

PSY9511: Seminar 5

Beyond linearity: Extensions of linear models and
tree-based models

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Outline

1. Exercise 3
2. Exercise 4
3. Recap
4. Extensions of linear models
 - 4.1 Generalized linear models (GLMs)
 - 4.2 Generalized additive models (GAMs)
5. Tree-based models
 - 5.1 Decision trees
 - 5.2 Random forests
 - 5.3 Gradient boosting (XGBoost)
6. Exercise 5



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5. Tree-based models
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 - 5.3 Gradient boosting (XGBoost)
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Exercise solutions

<https://uiuo.instructure.com/courses/59550>



Exercise 3: Solution

<http://localhost:8888/notebooks/notebooks/Assignment%203.ipynb>



Exercise 4: Stratification

<http://localhost:8888/notebooks/notebooks%2FStratification.ipynb>



Exercise 4: Solution

<http://localhost:8888/notebooks/notebooks/Assignment%204.ipynb>



Whenever a modelling choice is made on the basis of performance in a dataset, **we have to assume the performance achieved by the chosen model is inflated**

- ! Don't use a single train/validation split, cross-validation or bootstrapping with only train/validation sets to select the best model and then reports its performance
- Hold out a test set
- Use a nested cross-validation



Extensions of linear models



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Non-linear models: Nothing to worry about

```
formula <- ...
result <- glm(formula, family=Gamma(link="log"), data=data)
```

```
formula <- ...
result <- gam(formula, data=data)
```

```
In[2]: from xgboost import XGBClassifier

model = XGBClassifier()
model.fit(X, y)
```

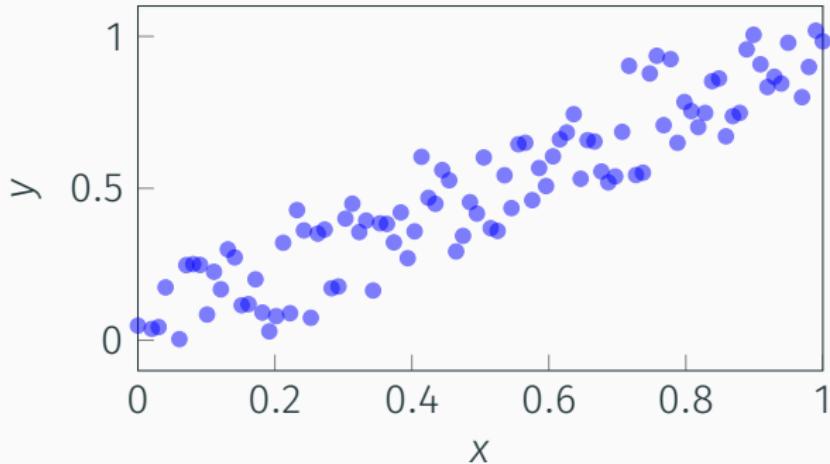


Extensions of linear models: Generalized linear models

$$\hat{y} = \beta_0 + \sum_{i=0}^p \beta_i x_i$$



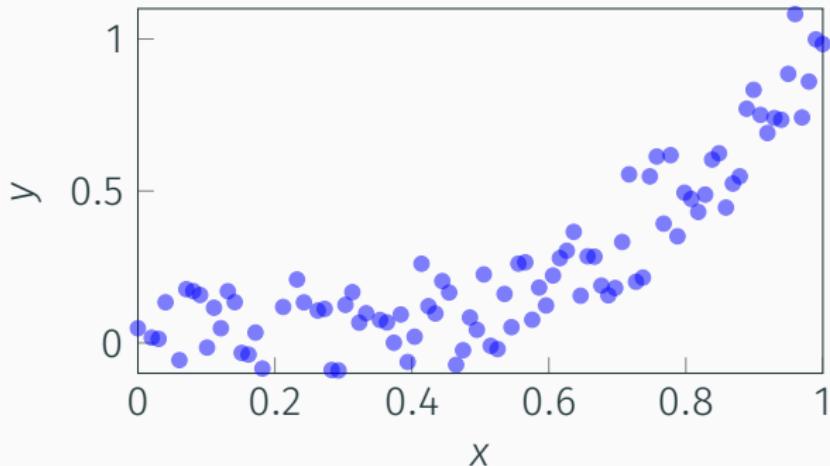
Extensions of linear models: Generalized linear models



$$\hat{y} = \beta_0 + \sum_{i=0}^p \beta_i x_i$$



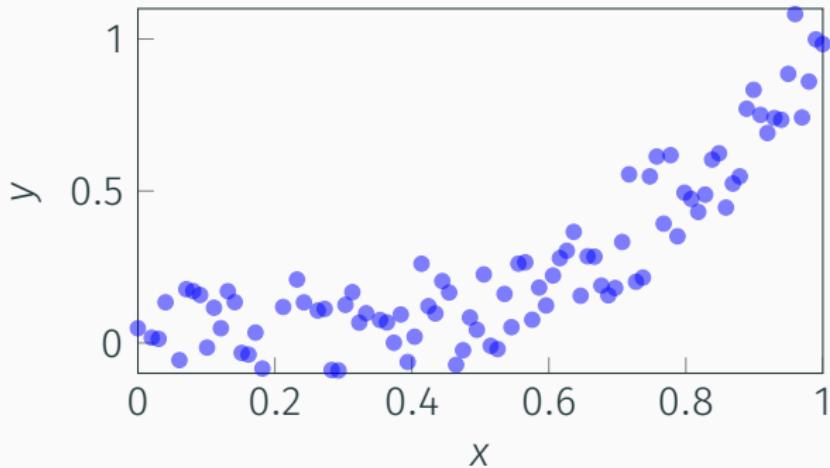
Extensions of linear models: Generalized linear models



$$\hat{y} = \beta_0 + \sum_{i=0}^p \beta_i x_i$$



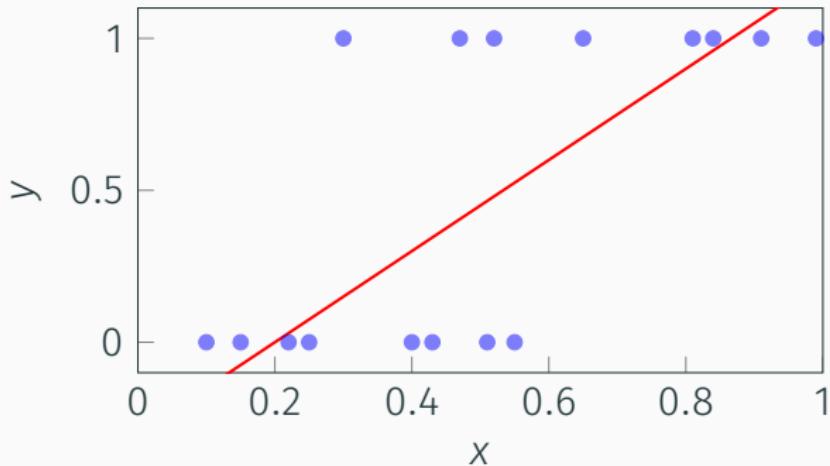
Extensions of linear models: Generalized linear models



$$\hat{y} = \beta_0 + \sum_{i=0}^p \beta_i x_i$$



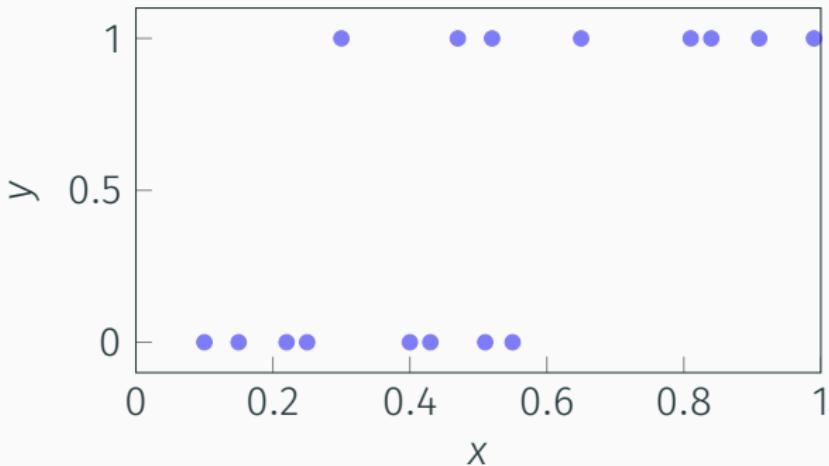
Extensions of linear models: Generalized linear models



$$\hat{y} = \beta_0 + \sum_{i=0}^p \beta_i x_i$$



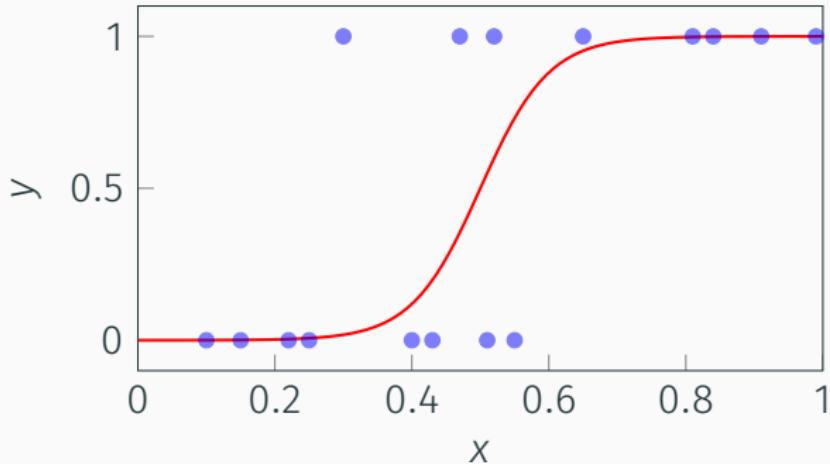
Extensions of linear models: Generalized linear models



$$\log \left(\frac{p(y = 1 | X)}{1 - p(y = 1 | X)} \right) = \beta_0 + \sum_{i=0}^p \beta_i X_i$$



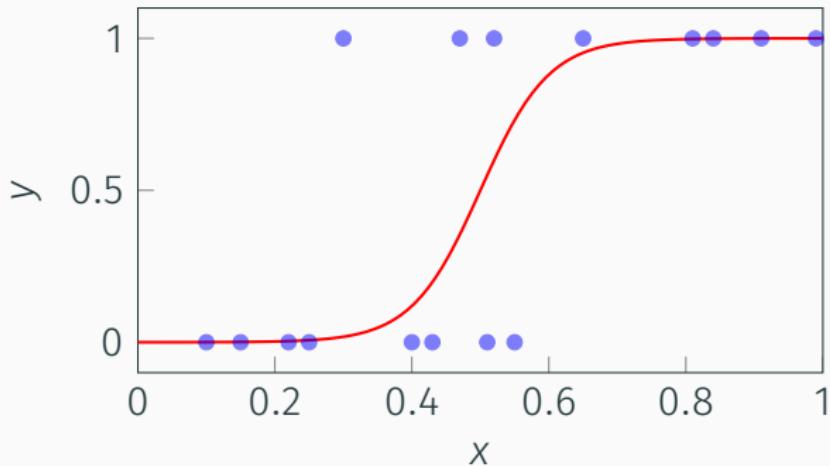
Extensions of linear models: Generalized linear models



$$p(y = 1 | X) = \frac{e^{\left(\beta_0 + \sum_{i=0}^p \beta_i x_i\right)}}{1 + e^{\left(\beta_0 + \sum_{i=0}^p \beta_i x_i\right)}}$$



Extensions of linear models: Generalized linear models



$$p(y = 1 | X) = \frac{e^{\left(\beta_0 + \sum_{i=0}^p \beta_i x_i\right)}}{1 + e^{\left(\beta_0 + \sum_{i=0}^p \beta_i x_i\right)}}$$



Extensions of linear models: Generalized linear models

$$\hat{y} = f(\beta_0 + \sum_{i=0}^p \beta_i x_i)$$

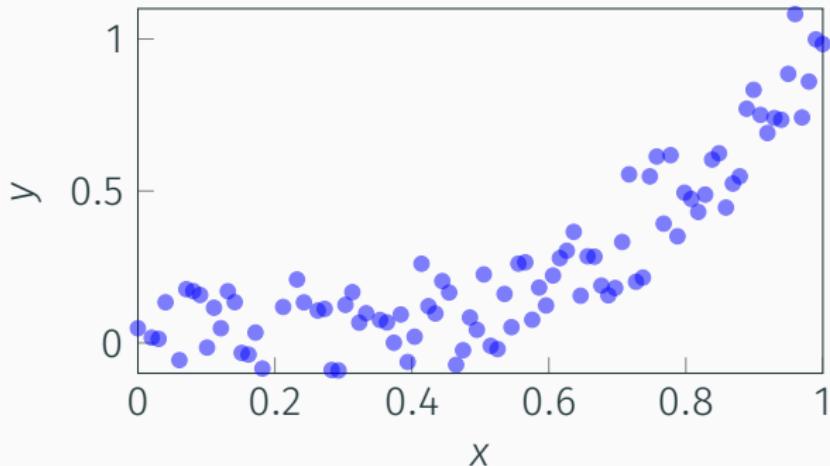


Extensions of linear models: Generalized linear models

$$f(\hat{y}) = \beta_0 + \sum_{i=0}^p \beta_i x_i$$



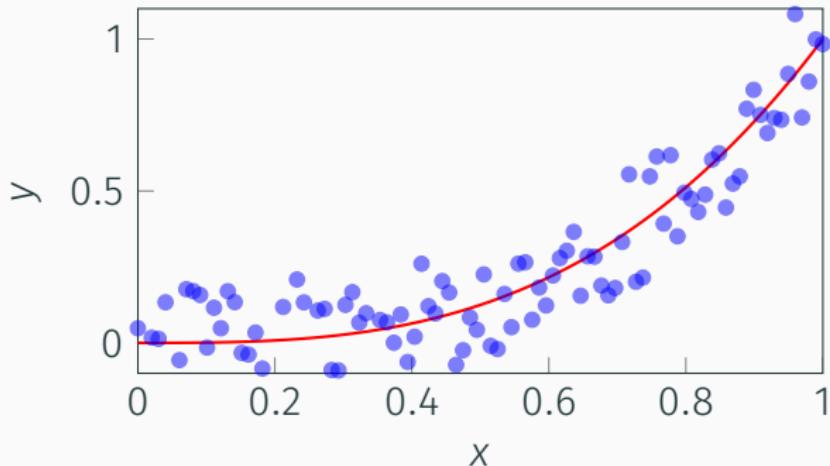
Extensions of linear models: Generalized linear models



$$f(\hat{y}) = \beta_0 + \sum_{i=0}^p \beta_i x_i$$



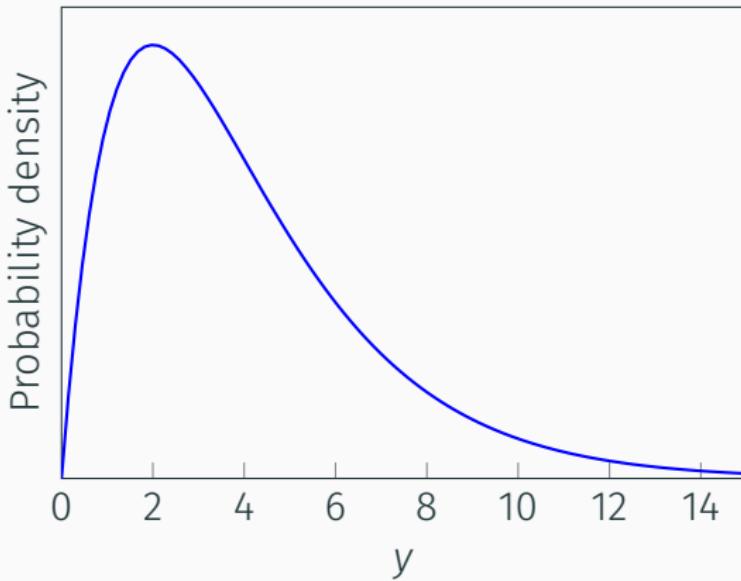
Extensions of linear models: Generalized linear models



$$\log(\hat{y}) = \beta_0 + \sum_{i=0}^p \beta_i x_i$$



Extensions of linear models: Generalized linear models



Generalized linear models (GLMs):

Extends the regular linear model by associating the predictors to the response via a non-linear link function f .

- Requires us to specify f (often determined by investigating the distribution of the response).



Extensions of linear models: Generalized linear models

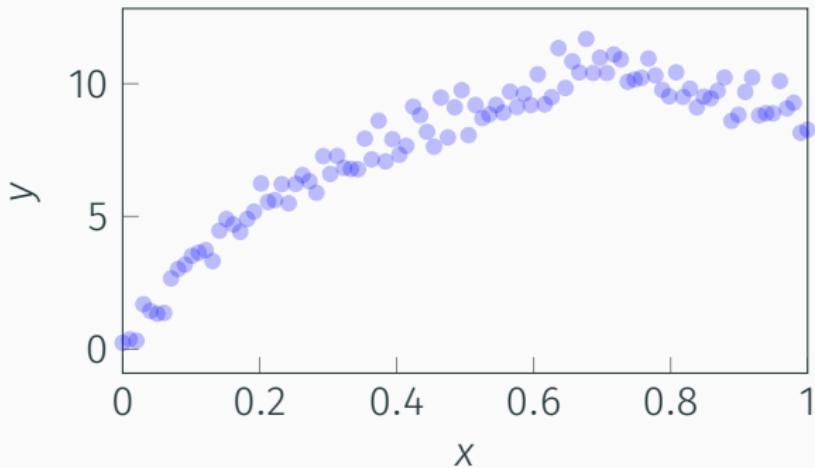
```
In[1]: from sklearn.linear_model import GammaRegressor  
  
model = GammaRegressor()  
model.fit(X, y)
```

```
In[2]: import statsmodels.api as sm  
  
link = sm.genmod.families.links.Log()  
model = sm.GLM(y, X, family=sm.families.Gamma(link=link))  
model.fit()
```

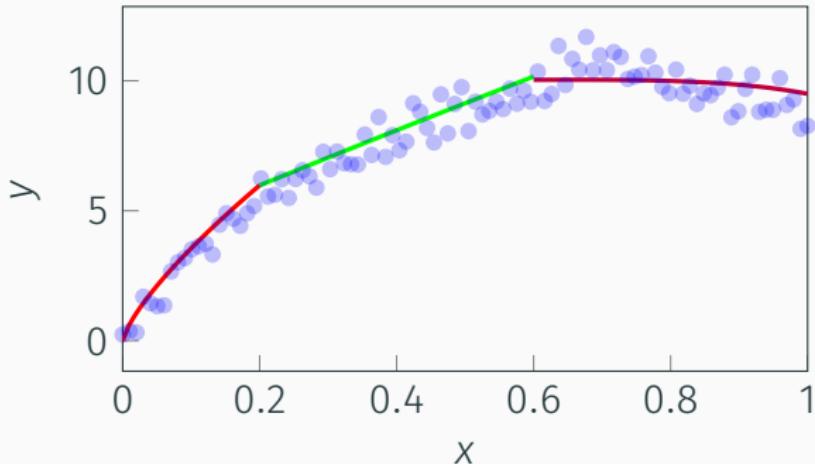
```
formula <- ...  
result <- glm(formula, family=Gamma(link="log"), data=data)
```



Extensions of linear models: Generalized additive models



Extensions of linear models: Generalized additive models

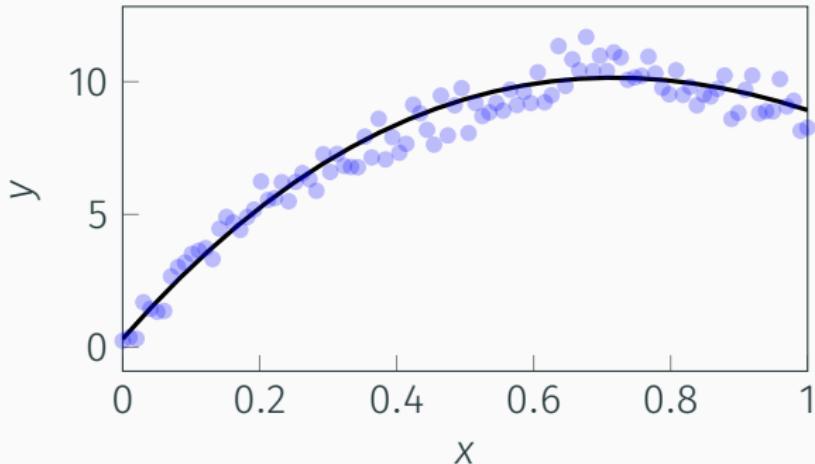


Piecewise polynomial functions:

- Regression splines (ISL, Chapter 7.4)



Extensions of linear models: Generalized additive models



A single, complex, polynomial function:

- Smoothing splines (This lecture)



Extensions of linear models: Generalized additive models

$$\sum_{i=1}^n (y_i - g(x_i))^2 + \lambda \int g''(t)^2 dt$$



Extensions of linear models: Generalized additive models

$$\hat{y}_i = g(x_i)$$
$$\sum_{i=1}^n (y_i - g(x_i))^2 + \lambda \int g''(t)^2 dt$$



Extensions of linear models: Generalized additive models

$$\sum_{i=1}^n (y_i - g(x_i))^2 + \lambda \int g''(t)^2 dt$$

\uparrow

$$\sum (y_i - \hat{y}_i)^2$$



Extensions of linear models: Generalized additive models

$$\sum_{i=1}^n (y_i - g(x_i))^2 + \lambda \int g''(t)^2 dt$$

↑
Large when g is wiggly



Extensions of linear models: Generalized additive models

Balances what is
most important



$$\sum_{i=1}^n (y_i - g(x_i))^2 + \lambda \int g''(t)^2 dt$$

↑
Tries to
fit data

↑
Tries to
simplify g



Extensions of linear models: Generalized additive models

<http://localhost:8888/notebooks/notebooks/Smoothing%20spline.ipynb>



Extensions of linear models: Generalized additive models

$$\hat{y} = \beta_0 + \beta_1 x$$

$$\hat{y} = g(x)$$



Extensions of linear models: Generalized additive models

$$\hat{y} = \beta_0 + \beta_1 x$$



$$\hat{y} = \beta_0 + \sum_{j=1}^p \beta_j x_j$$

$$\hat{y} = g(x)$$



$$\hat{y} = \beta_0 + \sum_{j=1}^p f_j(x_j)$$



Generalized additive models (GAMs):

Extends the linear model by allowing non-linear functions f_j (i.e. splines) to be fitted for each predictor x_j .

- Does not allow for interactions between predictors.

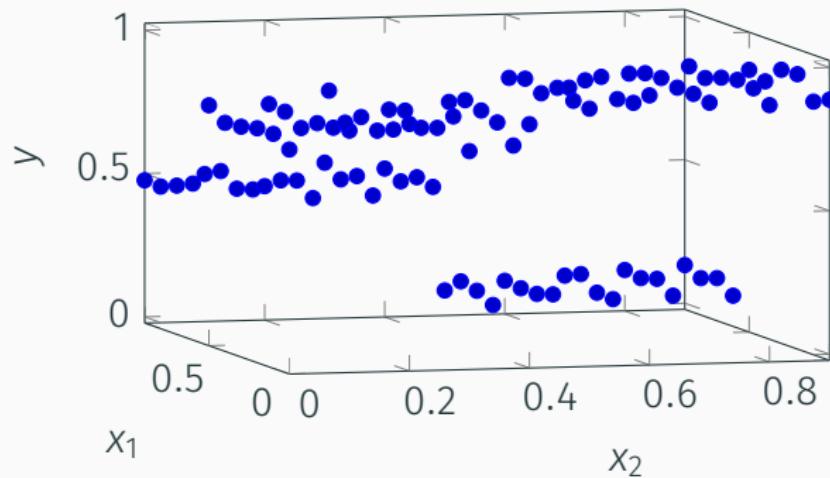


Tree-based models

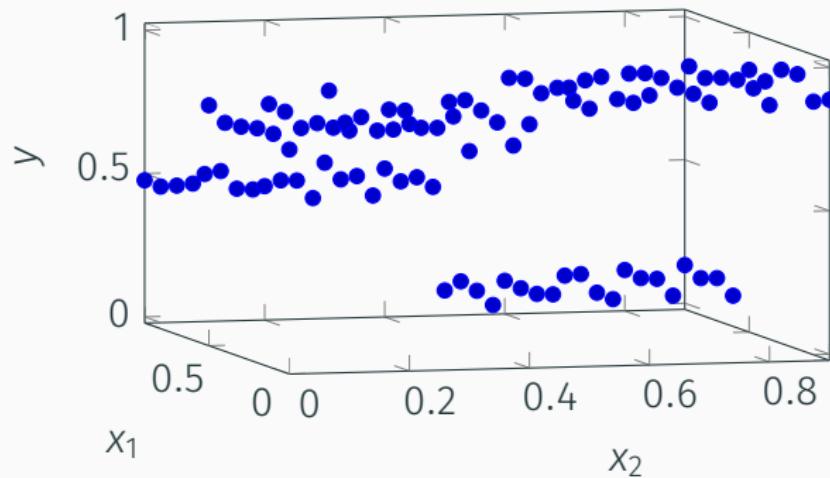


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Tree-based models: Motivation



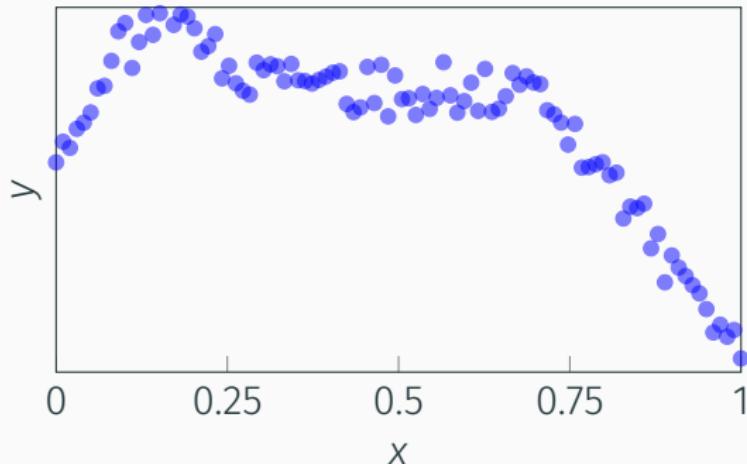
Tree-based models: Motivation



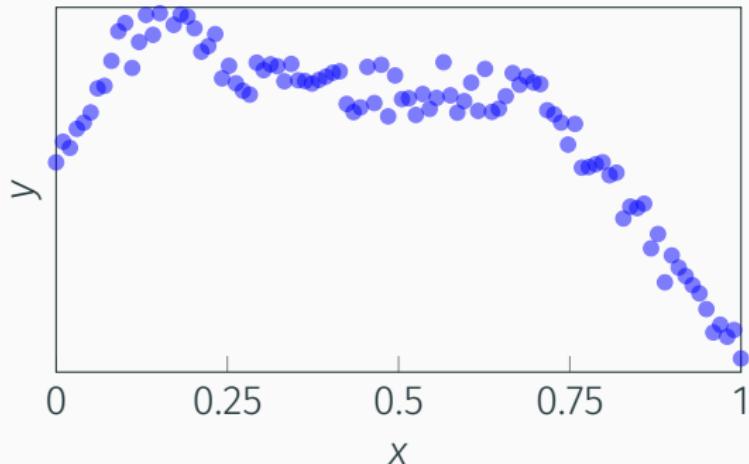
$$y = \begin{cases} 0.8 & x_1 \leq 0.6 \text{ & } x_2 \leq 0.5 \\ 0.9 & x_1 \leq 0.6 \text{ & } x_2 > 0.5 \\ 0.5 & x_1 > 0.5 \text{ & } x_2 \leq 0.5 \\ 0.1 & x_1 > 0.5 \text{ & } x_2 > 0.5 \end{cases}$$



Tree-based models: Motivation



Tree-based models: Motivation



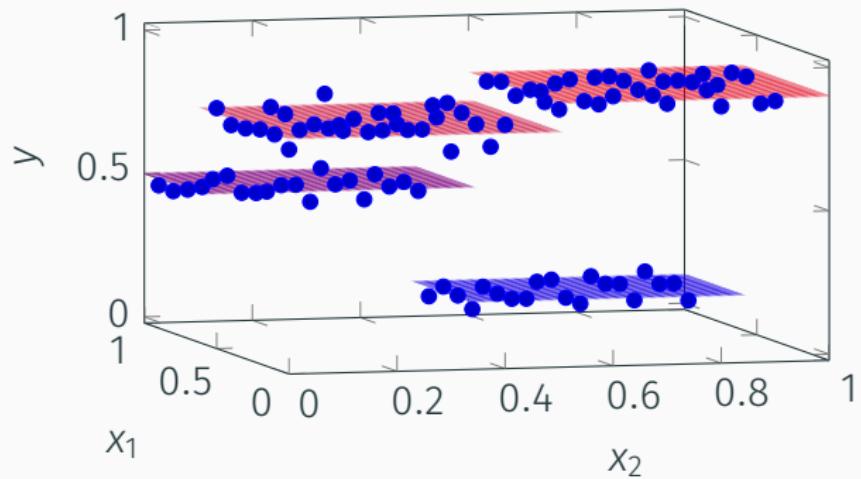
$$y = ?x^5 + ?x^4 + ?x^3 + ?x^2 + ?x + ?$$



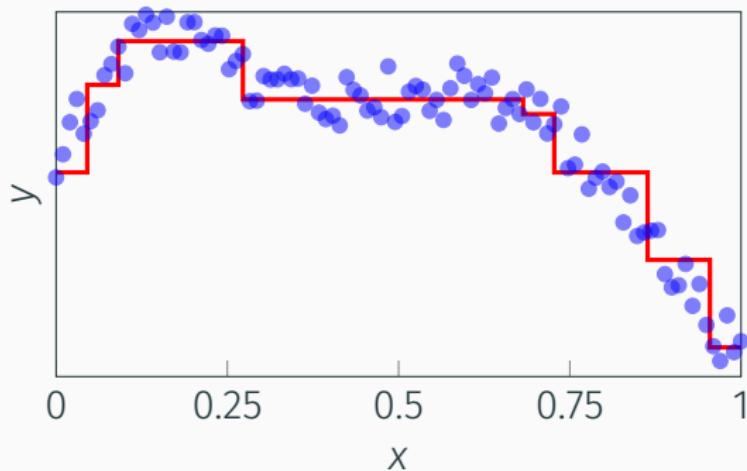
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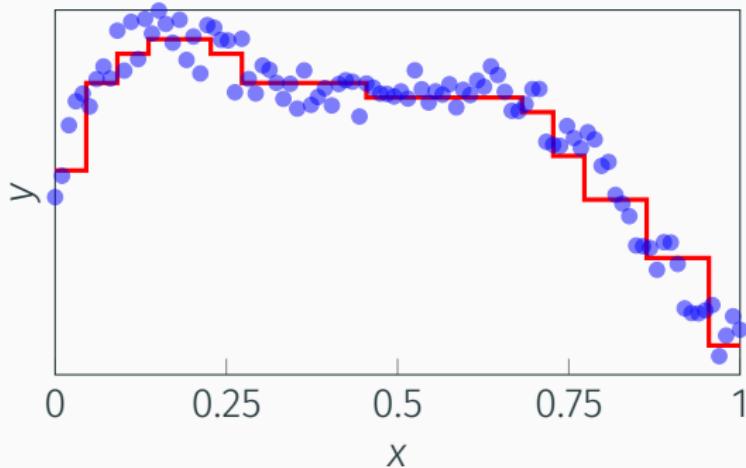
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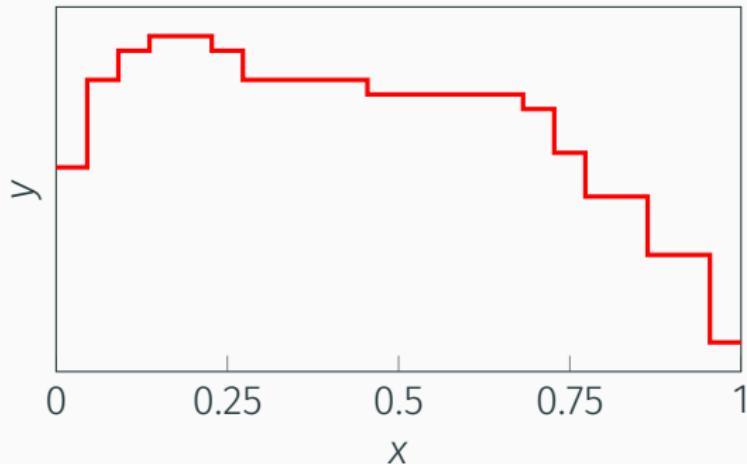
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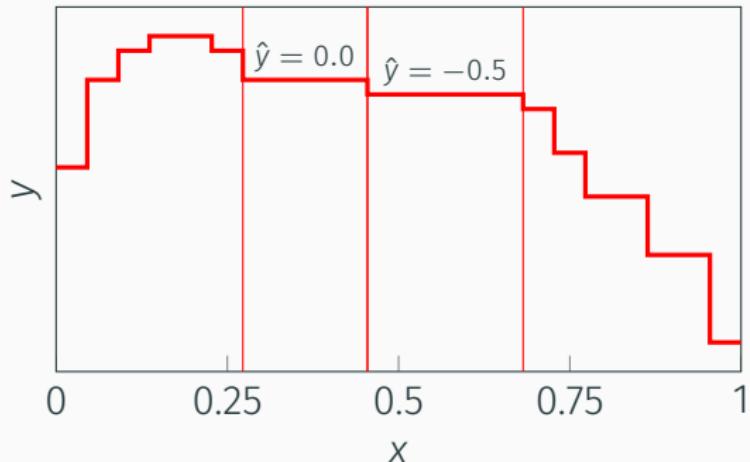
Tree-based models: Motivation



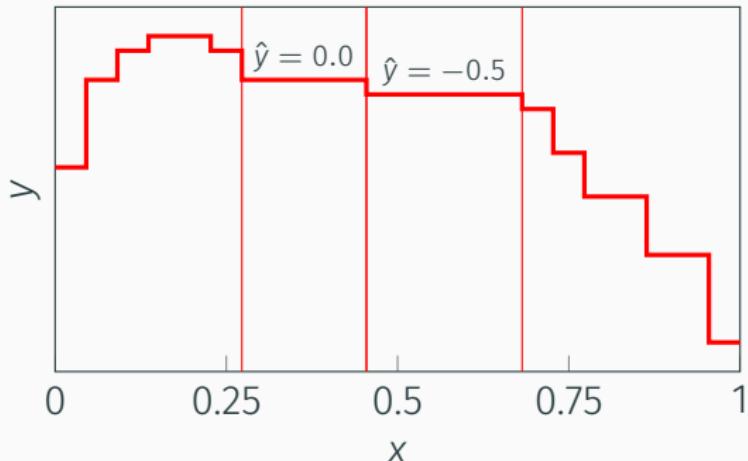
Piecewise constant function



Tree-based models: Motivation



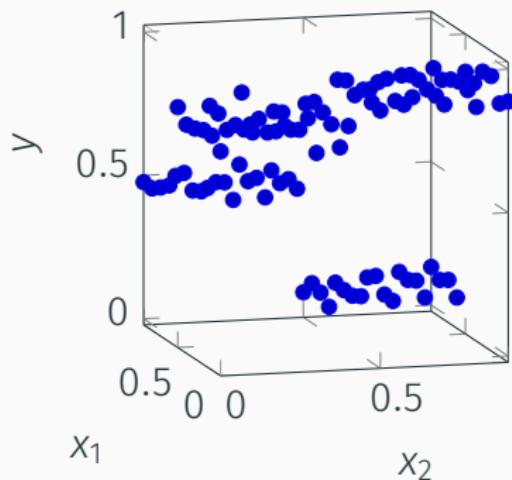
Tree-based models: Motivation



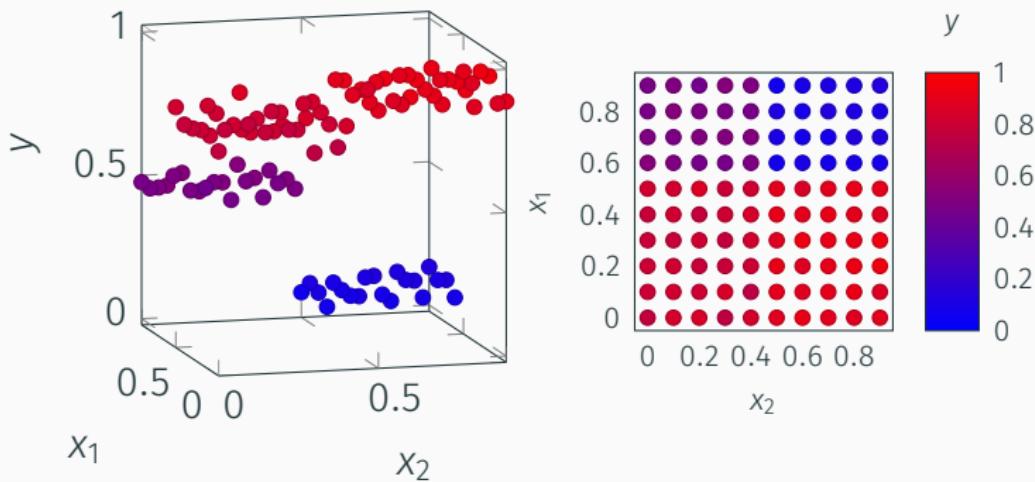
$$\hat{y} = \begin{cases} \dots \\ 0.0 & x \geq 0.27 \text{ \& } x < 0.45 \\ -0.5 & x \geq 0.45 \text{ \& } x < 0.69 \\ \dots \end{cases}$$



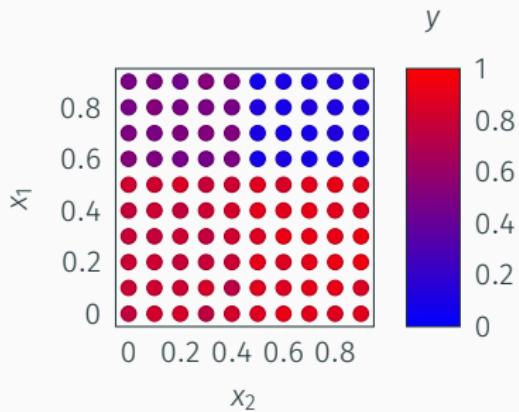
Tree-based models: Decision trees



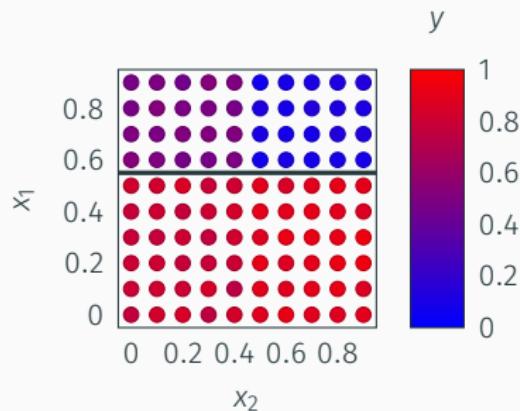
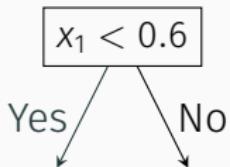
Tree-based models: Decision trees



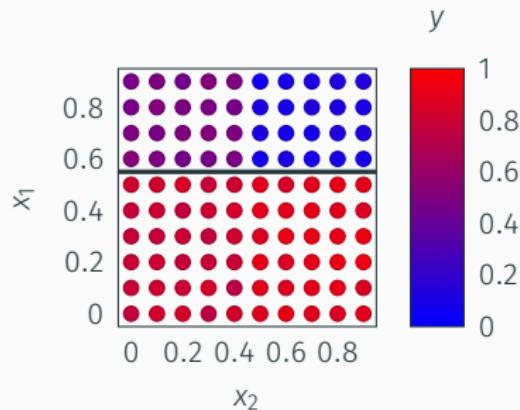
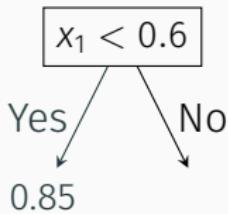
Tree-based models: Decision trees



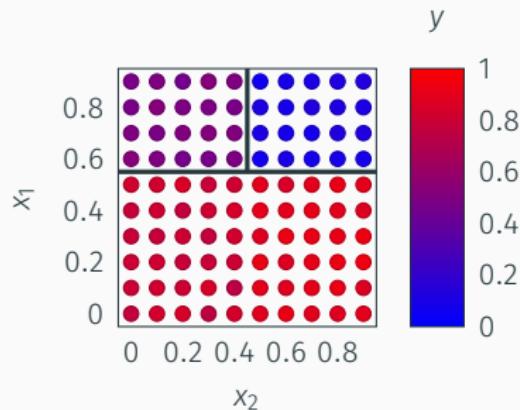
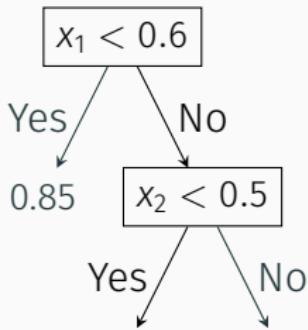
Tree-based models: Decision trees



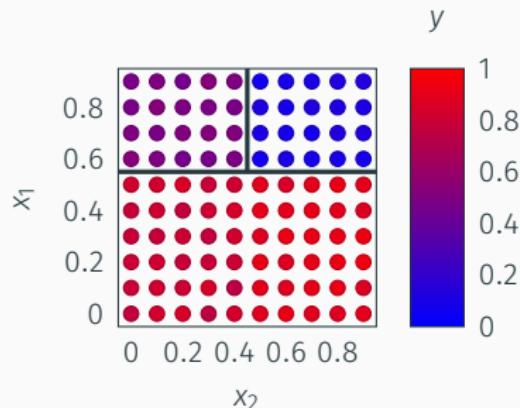
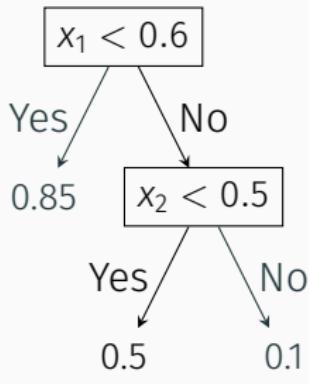
Tree-based models: Decision trees



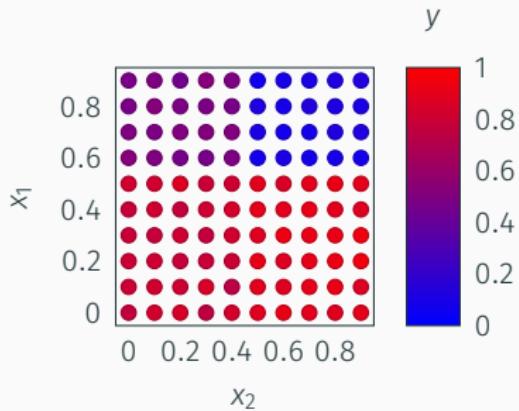
Tree-based models: Decision trees



Tree-based models: Decision trees

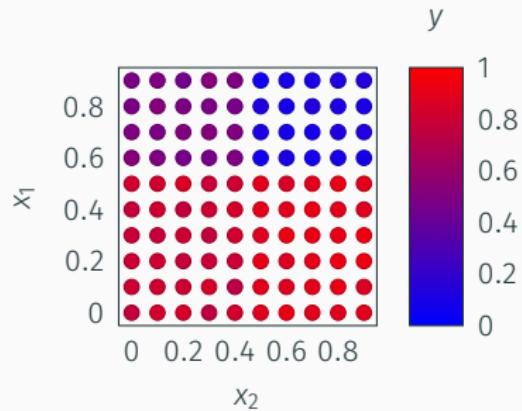


Tree-based models: Decision trees



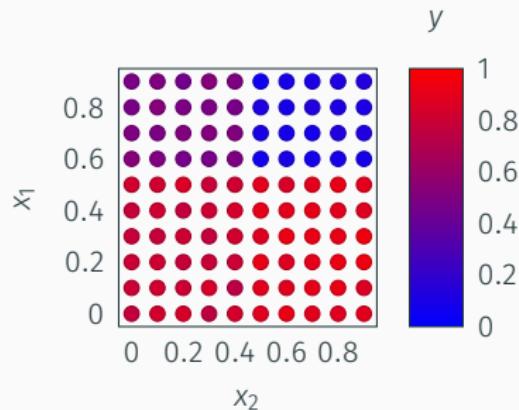
Tree-based models: Decision trees

x_1 or x_2 ?

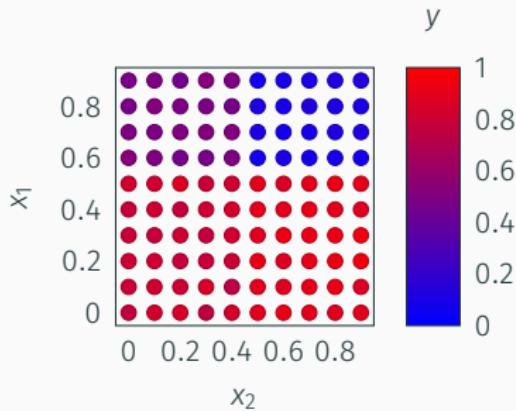
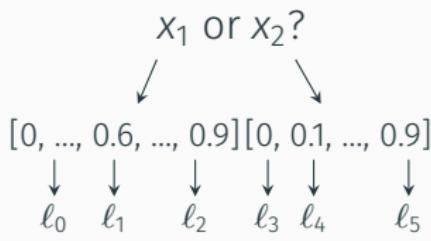


Tree-based models: Decision trees

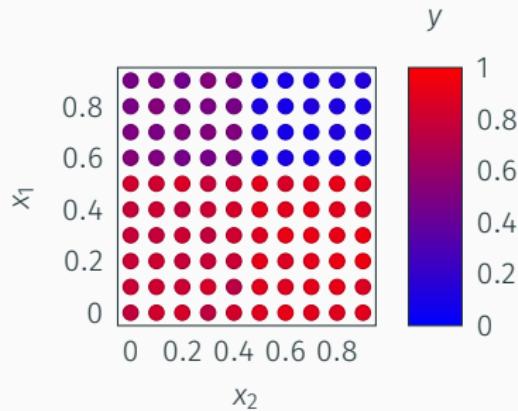
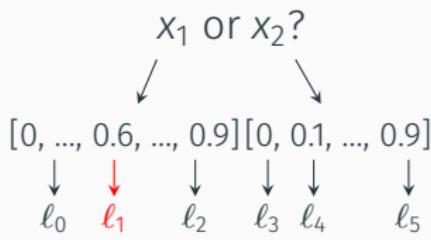
x_1 or x_2 ?
[0, ..., 0.6, ..., 0.9] [0, 0.1, ..., 0.9]



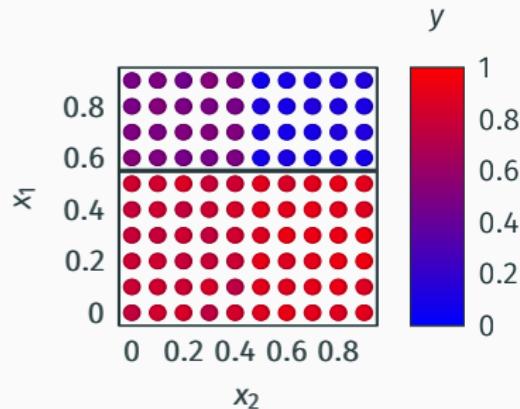
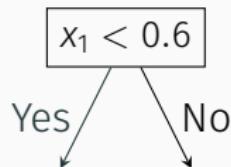
Tree-based models: Decision trees



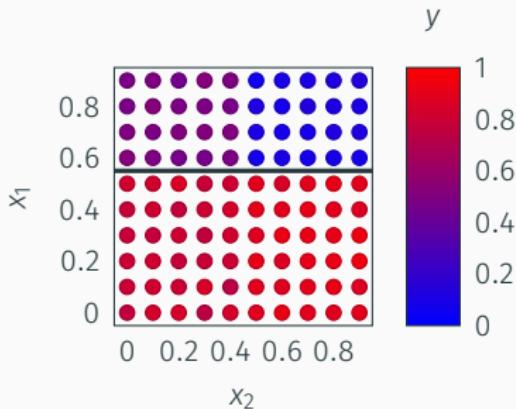
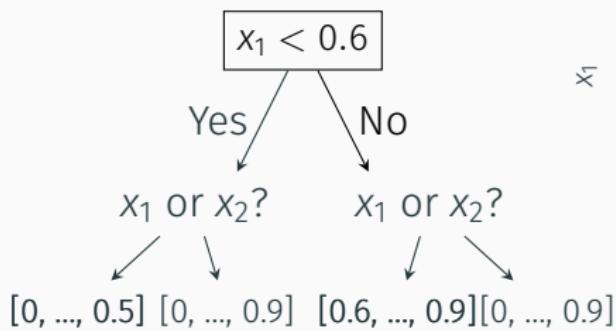
Tree-based models: Decision trees



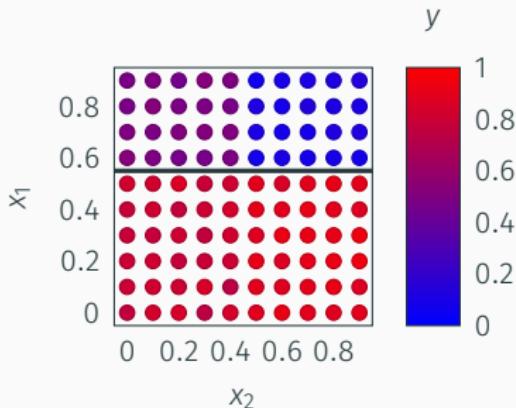
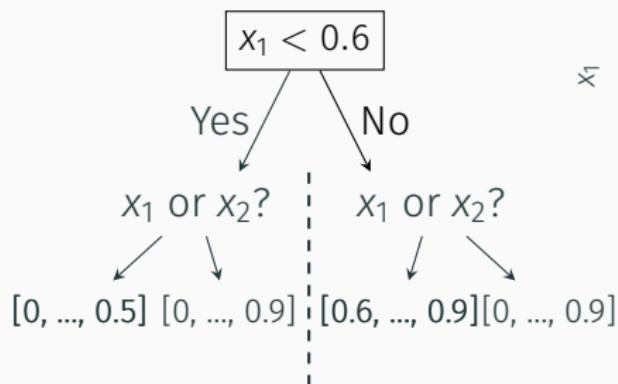
Tree-based models: Decision trees



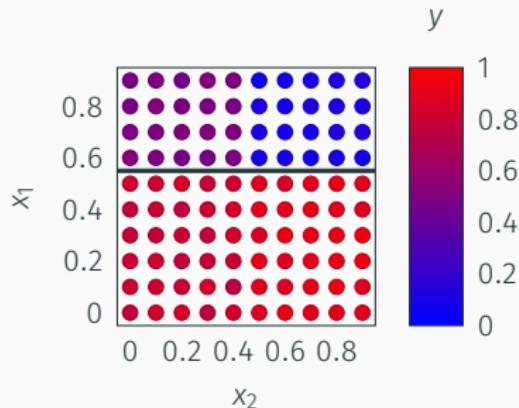
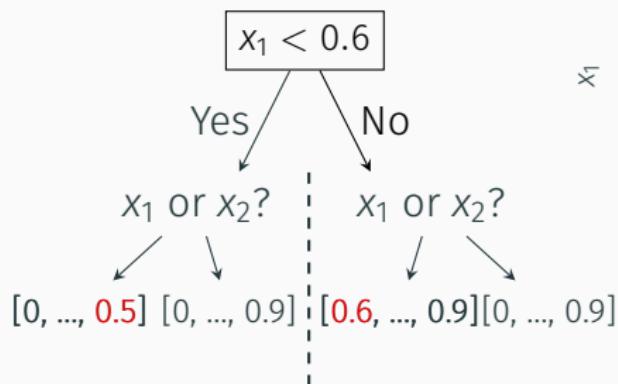
Tree-based models: Decision trees



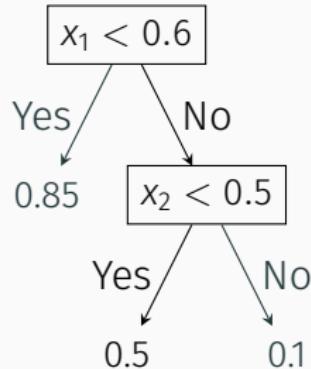
Tree-based models: Decision trees



Tree-based models: Decision trees

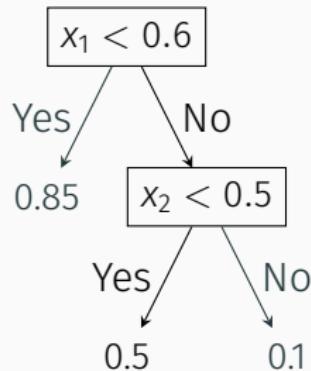


Tree-based models: Decision trees



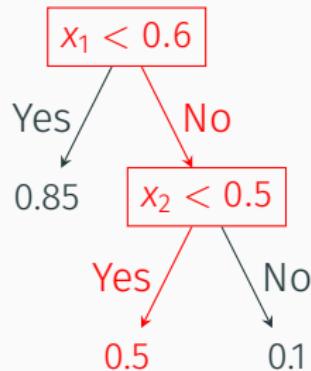
Tree-based models: Decision trees

$$x = (0.8, 0.3)$$



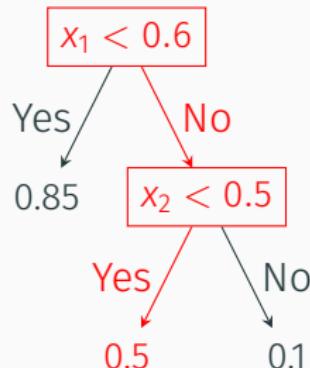
Tree-based models: Decision trees

$$x = (0.8, 0.3)$$



Tree-based models: Decision trees

$$x = (0.8, 0.3)$$

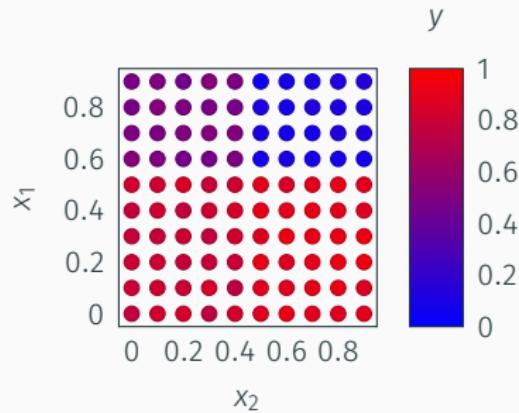


Why was $\hat{y} = 0.5$?

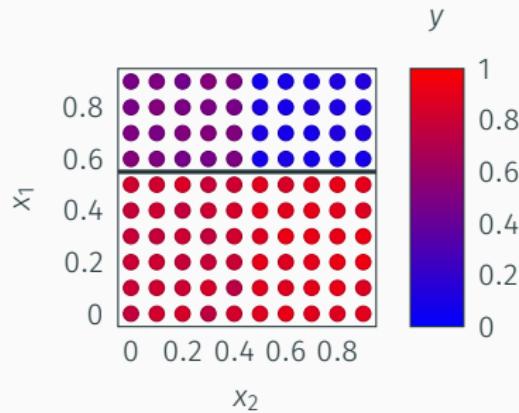
Because $x_1 \geq 0.6$ and $x_2 < 0.5$.



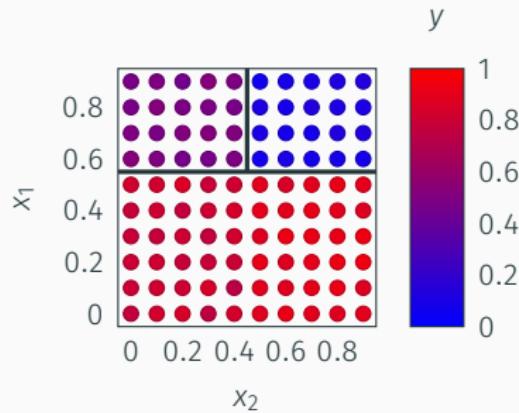
Tree-based models: Decision trees



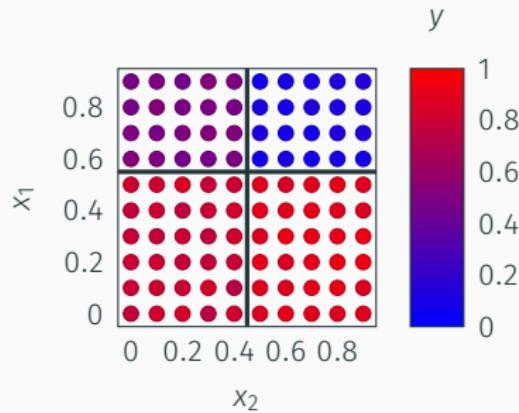
Tree-based models: Decision trees



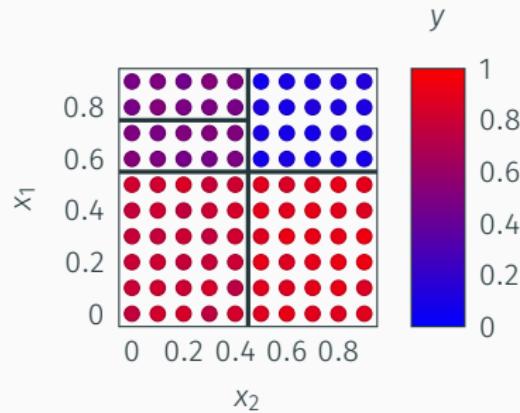
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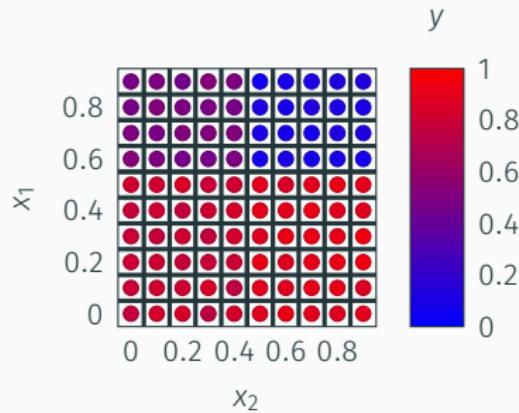
Tree-based models: Decision trees



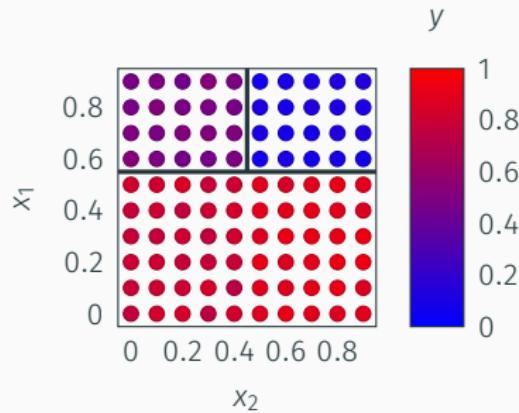
Tree-based models: Decision trees



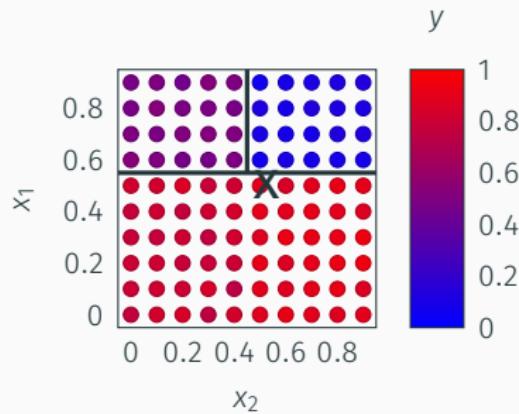
Tree-based models: Decision trees



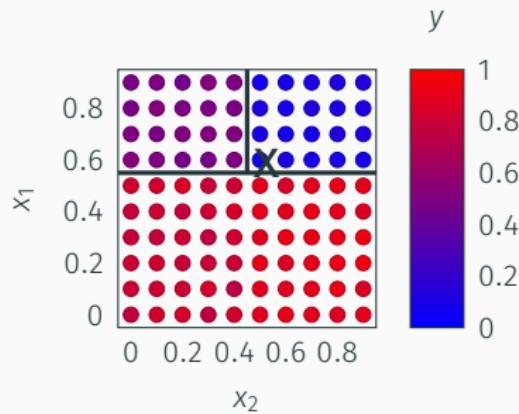
Tree-based models: Decision trees



Tree-based models: Decision trees



Tree-based models: Decision trees



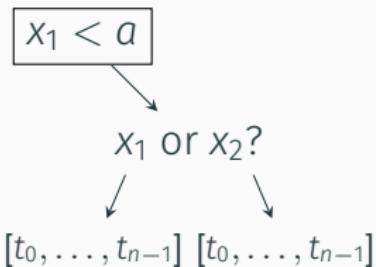
$$\hat{y} = 0.1$$



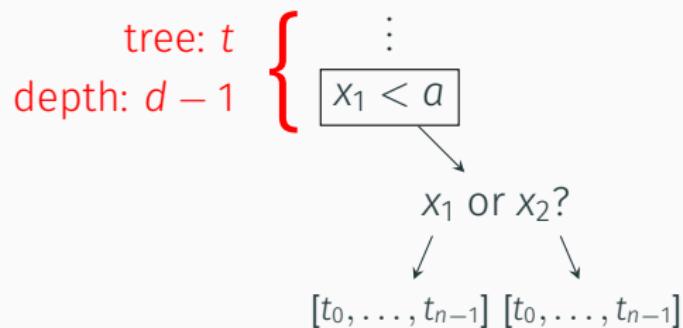
Tree-based models: Decision trees



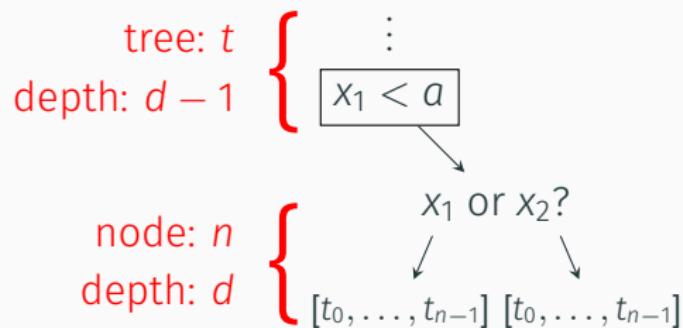
Tree-based models: Decision trees



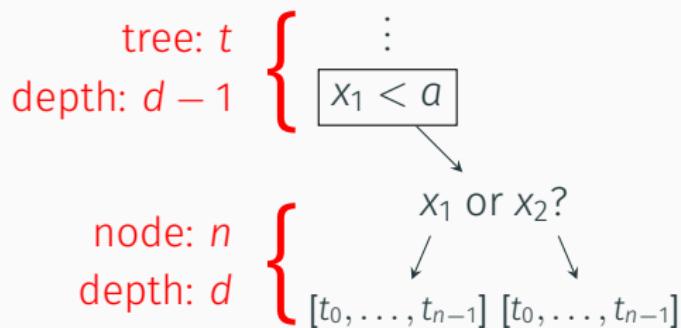
Tree-based models: Decision trees



Tree-based models: Decision trees



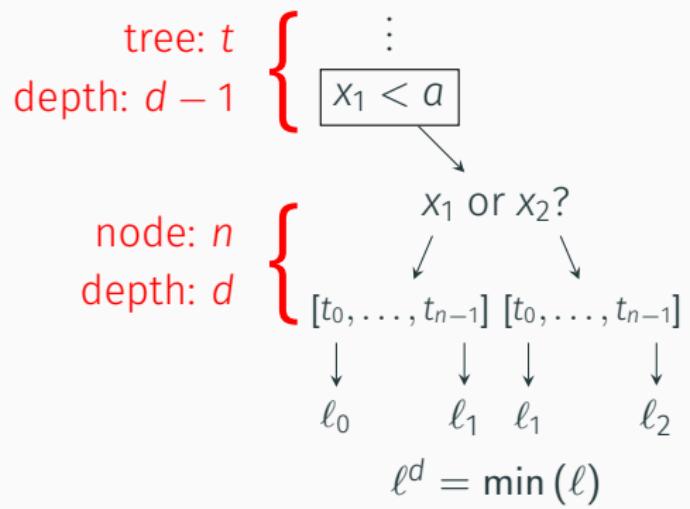
Tree-based models: Decision trees



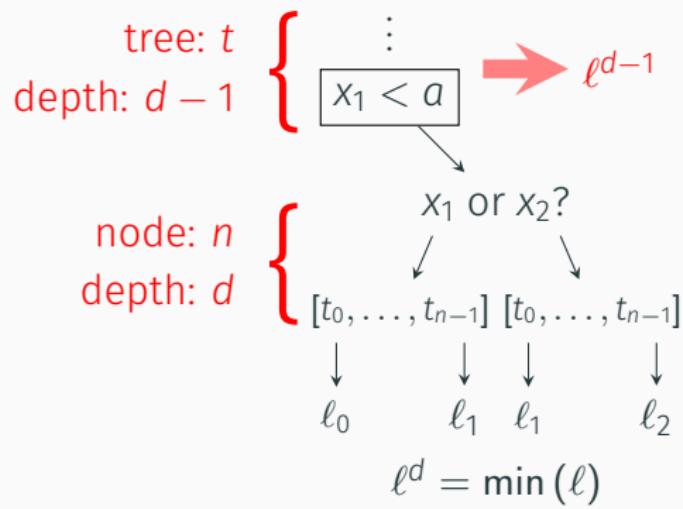
```
if  $d > \text{max\_depth}$ :  
    stop  
else:  
    continue
```



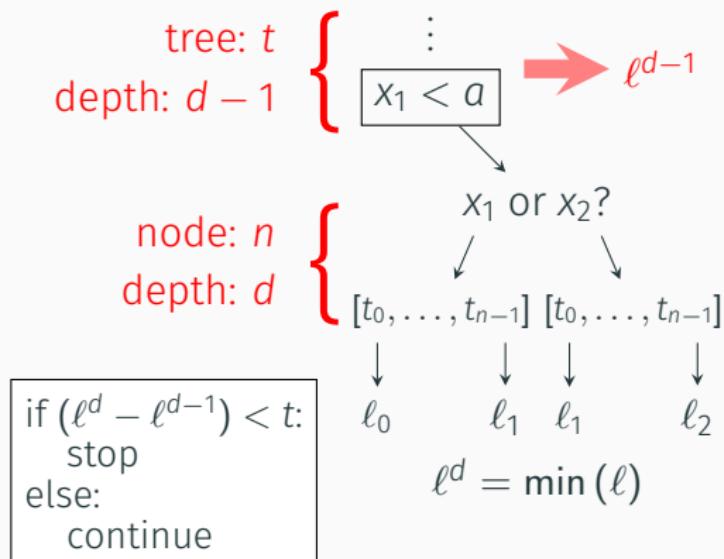
Tree-based models: Decision trees



Tree-based models: Decision trees



Tree-based models: Decision trees

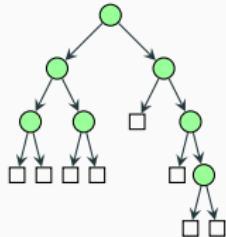


Tree-based models: Decision trees

<https://scikit-learn.org/dev/modules/generated/sklearn.tree.DecisionTreeClassifier.html>



Tree-based models: Decision trees



Tree-based models: Decision trees

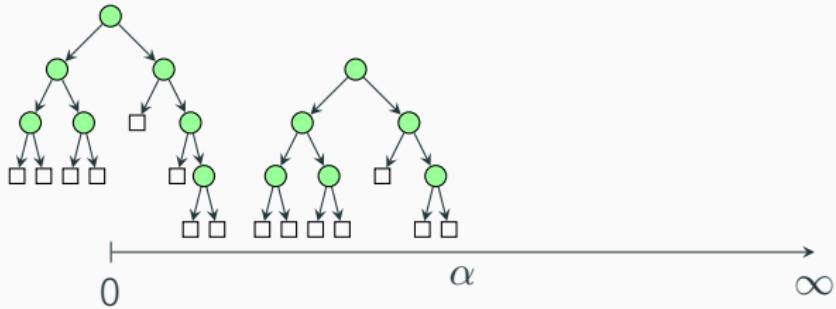


$$\sum_{i=0}^n (y_i - \hat{y}_i)^2 + \alpha|T|,$$

$|T|$ =number of leaf nodes



Tree-based models: Decision trees

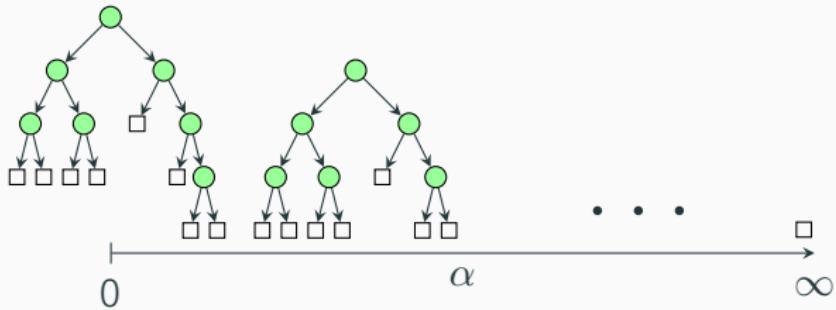


$$\sum_{i=0}^n (y_i - \hat{y}_i)^2 + \alpha|T|,$$

$|T|$ =number of leaf nodes



Tree-based models: Decision trees

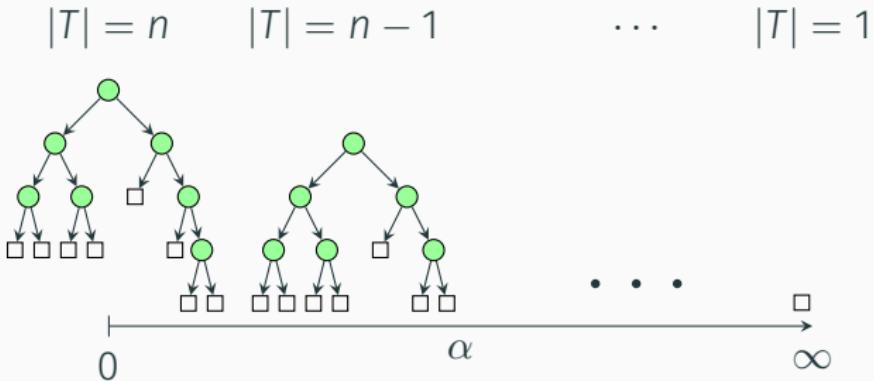


$$\sum_{i=0}^n (y_i - \hat{y}_i)^2 + \alpha |T|,$$

$|T|$ =number of leaf nodes



Tree-based models: Decision trees

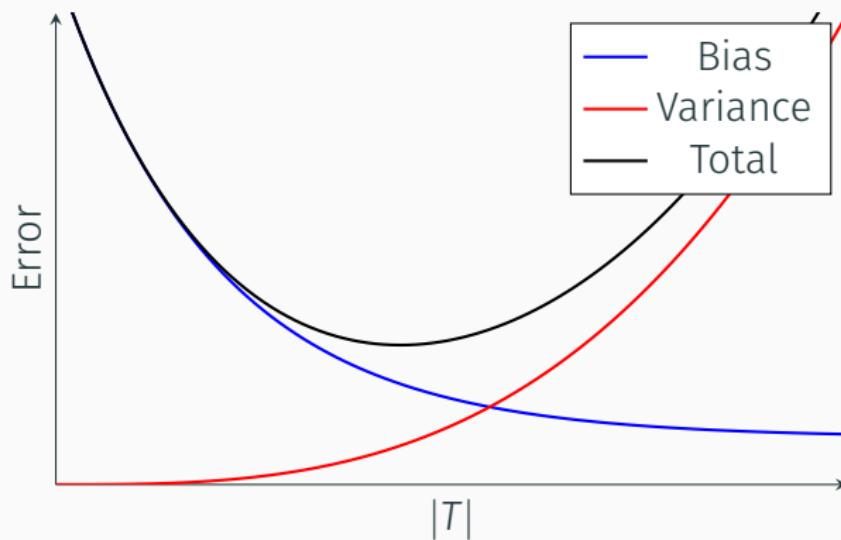


$$\sum_{i=0}^n (y_i - \hat{y}_i)^2 + \alpha|T|,$$

$|T|$ =number of leaf nodes



Tree-based models: Decision trees



Tree-based models: Decision trees

Decision trees: Segments the feature space into regions via the application of multiple binary splits.

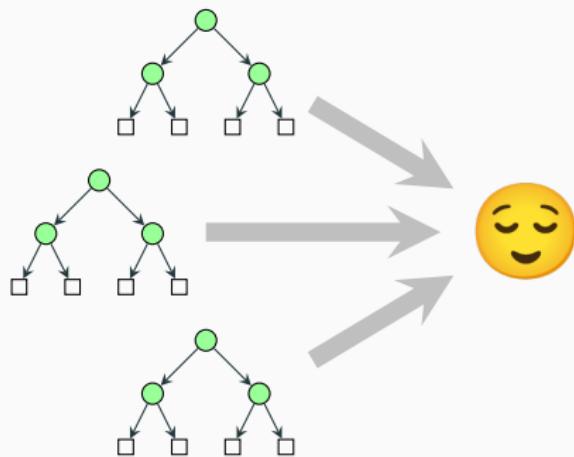
- Can be used for classification and regression
- + Very interpretable
- Quite bad
 - Prone to overfitting
 - Small changes in x can yield large differences in \hat{y}
- Complexity can be controlled via limiting tree size during fitting
- Can be pruned afterwards to determine optimal size



Tree-based methods: Ensembles



Tree-based methods: Ensembles



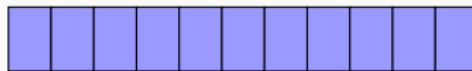
Tree-based methods: Ensembles

Ensembling: Combining multiple weak learners can improve performance.

- A single prediction is achieved by averaging or majority voting
- Models should ideally be diverse



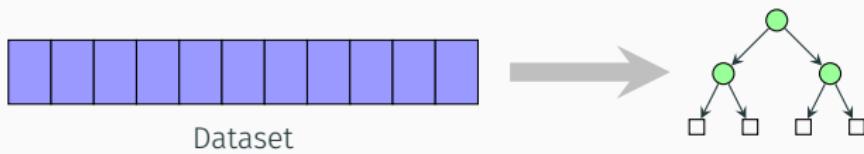
Tree-based methods: Ensembles



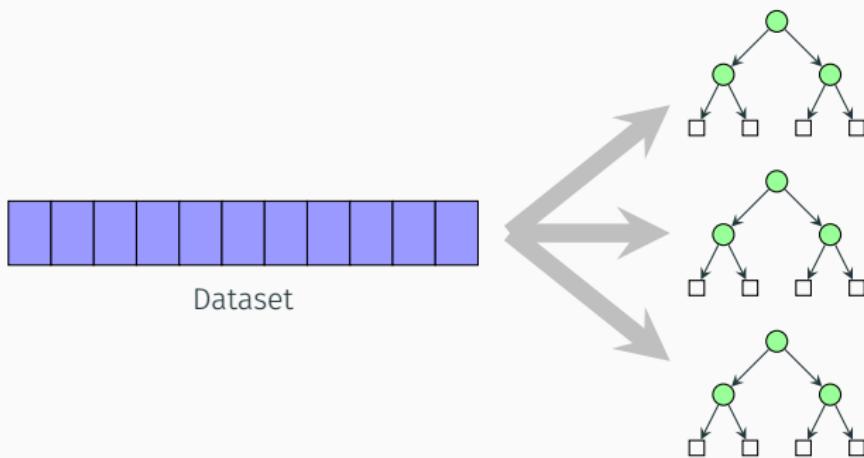
Dataset



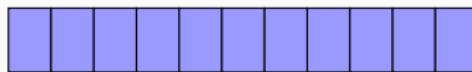
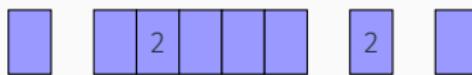
Tree-based methods: Ensembles



Tree-based methods: Ensembles



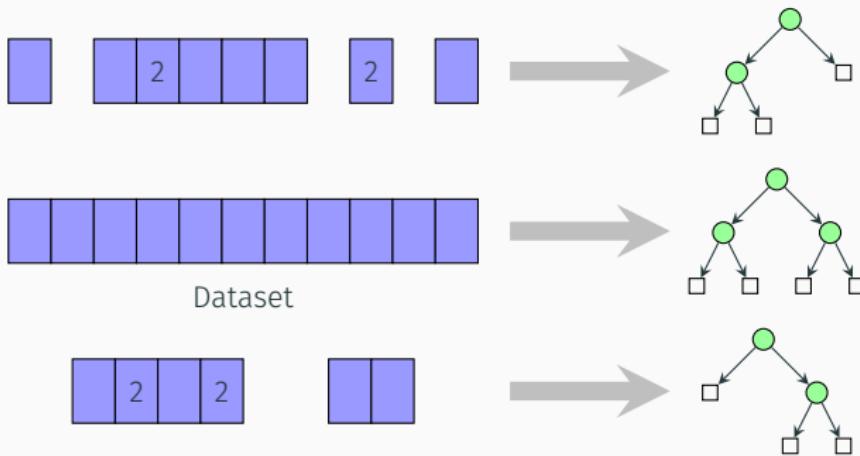
Tree-based methods: Ensembles



Dataset



Tree-based methods: Ensembles



Tree-based methods: Ensembles

Bootstrap aggregation (bagging): Create a set of datasets by sampling with replacement, and train individual models on each.



Tree-based methods: Ensembles

x_1	x_2	x_3	x_4	x_5	x_6	y
0.1	0.7	0.3	0.2	0.8	0.4	0.5
0.4	0.5	0.9	0.3	0.8	0.2	0.6
0.2	0.3	0.8	0.1	0.7	0.5	0.7
0.3	0.6	0.2	0.4	0.9	0.1	0.8
0.5	0.8	0.1	0.6	0.2	0.3	0.9
0.6	0.9	0.7	0.5	0.1	0.4	0.8



Tree-based methods: Ensembles

x_1	x_2	x_3	x_4	x_5	x_6	y
0.1	0.7	0.3	0.2	0.8	0.4	0.5
0.4	0.5	0.9	0.3	0.8	0.2	0.6
0.2	0.3	0.8	0.1	0.7	0.5	0.7
0.3	0.6	0.2	0.4	0.9	0.1	0.8
0.5	0.8	0.1	0.6	0.2	0.3	0.9
0.6	0.9	0.7	0.5	0.1	0.4	0.8



Tree-based methods: Ensembles

x_1	x_2	x_3	x_4	x_5	x_6	y
0.1	0.7	0.3	0.2	0.8	0.4	0.5
0.4	0.5	0.9	0.3	0.8	0.2	0.6
0.2	0.3	0.8	0.1	0.7	0.5	0.7
0.3	0.6	0.2	0.4	0.9	0.1	0.8
0.5	0.8	0.1	0.6	0.2	0.3	0.9
0.6	0.9	0.7	0.5	0.1	0.4	0.8



Tree-based methods: Ensembles

x_1	x_2	x_3	x_4	x_5	x_6	y
0.1	0.7	0.3	0.2	0.8	0.4	0.5
0.4	0.5	0.9	0.3	0.8	0.2	0.6
0.2	0.3	0.8	0.1	0.7	0.5	0.7
0.3	0.6	0.2	0.4	0.9	0.1	0.8
0.5	0.8	0.1	0.6	0.2	0.3	0.9
0.6	0.9	0.7	0.5	0.1	0.4	0.8



Tree-based methods: Ensembles

x_1	x_2	x_3	x_4	x_5	x_6	y
0.1	0.7	0.3	0.2	0.8	0.4	0.5
0.4	0.5	0.9	0.3	0.8	0.2	0.6
0.2	0.3	0.8	0.1	0.7	0.5	0.7
0.3	0.6	0.2	0.4	0.9	0.1	0.8
0.5	0.8	0.1	0.6	0.2	0.3	0.9
0.6	0.9	0.7	0.5	0.1	0.4	0.8



Tree-based methods: Ensembles

x_1	x_2	x_3	x_4	x_5	x_6	y
0.1	0.7	0.3	0.2	0.8	0.4	0.5
0.4	0.5	0.9	0.3	0.8	0.2	0.6
0.2	0.3	0.8	0.1	0.7	0.5	0.7
0.3	0.6	0.2	0.4	0.9	0.1	0.8
0.5	0.8	0.1	0.6	0.2	0.3	0.9
0.6	0.9	0.7	0.5	0.1	0.4	0.8

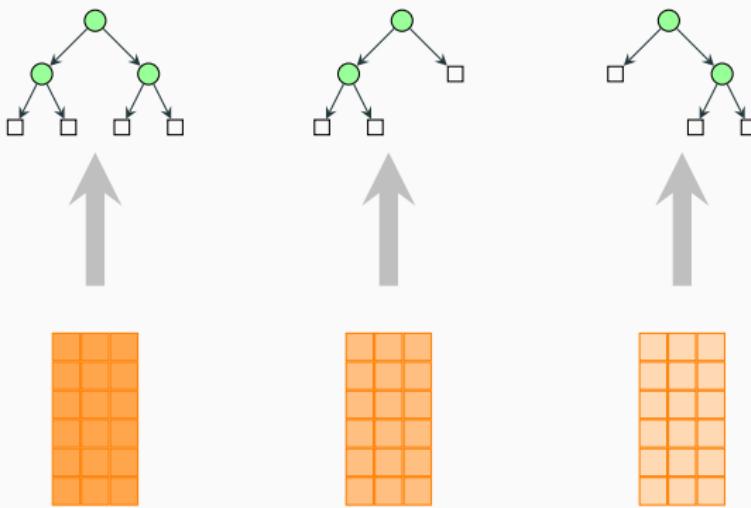


Tree-based methods: Ensembles

x_1	x_2	x_3	x_4	x_5	x_6	y
0.1	0.7	0.3	0.2	0.8	0.4	0.5
0.4	0.5	0.9	0.3	0.8	0.2	0.6
0.2	0.3	0.8	0.1	0.7	0.5	0.7
0.3	0.6	0.2	0.4	0.9	0.1	0.8
0.5	0.8	0.1	0.6	0.2	0.3	0.9
0.6	0.9	0.7	0.5	0.1	0.4	0.8



Tree-based methods: Ensembles



Tree-based methods: Ensembles

Random forests: Fits multiple decision trees on random subsets of **both** predictors and observations.

- Produces more diverse models than bagging
- The number of predictors per tree are usually $\approx \sqrt{p}$
- Requires us to select the number of trees. More trees \rightarrow better performance (\approx same bias, less variance)



Tree-based methods: Boosting

<https://kaggle.com>



Tree-based methods: Boosting

1st place solution

Thanks to Ennfit and Kaggle for hosting this great competition. And thanks to the great notebooks and discussions, I learned a lot. I have a feeling this comp is going to be slightly more volatile than optiver-one-and-wish-i-could-end-up-in-top-three. I am so happy to win my third solo win! 😊😊

Overview

My final solution consists of 4 XGBoost models (for two targets and two ls_consumption) and 2 GRU models (for two targets). These models share same 600 features.

model name	validation set	test set w/o online training	test set w/ update one time	test set w/ update three times
XGBoost	41.5912 (2 targets: 42.2531 / 42.8607)	55.6231	54.1343	54.1213

[1st Place Solution] My Betting Strategy

Foreword

Hello, this is yuuniee.

Thank you to the competition host Home Credit and staff, Kaggle staff and everyone involved.

The image below is a summary of my solution.

LGBM	DNN	CATBOOST
644 Features	661 Features	661 Features
CV : 0.707	CV : 0.699	CV : 0.706
Public LB : 0.58	Public LB : 0.56	Public LB : 0.59
0.4	0.24	0.36

1st place solution

Thanks to Optiver and Kaggle for hosting this great financial competition. And thanks to the great notebooks and discussions, I learned a lot. I am so happy to win my second solo win! 😊😊

Overview

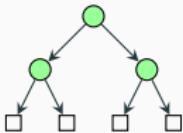
My final model [CV]/Private LB of 5.8117/5.4030 was a combination of CatBoost (5.8240/5.4165), GRU (5.8481/5.4259), and Transformer (5.8610/5.4795), with respective weights of 0.3, 0.3, 0.2 searched from validation set. And these models share same 300 features.

Besides, online learning(DL) and post-processing(PP) also play an important role in my final submission.

model name	validation set w/o PP	validation set w/ PP	test set w/o OL w/o PP	test set w/ OL one time w/o PP	test set w/ OL five times w/o PP
CatBoost	5.8287	5.8240	5.4523	5.4291	5.4165



Tree-based methods: Boosting



$$f_0 : \min_f \sum (y_i - f(x_i))^2$$



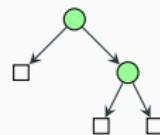
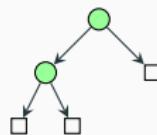
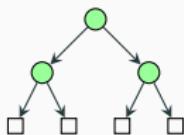
Tree-based methods: Boosting



$$f_0 : \min_f \sum (y_i - f(x_i))^2 \quad f_1 : \min_f \sum (y_i - f(x_i))^2$$



Tree-based methods: Boosting



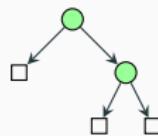
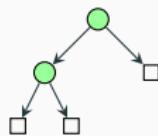
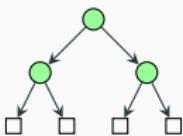
$$f_0 : \min_f \sum (y_i - f(x_i))^2$$

$$f_1 : \min_f \sum (y_i - f(x_i))^2$$

$$f_2 : \min_f \sum (y_i - f(x_i))^2$$



Tree-based methods: Boosting

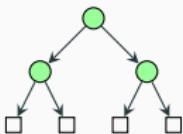


$$f_0 : \min_f \sum (y_i - f(x_i))^2 \quad f_1 : \min_f \sum (y_i - f(x_i))^2 \quad f_2 : \min_f \sum (y_i - f(x_i))^2$$

$$\hat{y} = \frac{1}{3} \sum_{i=0}^3 f_i(x)$$



Tree-based methods: Boosting



$$f_0 : \min_f \sum (y_i - f(x_i))^2$$

$$\hat{f}(x) = \lambda f_0(x)$$

$$y^* = y - (\hat{f}(x))$$



Tree-based methods: Boosting



$$f_0 : \min_f \sum (y_i - f(x_i))^2$$

$$\hat{f}(x) = \lambda f_0(x)$$

$$y* = y - (\hat{f}(x))$$

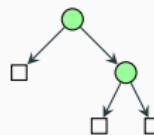
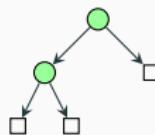
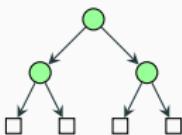
$$f_1 : \min_f \sum (y* - f(x_i))^2$$

$$\hat{f}(x) = \lambda(f_0(x) + f_1(x))$$

$$y* = y - (\hat{f}(x))$$



Tree-based methods: Boosting



$$f_0 : \min_f \sum (y_i - f(x_i))^2$$

$$\hat{f}(x) = \lambda f_0(x)$$

$$y^* = y - (\hat{f}(x))$$

$$f_1 : \min_f \sum (y_{*i} - f(x_i))^2$$

$$\hat{f}(x) = \lambda(f_0(x) + f_1(x))$$

$$y^* = y - (\hat{f}(x))$$

$$f_2 : \min_f \sum (y_{*i} - f(x_i))^2$$

$$\hat{f}(x) = \lambda(f_0(x) + f_1(x) + f_2(x))$$

$$y^* = y - (\hat{f}(x))$$



Tree-based methods: Boosting

Boosting: Fits a sequence of trees t_0, \dots, t_n . Each tree t_i tried to predict the residual of the preceding model, e.g. finding the signal that the model so far has missed.

- Works *really* well

$$f_0 : \min_f \sum (y_i - f(x_i))^2$$



Tree-based methods: Boosting



Tree-based methods: Boosting

<https://xgboost.readthedocs.io/en/stable/parameter.html#parameters-for-tree-booster>

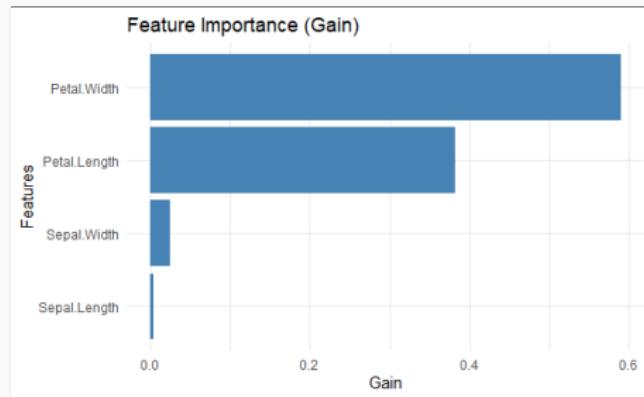


Tree-based methods: Boosting

https://xgboost.readthedocs.io/en/latest/python/examples/cross_validation.html



Tree-based methods: Boosting



https://xgboost.readthedocs.io/en/stable/python/python_api.html#module-xgboost.plotting



Exercise 5

1. Download the Credit.csv dataset from ISL
2. Preprocess the dataset as you see fit.
3. Fit one or more non-linear models, choosing model types and hyperparameters according to what you've learned
4. Report the performance of the best model
5. Reflect on the choices you made
6. Reflect upon the performance of the model(s)

