Homework 2

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Exercise 1

To prove that $f(z) = e^{\bar{z}}$ is not differentiable anywhere, we can show that the partial derivates for Cauchy-Reimann to hold are not equal. We can start by expanding out the function as follows

$$\begin{aligned} e^{\bar{z}} &= e^x * e^{-iy} \\ &= e^x(cos(y) + isin(-y) \\ &= e^x(cos(y) - isin(y)) \\ &= e^x cos(y) - e^x isin(y) \end{aligned}$$

From here, we can get the following equaitons

$$\begin{aligned} u &= e^x cos(y) \\ v &= -e^x sin(y) \end{aligned}$$

We can then use u and v for the partial derivatives as follows

$$\begin{aligned} \frac{\partial u}{\partial x} &= e^x cos(y) \\ \frac{\partial v}{\partial y} &= -e^x sin(y) \end{aligned}$$

Thus, we see that the partial derivatives are not equal and thus the function is not analytical anywhere

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Exercise 2

The definition for the principal log is as follows

$$Log(z) = ln(|z|) + iarg(z)$$

Thus, we can use both i^3 and i to see if the results differ. First, we can start with $Log(i^3)$ as follows.

$$\begin{aligned} Log(i^3) &= ln(|i^3|) + iarg(i^3) \\ &= ln(i) - \frac{3i\pi}{2} \end{aligned}$$

We can now try with $3Log(i)$ as follows

$$\begin{aligned} 3Log(i) &= 3(ln(|i|) + iarg(i)) \\ &= 3ln(i) + \frac{3i\pi}{2} \end{aligned}$$

Therefore, we can see that

$$3ln(i) + \frac{3i\pi}{2} \neq ln(i) - \frac{3i\pi}{2}$$

and therefore we cannot factor out the exponent as with regular log functions for non-complex functions.

Exercise 3

a)

To begin, we want to prove $sin(z) = sin(x+iy) = sin(x)cosh(y)+icos(x)sinh(y)$ and to do this we can use the following identities

$$\begin{aligned}
&sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz}) \\
&sinh(y) = \frac{1}{2}(e^{y} - e^{-y}) \\
&cosh(y) = \frac{1}{2}(e^{y} + e^{y}) \\
&e^{ix} = cos(x) + isin(x)
\end{aligned} \tag{1}$$

We can expand out sin(z) using (1) and then expand out z as $z = x + iy$ to get the following equation

$$\frac{1}{2i}(e^{i(x+iy)} - e^{-i(x+iy)}$$

From here, we can begin to factor and combine like terms as follows

$$\begin{aligned}
&\frac{1}{2i}(e^{ix}e^{-y} - e^{-ix}e^{y}) \\
&\frac{1}{2i}([cos(x) + isin(x)]e^{-y} - [cos(x) - isin(x)]e^{y}) \\
&(\frac{e^{-y} - e^{y}}{2i})cos(x) + isin(x)(\frac{e^{y} + e^{-y}}{2}) \\
&sin(x)cosh(y) + icos(x)sinh(y)
\end{aligned}$$

b)

We now want to prove that $|sin(z)|^2 = sin^2(x) + sinh^2(y)$. To begin, we can expand out sin(z) using the previously found identity

$$\begin{aligned}|sin(z)|^2 &= |sin(x)cosh(y) + icos(x)sinh(y)|^2\\ &= |sin(x)cosh(y)|^2 + |cos(x)sinh(y)|^2\\ &= sin^2(x)cosh(y)^2 + cos^2(x)sinh^2(y)\\ &= sin^2(x)cosh^2(y) + cos^2(x)(cosh^2(y) - 1)\\ &= cosh^2(y) - (1 - sin^2(x))\\ &= (sinh^2(y) + 1) - (1 - sin^2(y))\\ &= sin^2(x) + sinh^2(y)\end{aligned}$$

c)

If we want to find the zeroes of the complex function sin(z) which is in the form of $sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz}$, we want to show that $sin(z) = 0$. This is only possible when $e^{iz} - e^{-iz} = 0$ and thus we need to see where $e^{iz} = e^{-iz}$ and thus $e^{2iz} = 1$ For this to be the case, we must have the following constraints $z = n\pi, n \in \mathbb{Z}$

Exercise 4

a)

To find that $sin(\bar{z})$ is not analytic anywhere, we can use the identity found in exercise 3 as follows

$$\begin{aligned}sin(\bar{z}) &= sin(x)cosh(y) - icos(x)sinh(y)\\ u &= sin(x)cosh(y)\\ v &= -cos(x)sinh(y)\end{aligned}$$

We can then use Cauchy-Reimann as follows

$$\begin{aligned}\frac{\partial u}{\partial x} &= cos(x)cosh(y)\\ \frac{\partial v}{\partial y} &= -cos(x)cosh(y)\end{aligned}$$

We can therefore see that the two partial derivates are not equal and therefore Cauchy-Reimann does not hold

b)

A similar identity for cos(z) exists $cos(z) = cos(x)cosh(y) - isin(x)sinh(y)$ which we can use to find the conjugate and take partial derivates for

$$
\begin{aligned}
cos(\bar{z}) &= cos(x)cosh(y) + isin(x)sinh(y) \\
u &= cos(x)cosh(y) \\
v &= sin(x)sinh(y)
\end{aligned}
$$

Taking the partials yields the following results

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= -sin(x)cosh(y) \\
\frac{\partial v}{\partial y} &= sin(x)cosh(y)
\end{aligned}
$$

And thus the partials are once again not equal therefore $cos(\bar{z})$ is not analytical anywhere

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Exercise 5

We are told to use the following definitions

$$|\sin z|^2 = \sin^2 x + \sinh^2 y, \quad (15)$$

$$|\cos z|^2 = \cos^2 x + \sinh^2 y. \quad (16)$$

to show that $|sin(z)| \geq |sin(x)|$ and $|cos(z)| \geq |cos(x)|$

a)

We can use (15) to show that $|sin(z)|^2$ must be greater than or equal to $|sin(x)|^2$ by using modulus properties. Using modulus properties, we can rewrite both funtions as follows

$$
\begin{aligned}
|sin(z)|^2 &= Re(sin(z))^2 + Im(sin(z))^2 \\
|sin(x)|^2 &= Re(sin(x))^2 + Im(sin(x))^2
\end{aligned}
$$

Using identity 15 and the fact that there is no imaginary part of sin(x), we can rewrite as follows

$$
\begin{aligned}
|sin(z)|^2 &= sin^2(x) + sinh^2(y) \\
|sin(x)|^2 &= sin^2(x)
\end{aligned}
$$

Because we know that $sinh^2(y) \geq 0$, we can therefore state that $|sin(z)|^2 \geq |sin(x)|^2$ and therefore $|sin(z)| \geq |sin(x)|$

b)

We can now begin to prove $|cos(z)| \geq |cos(x)|$ using similar logic as follows.

$$|cos(z)|^2 = cos^2(x) + sinh^2(y)$$
$$|cos(x)|^2 = cos^2(x)$$

We can apply the same logic as before that $cosh^2(y) \geq 0$ because the function is squared and therefore it follows that $|cos(z)|^2 \geq |cos(x)|^2$ thus $|cos(z)| \geq |cos(x)|$

Exercise 6

a)

Given: $|sinh(y)| \leq |sin(z)| \leq |cosh(y)|$ Proof:
We are given that $|sin(z)|^2 = sin^2(x) + sinh^2(y)$ and similar to the last proof we can see that because $|sin(z)|^2$ includes $sinh^2(y)$, it must be greather than or equal to $sinh^2(y)$ because $sin^2(x)$ is positive everywhere. We now must use the modulus definition as follows

$$\begin{aligned} |sin(z)|^2 &= sin^2(x) + sinh^2(y) \\ &= sin^2(x) + cosh^2(y) - 1 \\ &= cosh^2(y) - (1 - sin^2(x)) \\ &= cosh^2(y) - cos^2(x) \end{aligned}$$

We know that $cos^2(x) \geq 0$ and therefore we know that $cosh^2(y)$ must be greater than $|sin(z)|^2$ for this inequality to hold. Therefore, we can combine all three parts to get

$$|sinh(y)| \leq |sin(z)|^2 \leq cosh(y)$$

b)

Given: $|sinh(y)| \leq |cos(z)| \leq |cosh(y)|$ Proof:
We are given that $|cos(z)|^2 = cos^2(x) + sinh^2(y)$ and therefore we can once again see that $|cos(z)|$ will be larger as $|cos(z)|^2 \geq sinh^2(y)$. We can also show that $cosh(y)$ as follows

$$\begin{aligned} |cos(z)|^2 &= cos^2(x) + sinh^2(y) \\ &= cos^2(x) + cosh^2(y) - 1 \\ &= cosh^2(y) - (1 - cos^2(x)) \\ &= cosh^2(y) - sin^2(x) \end{aligned}$$

We once again see that for this inequality to hold, $cosh^2(y)$ must be larger since $sin^2(x) \geq 0$. Therefore, we get

$$|sinh(y)| \leq |cos(z)| \leq |cosh(y)|$$

Exercise 7

a)

Identity: $-isinh(iz) = sin(z)$
Proof:

$$\begin{aligned}
-isinh(z) &= \frac{-i}{2}(e^{iz} - e^{-iz}) \\
&= \frac{-i}{2}([cos(z) + isin(z)] - [cos(z) - isin(z)]) \\
&= \frac{-i}{2}(2isin(z)) \\
&= sin(z)
\end{aligned}$$

b)

Identity: $-isin(iz) = sinh(z)$
Proof:

$$\begin{aligned}
-isin(iz) &= \frac{i}{2i}(e^{i^2 z} - e^{-i^2 z}) \\
&= \frac{-1}{2}(e^{-z} - e^{z}) \\
&= \frac{1}{2}(e^{z} - e^{-z}) \\
&= sinh(z)
\end{aligned}$$

c)

Identity: $cos(iz) = cosh(z)$
Proof:

$$\begin{aligned}
cos(iz) &= \frac{1}{2}(e^{i^2 z} + e^{-i^2 z}) \\
&= \frac{1}{2}(e^{z} + e^{-z}) \qquad = cosh(z)
\end{aligned}$$

Exercise 8

a)

Identity: $sinh(z) = sinh(x)cos(y) + icosh(x)sin(y)$
Proof:

$$\begin{aligned}
sinh(z) &= \frac{1}{2}(e^z - e^{-z}) \\
&= \frac{1}{2}(e^x e^{iy} - e^{-x}e^{-iy}) \\
&= \frac{1}{2}(e^x(cos(y) + isin(y)) - e^{-x}(cos(y) - isin(y))) \\
&= \frac{1}{2}((e^x - e^{-x})cos(y) + (e^x + e^{-x})isin(y)) \\
&= \frac{(e^x - e^{-x})cos(y)}{2} + \frac{(e^x + e^{-x})isin(y)}{2} \\
&= sinh(x)cos(y) + icosh(x)sin(y)
\end{aligned}$$

b)

Identity: $cosh(z) = cosh(x)cos(y) + isinh(x)sin(y)$
Proof:

$$\begin{aligned}
cosh(z) &= \frac{1}{2}(e^z + e^{-z}) \\
&= \frac{1}{2}(e^x e^{iy} + e^{-x}e^{-iy}) \\
&= \frac{1}{2}(e^x(cos(y) + isin(y)) + e^{-x}(cos(y) - isin(y))) \\
&= \frac{1}{2}((e^x + e^{-x})cos(y) + (e^x - e^{-x})isin(y)) \\
&= \frac{(e^x + e^{-x})cos(y)}{2} + \frac{(e^x - e^{-x})isin(y)}{2} \\
&= cosh(x)cos(y) + isinh(x)sin(y)
\end{aligned}$$

c)

Identity: $|sinh(z)|^2 = sinh^2(x) + sin^2(x)$
Useful Identity: $|z|^2 = Re(z)^2 + Im(z)^2$ Proof:

$$
\begin{aligned}
|sinh(z)|^2 &= (sinh(x)cos(y))^2 + (cosh(x)sin(y))^2 \\
&= sinh^2(x)cos^2(y) + cosh^2(x)sin^2(y) \\
&= (cosh^2(x) - 1)cos^2(y) + sin^2(y)cosh^2(x) \\
&= cosh^2(x)(cos^2(y) + sin^2(x)) - cos^2(y) \\
&= cosh^2(x) - cos^2(y) \\
&= sinh^2(x) + (1 - cos^2(y)) \\
&= sinh^2(x) + sin^2(y)
\end{aligned}
$$

d)

Identity: $|cosh(z)|^2 = sinh^2(x) + cos^2(y)$
Proof:

$$
\begin{aligned}
|cosh(z)|^2 &= (cosh(x)cos(y))^2 + (sinh(x)sin(y))^2 \\
&= coshh^2(x)cos^2(y) + sinh^2(x)sin^2(y) \\
&= (sinh^2(x) + 1)cos^2(y) + sin^2(y)sinh^2(x) \\
&= sinh^2(x)(sin^2(y) + sin^2(y)) + cos^2(y) \\
&= sinh^2(x) + cos^2(y)
\end{aligned}
$$

Exercise 9

To take the integral of a complex function, we must use the following form

$$\int_a^b u(t) + iv(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$$

a)

Problem: $\int_0^1 (1+it)^2 dt$ Solution:

$$\begin{aligned}
\int_0^1 (1+it)^2 dt &= \int_0^1 1 + 2it - t^2 dt \\
&= \int_0^1 1 - t^2 dt + i \int_0^1 2t dt \\
&= \left[t - \frac{t^3}{3} \right]_0^1 + i \left[\frac{2t^2}{2} \right]_0^1 \\
&= \frac{2}{3} + i
\end{aligned}$$

b)

Problem: $\int_1^2 (\frac{1}{t} - i)^2 dt$ Solution:

$$\begin{aligned}
\int_1^2 (\frac{1}{t} - i)^2 dt &= \int_1^2 \frac{1}{t^2} - 1 - \frac{2}{t} \\
&= \int_1^2 \frac{1}{t^2} - 1 dt + i \int_1^2 -\frac{2}{t} dt \\
&= \left[-(\frac{1}{t} + t) \right]_1^2 - i \left[ln(t^2) \right]_1^2 \\
&= -\frac{1}{2} - iln(4)
\end{aligned}$$

c)

Problem: $\int_0^\pi e^{i2t} dt$ Solution:

$$\begin{aligned}
\int_0^\pi e^{i2t} dt &= \int_0^\pi cos(2t) + isin(2t) dt \\
&= \int_0^\pi cos(2t) dt + i \int_0^\pi sin(2t) dt \\
&= \left[\frac{sin(2t)}{2} \right]_0^{\pi/6} - i \left[\frac{cos(2t)}{2} \right]_0^{\pi/6} \\
&= \frac{\sqrt{3}}{4} + \frac{i}{4}
\end{aligned}$$

d)

Problem: $\int_0^\infty e^{-zt}dt$ Solution:

$$\begin{aligned}
\int_0^\infty e^{-zt}dt &= \left[-\frac{e^{-zt}}{z}\right]_0^\infty \\
&= \frac{-e^{-\infty}}{z} + \frac{e^0}{z} \\
&= \frac{1}{z}
\end{aligned}$$

Exercise 10

To prove the given conjecture, we need to examine the integral when $m = n$ and when $m \neq n$

m=n

$$\begin{aligned}
\int_0^{2\pi} e^{im\theta}e^{-im\theta}d\theta &= \int_0^{2\pi} e^{im\theta-im\theta}d\theta \\
&= \int_0^{2\pi} e^0 d\theta \\
&= \int_0^{2\pi} 1 d\theta \\
&= [\theta]_0^{2\pi} \\
&= 2\pi
\end{aligned}$$

m not equal n

$$
\begin{aligned}
\int_0^{2\pi} e^{im\theta}e^{-in\theta}d\theta &= \int_0^{2\pi} e^{im\theta-in\theta}d\theta \\
&= \int_0^{2\pi} e^{i(m-n)\theta}d\theta \\
&= \left[\frac{e^{i(m-n)\theta}}{i(m-n)}\right]_0^{2\pi} \\
&= \frac{1}{1(m-n)}\left[cos((m-n)\theta)+isin((m-n)\theta)\right]_0^{2\pi} \\
&= \frac{1}{1(m-n)}(cos((m-n)2\pi)+isin((m-n)2\pi)-cos(0)-isin(0)) \\
&= \frac{1}{1(m-n)}(1+0-1-0) \\
&= 0
\end{aligned}
$$

Thus we have shown the conjecture to hold