

R-Notes

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OR-Notes are a series of introductory notes on topics that fall under the broad heading of the field of operations research (OR). They were originally used by me in an introductory OR course I give at Imperial College. They are now available for use by any students and teachers interested in OR subject to the following [conditions](#).

A full list of the topics available in OR-Notes can be found [here](#).

Linear programming formulation examples

Linear programming example 1996 MBA exam

A cargo plane has three compartments for storing cargo: front, centre and rear. These compartments have the following limits on both weight and space:

Compartment	Weight (tonnes)	Space capacity (cubic metres)
Front	20	6000
Centre	26	8700
Rear	20	5900

Furthermore, the weight of the cargo in the respective compartments must be the same proportion of that compartment's weight capacity to maintain the balance of the plane.

The following four cargoes are available for shipment on the next flight:

cargo	weight (tonnes)	volume (cubic metres/tonne)	profit (£/tonne)
C1	18	400	310
C2	15	450	380
C3	23	480	350
C4	12	180	290

Any proportion of these cargoes can be accepted. The objective is to determine how much (if any) of each cargo C1, C2, C3 and C4 should be accepted and how to distribute each among the compartments so that the total profit for the flight is maximised.

- Formulate the above problem as a linear program
- What assumptions are made in formulating this problem as a linear program?
- Briefly describe the advantages of using a software package to solve the above linear program, over a judgemental approach to this problem.

Solution

Variables

We need to decide how much of each of the four cargoes to put in each of the three compartments. Hence let:

x_{ij} be the number of tonnes of cargo ($i=1,2,3,4$ for C1, C2, C3 and C4 respectively) that is put into compartment j ($j=1$ for Front, $j=2$ for Centre and $j=3$ for Rear) where $x_{ij} \geq 0$ $i=1,2,3,4$; $j=1,2,3$

Note here that we are explicitly told we can split the cargoes into any proportions (fractions) that we like.

Constraints

- cannot pack more of each of the four cargoes than we have available

$$\begin{aligned}x_{11} + x_{21} + x_{31} + x_{41} &\leq 18 \\x_{21} + x_{32} + x_{42} &\leq 15 \\x_{31} + x_{32} + x_{33} &\leq 23 \\x_{41} + x_{42} + x_{43} &\leq 12\end{aligned}$$

- the weight capacity of each compartment must be respected

$$\begin{aligned}x_{11} + x_{21} + x_{31} + x_{41} &\leq 10 \\x_{12} + x_{22} + x_{32} + x_{42} &\leq 16 \\x_{13} + x_{23} + x_{33} + x_{43} &\leq 8\end{aligned}$$

- the volume (space) capacity of each compartment must be respected

$$\begin{aligned}400x_{11} + 650x_{21} + 500x_{31} + 390x_{41} &\leq 6000 \\400x_{12} + 650x_{22} + 500x_{32} + 390x_{42} &\leq 8700 \\400x_{13} + 650x_{23} + 500x_{33} + 390x_{43} &\leq 5900\end{aligned}$$

- the weight of the cargo in the respective compartments must be the same proportion of that compartment's weight capacity to maintain the balance of the plane

$$[x_{11} + x_{21} + x_{31} + x_{41}]/10 = [x_{12} + x_{22} + x_{32} + x_{42}]/16 = [x_{13} + x_{23} + x_{33} + x_{43}]/8$$

Objective

The objective is to maximise total profit, i.e.

$$\text{maximise } 310(x_{11} + x_{12} + x_{13}) + 380(x_{21} + x_{22} + x_{23}) + 350(x_{31} + x_{32} + x_{33}) + 290(x_{41} + x_{42} + x_{43})$$

The basic assumptions are:

- that each cargo can be split into whatever proportions/fractions we desire
 - that each cargo can be split between two or more compartments if we so desire
 - that the cargo can be packed into each compartment (for example if a cargo was spherical it would not be possible to pack a compartment to volume capacity, some free space is inevitable in sphere packing)
 - that the results from the algorithm seem reasonable (or if they are surprising we can at least understand why they are surprising).
- The advantages of using a software package to solve the above linear program, rather than a judgemental approach are:
- actually maximise profit, rather than just believing that our judgemental solution maximises profit (we may have had judgement, even if we have an MBA!)
 - makes the cargo loading the decision one that we can solve in a routine operational manner on a computer, rather than having to exercise judgement each and every time we want to solve it
 - problems that can be appropriately formulated as linear programs are almost always better solved by computers than by people
 - can perform sensitivity analysis very easily using a computer

Linear programming example 1995 MBA exam

Briefly describe the main steps in using mathematical modelling to support management.

A canning company operates two canning plants. The growers are willing to supply fresh fruits in the following amounts:

- S1: 200 tonnes at £11/tonne
- S2: 310 tonnes at £10/tonne
- S3: 420 tonnes at £9/tonne

Shipping costs in £ per tonne are:

	To: Plant A	Plant B
From: S1	3	3.5
S2	2	2.5
S3	6	4

Plant capacities and labour costs are:

	Plant A	Plant B
Capacity	480 tonnes	560 tonnes
Labour cost	£20/tonne	£25/tonne

The canned fruits are sold at £50/tonne to the distributors. The company can sell at this price all they can produce.

The objective is to find the best mixture of the quantities supplied by the three growers to the two plants so that the company maximises its profits.

- Formulate the problem as a linear program and explain it
- Explain the meaning of the dual values associated with the supply and plant capacity constraints
- What assumptions have you made in expressing the problem as a linear program

Solution

The main steps in using mathematical modelling to support management are:

- Problem identification
 - Diagnosis of the problem from its symptoms if not obvious (i.e. what is the problem?)
 - Delineation of the subproblem to be studied. Often we have to ignore parts of the entire problem.
 - Establishment of objectives, limitations and requirements.
- Solution of the problem as a mathematical model
- Model validation (or algorithm validation)
 - Model validation involves running the algorithm for the model on the computer in order to ensure:
 - the input data is free from errors
 - the computer program is bug-free (or at least there are no outstanding bugs)
 - the computer program correctly represents the model we are attempting to validate
 - the results from the algorithm seem reasonable (or if they are surprising we can at least understand why they are surprising).
- Solution of the model
 - Standard computer packages, or specially developed algorithms, can be used to solve the model.
 - In practice, a "solver" often involves very many solutions under varying assumptions to establish sensitivity.
- Implementation
 - This phase may involve the implementation of the results of the study or the implementation of the algorithm for solving the model as an operational tool (usually in a computer package).

To formulate the problem given in the question as a linear program we need to define:

- variables
- constraints
- objective

Variables

We need to decide how much to supply from each of the three growers to each of the two canning plants. Hence let x_{ij} be the number of tonnes supplied from grower i ($i=1,2,3$ for S1, S2 and S3 respectively) to plant j ($j=1$ for Plant A and $j=2$ for Plant B) where $x_{ij} \geq 0$ $i=1,2,3$; $j=1,2$

Constraints

- cannot supply more than a grower has available - a supply constraint

$$\begin{aligned}x_{11} + x_{12} &\leq 200 \\x_{21} + x_{22} &\leq 310 \\x_{31} + x_{32} &\leq 420\end{aligned}$$

- the capacity of each plant must be respected - a capacity constraint

$$\begin{aligned}x_{11} + x_{21} + x_{31} &\leq 480 \\x_{12} + x_{22} + x_{32} &\leq 560\end{aligned}$$

Objective

The objective is to maximise total profit, i.e.

$$\text{maximise revenue} - \text{grower supply cost} - \text{grower shipping cost} - \text{plant labour cost}$$

$$\text{maximise } 50\text{SUM}(i=1,2,3)\text{SUM}(j=1,2)x_{ij} - 11(x_{11} + x_{12}) - 10(x_{21} + x_{22}) - 9(x_{31} + x_{32}) - 3x_{11} - 2x_{21} - 6x_{31} - 3.5x_{12} - 2.5x_{22} - 4x_{32} - 20\text{SUM}(i=1,2,3)x_{i1} - 25\text{SUM}(i=1,2,3)x_{i2}$$

The dual values associated with the supply and plant capacity constraints in the optimal solution of the above linear program tell us by how much the optimal objective function value will change if we change the right-hand side of the constraints.

The basic assumptions are:

- can ship from a grower any quantity we desire
- no loss in weight in processing at the plant
- no loss in weight in shipping
- can sell all we produce
- all the data/numbers given are accurate

Linear programming example 1993 UG exam

The production manager of a chemical plant is attempting to devise a shift pattern for his workforce. Each day of every working week is divided into three eight-hour shift periods (00:01-08:00, 08:01-16:00, 16:01-24:00) denoted by night, day and late respectively. The plant must be manned at all times and the minimum number of workers required for each of these shifts over any working week is as below:

	Mon	Tues	Wed	Thur	Fri	Sat	Sun
Night	5	3	2	4	3	2	2
Day	6	8	9	5	7	2	5
Late	9	18	18	7	11	2	2

The union agreement governing acceptable shifts for workers is as follows:

- Each worker is assigned to work either a night shift or a day shift or a late shift and once a worker has been assigned to a shift they must remain on the same shift every day that they work.
- Each worker works four consecutive days during any seven day period.

In total there are currently 60 workers.

- Formulate the production manager's problem as a linear program.
- Comment upon the advantages/disadvantages you foresee of formulating and solving this problem as a linear program.

Solution

Variables

The union agreement is such that any worker can only start their four consecutive work days on one of the seven days (Mon to Sun) and in one of the three eight-hour shifts (night, day, late).

Monday be day 1, Tuesday be day 2, ..., Sunday be day 7
Night be shift 1, Day be shift 2, Late be shift 3
then the variables are:

N_{ij} the number of workers starting their four consecutive work days on day i ($i=1,...,7$) and shift j ($j=1,...,3$)

Note here that strictly these variables should be integer but, as we are explicitly told to formulate the problem as a linear program in part (a) of the question, we allow them to take fractional values.

Constraints

- upper limit on the total number of workers of 60

$$\text{SUM}(i=1 \text{ to } 7) \text{SUM}(j=1 \text{ to } 3) N_{ij} \leq 60$$

since each worker can start his working week only once during the seven day, three shift, week

- lower limit on the total number of workers required for each day/shift period

let D_{ij} be the (known) number of workers required on day i ($i=1,...,7$) and shift period j ($j=1,...,3$) e.g. $D_{23}=11$ (Friday, Late)

then the constraints are

$$\begin{aligned}\text{Monday: } &N_{11} + N_{21} + N_{31} + N_{41} \geq D_{11} \quad j=1, \dots, 3 \\ \text{Tuesday: } &N_{21} + N_{31} + N_{41} + N_{51} \geq D_{21} \quad j=1, \dots, 3 \\ \text{Wednesday: } &N_{31} + N_{41} + N_{51} + N_{61} \geq D_{31} \quad j=1, \dots, 3 \\ \text{Thursday: } &N_{41} + N_{51} + N_{61} + N_{71} \geq D_{41} \quad j=1, \dots, 3 \\ \text{Friday: } &N_{51} + N_{61} + N_{71} + N_{81} \geq D_{51} \quad j=1, \dots, 3 \\ \text{Saturday: } &N_{61} + N_{71} + N_{81} + N_{91} \geq D_{61} \quad j=1, \dots, 3 \\ \text{Sunday: } &N_{71} + N_{81} + N_{91} + N_{101} \geq D_{71} \quad j=1, \dots, 3\end{aligned}$$

The logic here is straightforward, for example for Wednesday (day 3) the workers working shift 1 on day 3 either started on Wednesday (day 3, N_{31}) or on Tuesday (day 2, N_{21}) or on Sunday (day 7, N_{71}) - so the sum of these variables is the total number of workers on day 3 in shift 1 and this must be at least the minimum number required (D_{31}).

Objective

It appears from the question that the production manager's objective is simply to find a feasible schedule so any objective is possible. Logically however he might be interested in reducing the size of the workforce so the objective function could be:

$$\text{minimise SUM}(i=1 \text{ to } 7) \text{SUM}(j=1 \text{ to } 3) N_{ij}$$

where all variables $N_{ij} \geq 0$ and continuous (i.e. can take fractional values).

This completes the formulation of the problem as a linear program.

Some of the advantages and disadvantages of solving this problem as a linear program are:

- really need variable values which are integer
- some workers will always end up working weekends
- how do we choose the workers to use, e.g. if $N_{43}=7$ which 7 workers do we choose to begin their work week on day 4 working shift 3
- in practice, a "solver" often involves very many solutions under varying assumptions to establish sensitivity
- the approach above enables us to deal with the problem in a systematic fashion
- have the potential to reduce the size of the workforce by more effectively matching the resources to the needs
- able to investigate changes (e.g. in shift patterns, workers needed per day, etc) very easily.

Linear programming example 1991 UG exam

A company manufactures four products (1,2,3,4) on two machines (X and Y). The time (in minutes) to process one unit of each product on each machine is shown below:

	Machine	
	X	Y
Product 1	3	10
Product 2	1	12
Product 3	2	13
Product 4	5	23

The profit per unit for each product (1,2,3,4) is £10, £12, £17 and £8 respectively. Product 1 must be produced on both machines X and Y but products 2, 3 and 4 can be produced on either machine.

The factory is very small and all this means that floor space is very limited. Only one week's production is stored in 50 square metres of floor space where the floor space taken up by each product is 0.1, 0.15, 0.5 and 0.05 (square metres) for products 1, 2, 3 and 4 respectively.

Customer requirements mean that the amount of product 3 produced should be related to the amount of product 2 produced. Over a week approximately twice as many units of product 2 should be produced as product 3.

Machine X is out of action (for maintenance/because of breakdown) 5% of the time and machine Y 7% of the time.

Assuming a working week 35 hours long formulate the problem of how to manufacture these products as a linear program.

Solution

Variables

Essentially we are interested in the amount produced on each machine. Hence let:

x_i = amount of product i ($i=1,2,3,4$) produced on machine X per week

y_i = amount of product i ($i=2,3,4$) produced on machine Y per week

where $x_i \geq 0$ $i=1,2,3,4$ and $y_i \geq 0$ $i=2,3,4$

Note here that as product 1 must be processed on both machines X and Y we do not define y_1 .

Constraints

- floor space

$$0.1x_1 + 0.15(x_2 + y_2) + 0.5(x_3 + y_3) + 0.05(x_4 + y_4) \leq 50$$

- customer requirements

$$x_2 + y_2 \geq 2(x_3 + y_3)$$

Note here that as this is only an approximate ($\pm 5\%$ say) constraint we might do better to express this constraint as

$$0.95(2x_3 + y_3) \leq x_2 + y_2 \leq 1.05(2x_3 + y_3)$$

- available time

$$10x_1 + 12x_2 + 13x_3 + 8x_4 \leq 0.95(35)(60) \text{ (machine X)}$$

$$27x_1 + 19y_2 + 33y_3 + 23y_4 \leq 0.95(35)(60) \text{ (machine Y)}$$

Objective

maximise profit, i.e.

$$\text{maximise } 10x_1 + 12(x_2 + y_2) + 17(x_3 + y_3) + 8(x_4 + y_4)$$

Linear programming example 1987 UG exam

A company is planning its production schedule over the next six months (it is currently the end of month 2). The demand (in units) for its product over that timescale is as shown below:

Month	1	2	3	4	5	6	7	8
Demand	5000	4	6000	4500	6500	7000	8000	9500

The company currently has in stock: 1000 units which were produced in month 2; 2000 units which were produced in month 1; 500 units which were produced in month 0.

The company can only produce up to 6000 units per month and the managing director has stated that stocks must be built up to help meet demand in months 5, 6, 7 and 8. Each unit produced costs £15 and the cost of holding stock is estimated to be £0.75 per unit per month (based upon the stock held at the beginning of each month).

The company has a major problem with deterioration of stock in that the stock inspection which takes place at the end of each month regularly identifies ruined stock (costing the company £25 per unit). It is estimated that, on average, 1% of the stock at the end of month t will show that 11% of the units in stock which were produced in month t are ruined, 47% of the units in stock which were produced in month $t-1$ are ruined; 180% of the units in stock which were produced in month $t-2$ are ruined. The stock inspection for month 2 is just about to take place.

The company wants a production plan for the next six months that avoids stockouts. Formulate their problem as a linear program.

Because of the stock deterioration problem the managing director is thinking of directing that customers should always be supplied with the oldest stock available. How would this affect your formulation of the problem?

Solution

Variables

Let

P_t be the production (units) in month t ($t=3,...,8$)

I_t be the number of units in stock at the end of month t which were produced in month i ($i=t-1, t-2$)

S_t be the number of units in stock at the beginning of month t which were produced in month i ($i=t-1, t-2$)

d_t be the demand in month t met from units produced in month i ($i=t-1, t-2$)

Constraints

- production limit

$$P_t \leq 6000$$

- initial stock position

$$I_2 = 1000$$

$$I_3 = 2000$$

$$I_4 = 500$$

- relate opening stock in month t to closing stock in previous months

$$S_{t+1} = 0.89(1-I_{t-1})$$

$$S_{t+2} = 0.53(1-I_{t-2})$$

- inventory continuity equation where we assume we can meet demand in month t from production in month t . Let D_t represent the (known) demand for the product in month t ($t=3,4,...,8$) then closing stock = opening stock + production - demand

and we have

$$I_t = 0 + P_t - I_{t-1}$$

$$I_{t+1} = S_{t+1} + 0 - d_{t+1}$$

$$I_{t+2} = S_{t+2} + 0 - d_{t+2}$$

where

$$d_t = d_{t-1} + d_{t-2} = D_t$$

- no stockouts

all inventory (I_5) and d variables ≥ 0

Objective

Presumably to minimise cost and this is given by

$$\text{minimise } 15 \text{SUM}(t=3 \text{ to } 8) P_t + 0.75 \text{SUM}(t=3 \text{ to } 8) I_t + 0.75 \text{SUM}(t=3 \text{ to } 8) S_t + 0.75 \text{SUM}(t=3 \text{ to } 8) d_t$$

Note because we are told to formulate this problem as a linear program we assume all variables are fractional - in reality they are likely to be quite large and so this is a reasonable approximation to make (also a problem occurs with finding integer values which may not be possible for some of the d_t variables) the oldest stock first to satisfy demand (although the managing director needs to tell the employees to issue the oldest stock first).

Linear programming example 1986 UG exam

A company assembles four products (1, 2, 3, 4) from delivered components. The profit per unit for each product (1, 2, 3, 4) is £10, £15, £22 and £17 respectively. The maximum demand in the next week for each product (1, 2, 3, 4) is 50, 85 and 70 units respectively.

There are three stages (A, B, C) in the manual assembly of each product and the man-hours needed for each stage per unit of product are shown below:

Stage	A	B	C
Product 1	3	2	3
Product 2	4	1	2
Product 3	2	6	1
Product 4	5	2	5

The nominal time available in the next week for assembly at each stage (A, B, C) is 160, 200 and 80 man-hours respectively.

It is possible to vary the man-hours spent on assembly at each stage such that workers previously employed on stage B assembly could spend up to 20% of their time on stage A assembly and workers previously employed on stage C assembly could spend up to 30% of their time on stage A assembly.