1 Motion vectors correlation

Here we show how to compute a correlation r for two sequences of vectors, $\vec{a}$ and $\vec{b}$, so that the result is a scalar between -1 and 1. It should tend towards $r = 1$ when corresponding vectors point to a similar direction, and towards $r = -1$ when pointing to opposite directions instead. It should be close to $r = 0$ when no such relationship exists between the two sequences.

Let $\vec{a}_i = (a_{i,1}, a_{i,2}, a_{i,3})$ for $i = 1..n$, and $\vec{b}_i = (b_{i,1}, b_{i,2}, b_{i,3})$ for $i = 1..n$. For implicity, we asume that

$$\vec{\mu}_a = \sum_{i=1}^{n} \vec{a}_i = \vec{0}$$

$$\vec{\mu}_b = \sum_{i=1}^{n} \vec{b}_i = \vec{0}$$

We define r_{ab}, the correlation between $\vec{a}$ and $\vec{b}$, as follows.

$$r_{ab} = \frac{\sum_{i=1}^{n} (\vec{a}_i - \vec{\mu}_b) \cdot (\vec{b}_i - \vec{\mu}_b)}{\sqrt{\sum_{i=1}^{n} |\vec{a}_i - \vec{\mu}_a|^2} \sqrt{\sum_{i=1}^{n} |\vec{b}_i - \vec{\mu}_b|^2}}$$

$$= \frac{\sum_{i=1}^{n} \vec{a}_i \cdot \vec{b}_i}{\sqrt{\sum_{i=1}^{n} |\vec{a}_i|^2} \sqrt{\sum_{i=1}^{n} |\vec{b}_i|^2}}$$

Note that

$$r_{ab} = \begin{cases} r_{ab} = 1 & \text{if } \vec{a} = \vec{b} \\ r_{ab} = -1 & \text{if } \vec{a} = -\vec{b} \\ -1 < r_{ab} < 1 & \text{otherwise} \end{cases}$$

Also,

$$r_{ab} = \frac{\sum_{i=1}^{n}(a_{i,1}b_{i,1} + a_{i,2}b_{i,2} + a_{i,3}b_{i,3})}{\sqrt{\sum_{i=1}^{n}(a_{i,1}^2 + a_{i,2}^2 + a_{i,3}^2)}\sqrt{\sum_{i=1}^{n}(b_{i,1}^2 + b_{i,2}^2 + b_{i,3}^2)}}$$

Therefore if components of $\vec{a}$ and $\vec{b}$ are uncorrelated we have

$$\mathbf{E}\left(\sum_{i=1}^{n} a_{i,j}b_{i,j}\right) = 0 \text{ for } j = 1..3$$

$$\mathbf{E}(r_{ab}) = 0$$

Finally, in order to compute a correlation we have

$$r_{ab} = \frac{\sum_{i=1}^{n}\sum_{j=1}^{3} a_{i,j}b_{i,j}}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{3} a_{i,j}^2}\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{3} b_{i,j}^2}}$$

2 Combining correlations from several sequences

The expression for r_{ab} in the previous section can be written like this:

$$r_{ab} = \frac{s_{ab}}{\sqrt{ss_a ss_b}}$$

Where

$$s_{ab} = \sum_{i=1}^{n} \vec{a}_i \cdot \vec{b}_i$$

$$ss_a = \sum_{i=1}^{n} |\vec{a}_i|^2$$

$$ss_b = \sum_{i=1}^{n} |\vec{b}_i|^2$$

In short ss_a is the sum of squares of $\vec{a}$, ss_b is the same for $\vec{b}$, and s_{ab} is the sum of products. We could append more ab data pairs from additional sequences this way

$$r_{ab} = \frac{s_{ab,1} + s_{ab,2} + ... + s_{ab,n}}{\sqrt{(ss_{a,1} + ss_{a,2} + ... + ss_{a,n})(ss_{b,1} + ss_{b,2} + ... + ss_{b,n})}}$$

only keeping in mind that residuals in each (s_{ab}, ss_a, ss_b) trio are subject to the restriction of having a zero sum, which means there are $n-1$ degrees of freedom times 3 vector components. The total degrees of freedom for a combined r_{ab} of m sequences would be

$$df = \sum_{j=1}^{m} (3 * n_j - 3)$$