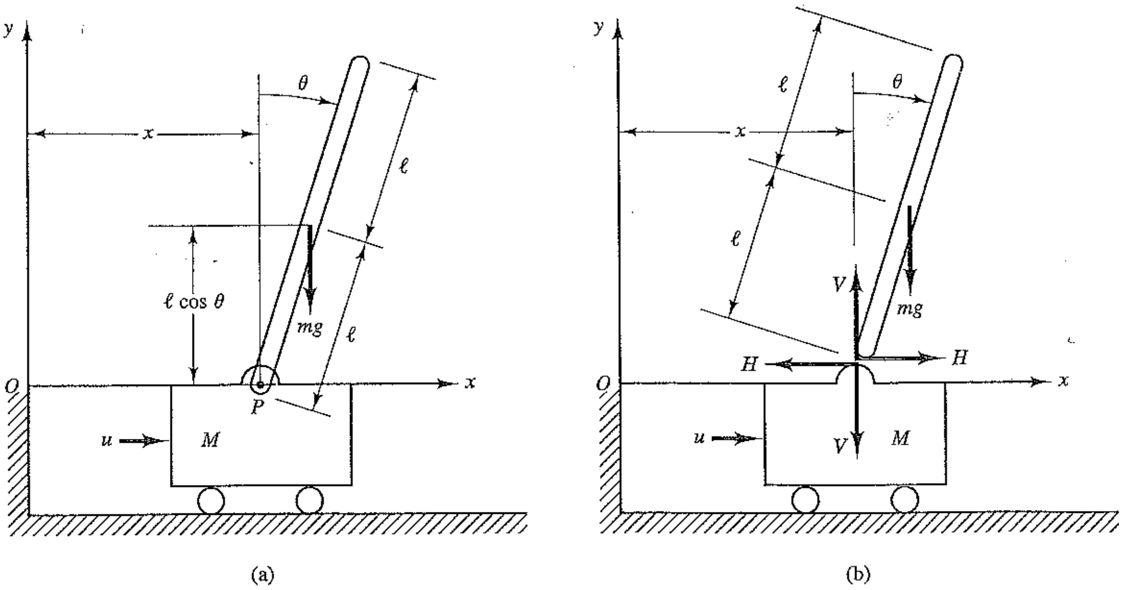

Figure 3–16
(a) Inverted pendulum system; (b) free-body diagram.

$$m = 0.1\,\text{kg}, \qquad M = 2\,\text{kg}, \qquad 2l = 1\,\text{m}$$

Define the angle of the rod from the vertical line as θ. Define also the (x, y) coordinates of the center of gravity of the pendulum rod as (x_G, y_G). Then

$$x_G = x + l \sin \theta$$

$$y_G = l \cos \theta$$

To derive the equations of motion for the system, consider the free-body diagram shown in Figure 3–16(b). The rotational motion of the pendulum rod about its center of gravity can be described by

$$I\ddot{\theta} = Vl \sin \theta - Hl \cos \theta \tag{3–47}$$

where I is the moment of inertia of the rod about its center of gravity.

The horizontal motion of center of gravity of pendulum rod is given by

$$m \frac{d^2}{dt^2}(x + l \sin \theta) = H \tag{3–48}$$

The vertical motion of center of gravity of pendulum rod is

$$m \frac{d^2}{dt^2}(l \cos \theta) = V - mg \tag{3–49}$$

The horizontal motion of cart is described by

$$M \frac{d^2 x}{dt^2} = u - H \tag{3–50}$$

Equations (3–47) through (3–50) describe the motion of the inverted-pendulum-on-the-cart system. Because these equations involve $\sin\theta$ and $\cos\theta$, they are nonlinear equations.

If we assume angle θ to be small, Equations (3–47) through (3–50) may be linearized as follows:

$$I\ddot{\theta} = Vl\theta - Hl \tag{3–51}$$

$$m(\ddot{x} + l\ddot{\theta}) = H \tag{3–52}$$

$$0 = V - mg \tag{3–53}$$

$$M\ddot{x} = u - H \tag{3–54}$$

From Equations (3–52) and (3–54), we obtain

$$(M + m)\ddot{x} + ml\ddot{\theta} = u \tag{3–55}$$

From Equations (3–51) and (3–53), we have

$$\begin{aligned} I\ddot{\theta} &= mgl\theta - Hl \\ &= mgl\theta - l(m\ddot{x} + ml\ddot{\theta}) \end{aligned}$$

or

$$(I + ml^2)\ddot{\theta} + ml\ddot{x} = mgl\theta \tag{3–56}$$

Equations (3–55) and (3–56) describe the motion of the inverted-pendulum-on-the-cart system. They constitute a mathematical model of the system. (Later in Chapters 12 and 13, we design controllers to keep the pendulum upright in the presence of disturbances.)

Mathematical modeling. Referring to Section 3–6, we derived a mathematical model for the inverted-pendulum system shown in Figure 3–16(a). When angle θ is assumed small, the equations describing the dynamics of the system are given by Equations (3–55) and (3–56), rewritten next:

$$(M + m)\ddot{x} + ml\ddot{\theta} = u$$

$$(I + ml^2)\ddot{\theta} + ml\ddot{x} = mgl\theta$$

where I is the moment of inertia of the pendulum rod about its center of gravity. Since in this system the mass is concentrated at the top of the rod, the center of gravity is the center of the pendulum ball. In this analysis, we assume that the moment of inertia of the pendulum about its center of gravity is zero, or $I = 0$. Then the mathematical model becomes as follows:

$$(M + m)\ddot{x} + ml\ddot{\theta} = u \tag{12–19}$$

$$ml^2\ddot{\theta} + ml\ddot{x} = mgl\theta \tag{12–20}$$

Equations (12–19) and (12–20) define a mathematical model of the inverted-pendulum system shown in Figure 12–2. (These linearized equations are valid as long as θ is small.)

Equations (12–19) and (12–20) can be modified to

$$Ml\ddot{\theta} = (M + m)g\theta - u \tag{12–21}$$

$$M\ddot{x} = u - mg\theta \tag{12–22}$$

Equation (12–21) was obtained by eliminating $\ddot{x}$ from Equations (12–19) and (12–20). Equation (12–22) was obtained by eliminating $\ddot{\theta}$ from Equations (12–19) and (12–20). From Equation (12–21) we obtain the plant transfer function to be

$$\frac{\Theta(s)}{-U(s)} = \frac{1}{Mls^2 - (M + m)g}$$

By substituting the given numerical values and noting that $g = 9.81\ \mathrm{m/sec^2}$, we have

$$\frac{\Theta(s)}{-U(s)} = \frac{1}{s^2 - 20.601} = \frac{1}{s^2 - (4.539)^2}$$

The inverted-pendulum plant has one pole on the negative real axis ($s = -4.539$) and another on the positive real axis ($s = 4.539$). Hence, the plant is open-loop unstable.

Define state variables x_1, x_2, x_3, and x_4 by

$$x_1 = \theta$$
$$x_2 = \dot{\theta}$$
$$x_3 = x$$
$$x_4 = \dot{x}$$

Note that angle θ indicates the rotation of the pendulum rod about point P, and x is the location of the cart. We consider θ and x as the outputs of the system, or

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \theta \\ x \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}$$

(Notice that both θ and x are easily measurable quantities.) Then, from the definition of the state variables and Equations (12–21) and (12–22), we obtain

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{M+m}{Ml} g x_1 - \frac{1}{Ml} u$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = -\frac{m}{M} g x_1 + \frac{1}{M} u$$

In terms of vector-matrix equations, we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{M+m}{Ml} g & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{m}{M} g & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{Ml} \\ 0 \\ \frac{1}{M} \end{bmatrix} u \tag{12–23}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{12–24}$$

Equations (12–23) and (12–24) give a state-space representation of the inverted pendulum system. (Note that state-space representation of the system is not unique. There are infinitely many such representations.)

By substituting the given numerical values for M, m, and l, we have

$$\frac{M+m}{Ml} g = 20.601, \qquad \frac{m}{M} g = 0.4905, \qquad \frac{1}{Ml} = 1, \qquad \frac{1}{M} = 0.5$$