Russell's Paradox

Esten H. Leonardsen

10.02.25

UNIVERSITY OF OSLO

Background

Background

$$\{S | S \notin S\}$$

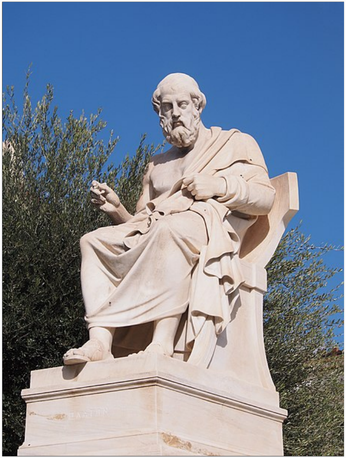
Background

Source: A guy I met at a party once

Background

Disclaimer: The contents of this talk will be approximately true

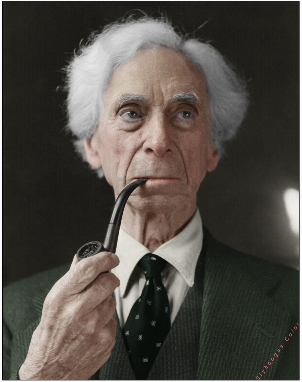
Historical underpinnings



Historical underpinnings

Historical underpinnings

Historical underpinnings



The project

$$0, 1, 2, 3, 4, \ldots$$

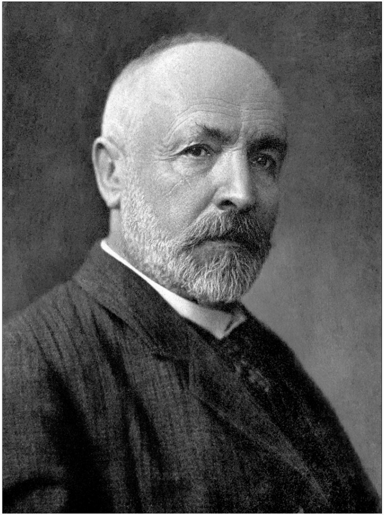
$$x + y = z$$
$$x - y = z$$
$$\ldots$$

The project

$$0, 1, 2, 3, 4, \ldots \qquad \Longrightarrow \qquad$$ 🤔

$$\begin{gathered} x + y = z \\ x - y = z \\ \ldots \end{gathered} \qquad \Longrightarrow \qquad$$ ✅

Set theory



Set theory

{🤓, 👩, 👧, 🐶}

{🤓, 😎, 🧐, 👩‍🔬, 👩‍💻, 👩‍🏫}

{monday, tuesday, wednesday, thursday, friday, saturday, sunday}

{💻, 📱}

Set theory

{🤓, 👩, 👧}

{⚽, 🏀, 🏐}

{tuesday, thursday, saturday}

...

Set theory

$$\{1, 3, 5\}$$

$$\{1, 10, 100\}$$

$$\{\}$$

Set theory

$$\{1, 2, 3, \ldots\}$$

Set theory

$$\{1, 2, 3, \ldots\}$$

$$\{x \mid x > 0\}$$

Set theory

$$\{1, 3, 5, \ldots\}$$

$$\{x \mid x \,\%\, 2 \neq 0\}$$

Set theory

{{🤓, 👩, 👧, 🐶}, {🤓, 😎, 🧐, 👩‍🔬, 👩‍💻, 👩‍🏫},
{monday, tuesday, wednesday, thursday, friday, saturday, sunday},
{💻, 📱}, $\{1, 3, 5\}$, $\{1, 10, 100\}$, $\{\}$, $\{x \mid x > 0\}$, $\{x \mid x \,\%\, 2 \neq 0\}$}

Set theory

{{🤓, 👩, 👧, 🐶}, {🤓, 😎, 🧐, 👩‍🔬, 👩‍💻, 👩‍🏫},
{monday, tuesday, wednesday, thursday, friday, saturday, sunday},
{💻, 📱}, $\{1, 3, 5\}$, $\{1, 10, 100\}$, $\{\}$, $\{x \mid x > 0\}$, $\{x \mid x \,\%\, 2 \neq 0\}$}

$$\{\{\}, \{0\}, \{1\}, \{0, 1\}, \ldots\}$$

Set theory

{{🤓, 👱‍♀️, 👧, 🐶}, {🤓, 😎, 🧐, 👩‍🔬, 👩‍💻, 👩‍🏫},
{monday, tuesday, wednesday, thursday, friday, saturday, sunday},
{💻, 📱}, $\{1, 3, 5\}$, $\{1, 10, 100\}$, $\{\}$, $\{x \mid x > 0\}$, $\{x \mid x \% 2 \neq 0\}$}

$$\{\{\}, \{0\}, \{1\}, \{0, 1\}, \dots\}$$

$$\{x \mid x \text{ is a set}\}$$

Set theory

{{🤓, 👩, 👧, 🐶}, {🤓, 😎, 🧐, 👩‍🔬, 👩‍💻, 👩‍🏫},
{monday, tuesday, wednesday, thursday, friday, saturday, sunday},
{💻, 📱}, $\{1, 3, 5\}$, $\{1, 10, 100\}$, $\{\}$, $\{x \mid x > 0\}$, $\{x \mid x \,\%\, 2 \neq 0\}$}

$$\{\{\}, \{0\}, \{1\}, \{0, 1\}, \ldots\}$$

$$S = \{x \mid x \text{ is a set}\}$$

Set theory

{{🤓, 👩, 👧, 🐶}, {🤓, 😎, 🧐, 👩‍🔬, 👩‍💻, 👩‍🏫},
{monday, tuesday, wednesday, thursday, friday, saturday, sunday},
{💻, 📱}, $\{1, 3, 5\}$, $\{1, 10, 100\}$, $\{\}$, $\{x \mid x > 0\}$, $\{x \mid x \,\%\, 2 \neq 0\}$}

$$\{\{\}, \{0\}, \{1\}, \{0, 1\}, \ldots\}$$

$$S = \{x \mid x \text{ is a set}\}$$

$$S \in S\ ?$$

Set theory

$$S = \{\{\}, \{0\}, \{1\}, \{0, 1\}, \ldots, S, \ldots\}$$

$$S \in S$$

Set theory

$$S = \{\{\}, \{0\}, \{1\}, \{0, 1\}, \ldots, S, \ldots\}$$

$$S \in S$$

$$S = \{1, 3, 5\}$$

$$S \notin S$$

Set theory

$$S = \{\{\}, \{0\}, \{1\}, \{0, 1\}, \ldots, S, \ldots\} \qquad S = \{1, 3, 5\}$$

$$S \in S \qquad S \notin S$$

$$\{S \mid S \in S\}$$

Set theory

$$S = \{\{\}, \{0\}, \{1\}, \{0, 1\}, \ldots, S, \ldots\} \qquad S = \{1, 3, 5\}$$

$$S \in S \qquad S \notin S$$

$$\{S \mid S \in S\}$$
$$\{S \mid S \notin S\}$$

Set theory

$$T = \{S \mid S \notin S\}$$

Set theory

$$T = \{S \mid S \notin S\}$$

$$T \in T\ ?$$

Set theory

$$T = \{S \mid S \notin S\}$$

$$T \in T\ ?$$

No ($T \notin T$)

$$T = \{S \mid S \notin S\}$$

$$T \in T\ ?$$

$$\swarrow$$

No ($T \notin T$)

$$\downarrow$$

Yes

Set theory

$$T = \{S \mid S \notin S\}$$

$$\begin{tikzcd}
 & T \in T \ ? \arrow[dl] \arrow[dr] & \\
\text{No } (T \notin T) \arrow[d] & & \text{Yes } (T \in T) \\
\text{Yes} & &
\end{tikzcd}$$

Set theory

$$T = \{S \mid S \notin S\}$$

$$\begin{tikzcd}[column sep=small]
& T \in T\ ? \arrow[dl] \arrow[dr] & \\
\text{No } (T \notin T) \arrow[d] & & \text{Yes } (T \in T) \arrow[d] \\
\text{Yes} & & \text{No}
\end{tikzcd}$$

Set theory

”Hardly anything more unfortunate can befall a scientific writer than to have one of the foundations of his edifice shaken after the work is finished. This was the position I was placed in by a letter of Mr. Bertrand Russell, just when the printing of this volume was nearing its completion.”

Gödels incompleteness theorem

Gödels incompleteness theorem

1. 0 is a natural number
2. If n is a natural number, $S(n)$ is a natural number
3. No numbers have 0 as their successor

$$0$$

$$1 = S(0)$$

$$2 = S(S(0))$$

$$\cdots$$

Gödels incompleteness theorem

1. 0 is a natural number
2. If n is a natural number, $S(n)$ is a natural number
3. No numbers have 0 as their successor
4. If $f(0)$ exists and $f(x+1)$ holds whenever $f(x)$ holds, f is a valid function

$$0$$
$$1 = S(0)$$
$$2 = S(S(0))$$
$$\cdots$$

$$x + 0 = x$$
$$x + S(y) = S(x + y)$$

Gödels incompleteness theorem

$$
\begin{aligned}
0 &:= 2 \\
S(0) &:= 12965404 \\
S(S(0)) &:= 24300749247875100
\end{aligned}
$$

Gödel number

Gödels incompleteness theorem

A valid gödel number

$\underset{\uparrow\ \text{A valid gödel number}}{x}$ is not provable

Gödels incompleteness theorem

A valid gödel number

x is not provable

x

The halting problem

The halting problem

Given a program P and an input i, determine if P halts on i

The halting problem

Given a program P and an input i, determine if P halts on i

```
print('Hello world')
```

↓

Halts

```
while True:
    print('Hello world')
```

↓

Does not halt

The halting problem

In[1]:
```
def decider(program, input):
    """Magical function that decides whether the program
    halts on the input"""
    ...
```

The halting problem

In[1]:

```python
def decider(program, input):
    """Magical function that decides whether the program
    halts on the input"""
    ...

def f(program):
    if decider(program, program):
       while True: pass
    else:
       return

f(f)
```

The halting problem

In[1]:

```python
def decider(program, input):
    """Magical function that decides whether the program
    halts on the input"""
    ...

def f(program):
    if decider(program, program):
       while True: pass
    else:
       return

f(f)
```

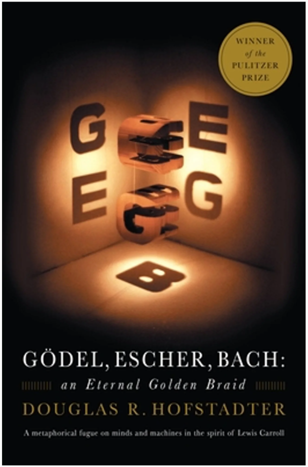
Summary

There are limits to what we can do with formal approaches

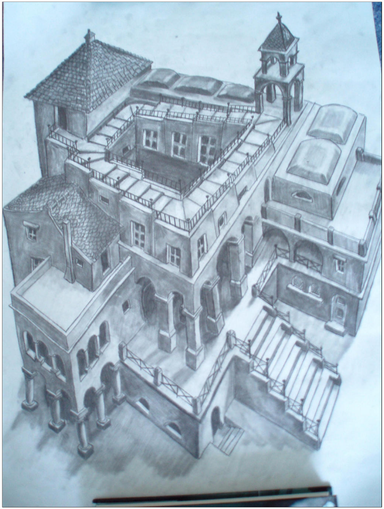
- These typically break down in the presence of self-references

Are we all strange loops?

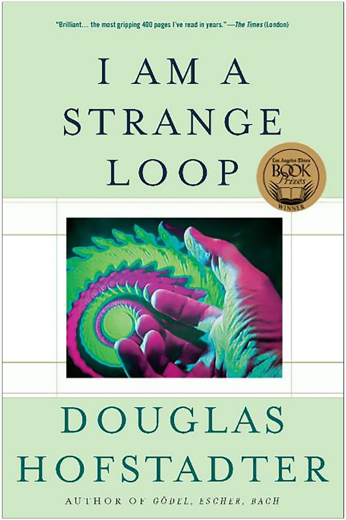
Are we all strange loops?



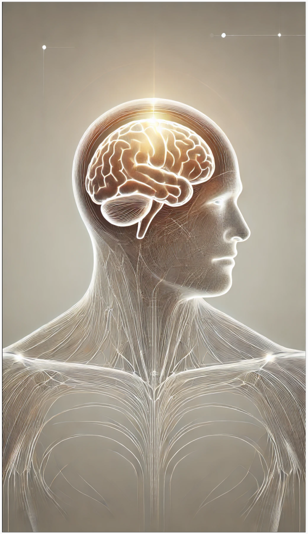
Are we all strange loops?



Are we all strange loops?



Are we all strange loops?



Are we all strange loops?