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# Chapter 5

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## Fundamentals of Traffic Flow and Queuing Theory

### 5.1 INTRODUCTION

It is important to realize that the primary function of a highway is to provide mobility. This mobility must be provided with safety in mind while achieving an acceptable level of performance (such as acceptable vehicle speeds). Many of the safety-related aspects of highway design were discussed in Chapter 3, and focus is now shifted to measures of performance.

The analysis of vehicle traffic provides the basis for measuring the operating performance of highways. In undertaking such an analysis, the various dimensions of traffic, such as number of vehicles per unit time (flow), vehicle types, vehicle speeds, and the variation in traffic flow over time, must be addressed because they all influence highway design (the selection of the number of lanes, pavement types, and geometric design) and highway operations (selection of traffic control devices, including signs, markings, and traffic signals), both of which impact the performance of the highway. In light of this, it is important for the analysis of traffic to begin with theoretically consistent quantitative techniques that can be used to model traffic flow, speed, and temporal fluctuations. The intent of this chapter is to focus on models of traffic flow and queuing, thus providing the groundwork for quantifying measures of performance (and levels of service, which will be discussed in Chapters 6 and 7).

### 5.2 TRAFFIC STREAM PARAMETERS

Traffic streams can be characterized by a number of different operational performance measures. Before commencing with a discussion of the specific measures, it is important to provide definitions for the contexts in which these measures apply. A traffic stream that operates free from the influence of such traffic control devices as signals and stop signs is classified as uninterrupted flow. This type of traffic flow is influenced primarily by roadway characteristics and the interactions of the vehicles in the traffic stream. Freeways, multilane highways, and two-lane highways often operate under uninterrupted flow conditions. Traffic streams that operate under the influence of signals and stop signs are classified as interrupted flow. Although all the concepts in this chapter are generally applicable to both types of flow, there are some additional

complexities involved with the analysis of traffic flow at signalized and unsignalized intersections. Chapter 7 will address the additional complexities relating to the analysis of traffic flow at signalized intersections. For details on the analysis of traffic flow at unsignalized intersections, refer to other sources [Transportation Research Board 1975, 2000]. It should be noted that environmental conditions (day vs. night, sunny vs. rainy, etc.) can also affect the flow of traffic, but this issue is beyond the scope of this book.

### 5.2.1 Traffic Flow, Speed, and Density

Traffic flow, speed, and density are variables that form the underpinnings of traffic analysis. To begin the study of these variables, the basic definitions of traffic flow, speed, and density must be presented. Traffic flow is defined as

$$q = \frac{n}{t} \quad (5.1)$$

where

$q$  = traffic flow in vehicles per unit time,

$n$  = number of vehicles passing some designated roadway point during time  $t$ , and  
 $t$  = duration of time interval.

Flow is often measured over the course of an hour, in which case the resulting value is typically referred to as volume. Thus, when the term "volume" is used, it is generally understood that the corresponding value is in units of vehicles per hour (veh/h). The definition of flow is more generalized to account for the measurement of vehicles over any period of time. In practice, the analysis flow rate is usually based on the peak 15-minute flow within the hour of interest. This aspect will be described in more detail in Chapter 6.

Aside from knowing the total number of vehicles passing a point in some time interval, the amount of time between the passing of successive vehicles (or time between the arrival of successive vehicles) is also of interest. The time between the passage of the front bumpers of successive vehicles, at some designated highway point, is known as the time headway. The time headway is related to  $t$ , as defined in Eq. 5.1, by

$$t = \sum_{i=1}^n h_i \quad (5.2)$$

where

$t$  = duration of time interval,

$h_i$  = time headway of the  $i$ th vehicle (the elapsed time between the arrivals of vehicles  $i$  and  $i - 1$ ), and

$n$  = number of measured vehicle time headways at some designated roadway point.

Substituting Eq. 5.2 into Eq. 5.1 gives

$$q = \frac{n}{\sum_{i=1}^n h_i} \quad (5.3)$$

or

$$q = \frac{1}{\bar{h}} \quad (5.4)$$

where  $\bar{h}$  = average time headway ( $\sum h_i/n$ ) in unit time per vehicle. The importance of time headways in traffic analysis will be given additional attention in forthcoming sections of this chapter.

The average traffic speed is defined in two ways. The first is the arithmetic mean of the vehicle speeds observed at some designated point along the roadway. This is referred to as the time-mean speed and is expressed as

$$\bar{u}_t = \frac{\sum_{i=1}^n u_i}{n} \quad (5.5)$$

where

$\bar{u}_t$  = time-mean speed in unit distance per unit time,

$u_i$  = spot speed (the speed of the vehicle at the designated point on the highway, as might be obtained using a radar gun) of the  $i$ th vehicle, and

$n$  = number of measured vehicle spot speeds.

The second definition of speed is more useful in the context of traffic analysis and is determined on the basis of the time necessary for a vehicle to travel some known length of roadway. This measure of average traffic speed is referred to as the space-mean speed and is expressed as (assuming that the travel time for all vehicles is measured over the same length of roadway)

$$\bar{u}_s = \frac{l}{\bar{t}} \quad (5.6)$$

where

$\bar{u}_s$  = space-mean speed in unit distance per unit time,

$l$  = length of roadway used for travel time measurement of vehicles, and

$\bar{t}$  = average vehicle travel time, defined as

$$\bar{t} = \frac{1}{n} \sum_{i=1}^n t_i \quad (5.7)$$

where

$t_i$  = time necessary for vehicle  $i$  to travel a roadway section of length  $l$ , and  
 $n$  = number of measured vehicle travel times.

Substituting Eq. 5.7 into Eq. 5.6 yields

$$\bar{u}_s = \frac{l}{\frac{1}{n} \sum_{i=1}^n t_i} \quad (5.8)$$

or

$$\bar{u}_s = \frac{1}{\frac{1}{n} \sum_{i=1}^n \left[ \frac{1}{l/t_i} \right]} \quad (5.9)$$

which is the harmonic mean of speed (space-mean speed). Space-mean speed is the speed variable used in traffic models.

### EXAMPLE 5.1

The speeds of five vehicles were measured (with radar) at the midpoint of a 0.5-mile (0.8-km) section of roadway. The speeds for vehicles 1, 2, 3, 4, and 5 were 44, 42, 51, 49, and 46 mi/h (70.8, 67.6, 82.1, 78.8, and 74 km/h), respectively. Assuming all vehicles were traveling at constant speed over this roadway section, calculate the time-mean and space-mean speeds.

#### SOLUTION

For the time-mean speed, Eq. 5.5 is applied, giving

$$\begin{aligned} \bar{u}_t &= \frac{\sum_{i=1}^n u_i}{n} \\ &= \frac{44 + 42 + 51 + 49 + 46}{5} \\ &= \underline{\underline{46.4 \text{ mi/h}}} \end{aligned}$$

For the space-mean speed, Eq. 5.9 will be applied. This equation is based on travel time; however, because it is known that the vehicles were traveling at constant speed, we can rearrange this equation to utilize the measured speed, knowing that distance,  $l$ , divided by travel time,  $t$ , is equal to speed ( $l/t_i = u$ ).

$$\begin{aligned}
 \bar{u}_s &= \frac{1}{\frac{1}{n} \sum_{i=1}^n \left[ \frac{1}{l/t_i} \right]} = \frac{1}{\frac{1}{n} \sum_{i=1}^n \left[ \frac{1}{u_i} \right]} \\
 &= \frac{1}{\frac{1}{5} \left( \frac{1}{44} + \frac{1}{42} + \frac{1}{51} + \frac{1}{49} + \frac{1}{46} \right)} \\
 &= \frac{1}{0.02166} \\
 &= \underline{\underline{46.17 \text{ mi/h}}}
 \end{aligned}$$

Note that the space-mean speed will always be lower than the time-mean speed, unless all vehicles are traveling at the exact same speed, in which case the two measures will be equal.

Finally, traffic density is defined as

$$k = \frac{n}{l} \quad (5.10)$$

where

$k$  = traffic density in vehicles per unit distance,

$n$  = number of vehicles occupying some length of roadway at some specified time, and

$l$  = length of roadway.

The density can also be related to the individual spacing between successive vehicles (measured from front bumper to front bumper). The roadway length,  $l$ , in Eq. 5.10 can be defined as

$$l = \sum_{i=1}^n s_i \quad (5.11)$$

where

$s_i$  = spacing of the  $i$ th vehicle (the distance between vehicles  $i$  and  $i - 1$ , measured from front bumper to front bumper), and

$n$  = number of measured vehicle spacings.

Substituting Eq. 5.11 into Eq. 5.10 gives

$$k = \frac{n}{\sum_{i=1}^n s_i} \quad (5.12)$$

or

$$k = \frac{1}{\bar{s}} \quad (5.13)$$

where  $\bar{s}$  = average spacing ( $\sum s_i/n$ ) in unit distance per vehicle.

Time headway and spacing are referred to as microscopic measures because they describe characteristics specific to individual pairs of vehicles within the traffic stream. Measures that describe the traffic stream as a whole, such as flow, average speed, and density, are referred to as macroscopic measures. As indicated by the preceding equations, the microscopic measures can be aggregated and related to the macroscopic measures.

Based on the definitions presented, a simple identity provides the basic relationship among traffic flow, speed (space-mean), and density (denoting space-mean speed,  $\bar{u}_s$ , as simply  $u$  for notational convenience):

$$q = uk \quad (5.14)$$

where

$q$  = flow, typically in units of veh/h,

$u$  = speed (space-mean speed), typically in units of mi/h (km/h), and

$k$  = density, typically in units of veh/mi (veh/km).

### EXAMPLE 5.2

Vehicle time headways and spacings were measured at a point along a highway, from a single lane, over the course of an hour. The average values were calculated as 2.5 s/veh for headway and 200 ft/veh (61 m/veh) for spacing. Calculate the average speed of the traffic.

#### SOLUTION

To calculate the average speed of the traffic, the fundamental relationship in Eq. 5.14 is used. To begin, the flow and density need to be calculated from the headway and spacing data. Flow is determined from Eq. 5.4 as

$$\begin{aligned} q &= \frac{1}{2.5 \text{ s/veh}} \\ &= 0.40 \text{ veh/s} \end{aligned}$$

or, because the data were collected for an hour,

$$\begin{aligned} q &= 0.40 \text{ veh/s} \times 3600 \text{ s/h} \\ &= 1440 \text{ veh/h} \end{aligned}$$

Density is determined from Eq. 5.13 as

$$\begin{aligned} k &= \frac{1}{200 \text{ ft/veh}} \\ &= 0.005 \text{ veh/ft} \end{aligned}$$

5.3

5.3.1

or, applying this spacing over the course of one mile,

(5.13)

$$\begin{aligned} k &= 0.005 \text{ veh/ft} \times 5280 \text{ ft/mi} \\ &= 26.4 \text{ veh/mi} \end{aligned}$$

Now applying Eq. 5.14, after rearranging to solve for speed, gives

$$\begin{aligned} u &= \frac{q}{k} \\ &= \frac{1440 \text{ veh/h}}{26.4 \text{ veh/mi}} \\ &= \underline{\underline{54.5 \text{ mi/h}}} \end{aligned}$$

Note that the average speed of traffic can be determined directly from the average headway and spacing values, as follows:

$$\begin{aligned} u &= \frac{\bar{s}}{\bar{h}} \\ &= \frac{200 \text{ ft/veh}}{2.5 \text{ s/veh}} \\ &= \underline{\underline{80 \text{ ft/s (54.5 mi/h)}}} \end{aligned}$$

### 5.3 BASIC TRAFFIC STREAM MODELS

While the preceding definitions and relationships provide the basis for the measurement and calculation of traffic stream parameters, it is essential to also understand the interaction of the individual macroscopic measures in order to fully analyze the operational performance of the traffic stream. The models that describe these interactions are discussed in the following sections, and it will be shown that Eq. 5.14 serves the important function of linking specific models of traffic into a consistent, generalized model.

#### 5.3.1 Speed-Density Model

The most intuitive starting point for developing a consistent, generalized traffic model is to focus on the relationship between speed and density. To begin, consider a section of highway with only a single vehicle on it. Under these conditions, the density (veh/mi) will be very low and the driver will be able to travel freely at a speed close to the design speed of the highway. This speed is referred to as the free-flow speed because vehicle speed is not inhibited by the presence of other vehicles. As more and more vehicles begin to use a section of highway, the traffic density will increase and the average operating speed of vehicles will decline from the free-flow value as drivers slow to allow for the maneuvers of

other vehicles. Eventually, the highway section will become so congested (will have such a high density) that the traffic will come to a stop ( $u = 0$ ), and the density will be determined by the length of the vehicles and the spaces that drivers leave between them. This high-density condition is referred to as the jam density.

One possible representation of the process described above is the linear relationship shown in Fig. 5.1. Mathematically, such a relationship can be expressed as

$$u = u_f \left(1 - \frac{k}{k_j}\right) \quad (5.15)$$

where

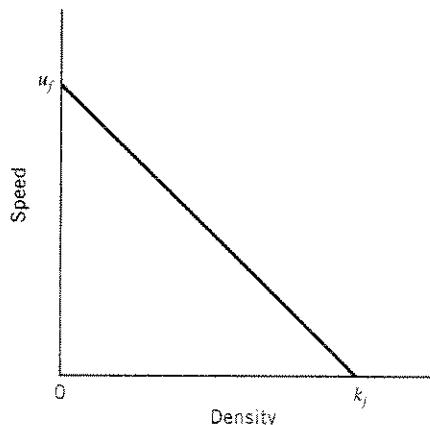
$u$  = space-mean speed in mi/h (km/h),

$u_f$  = free-flow speed in mi/h (km/h),

$k$  = density in veh/mi (veh/km), and

$k_j$  = jam density in veh/mi (veh/km).

The advantage of using a linear representation of the speed-density relationship is that it provides a basic insight into the relationships among traffic flow, speed, and density interactions without having these insights clouded by the additional complexity that a nonlinear speed-density relationship introduces. However, it is important to note that field studies have shown that the speed-density relationship tends to be nonlinear at low densities and high densities (those that approach the jam density). In fact, the overall speed-density relationship is better represented by three relationships: (1) a nonlinear relationship at low densities that has speed slowly declining from the free-flow value, (2) a linear relationship over the large medium-



**Figure 5.1** Illustration of a typical linear speed-density relationship.

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(5.15)

density region (speed declining linearly with density as shown in Eq. 5.15), and (3) a nonlinear relationship near the jam density as the speed asymptotically approaches zero with increasing density. For the purposes of exposition, we present only traffic stream models that are based on the assumption of a linear speed-density relationship. Examples of nonlinear speed-density relationships are provided elsewhere [Pipes 1967; Drew 1965].

### 5.3.2 Flow-Density Model

Using the assumption of a linear speed-density relationship as shown in Eq. 5.15, a parabolic flow-density model can be obtained by substituting Eq. 5.15 into Eq. 5.14:

$$q = u_f \left( k - \frac{k^2}{k_j} \right) \quad (5.16)$$

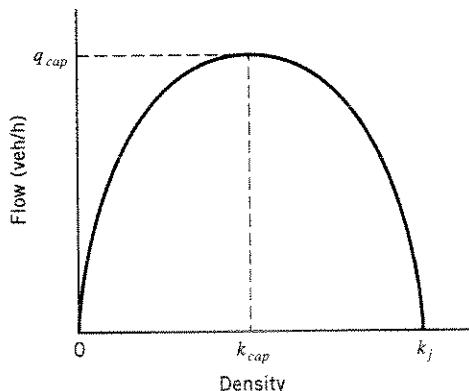
where all terms are as defined previously.

The general form of Eq. 5.16 is shown in Fig. 5.2. Note in this figure that the maximum flow rate,  $q_{cap}$ , represents the highest rate of traffic flow that the highway is capable of handling. This is referred to as the traffic flow at capacity, or simply the capacity of the roadway. The traffic density that corresponds to this capacity flow rate is  $k_{cap}$ , and the corresponding speed is  $u_{cap}$ . Equations for  $q_{cap}$ ,  $k_{cap}$ , and  $u_{cap}$  can be derived by differentiating Eq. 5.16, because at maximum flow

$$\frac{dq}{dk} = u_f \left( 1 - \frac{2k}{k_j} \right) = 0 \quad (5.17)$$

and because the free-flow speed ( $u_f$ ) is not equal to zero,

$$k_{cap} = \frac{k_j}{2} \quad (5.18)$$



**Figure 5.2** Illustration of the parabolic flow-density relationship.

Substituting Eq. 5.18 into Eq. 5.15 gives

$$\begin{aligned} u_{cap} &= u_f \left(1 - \frac{k_j}{2k_j}\right) \\ &= \frac{u_f}{2} \end{aligned} \quad (5.19)$$

and using Eq. 5.18 and Eq. 5.19 in Eq. 5.14 gives

$$\begin{aligned} q_{cap} &= u_{cap} k_{cap} \\ &= \frac{u_f k_j}{4} \end{aligned} \quad (5.20)$$

### 5.3.3 Speed-Flow Model

Again returning to the linear speed-density model (Eq. 5.15), a corresponding speed-flow model can be developed by rearranging Eq. 5.15 to

$$k = k_j \left(1 - \frac{u}{u_f}\right) \quad (5.21)$$

and by substituting Eq. 5.21 into Eq. 5.14,

$$q = k_j \left(u - \frac{u^2}{u_f}\right) \quad (5.22)$$

The speed-flow model defined by Eq. 5.22 again gives a parabolic function, as shown in Fig. 5.3. Note that Fig. 5.3 shows that two speeds are possible for flows,  $q$ , up to the highway's capacity,  $q_{cap}$  (this follows from the two densities possible for given flows as shown in Fig. 5.2). It is desirable, for any given flow, to keep the average space-mean speed on the upper portion of the speed-flow curve (above  $u_{cap}$ ). When speeds drop below  $u_{cap}$ , traffic is in a highly congested and unstable condition.

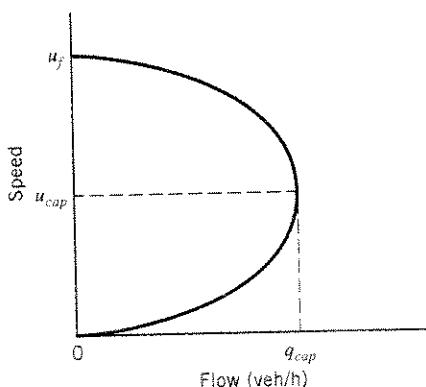
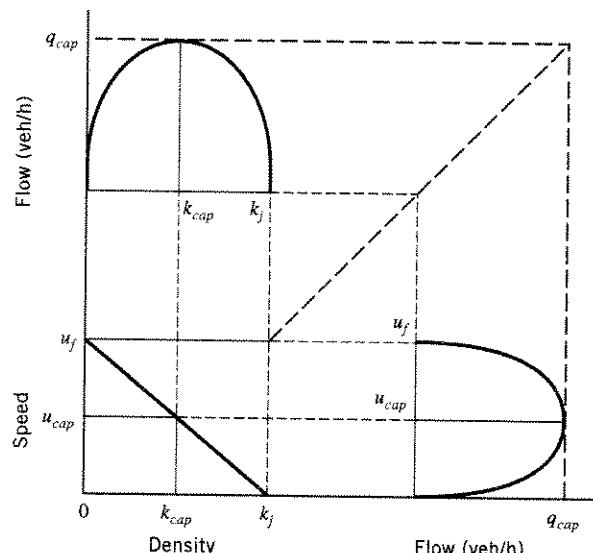


Figure 5.3 Illustration of the parabolic speed-flow relationship.



**Figure 5.4** Flow-density, speed-density, and speed-flow relationships (assuming a linear speed-density model).

All of the flow, speed, and density relationships and their interactions are graphically represented in Fig. 5.4.

### EXAMPLE 5.3

A section of highway is known to have a free-flow speed of 55 mi/h (88.5 km/h) and a capacity of 3300 veh/h. In a given hour, 2100 vehicles were counted at a specified point along this highway section. If the linear speed-density relationship shown in Eq. 5.15 applies, what would you estimate the space-mean speed of these 2100 vehicles to be?

#### SOLUTION

The jam density is first determined from Eq. 5.20 as

$$\begin{aligned} k_j &= \frac{4q_{cap}}{u_f} \\ &= \frac{4 \times 3300}{55} \\ &= 240.0 \text{ veh/mi} \end{aligned}$$

Rearranging Eq. 5.22 to solve for  $u$ ,

$$\frac{k_j}{u_f} u^2 - k_j u + q = 0$$

Substituting,

$$\frac{240.0}{55} u^2 - 240.0 u + 2100 = 0$$

which gives  $u = \underline{44.08 \text{ mi/h}}$  or  $\underline{10.92 \text{ mi/h}}$ . Both of these speeds are feasible, as shown in Fig. 5.3.

## 5.4 MODELS OF TRAFFIC FLOW

With the basic relationships among traffic flow, speed, and density formalized, attention can now be directed toward a more microscopic view of traffic flow. That is, instead of simply modeling the number of vehicles passing a specified point on a highway in some time interval, there is considerable analytic value in modeling the time between the arrivals of successive vehicles (the concept of vehicle time headway presented earlier). The most simplistic approach to vehicle arrival modeling is to assume that all vehicles are equally or uniformly spaced. This results in what is termed a deterministic, uniform arrival pattern. Under this assumption, if the traffic flow is 360 veh/h, the number of vehicles arriving in any 5-minute time interval is 30 and the headway between all vehicles is 10 seconds (because  $h$  will equal  $3600/q$ ). However, actual observations show that such uniformity of traffic flow is not always realistic because some 5-minute intervals are likely to have more or less traffic flow than other 5-minute intervals. Thus a representation of vehicle arrivals that goes beyond the deterministic, uniform assumption is often warranted.

### 5.4.1 Poisson Model

Models that account for the nonuniformity of flow are derived by assuming that the pattern of vehicle arrivals (at a specified point) corresponds to some random process. The problem then becomes one of selecting a probability distribution that is a reasonable representation of observed traffic arrival patterns. An example of such a distribution is the Poisson distribution (the limitations of which will be discussed later), which is expressed as

$$P(n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!} \quad (5.23)$$

where

$P(n)$  = probability of having  $n$  vehicles arrive in time  $t$ ,

$\lambda$  = average vehicle flow or arrival rate in vehicles per unit time,

$t$  = duration of the time interval over which vehicles are counted, and

$e$  = base of the natural logarithm ( $e = 2.718$ ).

#### EXAMPLE 5.4

An observer counts 360 veh/h at a specific highway location. Assuming that the arrival of vehicles at this highway location is Poisson distributed, estimate the probabilities of having 0, 1, 2, 3, 4, and 5 or more vehicles arriving over a 20-second time interval.

s shown

### SOLUTION

The average arrival rate,  $\lambda$ , is 360 veh/h, or 0.1 vehicles per second (veh/s). Using this in Eq. 5.23 with  $t = 20$  seconds, the probabilities of having exactly 0, 1, 2, 3, and 4 vehicles arrive are

$$P(0) = \frac{(0.1 \times 20)^0 e^{-0.1(20)}}{0!} = 0.135$$

$$P(1) = \frac{(0.1 \times 20)^1 e^{-0.1(20)}}{1!} = 0.271$$

$$P(2) = \frac{(0.1 \times 20)^2 e^{-0.1(20)}}{2!} = 0.271$$

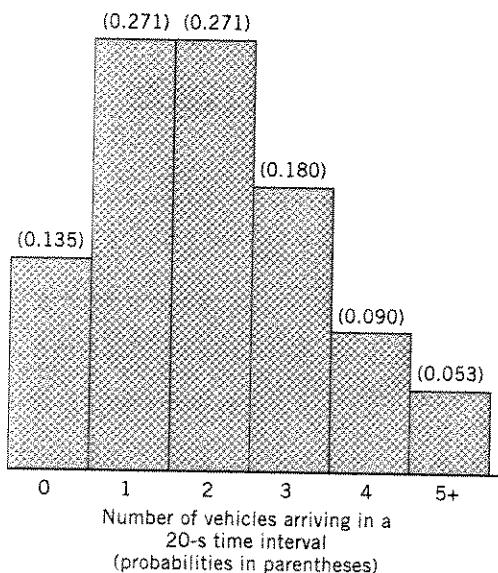
$$P(3) = \frac{(0.1 \times 20)^3 e^{-0.1(20)}}{3!} = 0.180$$

$$P(4) = \frac{(0.1 \times 20)^4 e^{-0.1(20)}}{4!} = 0.090$$

For five or more vehicles,

$$\begin{aligned} P(n \geq 5) &= 1 - P(n < 5) \\ &= 1 - 0.135 - 0.271 - 0.271 - 0.180 - 0.090 \\ &= 0.053 \end{aligned}$$

A histogram of these probabilities is shown in Fig. 5.5.



**Figure 5.5** Histogram of the Poisson distribution for  $\lambda = 0.1$  vehicles per second.

**EXAMPLE 5.5**

Traffic data are collected in 60-second intervals at a specific highway location as shown in Table 5.1. Assuming the traffic arrivals are Poisson distributed and continue at the same rate as that observed in the 15 time periods shown, what is the probability that six or more vehicles will arrive in each of the next three 60-second time intervals (12:15 P.M. to 12:16 P.M., 12:16 P.M. to 12:17 P.M., and 12:17 P.M. to 12:18 P.M.)?

**SOLUTION**

Table 5.1 shows that a total of 101 vehicles arrive in the 15-minute period from 12:00 P.M. to 12:15 P.M. Thus the average arrival rate,  $\lambda$ , is 0.112 veh/s (101/900). As in Example 5.4, Eq. 5.23 is applied to find the probabilities of exactly 0, 1, 2, 3, 4, and 5 vehicles arriving.

Applying Eq. 5.23, with  $\lambda = 0.112$  veh/s and  $t = 60$  seconds, the probabilities of having 0, 1, 2, 3, 4, and 5 vehicles arriving in a 60-second time interval are (using  $\lambda t = 6.733$ )

$$P(0) = \frac{(6.733)^0 e^{-6.733}}{0!} = \underline{\underline{0.0012}}$$

$$P(1) = \frac{(6.733)^1 e^{-6.733}}{1!} = \underline{\underline{0.008}}$$

$$P(2) = \frac{(6.733)^2 e^{-6.733}}{2!} = \underline{\underline{0.027}}$$

$$P(3) = \frac{(6.733)^3 e^{-6.733}}{3!} = \underline{\underline{0.0606}}$$

$$P(4) = \frac{(6.733)^4 e^{-6.733}}{4!} = \underline{\underline{0.102}}$$

$$P(5) = \frac{(6.733)^5 e^{-6.733}}{5!} = \underline{\underline{0.137}}$$

The summation of these probabilities is the probability that 0 to 5 vehicles will arrive in any given 60-second time interval, which is

$$\begin{aligned} P(n \leq 5) &= \sum_{n=0}^5 P(n) \\ &= 0.0012 + 0.008 + 0.027 + 0.0606 + 0.102 + 0.137 \\ &= 0.3358 \end{aligned}$$

So 1 minus  $P(n \leq 5)$  is the probability that 6 or more vehicles will arrive in any 60-second time interval, which is

**Table 5.1** Observed Traffic Data for Example 5.5

Time period	Observed number of vehicles
12:00 P.M. to 12:01 P.M.	3
12:01 P.M. to 12:02 P.M.	5
12:02 P.M. to 12:03 P.M.	4
12:03 P.M. to 12:04 P.M.	10
12:04 P.M. to 12:05 P.M.	7
12:05 P.M. to 12:06 P.M.	4
12:06 P.M. to 12:07 P.M.	8
12:07 P.M. to 12:08 P.M.	11
12:08 P.M. to 12:09 P.M.	9
12:09 P.M. to 12:10 P.M.	5
12:10 P.M. to 12:11 P.M.	3
12:11 P.M. to 12:12 P.M.	10
12:12 P.M. to 12:13 P.M.	9
12:13 P.M. to 12:14 P.M.	7
12:14 P.M. to 12:15 P.M.	6

$$\begin{aligned}
 P(n \geq 6) &= 1 - P(n \leq 5) \\
 &= 1 - 0.3358 \\
 &= 0.6642
 \end{aligned}$$

The probability that 6 or more vehicles will arrive in three successive time intervals ( $t_1$ ,  $t_2$ , and  $t_3$ ) is simply the product of probabilities, which is

$$\begin{aligned}
 P(n \geq 6) \text{ for three successive time periods} &= \prod_{t_i=1}^3 P(n \geq 6) \\
 &= (0.6642)^3 \\
 &= \underline{\underline{0.293}}
 \end{aligned}$$

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The assumption of Poisson vehicle arrivals also implies a distribution of the time intervals between the arrivals of successive vehicles (time headway). To show this, note that the average arrival rate is

$$\lambda = \frac{q}{3600} \quad (5.24)$$

where

$\lambda$  = average vehicle arrival rate in veh/s,

$q$  = flow in veh/h, and

3600 = number of seconds per hour.

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Substituting Eq. 5.24 into Eq. 5.23 gives

$$P(n) = \frac{(qt/3600)^n e^{-qt/3600}}{n!} \quad (5.25)$$

Note that the probability of having no vehicles arrive in a time interval of length  $t$ ,  $P(0)$ , is equivalent to the probability of a vehicle headway,  $h$ , being greater than or equal to the time interval  $t$ . So from Eq. 5.25,

$$\begin{aligned} P(0) &= P(h \geq t) \\ &= e^{-qt/3600} \end{aligned} \quad (5.26)$$

This distribution of vehicle headways is known as the negative exponential distribution and is often simply referred to as the exponential distribution.

### EXAMPLE 5.6

Consider the traffic situation in Example 5.4 (360 veh/h). Again assume that the vehicle arrivals are Poisson distributed. What is the probability that the gap between successive vehicles will be less than 8 seconds, and what is the probability that the gap between successive vehicles will be between 8 and 10 seconds?

#### SOLUTION

By definition,  $P(h < t) = 1 - P(h \geq t)$ . This expression gives the probability that the gap will be less than 8 seconds as

$$\begin{aligned} P(h < 8) &= 1 - e^{-360(8)/3600} \\ &= 1 - 0.449 \\ &= \underline{\underline{0.551}} \end{aligned}$$

To determine the probability that the gap will be between 8 and 10 seconds, compute the probability that the gap will be greater than or equal to 10 seconds:

$$\begin{aligned} P(h \geq 10) &= e^{-360(10)/3600} \\ &= 0.368 \end{aligned}$$

So the probability that the gap will be between 8 and 10 seconds is 0.081 ( $1 - 0.551 - 0.368$ ).

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To help in visualizing the shape of the exponential distribution, Fig. 5.6 shows the probability distribution implied by Eq. 5.26, with the flow,  $q$ , equal to 360 veh/h as in Example 5.4.

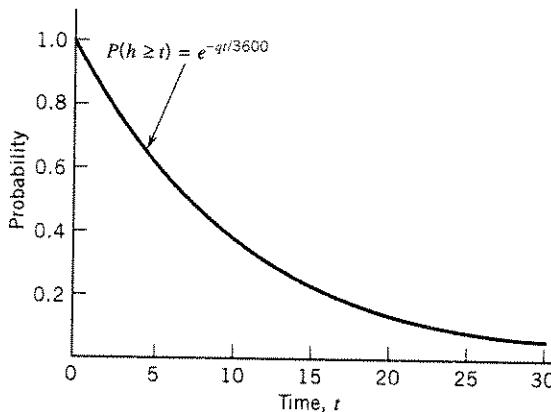


Figure 5.6 Exponentially distributed probabilities of headways greater than or equal to  $t$ , with  $q = 360$  veh/h.

#### 5.4.2 Limitations of the Poisson Model

Empirical observations have shown that the assumption of Poisson-distributed traffic arrivals is most realistic in lightly congested traffic conditions. As traffic flows become heavily congested or when traffic signals cause cyclical traffic stream disturbances, other distributions of traffic flow become more appropriate. The primary limitation of the Poisson model of vehicle arrivals is the constraint imposed by the Poisson distribution that the mean of period observations equals the variance. For example, the mean of period-observed traffic in Example 5.5 is 6.733 and the corresponding variance,  $\sigma^2$ , is 7.210. Because these two values are close, the Poisson model was appropriate for this example. If the variance is significantly greater than the mean, the data are said to be overdispersed, and if the variance is significantly less than the mean, the data are said to be underdispersed. In either case the Poisson distribution is no longer appropriate, and another distribution should be used. Such distributions are discussed in detail in more specialized sources [Transportation Research Board 1975; Poch and Mannerling 1996].

### 5.5 QUEUING THEORY AND TRAFFIC FLOW ANALYSIS

The formation of traffic queues during congested periods is a source of considerable time delay and results in a loss of highway performance. Under extreme conditions, queuing delay can account for 90% or more of a motorist's total trip travel time. Given this, it is essential in traffic analysis that one develop a clear understanding of the characteristics of queue formation and dissipation along with mathematical formulations that can predict queuing-related elements.

As is well known, the problem of queuing is not unique to traffic analysis. Many non-transportation fields, such as the design and operation of industrial plants, retail stores, service-oriented industries, and computer networks, must also give serious consideration to the problem of queuing. The impact that queues have on performance

and productivity in manufacturing, retailing, and other fields has led to numerous theories of queuing behavior (the process by which queues form and dissipate). As will be shown, the models of traffic flow presented earlier (uniform, deterministic arrivals and Poisson arrivals) will form the basis for studying traffic queues within the more general context of queuing theory.

### 5.5.1 Dimensions of Queuing Models

The purpose of traffic queuing models is to provide a means to estimate important measures of highway performance, including vehicle delay and traffic queue lengths. Such estimates are critical to roadway design (the required length of left-turn bays and the number of lanes at intersections) and traffic operations control, including the timing of traffic signals at intersections.

Queuing models are derived from underlying assumptions regarding arrival patterns, departure characteristics, and queue disciplines. Traffic arrival patterns were explored in Section 5.4, where, given an average vehicle arrival rate ( $\lambda$ ), two possible distributions of the time between the arrival of successive vehicles were considered:

1. Equal time intervals (derived from the assumption of uniform, deterministic arrivals)
2. Exponentially distributed time intervals (derived from the assumption of Poisson-distributed arrivals)

In addition to vehicle arrival assumptions, the derivation of traffic queuing models requires assumptions relating to vehicle departure characteristics. Of particular interest is the distribution of the amount of time it takes a vehicle to depart—for example, the time to pass through an intersection at the beginning of a green signal, the time required to pay a toll at a toll booth, or the time a driver takes before deciding to proceed after stopping at a stop sign. As was the case for arrival patterns, given an average vehicle departure rate (denoted as  $\mu$ , in vehicles per unit time), the assumption of a deterministic or exponential distribution of departure times is appropriate.

Another important aspect of queuing models is the number of available departure channels. For most traffic applications only one departure channel will exist, such as a highway lane or group of lanes passing through an intersection. However, multiple departure channels are encountered in some traffic applications, such as at toll booths on turnpikes and at entrances to bridges.

The final necessary assumption relates to the queue discipline. In this regard, two options have been popularized in the development of queuing models: first-in, first-out (FIFO), indicating that the first vehicle to arrive is the first to depart; and last-in, first-out (LIFO), indicating that the last vehicle to arrive is the first to depart. For virtually all traffic-oriented queues, the FIFO queuing discipline is the more appropriate of the two.

Queuing models are often identified by three alphanumeric values. The first value indicates the arrival rate assumption, the second value gives the departure rate assumption, and the third value indicates the number of departure channels. For traffic arrival and departure assumptions, the uniform, deterministic distribution is denoted  $D$ .

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and the exponential distribution is denoted  $M$ . Thus a  $D/D/1$  queuing model assumes deterministic arrivals and departures with one departure channel. Similarly, an  $M/D/1$  queuing model assumes exponentially distributed arrival times, deterministic departure times, and one departure channel.

### 5.5.2 $D/D/1$ Queuing

The case of deterministic arrivals and departures with one departure channel ( $D/D/1$  queue) is an excellent starting point in understanding queuing models because of its simplicity. The  $D/D/1$  queue lends itself to an intuitive graphical or mathematical solution that is best illustrated by an example.

#### EXAMPLE 5.7

Vehicles arrive at an entrance to a recreational park. There is a single gate (at which all vehicles must stop), where a park attendant distributes a free brochure. The park opens at 8:00 A.M., at which time vehicles begin to arrive at a rate of 480 veh/h. After 20 minutes the arrival flow rate declines to 120 veh/h, and it continues at that level for the remainder of the day. If the time required to distribute the brochure is 15 seconds, and assuming  $D/D/1$  queuing, describe the operational characteristics of the queue.

#### SOLUTION

Begin by putting arrival and departure rates into common units of vehicles per minute:

$$\lambda = \frac{480 \text{ veh/h}}{60 \text{ min/h}} = 8 \text{ veh/min} \quad \text{for } t \leq 20 \text{ min}$$

$$\lambda = \frac{120 \text{ veh/h}}{60 \text{ min/h}} = 2 \text{ veh/min} \quad \text{for } t > 20 \text{ min}$$

$$\mu = \frac{60 \text{ s/min}}{15 \text{ s/veh}} = 4 \text{ veh/min} \quad \text{for all } t$$

Equations for the total number of vehicles that have arrived and departed up to a specified time,  $t$ , can now be written. Define  $t$  as the number of minutes after the start of the queuing process (in this case the number of minutes after 8:00 A.M.). The total number of vehicle arrivals at time  $t$  is equal to

$$8t \quad \text{for } t \leq 20 \text{ min}$$

and

$$160 + 2(t - 20) \quad \text{for } t > 20 \text{ min}$$

Similarly, the number of vehicle departures is

$$4t \quad \text{for all } t$$

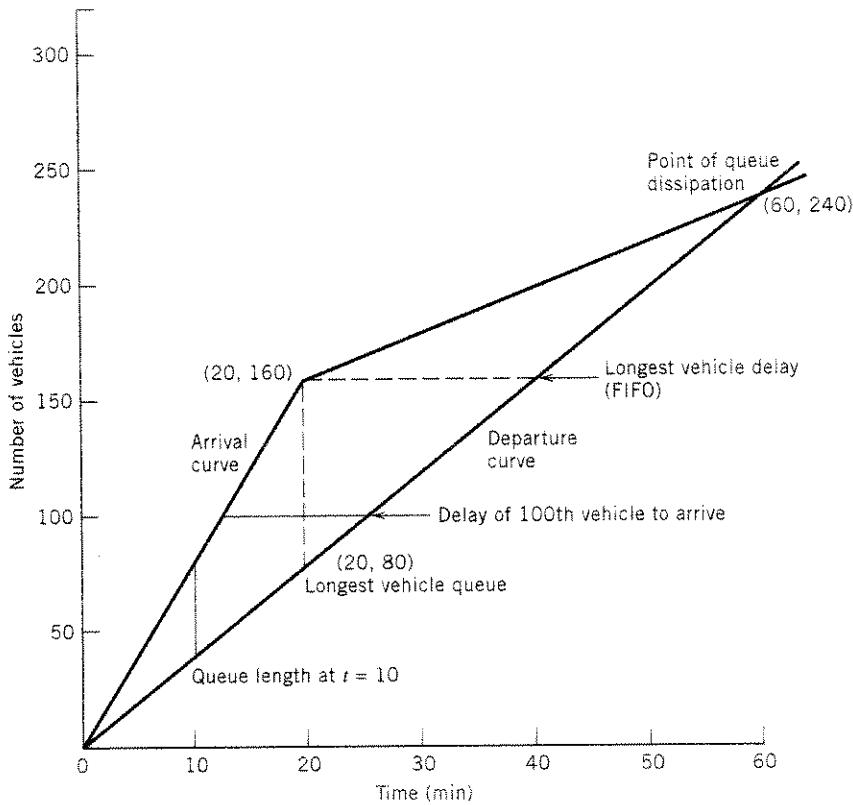


Figure 5.7 *D/D/1* queuing diagram for Example 5.7.

The preceding equations can be illustrated graphically as shown in Fig. 5.7. When the arrival curve is above the departure curve, a queue condition exists. The point at which the arrival curve meets the departure curve is the moment when the queue dissipates (no more queue exists). In this example, the point of queue dissipation can be determined graphically by inspection of Fig. 5.7, or analytically by equating appropriate arrival and departure equations, that is,

$$160 + 2(t - 20) = 4t$$

Solving for  $t$  gives  $t = 60$  minutes. Thus the queue that began to form at 8:00 A.M. will dissipate 60 minutes later (9:00 A.M.), at which time 240 vehicles will have arrived and departed ( $4 \text{ veh/min} \times 60 \text{ min}$ ).

Another aspect of interest is individual vehicle delay. Under the assumption of a FIFO queuing discipline, the delay of an individual vehicle is given by the horizontal distance between arrival and departure curves starting from the time of the vehicle's arrival in the queue. So, by inspection of Fig. 5.7, the 160th vehicle to arrive will have the longest delay, 20 minutes (the longest horizontal distance between arrival and departure curves), and vehicles arriving after the 239th vehicle will encounter no

queue delay because the queue will have dissipated and the departure rate will continue to exceed the arrival rate. It follows that with the LIFO queuing discipline, the first vehicle to arrive would have to wait until the entire queue clears (60 minutes of delay).

The total length of the queue at a specified time, expressed as the number of vehicles, is given by the vertical distance between arrival and departure curves at that time. For example, at 10 minutes after the start of the queuing process (8:10 A.M.) the queue is 40 vehicles long, and the longest queue (longest vertical distance between arrival and departure curves) will occur at  $t = 20$  minutes and is 80 vehicles long (see Fig. 5.7).

Total vehicle delay, defined as the summation of the delays for the individual vehicles, is given by the total area between the arrival and departure curves (see Fig. 5.7) and, in this case, is in units of vehicle-minutes. In this example, the area between the arrival and departure curves can be determined by summing triangular areas, giving total delay,  $D_t$ , as

$$\begin{aligned} D_t &= \frac{1}{2}(80 \times 20) + \frac{1}{2}(80 \times 40) \\ &= \underline{\underline{2400 \text{ veh-min}}} \end{aligned}$$

Finally, because 240 vehicles encounter queuing delay (as previously determined), the average delay per vehicle is 10 minutes (2400 veh-min/240 veh), and the average queue length is 40 vehicles (2400 veh-min/60 min).

### EXAMPLE 5.8

After observing arrivals and departures at a highway toll booth over a 60-minute time period, the observer notes that the arrival and departure rates (or service rates) are deterministic, but instead of being uniform, they change over time according to a known function. The arrival rate is given by the function  $\lambda(t) = 2.2 + 0.17t - 0.0032t^2$ , and the departure rate is given by  $\mu(t) = 1.2 + 0.07t$ , where  $t$  is in minutes after the beginning of the observation period and  $\lambda(t)$  and  $\mu(t)$  are in vehicles per minute. Determine the total vehicle delay at the toll booth and the longest queue, assuming  $D/D/1$  queuing.

#### SOLUTION

Note that this problem is an example of a time-dependent deterministic queue because the deterministic arrival and departure rates change over time. Begin by computing the time to queue dissipation by equating vehicle arrivals and departures:

$$\begin{aligned} \int_0^t 2.2 + 0.17t - 0.0032t^2 dt &= \int_0^t 1.2 + 0.07t dt \\ 2.2t + 0.085t^2 - 0.00107t^3 &= 1.2t + 0.035t^2 \\ -0.00107t^3 + 0.05t^2 + t &= 0 \end{aligned}$$

which gives  $t = 61.8$  minutes. Therefore, the total vehicle delay (the area between the arrival and departure functions) is

$$\begin{aligned} D_t &= \int_0^{61.8} 2.2t + 0.085t^2 - 0.00107t^3 dt - \int_0^{61.8} 1.2t + 0.035t^2 dt \\ &= 1.1t^2 + 0.0283t^3 - 0.0002675t^4 - 0.6t^2 - 0.0117t^3 \Big|_0^{61.8} \\ &= -0.0002675(61.8)^4 + 0.0166(61.8)^3 + 0.5(61.8)^2 \\ &= \underline{\underline{1925.8 \text{ veh-min}}} \end{aligned}$$

The queue length (in vehicles) at any time  $t$  is given by the function

$$\begin{aligned} Q(t) &= \int_0^t 2.2 + 0.17t - 0.0032t^2 dt - \int_0^t 1.2 + 0.07t dt \\ &= -0.00107t^3 + 0.05t^2 + t \end{aligned}$$

Solving for the time at which the maximum queue length occurs,

$$\begin{aligned} \frac{dQ(t)}{dt} &= -0.00321t^2 + 0.1t + 1 = 0 \\ t &= \underline{\underline{39.12 \text{ min}}} \end{aligned}$$

Substituting with  $t = 39.12$  minutes gives the maximum queue length:

$$\begin{aligned} Q(39.12) &= -0.00107t^3 + 0.05t^2 + t \Big|_0^{39.12} \\ &= -0.00107(39.12)^3 + 0.05(39.12)^2 + 39.12 \\ &= \underline{\underline{51.58 \text{ veh}}} \end{aligned}$$

### 5.5.3 M/D/1 Queuing

The assumption of exponentially distributed times between the arrivals of successive vehicles (Poisson arrivals) will, in some cases, give a more realistic representation of traffic flow than the assumption of uniformly distributed arrival times. Therefore, the  $M/D/1$  queue (exponentially distributed arrivals, deterministic departures, and one departure channel) has some important applications within the traffic analysis field. Although a graphical solution to an  $M/D/1$  queue is difficult, a mathematical solution is straightforward. Defining a new term (traffic intensity) for the ratio of average arrival to departure rates as

$$\rho = \frac{\lambda}{\mu} \quad (5.27)$$

een the

where

$\rho$  = traffic intensity, unitless,

$\lambda$  = average arrival rate in vehicles per unit time, and

$\mu$  = average departure rate in vehicles per unit time,

and assuming that  $\rho$  is less than 1, it can be shown that for an  $M/D/1$  queue the following queuing performance equations apply:

$$\bar{Q} = \frac{\rho^2}{2(1-\rho)} \quad (5.28)$$

$$\bar{w} = \frac{\bar{Q}}{\lambda} = \frac{\rho}{2\mu(1-\rho)} \quad (5.29)$$

$$\bar{t} = \bar{W} + \frac{1}{\mu} = \frac{\rho + 2 - 2\rho}{2\mu(1-\rho)} \quad \bar{t} = \frac{2 - \rho}{2\mu(1-\rho)} \quad (5.30)$$

$$\text{where } = \frac{2 - \rho}{2\mu(1-\rho)}$$

$\bar{Q}$  = average length of queue in vehicles,

$\bar{w}$  = average waiting time in the queue, in unit time per vehicle,

$\bar{t}$  = average time spent in the system (the summation of average waiting time in the queue and average departure time), in unit time per vehicle, and

Other terms are as defined previously.

It is important to note that under the assumption that the traffic intensity is less than 1 ( $\lambda < \mu$ ), the  $D/D/1$  queue will predict no queue formation. However a queuing model that is derived based on random arrivals or departures, such as the  $M/D/1$  queuing model, will predict queue formations under such conditions. Also, note that the  $M/D/1$  queuing model presented here is based on steady-state conditions (constant average arrival and departure rates), with randomness arising from the assumed probability distribution of arrivals. This contrasts with the time-varying deterministic queuing case, presented in Example 5.8, in which arrival and departure rates changed over time, but randomness was not present.

### EXAMPLE 5.9

Consider the entrance to the recreational park described in Example 5.7. However, let the average arrival rate be 180 veh/h and Poisson distributed (exponential times between arrivals) over the entire period from park opening time (8:00 A.M.) until closing at dusk. Compute the average length of queue (in vehicles), average waiting time in the queue, and average time spent in the system, assuming  $M/D/1$  queuing.

(5.27)

**SOLUTION**

Putting arrival and departure rates into common units of vehicles per minute gives

$$\lambda = \frac{180 \text{ veh/h}}{60 \text{ min/h}} = 3 \text{ veh/min} \text{ for all } t$$

$$\mu = \frac{60 \text{ s/min}}{15 \text{ s/veh}} = 4 \text{ veh/min} \text{ for all } t$$

and

$$\rho = \frac{\lambda}{\mu} = \frac{3 \text{ veh/min}}{4 \text{ veh/min}} = 0.75$$

For the average length of queue (in vehicles), Eq. 5.28 is applied:

$$\begin{aligned}\bar{Q} &= \frac{0.75^2}{2(1 - 0.75)} \\ &= \underline{1.125 \text{ veh}}\end{aligned}$$

For average waiting time in the queue, Eq. 5.29 gives

$$\begin{aligned}\bar{w} &= \frac{0.75}{2(4)(1 - 0.75)} \\ &= \underline{0.375 \text{ min/veh}}\end{aligned}$$

For average time spent in the system [queue time plus departure (service) time], Eq. 5.30 is used:

$$\begin{aligned}\bar{t} &= \frac{2 - 0.75}{2(4)(1 - 0.75)} \\ &= \underline{0.625 \text{ min/veh}}\end{aligned}$$

or, alternatively, because the departure (service) time is  $1/\mu$  (the 0.25 minutes it takes the park attendant to distribute the brochure),

$$\begin{aligned}\bar{t} &= \bar{w} + \frac{1}{\mu} \\ &= 0.375 + \frac{1}{4} \\ &= \underline{0.625 \text{ min/veh}}\end{aligned}$$

#### 5.5.4 M/M/1 Queuing

A queuing model that assumes one departure channel and exponentially distributed departure times in addition to exponentially distributed arrival times (an *M/M/1* queue) is applicable in some traffic applications. For example, exponentially distributed

uted departure patterns might be a reasonable assumption at a toll booth, where some arriving drivers have the correct toll and can be processed quickly, and others do not have the correct toll, producing a distribution of departures about some mean departure rate. Under standard  $M/M/1$  assumptions, it can be shown that the following queuing performance equations apply (again assuming that  $\rho$  is less than 1):

$$\bar{Q} = \frac{\rho^2}{1-\rho} \quad (5.31)$$

$$\bar{w} = \frac{\lambda}{\mu(\mu-\lambda)} \quad (5.32)$$

$$\bar{t} = \frac{1}{\mu-\lambda} \quad (5.33)$$

where

$\bar{Q}$  = average length of queue in vehicles,

$\bar{w}$  = average waiting time in the queue, in unit time per vehicle,

$\bar{t}$  = average time spent in the system ( $\bar{w} + 1/\mu$ ), in unit time per vehicle, and  
Other terms are as defined previously.

### EXAMPLE 5.10

Assume that the park attendant in Examples 5.7 and 5.9 takes an average of 15 seconds to distribute brochures, but the distribution time varies depending on whether park patrons have questions relating to park operating policies. Given an average arrival rate of 180 veh/h as in Example 5.9, compute the average length of queue (in vehicles), average waiting time in the queue, and average time spent in the system, assuming  $M/M/1$  queuing.

#### SOLUTION

Using the average arrival rate, departure rate, and traffic intensity as determined in Example 5.9, the average length of queue is (from Eq. 5.31)

$$\begin{aligned}\bar{Q} &= \frac{0.75^2}{1-0.75} \\ &= \underline{\underline{2.25 \text{ veh}}}\end{aligned}$$

the average waiting time in the queue is (from Eq. 5.32)

$$\begin{aligned}\bar{w} &= \frac{3}{4(4-3)} \\ &= \underline{\underline{0.75 \text{ min/veh}}}\end{aligned}$$

and the average time spent in the system is (from Eq. 5.33)

$$\bar{t} = \frac{1}{4-3}$$

$$= \underline{\underline{1 \text{ min/veh}}}$$


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### 5.5.5 M/M/N Queuing

A more general formulation of the  $M/M/1$  queue is the  $M/M/N$  queue, where  $N$  is the total number of departure channels.  $M/M/N$  queuing is a reasonable assumption at toll booths on turnpikes or at toll bridges, where there is often more than one departure channel available (more than one toll booth open). A parking lot is another example, with  $N$  being the number of parking stalls in the lot and the departure rate,  $\mu$ , being the exponentially distributed times of parking duration.  $M/M/N$  queuing is also frequently encountered in non-transportation applications such as checkout lines at retail stores, security checks at airports, and so on.

The following equations describe the operational characteristics of  $M/M/N$  queuing. Note that unlike the equations for  $M/D/1$  and  $M/M/1$ , which require that the traffic intensity,  $\rho$ , be less than 1, the following equations allow  $\rho$  to be greater than 1 but apply only when  $\rho/N$  (which is called the utilization factor) is less than 1.

$$P_0 = \frac{1}{\sum_{n_c=0}^{N-1} \frac{\rho^{n_c}}{n_c!} + \frac{\rho^N}{N!(1-\rho/N)}} \quad (5.34)$$

$$P_n = \frac{\rho^n P_0}{n!} \quad \text{for } n \leq N \quad (5.35)$$

$$P_n = \frac{\rho^n P_0}{N^{n-N} N!} \quad \text{for } n \geq N \quad (5.36)$$

$$P_{n>N} = \frac{P_0 \rho^{N+1}}{N! N(1-\rho/N)} \quad (5.37)$$

where

$P_0$  = probability of having no vehicles in the system,

$P_n$  = probability of having  $n$  vehicles in the system,

$P_{n>N}$  = probability of waiting in a queue (the probability that the number of vehicles in the system is greater than the number of departure channels),

$n$  = number of vehicles in the system,

$N$  = number of departure channels,

$n_c$  = departure channel number, and

$\rho$  = traffic intensity ( $\lambda/\mu$ ).

$$\bar{Q} = \frac{P_0 \rho^{N+1}}{N! N} \left[ \frac{1}{(1 - \rho/N)^2} \right] \quad (5.38)$$

$$\bar{w} = \frac{\rho + \bar{Q}}{\lambda} - \frac{1}{\mu} \quad (5.39)$$

$$\bar{t} = \frac{\rho + \bar{Q}}{\lambda} \quad (5.40)$$

where

$\bar{Q}$  = average length of queue (in vehicles),

$\bar{w}$  = average waiting time in the queue, in unit time per vehicle,

$\bar{t}$  = average time spent in the system, in unit time per vehicle, and

Other terms are as defined previously.

### EXAMPLE 5.11

(5.35) At an entrance to a toll bridge, four toll booths are open. Vehicles arrive at the bridge at an average rate of 1200 veh/h, and at the booths, drivers take an average of 10 seconds to pay their tolls. Both the arrival and departure rates can be assumed to be exponentially distributed. How would the average queue length, time in the system, and probability of waiting in a queue change if a fifth toll booth were opened?

### SOLUTION

(5.37) Using the equations for  $M/M/N$  queuing, we first compute the four-booth case. Note that  $\mu = 6$  veh/min and  $\lambda = 20$  veh/min, and therefore  $\rho = 3.333$ . Also, because  $\rho/N = 0.833$  (which is less than 1), Eqs. 5.34 to 5.40 can be used. The probability of having no vehicles in the system with four booths open (using Eq. 5.34) is

$$P_0 = \frac{1}{1 + \frac{3.333}{1!} + \frac{3.333^2}{2!} + \frac{3.333^3}{3!} + \frac{3.333^4}{4!(0.1667)}} \\ = 0.0213$$

The average queue length is (from Eq. 5.38)

$$\bar{Q} = \frac{0.0213(3.333)^5}{4!4} \left[ \frac{1}{(0.1667)^2} \right] \\ = 3.287 \text{ veh}$$

The average time spent in the system is (from Eq. 5.40)

$$\bar{t} = \frac{3.333 + 3.287}{20} \\ = 0.331 \text{ min/veh}$$

And the probability of having to wait in a queue is (from Eq. 5.37)

$$P_{n>N} = \frac{0.0213(3.333)^5}{4!4(0.1667)} \\ = 0.548$$

With a fifth booth open, the probability of having no vehicles in the system is (from Eq. 5.34)

$$P_0 = \frac{1}{1 + \frac{3.333}{1!} + \frac{3.333^2}{2!} + \frac{3.333^3}{3!} + \frac{3.333^4}{4!} + \frac{3.333^5}{5!(0.3333)}} \\ = 0.0318$$

The average queue length is (from Eq. 5.38)

$$\bar{Q} = \frac{0.0318(3.333)^6}{5!5} \left[ \frac{1}{(0.3333)^2} \right] \\ = 0.654 \text{ veh}$$

The average time spent in the system is (from Eq. 5.40)

$$\bar{t} = \frac{3.333 + 0.654}{20} \\ = 0.199 \text{ min/veh}$$

And the probability of having to wait in a queue is (from Eq. 5.37)

$$P_{n>N} = \frac{0.0318(3.333)^6}{5!5(0.3333)} \\ = 0.218$$

So opening a fifth booth would reduce the average queue length by 2.633 veh ( $3.287 - 0.654$ ), the average time in the system by 0.132 min/veh ( $0.331 - 0.199$ ), and the probability of waiting in a queue by 0.330 ( $0.548 - 0.218$ ).

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### EXAMPLE 5.12

A convenience store has four available parking spaces. The owner predicts that the duration of customer shopping (the time that a customer's vehicle will occupy a parking space) is exponentially distributed with a mean of 6 minutes. The owner knows that in the busiest hour customer arrivals are exponentially distributed with a mean arrival rate of 20 customers per hour. What is the probability that a customer will not find an open parking space when arriving at the store?

#### SOLUTION

Putting mean arrival and departure rates in common units gives  $\mu = 10 \text{ veh/h}$  and  $\lambda = 20 \text{ veh/h}$ . So  $\rho = 2.0$ , and because  $\rho/N = 0.5$  (which is less than 1), Eqs. 5.34 to 5.40 can be used. The probability of having no vehicles in the system with four parking spaces available (using Eq. 5.34) is

$$\begin{aligned} P_0 &= \frac{1}{1 + \frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!}(0.5)} \\ &= 0.1304 \end{aligned}$$

Thus the probability of not finding an open parking space upon arrival is (from Eq. 5.37)

$$\begin{aligned} P_{n>N} &= \frac{0.1304(2)^5}{4!4(0.5)} \\ &= \underline{\underline{0.087}} \end{aligned}$$


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## 5.6 TRAFFIC ANALYSIS AT HIGHWAY BOTTLENECKS

Some of the most severe congestion problems occur at highway bottlenecks, which are defined as a portion of highway with a lower capacity ( $q_{cap}$ ) than the incoming section of highway. This reduction in capacity can originate from a number of sources, including a decrease in the number of highway lanes and reduced shoulder widths (which tend to cause drivers to slow and thus effectively reduce highway capacity, as will be discussed in Chapter 6). There are two general types of traffic bottlenecks—those that are recurring and those that are incident induced. Recurring bottlenecks exist where the highway itself limits capacity—for example, by a physical reduction in the number of lanes. Traffic congestion at such bottlenecks results

from recurring traffic flows that exceed the vehicle capacity of the highway in the bottleneck area. In contrast, incident-induced bottlenecks occur as a result of vehicle breakdowns or accidents that effectively reduce highway capacity by restricting the through movement of traffic. Because incident-induced bottlenecks are unanticipated and temporary in nature, they have features that distinguish them from recurring bottlenecks, such as the possibility that the capacity resulting from an incident-induced bottleneck may change over time. For example, an accident may initially stop traffic flow completely, but as the wreckage is cleared, partial capacity (one lane open) may be provided for a period of time before full capacity is eventually restored. A feature shared by recurring and incident-induced bottlenecks is the adjustment in traffic flow that may occur as travelers choose other routes and/or different trip departure times, to avoid the bottleneck area, in response to visual information or traffic advisories.

The analysis of traffic flow at bottlenecks can be undertaken using the queuing models discussed in Section 5.5. The most intuitive approach to analyzing traffic congestion at bottlenecks is to assume  $D/D/1$  queuing.

### EXAMPLE 5.13

An incident occurs on a freeway that has a capacity in the northbound direction, before the incident, of 4000 veh/h and a constant flow of 2900 veh/h during the morning commute (no adjustments to traffic flow result from the incident). At 8:00 A.M. a traffic accident closes the freeway to all traffic. At 8:12 A.M. the freeway is partially opened with a capacity of 2000 veh/h. Finally, the wreckage is removed, and the freeway is restored to full capacity (4000 veh/h) at 8:31 A.M. Assume  $D/D/1$  queuing to determine time of queue dissipation, longest queue length, total delay, average delay per vehicle, and longest wait of any vehicle (assuming FIFO).

#### SOLUTION

Let  $\mu$  be the full-capacity departure rate and  $\mu_r$  be the restrictive partial-capacity departure rate. Putting arrival and departure rates in common units of vehicles per minute,

$$\mu = \frac{4000 \text{ veh/h}}{60 \text{ min/h}} = 66.67 \text{ veh/min}$$

$$\mu_r = \frac{2000 \text{ veh/h}}{60 \text{ min/h}} = 33.33 \text{ veh/min}$$

$$\lambda = \frac{2900 \text{ veh/h}}{60 \text{ min/h}} = 48.33 \text{ veh/min}$$

The arrival rate is constant over the entire time period, and the total number of vehicles is equal to  $\lambda t$ , where  $t$  is the number of minutes after 8:00 A.M. The total number of departing vehicles is

$$\begin{aligned}
 0 & \quad \text{for } t \leq 12 \text{ min} \\
 \mu_r(t - 12) & \quad \text{for } 12 \text{ min} < t \leq 31 \text{ min} \\
 633.33 + \mu(t - 31) & \quad \text{for } t > 31 \text{ min}
 \end{aligned}$$

Note that the value of 633.33 in the departure rate function for  $t > 31$  is based on the preceding departure rate function [ $33\frac{1}{3}(31 - 12)$ ]. These arrival and departure rates can be represented graphically as shown in Fig. 5.8. As discussed in Section 5.5, for  $D/D/1$  queuing, the queue will dissipate at the intersection point of the arrival and departure curves, which can be determined as

$$\lambda t = 633.33 + \mu(t - 31) \quad \text{or} \quad t = 78.16 \text{ min (just after 9:18 A.M.)}$$

At this time a total of 3777.5 vehicles ( $48.33 \times 78.16$ ) will have arrived and departed (for the sake of clarity, fractions of vehicles are used). The longest queue (longest vertical distance between arrival and departure curves) occurs at 8:31 A.M. and is

$$\begin{aligned}
 Q_{max} &= \lambda t - \mu_r(t - 12) \\
 &= 48.33(31) - 33.33(19) \\
 &= 865 \text{ veh}
 \end{aligned}$$

Total vehicle delay is (using equations for triangular and trapezoidal areas to calculate the total area between the arrival and departure curves)

$$\begin{aligned}
 D_t &= \frac{1}{2}(12)(580) + \frac{1}{2}(580 + 1498.33)(19) - \frac{1}{2}(19)(633.33) \\
 &\quad + \frac{1}{2}(1498.33 - 633.33)(78.16 - 31) \\
 &= 37,604.2 \text{ veh-min}
 \end{aligned}$$

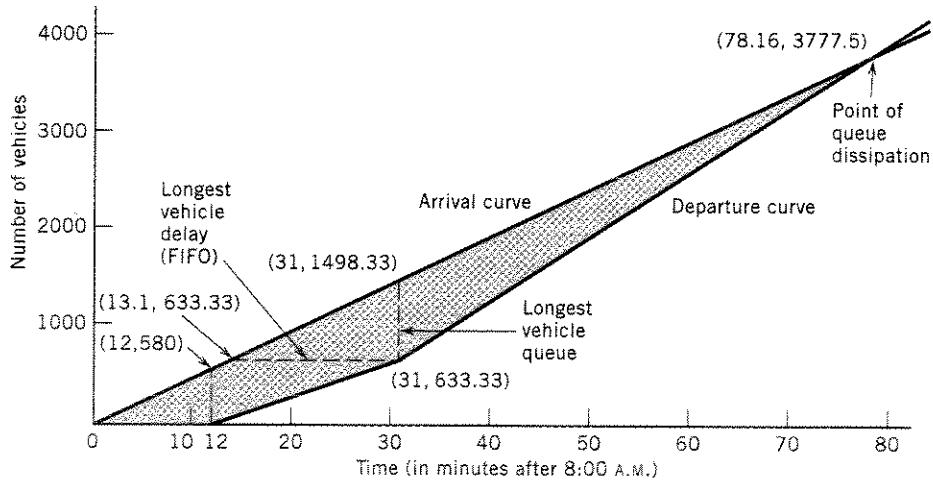


Figure 5.8  $D/D/1$  queuing diagram for Example 5.13.

The average delay per vehicle is 9.95 min ( $37,604.2/3777.5$ ). The longest wait of any vehicle (the longest horizontal distance between the arrival and departure curves), assuming a FIFO queuing discipline, will be the delay time of the 633.33rd vehicle to arrive. This vehicle will arrive 13.1 minutes ( $633.33/48.33$ ) after 8:00 A.M. and will depart at 8:31 A.M., being delayed a total of 17.9 min.

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## NOMENCLATURE FOR CHAPTER 5

$D$	deterministic arrivals or departures	$Q_{max}$	maximum length of queue
$D_i$	total vehicle delay	$s$	vehicle spacing
$h$	vehicle time headway	$t$	time
$k$	traffic density	$\bar{t}$	average time spent in the system
$k_j$	traffic jam density	$u$	space-mean speed (also denoted $\bar{u}_s$ )
$k_{cap}$	traffic density at capacity	$u_i$	spot speed for vehicle $i$
$l$	roadway length	$u_f$	free-flow speed
$M$	exponentially distributed arrivals or departures	$u_{cap}$	speed at capacity
$n$	number of vehicles	$\bar{u}_s$	space-mean speed (also denoted simply as $u$ )
$n_c$	departure channel number	$\bar{u}_t$	time-mean speed
$N$	total number of departure channels	$\bar{w}$	average time waiting in the queue
$q$	traffic flow	$\lambda$	arrival rate
$q_{cap}$	traffic flow at capacity (maximum traffic flow)	$\mu$	departure rate
$Q$	length of queue	$\rho$	traffic intensity
$\bar{Q}$	average length of queue		

## REFERENCES

- Drew, D. R. "Deterministic Aspects of Freeway Operations and Control." *Highway Research Record*, 99, 1965.
- Pipes, L. A. "Car Following Models and the Fundamental Diagram of Road Traffic." *Transportation Research*, vol. 1, no. 1, 1967.
- Poch, M., and F. Mannerling. "Negative Binomial Analysis of Intersection-Accident Frequencies," *Journal of Transportation Engineering*, vol. 122, no. 2, March/April 1996.

- Transportation Research Board. *Traffic Flow Theory: A Monograph*. Special Report 165. Washington, DC: National Research Council, 1975.
- Transportation Research Board. *Highway Capacity Manual*. Washington, DC: National Research Council, 2000.

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## PROBLEMS

**5.1** On a specific westbound section of highway, studies show that the speed-density relationship is

$$u = u_f \left[ 1 - \left( \frac{k}{k_j} \right)^{3.5} \right]$$

It is known that the capacity is 3800 veh/h and the jam density is 225 veh/mi. What is the space-mean speed of the traffic at capacity, and what is the free-flow speed?

**5.2** A section of highway has a speed-flow relationship of the form

$$q = au^2 + bu$$

It is known that at capacity (which is 2900 veh/h) the space-mean speed of traffic is 30 mi/h. Determine the speed when the flow is 1400 veh/h and the free-flow speed.

**5.3** A section of highway has the following flow-density relationship:

$$q = 50k - 0.156k^2$$

What is the capacity of the highway section, the speed at capacity, and the density when the highway is at one-quarter of its capacity?

**5.4** Assume you are observing traffic in a single lane of a highway at a specific location. You measure the average headway and average spacing of passing vehicles as 3 seconds and 150 ft, respectively. Calculate the flow, average speed, and density of the traffic stream in this lane.

**5.5** Assume you are an observer standing at a point along a three-lane roadway. All vehicles in lane 1 are traveling at 30 mi/h, all vehicles in lane 2 are traveling at 45 mi/h, and all vehicles in lane 3 are traveling at 60 mi/h. There is also a constant spacing of 0.5 mile between vehicles. If you collect spot speed data for all vehicles as they cross your observation point, for 30 minutes, what will be the time-mean speed and space-mean speed for this traffic stream?

**5.6** Four race cars are traveling on a 2.5-mile tri-oval track. The four cars are traveling at constant speeds of 195 mi/h, 190 mi/h, 185 mi/h, and 180 mi/h, respectively. Assume you are an observer standing at a point on the track for a period of 30 minutes and are recording the instantaneous speed of each vehicle as it crosses your point. What is the time-mean speed and

space-mean speed for these vehicles for this time period? (Note: Be careful with rounding.)

**5.7** For Problem 5.6, calculate the space-mean speed assuming you were provided with only an aerial photo of the circling race cars and the constant travel speed of each of the vehicles.

**5.8** An observer has determined that the time headways between successive vehicles on a section of highway are exponentially distributed and that 60% of the headways between vehicles are 13 seconds or greater. If the observer decides to count traffic in 30-second time intervals, estimate the probability of the observer counting exactly four vehicles in an interval.

**5.9** At a specified point on a highway, vehicles are known to arrive according to a Poisson process. Vehicles are counted in 20-second intervals, and vehicle counts are taken in 120 of these time intervals. It is noted that no cars arrive in 18 of these 120 intervals. Approximate the number of these 120 intervals in which exactly three cars arrive.

**5.10** For the data collected in Problem 5.9, estimate the percentage of time headways that will be 10 seconds or greater and those that will be less than 6 seconds.

**5.11** A vehicle pulls out onto a single-lane highway that has a flow rate of 280 veh/h (Poisson distributed). The driver of the vehicle does not look for oncoming traffic. Road conditions and vehicle speeds on the highway are such that it takes 1.5 seconds for an oncoming vehicle to stop once the brakes are applied. Assuming a standard driver reaction time of 2.5 seconds, what is the probability that the vehicle pulling out will get in an accident with oncoming traffic?

**5.12** Consider the conditions in Problem 5.11. How short would the driver reaction times of oncoming vehicles have to be for the probability of an accident to equal 0.15?

**5.13** A toll booth on a turnpike is open from 8:00 A.M. to 12 midnight. Vehicles start arriving at 7:45 A.M. at a uniform deterministic rate of six per minute until 8:15 A.M. and from then on at two per minute. If vehicles are processed at a uniform deterministic rate of six per minute, determine when the queue will dissipate, the total delay, the maximum queue length (in vehicles), the longest vehicle delay under FIFO, and the longest vehicle delay under LIFO.

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**5.14** Vehicles begin to arrive at a parking lot at 6:00 A.M. at a rate of eight per minute. Due to an accident on the access highway, no vehicles arrive from 6:20 to 6:30 A.M. From 6:30 A.M. on, vehicles arrive at a rate of two per minute. The parking lot attendant processes incoming vehicles (collects parking fees) at a rate of four per minute throughout the day. Assuming D/D/1 queuing, determine total vehicle delay.

**5.15** The arrival rate at a parking lot is 6 veh/min. Vehicles start arriving at 6:00 P.M., and when the queue reaches 36 vehicles, service begins. If company policy is that total vehicle delay should be equal to 500 veh-min, what is the departure rate? (Assume D/D/1 queuing and a constant service rate.)

**5.16** Vehicles begin to arrive at a toll booth at 8:50 A.M. with an arrival rate of  $\lambda(t) = 4.1 + 0.01t$  [with  $t$  in minutes and  $\lambda(t)$  in vehicles per minute]. The toll booth opens at 9:00 A.M. and processes vehicles at a rate of 12 per minute throughout the day. Assuming D/D/1 queuing, when will the queue dissipate and what will be the total vehicle delay?

**5.17** Vehicles begin to arrive at a toll booth at 7:50 A.M. with an arrival rate of  $\lambda(t) = 5.2 - 0.01t$  (with  $t$  in minutes after 7:50 A.M. and  $\lambda$  in vehicles per minute). The toll booth opens at 8:00 A.M. and serves vehicles at a rate of  $\mu(t) = 3.3 + 2.4t$  (with  $t$  in minutes after 8:00 A.M. and  $\mu$  in vehicles per minute). Once the service rate reaches 10 veh/min, it stays at that level for the rest of the day. If queuing is D/D/1, when will the queue that formed at 7:50 A.M. be cleared?

**5.18** Vehicles arrive at a freeway on-ramp meter at a constant rate of six per minute starting at 6:00 A.M. Service begins at 6:00 A.M. such that  $\mu(t) = 2 + 0.5t$ , where  $\mu(t)$  is in veh/min and  $t$  is in minutes after 6:00 A.M. What is the total delay and the maximum queue length (in vehicles)?

**5.19** Vehicles arrive at a tollbooth according to the function  $\lambda(t) = 5.2 - 0.20t$ , where  $\lambda(t)$  is in vehicles per minute and  $t$  is in minutes. The toll booth operator processes one vehicle every 20 seconds. Determine total delay, maximum queue length, and the time that the 20th vehicle to arrive waits from its arrival to its departure.

**5.20** There are 10 vehicles in a queue when an attendant opens a toll booth. Vehicles arrive at the booth at a rate of 4 per minute. The attendant opens the booth and improves the service rate over time following the function  $\mu(t) = 1.1 + 0.30t$ , where  $\mu(t)$  is in vehicles

per minute and  $t$  is in minutes. When will the queue clear, what is the total delay, and what is the maximum queue length?

**5.21** Vehicles begin to arrive at a parking lot at 6:00 A.M. with an arrival rate function (in vehicles per minute) of  $\lambda(t) = 1.2 + 0.3t$ , where  $t$  is in minutes. At 6:10 A.M. the parking lot opens and processes vehicles at a rate of 12 per minute. What is the total delay and the maximum queue length?

**5.22** At a parking lot, vehicles arrive according to a Poisson process and are processed (parking fee collected) at a uniform deterministic rate at a single station. The mean arrival rate is 4 veh/min and the processing rate is 5 veh/min. Determine the average length of queue, the average time spent in the system, and the average waiting time in the queue.

**5.23** Consider the parking lot and conditions described in Problem 5.22. If the rate at which vehicles are processed became exponentially distributed (instead of deterministic) with a mean processing rate of 5 veh/min, what would be the average length of queue, the average time spent in the system, and the average waiting time in the queue?

**5.24** Vehicles arrive at a toll booth with a mean arrival rate of 2 veh/min (the time between arrivals is exponentially distributed). The toll booth operator processes vehicles (collects tolls) at a uniform deterministic rate of one every 20 seconds. What is the average length of queue, the average time spent in the system, and the average waiting time in the queue?

**5.25** A business owner decides to pass out free transistor radios (along with a promotional brochure) at a booth in a parking lot. The owner begins giving the radios away at 9:15 A.M. and continues until 10:00 A.M. Vehicles start arriving for the radios at 8:45 A.M. at a uniform deterministic rate of 4 per minute and continue to arrive at this rate until 9:15 A.M. From 9:15 to 10:00 A.M. the arrival rate becomes 8 per minute. The radios and brochures are distributed at a uniform deterministic rate of 11 cars per minute over the 45-minute time period. Determine total delay, maximum queue length, and longest vehicle delay assuming FIFO and LIFO.

**5.26** Consider the conditions described in Problem 5.25. Suppose the owner decides to accelerate the radio-brochure distribution rate (in veh/min) so that the queue that forms will be cleared by 9:45 A.M. What would this new distribution rate be?

**5.27** A ferryboat queuing lane holds 30 vehicles. If vehicles are processed (tolls collected) at a uniform deterministic rate of 4 vehicles per minute and processing begins when the lane reaches capacity, what is the uniform deterministic arrival rate if the vehicle queue is cleared 30 minutes after vehicles begin to arrive?

**5.28** At a toll booth, vehicles arrive and are processed (tolls collected) at uniform deterministic rates  $\lambda$  and  $\mu$ , respectively. The arrival rate is 2 veh/min. Processing begins 13 minutes after the arrival of the first vehicle, and the queue dissipates  $t$  minutes after the arrival of the first vehicle. Letting the number of vehicles that must actually wait in a queue be  $x$ , develop an expression for determining processing rates in terms of  $x$ .

**5.29** Vehicles arrive at a recreational park booth at a uniform deterministic rate of 4 veh/min. If uniform deterministic processing of vehicles (collecting of fees) begins 30 minutes after the first arrival and the total delay is 3600 veh-min, how long after the arrival of the first vehicle will it take for the queue to be cleared?

**5.30** Trucks begin to arrive at a truck weigh station (with a single scale) at 6:00 A.M. at a deterministic but time-varying rate of  $\lambda(t) = 4.3 - 0.22t$  [ $\lambda(t)$  is in veh/min and  $t$  is in minutes]. The departure rate is a constant 2 veh/min (time to weigh a truck is 30 seconds). When will the queue that forms be cleared, what will be the total delay, and what will be the maximum queue length?

**5.31** Vehicles begin to arrive at a remote parking lot after the start of a major sporting event. They are arriving at a deterministic but time-varying rate of  $\lambda(t) = 3.3 - 0.1t$  [ $\lambda(t)$  is in veh/min and  $t$  is in minutes]. The parking lot attendant processes vehicles (assigns spaces and collects fees) at a deterministic rate at a single station. A queue exceeding four vehicles will back up onto a congested street, and is to be avoided.

How many vehicles per minute must the attendant process to ensure that the queue does not exceed four vehicles?

**5.32** A truck weighing station has a single scale. The time between truck arrivals at the station is exponentially distributed with a mean arrival rate of 1.5 veh/min. The time it takes vehicles to be weighed is exponentially distributed with a mean rate of 2 veh/min. When more than 5 trucks are in the system, the queue backs up onto the highway and interferes with through traffic. What is the probability that the number of trucks in the system will exceed 5?

**5.33** Consider the convenience store described in Example 5.12. The owner is concerned about customers not finding an available parking space when they arrive during the busiest hour. How many spaces must be provided for there to be less than a 1% chance of an arriving customer not finding an open parking space?

**5.34** Vehicles arrive at a toll bridge at a rate of 430 veh/h (the time between arrivals is exponentially distributed). Two toll booths are open and each can process arrivals (collect tolls) at a mean rate of 10 seconds per vehicle (the processing time is also exponentially distributed). What is the total time spent in the system by all vehicles in a 1-hour period?

**5.35** Vehicles leave an airport parking facility (arrive at parking fee collection booths) at a rate of 500 veh/h (the time between arrivals is exponentially distributed). The parking facility has a policy that the average time a patron spends in a queue waiting to pay for parking is not to exceed 5 seconds. If the time required to pay for parking is exponentially distributed with a mean of 15 seconds, what is the fewest number of payment processing booths that must be open to keep the average time spent in a queue below 5 seconds?