Ex. 9 Justin Estee

$$|\phi_i^a\rangle = a_a^\dagger a_i |\phi_0\rangle$$

$$|\phi_{ij}^{ab}\rangle = a_a^\dagger a_b^\dagger a_i a_j |\phi_0\rangle$$

b.) Generalized Wick's Theorem states we must only construct strings between normal ordered strings of operators

i.e. $a_q \{a_p^\dagger a_q^\dagger a_s a_r\} a_a^\dagger = 0$

will give ϕ $\Downarrow$ ϕ

$$\langle\phi_0| \hat{F} |\phi_0\rangle = \sum_{pq} \langle p|f|q\rangle \langle\phi_0| \{a_p^\dagger a_q\} |\phi_0\rangle$$

$$= \phi$$

$$\langle\phi_0| \hat{G} |\phi_0\rangle = \frac{1}{4}\sum_{pqrs} \langle pq|g|rs\rangle_{AS} \langle\phi_0| \{a_p^\dagger a_q^\dagger a_s a_r\} |\phi_0\rangle$$

both contractions of normalized order give zero.

$$\langle\phi_0| \hat{G} |\phi_0\rangle = \phi$$

c.) $\langle\phi_0| \hat{F}_N |\phi_i^a\rangle = \sum_{pq} \langle p|f|q\rangle \langle\phi_0| \{a_p^\dagger a_q\} a_a^\dagger a_i |\phi_0\rangle$

$$= \sum_{pq} \langle p|f|q\rangle \quad \delta_{pi<F}\, \delta_{aq>F}$$

c.) cont.)

$$\langle \Phi_0 | \hat{G}_N | \Phi_i^a \rangle = \langle \Phi_0 | \{a_p^\dagger a_q^\dagger a_s a_r\} a_a^\dagger a_i | \Phi_0 \rangle \sum_{pqrs} \langle pq|g|rs\rangle_{AS}$$
$$= \emptyset$$

Zero because all possible contractions with the normal ordered string would lead to zero

d.) $$\langle \Phi_0 | \hat{F}_N | \Phi_{ij}^{ab} \rangle = \sum_{pq} \langle p|f|q\rangle \langle \Phi_0 | \{a_p^\dagger a_q\} a_a^\dagger a_b^\dagger a_i a_j | \Phi_0 \rangle$$
$$= \emptyset$$

We can only make two contractions with the normal ordered pair. But one pair is left out

$$\{a_p^\dagger a_q\} a_a^\dagger a_b^\dagger a_i a_j$$

$$\left.\begin{array}{l} \to \emptyset \\ \to \emptyset \\ \to \emptyset \\ \to \emptyset \end{array}\right\} \begin{array}{l} \delta_{bi} \\ \delta_{ai} \\ \delta_{bj} \\ \delta_{ab} \end{array}$$

all these contractions give $\emptyset$ since $a b$ are above the Fermi level and i, j below.

$$\langle \Phi_0 | \hat{G}_N | \Phi_{ij}^{ab} \rangle = \frac{1}{4} \sum_{pqrs} \langle pq|g|rs\rangle_{AS} \langle \Phi_0 | \{a_p^\dagger a_q^\dagger a_s a_r\} a_a^\dagger a_b^\dagger a_i a_j | \Phi_0 \rangle$$

d.) cont.)

$$\{a_p^\dagger a_q^\dagger a_s a_r\} a_a^\dagger a_b^\dagger a_i a_j$$

$\delta_{pj}\delta_{qi}\delta_{sb}\delta_{ra}$

$-\delta_{pj}\delta_{qi}\delta_{sa}\delta_{rb}$

$-\delta_{pi}\delta_{qj}\delta_{sb}\delta_{ra}$

$\delta_{pi}\delta_{qj}\delta_{sa}\delta_{rb}$

We have 4 terms

$$\langle ji|\hat{g}|ab\rangle_{AS} - \langle ji|\hat{g}|ba\rangle_{AS}$$

$$\langle ij|\hat{g}|ba\rangle_{AS} - \langle ij|\hat{g}|ab\rangle_{AS}$$

Since
$$\langle ji|\hat{g}|ab\rangle_{AS} = \langle ij|\hat{g}|ba\rangle_{AS}$$
$$\langle ij|\hat{g}|ab\rangle_{AS} = -\langle ji|\hat{g}|ab\rangle_{AS}$$
$$\langle ji|\hat{g}|ba\rangle_{AS} = -\langle ji|\hat{g}|ab\rangle_{AS}$$

Thus all 4 terms are equal and the matrix element is

$$\langle \Phi_0|\hat{G}|\Phi_{ij}^{ab}\rangle = \langle ji|\hat{g}|ab\rangle_{AS}$$

e.) $\langle \Phi_0|\hat{G}|\Phi_{ijk}^{abc}\rangle$ its clear for any terms greater than 2h-2p state on the right will lead to ϕ since $\hat{G}$ will contract all but one pair of $\Phi_{ijk...}^{abc...}$ and since hole & particle states are orthogonal we get zero.