

Unclustered BWTs of any Length over Non-Binary Alphabets

WCTA 2025

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September 11th, 2025



Outline

The Burrows-Wheeler Transform

Unclustered BWT: Previous results

De Bruijn words

The ternary case: existence

Extension to all alphabets and lengths

The ternary case: lower bounds

Looking back at the binary case

BWT: definition

$$u = 0100111$$

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$$\left(\begin{array}{c} 0100111 \\ 1010011 \\ 1101001 \\ 1110100 \\ 0111010 \\ 0011101 \\ 1001110 \end{array} \right)$$

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$$\text{BWT}(u) = 1100110$$

BWT and clusterization

The BWT is widely used in data compression because of its *clustering effect*

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Totally unclustered BWTs

We study the following problem:

Problem

Let $k \leq n$ be a pair of integers. Is there a word w of length n over k letters such that $\text{BWT}(w)$ has no run longer than 1?

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(Note: We can already ignore words that are not primitive, because $\text{BWT}(u^r)$ always have runs of length $\geq r$.)

Motivation: This is the worst case of BWT clustering. Great attention has been given to the best case, in which the BWT produces the fewest number of clusters. (Simpson and Puglisi, 2008; Restivo and Rosone, 2009; Mantaci et al., 2003; Ferenczi and Zamboni, 2013).

Inverse standard permutation: definition

Definition

The *inverse standard permutation* (or *FL mapping*) of a word s is the permutation π_s^{-1} defined by $\pi_s^{-1}(i) = p$ if p is the position of the letter with lexicographical rank p in s .

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Example

Let $w = 12111200\textcolor{red}{0}0$. Then, we have:

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Let $w = 12\mathbf{1}1120000$. Then, we have:

$$\pi_w^{-1} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 7 & 8 & 9 & 0 & 2 & & & & \end{pmatrix}$$

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Example

Let $w = 121\textcolor{red}{1}120000$. Then, we have:

$$\pi_w^{-1} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 7 & 8 & 9 & 0 & 2 & 3 & & & \end{pmatrix}$$

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Let $w = 1211\mathbf{1}20000$. Then, we have:

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Example

Let $w = 12111\textcolor{red}{2}0000$. Then, we have:

$$\pi_w^{-1} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 7 & 8 & 9 & 0 & 2 & 3 & 4 & 1 & 5 \end{pmatrix}$$

Inverse standard permutation and BWT

Lemma (Folklore)

A n -length word w is a BWT of some primitive word u if and only if π_u^{-1} is a cyclic permutation.

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Example

Let $w = 1211120000$. Then, we have:

$$\pi_w^{-1} = \begin{pmatrix} 0 & \color{red}{1} & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & \color{red}{7} & 8 & 9 & 0 & 2 & 3 & 4 & 1 & 5 \end{pmatrix}$$

$$\pi_w^{-1} = (0\ 6\ 3\ 9\ 5\ 2\ 8\ 1\ 7)$$

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Lemma (Folklore)

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Hence, w is the BWT of a word ($w = \text{BWT}(1010210201)$).

Inverse standard permutation and BWT

Example

Let $w = 210102$. Then, we have:

$$\pi_w^{-1} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 3 & 0 & 5 \end{pmatrix}$$

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Example

Let $w = 210102$. Then, we have:

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Hence, w is *not* the BWT of any word u .

Problem reformulation

We can reformulate our problem as follows:

Problem

Let $k \leq n$ be a pair of integers. Is there a word u of length n over k letters such that (i) π_u^{-1} is a cycle, and (ii) u has no run longer than 1?

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The binary case: Mantaci et al., 2017

Theorem (Mantaci et al., 2017)

There exists a $2n$ -length word with totally unclustered BWT if and only if $2n + 1$ is an odd prime and 2 generates the cyclic group \mathbb{Z}_{2n+1}^ .*

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Sketch.

A binary word starting with 0 or ending with 1 is not a BWT image.

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Our problem: what happens after the binary case?

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Lemma

For every $k \leq n$, there exists a deBruijn word of order n over Σ_k and it has length k^n .

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Theorem (Higgins, 2012)

A word u of length k^n is a deBruijn word if and only if $w = \text{BWT}(u)$ can be written as a concatenation of permutations of the alphabet.

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Example

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Example

$$u = 001022112$$

$$\text{BWT}(u) = 201021120$$

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Example

$$u = 001022112$$

$$\text{BWT}(u) = 201 \cdot 021 \cdot 120$$

De Bruijn words and unclustered BWT

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▶ $\text{BWT}(u) = 201 \cdot 021 \cdot 120$

De Bruijn words and unclustered BWT

DeBruijn words are promising, but 2 things remain to solve:

1. What about arbitrary lengths?
2. de Bruijn words do not *exactly* have totally unclustered BWT.

► $\text{BWT}(u) = 201 \cdot 02\mathbf{1} \cdot \mathbf{1}20$

DeBruijn words and graphs, revisited

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Length $n = 3$

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$$V = \{000, 001, 010, 011, 100, 101, 110, 111\}$$

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$$E = \{\text{pairs of strings with length } n - 1 = 2 \text{ overlap} \}$$

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100
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101

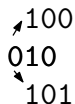
DeBruijn words and graphs, revisited

Alphabet size $k = 2$

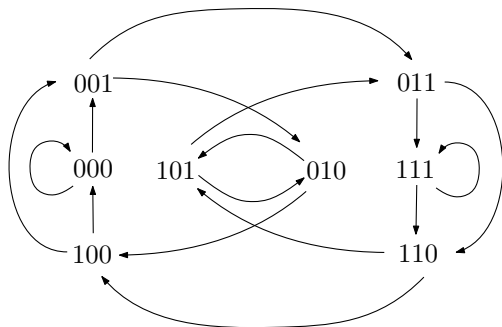
Length $n = 3$

$$V = \{000, 001, 010, 011, 100, 101, 110, 111\}$$

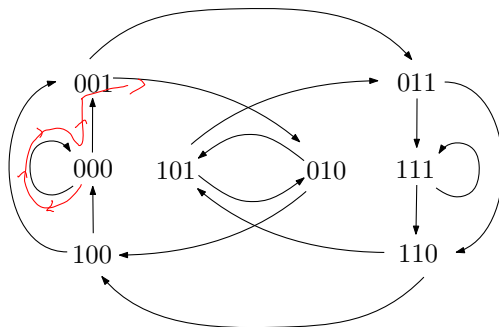
$$E = \{\text{pairs of strings with length } n - 1 = 2 \text{ overlap} \}$$



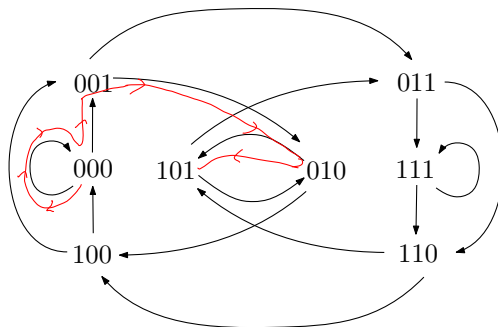
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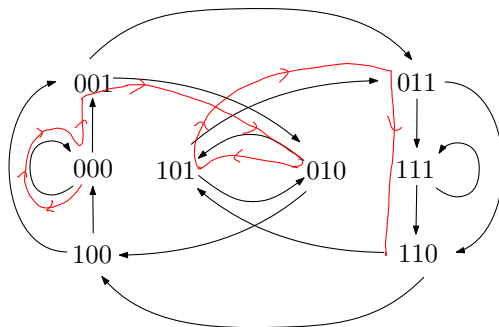
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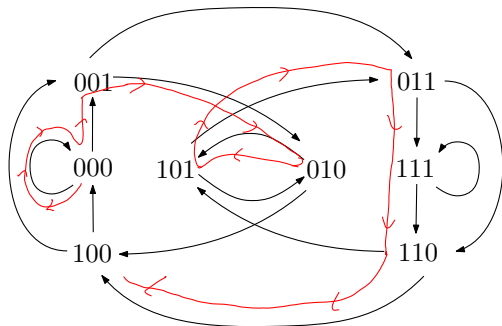
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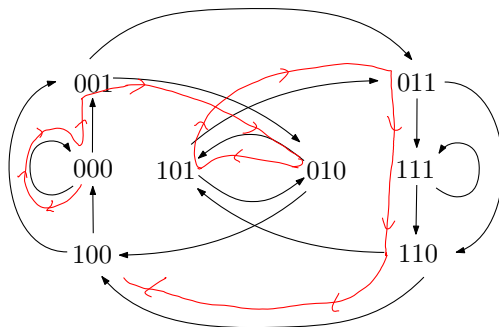
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DeBruijn words and graphs, revisited



DeBruijn words and graphs, revisited



De Bruijn Word $w = 0001011100$

DeBruijn words and graphs, revisited

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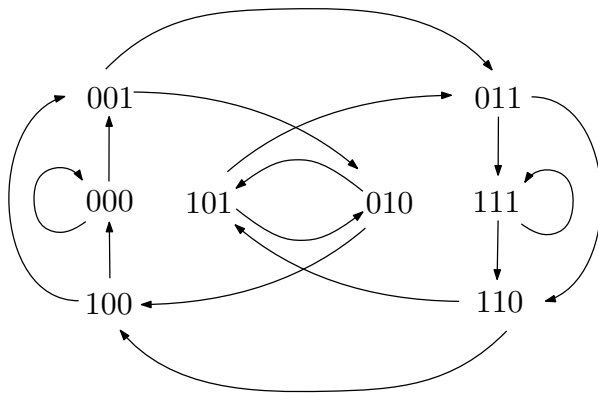
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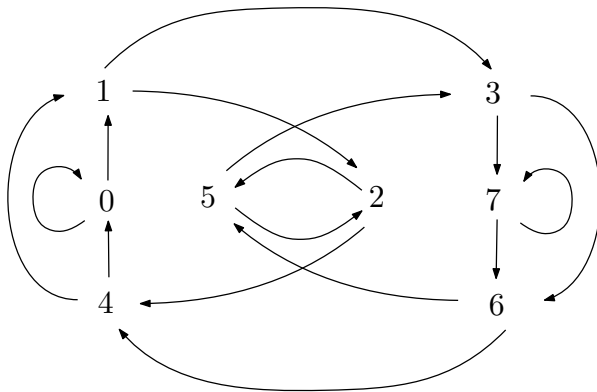
$$V = \{0, 1, 2, 3, 4, 5, 6, 7\}$$

$$E = \{(i, j), j \in \{2i, 2i + 1\} \bmod 8\}$$

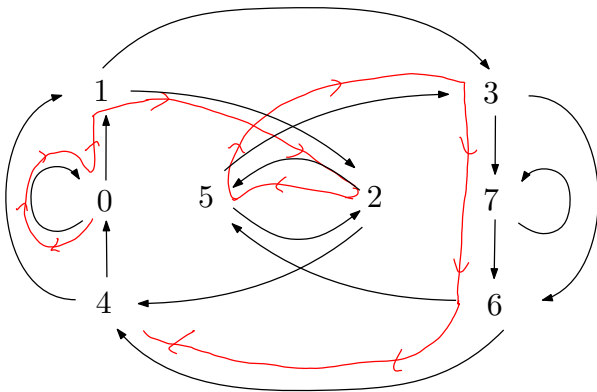
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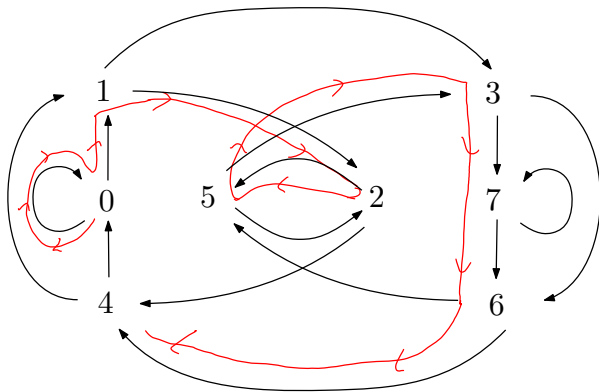
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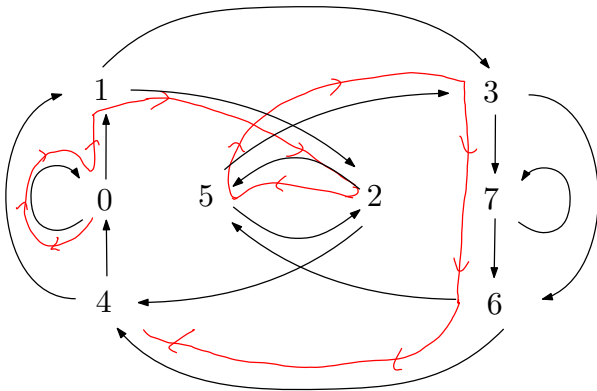


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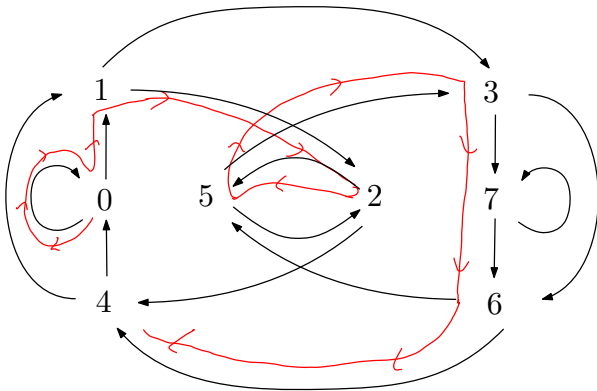
(0 1 2 5 3 7 6 4)

DeBruijn words and graphs, revisited



$w = 0001011100 \quad ? \quad (0 \ 1 \ 2 \ 5 \ 3 \ 7 \ 6 \ 4)$

DeBruijn words and graphs, revisited



$$\pi_{BWT(w)}^{-1} = (0 \ 1 \ 2 \ 5 \ 3 \ 7 \ 6 \ 4)$$

DeBruijn words and graphs, revisited

$$\pi_{BWT(w)}^{-1} = 01253764$$

Because the letters of the de Bruijn word w are 00001111, and in the graph, the strings starting with 0 are sent to $[0, 3]$, and the strings starting with 1 are sent to $[4, 7]$

DeBruijn words and graphs, revisited

Lemma

A word w is a de Bruijn word of order n over k letters if and only if $\pi_{\text{BWT}(w)}^{-1}$ (written as a cycle) is an Hamiltonian cycle of the de Bruijn graph $\text{DB}(k, kn)$.

Generalized de Bruijn words

Definition (Generalized de Bruijn graphs. Imase and Itoh, 1981; Reddy et al., 1980)

Let k, n be two integers. The *Generalized de Bruijn graph* $DB(k, n)$ is defined by:

- ▶ $V(DB(k, n)) = \{0, \dots, n-1\}$
- ▶ $E(DB(k, n)) = \{(i, ki + j \bmod n), i \in V, j \in \{0, \dots, k-1\}\}$

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Theorem (Fici and Gabory, 2025)

Let $w \in \Sigma_k^{kn}$. The following assertions are equivalent:

- ▶ The permutation $\pi_{\text{BWT}(w)}^{-1}$, written as a cycle, is an Hamiltonian cycle of the de Bruijn graph $DB(k, kn)$
- ▶ The word $\text{BWT}(w)$ is a concatenation of permutations of Σ_k

In that case, we say that w is a *generalized de Bruijn word*.

De Bruijn words and unclustered BWT

DeBruijn words are promising, but 2 things remain to solve:

1. What about arbitrary lengths?

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 - ▶ $\text{BWT}(u) = 201 \cdot 021 \cdot 120$.

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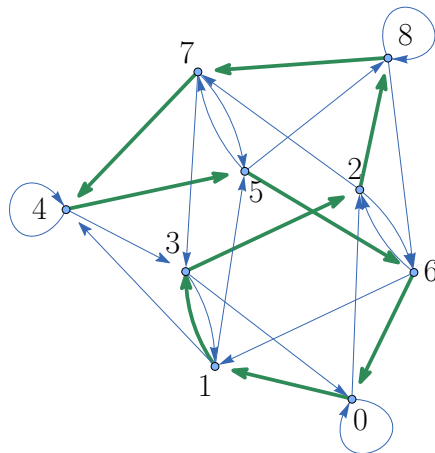
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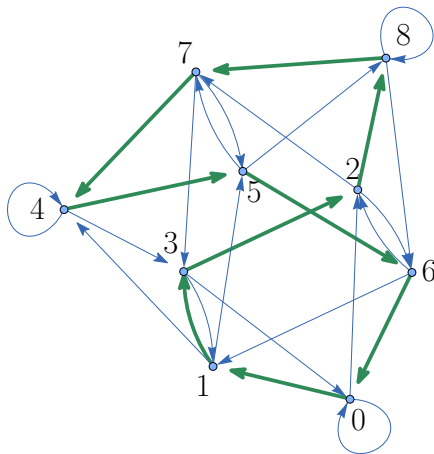


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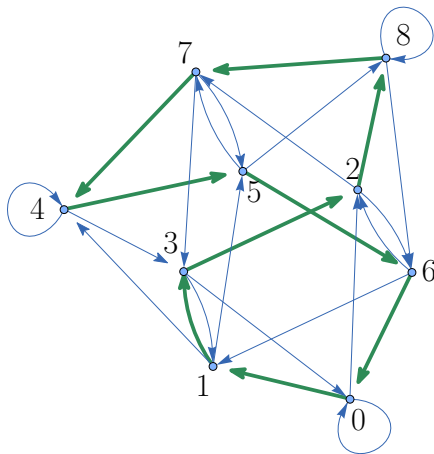
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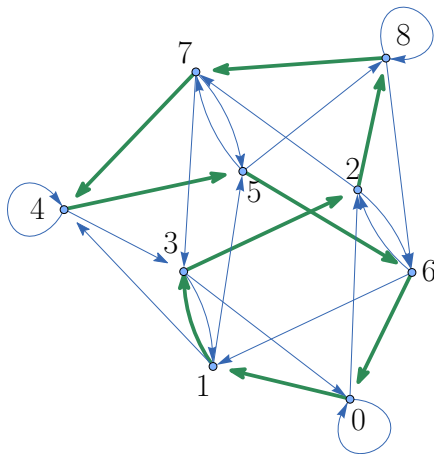
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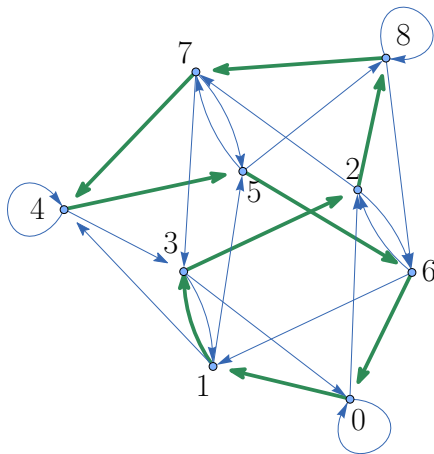
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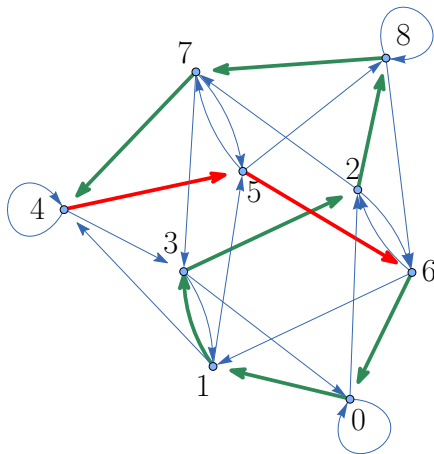
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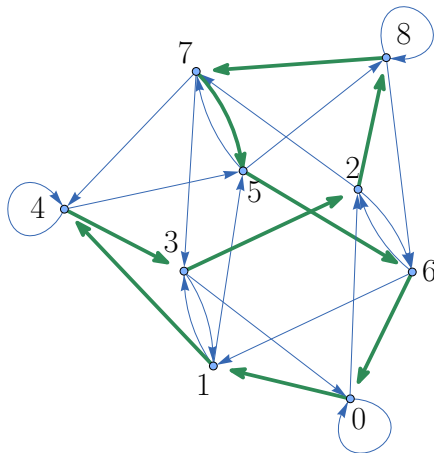
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$$u = [001102212]$$

$$w = \text{BWT}(u) = 201102120$$

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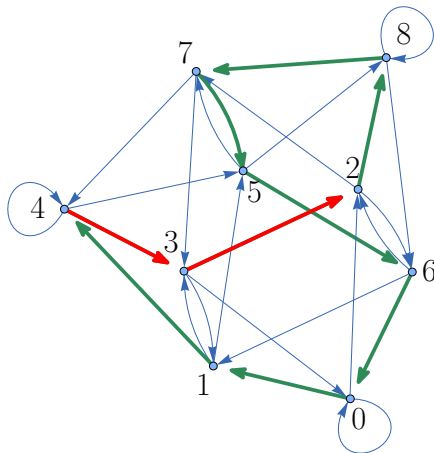
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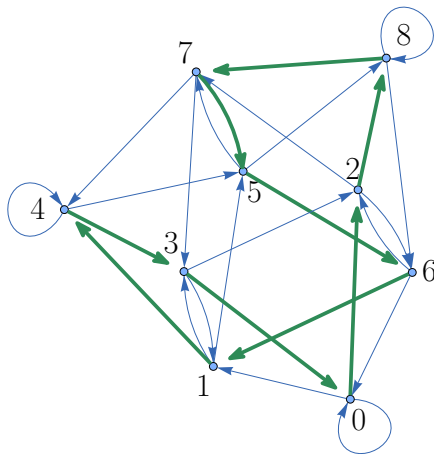
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$$u = [002212011]$$

$$w = \text{BWT}(u) = 120102120$$

$$\sigma = \pi_w^{-1} = (0, 2, 8, 7, 5, 6, 1, 4, 3)$$

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 8 & 0 & 3 & 6 & 1 & 5 & 7 \end{pmatrix}$$



Why does a rerouting always exists?

Sketch on the whiteboard

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DeBruijn words are promising, but 2 things remain to solve:

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Extension to all length

Let $n \geq 1$ and consider a de Bruijn word u with totally unclustered BWT, $w = \text{BWT}(u)$.

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Lastly, we can show that both cases occur for at least one word for every n . We finally obtained:

Theorem

For every integer $n > 0$, there exists a word u of length n over the alphabet Σ_3 having totally unclustered BWT.

Extending to full alphabets

0	0	1	0	2	1	1	2	2
0	1	0	2	1	1	2	2	0
0	2	1	1	2	2	0	0	1
1	0	2	1	1	2	2	0	0
1	1	2	2	0	0	1	0	2
1	2	2	0	0	1	0	2	1
2	0	0	1	0	2	1	1	2
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1	1	2	2	0	0	1	0	2
1	2	2	0	0	1	0	2	1
2	0	0	1	0	2	1	1	2
2	1	1	2	2	0	0	1	0
2	2	0	0	1	0	2	1	1

0	0	1	0	4	1	2	5	3
0	1	0	4	1	2	5	3	0
0	4	1	2	5	3	0	0	1
1	0	4	1	2	5	3	0	0
1	2	5	3	0	0	1	0	4
2	5	3	0	0	1	0	4	1
3	0	0	1	0	4	1	2	5
4	1	2	5	3	0	0	1	0
5	3	0	0	1	0	4	1	2

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Definition

Let p be a prime number. The generalized Euler's totient function counts the number of polynomials over \mathbb{F}_p of degree smaller than n and coprime with $X^n - 1$. It can be computed using the formula

$$\Phi_p(n) = p^n \prod_{d|(n/\lambda_p(n))} \left(1 - \frac{1}{p^{\text{ord}_p(d)}}\right)^{\frac{\phi(d)}{\text{ord}_p(d)}} \quad (1)$$

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Theorem (Fici and Gabory, 2025.)

The number of generalized de Bruijn words of length n over Σ_3 is
 $2^{n/3-1} \cdot \frac{\Phi_3(n/3)}{n/3}$

Finding the lower bound

When constructing generalized de Bruijn word with totally unclustered BWT, we untie blocks.

Finding the lower bound

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Theorem

For every integer $n > 0$, there exist at least $\Phi_3(\frac{n}{3})/2n = \Omega(2^{\frac{n}{3}}/n)$ word of length n over Σ_3 having a totally unclustered BWT.

Outline

The Burrows-Wheeler Transform

Unclustered BWT: Previous results

De Bruijn words

The ternary case: existence

Extension to all alphabets and lengths

The ternary case: lower bounds

Looking back at the binary case

Theorem (Mantaci et al., 2017)

There exists a $2n$ -length word with totally unclustered BWT if and only if $2n + 1$ is an odd prime and 2 generates the cyclic group \mathbb{Z}_{2n+1}^ .*

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This is interesting, because of its relation with the following conjecture:

Conjecture (Artin's conjecture)

Let a be an integer that is not a square number and not -1 . The set of prime number k such that k generates the cyclic group \mathbb{Z}_a^ is infinite.*

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Thank you! Questions?