3.1 The Homogeneous Transformation Matrix

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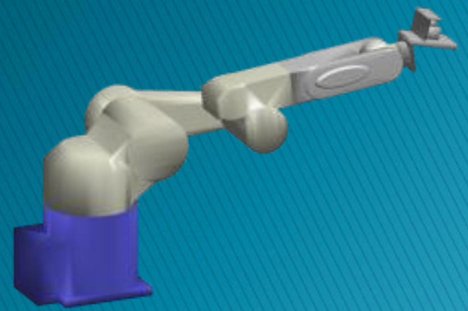


Roadmap

Rotation Matrices

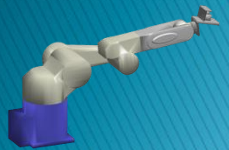
$$\mathbf{R} = [\hat{\mathrm{x}} \quad \hat{\mathrm{y}} \quad \hat{\mathrm{z}}]$$

$\rightarrow$

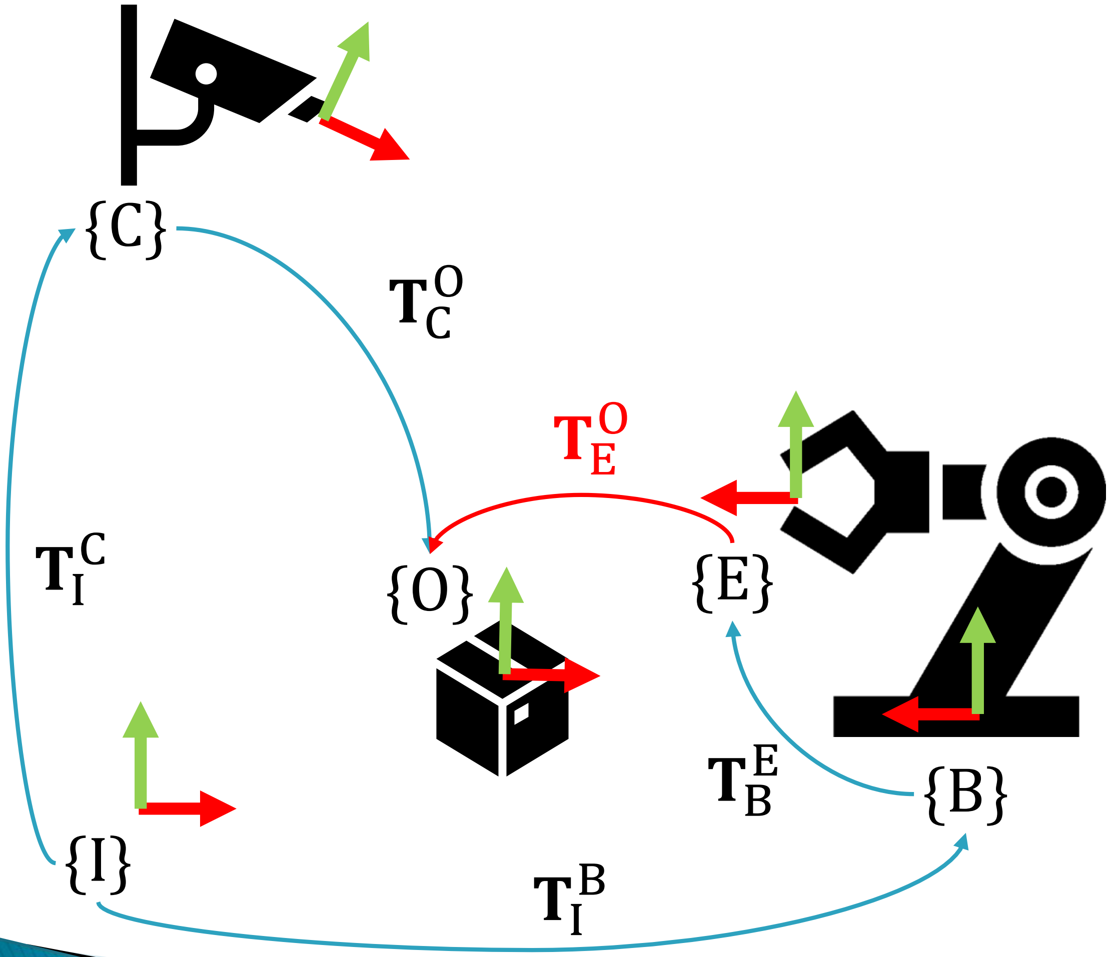
Transformation Matrices

$$\mathbf{T} = \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix}$$

How can we describe the relative position **and** orientation between reference frames (pose)?



How Should the Robot Position its End-Effector?



Given the following poses (rotation + translation):

- $\mathbf{T}_I^C$
- $\mathbf{T}_C^O$
- $\mathbf{T}_I^B$
- $\mathbf{T}_B^E$

What is $\mathbf{T}_E^O$?

Combining Rotation and Translation

Given 3 reference frames $\{1\}, \{2\}$ and $\{3\}$, how can the rotations $\mathbf{R} \in \mathbb{SO}(3)$ and translations $\mathbf{p} \in \mathbb{R}^3$ be combined mathematically?

It is not possible to concatenate the information like so:

$$\mathbf{T}_1^3 \neq [\mathbf{R}_1^2 \quad \mathbf{p}_1^2][\mathbf{R}_2^3 \quad \mathbf{p}_2^3]$$

As $[\mathbf{R}_1^2 \quad \mathbf{p}_1^2] \in \mathbb{R}^{3\times 4}$.

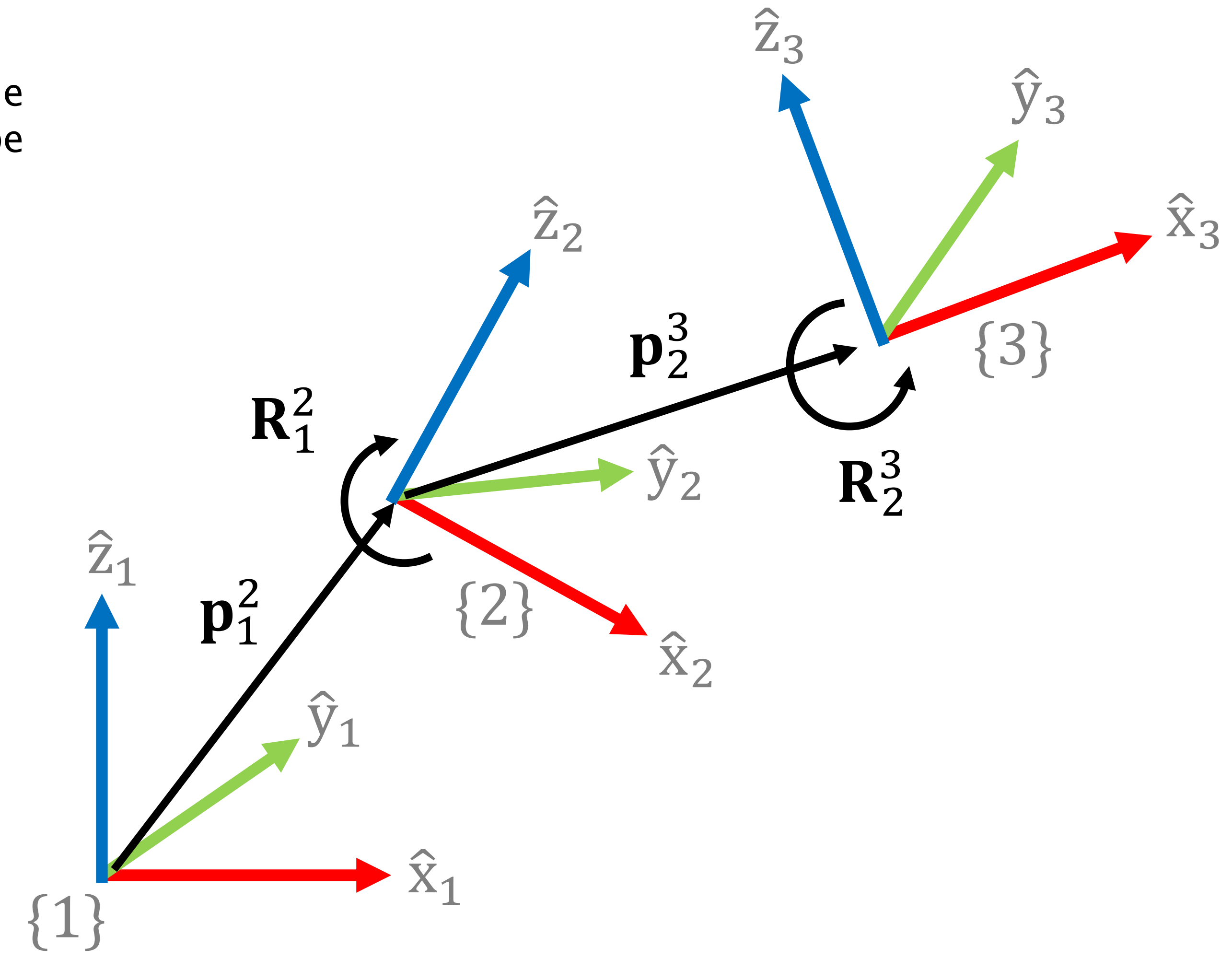
Instead, expand to 4×4:

$$\mathbf{T}_1^3 = \mathbf{T}_1^2\mathbf{T}_2^3 = \begin{bmatrix} \mathbf{R}_1^2 & \mathbf{p}_1^2 \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}\begin{bmatrix} \mathbf{R}_2^3 & \mathbf{p}_2^3 \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{R}_1^2\mathbf{R}_2^3 & \mathbf{R}_1^2\mathbf{p}_2^3 + \mathbf{p}_1^2 \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{R}_1^3 & \mathbf{p}_1^3 \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}$$

This format is called a homogeneous transformation matrix.

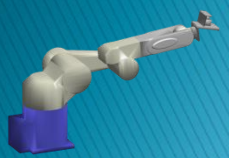


Homogeneous Transformation Matrix

- $\mathbf{R} \in \mathbb{SO}(3) \rightarrow$ Special Orthogonal Group
- $\mathbf{p} \in \mathbb{R}^3 \rightarrow$ Set of Real Values (3 dimensions)

- $\mathbf{T} = \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & x \\ r_{21} & r_{22} & r_{23} & y \\ r_{31} & r_{32} & r_{33} & z \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathbb{SE}(3)$

- $\mathbb{SE}$ is the *Special Euclidean Group*
- Describes both *rotation* and *translation* in 3-dimensional Euclidean space



Inverse of the Transformation Matrix

Transform from {1} to {2}:

$$\mathbf{T}_1^2 = \begin{bmatrix} \mathbf{R}_1^2 & \mathbf{p}_1^2 \\ \mathbf{0}_{1\times3} & 1 \end{bmatrix}$$

Then its opposite, {2} to {1} is:

$$\boxed{\mathbf{T}_2^1 = \begin{bmatrix} \left(\mathbf{R}_1^2\right)^{\mathrm{T}} & -\left(\mathbf{R}_1^2\right)^{\mathrm{T}}\mathbf{p}_1^2 \\ \mathbf{0}_{1\times3} & 1 \end{bmatrix}}$$

Such that:

$$\mathbf{T}_2^1 = \left(\mathbf{T}_1^2\right)^{-1}$$

Multiply the 2 transforms:

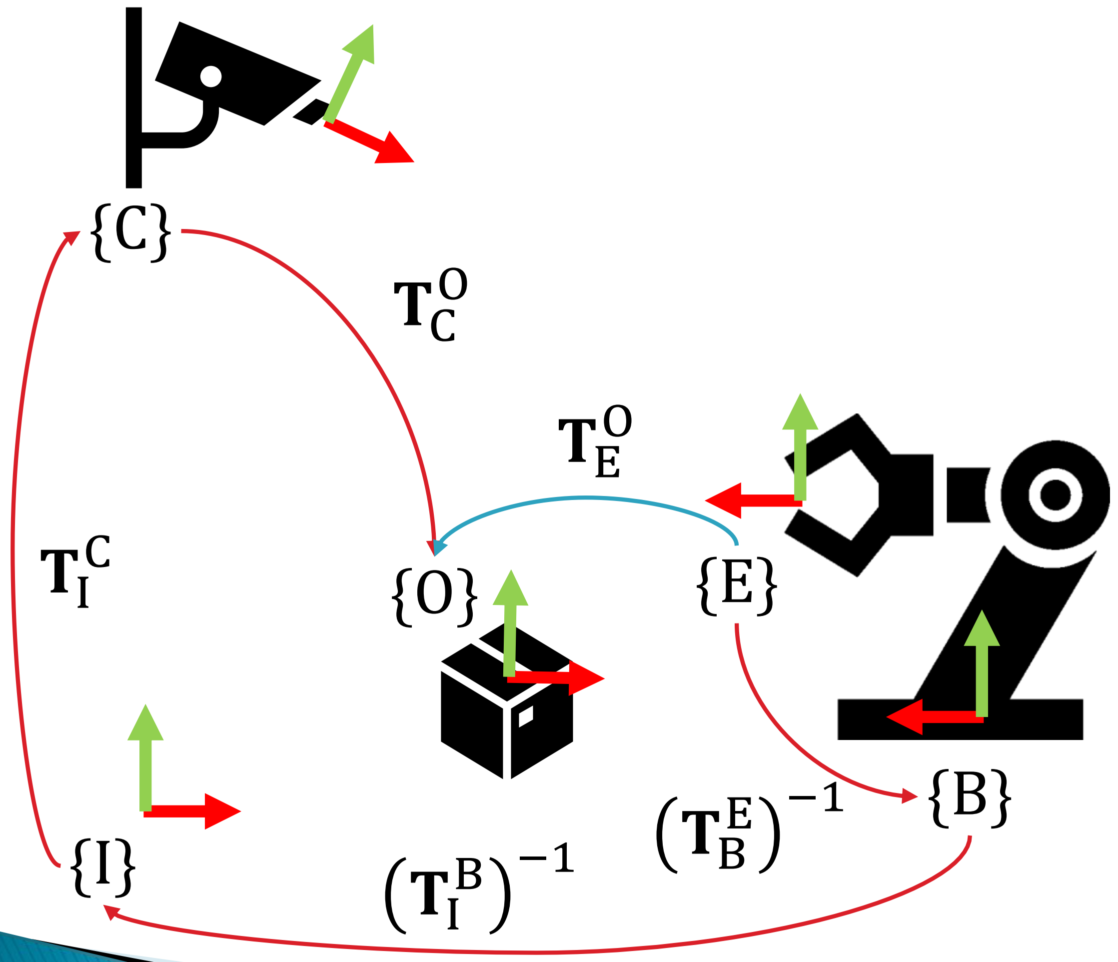
$$\mathbf{T}_1^2\mathbf{T}_2^1 = \begin{bmatrix} \mathbf{R}_1^2 & \mathbf{p}_1^2 \\ \mathbf{0}_{1\times3} & 1 \end{bmatrix}\begin{bmatrix} \left(\mathbf{R}_1^2\right)^{\mathrm{T}} & -\left(\mathbf{R}_1^2\right)^{\mathrm{T}}\mathbf{p}_1^2 \\ \mathbf{0}_{1\times3} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{R}_1^2\left(\mathbf{R}_1^2\right)^{\mathrm{T}} & -\mathbf{R}_1^2\left(\mathbf{R}_1^2\right)^{\mathrm{T}}\mathbf{p}_1^2 + \mathbf{p}_1^2 \\ \mathbf{0}_{1\times3} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{I} & -\mathbf{p}_1^2 + \mathbf{p}_1^2 \\ \mathbf{0}_{1\times3} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0}_{1\times3} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{I} \quad \checkmark$$

How Should the Robot Position its End-Effector?



First, write out the chain:

$$\mathbf{T}_{E}^{O} = \mathbf{T}_{E}^{B}\mathbf{T}_{B}^{I}\mathbf{T}_{I}^{C}\mathbf{T}_{C}^{O}$$

Then, invert the relevant transforms:

$$\mathbf{T}_{E}^{O} = \left(\mathbf{T}_{B}^{E}\right)^{-1}\left(\mathbf{T}_{I}^{B}\right)^{-1}\mathbf{T}_{I}^{C}\mathbf{T}_{C}^{O}$$

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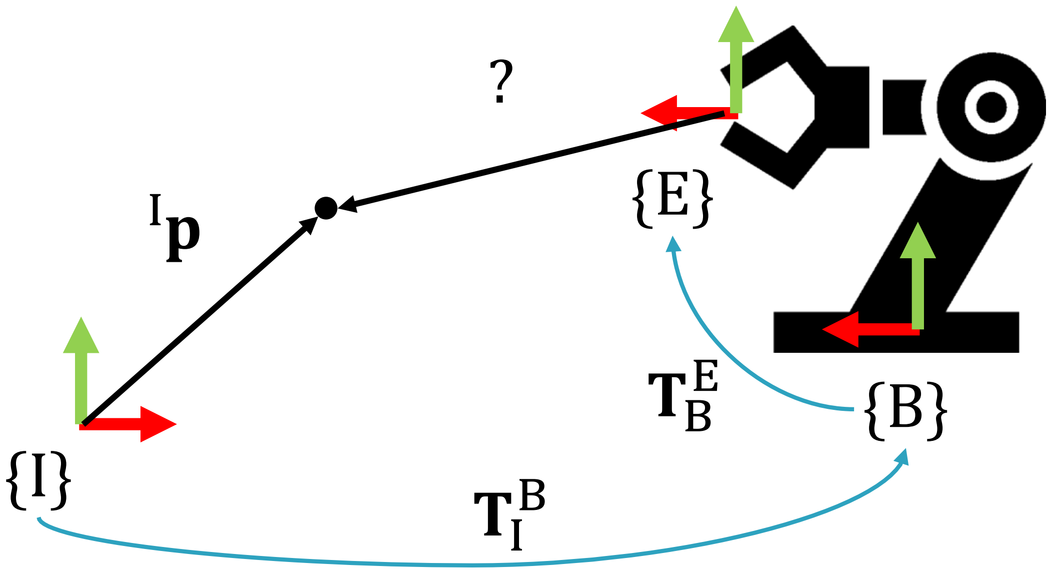
What is the Distance from the End-Effector to the Point?



Given:

- $^{I}\mathbf{p}$
- $\mathbf{T}_{I}^{B} = \begin{bmatrix} \mathbf{R}_{I}^{B} & \mathbf{p}_{I}^{B} \\ \mathbf{0}_{1\times3} & 1 \end{bmatrix}$
- $\mathbf{T}_{B}^{E} = \begin{bmatrix} \mathbf{R}_{B}^{E} & \mathbf{p}_{B}^{E} \\ \mathbf{0}_{1\times3} & 0 \end{bmatrix}$

How far is the point from the end-effector?

In which direction should the robot move?

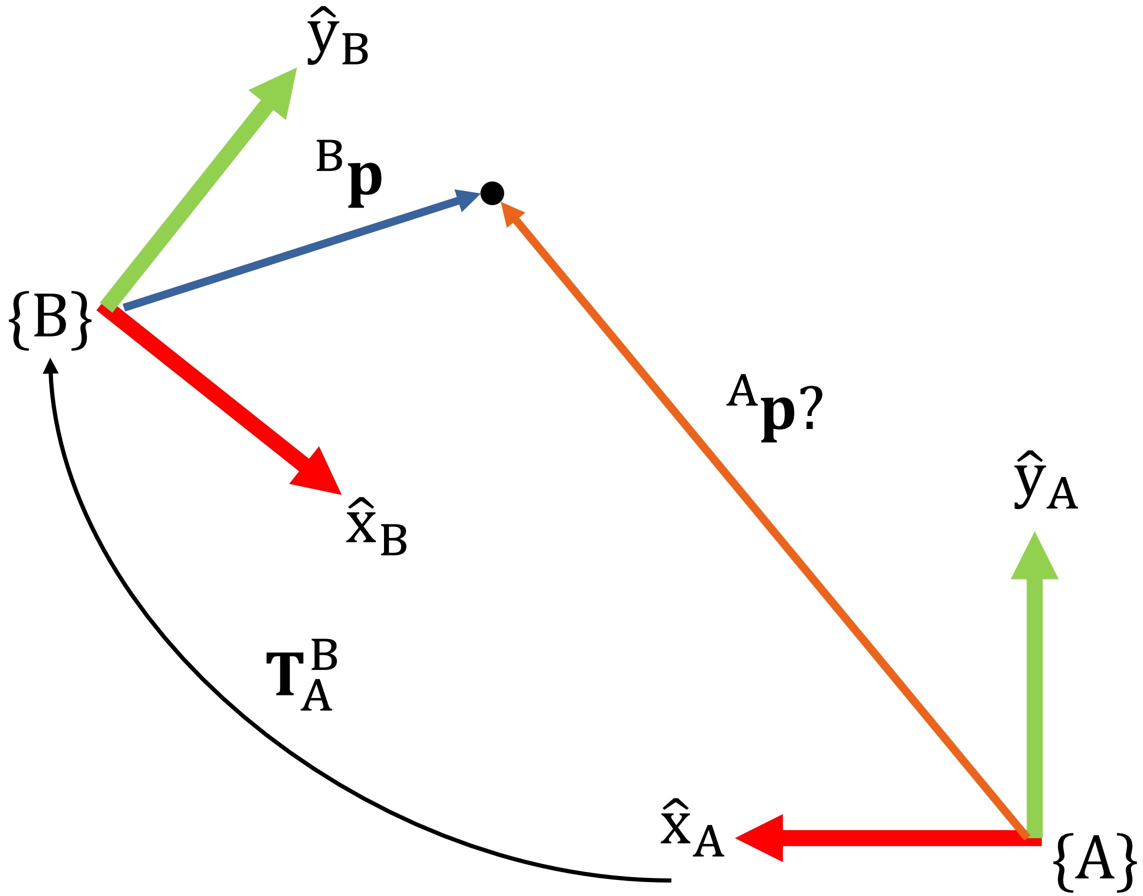
Point Transformations

${}^{B}\mathbf{p} \in \mathbb{R}^3 \rightarrow$ a point specified in frame $\{B\}$

$\mathbf{T}_A^B \in \mathbb{SE}(3) \rightarrow$ homogeneous transform from $\{A\} \rightarrow \{B\}$

${}^{A}\mathbf{p} =?$ how can we find the point w.r.t. $\{A\}$?

${}^{A}\mathbf{p} \neq \mathbf{T}_A^B \cdot {}^{B}\mathbf{p}$, since $\mathbf{T}_A^B \in \mathbb{R}^{4\times 4}$, and ${}^{B}\mathbf{p} \in \mathbb{R}^3$



Point Transformations

Use a homogeneous point transformation:

$$ {}^{A}\tilde{\mathbf{p}} = \mathbf{T}_{A}^{B} \cdot {}^{B}\tilde{\mathbf{p}} \in \mathbb{R}^4 $$

Expanding:

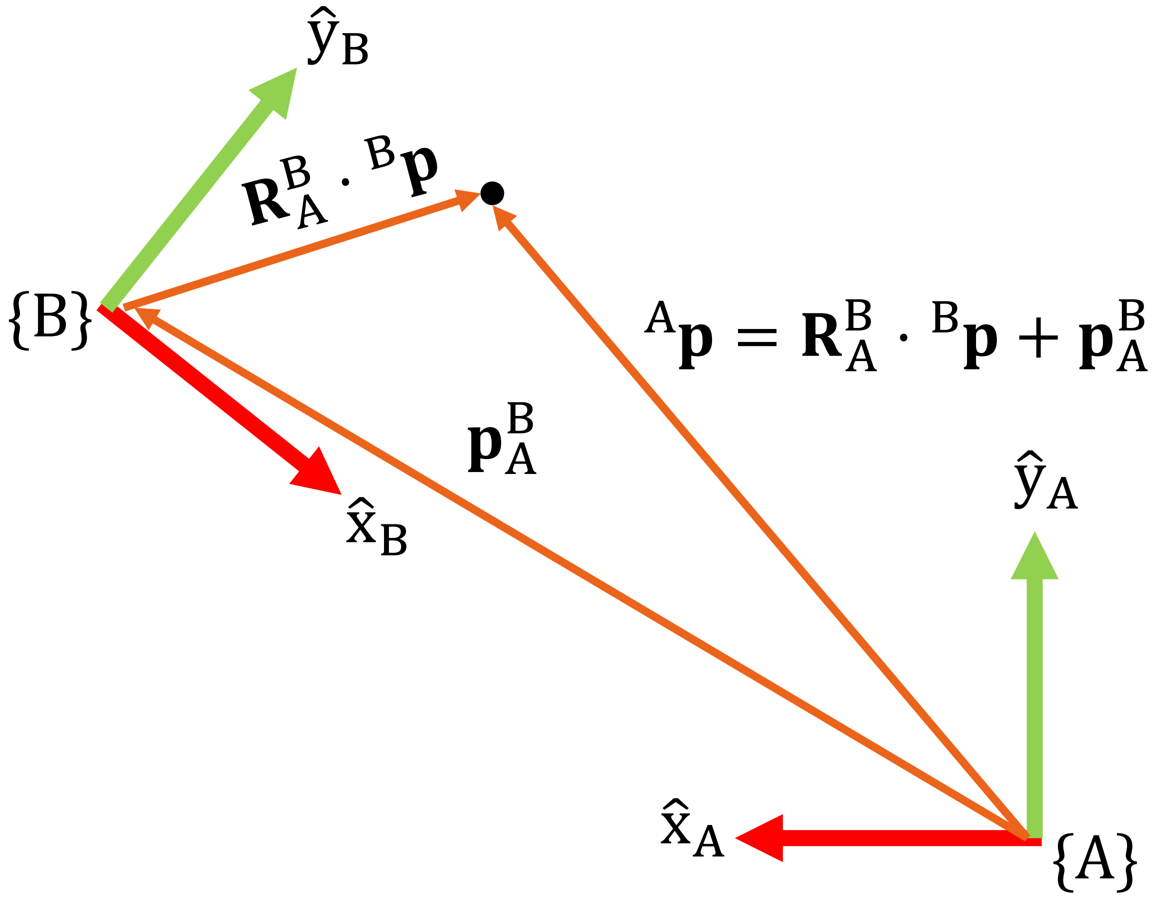
$$ \begin{bmatrix} {}^{A}\mathbf{p} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{A}^{B} & \mathbf{p}_{A}^{B} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \begin{bmatrix} {}^{B}\mathbf{p} \\ 1 \end{bmatrix} $$

$$ = \begin{bmatrix} \mathbf{R}_{A}^{B} \cdot {}^{B}\mathbf{p} + \mathbf{p}_{A}^{B} \\ 1 \end{bmatrix} $$

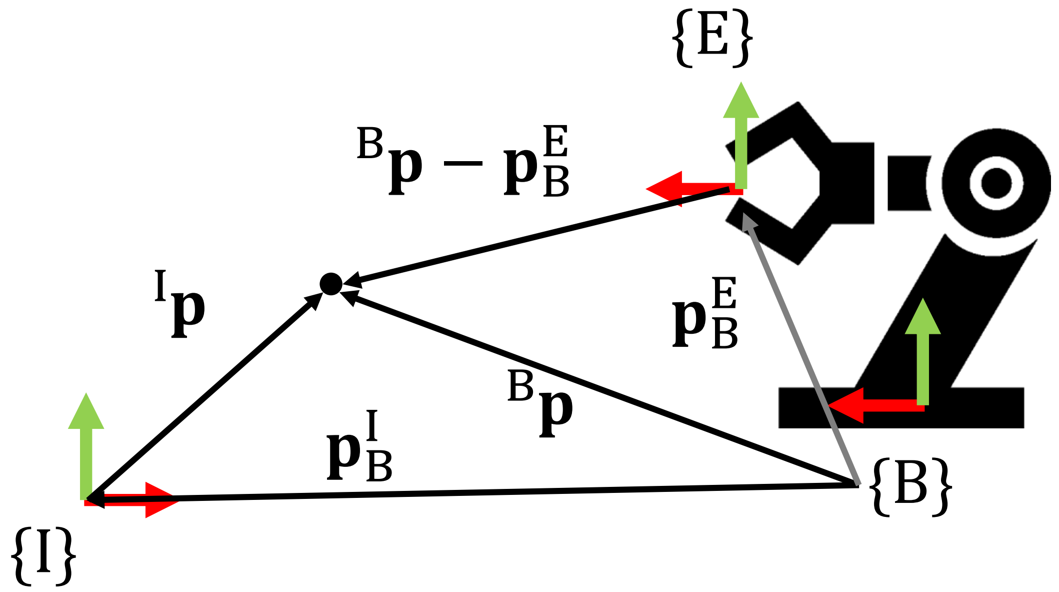
$$ \therefore {}^{A}\mathbf{p} = \mathbf{R}_{A}^{B} \cdot {}^{B}\mathbf{p} + \mathbf{p}_{A}^{B} $$

$\mathbf{R}_{A}^{B} \cdot {}^{B}\mathbf{p}$ gives the point $\mathbf{p}$ from the perspective of $\{A\}$.

Then add the translation from $\{A\}$ to $\{B\}$ $\mathbf{p}_{A}^{B}$.



What is the Distance from the End–Effector to the Point?



Transform from Inertial frame {I} to base frame {B}:

$$\mathbf{T}_{\mathrm{I}}^{\mathrm{B}} = \begin{bmatrix} \mathbf{R}_{\mathrm{I}}^{\mathrm{B}} & \mathbf{p}_{\mathrm{I}}^{\mathrm{B}} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}$$

Base {B} to inertial {I} frame:

$$\mathbf{T}_{\mathrm{B}}^{\mathrm{I}} = \begin{bmatrix} \left(\mathbf{R}_{\mathrm{I}}^{\mathrm{B}}\right)^{\mathrm{T}} & -\left(\mathbf{R}_{\mathrm{I}}^{\mathrm{B}}\right)^{\mathrm{T}}\mathbf{p}_{\mathrm{I}}^{\mathrm{B}} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}$$

Base to end–effector:

$$\mathbf{T}_{\mathrm{B}}^{\mathrm{E}} = \begin{bmatrix} \mathbf{R}_{\mathrm{B}}^{\mathrm{E}} & \mathbf{p}_{\mathrm{B}}^{\mathrm{E}} \\ \mathbf{0}_{1\times 3} & 0 \end{bmatrix}$$

Point in base frame {B}

$$\begin{aligned} ^{\mathrm{B}}\mathbf{p} &= \mathbf{R}_{\mathrm{B}}^{\mathrm{I}} \cdot {}^{\mathrm{I}}\mathbf{p} + \mathbf{p}_{\mathrm{B}}^{\mathrm{I}} \\ &= \left(\mathbf{R}_{\mathrm{I}}^{\mathrm{B}}\right)^{\mathrm{T}} \cdot {}^{\mathrm{I}}\mathbf{p} - \left(\mathbf{R}_{\mathrm{I}}^{\mathrm{B}}\right)^{\mathrm{T}}\mathbf{p}_{\mathrm{I}}^{\mathrm{B}} \end{aligned}$$

Distance from end–effector to the point, with respect to {B}:

$$\begin{aligned} \mathbf{d} &= {}^{\mathrm{B}}\mathbf{p} - \mathbf{p}_{\mathrm{B}}^{\mathrm{E}} \\ &= \left(\mathbf{R}_{\mathrm{I}}^{\mathrm{B}}\right)^{\mathrm{T}} \cdot {}^{\mathrm{I}}\mathbf{p} - \left(\mathbf{R}_{\mathrm{I}}^{\mathrm{B}}\right)^{\mathrm{T}}\mathbf{p}_{\mathrm{I}}^{\mathrm{B}} - \mathbf{p}_{\mathrm{B}}^{\mathrm{E}} \end{aligned}$$

Summary of Transformation Matrices

$\mathbf{T} = \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0}_{1\times3} & 1 \end{bmatrix}$	Homogeneous transformation matrix (4x4)
$\mathbf{T} \in \mathbb{SE}(3)$	In the Special Euclidean Group
$\mathbf{T}_\mathrm{A}^\mathrm{C} = \mathbf{T}_\mathrm{A}^\mathrm{B}\mathbf{T}_\mathrm{B}^\mathrm{C}$	Multiplying transforms produces another transform
$\mathbf{T}_\mathrm{B}^\mathrm{A} = \left(\mathbf{T}_\mathrm{A}^\mathrm{B}\right)^{-1} = \begin{bmatrix} \left(\mathbf{R}_\mathrm{A}^\mathrm{B}\right)^\mathrm{T} & -\left(\mathbf{R}_\mathrm{A}^\mathrm{B}\right)^\mathrm{T}\mathbf{p}_\mathrm{A}^\mathrm{B} \\ \mathbf{0}_{1\times3} & 1 \end{bmatrix}$	The inverse of a transform
${}^\mathrm{A}\tilde{\mathbf{p}} = \mathbf{T}_\mathrm{A}^\mathrm{B} \cdot {}^\mathrm{B}\tilde{\mathbf{p}}$	Homogeneous point transformation
${}^\mathrm{A}\mathbf{p} = \mathbf{R}_\mathrm{A}^\mathrm{B} \cdot {}^\mathrm{B}\mathbf{p} + \mathbf{p}_\mathrm{A}^\mathrm{B}$	Solution to transforming a point between frames