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$$[\hat{S}_z, \hat{H}_0] = \frac{\xi}{2} \sum_{q\tau} \tau a_{q\tau}^\dagger a_{q\tau} \sum_{p\sigma} (p-1) a_{p\sigma}^\dagger a_{p\sigma}$$

$$- \xi/2 \sum_{p\sigma} (p-1) a_{p\sigma}^\dagger a_{p\sigma} \sum_{q\tau} a_{q\tau}^\dagger a_{q\tau}$$

$$= \frac{\xi}{2} \left(\sum_{q\tau p\sigma} (p-1)\tau \left(a_{q\tau}^\dagger a_{q\tau} a_{p\sigma}^\dagger a_{p\sigma} - a_{p\sigma}^\dagger a_{p\sigma} a_{q\tau}^\dagger a_{q\tau} \right) \right) \quad (1.1)$$

Lets look only at the second operator ~~m~~ string in the double sum.

$$- a_{p\sigma}^\dagger a_{p\sigma} a_{q\tau}^\dagger a_{q\tau}$$

$$= -a_{p\sigma}^\dagger (\delta_{pq}\delta_{\sigma\tau} - a_{q\tau}^\dagger a_{p\sigma}) a_{q\tau}$$

$$= - a_{p\sigma}^\dagger a_{p\sigma} + a_{p\sigma}^\dagger a_{q\tau}^\dagger a_{p\sigma} a_{q\tau}$$

$$= - a_{p\sigma}^\dagger a_{p\sigma} - a_{q\tau}^\dagger a_{p\sigma}^\dagger a_{p\sigma} a_{q\tau}$$

$$= - a_{p\sigma}^\dagger a_{p\sigma} + a_{q\tau}^\dagger a_{p\sigma}^\dagger a_{q\tau} a_{p\sigma}$$

$$= - a_{p\sigma}^\dagger a_{p\sigma} + a_{q\tau}^\dagger (\delta_{pq}\delta_{\sigma\tau} - a_{q\tau} a_{p\sigma}^\dagger) a_{p\sigma}$$

$$= \underbrace{- a_{p\sigma}^\dagger \cancel{a_{p\sigma}} + a_{p\sigma}^\dagger \cancel{a_{p\sigma}}}_{\emptyset} - a_{q\tau}^\dagger a_{q\tau} a_{p\sigma}^\dagger a_{p\sigma}$$

$$= - a_{q\tau}^\dagger a_{q\tau} a_{p\sigma}^\dagger a_{p\sigma}$$

Which cancels with the first operator in equ. (1.1)

Thus $[\hat{S}_z, \hat{H}_0] \equiv 0$

$$[\hat{S}_z, V] = -\frac{g}{2}\sum_{p\sigma} \sigma\, a_{p\sigma}^\dagger a_{p\sigma} \sum_{qs} a_{q\uparrow}^\dagger a_{q\downarrow}^\dagger a_{s\uparrow} a_{s\downarrow}$$

$$+ \frac{g}{2}\sum_{qs} a_{q\uparrow}^\dagger a_{q\downarrow}^\dagger a_{s\uparrow} a_{s\downarrow} \sum_{p\sigma} \sigma\, a_{p\sigma}^\dagger a_{p\sigma}$$

$$= -\frac{g}{2}\sum_{qsp\sigma} \sigma \big(a_{p\sigma}^\dagger a_{p\sigma} a_{q\uparrow}^\dagger a_{q\downarrow}^\dagger a_{s\uparrow} a_{s\downarrow} - a_{q\uparrow}^\dagger a_{q\downarrow}^\dagger a_{s\uparrow} a_{s\downarrow} a_{p\sigma}^\dagger a_{p\sigma} \big) \quad (1.2)$$

Again lets simplify the second operator

$$a_{q\uparrow}^\dagger a_{q\downarrow} a_{s\uparrow} a_{s\downarrow} a_{p\sigma}^\dagger a_{p\sigma}$$

$$= a_{q\uparrow}^\dagger a_{q\downarrow}^\dagger a_{s\uparrow} (\delta_{sp}\delta_{\downarrow\sigma} - a_{p\sigma}^\dagger a_{s\downarrow}) a_{p\sigma}$$

$$= a_{q\uparrow}^\dagger a_{q\downarrow}^\dagger a_{s\uparrow} a_{s\downarrow} - a_{q\uparrow}^\dagger a_{q\downarrow}^\dagger a_{s\uparrow} a_{p\sigma}^\dagger a_{s\downarrow} a_{p\sigma}$$

$$= a_{q\uparrow}^\dagger a_{q\downarrow}^\dagger a_{s\uparrow} a_{s\downarrow} + a_{q\uparrow}^\dagger a_{q\downarrow}^\dagger a_{s\uparrow} a_{p\sigma}^\dagger a_{p\sigma} a_{s\downarrow}$$

$$= a_{q\uparrow}^\dagger a_{q\downarrow}^\dagger a_{s\uparrow} a_{s\downarrow} + a_{q\uparrow}^\dagger a_{q\downarrow}^\dagger (\delta_{sp}\delta_{\uparrow\sigma} - a_{p\sigma}^\dagger a_{s\uparrow}) a_{p\sigma} a_{s\downarrow}$$

$$= 2 a_{q\uparrow}^\dagger a_{q\downarrow}^\dagger a_{s\uparrow} a_{s\downarrow} - a_{q\uparrow}^\dagger a_{q\downarrow}^\dagger a_{p\sigma}^\dagger a_{s\uparrow} a_{p\sigma} a_{s\downarrow}$$

$$= 2 a_{q\uparrow}^\dagger a_{q\downarrow}^\dagger a_{s\uparrow} a_{s\downarrow} + a_{q\uparrow}^\dagger a_{q\downarrow}^\dagger a_{p\sigma}^\dagger a_{p\sigma} a_{s\uparrow} a_{s\downarrow}$$

$$= 2 a_{q\uparrow}^\dagger a_{q\downarrow}^\dagger a_{s\uparrow} a_{s\downarrow} - a_{q\uparrow}^\dagger a_{p\sigma}^\dagger a_{q\downarrow}^\dagger a_{p\sigma} a_{s\uparrow} a_{s\downarrow}$$

$$= 2 a_{q\uparrow}^\dagger a_{q\downarrow}^\dagger a_{s\uparrow} a_{s\downarrow} - a_{q\uparrow}^\dagger a_{p\sigma}^\dagger (\delta_{qp}\delta_{\downarrow\sigma} - a_{p\sigma} a_{q\downarrow}^\dagger) a_{s\uparrow} a_{s\downarrow}$$

$$= 2 a_{q\uparrow}^\dagger a_{q\downarrow}^\dagger a_{s\uparrow} a_{s\downarrow} - a_{q\uparrow}^\dagger a_{q\downarrow}^\dagger a_{s\uparrow} a_{s\downarrow} + a_{q\uparrow}^\dagger a_{p\sigma}^\dagger a_{p\sigma} a_{q\downarrow}^\dagger a_{s\uparrow} a_{s\downarrow}$$

$$= 2 a_{q\uparrow}^\dagger a_{q\downarrow}^\dagger a_{s\uparrow} a_{s\downarrow} - a_{q\uparrow}^\dagger a_{q\downarrow}^\dagger a_{s\uparrow} a_{s\downarrow} - a_{p\sigma}^\dagger a_{q\uparrow}^\dagger a_{p\sigma} a_{q\downarrow}^\dagger a_{s\uparrow} a_{s\downarrow}$$

$$= a_{q\uparrow}^\dagger a_{q\downarrow}^\dagger a_{s\uparrow} a_{s\downarrow} - a_{p\sigma}^\dagger (\delta_{qp}\delta_{\uparrow\sigma} - a_{p\sigma} a_{q\uparrow}^\dagger) a_{q\downarrow}^\dagger a_{s\uparrow} a_{s\downarrow}$$

$$= a_{q\uparrow}^\dagger a_{q\downarrow}^\dagger a_{s\uparrow} a_{s\downarrow} - a_{q\uparrow}^\dagger a_{q\downarrow}^\dagger a_{s\uparrow} a_{s\downarrow} + a_{p\sigma}^\dagger a_{p\sigma} a_{q\uparrow}^\dagger a_{q\downarrow}^\dagger a_{s\uparrow} a_{s\downarrow}$$

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Thus comparing this with the first operator string in equ. (1.2) we see

$$[\hat{S}_z, \hat{V}] = 0$$

We will exploit the following commutator

$$[\hat{S}^2, \hat{H}_0] = [\hat{S}_z^2 + \tfrac{1}{2}(\hat{S}_+\hat{S}_- + \hat{S}_-\hat{S}_+), \hat{H}_0]$$

$$= [\hat{S}_z^2, H_0] + \tfrac{1}{2}\left([S_+S_-, H_0] + [S_-S_+, H_0]\right)$$

$$[\hat{S}_z^2, H_0] = S_z^2 \hat{H}_0 - \hat{H}_0 S_z S_z$$

since $[S_z, H_0] = 0$ $\Rightarrow$ $= S_z^2 H_0 + S_z H_0 S_z$

$$= S_z^2 H_0 - S_z^2 H_0 = 0$$

$$[\hat{S}_+\hat{S}_-, \hat{H}_0] = \sum_{\substack{pr\\q\sigma}} (p-1)\big(a_{p+}^\dagger a_{p-} a_{r-}^\dagger a_{r+} a_{q\sigma}^\dagger a_{q\sigma} - a_{q\sigma}^\dagger a_{q\sigma} a_{p+}^\dagger a_{p-} a_{r-}^\dagger a_{r+}\big)$$

As before we look at the second string and simplify

$$- a_{q\sigma}^\dagger(\delta_{qp}\delta_{\sigma+} - a_{p+}^\dagger a_{q\sigma}) a_{p-} a_{r-}^\dagger a_{r+}$$

$$= - a_{p+}^\dagger a_{p-} a_{r-}^\dagger a_{r+} + a_{q\sigma}^\dagger a_{p+}^\dagger a_{q\sigma} a_{p-} a_{r-}^\dagger a_{r+}$$

$$= - a_{p+}^\dagger a_{p-} a_{r-}^\dagger a_{r+} - a_{q\sigma}^\dagger a_{p+}^\dagger a_{p-} a_{q\sigma} a_{r-}^\dagger a_{r+}$$

$$= - a_{p+}^\dagger a_{p-} a_{r-}^\dagger a_{r+} + a_{p+}^\dagger a_{q\sigma}^\dagger a_{p-} a_{q\sigma} a_{r-}^\dagger a_{r+}$$

$$= - a_{p+}^\dagger a_{p-} a_{r-}^\dagger a_{r+} + a_{p+}^\dagger(\delta_{pq}\delta_{\sigma-} - a_{p-} a_{q\sigma}^\dagger) a_{q\sigma} a_{r-}^\dagger a_{r+}$$

$$= - a_{p+}^\dagger a_{p-} a_{r-}^\dagger a_{r+} + a_{p+}^\dagger a_{p-} a_{r-}^\dagger a_{r+} - a_{p+}^\dagger a_{p-} a_{q\sigma}^\dagger a_{q\sigma} a_{r-}^\dagger a_{r+}$$

$$= - a_{p+}^\dagger a_{p-} a_{q\sigma}^\dagger(\delta_{qr}\delta_{\sigma-} - a_{r-}^\dagger a_{q\sigma}) a_{r+}$$

$$= - a_{p+}^\dagger a_{p-} a_{r-}^\dagger a_{r+} + a_{p+}^\dagger a_{p-} a_{q\sigma}^\dagger a_{r-}^\dagger a_{q\sigma} a_{r+}$$

$$= -a_{p+}^{\dagger}a_{p-}a_{r-}^{\dagger}a_{r+} - a_{p+}^{\dagger}a_{p-}a_{r-}^{\dagger}a_{q\sigma}^{\dagger}a_{q\sigma}a_{r+}$$

$$= -a_{p+}^{\dagger}a_{p-}a_{r-}^{\dagger}a_{r+} + a_{p+}^{\dagger}a_{p-}a_{r-}^{\dagger}a_{q\sigma}^{\dagger}a_{r+}a_{q\sigma}$$

$$= -a_{p+}^{\dagger}a_{p-}a_{r-}^{\dagger}a_{r+} + a_{p+}^{\dagger}a_{p-}a_{r-}^{\dagger}(\delta_{qr}\delta_{\sigma+} - a_{r+}a_{q\sigma}^{\dagger})a_{q\sigma}$$

$$= -a_{p+}^{\dagger}\underbrace{\cancel{a_{q\sigma}}}_{0}a_{r-}^{\dagger}a_{r+} + \underbrace{\cancel{a_{p+}^{\dagger}}}_{0}a_{p-}a_{r-}^{\dagger}a_{r+}$$
$$- a_{p}^{\dagger}a_{p-}a_{r-}^{\dagger}a_{r+}a_{q\sigma}^{\dagger}a_{q\sigma}$$

Thus comparing this with the first string, we see they cancel to give $[S_+S_-, H_0]=0$

$$[\hat{S}_-\hat{S}_+, \hat{H}_0] = \frac{1}{2}\sum_{\substack{pq\\ r\sigma}}(p-1)\Big(a_{r-}^{\dagger}a_{r+}a_{p+}^{\dagger}a_{p-}a_{q\sigma}^{\dagger}a_{q\sigma} - a_{q\sigma}^{\dagger}a_{q\sigma}a_{r-}^{\dagger}a_{r+}a_{p+}^{\dagger}a_{p-}\Big)$$

We simplify the second string

$$- a_{q\sigma}^{\dagger}a_{q\sigma}a_{r-}^{\dagger}a_{r+}a_{p+}^{\dagger}a_{p-}$$

$$= -a_{q\sigma}^{\dagger}(\delta_{qr}\delta_{\sigma-} - a_{r-}^{\dagger}a_{q\sigma})a_{r+}a_{p+}^{\dagger}a_{p-}$$

$$= -a_{r-}^{\dagger}a_{r+}a_{p+}^{\dagger}a_{p-} + a_{q\sigma}^{\dagger}a_{r-}^{\dagger}a_{q\sigma}a_{r+}a_{p+}^{\dagger}a_{p-}$$

$$= -a_{r-}^{\dagger}a_{r+}a_{p+}^{\dagger}a_{p-} + a_{r-}^{\dagger}a_{q\sigma}^{\dagger}a_{r+}a_{q\sigma}a_{p+}^{\dagger}a_{p-}$$

$$= -a_{r-}^{\dagger}a_{r+}a_{p+}^{\dagger}a_{p-} + a_{r-}^{\dagger}(\delta_{qr}\delta_{\sigma+} - a_{r+}a_{q\sigma}^{\dagger})a_{q\sigma}a_{p+}^{\dagger}a_{p-}$$

$$= -a_{r-}^{\dagger}a_{r+}a_{q\sigma}^{\dagger}a_{q\sigma}a_{p+}^{\dagger}a_{p-}$$

$$= -a_{r-}^{\dagger}a_{r+}a_{q\sigma}^{\dagger}(\delta_{pq}\delta_{\sigma+} - a_{p+}^{\dagger}a_{q\sigma})a_{p-}$$

$$= -a_{r-}^{\dagger}a_{r+}a_{p+}^{\dagger}a_{p-} + a_{r-}^{\dagger}a_{r+}a_{q\sigma}^{\dagger}a_{p+}^{\dagger}a_{q\sigma}a_{p-}$$

$$= -a_{r-}^{\dagger}a_{r+}a_{p+}^{\dagger}a_{p-} + a_{r-}^{\dagger}a_{r+}a_{p+}^{\dagger}a_{q\sigma}^{\dagger}a_{p-}a_{q\sigma}$$

$$= -a_{r-}^{\dagger}a_{r+}a_{p+}^{\dagger}a_{p-} + a_{r-}^{\dagger}a_{r+}a_{p+}^{\dagger}(\delta_{pq}\delta_{\sigma-} - a_{p-}a_{q\sigma}^{\dagger})a_{q\sigma}$$

$$= -a_{r-}^{\dagger}a_{r+}a_{p+}^{\dagger}a_{p-}a_{q\sigma}^{\dagger}a_{q\sigma}$$

Thus after simplifying the second string we find it cancels with the first string to give $[\hat{S}_- S_+, H_0] = 0$

Thus giving overall $[\hat{S}^2, \hat{H}_0] \equiv 0$

We will now show $[S^2, V] = 0$

$$[S^2, V] = [S_z^2, V] + \frac{1}{2}\left([S_+ S_-, V] + [S_- S_+, V]\right)$$

$$[S_z^2, V] = S_z^2 V - V S_z^2 = S_z^2 V + S_z V S_z$$

using $[S_z, V] = 0 \Rightarrow \; = S_z^2 V - S_z^2 V = 0$

$$[S_z^2, V] \equiv 0$$

Show $[S_+ S_-, V] = 0$

$$[S_+ S_-, V] = -g\left(\sum_{\substack{pq\\sr}}\left(a_{p+}^\dagger a_{p-} a_{q-}^\dagger a_{q+} a_{s+}^\dagger a_{s-}^\dagger a_{r+} a_{r-} + a_{s+}^\dagger a_{s-}^\dagger a_{r+} a_{r-} a_{p+}^\dagger a_{p-} a_{q-}^\dagger a_{q+}\right)\right)$$

Looking only at the second string op.

$$a_{s+}^\dagger a_{s-}^\dagger a_{r+} a_{r-} a_{p+}^\dagger a_{p-} a_{q-}^\dagger a_{q+}$$

$$= -a_{s+}^\dagger a_{s-}^\dagger a_{r+} a_{p+}^\dagger a_{r-} a_{p-} a_{q-}^\dagger a_{q+}$$

$$= a_{s+}^\dagger a_{s-} a_{r+} a_{p+}^\dagger a_{p-} a_{r-} a_{q-}^\dagger a_{q+}$$

$$= a_{s+}^\dagger a_{s-} (\delta_{rp}\delta_{++} - a_{p+}^\dagger a_{r+}) a_{p-} a_{r-} a_{q-}^\dagger a_{q+}$$

$$= a_{s+}^\dagger a_{s-}^\dagger \underbrace{a_{r-} a_{r-}} a_{q-}^\dagger a_{q+} - a_{s+}^\dagger a_{s-}^\dagger a_{p+}^\dagger a_{r+} a_{p-} a_{r-} a_{q-}^\dagger a_{q+}$$

notice $[a_{r-}, a_{r-}] = 0 \Rightarrow \hat{a}_{r-}\hat{a}_{r-} = \hat{\phi}$

$$= -a_{s+}^\dagger a_{s-}^\dagger a_{p+}^\dagger a_{r+} a_{p-} a_{r-} a_{q-}^\dagger a_{q+}$$

$$= a_{s+}^\dagger a_{s-}^\dagger a_{p+}^\dagger a_{p-} a_{r+} a_{r-} a_{q-}^\dagger a_{q+}$$

$$= a_{p+}^\dagger a_{s+}^\dagger a_{s-}^\dagger a_{p-} a_{r+} a_{r-} a_{q-}^\dagger a_{q+}$$

$$= a_{p+}^\dagger a_{s+}^\dagger \left(\delta_{sp} - a_{p-} a_{s-}^\dagger\right) a_{r+} a_{r-} a_{q-}^\dagger a_{q+}$$

$$= \underbrace{a_{s+}^\dagger a_{s+}^\dagger}_{\hat{\emptyset}} a_{r+} a_{r-} a_{q-}^\dagger a_{q+} - a_{p+}^\dagger a_{s+}^\dagger a_{p-} a_{s-}^\dagger a_{r+} a_{r-} a_{q-}^\dagger a_{q+}$$

$$= a_{p+}^\dagger a_{p-} a_{s+}^\dagger a_{s-}^\dagger a_{r+} a_{r-} a_{q-}^\dagger a_{q+}$$

$$= a_{p+}^\dagger a_{p-} a_{s+}^\dagger a_{s-}^\dagger a_{r+} \left(\delta_{rq} - a_{q-}^\dagger a_{r-}\right) a_{q+}$$

$$= a_{p+}^\dagger a_{p-} a_{s+}^\dagger a_{s-}^\dagger \underbrace{a_{q+} a_{q+}}_{\hat{\emptyset}} - a_{p+}^\dagger a_{p-} a_{s+}^\dagger a_{s-}^\dagger a_{r+} a_{q-}^\dagger a_{r-} a_{q+}$$

$$= - a_{p+}^\dagger a_{p-} a_{s+}^\dagger a_{s-}^\dagger a_{q-}^\dagger a_{r+} a_{q+} a_{r-}$$

$$= - a_{p+}^\dagger a_{p-} a_{q-}^\dagger a_{s+}^\dagger a_{s-}^\dagger a_{q+} a_{r+} a_{r-}$$

$$= a_{p+}^\dagger a_{p-} a_{q-}^\dagger \left(\delta_{sq} - a_{q+} a_{s+}^\dagger\right) a_{s-}^\dagger a_{r+} a_{r-}$$

$$= a_{p+}^\dagger a_{p-} \underbrace{a_{q-}^\dagger a_{q-}^\dagger}_{\hat{\emptyset}} a_{r+} a_{r-}$$

$$= - a_{p+}^\dagger a_{p-} a_{q-}^\dagger a_{q}^\dagger a_{s+}^\dagger a_{s-}^\dagger a_{r+} a_{r-}$$

which cancels with the first operator.
to give $[\hat{S}_+ \hat{S}_-, \hat{V}] = 0$

Show $[\hat{S}_- \hat{S}_+, V] = 0$

$$[S_- S_+, V] = -g \sum_{\substack{pq \\ sr}} \Big(a_{q-}^\dagger a_{q+} a_{p+}^\dagger a_{p-} a_{s+}^\dagger a_{s-}^\dagger a_{r+} a_{r-} - a_{s+}^\dagger a_{s-}^\dagger a_{r+} a_{r-} a_{q-}^\dagger a_{q+} a_{p+}^\dagger a_{p-} \Big)$$

$$- a_{s+}^\dagger a_{s-}^\dagger a_{r+} a_{r-} a_{q-}^\dagger a_{q+} a_{p+}^\dagger a_{p-}$$

$$= - a_{s+}^\dagger a_{s-}^\dagger a_{r+} (\delta_{rq}\delta_{--} - a_{q-}^\dagger a_{r-}) a_{q+} a_{p+}^\dagger a_{p-}$$

$$= - a_{s+}^\dagger a_{s-}^\dagger \underbrace{a_{q+} a_{q+}}_{\hat{\emptyset}} a_{p+}^\dagger a_{p-}$$

$$+ a_{s+}^\dagger a_{s-}^\dagger a_{r+} a_{q-}^\dagger a_{r-} a_{q+} a_{p+}^\dagger a_{p-}$$

$$= a_{s+}^\dagger a_{s-}^\dagger a_{q-}^\dagger a_{r+} a_{q+} a_{r-} a_{p+}^\dagger a_{p-}$$

$$= - a_{q-}^\dagger a_{s+}^\dagger a_{s-}^\dagger a_{q+} a_{r+} a_{r-} a_{p+}^\dagger a_{p-}$$

$$= a_{q-}^\dagger a_{s+}^\dagger a_{q+} a_{s-}^\dagger a_{r+} a_{r-} a_{p+}^\dagger a_{p-}$$

$$= a_{q-}^\dagger (\delta_{sq}\delta_{++} - a_{q+} a_{s+}^\dagger) a_{s-}^\dagger a_{r+} a_{r-} a_{p+}^\dagger a_{p-}$$

$$= \underbrace{a_{q-}^\dagger a_{q-}^\dagger}_{\hat{\emptyset}} a_{r+} a_{r-} a_{p+}^\dagger a_{p-} - a_{q-}^\dagger a_{q+} a_{s+}^\dagger a_{s-}^\dagger a_{r+} a_{r-} a_{p+}^\dagger a_{p-}$$

$$= a_{q-}^\dagger a_{q+} a_{s+}^\dagger a_{s-}^\dagger a_{r+} a_{p+}^\dagger a_{r-} a_{p-}$$

$$= - a_{q-}^\dagger a_{q+} a_{s+}^\dagger a_{s-}^\dagger (\delta_{rp} - a_{p+}^\dagger a_{r+}) a_{p-} a_{r-}$$

$$= - a_{q-}^\dagger a_{q+} a_{s+}^\dagger a_{s-}^\dagger \underbrace{a_{r-} a_{r-}}_{\hat{\emptyset}} + a_{q-}^\dagger a_{q+} a_{s+}^\dagger a_{s-}^\dagger a_{p+}^\dagger a_{r+} a_{p-} a_{r-}$$

$$= a_{q-}^\dagger a_{q+} a_{p+}^\dagger a_{s+}^\dagger a_{s-}^\dagger a_{r+} a_{p-} a_{r-}$$

$$= -a_{q-}^\dagger a_{q+} a_{p+}^\dagger a_{s+}^\dagger(\delta_{sp} - a_{p-} a_{s-}^\dagger) a_{r+} a_{r-}$$

$$= a_{q-}^\dagger a_{q+} \underbrace{a_{p+}^\dagger a_{p+}^\dagger}_{\hat{\emptyset}} + a_{q-}^\dagger a_{q+} a_{p+}^\dagger a_{s+}^\dagger a_{p-} a_{s-}^\dagger a_{r+} a_{r-}$$

$$= -a_{q-}^\dagger a_{q+} a_{p+}^\dagger a_{p-} a_{s+}^\dagger a_{s-}^\dagger a_{r+} a_{r-}$$

Thus canceling the first string of operators.

$[\hat{S}_- \hat{S}_+, V] = 0$

Thus totaling to give $[\hat{S}^2, V] = 0$

Prove $[P_r^+ P_q^-, H_0] = 0$

Neglecting the constants and the sums the problem reduces to simplifying

$$a_{r+}^\dagger a_{r-} a_{q-}^\dagger a_{q+} a_{p\sigma}^\dagger a_{p\sigma} - a_{p\sigma}^\dagger a_{p\sigma} a_{r+}^\dagger a_{r-} a_{q-}^\dagger a_{q+}$$

Like the problems before we simplify the last string.

$$-a_{p\sigma}^\dagger(\delta_{pr}\delta_{\sigma+} - a_{r+}^\dagger a_{p\sigma}) a_{r-} a_{q-}^\dagger a_{q+}$$

$$= -a_{r+}^\dagger a_{r-} a_{q-}^\dagger a_{q+} + a_{p\sigma}^\dagger a_{r+}^\dagger a_{p\sigma} a_{r-} a_{q-}^\dagger a_{q+}$$

$$= -a_{r+}^\dagger a_{r-} a_{q-}^\dagger a_{q+} + a_{r+}^\dagger a_{p\sigma}^\dagger a_{r-} a_{p\sigma} a_{q-}^\dagger a_{q+}$$

$$= -a_{r+}^\dagger a_{r-} a_{q-}^\dagger a_{q+} + a_{r+}^\dagger(\delta_{pr}\delta_{\sigma-} - a_{r-} a_{p\sigma}^\dagger) a_{p\sigma} a_{q-}^\dagger a_{q+}$$

$$= -a_{r+}^\dagger \underbrace{a_{r-} a_{q-}^\dagger}_{\phi} a_{q+} + a_{r+}^\dagger \underbrace{a_{r-} a_{q-}^\dagger}_{\phi} a_{q+}$$

$$- a_{r+}^\dagger a_{r-} a_{p\sigma}^\dagger a_{p\sigma} a_{q-}^\dagger a_{q+}$$

$$= -a_{r+}^\dagger a_{r-} a_{p\sigma}^\dagger(\delta_{pq}\delta_{\sigma-} - a_{q-}^\dagger a_{p\sigma}) a_{q+}$$

$$= -a_{r+}^\dagger a_{r-} a_{p-}^\dagger a_{p+} + a_{r+}^\dagger a_{r-} a_{p\sigma}^\dagger a_{q-}^\dagger a_{p\sigma} a_{q+}$$

$$= -a_{r+}^\dagger a_{r-} a_{p-}^\dagger a_{p+} + a_{r+}^\dagger a_{r-} a_{q-}^\dagger a_{p\sigma}^\dagger a_{q+} a_{p\sigma}$$

$$= -a_{r+}^\dagger a_{r-} a_{p-}^\dagger a_{p+} + a_{r+}^\dagger a_{r-} a_{q-}^\dagger (\delta_{pq}\delta_{\sigma+} - a_{q+} a_{p\sigma}^\dagger) a_{p\sigma}$$

$$= -a_{r+}^\dagger a_{r-} \cancel{a_{p-}^\dagger a_{p+}} + a_{r+}^\dagger a_{r-} \cancel{a_{p-}^\dagger a_{p+}} - a_{r+}^\dagger a_{r-} a_{q-}^\dagger a_{q+} a_{p\sigma}^\dagger a_{p\sigma}$$

$$= -a_{r+}^\dagger a_{r-} a_{q-}^\dagger a_{q+} a_{p\sigma}^\dagger a_{p\sigma}$$

Thus cancels with the first operator leading to $[P^\dagger P_-, H_0] = 0$

Next I will aim to show $[P^\dagger P_-, V] = 0$

$$[P^\dagger P_-, V] = -g \sum_{\substack{qr \\ st}} \Big(a_{s+}^\dagger a_{s-}^\dagger a_{t+} a_{t-} a_{q+}^\dagger a_{q-}^\dagger a_{r+} a_{r-} - a_{q+}^\dagger a_{q-}^\dagger a_{r+} a_{r-} a_{s+}^\dagger a_{s-}^\dagger a_{t+} a_{t-} \Big)$$

Simplifying the second term

$$= -a_{q+}^\dagger a_{q-}^\dagger a_{r+} a_{r-} a_{s+}^\dagger a_{s-}^\dagger a_{t+} a_{t-}$$

$$= a_{q+}^\dagger a_{q-}^\dagger a_{r+} a_{s+}^\dagger a_{r-} a_{s-}^\dagger a_{t+} a_{t-}$$

$$= a_{q+}^\dagger a_{q-}^\dagger (\delta_{sr} - a_{s+}^\dagger a_{r+}) a_{r-} a_{s-}^\dagger a_{t+} a_{t-}$$

$$= \boxed{a_{q+}^\dagger a_{q-}^\dagger a_{s-} a_{s-}^\dagger a_{t+} a_{t-}} - a_{q+}^\dagger a_{q-}^\dagger a_{s+}^\dagger a_{r+} a_{r-} a_{s-}^\dagger a_{t+} a_{t-}$$

$$= -a_{q+}^\dagger a_{q-}^\dagger a_{s+}^\dagger a_{r+} (\delta_{rs} - a_{s-}^\dagger a_{r-}) a_{t+} a_{t-}$$

$$= \boxed{-a_{q+}^\dagger a_{q-}^\dagger a_{r+}^\dagger a_{r+} a_{t+} a_{t-}}$$

$$+ a_{q+}^\dagger a_{q-}^\dagger a_{s+}^\dagger a_{r+} a_{s-}^\dagger a_{r-} a_{t+} a_{t-}$$

$$= -a_{q+}^\dagger a_{q-}^\dagger a_{s+}^\dagger a_{s-}^\dagger a_{r+} a_{r-} a_{t+} a_{t-}$$

$$= a_{s+}^\dagger a_{q+}^\dagger a_{q-}^\dagger a_{s-}^\dagger a_{r+} a_{r-} a_{t+} a_{t-}$$

$$= -a_{s+}^\dagger a_{s-}^\dagger a_{q+}^\dagger a_{q-}^\dagger a_{t+} a_{t-} a_{r+} a_{r-}$$

$$= a_{s+}^\dagger a_{s-}^\dagger a_{q+}^\dagger a_{t+} a_{q-}^\dagger a_{t-} a_{r+} a_{r-}$$

$$= a_{s+}^\dagger a_{s-}^\dagger (\delta_{qt} - a_{t+} a_{q+}^\dagger) a_{q-}^\dagger a_{t-} a_{r+} a_{r-}$$

$$= \boxed{a_{s+}^\dagger a_{s-}^\dagger a_{q-}^\dagger a_{q-} a_{r+} a_{r-}}$$

$$- a_{s+}^\dagger a_{s-}^\dagger a_{t+} a_{q+}^\dagger a_{q-}^\dagger a_{t-} a_{r+} a_{r-}$$

$$= - a_{s+}^\dagger a_{s}^\dagger a_{t+} a_{q+}^\dagger (\delta_{qt} - a_{t-} a_{q-}^\dagger) a_{r+} a_{r-}$$

$$= \boxed{- a_{s+}^\dagger a_{s-}^\dagger a_{q+} a_{q+}^\dagger a_{r+} a_{r-}}$$

$$+ a_{s+}^\dagger a_{s-}^\dagger a_{t+} a_{q+}^\dagger a_{t-} a_{q-}^\dagger a_{r+} a_{r-}$$

If we look at the elements in the boxes

$$a_{s+}^\dagger a_{s-}^\dagger a_{q-}^\dagger a_{q-} a_{r+} a_{r-} \quad - a_{s+}^\dagger a_{s-}^\dagger a_{q+} a_{q+}^\dagger a_{r+} a_{r-}$$

$$a_{q+}^\dagger a_{q-}^\dagger a_{s-}^\dagger a_{s-}^\dagger a_{t+} a_{t-} \quad - a_{q+}^\dagger a_{q-}^\dagger a_{r+}^\dagger a_{r+} a_{t+} a_{t-}$$

By changing the dumy indricies and noting that

$$a_{q-}^\dagger a_{q-} = - a_{q-} a_{q-}^\dagger$$

$$\& \quad a_{q+} a_{q+}^\dagger = - a_{q+}^\dagger a_{q+}$$

Thus these 4 tems cancel giving only

$$= a_{s+}^\dagger a_{s-}^\dagger a_{t+} a_{q+}^\dagger a_{t-} a_{q-}^\dagger a_{r+} a_{r-}$$

$$= - a_{s+}^\dagger a_{s-}^\dagger a_{t+} a_{t-} a_{q+}^\dagger a_{q-}^\dagger a_{r+} a_{r-}$$

which cancels with the first operator

giving $\quad [P^+ P_-, V] = 0$

b.) $SD_1 = a_{1+}^\dagger a_{1-}^\dagger |0\rangle$

$SD_2 = a_{2+}^\dagger a_{2-}^\dagger |0\rangle$

Using Wick's thm we calculate only the fully contracted terms to calculate the matrix elements.

$$\langle SD_1 | \hat{H}_0 | SD_1\rangle = \langle 0 | a_{1-} a_{1+} \sum_p (p-1) a_{p\sigma}^\dagger a_{p\sigma} a_{1+}^\dagger a_{1-}^\dagger |0\rangle$$

$$a_{1-} a_{1+} a_{p\sigma}^\dagger a_{p\sigma} a_{1+}^\dagger a_{1-}^\dagger$$

giving rise to two terms

$$\delta_{p1}\delta_{\sigma 1} + \delta_{1p}\delta_{\sigma +}$$

This is just the sum over the sp energies as expected. Thus $\langle SD_1 | \hat{H}_0 | SD_1\rangle = \sum_p (p-1) \delta_{p1}\delta_{\sigma -} + \delta_{p1}\delta_{\sigma +}$

$$= \emptyset$$

$$\langle SD_2 | \hat{H}_0 | SD_2\rangle = \sum_p (p-1)(\delta_{p2}\delta_{\sigma -} + \delta_{p2}\delta_{\sigma +}) = 2$$

The off diagonal elements are symmetric. Thus we calculate $\langle SD_1 | \hat{V} | SD_2\rangle = \langle SD_2 | \hat{V} | SD_1\rangle$

$$a_{1-} a_{1+} a_{p+}^\dagger a_{p-}^\dagger a_{q-} a_{q+} a_{2+}^\dagger a_{2-}^\dagger$$

$$\delta_{q1}\delta_{p2}$$

$$\langle SD_1 | \hat{V} | SD_2\rangle = -g \sum_{pq} \delta_{q1}\delta_{p2} = -g$$

$$\langle SD_1 | \hat{V} | SD_1 \rangle = -g \sum_{pq} \delta_{p1}\delta_{q1} = -g$$

$$a_{1-}\, a_{1+}\, a_{p+}^{\dagger} a_{p-}^{\dagger}\, a_{q-}\, a_{q+}\, a_{1+}^{\dagger} a_{1-}$$

$$\langle SD_2 | \hat{V} | SD_2 \rangle = -g \sum_{pq} \delta_{p2}\delta_{q2} = -g$$

$$a_{2-}\, a_{2+}\, a_{p+}^{\dagger} a_{p-}^{\dagger}\, a_{q-}\, a_{q}\, a_{2+}^{\dagger} a_{2-}$$

$$\hat{H} = \begin{pmatrix} -g & -g \\ -g & 2\xi - g \end{pmatrix}$$

Assuming $\xi = 1$

Eigen values $\quad 1 - g \pm \sqrt{1+g^2}$

Eigen vectors $\quad \begin{pmatrix} \dfrac{-1 \pm \sqrt{1+g^2}}{g} \\ 1 \end{pmatrix}$

My discussion of the eigen values for parts b.) & c.) will be on a next page c.) 4 particle 4-level calculation

$$\# \text{ of SD's} = \frac{4!}{(4-2)!\,2!} = \frac{4\cdot 3}{2} = 6 \text{ SD } (S=0)$$

$$SD_1 = a_{2+}^\dagger a_{2-}^\dagger a_{1+}^\dagger a_{1-}^\dagger |0\rangle \qquad a_{4+}^\dagger a_{4-}^\dagger a_{3+}^\dagger a_{3-}^\dagger |0\rangle$$

$$SD_2 = a_{3+}^\dagger a_{3-}^\dagger a_{1+}^\dagger a_{1-}^\dagger |0\rangle \qquad SD_6 = a_{4+}^\dagger a_{4-}^\dagger a_{2+}^\dagger a_{2-}^\dagger |0\rangle$$

$$a_{4+}^\dagger a_{4-}^\dagger a_{1+}^\dagger a_{1-}^\dagger |0\rangle$$

$$a_{3+}^\dagger a_{3-}^\dagger a_{2+}^\dagger a_{2-}^\dagger |0\rangle$$

Let's use wick's theorem and calculate the fully contracted terms.

$$\langle SD_1 | \hat{H}_0 | SD_1 \rangle = \sum_p (p-1) \langle 0 | a_{1-} a_{1+} a_{2-} a_{2+} a_{p\sigma}^\dagger a_{p\sigma} \ldots a_{2+}^\dagger a_{2-}^\dagger a_{1+}^\dagger a_{1-}^\dagger |0\rangle$$

$$a_{1-} a_{1+} a_{2-} a_{2+} a_{p\sigma}^\dagger a_{p\sigma} a_{2+}^\dagger a_{2-}^\dagger a_{1+}^\dagger a_{1-}^\dagger$$

$$= \xi \sum_p (p-1) \left(\delta_{2p}\delta_{\sigma+} + \delta_{2p}\delta_{\sigma-} + \delta_{1p}\delta_{\sigma+} + \delta_{1p}\delta_{\sigma-} \right)$$

$$= 2\xi$$

As we expect the diagonal elements are just the sum of the sp energies. Since H_0 is diagonal, the term can only connect diagonal states.

we will find the diagonal elements in this way

$$\hat{H}_0 = \begin{pmatrix} 2\xi & & & & & \\ & 4\xi & & & 0 & \\ & & 6\xi & & & \\ & & & 6\xi & & \\ & 0 & & & 8\xi & \\ & & & & & 10\xi \end{pmatrix}$$

$$\langle SD_1 | \hat{V} | SD_1 \rangle$$

$$= -g \sum_{pq} \langle 0 | a_{1-} a_{1+} a_{2-} a_{2+} a_{p+}^{\dagger} a_{p-}^{\dagger} a_{q-} a_{q+} a_{2+}^{\dagger} a_{2-}^{\dagger} a_{1+}^{\dagger} a_{1-}^{\dagger} | 0 \rangle$$

$$a_{1-} a_{1+} a_{2-} a_{2+} a_{p+}^{\dagger} a_{p-}^{\dagger} a_{q-} a_{q+} a_{2+}^{\dagger} a_{2-}^{\dagger} a_{1+}^{\dagger} a_{1-}^{\dagger}$$

$$= -g \sum_{pq} \left(\delta_{p2}\delta_{q2} + \delta_{p1}\delta_{q1} \right) = -2g$$

Thank fully due to wick's thm there are only two non-zero, full contractions, that give rise to the diagonal elements. Since the interaction only depends on a constant and, for the diagonal matrix elements, counts the number of levels. It does not depend on the level number explicitly. Thus all diagonal elements get $-2g$.

We will calculate the first of diagonal element.

$$\langle SD_1 | \hat{V} | SD_2 \rangle = -g \sum_{pq} \langle 0 | a_{1-} a_{1+} a_{2-} a_{2+} a_{p+}^\dagger a_{p-}^\dagger a_{q-} a_{q+} a_{2+}^\dagger \cdots$$

$$a_{1-}\, a_{1+}\, a_{2-} a_{2+}\, a_{p+}^\dagger a_{p-}^\dagger\, a_{q-} a_{q+}\, a_{3+}^\dagger a_{3-}^\dagger\, a_{1+}^\dagger a_{1-}^\dagger$$

$$-g \sum_{qp} \delta_{q3} \delta_{p2}$$

According to wick's theorem only one term survives that is non-zero. We expect this as the 2-body interaction in this case creates & destroys pairs. Thus it can only connect two dissimilar pairs. The other two pairs must be the same in each SD.

i.e. $\langle a_{2+}^\dagger a_{2-}^\dagger a_{1+}^\dagger a_{-}^\dagger | \hat{V} | a_{3+}^\dagger a_{3-}^\dagger a_{1+}^\dagger a_{-}^\dagger \rangle \neq 0$

$\langle a_{2}^\dagger a_{2-}^\dagger a_{4+}^\dagger a_{4-}^\dagger | \hat{V} | a_{3+}^\dagger a_{3-}^\dagger a_{1+}^\dagger a_{1-}^\dagger \rangle \equiv \emptyset$

$\Delta 1$ in state $\quad \Delta$ 3 in state

We are only allowed 1 difference in the states

Thus the final hamiltonian is listed on the back.

$$|a_{2+}^{\dagger} a_{2-}^{\dagger} a_{1+}^{\dagger} a_{1-}^{\dagger}|0\rangle = |12\rangle$$

$$\hat{H} = \begin{array}{c|cccccc}
 & 12 & 13 & 14 & 23 & 24 & 34 \\
\hline
12 & -2g & -g & -g & -g & -g & 0 \\
13 & -g & 4-2g & -g & -g & 0 & -g \\
14 & -g & -g & 6-2g & 0 & -g & -g \\
23 & -g & -g & 0 & 6-2g & -g & -g \\
24 & -g & 0 & -g & -g & 8-2g & -g \\
34 & 0 & -g & -g & -g & -g & 10-2g
\end{array}$$

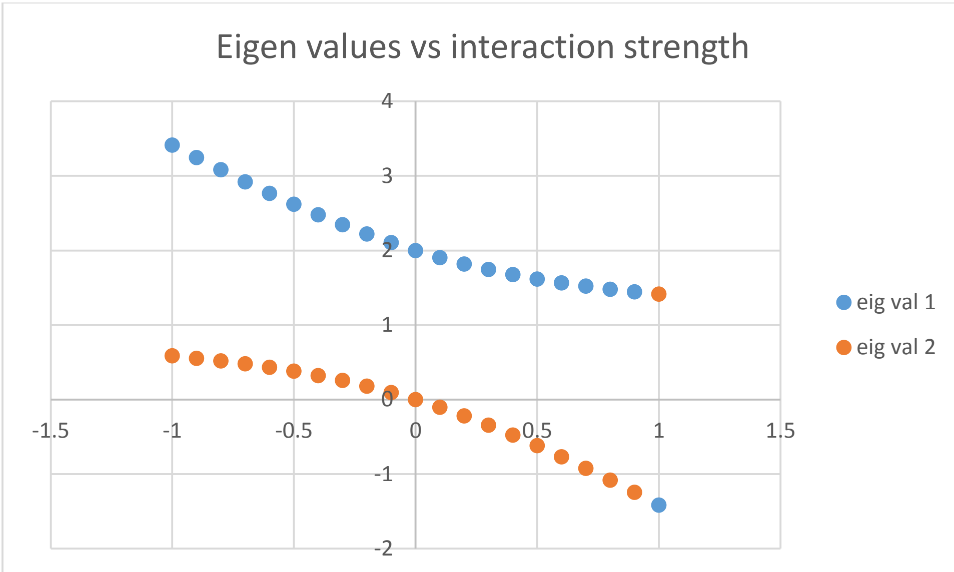

Figure 1.1 showing the two eigenvalues from a calculation of a 2-particle 2-level scheme. The x-axis is the interaction term, g, which was varied from -1 to 1 in steps of .1.

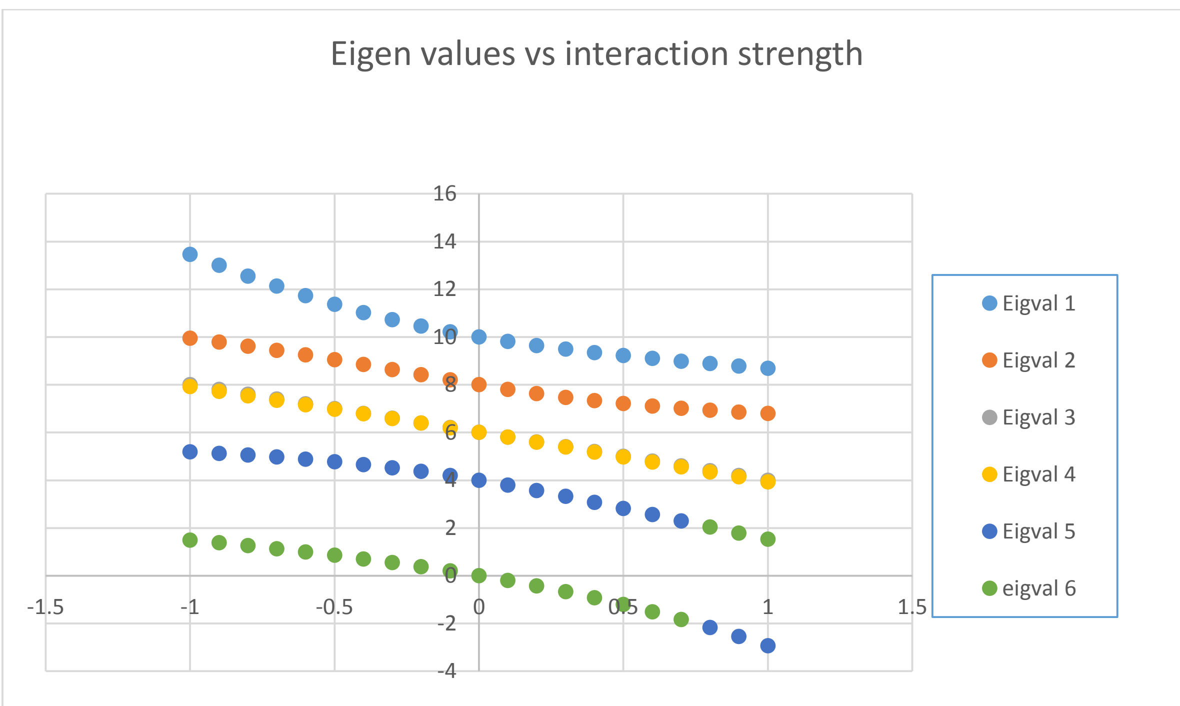

Figure 1.2 showing the eigenvalues from a calculation of a 4-particle 4-level scheme. The x-axis is the interaction term, g, which was varied from -1 to 1 in steps of .1.

In both figures you can see the ground state is primarily one eigenvector for most of the range of the parameter g. Then near the end the eigenvectors which produce the ground state function switch nearing g approximately equal to 1. Also another feature is the states are smooth and continuous. Also we see level repulsion for both cases. I have also checked this for the states which are representing eigenvalues 4 and 3 in Fig 1.2. Where when g is non-zero they repel and when g is zero they are degenerate. We expect this type of level repulsion when introducing an interaction term.