Computers Math. Applic. Vol. 26, No. 5, pp. 35-43, 1993
Printed in Great Britain.
0898-1221/93 $6.00 + 0.00

Bertrand's Postulate for Primes in Arithmetical Progressions

P. MOREE
Mathematical Institute, Leiden University, P.O. Box 9512
2300 RA Leiden, The Netherlands

Dedicated to P. Erdős

(Received October 1992; accepted February 1993)

Abstract—Bertrand's Postulate is the theorem that the interval $(x, 2x)$ contains at least one prime for $x > 1$. We prove, building on work of Erdős, analogues of this result, in which the interval is of the form (x, zx) and there are at least m primes $\equiv a(\bmod\ d)$ required to be contained in this interval, and where z, a and d have to satisfy some conditions. For the case $m = 1$ the results are worked out using a computer. They can be found in Table 1.

Keywords—Arithmetical progression, Prime, Interval.

1. INTRODUCTION

In this paper, we consider the distribution of primes in arithmetical progressions from a non-asymptotical viewpoint. Let $a, a + d, a + 2d, \ldots$ be an arithmetical progression. By a classical result of Dirichlet there are $\varphi(d)$ arithmetical progressions with $1 \le a < d$ containing infinitely many primes, where φ is Euler's totient function. The Prime Number Theorem for arithmetical progressions states that $\pi(x; d, a)$, the number of primes $\le x$ in the progression $a, a + d, \ldots$ with $(a, d) = 1$ satisfies

$$\lim_{x \to \infty} \frac{\pi(x; d, a)\varphi(d) \log x}{x} = 1,$$

or, equivalently,

$$\pi(x; d, a) \sim \frac{x}{\varphi(d) \log x}$$

as x tends to infinity. From this result, it follows that for every fixed $d \ge 3$ the primes are roughly equipartitioned over the arithmetical progressions with difference $d, 1 \le a < d$ and $(a, d) = 1$.

For arbitrary natural numbers m and $d(\ge 2)$ and real $z > 1$, we define

$$B_m(z, d) = \liminf\{c : \text{For every } x \ge c \text{ the interval } (x, zx) \text{ contains at least } m \text{ primes } \equiv a(\bmod\ d) \text{ for every integer } a \text{ satisfying } (a, d) = 1\}.$$

By the Prime Number Theorem for arithmetical progressions $B_m(z, d)$ exists. Note that if there are a and d with $(a, d) = 1$, such that we only have an asymptotic result for $\pi(x; d, a)$, it is not possible to deduce on this basis an upper bound for $B_m(z; d)$. In this paper, a method of Erdős (and Ricci) will be worked out that will enable us to obtain upper bounds for $B_m(z, d)$ for various z and d (Theorem 1). Having small upper bounds is usually enough; for applications the precise value of $B_m(z, d)$ is not of importance. Furthermore, given a small upper bound for $B_m(z, d)$, it is a matter of simple computation to obtain the precise value of $B_m(z, d)$.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

CAMWA 26-5-D

The problem of determining upper bounds for $B_m(z,d)$ goes back at least to Bertrand. In 1845 he postulated, in our notation, that $B_1(2,2)=1$. The problem of establishing Bertrand's Postulate was a focal point of much effort in the mid nineteenth century. It was finally proved by Tschebycheff in 1852. A result of the form $B_m(z,d)\le c$ is called a Bertrand's Postulate type Theorem (B.P.T.). For several non-asymptotic problems (see e.g. [1–3, 4 p. 491, 5–7]) B.P.T.'s are helpful tools. Vaidya [8] gives some further applications of Bertrand's Postulate.

There are two quite different approaches in proving B.P.T.'s; one involving complex analysis and one involving elementary analysis. The first approach is based on the calculation of zeros of Dirichlet L-functions formed with characters modulo d. This information is then used in about the same way Rosser and Schoenfeld used it for their estimates of $\psi(x)$ and $\theta(x)$ [9]. However, there are many hurdles, e.g., exceptional moduli, to overcome and there is a lot of calculation involved. Work in this direction has been carried out by McCurley [10,11] and more recently by Ramaré [12] who based his results on computations of Rumely [13]. The methods involving only elementary analysis are far less complicated and require less computation. If they can be applied, the resulting c ('the starting point') is usually far less than the c that results from using the first method. For example in [10] McCurley implicitly proves that $B_1((1+\varepsilon)/(1-\varepsilon),d)\le e^{c\log^2 d}$, where $d\ge 10^b$, d is a non-exceptional modulus (any $d\le 986$ is a non-exceptional modulus by [14]) and (b,ε,c) is in his Table 1. For $\varepsilon\le 1/2$ it appears that his starting point, $e^{c\log^2 d}$, exceeds 10^{94}. If z is large enough with respect to d, elementary methods usually result in starting points which are polynomial in d. (To do justice to the work of McCurley, it should be said that in [11], where he limits his d range to just 3, he gets lower starting points. For $d=3$ he implicitly proves that $B_1((1+2\varepsilon)/(1-2\varepsilon),3)\le E^L$, where (L,ε) is a pair from his Table 4 with $L\ge 13.815$). The non-elementary method has the advantage that the (z,m,d) domain that can be covered, is considerably larger. Since neither method surpasses the other in all aspects, both deserve to be studied in the author's opinion.

From one B.P.T., infinitely many others can be derived by using the following trivial lemma:

LEMMA 1. *Let z, z_1 and z_2 be arbitrary real numbers >1. Let m, m_1, m_2, s and q be arbitrary natural numbers.*

(i) $B_m(z_1,d)\le B_m(z_2,d)$ *if* $z_1<z_2$.
(ii) $B_{m_1}(z,d)\le B_{m_2}(z,d)$ *if* $m_1<m_2$.
(iii) $B_{sm}(z^s,d)\le B_m(z,d)$.
(iv) *If q is a natural number, write $q=q_1q_2$ with q_1 the greatest divisor of q satisfying $(q_1,d)=1$. Then* $B_{\varphi(q_1)q_2m}(z,d)\le B_m(z,qd)$.

Given a set of B.P.T.'s, only those assertions are of interest that cannot be derived from the other ones by using Lemma 1.

In the literature, surprisingly few B.P.T.'s are explicitly stated and proved. Breusch [15] showed that $B_1(9/8,2)\le 48, B_1(2,3)\le 7$ and $B_1(2,4)\le 7$ and Molsen [16] showed that $B_1(8/7,3)\le 199$ and $B_1(4/3,12)\le 118$. Rohrbach and Weis [17] showed that $B_1(1.073,2)\le 119$ and Erdős [4] $B_1(2,6)\le 13/2$ and $B_1(2,4)\le 7/2$, where in the definition of $B_m(z,d)$, (x,zx) is replaced by $(x,zx]$. It is the purpose of this note to extend this list considerably. The B.P.T. method of proof that will be used here is elementary. It is based on an effective version of a theorem of Erdős [4, Satz 3].

2. ERDŐS' 'ÜBER DIE PRIMZAHLEN IN GEWISSER ARITHMETISCHER REIHEN' REVISITED

In a beautiful paper published in 1935 [4], Erdős proved, using elementary methods only, some results on primes in arithmetical progressions that were obtained previously by the use of deep analytical methods [15,18]. In particular, he proved some B.P.T.'s and he showed that for several d the arithmetical progressions with difference d contains an infinitude of primes. To do justice to history, it should be remarked that, about the same year, similar results were obtained independently by Ricci, who used similar methods [19,20]. Further references and a somewhat alternative account of the contents of Erdős' paper can be found in [21]. Erdős' method of proof is a generalization of the method he used to prove Bertrand's Postulate [22] (an account of which can also be found in most textbooks on elementary number theory).

Let d be an integer ≥ 2. Let $p_1, p_2, \ldots, p_h$ denote the primes $< d$ that do not divide d. Put

$$\sigma(d) = \sum_{i=1}^{h} \frac{1}{p_i}.$$

Erdős proved (Satz 3) that if $\sigma(d) < 1$ and $z > d(d-1)^{-1}(1-\sigma(d))^{-1}$, there is at least one prime from each primitive congruence class modulo d in the interval $(x, zx]$ for all x sufficiently large. We will make this result effective (Theorem 1). Erdős did this only in the case $d = 4$ and $d = 6$. In Section 5, it is shown that there are only finitely many d such that $\sigma(d) < 1$ and all these d's are determined, the largest one being $d = 840$.

We start by recalling some more notation and results from [4]. Let d be an integer. Throughout we assume $d \geq 2$. Let a be any integer in $[1, d)$ with $(a, d) = 1$. For $i = 1, \ldots, h$ let q_i be the unique number in $(0, d)$ satisfying $p_i q_i \equiv a (\operatorname{mod} d)$. Put

$$\alpha(d) = d \prod_{p|d} p^{1/(p-1)}, \quad \beta(d) = \prod_{p|d} p^{(p-2)/(p-1)}, \quad \gamma(d) = \prod_{i=1}^{h} p_i$$

$$P_n(a,d) = \frac{1}{n!} \prod_{p|d} p^{[n/(p-1)]} \prod_{m=1}^{n} (a + md)$$

$$\Pi_n(a,d) = \frac{P_n(a,d)}{\prod_{i=1}^{h} P_{\left[\frac{n}{p_i}\right]}(q_i, d)} \quad \text{and}$$

$$\Phi_n(a, d; z) = \frac{\Pi_{[zn]}(a,d)}{P_n(a,d)} \qquad z > 1.$$

If $\sigma(d) < 1$, put $z_{\min} = d(d-1)^{-1}(1-\sigma(d))^{-1}$. If n is an integer, let $\omega(n)$ denote the number of different prime factors of n. Let $\pi(x)$ denote the number of primes $\leq x$.

LEMMA 2. [4, p. 474]. *$P_n(a,d)$ is an integer.*

LEMMA 3. [4, (5)]. *$\alpha(d)^n/\beta(d) \leq P_n(a,d) \leq (n+1)\alpha(d)^n$.*

LEMMA 4. [4, Hilfssatz 2] *If $p^w \parallel P_n(a,d)$, then $p^w \leq (n+1)d$.*

LEMMA 5. *Suppose that p does not divide d, $p > \sqrt{(n+1)\,d}$ and $pb \equiv a (\operatorname{mod} d)$ with $0 < b < d$.*

(i) *If there exists an $m \in \mathbb{N} \cup \{0\}$ such that p is in $\left(\frac{a+nd}{b+md}, \frac{n}{m}\right]$, then p does not divide $P_n(a,d)$.*

(ii) *If p does not divide a and there exists an $m \in \mathbb{N} \cup \{0\}$ such that p is in $\left(\frac{n}{m+1}, \frac{a+(n+1)d}{b+md}\right]$, then $p \parallel P_n(a,d)$.*

PROOF. The proof is a straightforward extension of the proof of [4, Hilfssatz 3].

LEMMA 6. *If $n \geq d$ and $p > \sqrt{(n+1)\,d}$, then $p \mid \Pi_n(a,d)$ implies $p \equiv a (\operatorname{mod} d)$.*

PROOF. If one uses Lemma 5 instead of [4, Hilfssatz 3] the proof becomes a straightforward extension of the proof of [4, Hilfssatz 5].

LEMMA 7. [4, Satz 7]. *For $x \geq 1$ we have*

$$\prod_{p \leq x,\ p \equiv a (\operatorname{mod} d)} p \leq dx\, \alpha(d)^{x/(d-1)}.$$

LEMMA 8. [23, (3.6)].

$$\pi(x) < 1.256\,\frac{x}{\log x}, \qquad \text{for } x > 1.$$

The Lemmas 2–8 form the ingredients in the proof of the following effective version of Satz 3.

THEOREM 1. *If $\sigma(d) < 1$ and $z > d(d-1)^{-1}(1-\sigma(d))^{-1}(= z_{\min})$, then for each $m \geq 1$ there exists a real number y_m such that*

(i)

$$y_m \geq d \text{ and}$$

(ii)

$$y_m\left(z - 1 - \sigma(d)z - \frac{1}{d-1}\right)\log\alpha(d) > 2.152\sqrt{d(zy_m+1)} + (\pi(d)+1+m)\log(d(zy_m+1)).$$

Further, for every $x \geq x_m := (y_m+1)\,d$, the interval (x, zx) contains at least m primes from each primitive congruence class modulo d, that is $B_m(z,d) \leq x_m$.

PROOF. Assume that the hypothesis of the theorem is satisfied. Then $z-1-\sigma(d)z-1/(d-1) > 0$, and the existence of y_m is obvious. Suppose $\underline{a}$ is a primitive congruence class modulo d, where the representative a is in $(0,d)$. Let n be any natural number $\geq y_m$. On writing out $\Phi_n(a,d;z)$ (which is defined for this choice of a and d) in terms of P functions and using Lemma 3, we obtain

$$\Phi_n(a,d;z) \geq \frac{\gamma(d)}{\alpha(d)\beta(d)}\,\frac{\alpha(d)^{(z-1-\sigma z)n}}{(n+1)\prod_{i=1}^{h}(zn+p_i)}. \tag{1}$$

By Lemma 4 and Lemma 6, we find that

$$\Pi_{[zn]}(a,d) \leq \prod_{p\leq\sqrt{d(zn+1)}} d(zn+1) \prod_{\sqrt{dzn}\leq p\leq a+znd,\,p\equiv a(\text{mod } d)} p. \tag{2}$$

Put

$$\Pi_1(n) = \prod_{a+(n+1)d\leq p\leq a+nzd,\ p\equiv a(\text{mod } d)} p.$$

Since $n \geq y_m \geq d$, it follows from part (ii) of Lemma 5 that if p is in $(n, a+(n+1)d)$ and $p \equiv a(\text{mod } d)$, then $p \mid P_n(a,d)$. Using this and (2), we find

$$\Phi_n(a,d;z) \leq \prod_{p\leq\sqrt{d(zn+1)}} d(zn+1) \prod_{p\leq n,\,p\equiv a(\text{mod } d)} p\,\Pi_1(n).$$

Estimating the second product using Lemma 7, it then follows that

$$\Phi_n(a,d;z) \leq (d(zn+1))^{\pi\left(\sqrt{d(zn+1)}\right)}\,dn\alpha(d)^{n/(d-1)}\Pi_1(n). \tag{3}$$

Notice that

$$\frac{d\gamma(d)}{\alpha(d)\beta(d)} \geq \frac{1}{\prod_{p\mid d} p} \geq \frac{1}{(d(zn+1))^{\omega(d)}}.$$

Using this, it follows from (1) and (3) that

$$\Pi_1(n) \geq \frac{\alpha^{(z-1-\sigma z-1/(d-1))n}}{(d(zn+1))^{\pi(\sqrt{d(zn+1)})+h+2+\omega(d)}}.$$

Replace in the left side of (ii) y_m by y, and consider it as a function f of y. In the same way, we obtain from the right side the function $g(y)$. It is is easy to show that $f(y_m) > g(y_m)$ implies

$f'(y) > g'(y)$ for every $y \geq y_m$ and so in particular $f(n) > g(n)$. Note that $h + \omega(d) = \pi(d)$. By using this, the inequality for $\Pi_1(n)$ and Lemma 8, it then follows that

$$\Pi_1(n) > (d(zn+1))^{m-1}, \qquad \text{for every } n \geq y_m. \tag{4}$$

Now suppose x is any real number $\geq x_m (= (y_m + 1)\,d)$. Then x can be written in the form $a + nd + r$ for some $r \in [0, d), r \in \mathbb{R}$, and some natural number $n \geq y_m$. Then

$$\Pi_1(n) \text{ divides } \prod_{\substack{a+nd<p\leq za+znd \\ p\equiv a(\text{mod } d)}} p \text{ and this product divides } \prod_{\substack{p\in(x,zx) \\ p\equiv a(\text{mod } d)}} p. \tag{5}$$

From (4) and the definition of $\Pi_1(n)$, we deduce that $\Pi_1(n)$ is divisible by at least m different primes $\equiv a(\text{mod } d)$. Finally, we conclude from (5) that for every $x \geq x_m$ the interval (x, zx) contains at least m primes $\equiv a(\text{mod } d)$. Since this argument holds true for any primitive congruence class modulo d, Theorem 1 is proved.

REMARK 1. Using only the results of Erdős, one arrives at the condition $(i') : y_m > d^2$. If z is large in comparison with d it turns out that condition (ii) can be satisfied while (i') is not satisfied. Furthermore if y_m has to satisfy condition (i') instead of (i), the starting point is at least $\geq (d^2+1)d$ instead of $(d+1)d$. So the improvement of condition (i') is valuable from the numerical point of view.

REMARK 2. Ramaré has pointed out to the author that Montgomery and Vaughan [24] improved Lemma 7. They showed that if $a(\text{mod } d)$ is a primitive congruence class modulo d, then, for $1 \leq d < x$,

$$\prod_{p\leq x, p\equiv a(\text{mod } d)} p < e^{(2/\phi(d))\,(x \log x / \log(x/d))}.$$

This improves Lemma 7 if d has not too many different prime factors. Use of the result of Montgomery and Vaughan, instead of Lemma 7, leads to a more complicated version of Theorem 1. The equivalent of $z_{\min}$ is then $z'_{\min} := \frac{2d}{(\log \alpha(d))\phi(d)(1-\sigma)}$. As the d in Table 1 are rather composite, it is not surprising that there are only two primitive d's (cf. Remark 2 of Section 4) for which $z'_{\min} \leq z_{\min}$. Because of this, it is better to use Lemma 7 instead of the result of Montgomery and Vaughan.

3. BERTRAND SEQUENCES

If z, m and d are such that the hypothesis of Theorem 1 is satisfied, we can effectively determine a c such that $B_m(z, d) \leq c$. For applications it is of importance that c is as small as possible. So the problem arises of reducing c in an efficient way. To this end, we introduce *Bertrand sequences.*

Let t be some real number. Let $M_1(t; a, d)$ be the largest prime $\equiv a(\text{mod } d)$ less than t. In general let $M_k(t; a, d)$ be the k^{th} largest prime $\equiv a(\text{mod } d)$ less than t. If there is no k^{th} largest prime $\equiv a(\text{mod } d)$ less than t, we put $M_k(t; a, d) = 0$. A sequence $q_1 < q_2 < \cdots < q_s$ of integers is called a Bertrand sequence with initial term q_1 and parameters z, m and d if

$$q_{i+1} = \min\{M_m(zq_i; a, d) : 1 \leq a < d, \quad (a, d) = 1\} \qquad (i \geq 1).$$

Notice that once q_1 is given, $q_2, q_3, \ldots,$ are determined.

LEMMA 9. *If $B_m(z, d) \leq c$ and there is a Bertrand sequence $q_1 < q_2 < \cdots < q_s$ with initial term q_1 and parameters $z'(\leq z), m$ and d such that $q_s \geq c$, then $B_m(z, d) \leq q_1$.*

PROOF. It suffices to show that for each $x \in [q_1, c)$, there are at least m primes $\equiv a(\text{mod } d)$ in the interval (x, zx) from each primitive congruence class modulo d. If $x \in [q_1, c)$, then there exists an i in $\{1, \ldots, s-1\}$ such that $x \in [q_i, q_{i+i})$. Then $zx \geq z'q_i$ and by definition of q_{i+1}

and the fact that $q_{i+1} > q_i$, there are from each primitive congruence class modulo d at least m primes $\equiv a(\mathrm{mod}\ d)$ in the interval $[q_{i+1}, z'q_i)$. Since this interval is contained in the interval (x, zx), the result follows.

As an example, we give a part of the Bertrand sequence with starting term 21 and parameters 3/2, 1 and 6 : 21, 29, 41, 59, 79, 109, 149, 197, 283, 419, 617, 911, 1327, 1979, 2953, 4421, 6599, 9883, 14813, 22189, 33203, 49787, 74609, 111893, 167801, 251663, 377477, 566179, 849221, 1273771, 1910611, 2865899.

Finding a suitable q_1 is a matter of trial and error. If one takes q_1 too small, it can happen that there is no term in the Bertrand sequence $\geq c$. If there is a term $q_s \geq c$, it is still possible that q_1 can be reduced further.

4. NUMERICAL RESULTS

In this section, it will be demonstrated that $B_1(z_2, d) \leq q_1$ for all triples (d, z_2, q_1) in Table 1 and so, by Lemma 1, $B_1(z, d) \leq q_1$ provided $z \geq z_2$.

If (d, z_2, q_1) is in Table 1 it can be shown by direct compution (e.g., with a computer) and the help of the information in the columns headed $\overline{\sigma(d)}, h_1, \underline{\log \alpha}$ and d in Table 1, that there is an x_1 which is less than the term q_s, which is in the Bertrand sequence with starting term q_1 and parameters $z_1, 1$ and d. Then it follows that $B_1(z_2, d) \leq q_1$ by using Theorem 1 and Lemma 9.

Table 1.

d	$\overline{\sigma(d)}$	$\pi(d)$	$\underline{\log \alpha}$	$\overline{z_{\min}}$	z_1	q_1	q_s	z_2
$2 \to 6$	.000000	1	1.386	2.00				
$3 \to 6$	.500000	2	1.647	3.00				
$4 \to 12$	.333334	2	2.079	2.00				
$5 \to 30$	.833334	3	2.011	7.50				
6	.200000	3	3.034	1.50	1.5	21	6448223	1.51
$8 \to 24$	.676191	4	2.772	3.53				
$9 \to 18$	.842858	4	2.746	7.16				
$10 \to 30$	.476191	4	3.398	2.13				
12	.433767	5	3.727	1.93	1.9	25	8465251	1.94
$14 \to 42$	.701166	6	3.656	3.61				
$15 \to 30$	.810690	6	3.659	5.66				
$16 \to 48$	.844023	6	3.465	6.84				
18	.569513	7	4.132	2.46	2.4	23	4726591	2.48
$20 \to 120$	.755478	8	4.091	4.31				
$21 \to 42$	.979288	8	3.918	50.7				
$22 \to 66$	.864569	8	4.023	7.74				
24	.665623	9	4.420	3.13	3.1	24	8584187	3.15
$26 \to 78$	.922033	9	4.164	13.4				
$28 \to 84$	.856099	9	4.349	7.21				
30	.500106	10	5.046	2.07	2.0	76	8287067	2.09
36	.732364	11	4.825	3.85	3.8	42	5991103	3.91
$40 \to 120$	.892724	12	4.784	9.57				
42	.640924	13	5.304	2.86	2.8	46	5345881	2.90

- If z is in $(z_{\min}, z_2)$, then this table does not yield a B.P.T. However the information in this table may be used to obtain a B.P.T. (see Section 4, Remark 3).
- If (d, z_1, q_1, q_s) is in this table, then q_s is a term in the Bertrand sequence with starting point q_1 and parameters $z_1, 1$ and d (see Section 3).
- If d is such that the Erdős theory yields only B.P.T.'s that can be derived too by working with a suitable multiple δ of d, this is indicated by '$\delta \to d$' (see Section 4, Remark 2).
- If (d, q_1, z_2) is a triple in some row above, then $B_1(z, d) \leq q_1$ for all $z \geq z_2$.

Table 1 is continued on the next page.

Table 1, continued.

d	$\overline{\sigma(d)}$	$\pi(d)$	$\underline{\log\alpha}$	$\overline{z_{\min}}$	z_1	q_1	q_s	z_2
48	.828314	15	5.113	5.95	5.9	30	6457271	6.09
$50 \to 150$	.961647	15	5.007	26.7				
54	.847182	16	5.231	6.67	6.6	27	4454141	6.89
60	.664131	17	5.739	3.03	3.0	88	7241747	3.09
66	.789615	18	5.671	4.83	4.8	48	7459717	4.95
$70 \to 210$	.885926	19	5.668	8.90				
72	.909534	20	5.519	11.3	11	18	3302449	12.0
78	.846310	21	5.812	6.60	6.5	36	2805161	6.96
84	.805082	23	5.997	5.20	5.1	74	4867409	5.39
90	.759175	24	6.144	4.20	4.1	95	4028539	4.37
96	.959175	24	5.806	24.8	10	32	5189311	27.2
102	.920562	26	6.044	12.8	12	36	1158457	14.8
108	.998440	28	5.924	648	10	58	6042149	1040
114	.963832	30	6.142	27.9	10	64	1497701	34.4
120	.816464	30	6.432	5.50	5.4	76	6659843	5.75
126	.873607	30	6.403	7.98	7.9	54	2598707	8.74
132	.941062	32	6.365	17.1	17	25	8606173	18.5
138	.995792	33	6.312	240	230	3	6941731	331
150	.853176	35	6.655	6.86	6.8	126	3265861	7.48
156	.982876	36	6.506	58.8	10	69	1087829	92.8
168	.935434	39	6.690	15.6	15	59	5330657	17.5
180	.889658	41	6.837	9.12	9.1	73	5587837	9.96
210	.772844	46	7.316	4.43	4.4	173	4045567	4.80
240	.942173	52	7.125	17.4	17	88	2366653	22.2
270	.961717	57	7.243	26.3	26	42	2831687	35.4
300	.979523	62	7.348	49.0	10	164	2114089	81.5
330	.901436	66	7.683	10.2	10	137	4387441	18.5
390	.946002	77	7.824	18.6	18	89	4846063	24.1
420	.889912	81	8.009	9.11	9.1	197	8772977	10.6
630	.954032	114	8.414	21.8	21	149	1450199	48.6
840	.997976	146	8.702	495	10	130	4082809	4630

- If z is in $(z_{\min}, z_2)$, then this table does not yield a B.P.T. However the information in this table may be used to obtain a B.P.T. (see Section 4, Remark 3).
- If (d, z_1, q_1, q_s) is in this table, then q_s is a term in the Bertrand sequence with starting point q_1 and parameters $z_1, 1$ and d (see Section 3).
- If d is such that the Erdős theory yields only B.P.T.'s that can be derived too by working with a suitable multiple δ of d, this is indicated by '$\delta \to d$' (see Section 4, Remark 2).
- If (d, q_1, z_2) is a triple in some row above, then $B_1(z, d) \le q_1$ for all $z \ge z_2$.

In working numerically with the conditions of Theorem 1, one should take care of rounding off the quantities correctly, e.g., $\sigma(d)$ upwards and $\log\alpha$ downwards. This is the reason that $\sigma(d)$ is rounded upwards in Table 1 (in the sixth decimal place) and $\log\alpha$ downwards (in the third decimal place). Furthermore $z_{\min}$ is rounded upwards also (in the second decimal place). So Theorem 1 can be applied if $z > \overline{z_{\min}}$.

REMARK 1. If (z_2, d) is in Table 1 and $z \ge z_2, q_1$ is certainly a starting point. It remains to find the optimal starting point. Since this strongly depends on the particular choice of z and d, I have to leave this to the reader.

REMARK 2. For some values of d the Erdős theory yields B.P.T.'s, all of which can be derived from the Erdős theory for some multiple of d by using Lemma 1. In this case, there is the entrance '$d \to \delta$' in the row headed d in Table 1. The number δ is the multiple of d for which $z_{\min}$ is minimal. Using the triple (δ, q_1, z_2) in the entry in Table 1 headed δ, you then find $B_1(z, \delta) \le q_1$, provided $z \geq z_2$. By part (iv) and (ii) of Lemma 1, this implies $B_1(z,d) \leq q_1$.

REMARK 3. If d is such that $\sigma(d) < 1$ and $z \in (z_{\min}, z_2)$, then again the hypothesis of Theorem 1 is satisfied and so x_m can be effectively computed. These cases are not covered by the results in Table 1 and I will describe how to proceed. In order to reduce x_m to some c it suffices to find a Bertrand sequence with initial term c and parameters $z'(z' \leq z), 1$ and d. The easiest way to lay your hands on such a sequence is to try to extend the Bertrand sequence in Table 1 with parameters $z_2(\leq z), 1$ and d (of which only the starting term, q_1 and the largest term I have calculated, q_s are given in Table 1). The term q_s has then to be used as a starting term. However there is no guarantee that the sequence is extendable in this way, although heuristically this is very likely.

5. ON THE COMPLETENESS OF TABLE 1

In this section it will be demonstrated that Table 1 is complete in the sense that it contains all d for which $\sigma(d) < 1$.

Let $q_1, q_2, \ldots$ denote the sequence of consecutive rational primes. For every $k \in \mathbb{N}$, we define $r_k = \prod_{i=1}^{k} q_i$ and $C_k = \{n \in \mathbb{N} : \omega(n) = k\}$. Let $\theta(x)$ denote the Tschebycheff θ function, that is the function defined by $\theta(x) = \sum_{p \leq x} \log p$.

LEMMA 10. *$\sigma(r_k) = \min\{\sigma(d) : d \in C_k\}$ for every $k \in \mathbb{N}$.*

PROOF. Let k be an arbitrary natural number. Notice that $\min C_k = r_k$. So if $d \in C_k$, then $\sum_{p \leq d} 1/p \geq \sum_{p \leq r_k} 1/p$. Furthermore, notice that $\max\{\sum_{p|d} 1/p : d \in C_k\} = \sum_{p|r_k} 1/p$. So if d is an element from C_k, then $\sigma(d) = \sum_{i=1}^{h} 1/p_i = \sum_{p \leq d} 1/p - \sum_{p|d} 1/p \geq \sum_{p \leq r_k} 1/p - \sum_{p|r_k} 1/p = \sigma(r_k)$. Since $r_k \in C_k$, the lemma follows.

LEMMA 11. *If $k \geq 6$ then $\theta(q_k) > .77 q_k$.*

PROOF. By [9, p. 268], it follows that $\theta(x) > .86x..77x$ for $x \geq 149$. Computation of $\theta(x)$ for x prime, $x \in [13, 149)$, shows that the desired inequality also holds true in this region.

LEMMA 12. [23, (3.19), (3.20)].

(i) $\sum_{p \leq x} 1/p > \log\log x$ *for* $x > 1$.
(ii) $\sum_{p \leq x} 1/p < \log\log x + .2615 + 1/\log^2 x$ *for* $x > 1$.

LEMMA 13. *$\sigma(r_k) > 1$ if and only if $k \geq 5$.*

PROOF. Using part (i) of Lemma 12 and Lemma 11 one finds that $\sum_{p \leq r_k} 1/p > \log(.77 q_k)$ if $k \geq 6$. Furthermore, $\sum_{p \leq q_k} 1/p < \log\log(q_k) + .2615 + 1/\log(q_k)$ for $k \geq 1$, by part (ii) of Lemma 12. Put $f(x) = \log x - \log\log x + \log(.77) - .2615 - 1/\log x (x > 1)$.
By noticing that $f(17) > 1$ and that f is increasing for $x \geq 17$, it follows that $\sigma(r_k) = \sum_{p \leq r_k} 1/p - \sum_{p \leq q_k} 1/p > f(q_k) > 1$ for $k \geq 7$. By direct computation, it follows that both $\sigma(r_5)$ and $\sigma(r_6) > 1$ and that $\sigma(r_k) < 1$ for $k \in \{1, \ldots, 4\}$.

THEOREM 2. *If $\sigma(d) < 1$, then d is in Table 1.*

PROOF. Let d be any natural number ≥ 2 satisfying $\sigma(d) < 1$. If $\omega(d) \geq 5$, then it follows by Lemma 10 and 13 that $\sigma(d) > 1$. So $\omega(d) \leq 4$ and therefore $\sum_{p|d} 1/p \leq 1/2 + 1/3 + 1/5 + 1/7$. Now if $d \geq 881$, then $\sigma(d) \geq \sum_{p \leq 881} 1/p - 1/2 - 1/3 - 1/5 - 1/7 > 1.004$ and so $d < 881$.
Finally, by direct computation, it follows that the $d < 881$ for which $\sigma(d) < 1$, are exactly the d in Table 1.

REMARK. The table of Erdős [4, p. 478] is far from complete and contains a misprint; the first digit after the decimal point in the tabulated value of $\sigma(15)$ should be an 8.

REFERENCES

1. L.K. Arnold, S.J. Benkoski and B.J. McCabe, The discriminator (a simple application of Bertrand's postulate), *Amer. Math. Monthly* **92**, 275–277 (1985).
2. E. Artin, Braids and permutations, *Annals of Mathematics* **48**, 643–649 (1947).
3. R.F. Coleman, On the Galois groups of the exponential Taylor polynomials, *Enseign. Math.* **33** (2), 183–189 (1987).
4. P. Erdős, Über die Primzahlen gewisser arithmetischer Reihen, *Math. Z.* **39**, 473–491 (1935).
5. P. Moree, A complete determination of some discriminators with even exponents, unpublished manuscript.
6. P. Schumer, On the incongruence of consecutive cubes, *Math. Student* **58**, 42–48 (1990).
7. P. Schumer and J. Steinig, On the incongruence of consecutive fourth powers, *Elem. Math.* **43**, 145–149 (1988).
8. A.M. Vaidya, Bertrand's postulate and some of its consequences, *Math. Student* **57**, 133–141 (1989).
9. J.B. Rosser and L. Schoenfeld, Sharper bounds for Chebyshev functions $\theta(x)$ and $\psi(x)$, *Math. Comp.* **29**, 243–269 (1975).
10. K.S. McCurley, Explicit estimates for the error term in the prime number theorem for arithmetic progressions, *Math. Comp.* **42**, 265–285 (1984).
11. K.S. McCurley, Explicit estimates for $\theta(x;3,1)$ and $\psi(x;3,1)$, *Math. Comp.* **42**, 287–296 (1984).
12. O. Ramaré, Contribution au problème de Goldbach: Tout entier supérieur à l'est somme d'au plus 13 nombres premiers, Ph.D. Thesis, Univ. Bordeaux, Bordeaux, France, (1991).
13. R. Rumely, Numerical computations concerning the ERH, *Math. Comp.* (to appear).
14. J. B. Rosser, unpublished manuscript.
15. R. Breusch, Zur Verallgemeinerung des Bertrandschen Postulates, dass zwischen x und $2x$ stets Primzahlen liegen, *Math. Z.* **34**, 505–526 (1932).
16. K. Molsen, Zur Verallgemeinerung des Bertrandschen Postulates, *Deutsche Math.* **6**, 248–256 (1941).
17. H. Rohrbach and J. Weis, Zum finiten Fall des Bertrandschen Postulates, *J. Reine Angew. Math.* **214/215**, 432–440 (1964).
18. P.G. Lejeune Dirichlet, Beweis des Satzes, dass jede unbegrentze arithmetische Progression, deren erstes Glied und Differenz ganz Zahlen ohne gemeinschaftlichen Factor sind, unendliche viele Primzahlen enthalt, *Werke* **I**, 315–342 (1837).
19. G. Ricci, Sul teorema di Dirichlet relativo alla progressione arithmetica, *Bolletino della Unione Mathematica Italiana* **12**, 304–309 (1933).
20. G. Ricci, Sui teoremi di Dirichlet e di Bertrand-Tchebychev relativi alla progressione arithmetica, *Bolletino della Unione Matematica Italiana* **13**, 1–11 (1934).
21. H.N. Shapiro, *Introduction to the Theory of Numbers*, pp. 363–399, Wiley-Interscience, New York, (1983).
22. P. Erdős, Beweis eines Satzes von Tschebyschef, *Acta Litt. Ac Scient. Regiae Univ. Hungaricae Francisco Josephinea* **5**, 194–198 (1930–1932).
23. J.B. Rosser and L. Schoenfeld, Approximate formulas for some functions of prime numbers, *Illinois J. Math.* **6**, 64–94 (1962).
24. H.L. Montgomery and R.C. Vaughan, The large sieve, *Mathematika* **20**, 119–134 (1973).