2.3 Rotations in 3D

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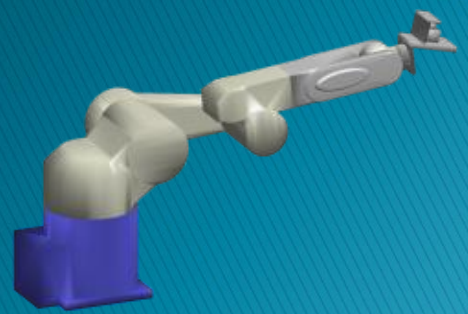


3D Rotation Matrix

Add a 3rd dimension to the rotation matrix:

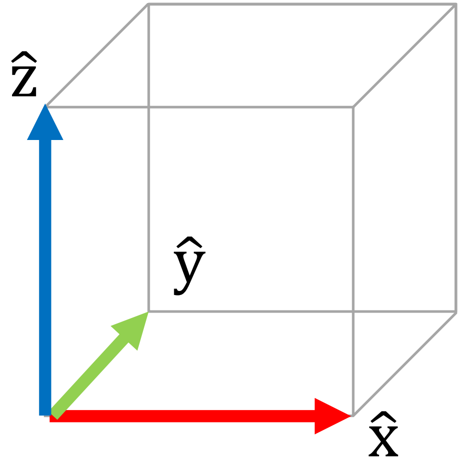
- $\mathbf{R} = [\hat{x} \quad \hat{y} \quad \hat{z}] \in \mathbb{SO}(3)$

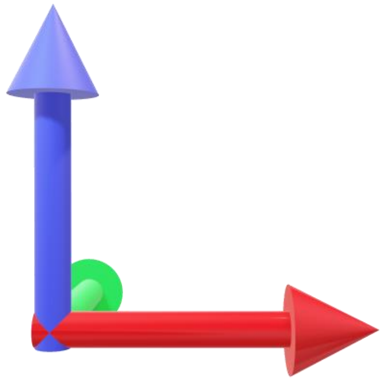
Each column is a 3D unit vector

- $\hat{x}, \hat{y}, \hat{z} \in \mathbb{R}^3$
- $\|\hat{x}\| = \|\hat{y}\| = \|\hat{z}\| = 1$

All columns are **orthonormal**

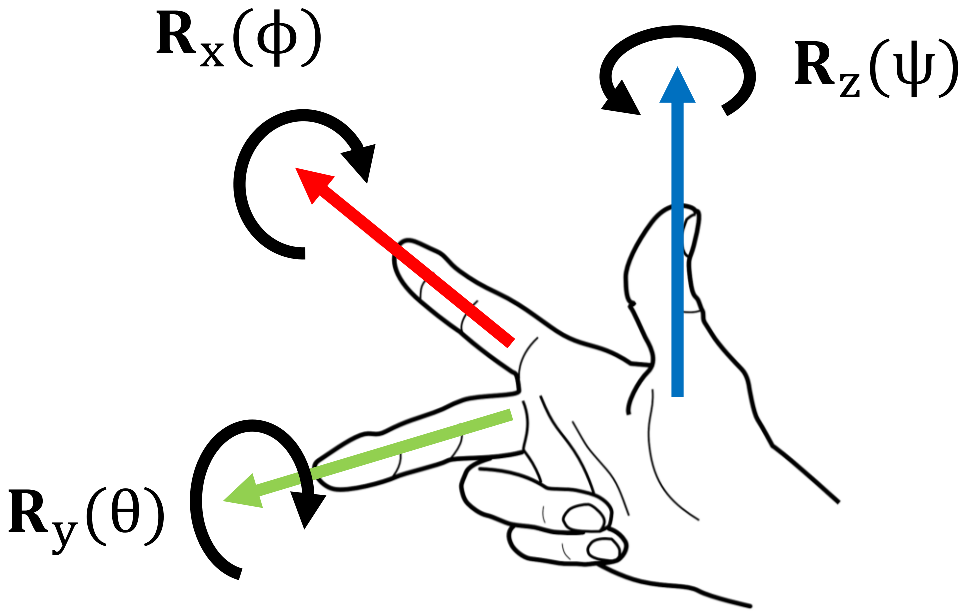
- $\hat{x}^T\hat{y} = \hat{x}^T\hat{z} = \hat{y}^T\hat{z} = 0$





Right Hand Rule

- x-axis forward, from the index finger
- y-axis left, from the middle finger
- z-axis up, from the thumb
- Roll ϕ about x-axis
- Pitch θ about y-axis
- Yaw ψ about z-axis

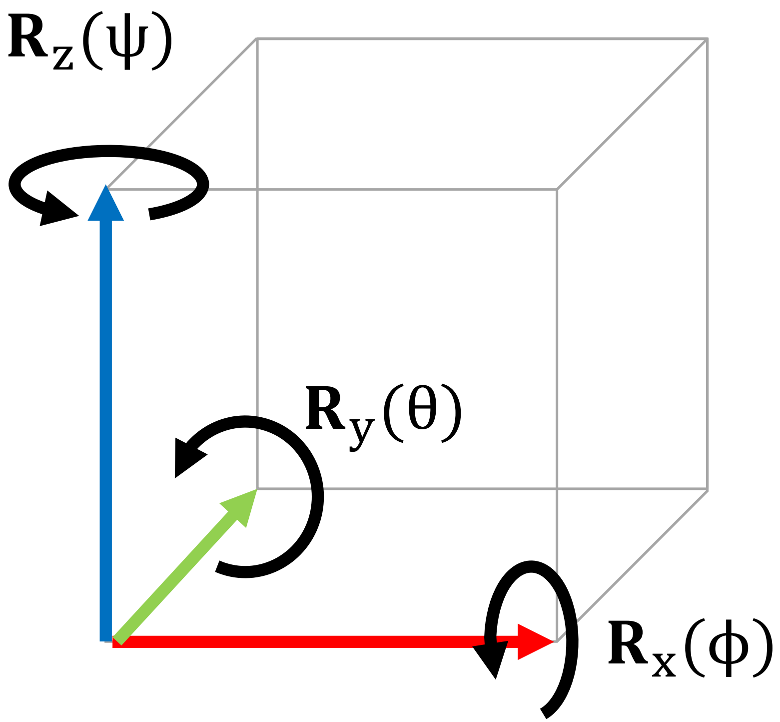


Elementary Rotations

$$\mathbf{R}_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi) & \cos(\phi) \end{bmatrix}$$

$$\mathbf{R}_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

$$\mathbf{R}_z(\psi) = \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Combinations of Elementary Rotations

A rotation matrix can be formed from a maximum of 3 sequential rotations about the primary axes.

$$\mathbf{R}(\phi, \theta, \psi)$$

These rotations can be in any sequence, but not the same axis in succession.

$$\mathbf{R}_x(\phi)\mathbf{R}_x(\phi) = \mathbf{R}_x(2\phi)$$

In total, there are $3 \times 2 \times 2 = 12$ total combinations.

Euler Angles:

Same axis twice

$3 \times 2 \times 1 = 6$ combinations

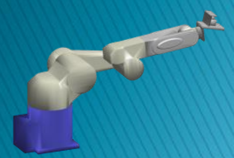
XYX	XYZ	XZX	XZY
YXY	YXZ	YZX	YZY
ZXY	ZXZ	ZYX	ZYZ

Cardan Angles:

All 3 axes

$3 \times 2 \times 1 = 6$ combinations

XYX	XYZ	XZX	XZY
YXY	YXZ	YZX	YZY
ZXY	ZXZ	ZYX	ZYZ



Order of Rotations Is Important!

$\mathbf{R_x R_y R_z \neq R_z R_y R_x}$

Matrices are **not** commutative.

Proof:

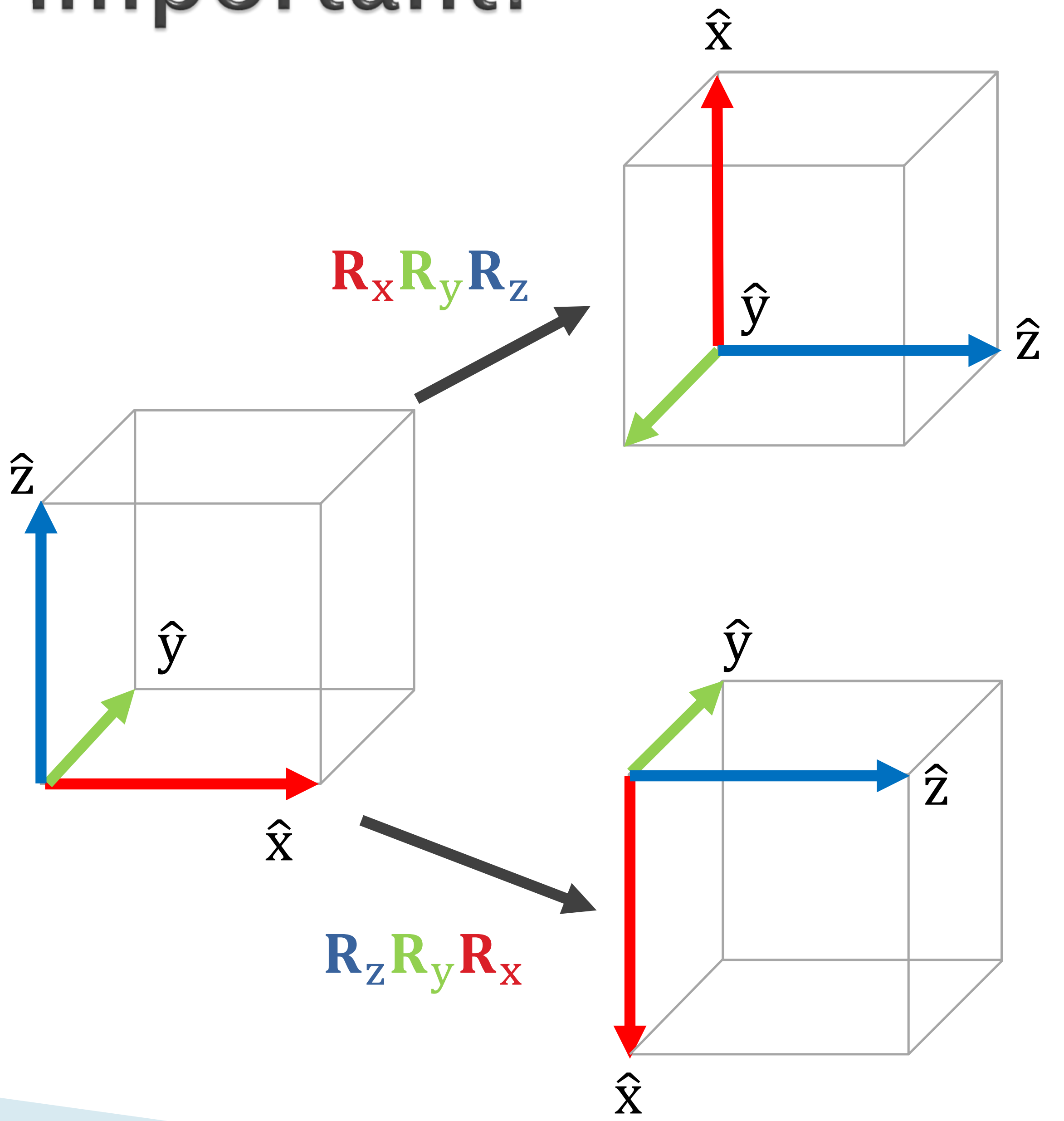
$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

Such that:

$$\begin{aligned} \mathbf{AB}(\mathbf{AB})^{-1} &= \mathbf{ABB}^{-1}\mathbf{A}^{-1} \\ &= \mathbf{AA}^{-1} \\ &= \mathbf{I} \end{aligned}$$

But,

$$\begin{aligned} \mathbf{BA}(\mathbf{AB})^{-1} &= \mathbf{BAB}^{-1}\mathbf{A}^{-1} \\ &\neq \mathbf{I} \\ \therefore \mathbf{AB} &\neq \mathbf{BA} \end{aligned}$$



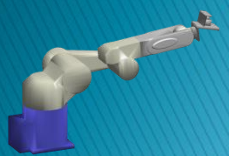
Roll, Pitch, and Yaw from Rotation

$$\mathbf{R}(\phi, \theta, \psi) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\phi = \operatorname{atan2}(r_{32}, r_{33})$$

$$\psi = \operatorname{atan2}(r_{21}, r_{11})$$

$$\theta = \begin{cases} \operatorname{atan2}\left(-r_{31}, \dfrac{r_{21}}{\sin(\psi)}\right) & \text{if } \cos(\psi) = 0 \\ \operatorname{atan2}\left(-r_{31}, \dfrac{r_{11}}{\cos(\psi)}\right) & \text{otherwise} \end{cases}$$



Gimbal Lock

Consider the case where the pitch angle $\theta = \pi/2$:

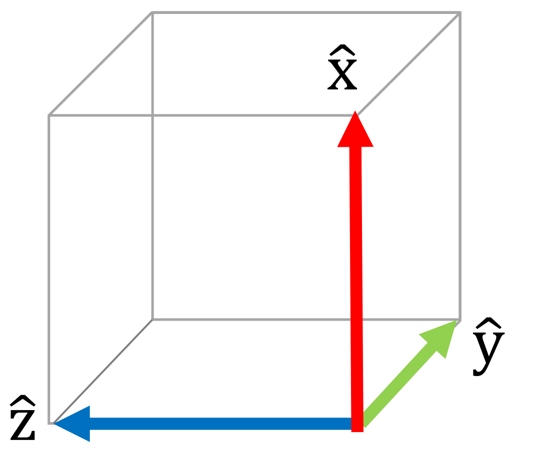
$$\mathbf{R}_y(\pi/2) = \begin{bmatrix} \cos(\pi/2) & 0 & \sin(\pi/2) \\ 0 & 1 & 0 \\ -\sin(\pi/2) & 0 & \cos(\pi/2) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

Then:

$$\mathbf{R}_x(\phi)\mathbf{R}_y(\pi/2)\mathbf{R}_z(\psi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi) & \cos(\phi) \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ \sin(\phi) & \cos(\phi) & 0 \\ -\cos(\phi) & \sin(\phi) & 0 \end{bmatrix} \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ \sin(\phi)\cos(\psi) + \cos(\phi)\sin(\psi) & -\sin(\phi)\sin(\psi) + \cos(\phi)\cos(\psi) & 0 \\ -\cos(\phi)\cos(\psi) + \sin(\phi)\sin(\psi) & \cos(\phi)\sin(\psi) + \sin(\phi)\cos(\psi) & 1 \end{bmatrix}$$



Gimbal Lock

$$\mathbf{R}_x(\phi)\mathbf{R}_y(\pi/2)\mathbf{R}_z(\psi) = \begin{bmatrix} 0 & 0 & 0 \\ \sin(\phi)\cos(\psi) + \cos(\phi)\sin(\psi) & -\sin(\phi)\sin(\psi) + \cos(\phi)\cos(\psi) & 0 \\ -\cos(\phi)\cos(\psi) + \sin(\phi)\sin(\psi) & \cos(\phi)\sin(\psi) + \sin(\phi)\cos(\psi) & 1 \end{bmatrix}$$

Using trigonometric identities:

$$\mathbf{R}_x(\phi)\mathbf{R}_y(\pi/2)\mathbf{R}_z(\psi) = \begin{bmatrix} 0 & 0 & 0 \\ \sin(\phi + \psi) & \cos(\phi + \psi) & 0 \\ -\cos(\phi + \psi) & \sin(\phi + \psi) & 1 \end{bmatrix}$$

Hence, when the pitch angle $\theta = \pm\frac{\pi}{2}$, the roll ϕ cannot be distinguished from yaw ψ.

Either:

- Set the standard operating conditions away from $\theta = \pm\frac{\pi}{2}$
- Avoid manoeuvres that pass through $\theta = \pm\frac{\pi}{2}$

Rotation Error

Define rotation error:

$$\mathbf{R}_e = \mathbf{R}_d\mathbf{R}^T$$

where:

- $\mathbf{R}_d$ is the desired rotation
- $\mathbf{R}$ is the actual rotation

If $\mathbf{R}_e = \mathbf{I}$, then $\mathbf{R}_d = \mathbf{R}$

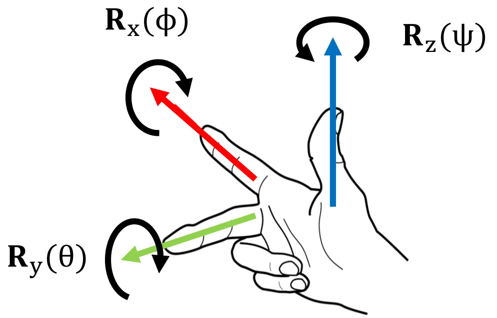
$$\mathbf{R}_e = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Extracting RPY angles (ϕ, θ, ψ) from rotation error:

$$\phi_e = \operatorname{atan2}(r_{32}, r_{33})$$
$$\psi_e = \operatorname{atan2}(r_{21}, r_{11})$$

$$\theta_e = \begin{cases} \operatorname{atan2}\left(-r_{31}, \dfrac{r_{21}}{\sin(\psi_e)}\right) & \text{if } \cos(\psi_e) = 0 \\ \operatorname{atan2}\left(-r_{31}, \dfrac{r_{11}}{\cos(\psi_e)}\right) & \text{otherwise} \end{cases}$$

Summary of Rotations in 3D



Right hand rule is used to denote axes and direction of rotations

$$\mathbf{R}(\phi, \theta, \psi) = \begin{cases} \mathbf{R}_x(\phi) \\ \mathbf{R}_y(\theta) \\ \mathbf{R}_z(\psi) \end{cases}$$

A rotation can be constructed from rotations about the primary axes

$\mathbf{R}_A\mathbf{R}_B \neq \mathbf{R}_B\mathbf{R}_A$ Order of rotations is important

$\mathbf{R}_y(\pi/2)$ Gimbal lock occurs when the pitch angle is 90°

$\mathbf{R}_e = \mathbf{R}_d\mathbf{R}^T$ Rotation error