

## 41014 Sensors and Control for Mechatronic Systems

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# 0. Quiz 2

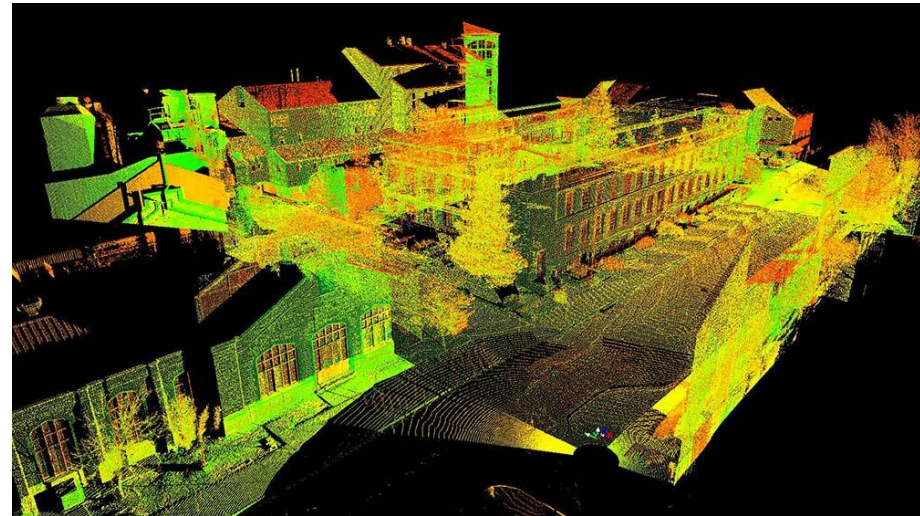
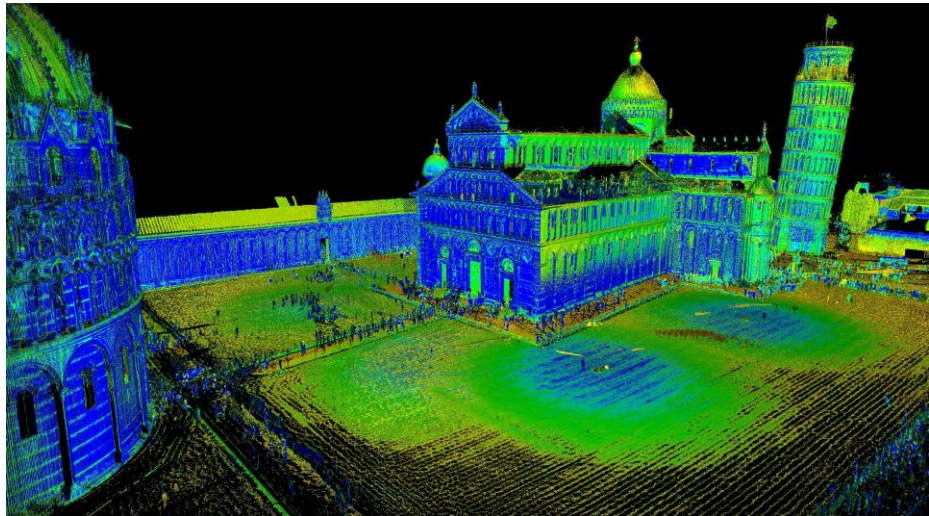
❖ 15%

❖ 40 mins

❖ Restrict open book: one hand writing A4 paper

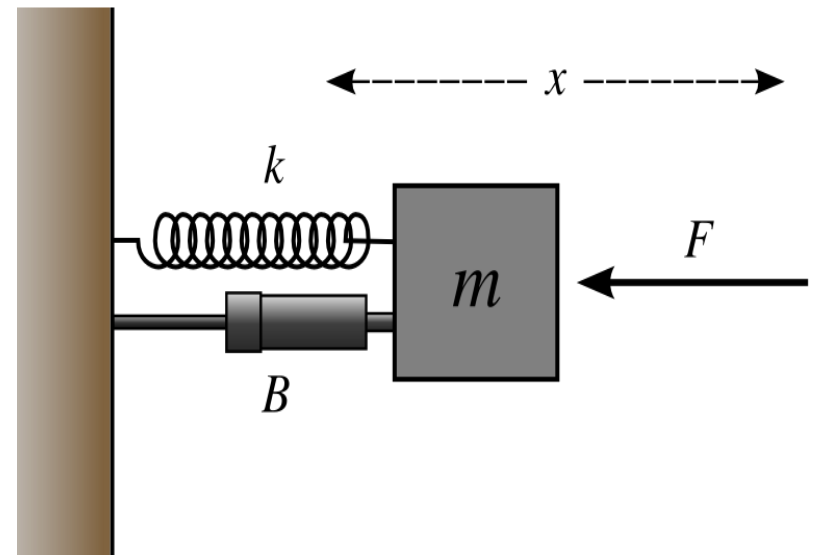
## ❖ Control:

- Linear Continuous-Time Systems
- Linear Discrete-Time Systems
- **Nonlinear Discrete-Time Systems**



## ❖ Control Part 1:

- Linear Continuous-Time Systems
- Formulation
- Stability
- Pole Placement
- Controllability
- Linear quadratic optimal control



## ❖ Lecture:

- Linear discrete-time Systems
- Discretization
- Stability
- State feedback control
- Optimal control

## ❖ Active hands on:

- Examples
- Solve problem using Matlab



## 41014 Sensors and Control for Mechatronic Systems

### Lecture-8: Control Part 3 Nonlinear Discrete-Time Systems

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## ❖ Introduction:

- A system of difference equations as

$$x(k + 1) = f(x(k), u(k))$$

- $f(x, u)$  is a vector function with  $n$  components:

$$f(x, u) = \begin{bmatrix} f_1(x, u) \\ \vdots \\ f_n(x, u) \end{bmatrix}$$

## ❖ Introduction:

- The functions  $f_i(x, u)$  are in turn functions of  $n + m$  variables, the components of the  $x$  and  $u$  vectors.

$$\begin{aligned}x_1(k+1) &= f_1(x_1(k), \dots, x_n(k), u_1(k), \dots, u_m(k)) \\x_2(k+1) &= f_2(x_1(k), \dots, x_n(k), u_1(k), \dots, u_m(k)) \\&\vdots \\x_n(k+1) &= f_n(x_1(k), \dots, x_n(k), u_1(k), \dots, u_m(k))\end{aligned}$$

- $f_i(x, u)$  is nonlinear.



## ❖ Introduction:

- The outputs of the model:

$$y(k) = h(x(k), u(k))$$

- can then be calculated from the internal variables  $x_i(k)$  and the inputs  $u_i(k)$ :

$$y_1(k) = h_1(x_1(k), \dots, x_n(k), u_1(k), \dots, u_m(k))$$

$$y_2(k) = h_2(x_1(k), \dots, x_n(k), u_1(k), \dots, u_m(k))$$

$$\vdots$$

$$y_p(k) = h_p(x_1(k), \dots, x_n(k), u_1(k), \dots, u_m(k))$$

- $h_i(x, u)$  is nonlinear.

## ❖ Compare to Linear discrete-time system:

- Consider the set of  $n$  first-order linear difference equations forced by the input  $u(k) \in \mathbb{R}$

$$\begin{cases} x_1(k+1) &= a_{11}x_1(k) + \dots + a_{1n}x_n(k) + b_1u(k) \\ x_2(k+1) &= a_{21}x_1(k) + \dots + a_{2n}x_n(k) + b_2u(k) \\ \vdots & \vdots \\ x_n(k+1) &= a_{n1}x_1(k) + \dots + a_{nn}x_n(k) + b_nu(k) \\ x_1(0) = x_{10}, \dots & x_n(0) = x_{n0} \end{cases}$$

- In compact matrix form:

$$\begin{cases} x(k+1) &= Ax(k) + Bu(k) \\ x(0) &= x_0 \end{cases}$$

$$\text{where } x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n.$$

## ❖ Nonlinear discrete-time state-space models:

- Nonlinear discrete-time model:

$$\begin{aligned}x(k+1) &= f(x(k), u(k)) & k = 0, 1, 2, \dots \\y(k) &= h(x(k), u(k))\end{aligned}$$

- $u(k)$ : input at time  $k$ , an  $m$ -dimensional column vector.
- $y(k)$ : output at time  $k$ , a  $p$ -dimensional column vector.
- $x(k)$ : state at time  $k$ , an  $n$ -dimensional column vector.
- The model is said to be  *$n$ -th* order.
- For a given initial value  $x(k_0) = x_0$ , always has a unique solution.

## 2. Equilibrium

- ❖ A constant trajectory, generated by a constant input function, is called *equilibrium*.
  - Equilibrium point -- a point where the system can stay forever without moving.
  
- ❖ Given a constant input  $\bar{u}$ , the equilibria are solutions of the following equations:

$$\bar{x} = f(\bar{x}, \bar{u})$$

$$\bar{y} = h(\bar{x}, \bar{u})$$

### ❖ How to calculate equilibria?

- Solve the solution of  $x(k+1)-x(k)=0$ .

### ❖ Example:

$$x(k + 1) = \frac{1}{4}x(k)^2 + x(k) - 2u(k)$$

- When  $u(k) \equiv 0.5$ , compute the equilibrium points.

### ❖ Solution:

$$x(k + 1) - x(k) = \frac{1}{4}x(k)^2 - 1 = 0$$

- The solutions are  $\bar{x} = \pm 2$ .

## 2. Equilibrium

❖ Activity 1:

❖ Calculate the equilibria?

$$x_1(k+1) = \frac{1}{4}x_1(k)^2 + x_1(k) - 2u_1(k)$$

$$x_2(k+1) = \frac{1}{4}x_2(k)^2 + x_1(k) - 2u_2(k)$$

- When  $u_1(k) \equiv 0.5$  and  $u_2(k) \equiv 1$ , compute the equilibrium points.



❖ Activity 1:

❖ Calculate the equilibria?

$$x_1(k+1) = \frac{1}{4}x_1(k)^2 + x_1(k) - 2u_1(k)$$

$$x_2(k+1) = \frac{1}{4}x_2(k)^2 + x_1(k) - 2u_2(k)$$

- When  $u_1(k) \equiv 0.5$  and  $u_2(k) \equiv 1$ , compute the equilibrium points.

❖ Solution:

$$x_1(k+1) - x_1(k) = \frac{1}{4}x_1(k)^2 - 1 = 0$$

$$x_2(k+1) - x_2(k) = \frac{1}{4}x_2(k)^2 + x_1 - x_2 - 2 = 0$$

## 2. Equilibrium

❖ Activity 1:

❖ Solution:

$$x_1(k+1) - x_1(k) = \frac{1}{4}x_1(k)^2 - 1 = 0$$

$$x_2(k+1) - x_2(k) = \frac{1}{4}x_2(k)^2 + x_1 - x_2 - 2 = 0$$

- The solutions are  $\bar{x}_1 = \pm 2$ .
- When  $\bar{x}_1 = 2$ .

$$\frac{1}{4}x_2(k)^2 - x_2 = 0$$

- The solutions are  $\bar{x}_2 = 0$  or  $\bar{x}_2 = 4$ .

## 2. Equilibrium

❖ Activity 1:

❖ Solution:

$$x_1(k+1) - x_1(k) = \frac{1}{4}x_1(k)^2 - 1 = 0$$

$$x_2(k+1) - x_2(k) = \frac{1}{4}x_2(k)^2 + x_1 - x_2 - 2 = 0$$

- The solutions are  $\bar{x}_1 = \pm 2$ .
- When  $\bar{x}_1 = -2$ .

$$\frac{1}{4}x_2(k)^2 - x_2 - 4 = 0$$

- The solutions are  $\bar{x}_2 = 2 \pm 2\sqrt{5}$ .

### 3. Linearization



Given a nonlinear time invariant system

$$\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k), \mathbf{u}(k)) \quad \mathbf{x} \in \mathbb{R}^n \quad \mathbf{u} \in \mathbb{R}^m$$

$$\mathbf{y}(k) = \mathbf{g}(\mathbf{x}(k), \mathbf{u}(k)) \quad \mathbf{y} \in \mathbb{R}^p$$

and an equilibrium

$$\mathbf{u}(k) = \bar{\mathbf{u}} \quad \mathbf{x}(k) = \bar{\mathbf{x}} \quad \mathbf{y}(k) = \bar{\mathbf{y}} \quad \forall k$$

We can locally approximate the nonlinear system, around the equilibrium, with the linearized system

$$\delta \mathbf{x}(k+1) = \mathbf{A} \delta \mathbf{x}(k) + \mathbf{B} \delta \mathbf{u}(k)$$

$$\delta \mathbf{y}(k) = \mathbf{C} \delta \mathbf{x}(k) + \mathbf{D} \delta \mathbf{u}(k)$$

where

$$\delta \mathbf{x}(k) = \mathbf{x}(k) - \bar{\mathbf{x}} \quad \delta \mathbf{u}(k) = \mathbf{u}(k) - \bar{\mathbf{u}} \quad \delta \mathbf{y}(k) = \mathbf{y}(k) - \bar{\mathbf{y}}$$

and

$$\mathbf{A} = \left. \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right|_{\bar{\mathbf{x}}, \bar{\mathbf{u}}} \quad \mathbf{B} = \left. \frac{\partial \mathbf{f}}{\partial \mathbf{u}} \right|_{\bar{\mathbf{x}}, \bar{\mathbf{u}}} \quad \mathbf{C} = \left. \frac{\partial \mathbf{g}}{\partial \mathbf{x}} \right|_{\bar{\mathbf{x}}, \bar{\mathbf{u}}} \quad \mathbf{D} = \left. \frac{\partial \mathbf{g}}{\partial \mathbf{u}} \right|_{\bar{\mathbf{x}}, \bar{\mathbf{u}}}$$

❖ Example:

❖ Linearize the nonlinear discrete-time system

$$\begin{aligned}x_1(k+1) &= \frac{1}{4}x_1(k)^2 + x_1(k) - 2u_1(k) \\x_2(k+1) &= \frac{1}{4}x_2(k)^2 + x_1(k) - 2u_2(k)\end{aligned}$$

❖ Solution:

$$\begin{aligned}\frac{\partial f_1}{\partial x_1} &= \frac{1}{2}x_1 + 1, \frac{\partial f_1}{\partial x_2} = 0 \\ \frac{\partial f_2}{\partial x_1} &= 1, \frac{\partial f_2}{\partial x_2} = \frac{1}{2}x_2\end{aligned}$$

### 3. Linearization

❖ Example:

❖ Solution:

$$\frac{\partial f_1}{\partial x_1} = \frac{1}{2}x_1 + 1, \frac{\partial f_1}{\partial x_2} = 0$$

$$\frac{\partial f_2}{\partial x_1} = 1, \frac{\partial f_2}{\partial x_2} = \frac{1}{2}x_2$$

- At equilibrium point  $\bar{x}_1 = 2, \bar{x}_2 = 4$ :

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}, B = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

- At equilibrium point  $\bar{x}_1 = 2, \bar{x}_2 = 0$ :

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$



## ❖ Activity 2:

$$x(k + 1) = \frac{1}{2}x(k)^2 + x(k) - 2u(k)$$

- When  $u(k) \equiv 0.5$ , the equilibrium points are  $\bar{x} = \pm\sqrt{2}$ .

## ❖ Linearization?

## ❖ Activity 2:

$$x(k+1) = \frac{1}{2}x(k)^2 + x(k) - 2u(k)$$

- When  $u(k) \equiv 0.5$ , the equilibrium points are  $\bar{x} = \pm\sqrt{2}$ .

## ❖ Linearization?

- $A = x(k) + 1, B = -2$
- When at equilibrium points  $\bar{x} = \sqrt{2}$ :

$$\delta x(k+1) = (1 + \sqrt{2})\delta x(k) - 2\delta u(k)$$

- When at equilibrium points  $\bar{x} = -\sqrt{2}$ :

$$\delta x(k+1) = (1 - \sqrt{2})\delta x(k) - 2\delta u(k)$$

- ❖ As the linearized system is a linear discrete-time system, we can assess the stability of the equilibrium point of the nonlinear system by analyzing the state matrix

$$\mathbf{A} = \left. \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right|_{\bar{\mathbf{x}}, \bar{\mathbf{u}}}$$

- ❖ We can state the following results:
  - if all the eigenvalues of matrix  $\mathbf{A}$  lie inside the unit circle (  $\lambda_i < 1$  ), the equilibrium point is asymptotically stable,
  - if at least one eigenvalue of matrix  $\mathbf{A}$  lies outside the unit circle (  $\exists i: \lambda_i > 1$  ), the equilibrium point is unstable
  - If the eigenvalues of matrix  $\mathbf{A}$  lie inside the unit circle and there is at least one eigenvalue on the circumference the linearization, that is a first order approximation, is too rough to assess the stability of the equilibrium point.

### ❖ Example:

$$x(k+1) = \frac{1}{2}x(k)^2 + x(k) - 2u(k)$$

- When  $u(k) \equiv 0.5$ , the equilibrium points are  $\bar{x} = \pm\sqrt{2}$ .

### ❖ Linearization?

- $A = x(k) + 1, B = -2$
- When at equilibrium points  $\bar{x} = \sqrt{2}$ : **(unstable)**

$$\delta x(k+1) = (1 + \sqrt{2})\delta x(k) - 2\delta u(k)$$

- When at equilibrium points  $\bar{x} = -\sqrt{2}$ : **(stable)**

$$\delta x(k+1) = (1 - \sqrt{2})\delta x(k) - 2\delta u(k)$$

### ❖ Matlab Example:

$$x(k + 1) = \frac{1}{2}x(k)^2 + x(k) - 2u(k)$$

- When at equilibrium points  $\bar{x} = \sqrt{2}$ : (**unstable**)
- When at equilibrium points  $\bar{x} = -\sqrt{2}$ : (**stable**)

## ❖ Example 1:

$$x(k + 1) = \frac{1}{2}x(k)^2 + x(k) - 2u(k)$$

- When at equilibrium points  $\bar{x} = \sqrt{2}$ : (**unstable**)

$$\delta x(k + 1) = (1 + \sqrt{2})\delta x(k) - 2\delta u(k)$$

- Apply state feedback control law

$$\delta u(k) = -K\delta x(k)$$

$$\delta x(k + 1) = (A - KB)\delta x(k) = (1 + \sqrt{2} + 2K)\delta x(k)$$

- Let  $(A - KB) = 1 - \sqrt{2}$ , then  $K = -\sqrt{2}$
- $\delta u(k) = -K\delta x(k) = \sqrt{2}\delta x(k)$



## ❖ Example 1:

$$x(k+1) = \frac{1}{2}x(k)^2 + x(k) - 2u(k)$$

- When at equilibrium points  $\bar{x} = \sqrt{2}$ : (**unstable**)

$$\delta x(k+1) = (1 + \sqrt{2})\delta x(k) - 2\delta u(k)$$

- Apply state feedback control law

- $\delta u(k) = -K\delta x(k) = \sqrt{2}\delta x(k)$

- As  $\delta u(k) = u(k) - \bar{u}$ ,  $\delta x(k) = x(k) - \bar{x}$ ,

$$u(k) - \bar{u} = -K(x(k) - \bar{x}), u(k) = -K(x(k) - \bar{x}) + \bar{u}$$

$$u(k) = \sqrt{2}x(k) - 1.5$$

- Matlab example

## ❖ Example 2:

$$x(k + 1) = \frac{1}{2}x(k)^2 + x(k) - 2u(k)$$

- When at equilibrium points  $\bar{x} = \sqrt{2}$ : (**unstable**)

$$\delta x(k + 1) = (1 + \sqrt{2})\delta x(k) - 2\delta u(k)$$

- Apply state feedback control law

$$\delta u(k) = -K\delta x(k)$$

$$\delta x(k + 1) = (A - KB)\delta x(k) = (1 + \sqrt{2} + 2K)\delta x(k)$$

- Let  $(A - KB) = 0$ , then  $u = ?$
- **Activity 3**

## ❖ Example 2:

$$x(k + 1) = \frac{1}{2}x(k)^2 + x(k) - 2u(k)$$

- When at equilibrium points  $\bar{x} = \sqrt{2}$ : (**unstable**)

$$\delta x(k + 1) = (1 + \sqrt{2})\delta x(k) - 2\delta u(k)$$

- Apply state feedback control law

$$\delta u(k) = -K\delta x(k)$$

$$\delta x(k + 1) = (A - KB)\delta x(k) = (1 + \sqrt{2} + 2K)\delta x(k)$$

- Let  $(A - KB) = 0$ , then  $K = -\frac{1+\sqrt{2}}{2}$

- $\delta u(k) = -K\delta x(k) = \frac{1+\sqrt{2}}{2}\delta x(k)$

## ❖ Example 2:

$$x(k+1) = \frac{1}{2}x(k)^2 + x(k) - 2u(k)$$

- When at equilibrium points  $\bar{x} = \sqrt{2}$ : (**unstable**)

$$\delta x(k+1) = (1 + \sqrt{2})\delta x(k) - 2\delta u(k)$$

- Apply state feedback control law

- $\delta u(k) = -K\delta x(k) = \frac{1+\sqrt{2}}{2}\delta x(k)$

- As  $\delta u(k) = u(k) - \bar{u}$ ,  $\delta x(k) = x(k) - \bar{x}$ ,

$$u(k) - \bar{u} = -K(x(k) - \bar{x}), u(k) = -K(x(k) - \bar{x}) + \bar{u}$$

$$u(k) = \frac{1 + \sqrt{2}}{2}x(k) - \frac{1 + \sqrt{2}}{2}$$

- Matlab example

## 41014 Sensors and Control for Mechatronic Systems

### Next Lectures: Integrating Sensor and Control

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# *THANK YOU*

## Questions?



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