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CENTRE FOR AUTONOMOUS SYSTEMS

41014 Sensors and Control for Mechatronic Systems

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0. Quiz 2

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- **15%**
- **40 mins**
- **Restrict open book: one hand writing A4 paper**

0. Lecture 6-8

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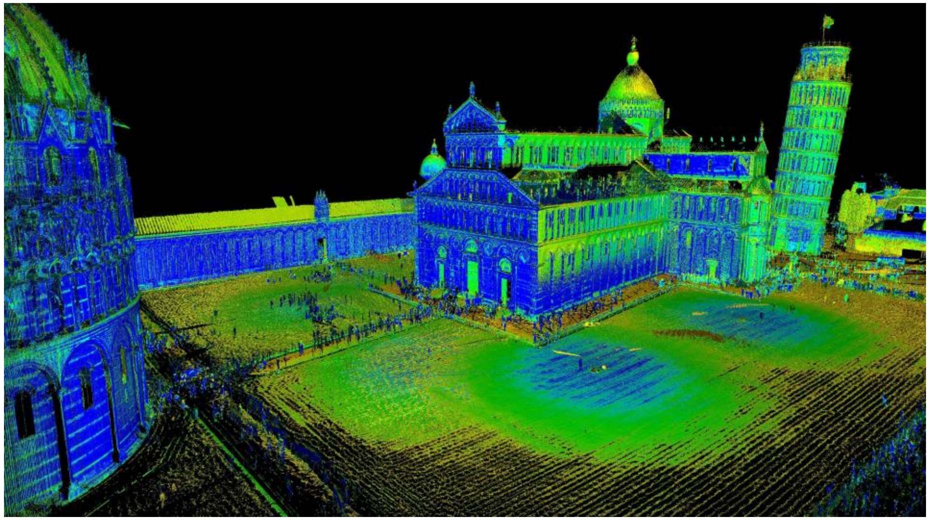
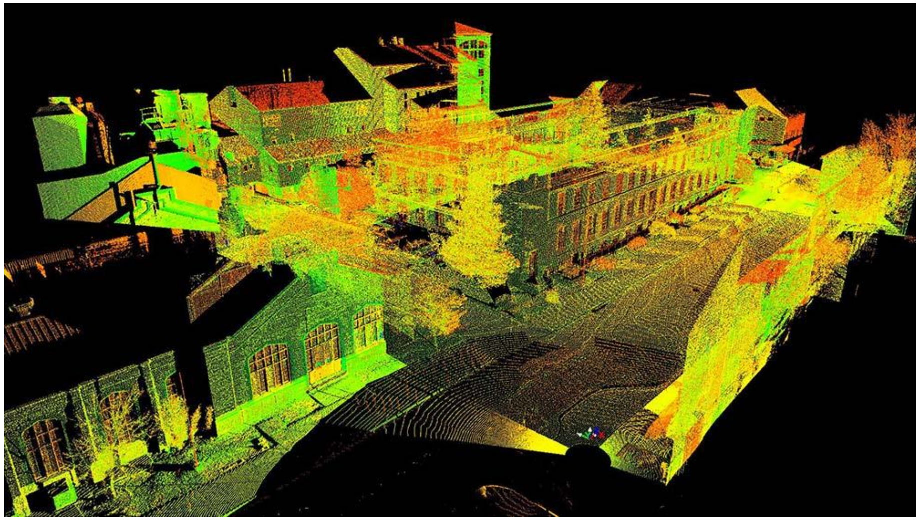
❖ Control:

- Linear Continuous-Time Systems
- Linear Discrete-Time Systems
- Nonlinear Discrete-Time Systems





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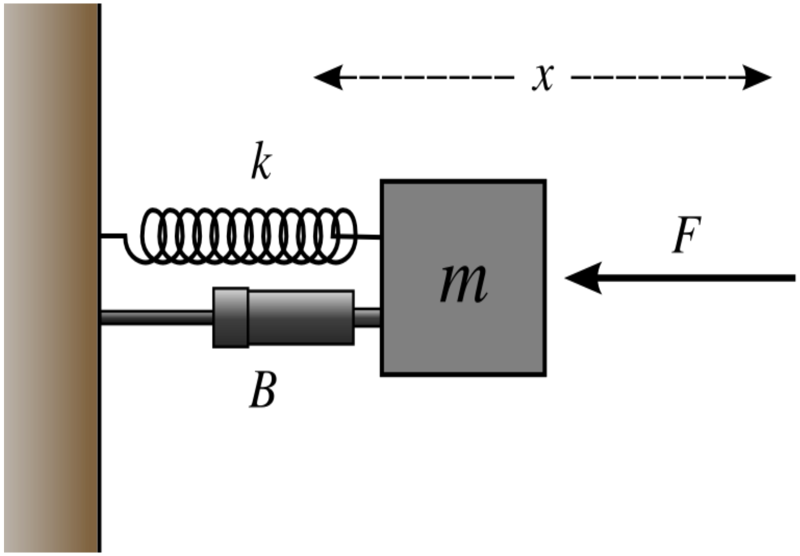
❖ Control Part 1:

- **Linear Continuous-Time Systems**
- Formulation
- Stability
- Pole Placement
- Controllability
- Linear quadratic optimal control



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41014 Sensors and Control for Mechatronic Systems

Lecture-7: Control Part 2
Linear Discrete-Time Systems

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0. Lecture-7

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Lecture:

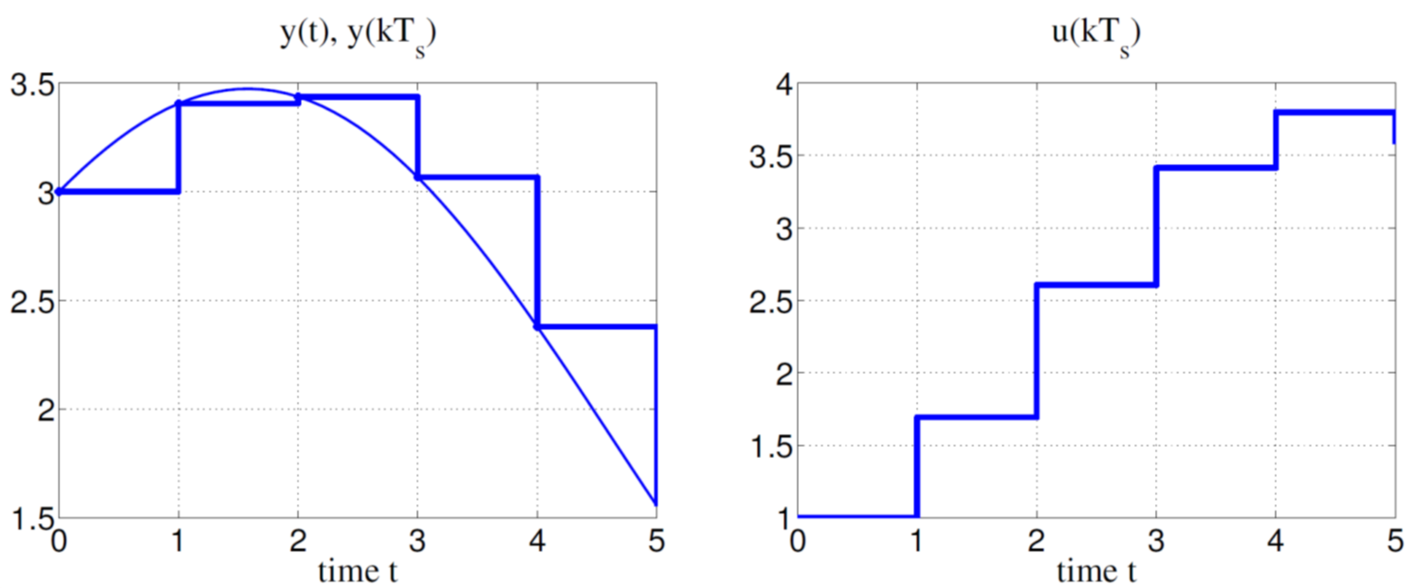
- Linear discrete-time Systems
- Discretization
- Stability
- State feedback control
- Optimal control

Active hands on:

- Examples
- Solve problem using Matlab

❖ Introduction:

- Discrete-time models describe relationships between **sampled** variables $x(kT_s)$, $u(kT_s)$, $y(kT_s)$, $k = 0, 1, \ldots$
- The value $x(kT_s)$ is kept **constant** during the sampling interval $[kT_s, (k + 1)T_s]$
- A discrete-time signal can either represent the sampling of a continuous-time signal, or be an intrinsically discrete



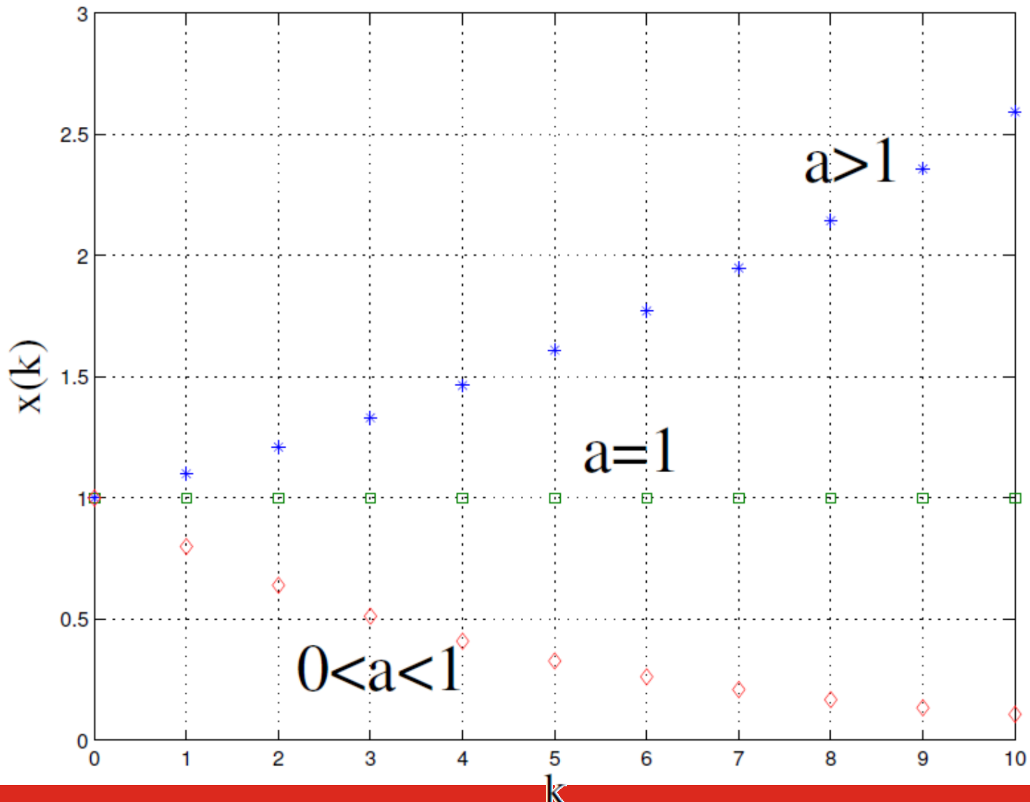
Difference equation:

- Consider the first order difference equation (autonomous system)

$$\begin{cases} \boldsymbol{x(k+1) = ax(k)} \\ \boldsymbol{x(0) = x_0} \end{cases}$$

- The solution

is $x(k) = a^k x_0$.



1. Linear Discrete-Time System

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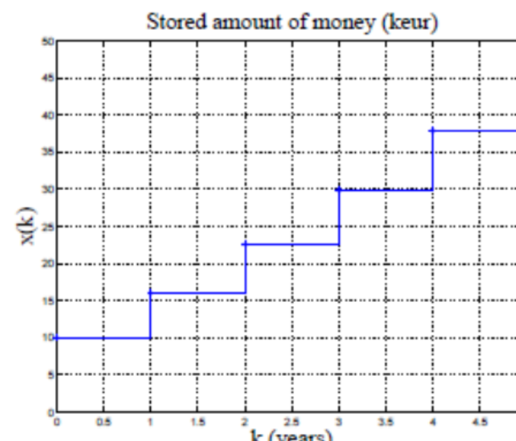
❖ Example - Wealth of a bank account:

- k: year counter; ρ: interest rate
- x(k): wealth at the beginning of year k
- u(k): money saved at the end of year k
- x0: initial wealth in bank account

Discrete-time model:

$$\begin{cases} x(k+1) &= (1+\rho)x(k)+u(k) \\ x(0) &= x_0 \end{cases}$$

x_0	10 k€
$u(k)$	5 k€
ρ	10 %



$$x(k) = (1.1)^k \cdot 10 + \frac{1-(1.1)^k}{1-1.1}5 = 60(1.1)^k - 50$$

1. Linear Discrete-Time System

Linear discrete-time system:

- Consider the set of n first-order linear difference equations forced by the input $u(k) \in R^n$

$$\begin{cases} x_1(k+1) &= a_{11}x_1(k) + \ldots + a_{1n}x_n(k) & +b_1u(k) \\ x_2(k+1) &= a_{21}x_1(k) + \ldots + a_{2n}x_n(k) & +b_2u(k) \\ \vdots & \vdots & \vdots \\ x_n(k+1) &= a_{n1}x_1(k) + \ldots + a_{nn}x_n(k) & +b_nu(k) \\ x_1(0) = x_{10}, \quad \ldots \quad x_n(0) = x_{n0} \end{cases}$$

- In compact matrix form:

$$\begin{cases} x(k+1) &= Ax(k) + Bu(k) \\ x(0) &= x_0 \end{cases}$$

where $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$.

1. Linear Discrete-Time System

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- Comparing to Linear continuous-time system:
  - Discrete-time system:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ x(0) = x_0 \end{cases}$$

  - Continuous-time system:

$$\boxed{\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \qquad x(t_0) = x_0 \\ y(t) &= Cx(t) + Du(t) \end{aligned}}$$

1. Linear Discrete-Time System

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- Comparing to Linear continuous-time system:

  - Continuous-time system:

$$\boxed{\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bu(t) \qquad x(t_0) = x_0 \\ y(t) & = & Cx(t) + Du(t) \end{array}}$$

  - Solution:

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

1. Linear Discrete-Time System

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- Linear discrete-time system:
  - In compact matrix form:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ x(0) = x_0 \end{cases}$$

  - The solution is

$$x(k) = \underbrace{A^k x_0}_{\text{natural response}} + \underbrace{\sum_{i=0}^{k-1} A^i B u(k-1-i)}_{\text{forced response}}$$

1. Linear Discrete-Time System

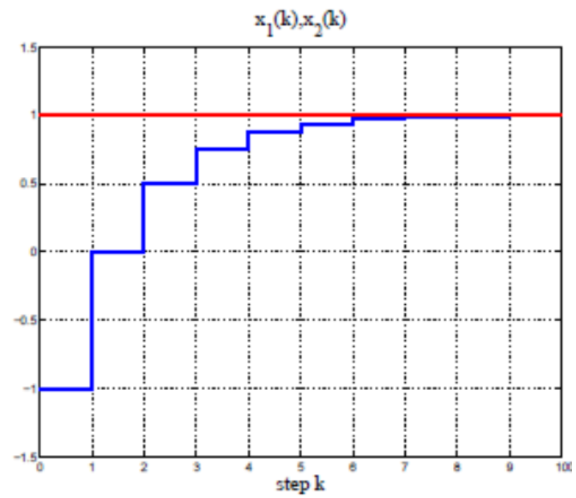
❖ Example:

- Consider the linear discrete-time system

$$\begin{cases} x_1(k+1) &= \frac{1}{2}x_1(k) + \frac{1}{2}x_2(k) \\ x_2(k+1) &= x_2(k) + u(k) \\ x_1(0) &= -1 \\ x_2(0) &= 1 \end{cases} \qquad \begin{cases} x(k+1) &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \\ x(0) &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{cases}$$

- Eigenvalues of A: $\lambda_1 = \frac{1}{2}, \lambda_2 = 1$. Solution:

$$\begin{aligned} x(k) &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix}^k \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \sum_{i=0}^{k-1} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix}^i \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k-1-i) \\ &= \begin{bmatrix} \frac{1}{2^k} & 1-\frac{1}{2^k} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \sum_{i=0}^{k-1} \begin{bmatrix} 1-\frac{1}{2^i} \\ 1 \end{bmatrix} u(k-1-i) \\ &= \underbrace{\begin{bmatrix} 1-\left(\frac{1}{2}\right)^{k-1} \\ 1 \end{bmatrix}}_{\text{natural response}} + \underbrace{\sum_{i=0}^{k-1} \begin{bmatrix} 1-\frac{1}{2^i} \\ 1 \end{bmatrix} u(k-1-i)}_{\text{forced response}} \end{aligned}$$



simulation for $u(k) \equiv 0$

1. Linear Discrete-Time System

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- Discrete-time linear system:

$$\begin{cases} \boldsymbol{x}(\boldsymbol{k}+\mathbf{1}) = \boldsymbol{A}\boldsymbol{x}(\boldsymbol{k}) + \boldsymbol{B}\boldsymbol{u}(\boldsymbol{k}) \\ \boldsymbol{y}(\boldsymbol{k}) = \boldsymbol{C}\boldsymbol{x}(\boldsymbol{k}) + \boldsymbol{D}\boldsymbol{u}(\boldsymbol{k}) \\ \boldsymbol{x}(\mathbf{0}) = \boldsymbol{x}_{\mathbf{0}} \end{cases}$$

  - Given the initial condition x(0) and the input sequence u(k), k∈N, it is possible to predict the entire sequence of states x(k) and outputs y(k)
  - The state x(0) summarizes all the past history of the system
  - The dimension n of the state x(k) ∈ R^n is called the order of the system
  - The system is called proper (or strictly causal) if D = 0
  - General multivariable case:

$$\begin{aligned} x(k) &\in \mathbb{R}^n \\ u(k) &\in \mathbb{R}^m \\ y(k) &\in \mathbb{R}^p \end{aligned} \qquad \begin{aligned} A &\in \mathbb{R}^{n\times n} \\ B &\in \mathbb{R}^{n\times m} \\ C &\in \mathbb{R}^{p\times n} \\ D &\in \mathbb{R}^{p\times m} \end{aligned}$$

1. Linear Discrete-Time System

- Activity-1: Student dynamics
- Problem Statement:
  - 3-years undergraduate course
  - percentages of students promoted, repeaters, and dropouts are roughly constant
  - direct enrollment in 2nd and 3rd academic year is not allowed
  - students cannot enroll for more than 3 years
- Notation:

k	Year
$x_i(k)$	Number of students enrolled in year i at year k, $i = 1, 2, 3$
$u(k)$	Number of freshmen at year k
$y(k)$	Number of graduates at year k
α_i	promotion rate during year i, $0 \leq \alpha_i \leq 1$
β_i	failure rate during year i, $0 \leq \beta_i \leq 1$

- Activity-1: Student dynamics
- 3rd-order linear discrete-time system:

$$\begin{cases} x_1(k+1) &= \beta_1 x_1(k) + u(k) \\ x_2(k+1) &= \alpha_1 x_1(k) + \beta_2 x_2(k) \\ x_3(k+1) &= \alpha_2 x_2(k) + \beta_3 x_3(k) \\ y(k) &= \alpha_3 x_3(k) \end{cases}$$

  - In matrix form

$$\begin{cases} x(k+1) &= \begin{bmatrix} \beta_1 & 0 & 0 \\ \alpha_1 & \beta_2 & 0 \\ 0 & \alpha_2 & \beta_3 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 0 & 0 & \alpha_3 \end{bmatrix} x(k) \end{cases}$$

1. Linear Discrete-Time System

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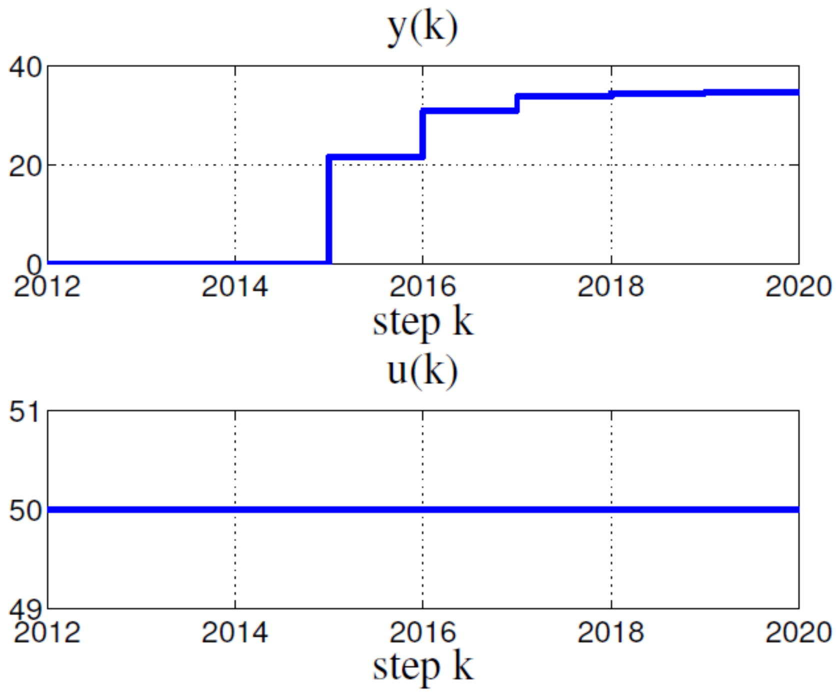
- Activity-1: Student dynamics

- Simulation and Solution:

$\alpha_1 = .60$	$\beta_1 = .20$
$\alpha_2 = .80$	$\beta_2 = .15$
$\alpha_3 = .90$	$\beta_3 = .08$

$$u(k) \equiv 50, k = 2012, \ldots$$



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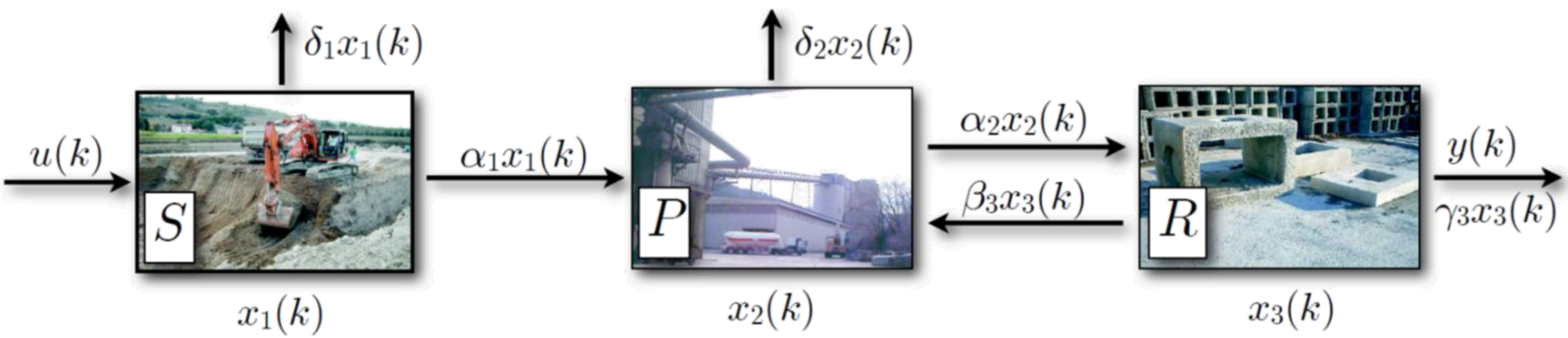
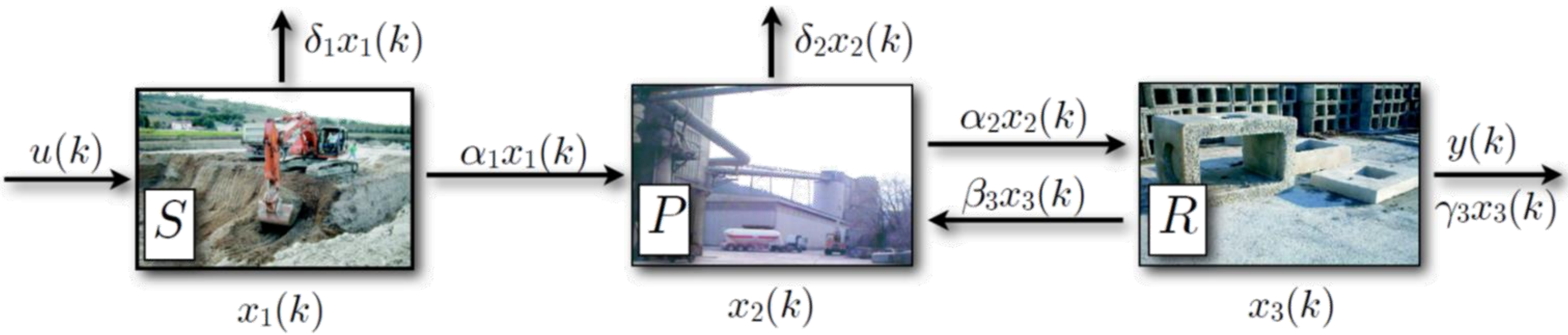
- Activity-2: Supply chain
- Problem Statement:
  - S purchases the quantity u(k) of raw material at each month k
  - A fraction δ_1 of raw material is discarded, a fraction α_1 is shipped to producer P
  - A fraction α_2 of product is sold by P to retailer R, a fraction δ_2 is discarded
  - retailer R returns a fraction β_3 of defective products every month, and sells a fraction γ_3 to customers



1. Linear Discrete-Time System

❖ Activity-2: Supply chain



$$
\begin{cases}
x_1(k+1) = (1-\alpha_1-\delta_1)x_1(k) + u(k) \\
x_2(k+1) = \alpha_1 x_1(k) + (1-\alpha_2-\delta_2)x_2(k) + \beta_3 x_3(k) \\
x_3(k+1) = \alpha_2 x_2(k) + (1-\beta_3-\gamma_3)x_3(k) \\
y(k) = \gamma_3 x_3(k)
\end{cases}
$$

k	month counter
$x_1(k)$	raw material in S
$x_2(k)$	products in P
$x_3(k)$	products in R
$y(k)$	products sold to customers

2. Discretization

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❖ Approximate sampling: Euler's method

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \qquad x(t_0) = x_0 \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ x(0) = x_0 \end{cases}$$

- Approximation

$$\dot{x}(kT_s) \approx \frac{x((k+1)T_s) - x(kT_s)}{T_s}$$

- Sampling

$$\dot{x}(t) = Ax(t) + Bu(t):$$

$$x((k+1)T_s) = (I + T_s A)x(kT_s) + T_s Bu(kT_s)$$

$$\bar{A} \triangleq I + AT_s, \quad \bar{B} \triangleq T_s B, \quad \bar{C} \triangleq C, \quad \bar{D} \triangleq D$$

2. Discretization

- Approximate sampling: Tustin's discretization method
  - Approximation by applying the trapezoidal rule

$$x(k+1) - x(k) = \int_{kT_s}^{(k+1)T_s} \dot{x}(t)dt = \int_{kT_s}^{(k+1)T_s} (Ax(t) + Bu(t))dt$$

$$\approx \frac{T_s}{2}(Ax(k) + Bu(k) + Ax(k+1) + Bu(k))$$ (trapezoidal rule)

  - Then

$$(I - \frac{T_s}{2}A)x(k+1) = (I + \frac{T_s}{2})x(k) + T_s Bu(k)$$

$$x(k+1) = \left(I - \frac{T_s}{2}A\right)^{-1}\left(I + \frac{T_s}{2}A\right)x(k) + \left(I - \frac{T_s}{2}A\right)^{-1} T_s Bu(k)$$

3. Stability

The continuous-time system

$$\dot{x}(t) = Ax(t) \qquad x(t_0) = x_0$$

is called asymptotically stable if for any initial state, the state x(t) converges to zero as t increases indefinitely.

Simple example 1

$$\dot{x} = -x, \quad x(0) = x_0$$

The solution is

$$x(t) = e^{-t}x_0$$

Simple example 2

$$\dot{x} = x, \quad x(0) = x_0$$

The solution is

$$x(t) = e^{t}x_0$$

Condition of stability: all the eigenvalues of A have negative real parts.

3. Stability

- Consider the first-order linear system

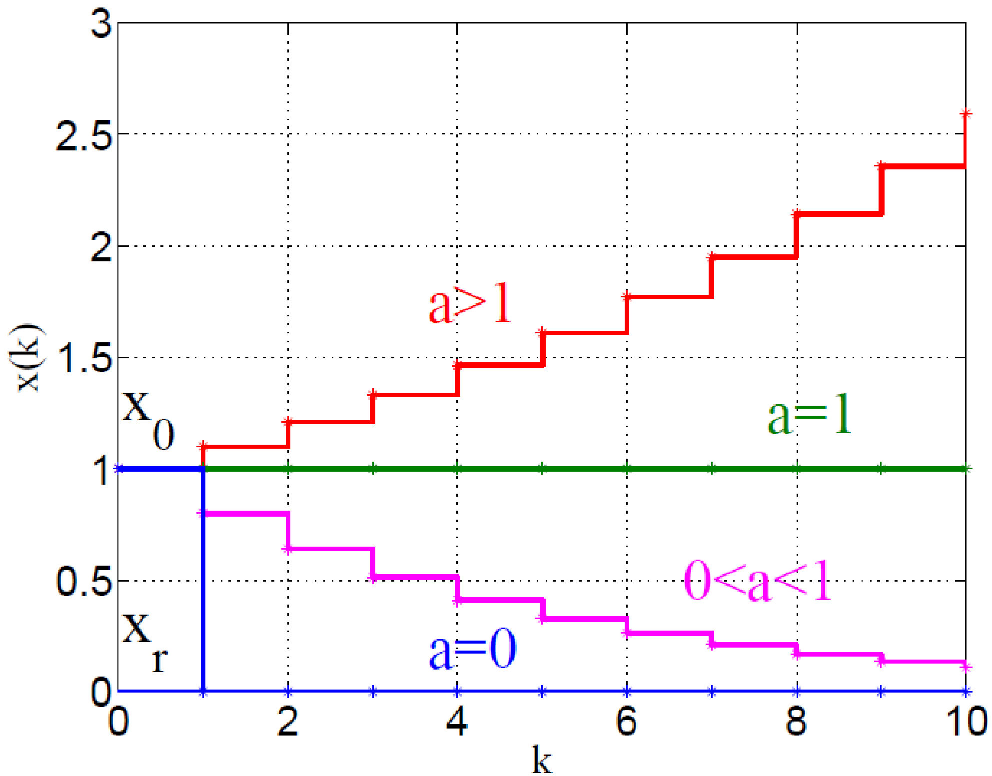
$$x(k+1) = ax(k) + bu(k)$$

  - For $u(k) \equiv 0, \forall k = 0,1, \ldots$ the solution is

$$x(k) = a^k x_0$$

  - The initial $x_0 = 1$ is
    - unstable if $|a| > 1$
    - stable if $|a| \leq 1$
    - asymptotically

    stable if $|a| < 1$



3. Stability

Stability of discrete-time linear systems

- Since the natural response of $x(k+1) = Ax(k) + Bu(k)$ is $x(k) = A^k x_0$ the stability properties depend only on A. We can therefore talk about system stability of a discrete-time linear system (A,B,C,D)

Theorem:

Let $\lambda_1, \ldots, \lambda_m$, $m \leq n$ be the eigenvalues of $A \in \mathbb{R}^{n \times n}$. The system $x(k+1) = Ax(k) + Bu(k)$ is

- asymptotically stable iff $|\lambda_i| < 1$, $\forall i = 1, \ldots, m$
- (marginally) stable if $|\lambda_i| \leq 1$, $\forall i = 1, \ldots, m$, and the eigenvalues with unit modulus have equal algebraic and geometric multiplicity [a]
- unstable if $\exists\ i$ such that $|\lambda_i| > 1$

[a] Algebraic multiplicity of λ_i = number of coincident roots λ_i of $\det(\lambda I - A)$. Geometric multiplicity of λ_i = number of linearly independent eigenvectors v_i, $Av_i = \lambda_i v_i$

- The stability properties of a discrete-time linear system only depend on the modulus of the eigenvalues of matrix A

3. Stability

❖ Activity-4:

$$\begin{cases} x(k+1) = \begin{bmatrix} 0 & 0 \\ 1 & \frac{1}{2} \end{bmatrix} x(k) \\ x(0) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \end{cases}$$

- Solution?
- Stability?

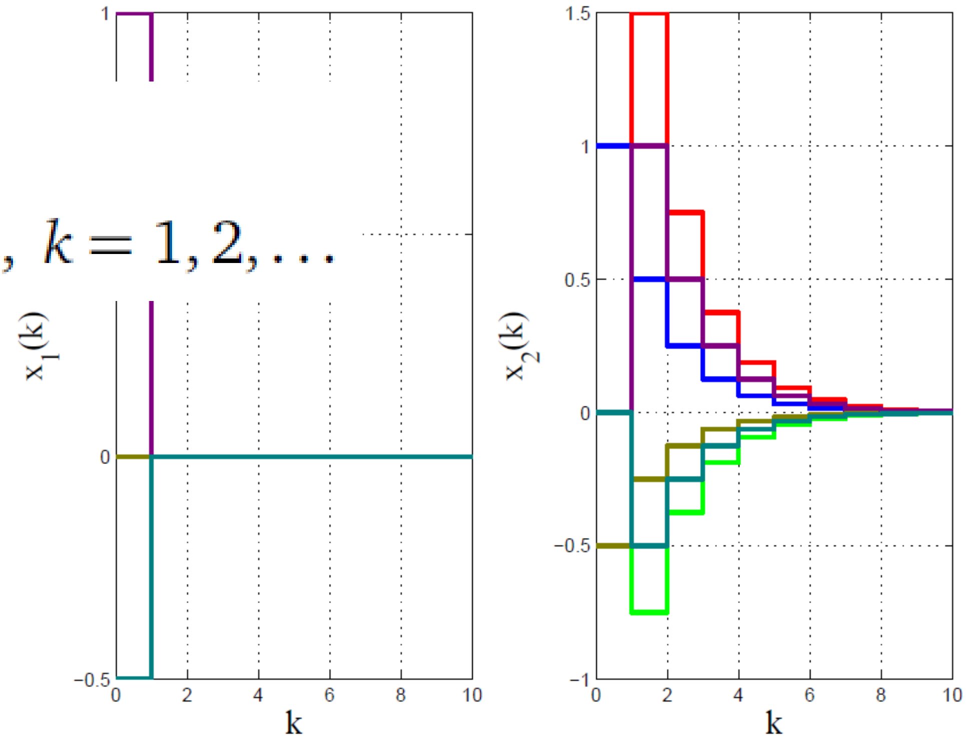
3. Stability

❖ Activity-4:

$$\begin{cases} x(k+1) = \begin{bmatrix} 0 & 0 \\ 1 & \frac{1}{2} \end{bmatrix} x(k) \\ x(0) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \end{cases}$$

- Solution?

$$\begin{cases} x_1(k) & = & 0,\ k = 1, 2, \ldots \\ x_2(k) & = & \left(\frac{1}{2}\right)^{k-1} x_{10} + \left(\frac{1}{2}\right)^{k} x_{20},\ k = 1, 2, \ldots \end{cases}$$



- Stability?
- Eigenvalues of *A*: {0,1/2}
- asymptotically stable

3. Stability

- Activity-5:

$$\begin{cases} x(k+1) = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} x(k) \\ x(0) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \end{cases}$$

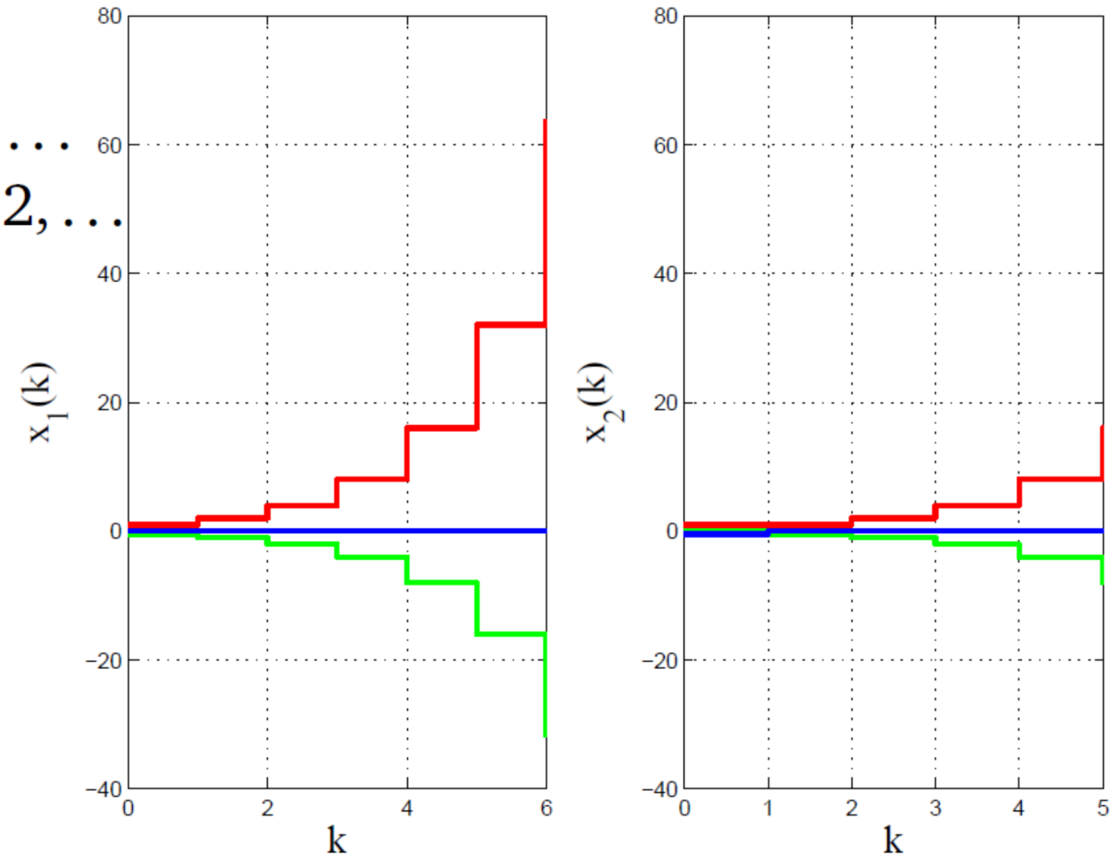
- Solution?
- Stability?

3. Stability

- Activity-5: $$\begin{cases} x(k+1) = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} x(k) \\ x(0) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \end{cases}$$

  - Solution?

$$\begin{cases} x_1(k) & = & 2^k x_{10},\ k = 0, 1, \ldots \\ x_2(k) & = & 2^{k-1} x_{10},\ k = 1, 2, \ldots \end{cases}$$



  - Stability?
  - Eigenvalues of *A*: {0,2}
  - unstable

3. Stability

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- Summary of stability conditions for linear systems

system		*continuous-time*	*discrete-time*
asympt. stable	$\forall i = 1, \ldots, n$	$\Re(\lambda_i) < 0$	$\|\lambda_i\| < 1$
unstable	$\exists i$ such that	$\Re(\lambda_i) > 0$	$\|\lambda_i\| > 1$
stable	$\forall i, \ldots, n$	$\Re(\lambda_i) \leq 0$	$\|\lambda_i\| \leq 1$
	and $\forall \lambda_i$ such that algebraic = geometric mult.	$\Re(\lambda_i) = 0$	$\|\lambda_i\| = 1$

4. Controllability

- Controllability:
  - *In order to be able to do whatever we want with the given dynamic system under control input, the system must be controllable.*
- Check for Controllability:
  - **Theorem:** The state space model

$$x(t+1) = Ax(t) + Bu(t),\ x(t) \in \mathbf{R}^n$$

  - is completely controllable if and only if the matrix

$$\mathcal{C}_t = \begin{bmatrix} B & AB & \cdots & A^{t-1}B \end{bmatrix}$$

  - has full row rank.

4. Controllability

- Activity-6:

$$x(t+1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

- Is the system controllable?

4. Controllability

- Activity-6:

$$x(t+1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

- Is the system controllable?

- Controllability Matrix

$$\mathcal{C} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

- System is not controllable.

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- Pole Placement:
  - Choose state feedback control law

$$u(k) = -Kx(k)$$

  - Then the closed-loop system becomes

$$x(k+1) = Ax(k) + Bu(k) = Ax(k) - BKx(t) = (A - BK)x(k)$$

  - We can design the closed-loop system to have good properties by assigning its poles --- the eigenvalues of matrix $A - BK$.

6. Optimal Control

- Linear continuous-time system

- Optimal Solution = Optimal K:

Solve P from the Riccati equation:

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

then let

$$K = R^{-1}B^T P$$

So the controller is

$$u = -R^{-1}B^T Px$$

and the minimal performance index

$$J = x^T(0)Px(0)$$

Optimal Control:

- A discrete-time linear system

$$x(k+1) = Ax(k) + Bu(k)$$

- Design state feedback control

$$u(k) = -Kx(k)$$

- Such that the ***Performance index***:

$$J(U) = \sum_{\tau=0}^{N-1} \left(x_\tau^T Q x_\tau + u_\tau^T R u_\tau \right) + x_N^T Q_f x_N$$

- is minimized.
- Where

$$U = (u_0, \ldots, u_{N-1})$$

$$Q = Q^T \geq 0, \qquad Q_f = Q_f^T \geq 0, \qquad R = R^T > 0$$

6. Optimal Control

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❖ Linear Quadratic Regulator (LQR):

- The algebraic Riccati equation (ARE)

$$P_{\mathrm{ss}} = Q + A^T P_{\mathrm{ss}} A - A^T P_{\mathrm{ss}} B (R + B^T P_{\mathrm{ss}} B)^{-1} B^T P_{\mathrm{ss}} A$$

- P can be found by iterating the Riccati recursion, or by direct methods

- LQR optimal input is approximately a linear, constant state feedback

$$u_t = K_{\mathrm{ss}} x_t, \qquad K_{\mathrm{ss}} = -(R + B^T P_{\mathrm{ss}} B)^{-1} B^T P_{\mathrm{ss}} A$$

- It is very widely used in practice.

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Next Lectures: Control Part 3
Nonlinear Discrete-Time Systems

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THANK YOU

Questions?