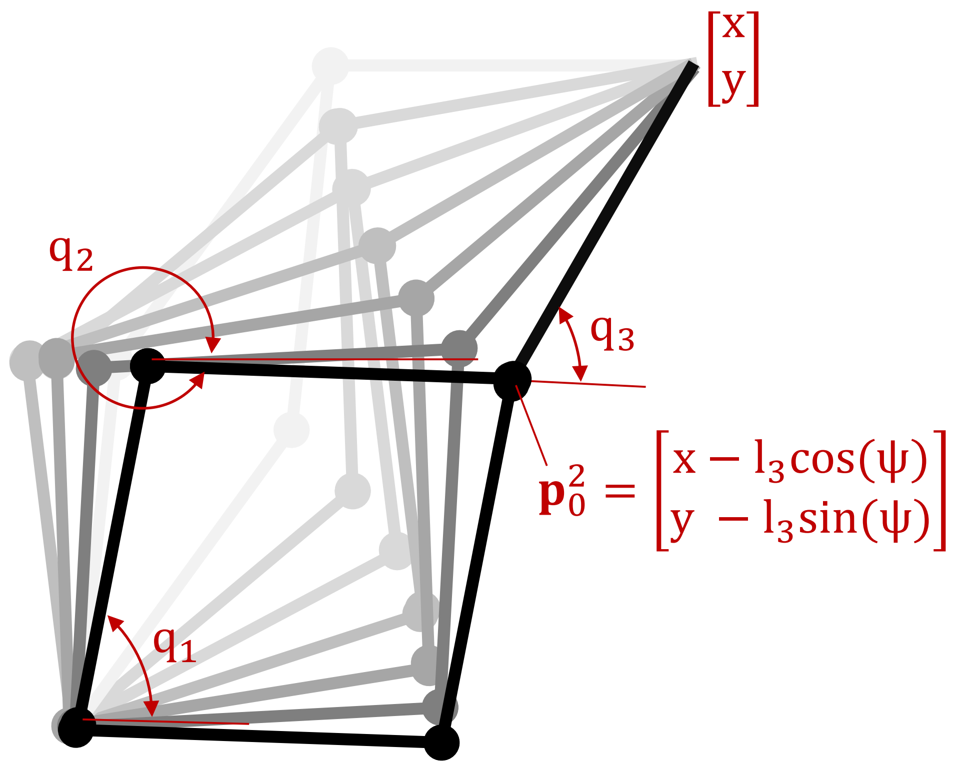
Given ψ and $\mathbf{p}_0^2$, then the first 2 joint angles of the manipulator are:

$$q_2 = \begin{cases} \pi - \cos^{-1}\left(\dfrac{l_1^2 + l_2^2 - \left(\mathbf{p}_0^2\right)^T \mathbf{p}_0^2}{2 l_1 l_2}\right) \\[2ex] \cos^{-1}\left(\dfrac{\left(\mathbf{p}_0^2\right)^T \mathbf{p}_0^2 - l_1^2 - l_2^2}{2 l_1 l_2}\right) \end{cases}$$

$$q_1 = \tan^{-1}\left(\frac{x_2}{y_2}\right) - \tan^{-1}\left(\frac{l_2 \sin(q_2)}{l_1 + l_2 \cos(q_2)}\right)$$

The 3rd joint is:

$$q_3 = \psi - (q_1 + q_2)$$



Insights in to the Inverse Kinematics Problem

For task space $\mathbf{x} \in \mathbb{R}^{m}$ and joint space $\mathbf{q} \in \mathbb{R}^{n}$:

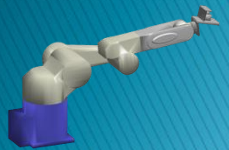
- If $m = n$ (same no. of joints to task space) there are **finite** joint configurations to achieve the end-effector pose*
- If $n > m$, (more joints than task space) there are **infinite** joint configurations to achieve the end-effector pose*

A robot with more degrees of freedom in the control space than required by the task space is said to be **redundant.**

Advantages of redundancy:

- Avoid joint limits
- Avoid collisions with obstacles
- Minimize joint velocities, torque, energy consumption…

* Actually, this is not 100% correct, but it is a good heuristic!



Inverse Kinematics for Complex Robot Arms

Very difficult, and sometimes an analytical solution may not even exist.

Certain robot geometry can be exploited, e.g. wrist-type robots.

Alternatively, we can use mathematical optimization to solve IK.

We can let computers do the work for us.

$$\mathbf{q} \in \mathbb{R}^7$$

Summary of Inverse Kinematics

- Given a desired end-effector pose $\mathbf{x} \in \mathbb{R}^{\mathrm{m}}$, find a feasible joint configuration $\mathbf{q} = \mathbf{f}^{-1}(\mathbf{x}) \in \mathbb{R}^{\mathrm{n}}$
- Two approaches:
  - Closed-form solutions
  - Optimization
- Finite solutions for $\mathrm{m} = \mathrm{n}$
- Infinite solutions for $\mathrm{m} < \mathrm{n}$