

# Variational Bayesian Inference for Unsupervised Lexicon Discovery

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## 1 Abstract

Speech is a defining feature of humankind, but little is known about the mental storage and processing of spoken language. Computational models can help elucidate these internal processes while allowing us to engineer improved speech technology. We present a variational Bayesian inference model for the fully unsupervised discovery of phones, sub-word units, and words directly from an acoustic input. Extending the state-of-the-art model for unsupervised lexicon discovery introduced by Lee et al. (2015), which relied on sampling, our framework permits parallelization and distribution to multiple cores, promising speed improvements. We introduce variational Bayesian methodology, re-framing the original model with this new paradigm. We highlight some relevant results of the original paper, and discuss improvements made to similar models through the application of variational Bayesian methods. Finally, we consider future experiments enabled by this system and suggest some possible applications of a completely unsupervised language-independent model such as the one we formulate.

## 2 Introduction

### 2.1 Background

Spoken language allows humans to communicate effectively with each other, making it a fundamental characteristic of our species. Studying the computational underpinnings of speech production and perception gives us a fuller understanding of the phenomenon, and of language as a whole, while helping us to engineer better computer-human interactions. Many current state-of-the-art spoken language systems focus on supervised learning—that is, training a system on copious amounts of labeled data. Producing such data is costly and time-consuming, meaning that sufficient quantities exist only for a small fraction of the world’s languages, and even then, the quality is often insufficient. This contributes to the underrepresentation of many world languages and language families, particularly those spoken in the developing world. In addition, a supervised approach does not offer an entirely accurate model of human language capabilities. Language acquisition is unsupervised in the machine-learning sense; we infer linguistic structure and rules implicitly, and from unlabeled data. These factors motivate an unsupervised approach which incorporates existing theories about language and cognition in order to eliminate the need for labeled training data. The practically unbounded amount of unlabeled speech data which exists in the form of videos,

audiobooks, and archival recordings (to name a few sources) particularly incentivizes unsupervised algorithms for speech processing, especially one that can interface directly with an acoustic signal.

## 2.2 The model

The specific language phenomenon we model is lexicon discovery, for which we implement an unsupervised learning algorithm. Extending the state-of-the-art framework introduced by Lee et al. (2015), our model constructs a complete hierarchy of linguistic units (phonemes, morphemes, and words) directly from an acoustic input. The *unsupervised lexicon discovery* model (ULD) presented by Lee et al. (2015) was the first to jointly model the induction of a full stack of hierarchical components, combining and building on earlier work in phoneme discovery (Lee and Glass, 2012) and in unsupervised syntactic (Johnson et al., 2007) and morphemic structure inference (O’Donnell, 2015). ULD is composed of three main components: a Dirichlet process hidden Markov model (DPHMM) (Lee and Glass, 2012) for segmenting continuous audio input and hypothesizing a sequence of reusable phone-like units (PLUs), an adaptor grammar (Johnson et al., 2007) which recognizes and stores frequently reused syntactic structures based on an underlying grammar—in this case, grouping PLUs into morphemes and words—and finally a noisy channel model, which allows substitutions, insertions, and deletions to occur between the inputs and outputs of the DPHMM and adaptor grammar components, approximating a phonological system. The noisy channel is crucial to the *joint learning* nature of the model in that it allows the DPHMM and adaptor grammar to constrain one another. This type of joint learning framework has been shown to improve accuracy (Johnson, 2008).

ULD uses nonparametric Bayesian inference to implement a fully unsupervised learning model. In Bayesian modeling, we posit unobserved (latent) variables which are conditionally dependent on the observed data. Defining the model in this way allows us to dynamically update our latent variables (our hypothesis) according to the data (our evidence) while incorporating prior theoretical assumptions about the problem, yielding a powerful method for unsupervised learning.

A Bayesian model is defined as follows: let  $Z$  be the set of latent variables, let  $X$  be the set of observed data, and let  $\Phi$  be a set of static model hyperparameters specified by the user. Then by Bayes’ rule we have:

$$P(Z|X, \Phi) = \frac{P(X|Z, \Phi)P(Z, \Phi)}{\sum_{z \in Z} P(X|Z, \Phi)P(Z, \Phi)}$$

The numerator is often called the *generative model*, and gives a joint distribution on data and latent variables. In general, we define Bayesian models in terms of a generative model, where the latent random variables are used to generate data. It consists of the conditional probability of the data given the latent variables defining our model (referred to as the *likelihood*), multiplied by the prior probability of our model (known as the *prior*). This gives us an unnormalized measure of the likelihood of generating the data from our model. However, in order to obtain a proper probability we need to divide by a normalizing constant—the probability of the data. We obtain this probability by summing over all possible ways of

obtaining the data—that is, all possible latent variable values. It is easy to see why, given a sufficiently complex model and a large amount of data, this sum becomes computationally intractable.

In absence of a way to directly compute the marginal probability of the data, there are two common approaches to doing Bayesian inference. The first, used in the original ULD model, is sampling, which capitalizes on the fact that given a generative model, we can approximate the *posterior* (the left-hand side of Bayes’ rule) by randomly sampling from the generative model, eliminating the need to calculate the *marginal likelihood*. This technique has played a major role in Bayesian inference, but is difficult to parallelize. This makes it nearly impossible to scale such algorithms to the types of large speech datasets available (Blei et al., 2017). The second method for avoiding the intractable computation required to obtain the marginal likelihood is variational Bayesian inference.

## 2.3 Variational Bayesian inference

Variational Bayesian inference re-casts the challenge of computing the posterior distribution on latent variables as an optimization problem. By iteratively maximizing a lower bound on the (incomputable) marginal likelihood, framed as a *variational distribution* over latent variables, the algorithm yields an approximation of the posterior. This intuition is clarified by (3). This strategy lends itself well to parallelization across multiple cores and interfacing with tools such as the MapReduce framework for cluster computation (Zhai et al., 2012). In order obtain the approximation of our posterior, we introduce a family of variational distributions  $q_\nu(Z)$  which have the same support as the posterior ( $p(Z|X, \Phi)$ ) indexed by variational parameter  $\nu$ , used to adjust the distribution  $q$ .

### 2.3.1 Computing the ELBO

Our goal is to find the  $q_\nu(Z)$  which minimizes the Kullback-Liebler (KL) divergence between  $q_\nu(Z)$  and  $p(Z|X, \Phi)$ , or  $D_{KL}(q_\nu(Z) || p(Z|X, \Phi))$ , where KL-divergence is a measure of the difference between to probability distributions. KL-divergence is given by

$$\begin{aligned} D_{KL}(q_\nu(Z) || p(Z | X, \Phi)) &= \mathbb{E}_q[\log \frac{q_\nu(Z)}{p(Z | X, \Phi)}] \\ &= \mathbb{E}_q[\log q_\nu(Z)] - \mathbb{E}_q[\log p(Z, X | \Phi)] + \log p(X | \Phi) \end{aligned} \quad (1)$$

where  $\mathbb{E}_q$  indicates taking the expected value with respect to  $q$  (see Appendix A for further explanation of this notation). Unfortunately, the third value  $\log p(X | \Phi)$  is the marginal likelihood, requiring the intractable computation we seek to avoid, so we cannot directly compute KL divergence. However, this equation does generate a valuable result: a lower bound on the marginal ( $\log p(X | \Phi)$ ) called the evidence lower bound (ELBO). To obtain this result, first note that because of what KL divergence represents, it can never be negative. This gives us:

$$\begin{aligned} 0 &\leq \mathbb{E}_q[\log q(Z)] - \mathbb{E}_q[\log p(Z, X | \Phi)] + \log p(X | \Phi) \\ -\log p(X | \Phi) &\leq \mathbb{E}_q[\log q(Z)] - \mathbb{E}_q[\log p(Z, X | \Phi)] \\ \log p(X | \Phi) &\geq \mathbb{E}_q[\log p(Z, X | \Phi)] - \mathbb{E}_q[\log q(Z)] \end{aligned} \quad (2)$$

Thus

$$\begin{aligned} ELBO(q) &= \mathbb{E}_q[\log p(Z, X | \Phi)] - \mathbb{E}_q[\log q(Z)] \\ &= \mathbb{E}_q[\log p(Z, X | \Phi)] + H(q) \end{aligned}$$

where  $H(q)$  is the entropy of the distribution  $q$ . This derivation yields an important fact:

$$\log p(X) - D_{KL}(q(Z) || p(Z | X | \Phi)) = ELBO(q) \quad (3)$$

(3) provides the explanation to the previous intuition that maximizing the ELBO allows us to minimize the KL divergence—the maximal ELBO is  $\log p(X)$ . When  $ELBO(q) = \log p(X | \Phi)$  the KL-divergence must be 0 (Blei et al., 2017). For an expanded derivation, as well as an equivalent derivation using Jensen’s inequality, see Appendix A.

### 2.3.2 Mean-field approximation

One of the fundamental reasons why we cannot compute the posterior directly is the presence of conditional dependencies between latent variables in it—variables that might have been independent in the generative process become conditionally dependent in the posterior. Since our variational distribution need only be an approximation, we make a mean-field assumption—that is, to assume that the variational distribution  $q_\nu(Z)$  has none of these conditional dependencies, or

$$q_\nu(Z) = \prod_{z_i \in Z} q_{\nu_i}(z_i)$$

This is a powerful assumption allowing us to optimize each variational distribution iteratively. While holding all other variational distributions constant, we can find the variational parameters for  $q_{\nu_i}(z_i)$  that maximize the marginal likelihood. By the chain rule, we derive the following lower bound for each variational distribution as:

$$\mathcal{L}_i(q_{\nu_i}(z_i)) = \mathbb{E}_q[\log p(z_i | Z_{-i}, X, \Phi)] - \mathbb{E}_q[\log q_{\nu_i}(z_i)] \quad (4)$$

where  $Z_{-i}$  indicates the set of all latent variables in  $Z$  which are not  $z_i$ .

### 2.3.3 Updates

Recall that our goal is to maximize our lower bound on the variational distribution over each latent variable  $\mathcal{L}_i$ , which is accomplished by adjusting each  $\nu_i$ . The value for  $\nu_i$  that locally maximizes the function  $\mathcal{L}_i$  is found by setting the first derivative with respect to  $\nu_i$  equal to 0 and solving for  $\nu_i$ . Taking the derivative of the objective function is costly; this cost is compounded by the need to recompute the derivative at every parameter update, and for every variational distribution parameter. Using exponential family random variables allows us to take advantage of some convenient mathematical facts and avoid this costly computation entirely. When each  $q_{\nu_i}(z_i)$  and each distribution in the generative model are in the exponential family, we obtain the following closed-form update for each  $\nu_i$ :

$$\nu_i = \mathbb{E}_q[g_i(Z_{-i}, X, \Phi)] = \mathbb{E}_q \left[ \frac{\phi_1 + \sum_{z_n \in Z_{-i}} t(x_n, z_n)}{\phi_2 + N} \right] \quad (5)$$

where  $g_i(Z_{-i}, X, \Phi)$  is a function which gives the natural parameters of the exponential family distribution *in the posterior*,  $\phi_1$  and  $\phi_2$  are the parameters for the exponential family distribution in the *prior*,  $N$  is the total number of datapoints, and  $t(x_n, z_n)$  is the *sufficient statistic* of the prior distribution—in many cases, this is simply a count of occurrences of  $z_n$ . For a more in-depth explanation of this result, see Appendix B.

### 2.3.4 Coordinate ascent

We now have a way of optimizing each variational distribution by setting the parameters to the expected value of the natural parameters in the posterior, conditioned on the other latent variables and the data. If our objective function could be formulated as a strictly convex function, then a single update of the variational parameters would be sufficient to find a solution, since a local optimum in a convex function is a global one. However, given the composite nature of most Bayesian models, this is seldom the case, and we use a non-convex optimization algorithm to iteratively find local maxima, with the ultimate goal of converging on the global maximum. The Coordinate Ascent Variational Inference (CAVI) provides an interface for optimizing our ELBO. It is worth noting that the CAVI algorithm is a generalization of the well-known Expectation-Maximization algorithm (Dempster et al., 1977); where the latter gives a point estimate of the posterior, the former returns an approximation of the full distribution—a more data-rich representation. In CAVI, we alternate between computing our objective function (analogous to the expectation step) and updating our variational parameters (analogous to the maximization step) (Neal and Hinton, 1998).

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**Algorithm 1:** The CAVI algorithm

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initialize each  $\nu_i$ 
while not ELBO converged do
    for each variational parameter  $\nu_i$  do
         $\nu_i = \mathbb{E}_q[g_i(Z_{-i}, X, \Phi)]$ 
    re-compute ELBO  $\mathcal{L}(q) = \mathbb{E}_q[\log p(Z, X)] + H(q)$ 

```

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The initialization step for each  $\nu_i$  can be random, but this is not required. Often, we let  $q_{\nu_i}(z_i)$  be a distribution of the same type as in the generative model, and initialize it with uniform or random parameters. However, we may choose the initial parameters more deliberately and encode some bias in the variational distribution, with the caveat (and occasionally the benefit) that varying initializations can lead to convergence on different local optima (Blei et al., 2017)

## 3 The Generative Model

The ULD generative model can be broken up into roughly three parts: the adaptor grammar, the noisy channel, and the Dirichlet process hidden Markov model (DPHMM). From a top-down perspective, the adaptor grammar parses a sequence of top-level phone-like units (PLUs) into morphemes and words, building a syntactic tree. The noisy channel, using a sequence of edit operations (insertion, deletion, and substitution) maps the yield of this tree (top-level PLUs) to bottom-level PLUs, modeling some of the phonological processes which

occur during speech production. Finally, the DPHMM takes these bottom-level PLUs and finds an acoustic signal (represented by 39-dimensional Mel frequency cepstral coefficient (MFCC) vectors) that could have generated them. It is necessary to point out that this top-down view of the model does not accurately reflect the complete flow of information in it. Because it is a joint learning model, every component affects every other—for example, the DPHMM first infers the PLU inventory which the adaptor grammar needs to generate trees, and makes adjustments to PLU boundaries which can affect the adaptor grammar’s parses. This complex generative model contains a multitude of inference steps, making it an ideal candidate for the application of variational Bayesian methods. The following sections offer more detailed descriptions of each model and their respective latent variables.

### 3.1 Adaptor grammars

First developed by Johnson et al. (2007), adaptor grammars take as input some context-free grammar and a set of strings which can be parsed by that grammar. Using a non-parametric distribution, they store derivational trees while biasing the reuse of frequently occurring trees; these stored fragments reveal patterns in the linguistic structure of the data. By increasing the likelihood of reusing a tree according to its frequency, adaptor grammars instantiate a “rich get richer” dynamic where common trees become more likely than rare ones. In ULD, we use adaptor grammars to group discovered PLUs into morphemes and words. Before formally defining adaptor grammars, we need to define context free grammars, probabilistic context free grammars, and the Pitman-Yor Process.

#### 3.1.1 Context-free Grammars and probabilistic context-free grammars

A context-free grammar (CFG) is a tuple  $(N, E, R, S)$  where  $N$  is a set of nonterminals symbols,  $E$  is a set of terminal symbols disjoint from  $N$ , and  $R$  is a set of rules of the form  $A \rightarrow \beta$  where  $A \in N$  and  $\beta \in (N \cup E)^*$  (i.e. any concatenation of symbols in  $N$  and  $E$ ). We constrain our CFGs to be in Chomsky normal form, meaning every rule is either of form  $A \rightarrow a$  where  $a \in E$  or  $A \rightarrow BC$  where  $B \in N$  and  $C \in N$ . Note that any CFG without epsilon productions (rules that go to the empty string) can be rewritten in Chomsky normal form. (Hopcroft et al., 2006)

Similar to a CFG, a probabilistic context-free grammar (PCFG) is a tuple  $(N, E, R, S, \theta)$  where  $N, E, R, S$  are the same as in a CFG, and  $\theta$  is a set of probability vectors such that  $\sum_{A \rightarrow \beta \in R_A} \theta_{A \rightarrow \beta} = 1$ , where  $R_A$  is the set of rules which have nonterminal  $A$  on the left-hand side.

#### 3.1.2 Pitman-Yor Process

The Pitman-Yor process (Pitman and Yor, 1997) can be thought of as a distribution on infinite-sided dice, or as generating a partition of integers. Perhaps more intuitively, a Pitman-Yor process defines a distribution over distributions—each draw from a Pitman-Yor process is itself a countably infinite distribution. There are multiple equivalent ways of defining a Pitman-Yor process; we use the stick-breaking construction, as it gives us an

iterative definition. Given a scale parameter  $a$ , a discount factor  $b$  and a base distribution  $G_0$ , a Pitman-Yor process which partitions  $[0, 1]$  into countably infinite segments is defined by algorithm 2:

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**Algorithm 2:** The Pitman-Yor process

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```

for  $i \in \{1, \dots\}$  do
    draw  $\nu_i \sim \text{Beta}(1 - b, a + ib)$ 
    sample atom  $z_i \sim G_0$ 
    define  $\pi_i \triangleq \nu_i \prod_{j=1}^{i-1} (1 - \nu_j)$ 

define  $G(z) \triangleq \sum_{i=1}^{\infty} \pi_i \delta(z_i, z)$  where  $\delta(z_i, z) = 1$  if  $z_i = z$ , 0 otherwise

```

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(Sethuraman, 1994)

Recall that a draw from a Beta distribution is a biased coin. Intuitively, each  $\nu_i$  is a coin that gives the probability of stopping at that stick, and  $1 - \nu_i$  is the probability of continuing to the next stick.  $\pi_i$ , the probability of being at stick  $i$ , is equivalent to the product of the probability of having passed sticks  $1, \dots, i - 1$  and the probability of stopping at stick  $i$ . The parameters  $a, b$  control the concentration and spread of the distribution, determining whether there will be few sticks with a lot of mass each or many sticks with little mass. Since  $\pi$  is normalized, it gives an infinite discrete probability distribution.

### 3.1.3 Adaptor grammar definition

With these components, we can formally define an adaptor grammar as a tuple  $(G, M, a, b, \alpha)$  where  $G$  is a CFG,  $M$  is a set of *adapted nonterminals*,  $a$  and  $b$  are Pitman-Yor process parameters, and  $\alpha$  is a set of Dirichlet distribution parameters indexed by each nonterminal in  $N$ . The adaptor grammar employs the Pitman-Yor process to generate a distribution over tree fragments (called *grammatons*) which biases the reuse of common fragments, increasing the probability of stopping at the stick associated with the grammaton when it appears. To formally define an adaptor grammar, let  $A_1, \dots, A_k$  be a reverse topological sorting of the adapted nonterminals in  $M$ , such that for all  $A_j \in M$  the children of  $A_j$  come before  $A_j$  in the ordering. Relying on this ordering, algorithms 3 and 4 give a formal definition of the adaptor grammar generative process.

This definition of adaptor grammars gives us the following latent variables:

- $z_i$ : the full derivational trees that yielded the data.
- $z_{A,i}$ : the stored sub-trees headed by adapted non-terminals
- $\nu$ : the set of stick-weight proportions for the Pitman-Yor process
- $\theta$ : the set of PCFG rule probabilities

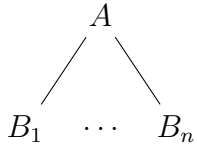
Our inference problem can be formalized as finding the posterior distribution on full derivational trees  $z_i$ —these depend on all the other latent variables, and reveal the inferred underlying linguistic structure. However, this inference is over an extremely large set of latent

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**Algorithm 3:** Building the grammar

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```
# constructing the PCFG
foreach nonterminal  $A$  in  $N$  do
  draw rule weights  $\theta_A \sim \text{Dir}(\alpha_A)$  ;
  # constructing the grammatons  $G_A$ 
  for  $A \in A_1, \dots, A_k$  do
    draw  $\pi_A \sim \text{PYP}(a_A, b_A)$ 
    # construct tree  $z_{A,i}$ 
    for  $i \in \{1, \dots\}$  do
      draw rule  $A \rightarrow B_1 \dots B_n$  from
         $R_A$ 
      set  $z_{A,i} =$ 



```

      graph TD
      A --- B1
      A --- Bn
      B1 --- dots[...]
      Bn --- dots
      
```

while  $z_{A,i}$  has nonterminals in
        leaves do
        choose a  $B$  from  $B_1 \dots B_n$ 
        if  $B$  is non-adapted
          nonterminal then
            expand  $B$  using the
              PCFG
          else
            expand  $B$  using  $G_B$ 
            # guaranteed to exist
            because of topological
            ordering
          for  $i \in \{1, \dots\}$  do
            set  $G_A(z_{A,i}) = \pi_{A,i}$ 
```

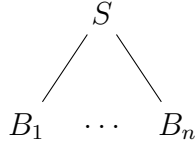
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**Algorithm 4:** Generating data

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```
# generating derivation trees  $z_i$ 
for  $i \in \{1, \dots\}$  do
  if  $S$  is adapted nonterminal then
    draw  $z_i \sim G_S$ 
  else
    draw  $S \rightarrow B_1 \dots B_n$  from  $R_S$ 
    set  $z_i =$ 



```

    graph TD
    S --- B1
    S --- Bn
    B1 --- dots[...]
    Bn --- dots
    
```

while  $z_i$  has nonterminals in leaves
      do
        choose a  $B$  from  $B_1 \dots B_n$ 
        if  $B$  is non-adapted nonterminal
          then
            expand  $B$  using the PCFG
          else
            expand  $B$  using  $G_B$ 
        set  $x_i$  to the yield of tree  $z_i$ 
```

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variables. In fact, in the current formalization, we cannot do inference over this set of latent variables, since some are countably infinite. To make this problem finite, we use a truncated stick-breaking representation, where after a sufficiently large  $i$  we let  $\nu_i = 1$ , so that the probability of continuing past that stick is 0. Beyond the large number of latent variables, we need to take into account the large number of potential parses for each sentence given the grammar. Indeed, averaging rule probabilities over all of these parses is the most costly portion of the algorithm.

### 3.2 Dirichlet process hidden Markov model

The goal of the DPHMM is to jointly learn the phonetic boundaries of the speech input, clusters of acoustically similar segments, and PLU identities. By using a Dirichlet process,



we do not bound the number of possible PLUs, but the reuse of existing PLUs is preferred (as in the adaptor grammar model). In the original model, defined by Lee and Glass (2012), a sampling approach was used. Extending this work, Ondel et al. (2016) implemented the model using variational Bayesian techniques. To describe the model, we first need to provide background on hidden Markov models and Gaussian mixture models.

### 3.2.1 Hidden Markov Models

A hidden Markov model (HMM) consists of a finite number of states combined with probability distribution over transitioning between states, dependent on the previous state. Observations are generated by such transitions, and the probability of emitting a certain observation is defined by a distribution which depends on the current state (the state transitioned to) (Rabiner and Juang, 1986). In the case of the DPHMM model, each PLU is modeled by its own three-state HMM, corresponding to the start, middle, and end of a phone. Each emission distribution is modeled by a Gaussian mixture model.

### 3.2.2 Gaussian Mixture Models

A multimodal continuous distribution can be modeled by a linear combination (mixture) of Gaussian distributions. Each Gaussian is known as a *component* with a mean  $\mu_k$  and a covariance  $\Sigma_k$ . In order to combine several Gaussians, we need *mixing coefficients*  $\pi_k$  such that  $\sum_{k=1}^K \pi_k = 1$ . Using these coefficients, we choose a Gaussian distribution and then sample a datapoint from its probability density function. The probability of datapoint  $x$  in a GMM is

$$p(x) = \sum_{k=1}^K \pi_k \mathcal{N}(x | \mu_k, \Sigma_k)$$

(Bishop, 2006)

From this, it is clear that the greater the mixing coefficient, the more often that component is chosen.

### 3.2.3 The DPHMM generative model

The DPHMM generative model first chooses a PLU label, or cluster, from the Dirichlet process. This cluster is associated with an 3-state HMM giving the start, beginning, and end of the phone. Sampling from the Gaussian mixture models at each state of the HMM produces a vector of MFCCs corresponding to the acoustic signal of the chosen PLU. Algorithm 5 gives a formal representation of this process:

This model has three sets of latent variables:

- $c_i$ : the cluster assignment of the  $i^{th}$  segment in the dataset
- $s_{ij}$ : the HMM state of the  $j^{th}$  frame for the  $i^{th}$  segment
- $m_{ij}$ : the GMM component of the  $j^{th}$  frame for the  $i^{th}$  segment

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**Algorithm 5:** Defining the DPHMM

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```
# Defining the mixture models
choose the GMM mixture weights  $\pi \sim \text{Dir}(\eta_0^{gmm})$ 
choose mean  $\mu$ , and covariance matrix  $\Sigma$  with diagonal  $\lambda$  by drawing from the
Normal-Gamma distribution parametrized by normal distribution hyper-parameters
 $\mu_0$  and  $(\kappa_0\lambda)^{-1}$ , and Gamma distribution hyper-parameters  $\alpha_0, \beta_0$ .
# Defining the HMM transition matrix
choose the rows of the transition matrix  $r_i \sim \text{Dir}(\eta_0^{hmm,i})$ 
# Sampling  $M$  possible PLUs
for  $i \in \{1, \dots, M\}$  do
  sample  $\nu_i \sim \text{Beta}(1, \gamma)$ 
  sample cluster label  $\theta_i \sim G_0$ , the Dirichlet process base distribution
# Sampling  $N$  datapoints
for  $i \in \{1, \dots, N\}$  do
  choose an cluster label  $\theta_i$  with probability  $\pi_i(\nu) = \nu_i \prod_{j=1}^{i-1} (1 - \nu_j)$ 
  # cluster labels correspond to HMMs
  sample a path  $S = s_1, \dots, s_n$  from the transition probability distribution
  for  $s_j \in S$  do
    choose a Gaussian component from mixture model
    sample datapoint from Gaussian density function
```

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### 3.3 The noisy channel

The final component to the joint model, the noisy channel, allows the DPHMM and adaptor grammar to interface by rewriting each other's outputs. For example, if there is a mistake in the acoustic input or in the DPHMM cluster assignment, the adaptor grammar could fix this by substituting in a more probable PLU. The full set of operations the noisy channel allows is: substitution, insertion, and deletion. These are the same exact operations as in the well-known *Levenshtein distance* algorithm (Levenshtein, 1966). This dynamic programming algorithm is typically used to find the minimum edit distance between two strings—that is, the minimum number of insertions, deletions, and substitutions required to rewrite one string as the other. In order to define a noisy channel, we leave the second string unspecified, and use the Levenshtein framework to enumerate all possible strings that the first string could be edited into. In order maintain the plausibility of our model, we limit the number of consecutive insertions and deletions, and strongly bias making no edit at all. This makes intuitive sense, as all communication would break down were a speaker to substitute every phoneme he intended to produce, insert an arbitrary number of extra ones, or delete all of them. In order to define this model, we need the following prior probabilities:

- operation probabilities  $o$ : this is a vector specifying the probability of doing an insertion, deletion, or substitution at all. It is drawn from a Dirichlet distribution.
- probability of inserting each phone  $I$ : given an alphabet of length  $k$ , we draw a vector

from a Dirichlet distribution which specifies the probability of inserting that phone into the produced string (assuming that insertion has already been picked)

- probability of substitution  $\zeta$ : since each phone can be substituted for each other phone, this is a  $k \times k$  matrix where each row sums to 1. Thus, we draw each row from a  $k$ -dimensional Dirichlet distribution.

These three random variables also comprise our set of latent variables in the noisy channel model.

## 4 Variational Updates

Having defined our model, we need to construct variational distributions which approximate the posterior for each latent variable. The full set of latent variables, listed with the hyperparameters of their prior distributions and the name of the variational parameter indexing its variational distribution  $q$  (if applicable) is given in Table 1. Note that the variational implementation of the DPHMM follows a slightly different paradigm, so we refer the reader to Ondel et al. (2016) for a full description of the variational parameter updates and derivations.

### 4.1 Adaptor grammar updates

The updates for the adaptor grammar are given as in Cohen et al. (2010):

$$\begin{aligned}
\gamma_{A,i}^1 &= 1 - b_A + \sum_{B \in M} \sum_{k=1}^{N_B} \tilde{f}(A \xrightarrow{*} s_{A,k}, s_{B,k}) \\
\gamma_{A,i}^2 &= a_A + ib_A + \sum_{j=1}^{i-1} \sum_{B \in M} \sum_{k=1}^{N_B} \tilde{f}(A \xrightarrow{*} s_{A,j}, s_{B,k}) \\
\tau_{A,A \rightarrow \beta} &= \sum_{B \in M} \sum_{k=1}^{N_B} \tilde{f}(A \rightarrow \beta, s_{B,k}) \\
\phi_{A,A \xrightarrow{*} s_{A,k}} &= \Phi(\gamma_{A,i}^1) - \Phi(\gamma_{A,i}^1 + \gamma_{A,i}^2) + \sum_{j=1}^{i-1} (\Phi(\gamma_{A,i}^1) - \Phi(\gamma_{A,i}^1 + \gamma_{A,i}^2)) \\
\phi_{A,A \rightarrow \beta} &= \Phi(\tau_{A,A \rightarrow \beta}) - \Phi(\sum_{\beta} \tau_{A,A \rightarrow \beta})
\end{aligned}$$

where  $\tilde{f}(r, s_{B,k})$  is the expected count of rule  $r$  in the derivation trees of string  $s_{B,k}$  which is headed by nonterminal  $B$  and spans  $k$  units, and  $A \xrightarrow{*} s_{A,k}$  indicates that non-terminal  $A$  expands to the string spanning  $i$  and corresponding to the yield of the grammaton headed by  $A$ . In Cohen et al. (2010) the value  $\tilde{f}(r, s_{B,k})$  is computed using the inside-outside algorithm and a preprocessing step to determine  $s_{B,k}$ . However, in the implementation proposed in

Latent variable	Description	variational parameter	hyperparameter
<b>Adaptor grammar</b>			
$z_i$	the full derivational trees that yielded the data	$\phi$	
$z_{A,i}$	the stored sub-trees headed by adapted non-terminals	$\phi_A$	
$\nu$	the set of stick-weight proportions for the Pitman-Yor process	$\gamma^1, \gamma^2$	$a, b$
$\theta$	the set of PCFG rule probabilities	$\tau$	$\alpha$
<b>DPHMM</b>			
$c_i$	the cluster assignment of the $i^{th}$ segment in the dataset		$\gamma$
$s_{ij}$	the HMM state of the $j^{th}$ frame for the $i^{th}$ segment		$\eta_0^{hmm}, G_0$
$m_{ij}$	the GMM component of the $j^{th}$ frame for the $i^{th}$ segment		$\mu_0, (\kappa_0 \lambda)^{-1}, \eta_0^{gmm}$
<b>Noisy channel</b>			
$o$	the operation probabilities	$\xi^{ops}$	$\varepsilon^{ops}$
$I$	the insertion probabilities	$\varphi^{ins}$	$\varsigma^{ins}$
$\zeta$	the substitution probabilities	$\sigma$	$\rho$

Table 1: latent variables with their respective variational parameters and hyperparameters

Zhai et al. (2014), this preprocessing step is avoided by sampling an approximating PCFG. In fact, the Zhai et al. (2014) model uses sampling to approximate both the tree fragments  $z_{A,i}$  and the full tree derivations  $z_i$ . Counter-intuitively, this speeds up the model, despite the sampling approach being slower in the general case. This speed increase emerges from the fact that the expectation and maximization steps of the CAVI algorithm can be equivalently defined in terms of local and global latent variables. Local variables, such as the stick-weight proportions and rule weights, must be computed for each data point. Global variables, like the set of derivation trees  $z_i$ , need to take all of these variables into account. The expectation step involves optimizing the global variables, while the maximization step optimizes the local variables. This second optimization can be easily distributed across multiple cores. However, these optimized local variables need to be collected again in order to recalculate the global variables in the expectation step, meaning that it of the algorithm is not easily parallelizable. Furthermore, in the original variational model, the run time was dominated by the inside-outside algorithm for calculating expected values of rule counts, which has a time-complexity of  $O(|N|^2 |x_i|^3 + |N|^3 |x_i|^2)$  where  $|x_i|$  is the length of the  $i^{th}$  input sequence (Cohen et al., 2010). By using sampling, Zhai et al. (2014) avoid some of the cost involved in this computation. We incorporate this faster implementation into the ULD framework. For an example derivation of a variational update see Appendix C.

## 4.2 Noisy channel updates

Let  $S$  be the set of all input strings to the noisy channel, let  $PLU(i)$  indicate the PLU with index  $i$ , and let  $O(i)$  indicate the operation indexed by  $i$ . Let  $\tilde{g}(op[p], s_n)$  be the expected number of times an operation  $op$  (which can be insertion or substitution) is applied to PLU parameter(s)  $p$  in the string  $s_n$ . Note the overloaded call to  $op[p]$  in the case of substitution, where it takes two parameters. The expected count  $\tilde{g}$  can be computed using a Forward-Backward style algorithm which sums over all entries in the expanded Levenshtein chart. With this value, we can derive the updates for the noisy channel’s variational distributions, using (29). They are:

$$\begin{aligned}\xi_i^{ops} &= \varepsilon_i^{ops} + \sum_{s_n \in S} \sum_{l=1}^k \tilde{g}\left((O(i)[PLU(l)]), s_n\right) \\ \phi_i^{ins} &= \varsigma_i^{ins} + \sum_{s_n \in S} \tilde{g}\left(ins[PLU(i)], s_n\right) \\ \sigma_{i,j} &= \rho_{i,j} + \sum_{s_n \in S} \tilde{g}\left(sub[PLU(i), PLU(j)], s_n\right)\end{aligned}$$

## 5 Summary of previous results

Lee et al. (2015) ran several variants of the ULD model on a set of lecture recordings from the MIT lecture corpus. These were: a full model where the number of distinct PLU types was inferred from the data, a truncated model where the PLU inventory size was upper-bounded by 50, a lesioned version where the acoustic model (the DPHMM component) was removed after discovering the initial PLU labels and boundaries, meaning that the joint model could

Lecture topic	Full50	HHMM	FullDP	DPHMM
Economics	74.4	74.6	74.6	75.0
Signal processing	76.2	76.0	76.0	76.3
Clustering	76.6	76.6	77.0	76.9
Speaker adaptation	76.5	76.9	76.7	76.9
Physics	75.9	74.9	75.7	75.8
Linear algebra	75.5	73.8	75.5	75.7

Table 2: F1 scores for phone segmentation for each system and their respective initialization systems (Lee et al., 2015)

no longer relabel or re-segment PLUs, and finally a version where the noisy channel and acoustic model were removed, splitting the joint model into two separate ones.

## 5.1 Phone segmentation results

The phone segmentation produced by the joint model was evaluated against forced alignments of each lecture, with a  $20ms$  tolerance margin (i.e. anything within  $20ms$  of the force-aligned gold standard would be considered correct). Note that forced alignment—aligning a transcription of an audio recording with the actual audio by determining word and phone boundaries—does not produce perfect alignment itself, so the gold standard against which Lee et al. (2015) evaluated their results most likely contained errors itself. Nevertheless, the F1 values reported by Lee et al. (2015) for phone segmentation, which can be seen in Table 2, show similar values for both the inferred PLU inventory system (FullDP) and the limited PLU inventory system (Full50), as well as the DPHMM system used to initialize the phone boundaries in the FullDP system, and the hierarchical hidden Markov model (HHMM) used to initialize the Full50 system.

## 5.2 Word segmentation

As Lee et al. (2015) mention, due to the lack of a gold standard alignment of the audio used in the experiments, defining and measuring word segmentation is itself an unclear task. Nevertheless, Table 3 has the F1 scores for the word segmentation task for both the truncated and the full PLU inventory systems run by Lee et al. (2015). These results show that the noisy channel was important for word segmentation—intuitively, this makes sense, as words of the same type but with different surface realizations cannot be labeled as the same if the noisy channel is not allowed to make edits to accommodate the variation. The 1.6% average improvement between the full system and the -AM lesioned version suggests that the joint learning nature of the model has a small positive effect on word segmentation.

Lee et al. (2015) also evaluated the number of top 20 term frequency-inverse document frequency words (a commonly-used measure of word importance in a set of documents) that the various systems identified. These values are reported in comparison with the number of terms identified by a baseline system (Park and Glass, 2008) and a state-of-the-art system (Zhang et al., 2013), the latter of which uses a much richer representation for audio data

Lecture topic	Full50	-AM	-NC	FullDP	-AM	-NC
Economics	15.4	15.4	14.5	16.1	14.9	13.8
Signal processing	17.5	16.4	12.1	18.3	17.0	14.5
Clustering	16.7	18.1	15.9	18.4	16.9	15.2
Speaker adaptation	17.3	17.4	15.4	18.7	17.6	16.2
Physics	17.7	17.9	15.6	20.0	18.0	15.2
Linear algebra	17.9	17.5	15.4	20.0	17.0	15.6

Table 3: F1 scores for word segmentation by each system and its lesioned versions(Lee et al., 2015)

Lecture topic	Full50	-AM	-NC	FullDP	-AM	-NC	Park&Glass	Zhang
Economics	12	4	2	12	9	6	11	14
Signal processing	16	16	5	20	19	14	15	19
Clustering	18	17	9	17	18	13	16	17
Speaker adaptation	14	14	8	19	17	13	13	19
Physics	20	14	12	20	18	16	17	18
Linear algebra	18	16	11	19	17	7	17	16

Table 4: Number of top 20 term frequency-inverse document frequency words discovered by each system (Lee et al., 2015)

than the MFCCs used in ULD. As can be seen in Table 4, both ULD systems frequently outperformed both the baseline and the state-of-the-art system, despite using a sparser data format to represent the audio than Zhang et al. (2013).

### 5.3 Qualitative results

In addition to these quantitative values, Lee et al. (2015) report several more qualitative results. For example, the ULD system discovered words such as *globalization* and *collaboration* which occurred frequently in the lectures; for both of these words, the system also discovered the productive *-ation* suffix. Because the purpose of adaptor grammars is to compactly store parse trees, certain frequently occurring morphemes like *-able* and *-ation* are saved. Simultaneously, certain sequences of words, like *the Arab Muslim word*, are identified as lexical items if they are common in the data. This calls into question the usefulness of words in evaluating an unsupervised system like ULD. There are sequences of words (such as some idioms) that almost always occur in that order, especially in a given context. Such sequences of words might reasonably be considered one lexical unit by a language learner presented with only an acoustic input—the grouping of two lexical items into one can be seen in the common malapropism *for all intensive purposes*. It is not impossible then that either through a misunderstanding, or due to the relative frequency of a phrase, we treat a sequence of words as one stored unit. Since our own storage and production process for lexical items is unclear, and in the case of some idioms and multi-word units, independent of orthographic word boundaries, there is no definitive way of knowing how closely the discovered lexicon corresponds to our internal lexica.

Such considerations tie closely into the overall linguistic question of balancing productivity and reuse; namely, how much of our language do we compute on the fly (productively) and how much do we store and reuse statically. Both productivity and reuse have their costs and benefits: computing everything is inefficient, especially for high-frequency terms, but it lets us avoid storing anything; storing everything, on the other hand, makes it very efficient to produce sentences and terms that have already been used, but precludes the creation of novel sentences or terms, and entails storing sentences which are never reused. The optimal solution is to reuse those linguistic units which occur often, and compute those larger ones which are rare or unique. By modeling this balancing act mathematically with the Pitman-Yor and Dirichlet processes, ULD offers a rare glimpse at the internal mechanism of a productivity-reuse system. As the storage of super-word units in the results of Lee et al. (2015) shows, the optimal balance may not dovetail perfectly with our conception of the units in question.

## 6 Variational improvements

Given the faster convergence rate and multiprocessing capabilities of our variational ULD framework, more experiments can be run in a shorter time-frame, and the system scales to large audio corpora. The following data shows the improvements that variational systems made over sampling approaches for both the DPHMM and adaptor grammar components of the ULD model.

### 6.1 DPHMM improvement

Ondel et al. (2016) found that the variational was both faster and more accurate than the same model using Gibbs sampling. While training the latter took approximately 11 hours on one core, it took less than 30 minutes to train the variational DPHMM on 300 cores. Additionally, the variational model had a better mutual information score between discovered phones and previously labeled phones.

### 6.2 Adaptor grammar improvement

Cohen (2011) replicated the word-segmentation experiments run by Johnson (2008), and found that the variational system converged in fewer iterations (full passes through the dataset). While the sampling approach took 2000 iterations to converge, the variational system only needed 40. In addition, the variational system was faster when run on multiple cores. Inference by sampling took 2 hours and 14 minutes. The variational adaptor grammar needed 2 hours and 34 minutes when run on a single core—however, once distributed to 20 cores, it finished in 47 minutes.

## 7 Future Work

With the variational implementation of ULD, we plan on running experiments which test lesioning different parts of the model; in Lee et al. (2015), the acoustic model was removed, and



then the noisy channel was removed from that. We are particularly interested in removing the noisy channel but keeping the acoustic model in place. Recall that in the variational setting, we introduce a new family of variational distributions, indexed by variational parameters which can be initialized randomly, but can also be given deliberately chosen values. Thus the introduction of variational methods creates an opportunity for new experiments which test the optimal initialization parameters for the variational distributions, and what those indicate about the phenomena being modeled. Additionally, the empirical Bayesian framework implemented in both the Zhai et al. (2014) and Ondel et al. (2016) models optimizes not only the variational parameters, but also finds the best value for the hyperparameters of the model, which can play an important role in future models. Thanks to the new multi-processing capabilities of our updated ULD model, we will be able to run larger experiments by distributing the computation to a cluster. For example, we will be able to test the full system on large speech databases with gold standard alignments (e.g. the TIMIT corpus (Consortium et al., 1993)) , apply the learning algorithm to a variety of languages.

Given its the language-independent nature, ULD provides a potential framework for generating linguistic and automatic speech recognition (ASR) resources such as pronunciation dictionaries, particularly for under-resourced languages. Pronunciation dictionaries, which map words to their phonetic transcriptions, are required for forced alignment as well as most ASR applications (Besacier et al., 2014). With improved accuracy, ULD might in aid in lowering the production cost associated with generating such dictionaries, while simultaneously allowing for the creation of accent-specific dictionaries. These resources this would be instrumental in both conducting additional linguistic research on these languages and developing ASR applications for them.

The utility of ULD’s complete learning framework is not limited to research and industrial development. In the developing world—where many of under-resourced languages can be found—literacy and computer literacy are major issues facing millions. While ASR applications have been credited with improving literacy (Adams, 2005) and increasing computer accessibility, Plauche et al. (2006) point to the prohibitive cost of producing the requisite resources as the main obstacle to developing these technologies. An unsupervised system such as ULD could break this barrier by increasing the speed at which production can take place and lowering the cost.

## 8 Conclusion

We have presented a language-independent variational Bayesian inference model for the fully unsupervised induction of a complete hierarchy of linguistic units directly of an acoustic input, based on the sampling-based approach to the same problem by Lee et al. (2015). Our variational model promises significant decreases in amount of time required to train the model by virtue of the ease with which it can be distributed to multiple cores. For the acoustic model and adaptor grammar, we discussed experimental results and speed improvements made by their existing variational Bayesian implementations (Ondel et al., 2016; Cohen et al., 2010; Zhai et al., 2014). Lastly, we discussed future experiments that our variational framework will enable us to conduct, as well as several real-world applications of our model.

# A Deriving the ELBO

## A.1 The problem

Given our generative model and our data, we would like to find a posterior distribution:  $P(Z | X)$ . Using Bayes Rule, we get:  $P(Z | X) = \frac{P(X|Z)P(Z)}{P(X)} = \frac{P(X|Z)P(Z)}{\sum_{\forall Z} P(X|Z)P(Z)}$  We call the numerator of the fraction on the right the generative model. It is composed of the product of the likelihood *of the hypothesis* ( $P(X | Z)$ ) and the prior probability of the hypothesis ( $P(Z)$ ). Note that the former is not a probability but a measure of how well our hypothesis fits the data. The denominator is the “marginal likelihood” of the data,  $P(X)$ . To find this, we need to marginalize out (sum over) all possible hypotheses. Because the hypotheses are the range of values for all of the latent variables in our model, this summation is computationally intractable. In order to obtain a posterior, we are forced to use some approximate means of finding this denominator. Often, a sampling approach is used. However, sampling can be very slow to converge and is not easily parallelizable across multiple cores. The variational Bayesian approach, on the other hand, treats the problem of finding an appropriate marginal distribution as an optimization problem.

- Let  $Z$  be our set of hidden variable collections:
- Let  $\Phi$  be the collection of all model parameters (Pitman-Yor parameters  $a, b$  and Dirichlet distribution parameter  $\alpha$ ).
- Let  $X$  be the set of observations. In the case of word segmentation, for example, these would be each string of unsegmented phonemes.
- Note that our goal is to find  $P(Z | X)$ , the posterior (where  $Z$  is the set of latent variables)
- recall  $P(Z | X) = \frac{P(X|Z,\Phi)P(Z|\Phi)}{\sum_{\forall Z} P(X|Z,\Phi)P(Z|\Phi)}$

## A.2 Important formulae

### A.2.1 Jensen’s inequality

Jensen’s inequality states that for a convex function  $f$  and random variable  $X$ :

$$f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)] \quad (6)$$

We are using the logarithm of the probability, so the function is actually concave. Jensen’s inequality works both ways, meaning we switch the direction of the inequality:

$$\log(\mathbb{E}[X]) \geq \mathbb{E}[\log(X)] \quad (7)$$

### A.2.2 Expected value

Note that for discrete random variables (which we are dealing with in this case)

$$\mathbb{E}_q(f(x)) = \sum_{\forall x} q(x)f(x) \quad (8)$$

### A.2.3 Logarithms

Throughout this derivation (and the variational literature as a whole) the logarithm of the probability is used. There are various reasons to do this. Firstly, taking the logarithm allows us to use information-theoretic measures such as entropy. Furthermore, logarithms allow us to transform expensive multiplication and division into cheaper addition and subtraction, and helps when working with probabilities below the floating-point precision bound. Recall some facts about logarithms:

- $\lim_{n \rightarrow 0} \log n = -\infty$
- $\log AB = \log A + \log B$
- $\log \frac{A}{B} = \log A - \log B$

## A.3 Derivation of variational bound

The value we are looking to approximate is our denominator, which is the likelihood of the data integrated over all hypotheses. Recall that our inference problem lies in finding the denominator to the Bayesian equation  $P(Z | X) = \frac{P(X|Z)P(Z)}{\int_{\forall Z} P(X|Z)P(Z)dZ}$ . Our hypotheses in this case are possible values for the latent variables in the model. This integral (or in the discrete case, summation) is often computationally intractable, so we use a variational approximation for it. One way we can do this is by using the Kullback Leibler (KL) divergence between this intractable integral and some variational distribution  $q$ .

1. Let  $q_\nu(Z)$  be a family of variational distributions with variational parameter  $\nu$ .
2. to get the marginal likelihood ( $\log p(X | \Phi)$ ) we take the KL divergence between  $q_\nu(Z)$  and  $p(Z | X, \Phi)$ .
3. KL divergence is given by:

$$\begin{aligned}
 D_{KL}(q_\nu(Z) || p(Z | X, \Phi)) &= \mathbb{E}_q[\log \frac{q_\nu(Z)}{p(Z | X, \Phi)}] \\
 &= \mathbb{E}_q[\log q_\nu(Z) - \log p(Z | X, \Phi)] \\
 &= \mathbb{E}_q[\log q_\nu(Z) - \log \frac{p(Z, X | \Phi)}{p(X | \Phi)}] \\
 &= \mathbb{E}_q[\log q_\nu(Z) - (\log p(Z, X | \Phi) - \log p(X | \Phi))] \\
 &= \mathbb{E}_q[\log q_\nu(Z)] - \mathbb{E}_q[\log p(Z, X | \Phi)] + \log p(X | \Phi)
 \end{aligned} \tag{9}$$

(Blei and Jordan, 2006)

If we think about what KL divergence represents, we can intuitively understand why it cannot be negative. From here, we can see how minimizing this equation is the same as maximizing the lower bound on  $\log p(X | \Phi)$ :

$$\begin{aligned}
0 &\leq \mathbb{E}_q[\log q_\nu(Z)] - \mathbb{E}_q[\log p(Z, X | \Phi)] + \log p(X | \Phi) \\
&\quad - \log p(X | \Phi) \leq \mathbb{E}_q[\log q_\nu(Z)] - \mathbb{E}_q[\log p(Z, X | \Phi)] \\
&\quad \log p(X | \Phi) \geq \mathbb{E}_q[\log p(Z, X | \Phi)] - \mathbb{E}_q[\log q_\nu(Z)]
\end{aligned} \tag{10}$$

Another way to reach this same equation is by using Jensen's inequality. Consider the log marginal likelihood:

$$\log p(X | \Phi) = \log \sum_{z \in \mathbf{Z}} p(X, z | \Phi) \tag{11}$$

The sum marginalizes out the hidden variables  $z$  in the joint probability distribution. Picking any variational distribution  $q(z)$  we can multiply them by  $\frac{q(z)}{q(z)}$ :

$$\log \sum_{\forall z \in \mathbf{Z}} (p(x, z | \Phi) * \frac{q(z)}{q(z)}) = \log \sum_{\forall z \in \mathbf{Z}} q(z) \frac{p(x, z | \Phi)}{q(z)} \tag{12}$$

Jensen's inequality implies

$$\log \sum_{\forall z \in \mathbf{Z}} q(z) \frac{p(x, z | \Phi)}{q(z)} \geq \sum_{\forall z \in \mathbf{Z}} q(z) \log \frac{p(x, z | \Phi)}{q(z)} \tag{13}$$

This equation can be broken into:

$$\begin{aligned}
\sum_{\forall z \in \mathbf{Z}} q(z) \log \frac{p(x, z | \Phi)}{q(z)} &= \sum_{\forall z \in \mathbf{Z}} q(z) (\log p(x, z | \Phi) - \log q(z)) = \\
&\quad \sum_{\forall z \in \mathbf{Z}} q(z) \log p(x, z | \Phi) - \sum_{\forall z \in \mathbf{Z}} q(z) \log q(z) = \\
&\quad \sum_{\forall z \in \mathbf{Z}} q(z) \log p(x, z | \Phi) + \mathcal{H}(q)
\end{aligned} \tag{14}$$

$$\tag{15}$$

where

$$\mathcal{H}(q) = - \sum_{\forall z \in \mathbf{Z}} q(z) \log q(z) \tag{16}$$

(Blei et al., 2017) This first term is of the form of our expected value definition, so our equation becomes:

$$\log p(x | \Phi) \geq \mathbb{E}_q[\log p(x, z | \Phi)] + \mathcal{H}(q) \tag{17}$$

This derivation yields an important fact:

$$\log p(X | \Phi) - KL(q(Z) || p(Z | X, \Phi)) = \mathbb{E}_q[\log p(z, x | \Phi)] + H(q) \quad (18)$$

From this equation, we can see why minimizing KL divergence gives us the best possible value for our marginal likelihood.

## B Deriving Variational Updates

### B.1 Mean Field Approximation

Recall our mean-field assumption to treat each variational distribution as conditionally independent:  $q(Z) = \prod_i q_i(z_i)$ ; also recall that our bound on the log marginal likelihood was:

$$\mathcal{L}(q) \geq \sum_{z_i \in Z} q(Z) \log p(X, Z | \Phi) + H(q)$$

For the sake of concise notation, let  $q_j = q_{\nu_j}(Z_j)$  and disregard the conditioner  $\Phi$  for now. Replace  $q(Z)$  with this approximating product:

$$\begin{aligned} \mathcal{L}(q) &\geq \sum_{z_i \in Z} \left( \prod_i q_i(z_i) \right) \log p(X, Z | \Phi) + H(q) \\ \mathcal{L}(q) &\geq \mathbb{E}_{\prod_i q_i(z_i)} \log p(X, Z | \Phi) [\log p(X, Z | \Phi)] + H(q) \end{aligned} \quad (19)$$

Using the chain rule and by expanding the entropy term, we can rewrite this expression as

$$\log p(X | \Phi) + \sum_{i=1}^{|Z|} \mathbb{E}_q[\log p(z_i | X, z_1, \dots, z_{i-1}, \Phi)] - \sum_{i=1}^{|Z|} \mathbb{E}_q[\log q_{\nu_i}(z_i)] \quad (20)$$

Since  $p(X | \Phi)$  does not depend on the variational parameter  $\nu_i$  it factors out as a constant (recall that this is a lower bound, not an exact equality). We can reorder the elements of  $Z$  in any way we wish. If we reorder them each time so that  $z_i$  comes last, we can say:

$$\mathcal{L}_i = \mathbb{E}_q[\log p(z_i | Z_{-i}, X, \Phi)] - \mathbb{E}_q[\log q_{\nu_i}(z_i)] \quad (21)$$

(Blei and Jordan, 2006)

Note that for any exponential family distribution  $q_{\nu_i}$ ,

$$q_{\nu_i}(z_i) = h(z_i) \exp \{ \nu_i^T z_i - a(\nu_i) \} \quad (22)$$

where  $a(\nu_i)$  is the cumulant function, which for the first three derivatives is equivalent to the corresponding derivatives of the moment generating function. We can rewrite our equation using this form for  $q_{\nu_i}(z_i)$ :

$$\begin{aligned} \mathcal{L}_i &= \mathbb{E}_q[\log p(z_i|Z_{-i}, X, \Phi)] - \mathbb{E}_q \left[ \log \left( h(z_i) \exp \{ \nu_i^T z_i - a(\nu_i) \} \right) \right] \\ &= \mathbb{E}_q[\log p(z_i|Z_{-i}, X, \Phi)] - \mathbb{E}_q \left[ \log (h(z_i)) + \nu_i^T z_i - a(\nu_i) \right] \\ &= \mathbb{E}_q[\log p(z_i|Z_{-i}, X, \Phi)] - \mathbb{E}_q[\log (h(z_i))] - \mathbb{E}_q[\nu_i^T z_i] + \mathbb{E}_q[a(\nu_i)] \\ &= \mathbb{E}_q[\log p(z_i|Z_{-i}, X, \Phi)] - \mathbb{E}_q[\log (h(z_i))] - \nu_i^T a'(\nu_i) + a(\nu_i) \end{aligned} \quad (23)$$

Note that  $\mathbb{E}_q[\nu_i^T z_i] = \nu_i^T a'(\nu_i)$  since  $E_q(z_i) = a'(\nu_i)$  and  $\nu_i^T$  factors out as a constant when taking the expectation with respect to  $q$ . The main premise of variational inference is to cast the intractable calculation of the posterior as an optimization problem. In most optimization problems, there are two general steps:

1. computing an objective function which allows us to
2. optimize the function by adjusting the parameters.

Recall that to avoid expensive computations, we employ exponential family distributions which allow us to simplify the problem. Our goal is to optimize the function by adjusting the variational parameters, so we take the partial derivative of our function with respect to  $\nu_i$ :

$$\begin{aligned} \frac{\delta}{\delta \nu_i} \mathcal{L}_i &= \frac{\delta}{\delta \nu_i} (\mathbb{E}_q[\log p(z_i|Z_{-i}, X, \Phi)] - \mathbb{E}_q[\log (h(z_i))] - \nu_i^T a'(\nu_i) + a(\nu_i)) \\ &= \frac{\delta}{\delta \nu_i} (\mathbb{E}_q[\log p(z_i|Z_{-i}, X, \Phi)] - \mathbb{E}_q[\log h(z_i)]) - (\nu_i^T a''(\nu_i) + a''(\nu_i)) + a''(\nu_i) \\ &= \frac{\delta}{\delta \nu_i} (\mathbb{E}_q[\log p(z_i|Z_{-i}, X, \Phi)] - \mathbb{E}_q[\log h(z_i)]) - \nu_i^T a''(\nu_i) \end{aligned} \quad (24)$$

Setting this to 0 we get:

$$\begin{aligned} 0 &= \frac{\delta}{\delta \nu_i} (\mathbb{E}_q[\log p(z_i|Z_{-i}, X, \Phi)] - \mathbb{E}_q[\log h(z_i)]) - \nu_i^T a''(\nu_i) \\ \nu_i^T a''(\nu_i) &= \frac{\delta}{\delta \nu_i} (\mathbb{E}_q[\log p(z_i|Z_{-i}, X, \Phi)] - \mathbb{E}_q[\log h(z_i)]) \\ \nu_i &= \left( \frac{\delta}{\delta \nu_i} \mathbb{E}_q[\log p(z_i|Z_{-i}, X, \Phi)] - \frac{\delta}{\delta \nu_i} \mathbb{E}_q[\log h(z_i)] \right) (a''(\nu_i))^{-1} \end{aligned} \quad (25)$$

If  $p(z_i|Z_{-i}, X, \Phi)$  is also a member of the exponential family, it can be rewritten:

$$p(z_i|Z_{-i}, X, \Phi) = h(z_i) \exp \{ g_i(Z_{-i}, X, \Phi)^T z_i - a(g_i(Z_{-i}, X, \Phi)) \} \quad (26)$$

where  $g_i(Z_{-i}, X, \Phi)$  is the natural parameter of distribution  $p$ . Replacing  $p(z_i|Z_{-i}, X, \Phi)$  (first in the expected values for the sake of readability) and taking the derivative gives us

$$\mathbb{E}_q[\log p(z_i|Z_{-i}, X, \Phi)] = \mathbb{E}_q[\log h(z_i)] + \mathbb{E}_q[g_i(Z_{-i}, X, \Phi)]^T a'(\nu_i) - \mathbb{E}_q[a(g_i(Z_{-i}, X, \Phi))] \quad (27)$$

$$\begin{aligned} \frac{\delta}{\delta \nu_i} \mathbb{E}_q[p(z_i|Z_{-i}, X, \Phi)] &= \frac{\delta}{\delta \nu_i} \mathbb{E}_q[\log h(z_i)] \\ &+ \left( \frac{\delta}{\delta \nu_i} (\mathbb{E}_q[g_i(Z_{-i}, X, \Phi)]^T) a'(\nu_i) + \mathbb{E}_q[g_i(Z_{-i}, X, \Phi)]^T a''(\nu_i) \right) - \frac{\delta}{\delta \nu_i} \mathbb{E}_q[a(g_i(Z_{-i}, X, \Phi))] \\ &= \frac{\delta}{\delta \nu_i} \mathbb{E}_q[\log h(z_i)] + \mathbb{E}_q[g_i(Z_{-i}, X, \Phi)]^T a''(\nu_i) \end{aligned} \quad (28)$$

Notice that many of the expectations drop out. Substituting this for  $\frac{\delta}{\delta \nu_i} (\mathbb{E}_q[\log p(z_i|Z_{-i}, X, \Phi)])$  in our first differentiation, we get:

$$\begin{aligned} \nu_i &= \left( \frac{\delta}{\delta \nu_i} \mathbb{E}_q[\log h(z_i)] + \mathbb{E}_q[g_i(Z_{-i}, X, \Phi)]^T a''(\nu_i) - \frac{\delta}{\delta \nu_i} \mathbb{E}_q[\log h(z_i)] \right) (a''(\nu_i))^{-1} \\ &= (\mathbb{E}_q[g_i(Z_{-i}, X, \Phi)]^T a''(\nu_i)) (a''(\nu_i))^{-1} \\ &= \mathbb{E}_q[g_i(Z_{-i}, X, \Phi)] \end{aligned} \quad (29)$$

So the optimal value (when the derivative is 0) of  $\nu_i$  is  $\nu_i = \mathbb{E}_q[g_i(Z_{-i}, X, \Phi)]$ .

## B.2 Conjugacy

Now we need to obtain a closed-form expression for  $\mathbb{E}_q[g_i(Z_{-i}, X, \Phi)]$ . Firstly, let  $\Phi$  be composed of 2 parts  $\phi_1$  and  $\phi_2$ , where  $\phi_1$  is the number of observations contributed by the prior, and  $\phi_2$  corresponds to the total effect of the observations on the sufficient statistic. Because of the factorization and exponential family assumptions we made earlier, we can say:

$$\begin{aligned} P_\pi(z_i|\phi_1, \phi_2) &= f(\phi_1, \phi_2) \exp \{ \eta^T \phi_1 - \phi_2 a(\eta) \} \\ &= f(\phi_1, \phi_2) g(\eta)^{\phi_2} \exp \{ \eta^T \phi_1 \} \\ &\propto g(\eta)^{\phi_2} \exp \{ \eta^T \phi_1 \} \end{aligned} \quad (30)$$

where  $\eta$  are the natural parameters for the distribution,  $a(\eta)$  is the cumulant function, and  $f(\phi_1, \phi_2)$  is a normalizing function.

Assuming the posterior over data and local hidden variables  $P(X, Z_{-i} | z_i)$  is also in the exponential family and factorizes, we can say that for one data point  $x_n$

$$\begin{aligned}
P(x_n, z_n \mid z_i) &= h(x_n, z_n)g(z_i) \exp \{z_i^T t(x_n, z_n)\} \\
\Rightarrow P(X, Z_{-i} \mid z_i) &= \prod_{z_n \in Z_{-i}} h(x_n, z_n)g(z_i)^N \exp \{z_i^T t(x_n, z_n)\}
\end{aligned} \tag{31}$$

where  $t(x_n, z_n)$  is the sufficient statistic, which in most cases is simply the count of occurrences of  $x_n$  or  $z_n$ . By Bayes rule, the distribution over the selected hidden variable rewrites as

$$\begin{aligned}
P(z_i \mid X, Z_{-i}, \Phi) &\propto P(X, Z_{-i} \mid z_i)P(z_i \mid \Phi) \\
&= \prod_{n=0}^N h(x_n, z_n)g(z_i) \exp \{z_i^T t(x_n, z_n)\} g(\eta)^{\phi_2} \exp \{\eta^T \phi_1\} \\
&\propto g(z_i)^N \exp \{z_i^T t(x_n, z_n)\} g(\eta)^{\phi_2} \exp \{\eta^T \phi_1\} \\
&\propto g(z_i)^{N+\phi_2} \exp \left\{ z_i^T \left( \phi_1 + \sum_{z_n \in Z_{-i}} t(x_n, z_n) \right) \right\}
\end{aligned} \tag{32}$$

Because all exponential family distributions have conjugate priors, this result implies that the posterior  $P(z_i \mid X, Z_{-i}, \Phi)$  is the same type of distribution as the prior with parameters:

$$P(z_i \mid X, Z_{-i}, \Phi) = P_\pi(z_i \mid \phi_1 + \sum_{z_n \in Z_{-i}} t(x_n, z_n), \phi_2 + N) \tag{33}$$

This means that the natural parameters of the posterior distribution on global hidden variables has the natural parameters  $\phi_2 + \sum_{z_n \in Z_{-i}} t(x_n, z_n)$  and  $\phi_2 + N$ , giving us a closed form for our expectation in (29):

$$\mathbb{E}_q[g_i(Z_{-i}, X, \Phi)] = \mathbb{E}_q \left[ \frac{\phi_1 + \sum_{z_n \in Z_{-i}} t(x_n, z_n)}{\phi_2 + N} \right] \tag{34}$$

(Hoffman et al., 2013)

### B.3 Alternative Form

We can derive an alternative form for an optimal setting of  $q$  without the exponential family requirements by following the method described in Bishop (2006). Recall that

$$\mathcal{L}(q) = \sum_{z_i \in Z} \left( \left( \prod_i q_i(z_i) \right) \log p(X, Z \mid \Phi) + \sum_i q_i(z_i) \right)$$



This can be rewritten as

$$\begin{aligned}\mathcal{L}(q) &= \sum_{\forall z_j} q_j(z_j) \left( \sum_{z_i \neq z_j} \log p(X, Z) \prod_{i \neq j} q_i(z_i) \right) - \sum_{\forall z_j} q_j(z_j) \log q_j + \text{const} \\ &= \sum_{\forall z_j} q_j(z_j) \log \tilde{p}(X, z_j) - \sum_{\forall z_j} q_j(z_j) \log q_j(z_j) + \text{const}\end{aligned}\quad (35)$$

where  $\log \tilde{p}(X, z_j) = \mathbb{E}_{i \neq j}[\log p(X, Z)] + \text{const}$  and  $\mathbb{E}_{i \neq j}$  is the expectation taken with respect to all distributions  $q$  except  $q_i$ . Maximizing (35) is equivalent to minimizing the KL divergence, with the minimum occurring when  $q_j(z_j) = \tilde{p}(X, z_j)$ . Thus the optimal distribution  $q_j^*(z_j)$  can be written:

$$\log q_j^*(z_j) = \mathbb{E}_{i \neq j}[\log p(X, Z)] + \text{const} \quad (36)$$

(Bishop, 2006)

## C Derivation of Updates

As an example of how variational updates can be found, we show the explicit derivation of the Beta distribution updates for the adaptor grammar portion of the ULD model.

Recall that in the generative process, each stick-weight proportion  $\nu_i$  is drawn from a Beta distribution prior parametrized as  $Beta(1 - b_A, a_A - ib_A)$ . Recall also that our updates take the general form

$$\mathbb{E}_q[g_i(Z_{-i}, X, \Phi)] = \mathbb{E}_q \left[ \frac{\phi_1 + \sum_{z_n \in Z_{-i}} t(x_n, z_n)}{\phi_2 + N} \right] \quad (37)$$

where  $\phi_1$  and  $\phi_2$  are the parameters of the prior. This implies that in this case,  $\phi_1 = 1 - b_A$  and  $\phi_2 = a_A - ib_A$ . The sufficient statistic  $t(x_n, z_n)$  is the number of times stick  $i$  was the final stick (i.e. the process did not continue after  $i$ ).  $N$  is the total number of times any stick was the final one, which is equivalent to the sum over all sticks of  $t(x_n, z_n)$ . Formally,

$$\sum_{z_n \in Z_{-i}} t(x_n, z_n) = \sum_{B \in M} \sum_{k=1}^{N_B} f(A \xrightarrow{*} s_{A,k}, z_k) \quad (38)$$

which is the sum over all adapted nonterminals of the sum up to the truncation level of that nonterminal (the maximal stick index) of the count of the grammar expansion of  $A$  to the substring  $s_{A,k}$  in the derivation tree  $z_k$ . As mentioned before,

$$N = \sum_{j=1}^{i-1} \sum_{B \in M} \sum_{k=1}^{N_B} f(A \xrightarrow{*} s_{A,j}, z_k) \quad (39)$$

Putting (37),(38), and (39) together, the full update becomes

$$\begin{aligned}
\begin{bmatrix} \gamma_{A,i}^1 \\ \gamma_{A,i}^2 \end{bmatrix} &= \mathbb{E}_q \begin{bmatrix} 1 - b_A + \sum_{B \in M} \sum_{k=1}^{N_B} f(A \xrightarrow{*} s_{A,k}, z_k) \\ a_A - ib_A + \sum_{j=1}^{i-1} \sum_{B \in M} \sum_{k=1}^{N_B} f(A \xrightarrow{*} s_{A,j}, z_k) \end{bmatrix} \\
&= \begin{bmatrix} 1 - b_A + \sum_{B \in M} \sum_{k=1}^{N_B} \tilde{f}(A \xrightarrow{*} s_{A,k}, s_{B,k}) \\ a_A + ib_A + \sum_{j=1}^{i-1} \sum_{B \in M} \sum_{k=1}^{N_B} \tilde{f}(A \xrightarrow{*} s_{A,j}, s_{B,k}) \end{bmatrix}
\end{aligned} \tag{40}$$

where  $\tilde{f}(r, s_{B,k})$  is the expected count of rule  $r$  in the derivation trees of string  $s_{B,k}$  which is headed by nonterminal  $B$  and spans  $k$  units, and  $A \xrightarrow{*} s_{A,k}$  indicates that non-terminal  $A$  expands to the string headed by  $A$  and spanning  $k$  in its grammar form (Cohen et al., 2010; Zhai et al., 2014).

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