OPTIMIZATION. HOMEWORK 3

OSCAR DALMAU

(1) Is the set $S = \{a \in \mathbb{R}^k | \ p(0) = 1, \ |p(t)| \leq 1 \text{ for } t \in [\alpha, \beta]\}$, where
$$p(t) \ = \ a_1 + a_2 t + \cdots + a_k t^{k-1},$$
convex?

(2) Suppose f is convex, $\lambda_1 > 0$ and $\lambda_2 \leq 0$ with $\lambda_1 + \lambda_2 = 1$, and let $x_1, \ x_2 \in \operatorname{dom} f$. Show that the inequality
$$f(\lambda_1 x_1 + \lambda_2 x_2) \ \geq \lambda_1 f(x_1) + \lambda_2 f(x_2)$$
always holds.

(3) Show that the following function $f : \mathbb{R}^n \to \mathbb{R}$ is convex.
$$f(x) \ = \ -\exp(-g(x))$$
where $g : \mathbb{R}^n \to \mathbb{R}$ has a convex domain and satisfies
$$\begin{bmatrix} \nabla^2 g(x) & \nabla g(x) \\ \nabla^T g(x) & 1 \end{bmatrix} \ \succeq \ 0$$
for $x \in \operatorname{dom} g$.

(4) Show that $f(x, y) = x^2/y, \ y > 0$ is convex.

(5) Consider the function $f(x_1, x_2) = (x_1 + x_2^2)^2$. At the point $x^T = [1, 0]$ we consider the search direction $p^T = [-1, 1]$. Show that p is a descent direction and find all minimizers of the function.

(6) Find all the values of the parameter a such that $[1, 0]^T$ is the minimizer or maximizer of the function
$$f(x_1, x_2) \ = \ a^3 x_1 e^{x_2} + 2a^2 \log(x_1 + x_2) - (a + 2)x_1 + 8a x_2 + 16 x_1 x_2.$$

(7) Consider the sequence $x_k = 1 + 1/k!, \ k = 0, 1, \cdots$. Does this sequence converge linearly to 1? Justify your response.

(8) Show that $f(\boldsymbol{x}) = \log \sum_{i=1}^n \exp(x_i)$ is convex.

(9) Show that $f(x) = \log \sum_{i=1}^n \exp(g_i(x)) : \mathbb{R} \to \mathbb{R}$ is convex if $g_i : \mathbb{R} \to \mathbb{R}$ are convex

(10) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. Show that f is convex over a nonempty convex set C if and only if
$$(\nabla f(x) - \nabla f(y))^T (x - y) \ \geq \ 0, \ \forall x, y \in C$$
Note: the proof we have is only for the case ($\Rightarrow$)