

PSY9511: Seminar 4

The basics of regression and classification

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Recap

What is statistical learning?



What is statistical learning?

Inferential view: Finding a function $\hat{f}(X)$ that describes the relationship between some input variables X and an output variable y .



Recap

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Inferential view: Finding a function $\hat{f}(X)$ that describes the relationship between some input variables X and an output variable y .

Predictive view: Finding a function $\hat{f}(X)$ that, when given a new set of inputs X allows us to predict an output y .



Recap

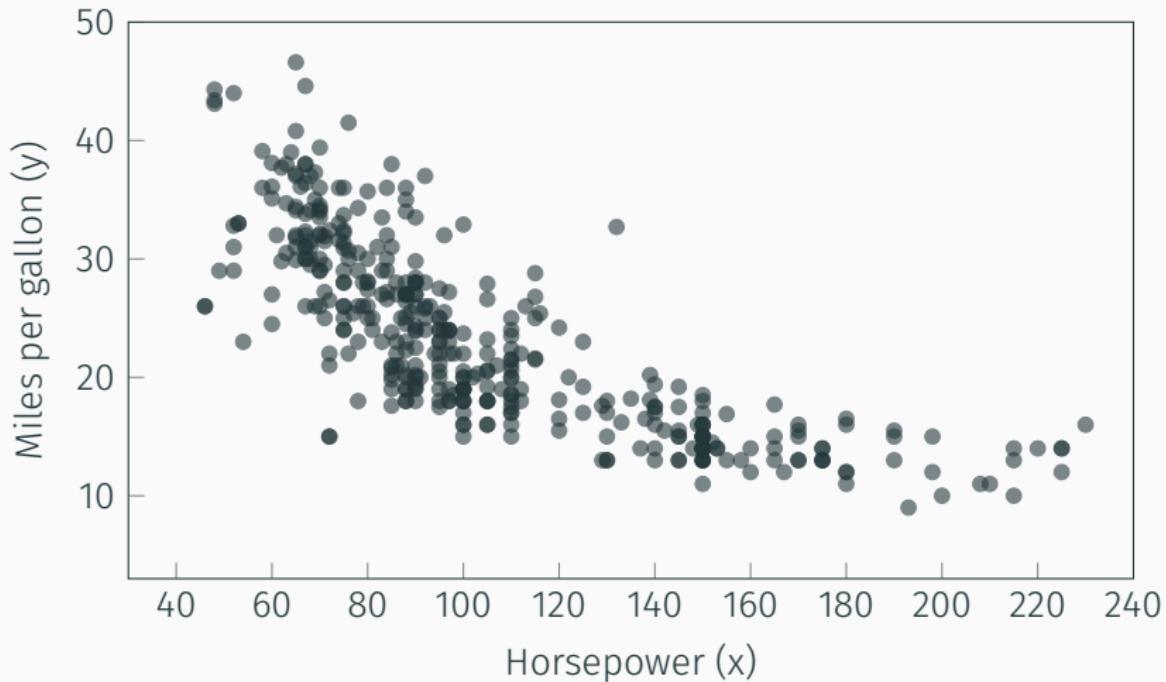
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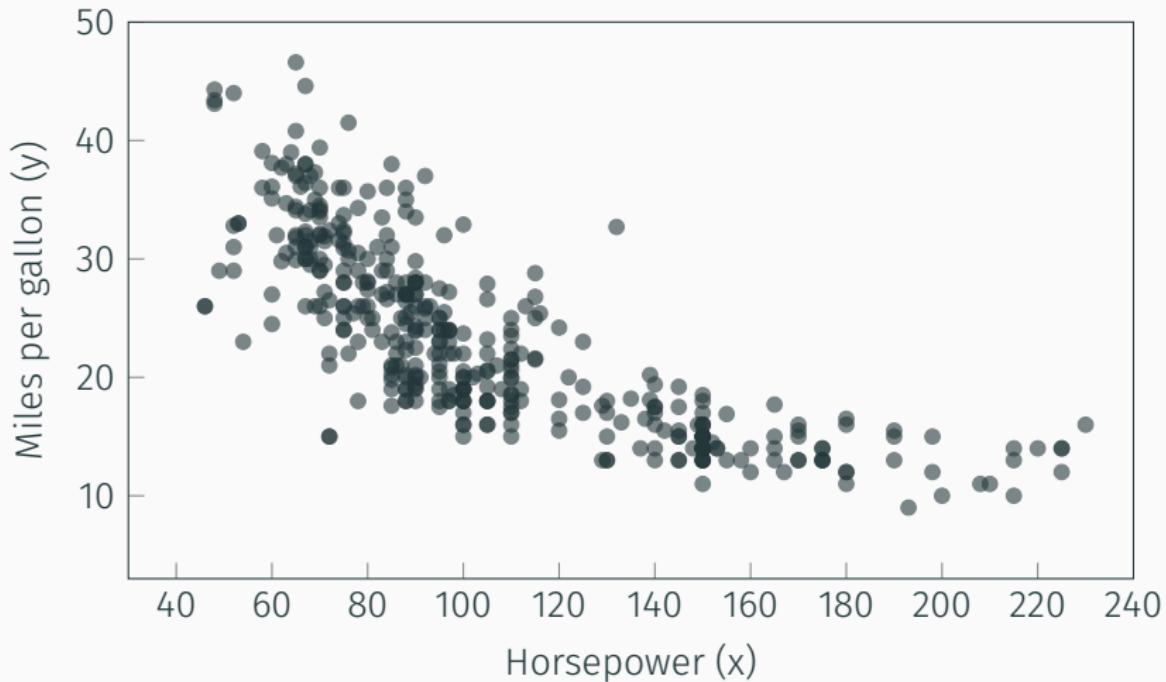
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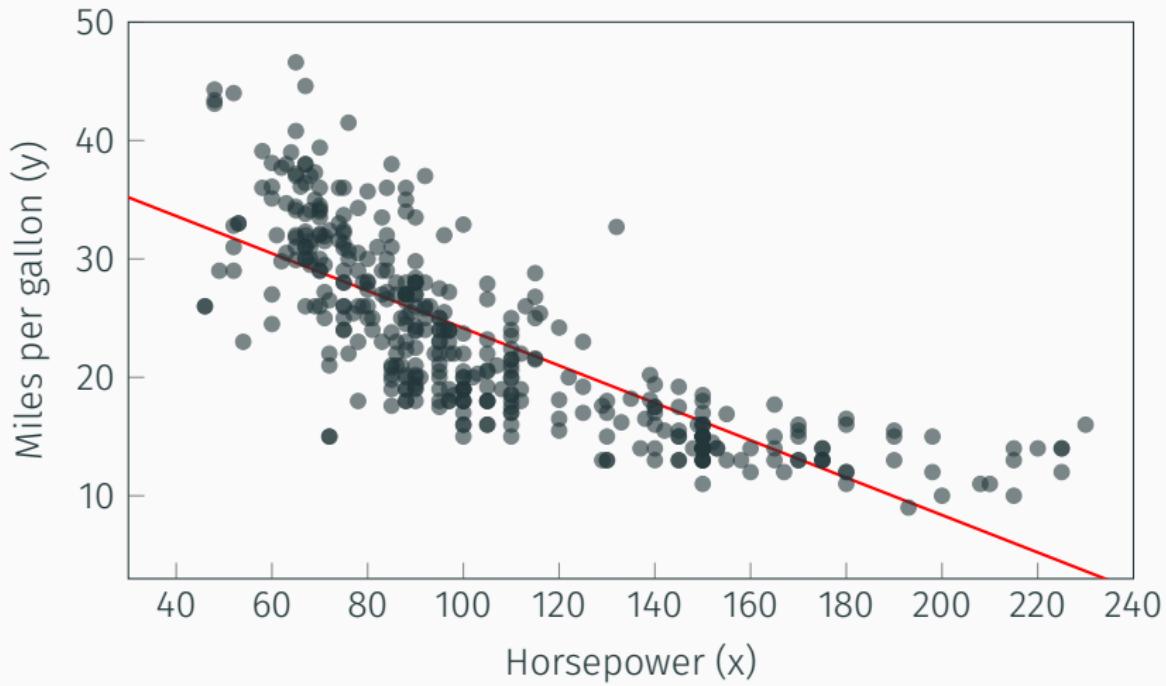
Recap



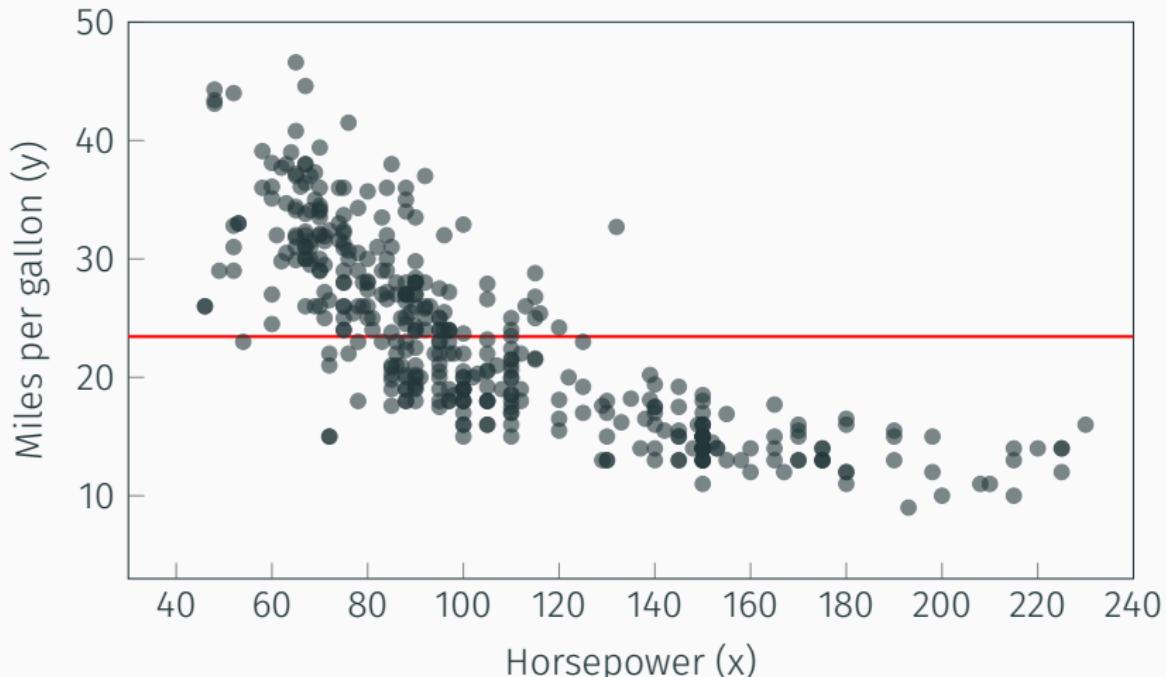
$$\hat{y} = \hat{f}(x)$$



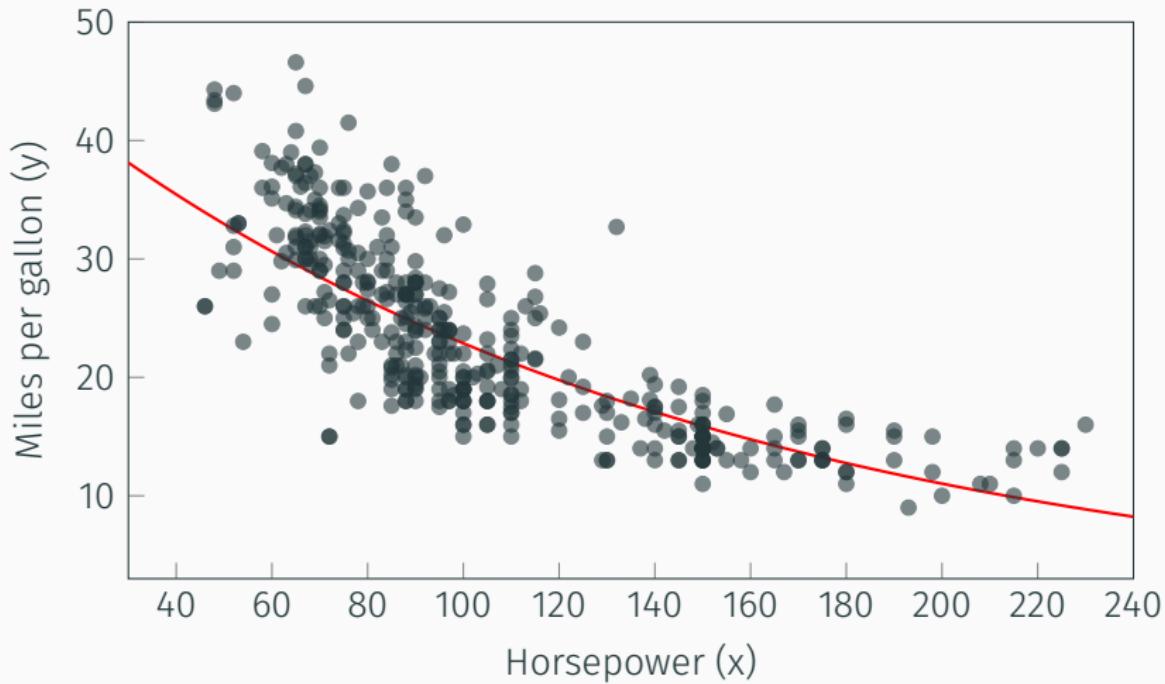
Recap



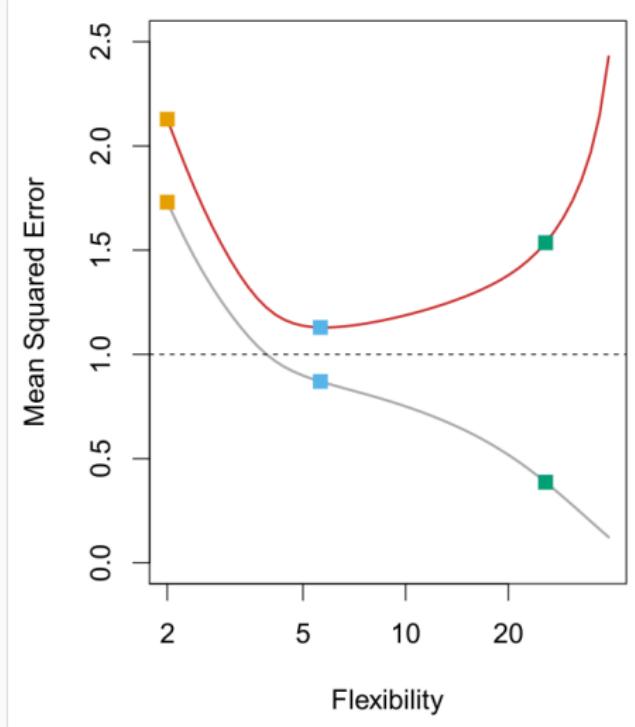
Recap



Recap



Recap



Outline

Plan for the day:

- Different types of outputs y : Regression vs classification
- Linear regression: Restricting the scope of $\hat{f}(X)$
 - Live coding
- k-Nearest Neighbours
- Logistic regression: Extending linear regression to classification
 - Live coding
- Generative models

Plan for future lectures:

- How do we evaluate how good our models are? (Lecture 3)
- Complex solutions to regression and classification problems (Lecture 4 and onwards)



Regression vs. classification

Weight	Manufacturer
3504	Chevrolet
3693	Ford
3436	Pontiac
3433	Pontiac
3449	Ford
4341	Ford
4354	Chevrolet
4312	Ford
4425	Pontiac
3850	Chevrolet



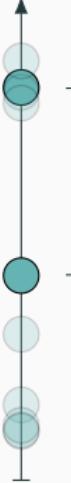
Regression vs. classification



Weight	Manufacturer
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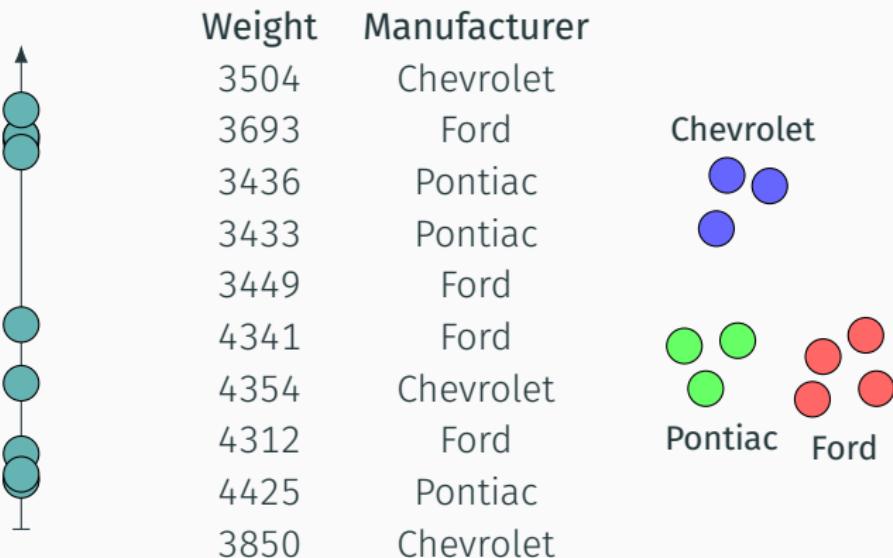
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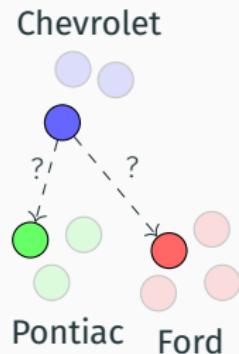
Regression vs. classification



Regression vs. classification



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Regression vs. classification

Mean squared error (MSE):

$$\frac{1}{n} \sum_{i=1}^n (y_i - \hat{y}_i)^2$$

Accuracy

$$\frac{1}{n} \sum_{i=1}^n \mathbb{1}_{eq}(y_i, \hat{y}_i),$$

$$\mathbb{1}_{eq}(a, b) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}$$



Regression vs. classification

Regression:

- Predicting reaction time on a cognitive task based on sleep scores
- Predicting the age of an individual based on a brain scan
- Predicting the anxiety scores based on questionnaire data

Classification:

- Predicting whether an individual is depressed based on cell phone usage data.
- Predicting if a patient has dementia based on a brain scan
- Predicting whether a patient is happy based on their facial expression



Regression vs. classification

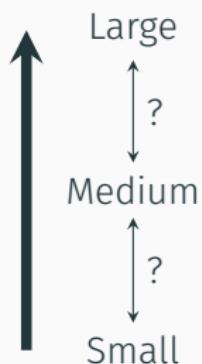
Large

Medium

Small



Regression vs. classification



Regression vs. classification

The quick brown fox jumps over the lazy



Regression vs. classification

The quick brown fox jumps over the lazy _____



Regression vs. classification

"Students taking
a machine learning
class"

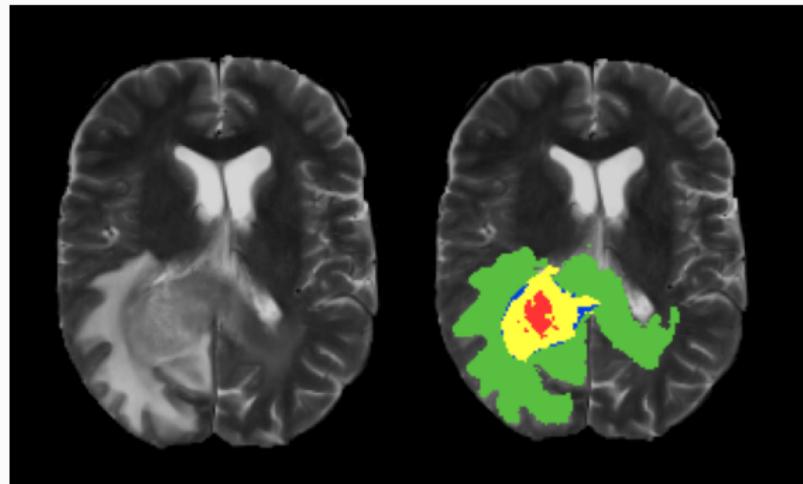


Regression vs. classification

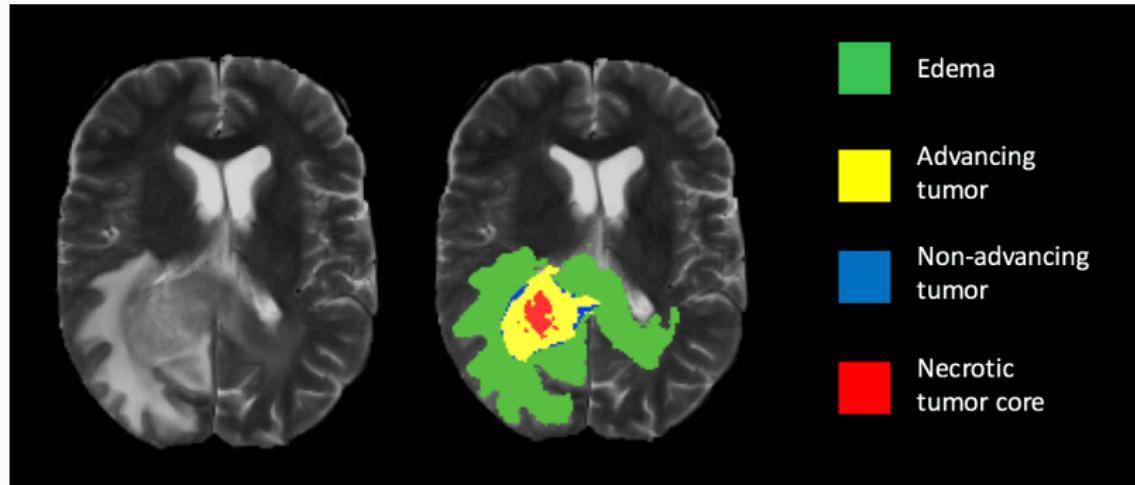
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Regression vs. classification



Regression vs. classification



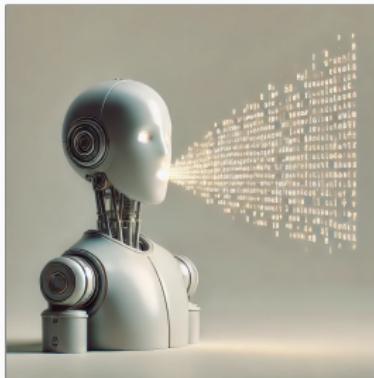
Regression vs. classification



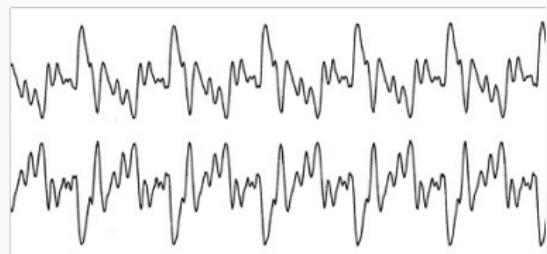
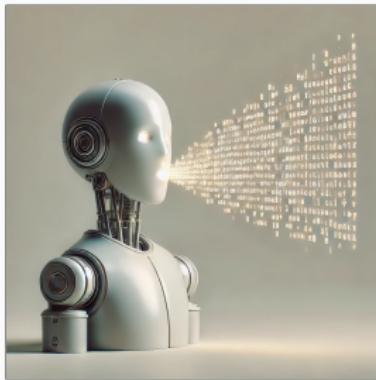
Regression vs. classification



Regression vs. classification



Regression vs. classification



Regression vs. classification

Different types of outputs y require us to use different mathematical formulations of the problem we want to solve.

- Problems with quantitative outputs are solved via regression, often by minimizing the mean squared error.
- Problems with qualitative outputs are solved by classification, often by maximizing accuracy.
- Ordinal regression falls between the two, with qualitative classes that have some kind of order.
- A variety of other types of problems can be seen as special cases of these two.

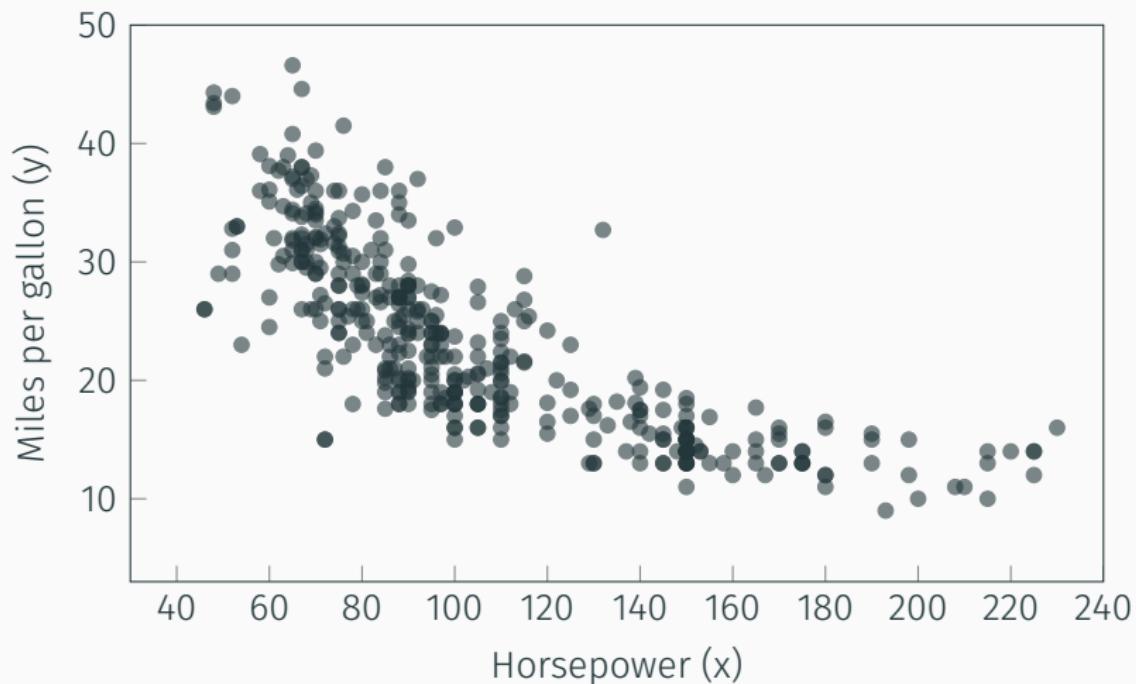


Linear regression (via ordinary least squares)

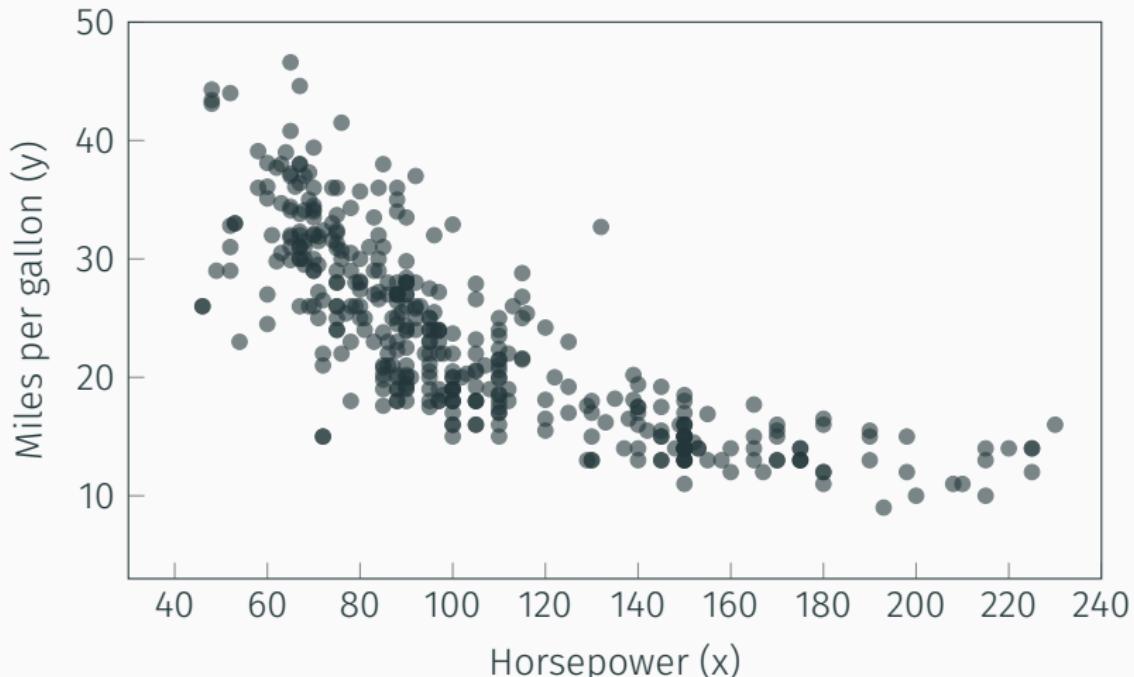


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Linear regression (via ordinary least squares)



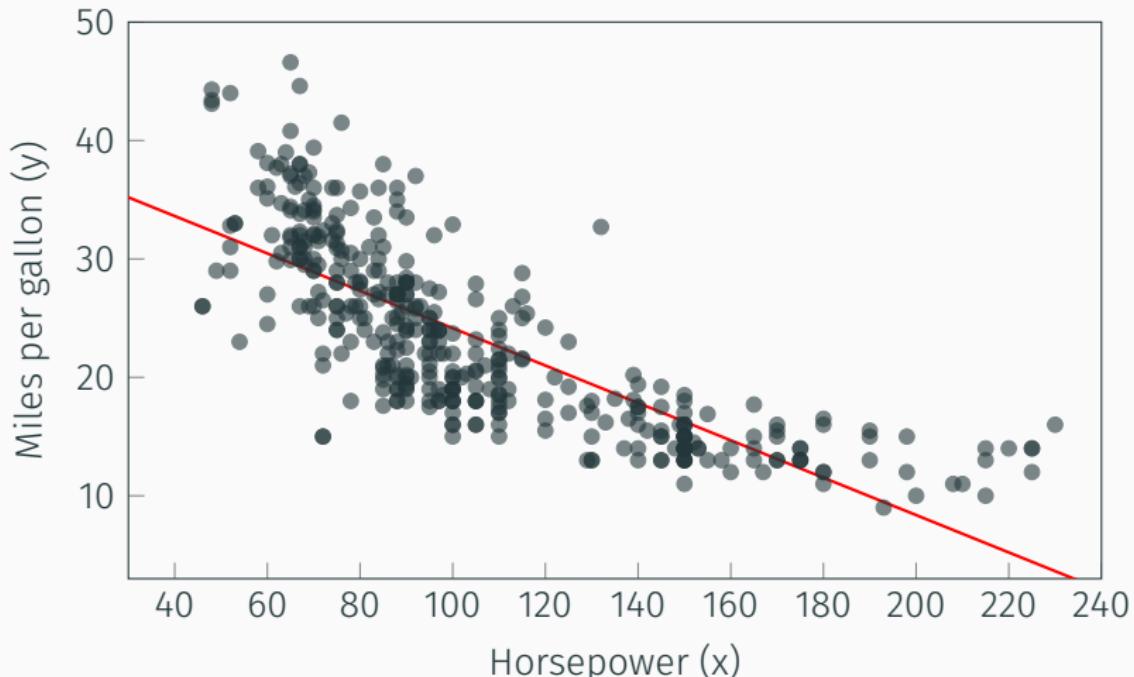
Linear regression (via ordinary least squares)



$$\hat{y} = \beta_0 - \beta_1 x$$



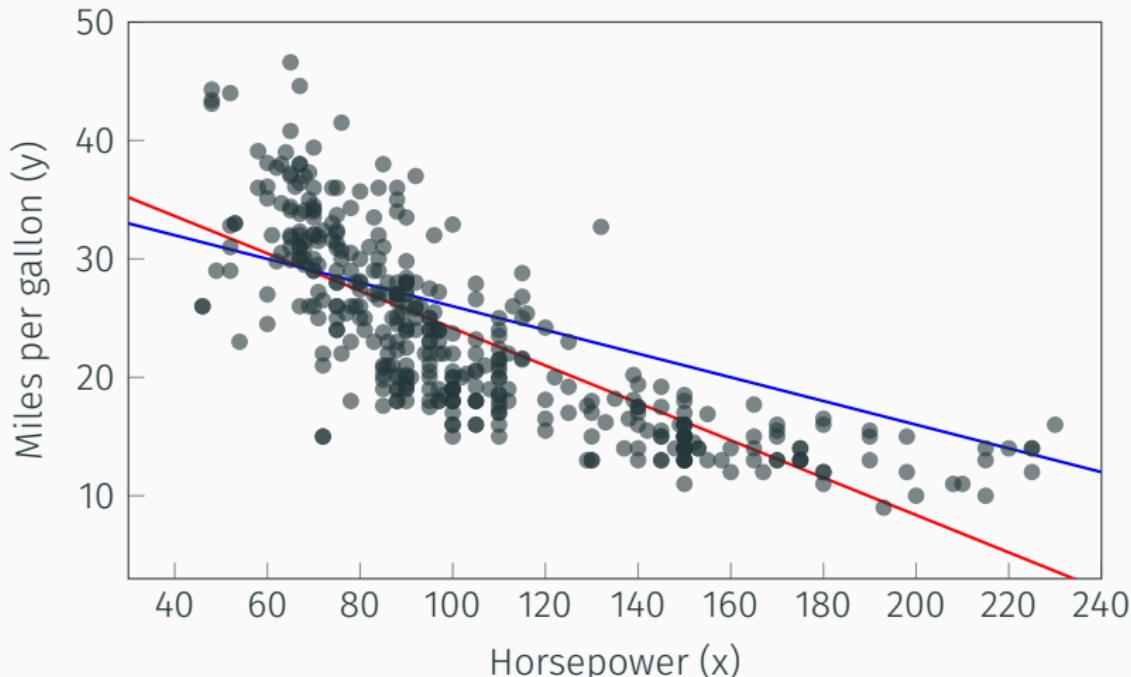
Linear regression (via ordinary least squares)



$$\hat{y} = 39.93 - 0.1578x$$



Linear regression (via ordinary least squares)

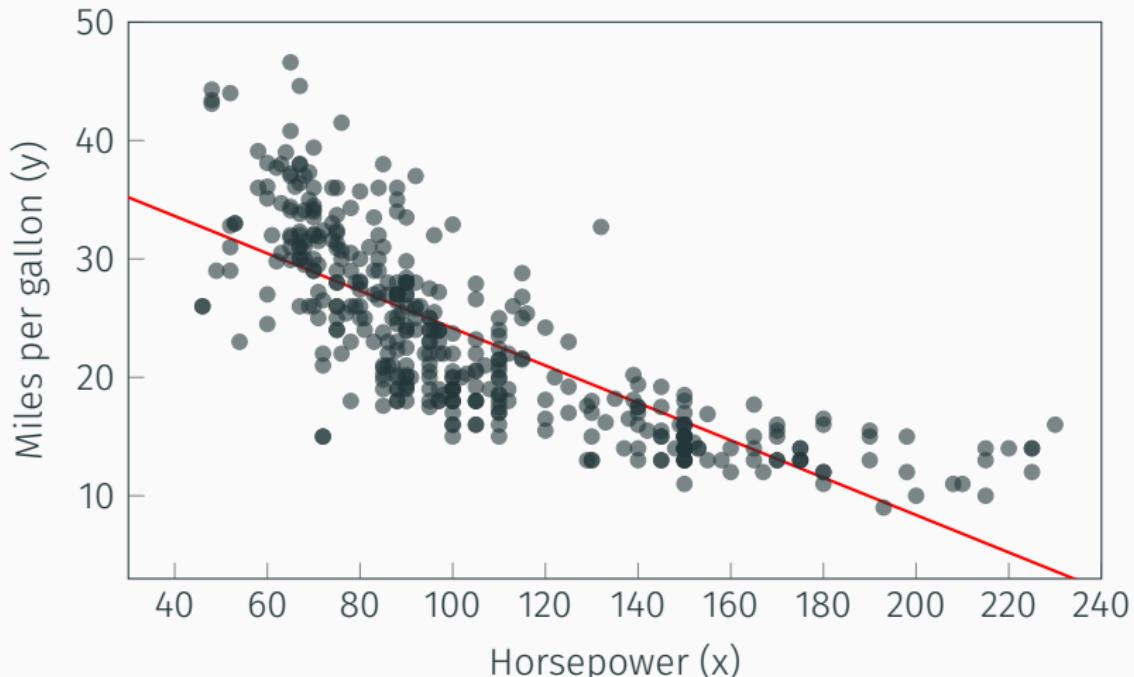


$$\hat{y} = 39.93 - 0.1578x$$

$$\hat{y} = 36 - 0.1x$$



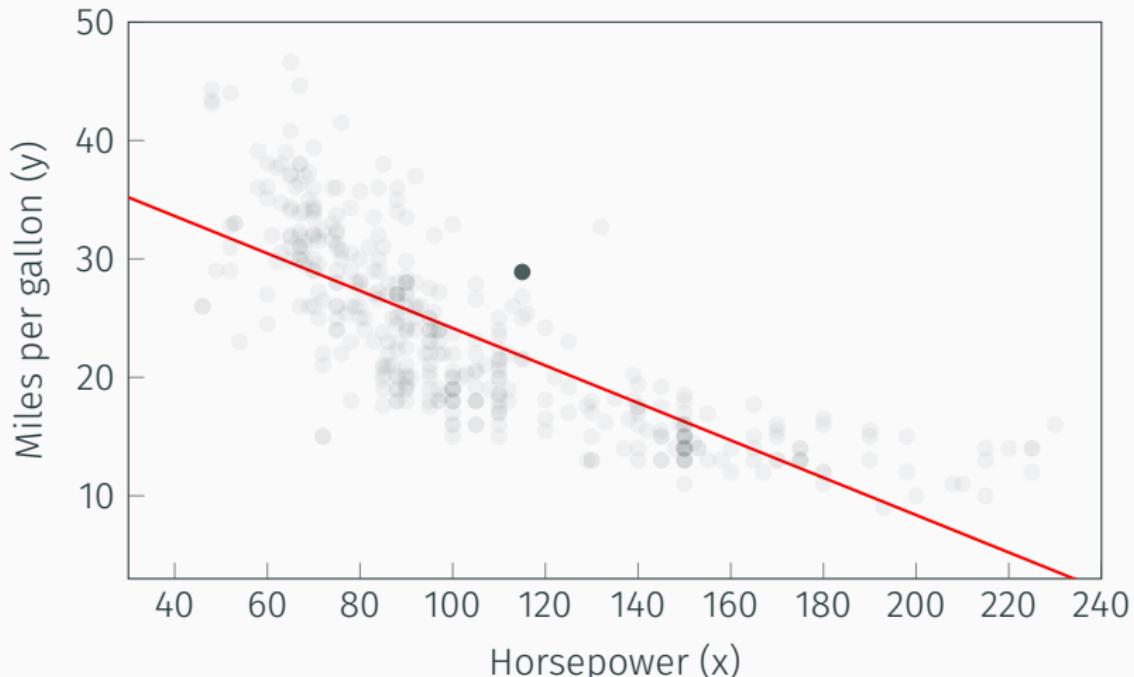
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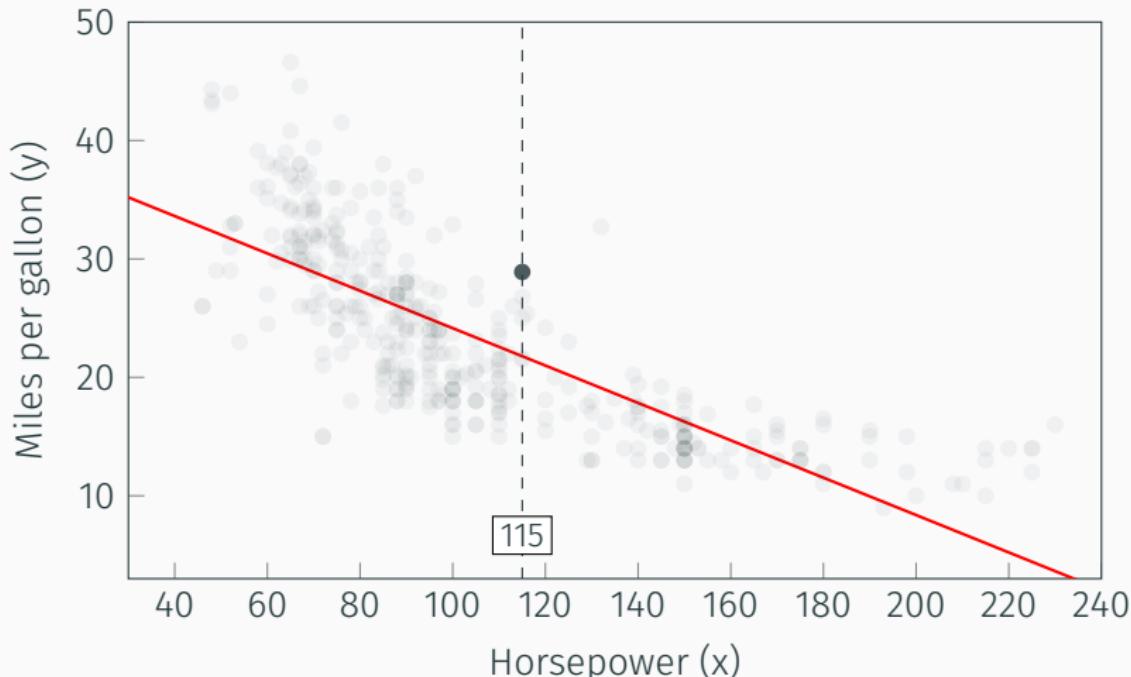
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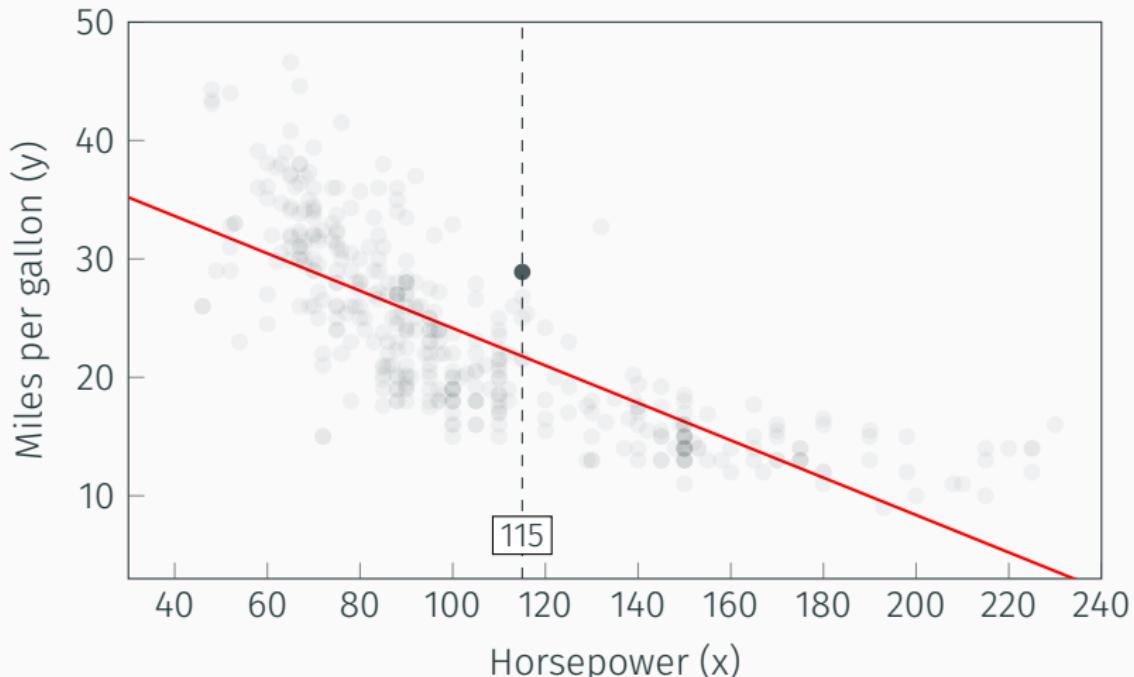
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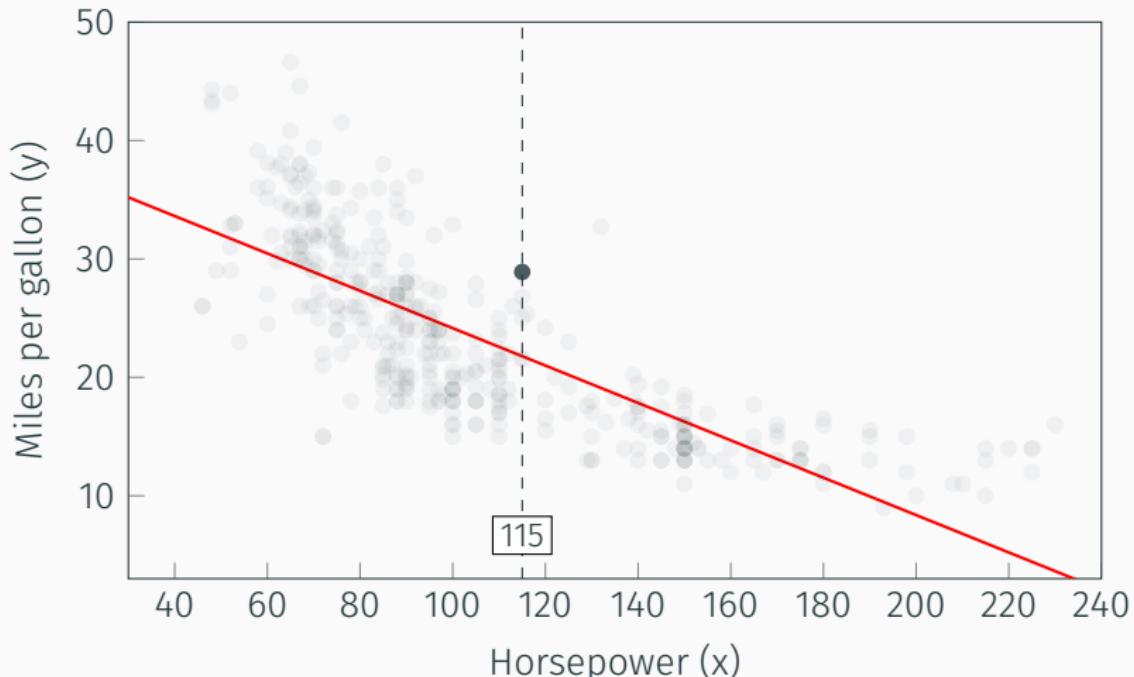
Linear regression (via ordinary least squares)



$$\hat{y} = 39.93 - 0.1578 * 115$$



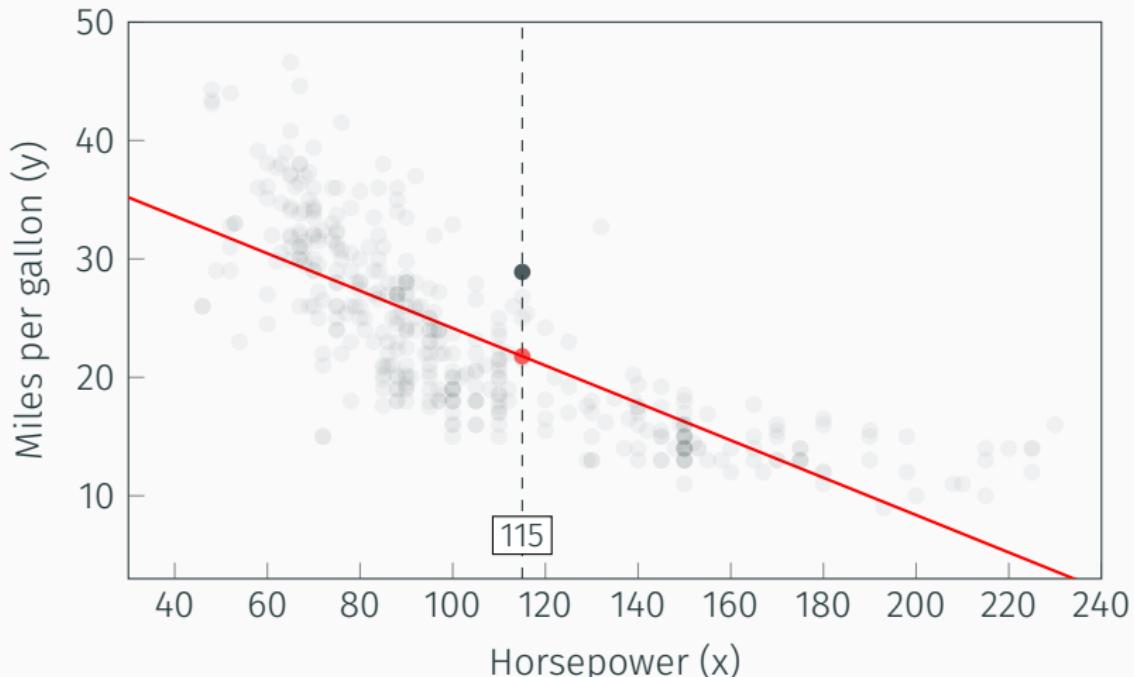
Linear regression (via ordinary least squares)



$$21.78 = 39.93 - 0.1578 * 115$$



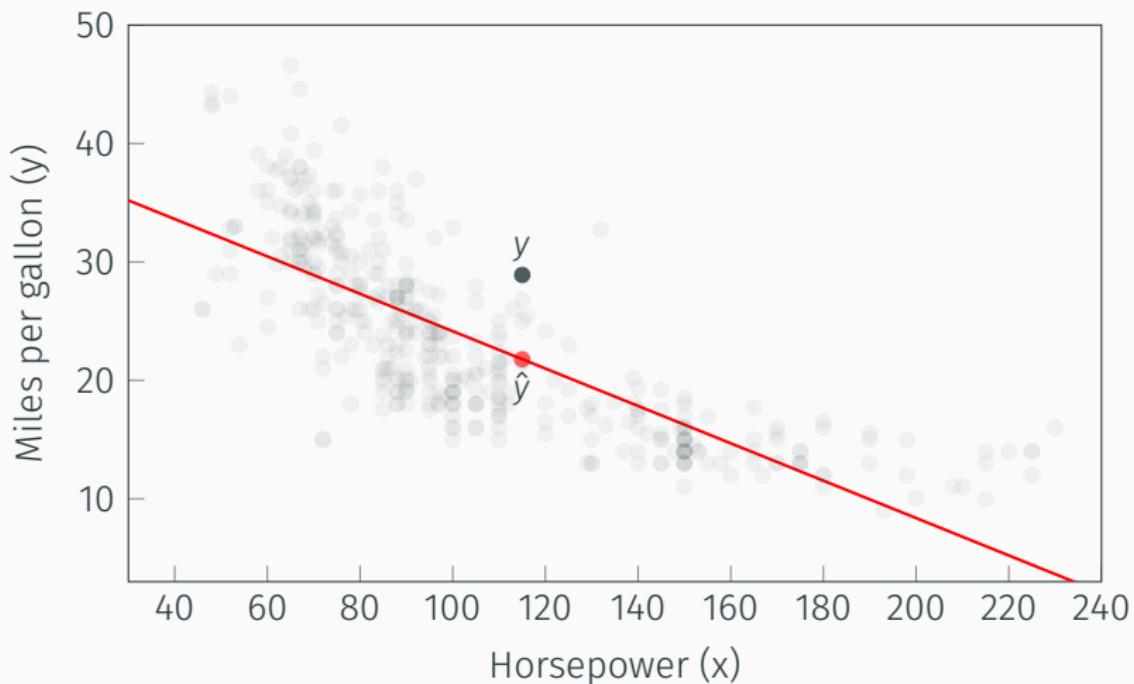
Linear regression (via ordinary least squares)



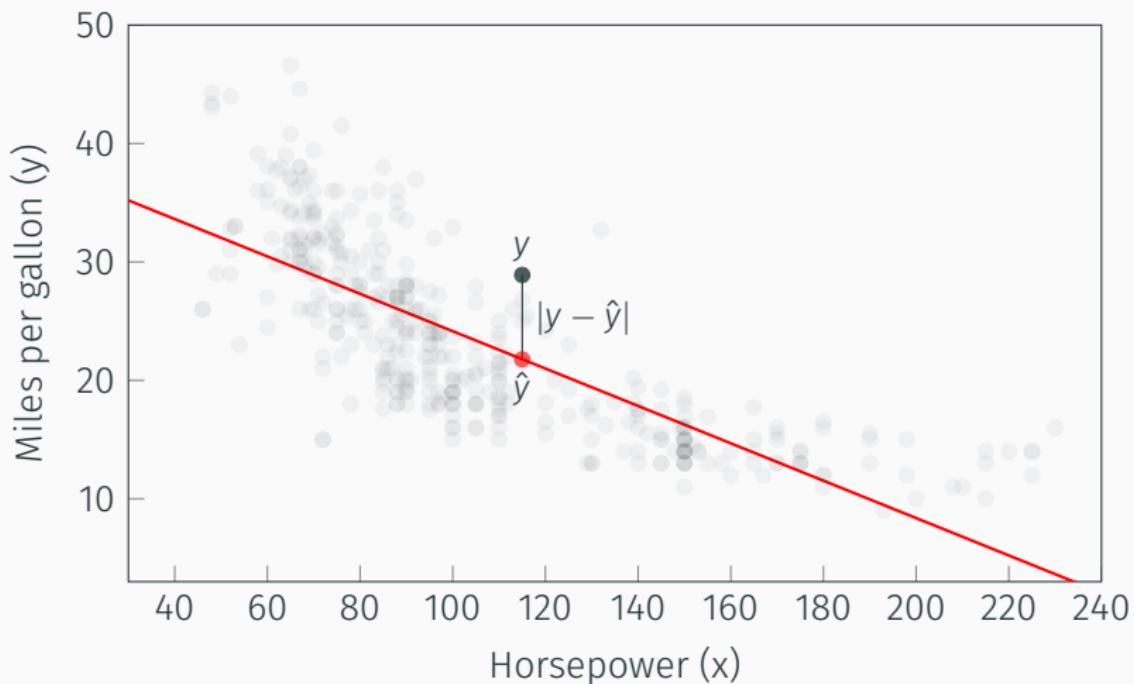
$$21.78 = 39.93 - 0.1578 * 115$$



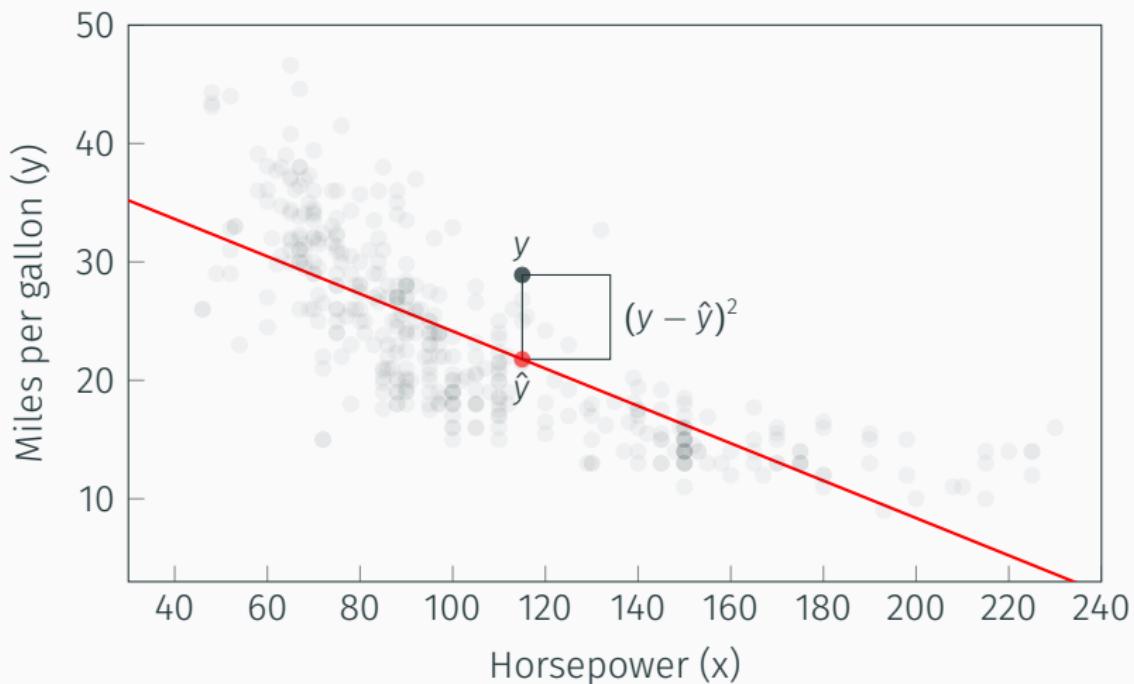
Linear regression (via ordinary least squares)



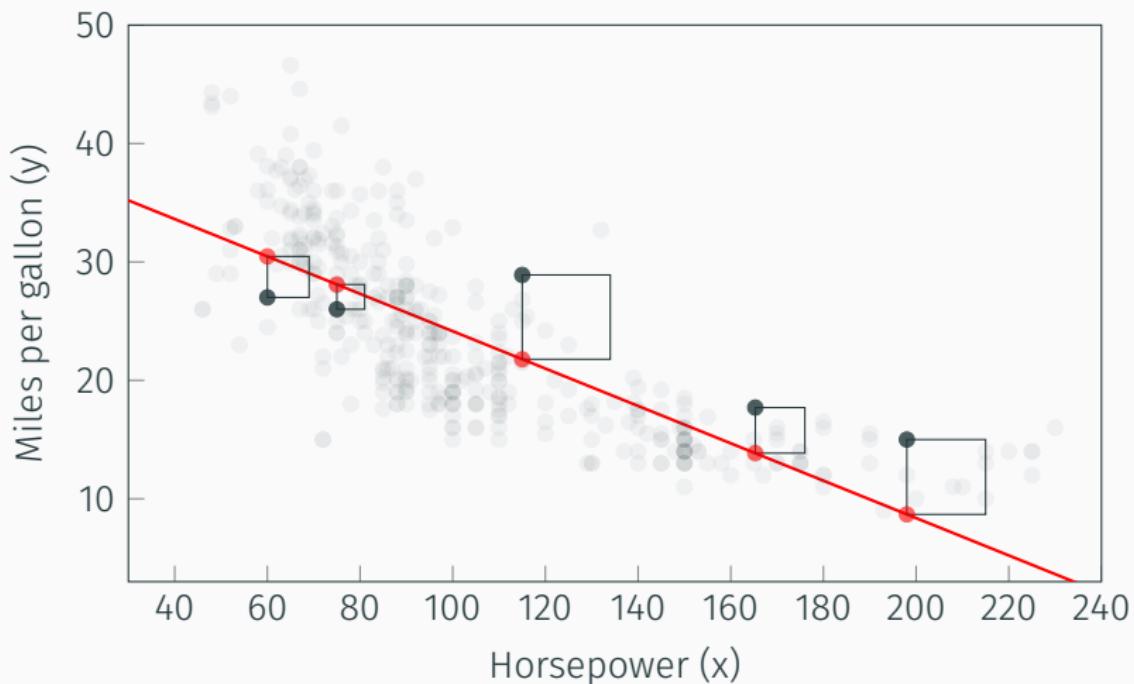
Linear regression (via ordinary least squares)



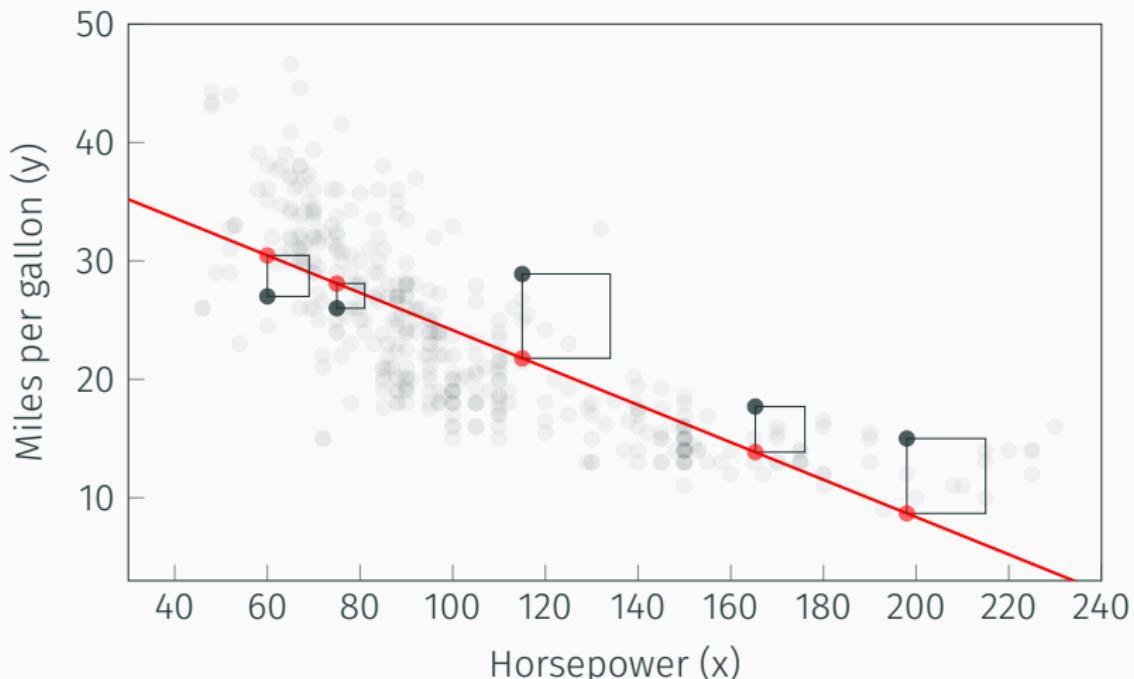
Linear regression (via ordinary least squares)



Linear regression (via ordinary least squares)



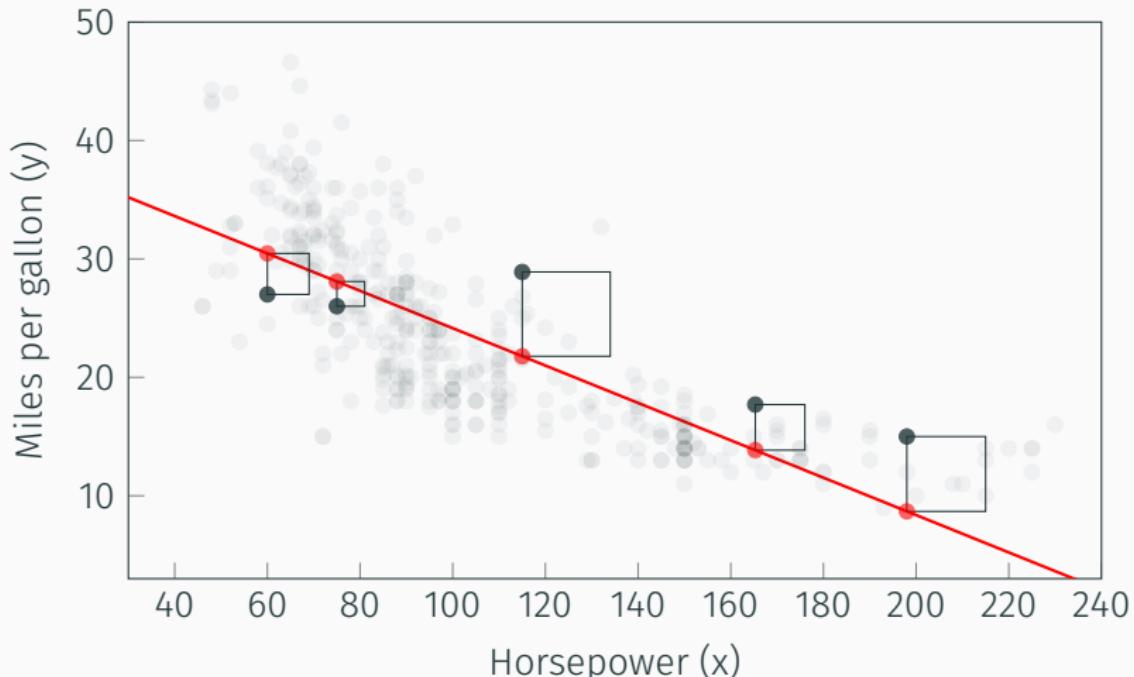
Linear regression (via ordinary least squares)



$$MSE = \frac{1}{n} \sum_{i=1}^n (y_i - \hat{y}_i)^2$$



Linear regression (via ordinary least squares)



$$RSS = \sum_{i=1}^n (y_i - \hat{y}_i)^2$$



Linear regression (via ordinary least squares)

$$\hat{y} = \beta_0 + \beta_1 x$$

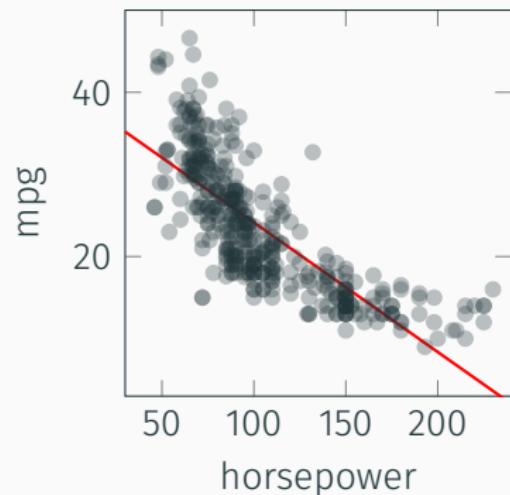


Linear regression (via ordinary least squares)

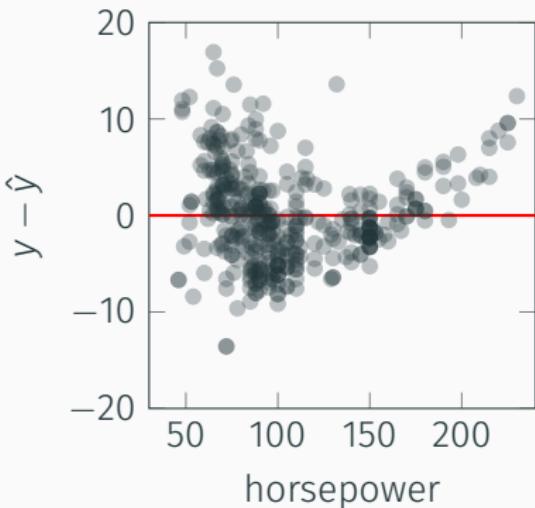
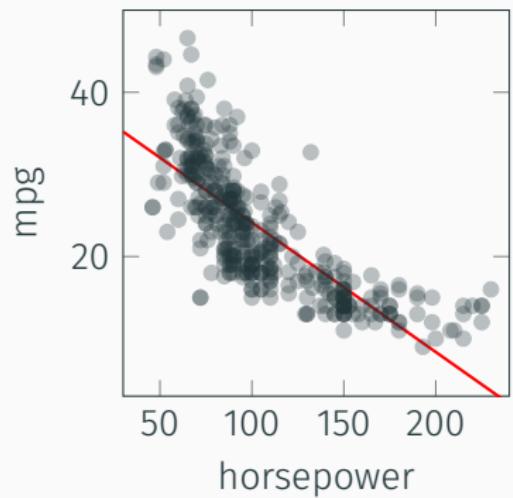
$$\hat{y} = \beta_0 + \beta_1 x$$



Linear regression (via ordinary least squares)



Linear regression (via ordinary least squares)



Linear regression (via ordinary least squares)

$$\hat{y} = \beta_0 + \beta_1 x$$



Linear regression (via ordinary least squares)

$$\hat{y} = \beta_0 + \beta_1 x$$

$$MSE = \frac{1}{n} \sum_{i=1}^n (y_i - \hat{y}_i)^2$$



Linear regression (via ordinary least squares)

$$MSE = \frac{1}{n} \sum_{i=1}^n (y_i - \beta_0 + \beta_1 x_i)^2$$



Linear regression (via ordinary least squares)

$$\hat{y} = \beta_0 + \beta_1 x$$

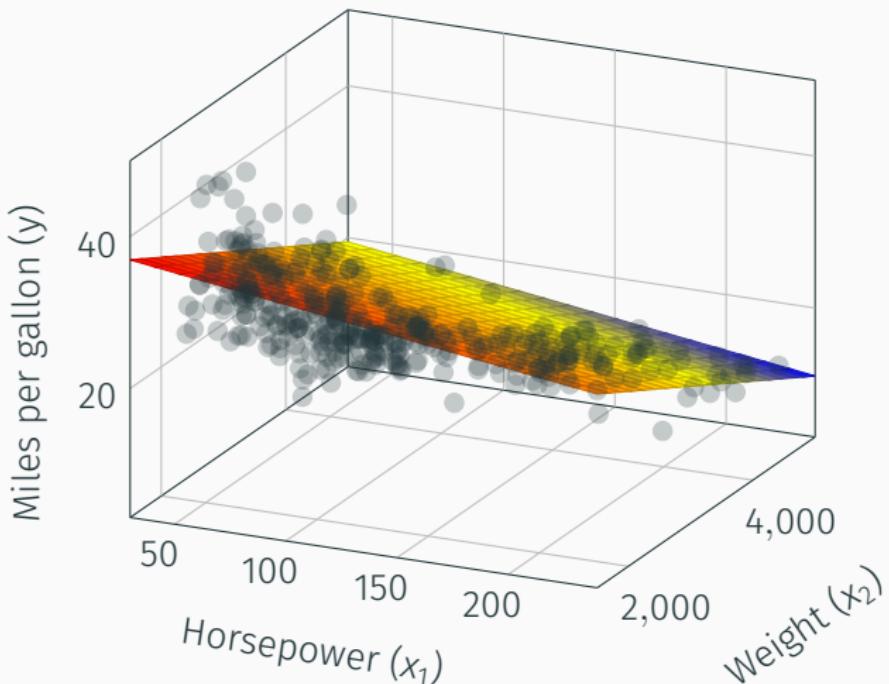


Linear regression (via ordinary least squares)

$$\hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_p x_p$$



Linear regression (via ordinary least squares)



Linear regression (via ordinary least squares)

mpg	manufacturer
36	Chevrolet
15	Ford
25	Chevrolet
26	Chevrolet
17	Ford
15	Ford
32	Chevrolet
14	Ford
14	Ford
28	Chevrolet

$$\widehat{\text{mpg}} = \beta_0 + \beta_1 \times \text{manufacturer}$$



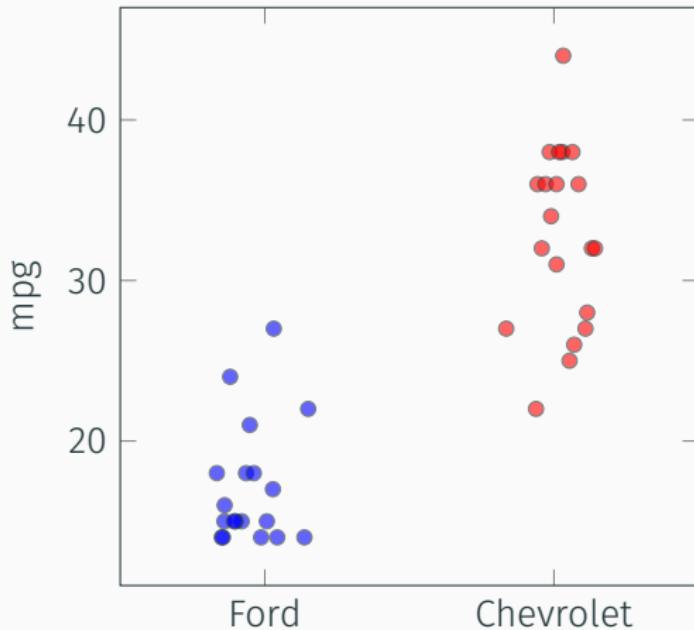
Linear regression (via ordinary least squares)

mpg	manufacturer	chevrolet
36	Chevrolet	1
15	Ford	0
25	Chevrolet	1
26	Chevrolet	1
17	Ford	0
15	Ford	0
32	Chevrolet	1
14	Ford	0
14	Ford	0
28	Chevrolet	1

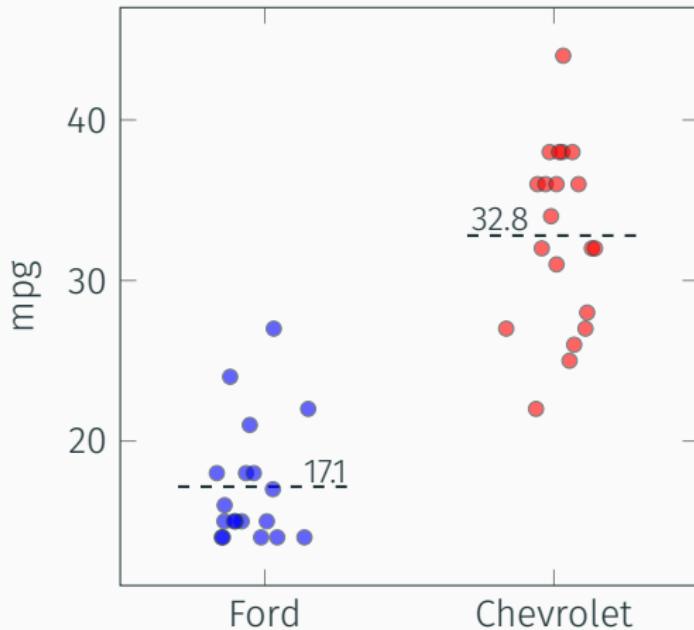
$$\widehat{\text{mpg}} = \beta_0 + \beta_1 \times \text{chevrolet}$$



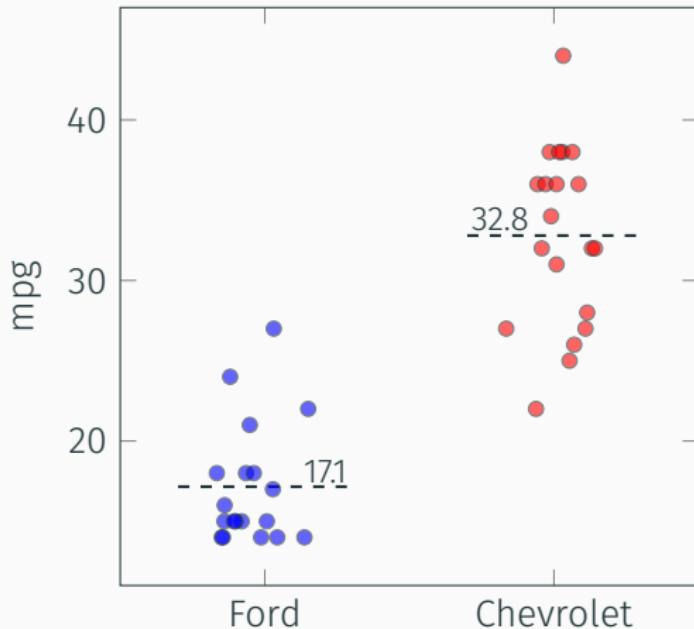
Linear regression (via ordinary least squares)



Linear regression (via ordinary least squares)



Linear regression (via ordinary least squares)



Blackboard!



Linear regression (via ordinary least squares)

mpg	manufacturer
36	Chevrolet
15	Ford
25	Chevrolet
26	Pontiac
17	Ford
15	Ford
32	Pontiac
14	Ford
14	Pontiac
28	Chevrolet

$$\widehat{\text{mpg}} = \beta_0 + \beta_1 \times \text{manufacturer}$$



Linear regression (via ordinary least squares)

mpg	manufacturer	chevrolet	pontiac
36	Chevrolet	1	0
15	Ford	0	0
25	Chevrolet	1	0
26	Pontiac	0	1
17	Ford	0	0
15	Ford	0	0
32	Pontiac	0	1
14	Ford	0	0
14	Pontiac	0	1
28	Chevrolet	1	0

$$\widehat{\text{mpg}} = \beta_0 + \beta_1 \times \text{chevrolet} + \beta_2 \times \text{pontiac}$$



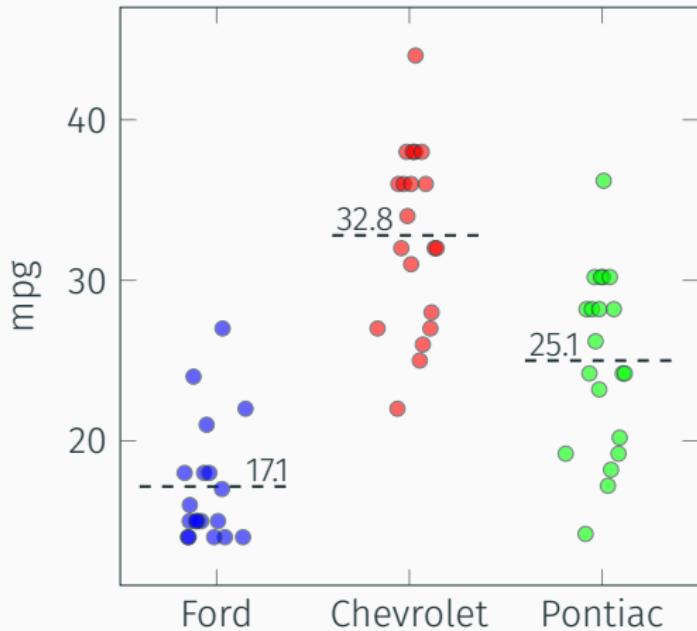
Linear regression (via ordinary least squares)

```
In[1]: import pandas as pd  
  
df = pd.DataFrame(...)  
print(f'Columns before: {df.columns.values}')  
df = pd.get_dummies(df)  
print(f'Columns after: {df.columns.values}')
```

```
Out[1]: Columns before: ['manufacturer']  
Columns after: ['manufacturer_chevrolet' 'manufacturer_ford']
```



Linear regression (via ordinary least squares)



Linear regression (via ordinary least squares)

mpg	chevrolet	horsepower
36	1	130
15	0	165
25	1	150
26	1	150
17	0	140
15	0	198
32	1	220
14	0	215
14	0	225
28	1	212

$$\widehat{\text{mpg}} = \beta_0 + \beta_1 \times \text{chevrolet} + \beta_2 \times \text{horsepower}$$



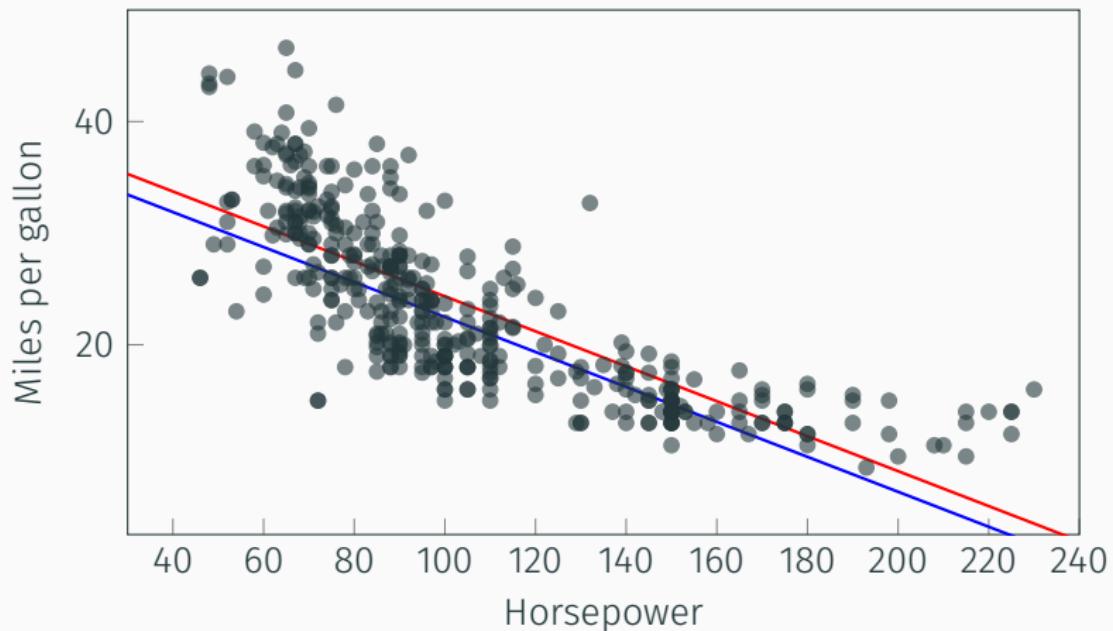
Linear regression (via ordinary least squares)

mpg	chevrolet	horsepower
36	1	130
15	0	165
25	1	150
26	1	150
17	0	140
15	0	198
32	1	220
14	0	215
14	0	225
28	1	212

$$\widehat{mpg} = \begin{cases} \beta_0 + \beta_1 + \beta_2 \times \text{horsepower} & \text{if chevrolet} \\ \beta_0 + \beta_2 \times \text{horsepower} & \text{else} \end{cases}$$



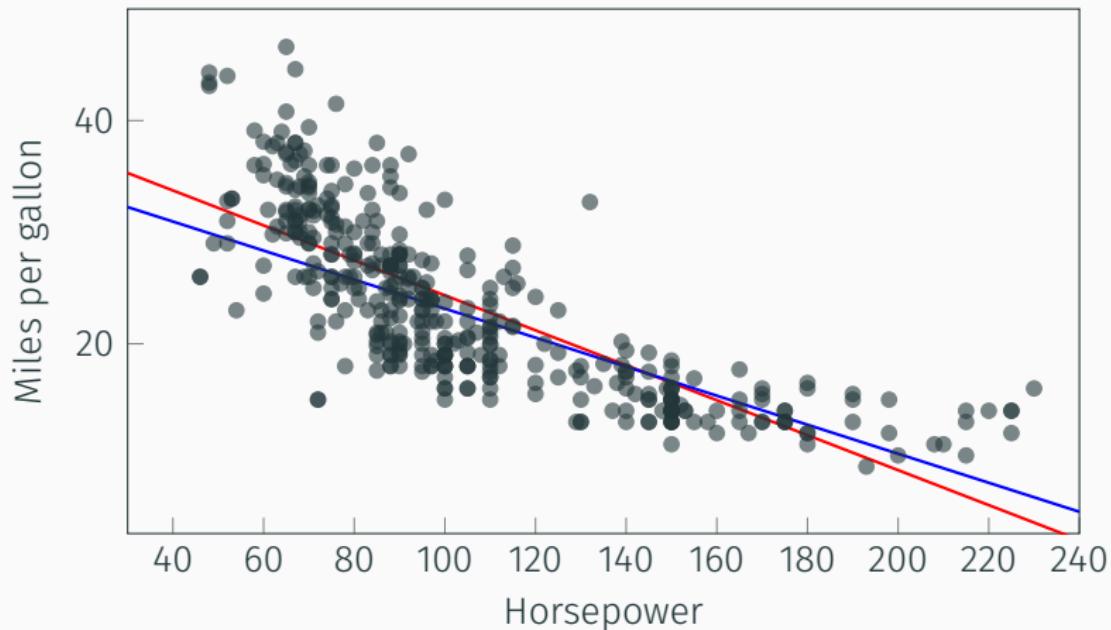
Linear regression (via ordinary least squares)



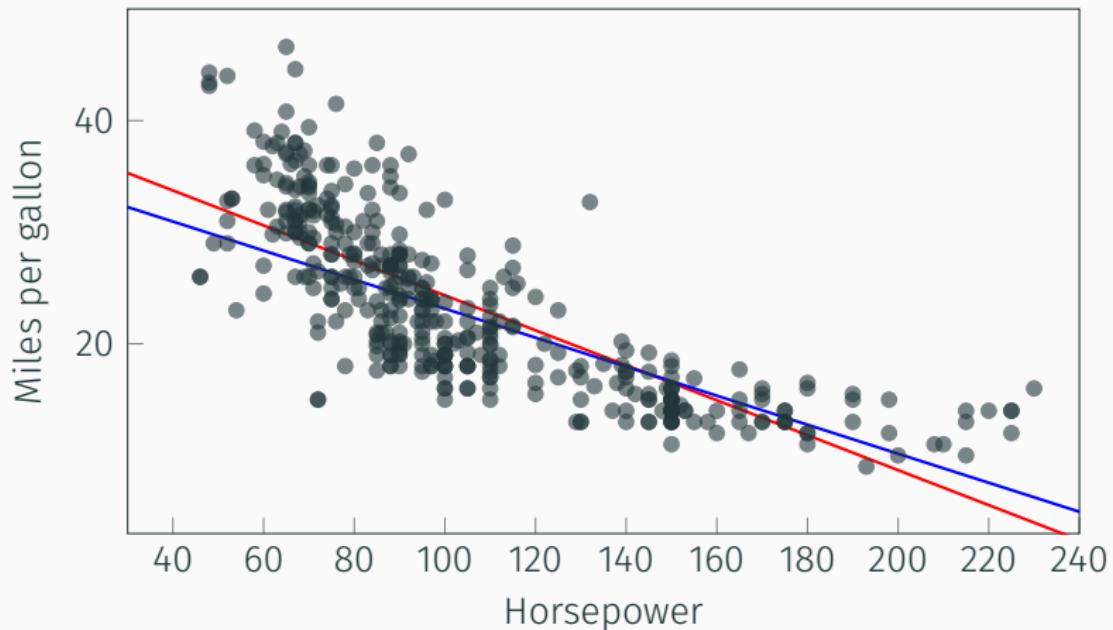
$$\widehat{mpg} = \begin{cases} \beta_0 + \beta_1 + \beta_2 \times \text{horsepower} & \text{if chevrolet} \\ \beta_0 + \beta_2 \times \text{horsepower} & \text{else} \end{cases}$$



Linear regression (via ordinary least squares)



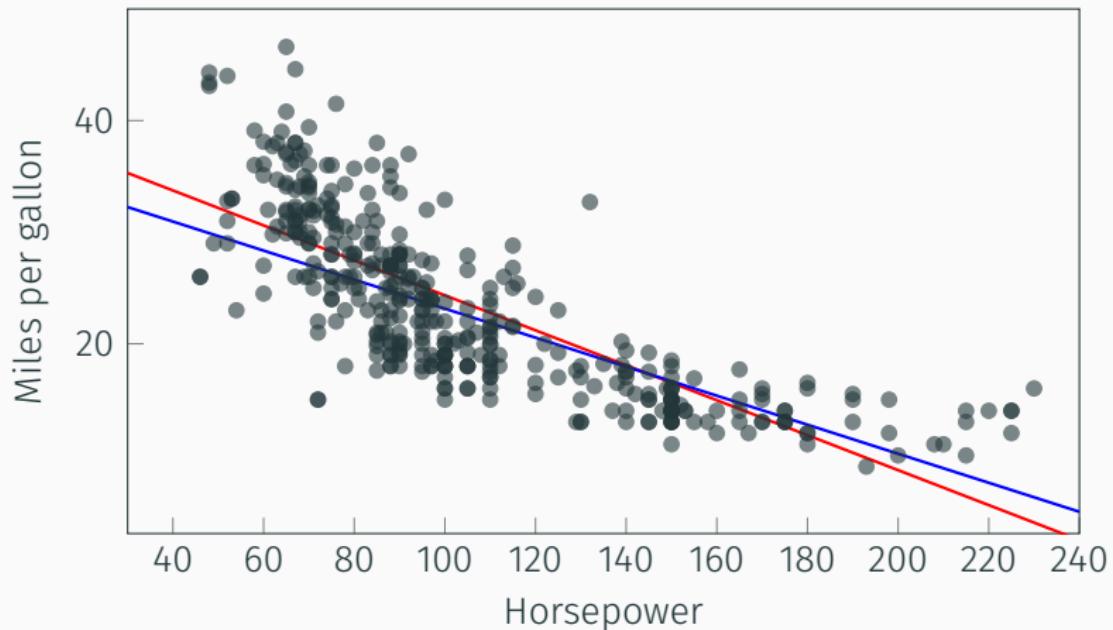
Linear regression (via ordinary least squares)



$$\widehat{mpg} = \beta_0 + \beta_1 \times \text{chevrolet} + \beta_2 \times \text{horsepower}$$



Linear regression (via ordinary least squares)



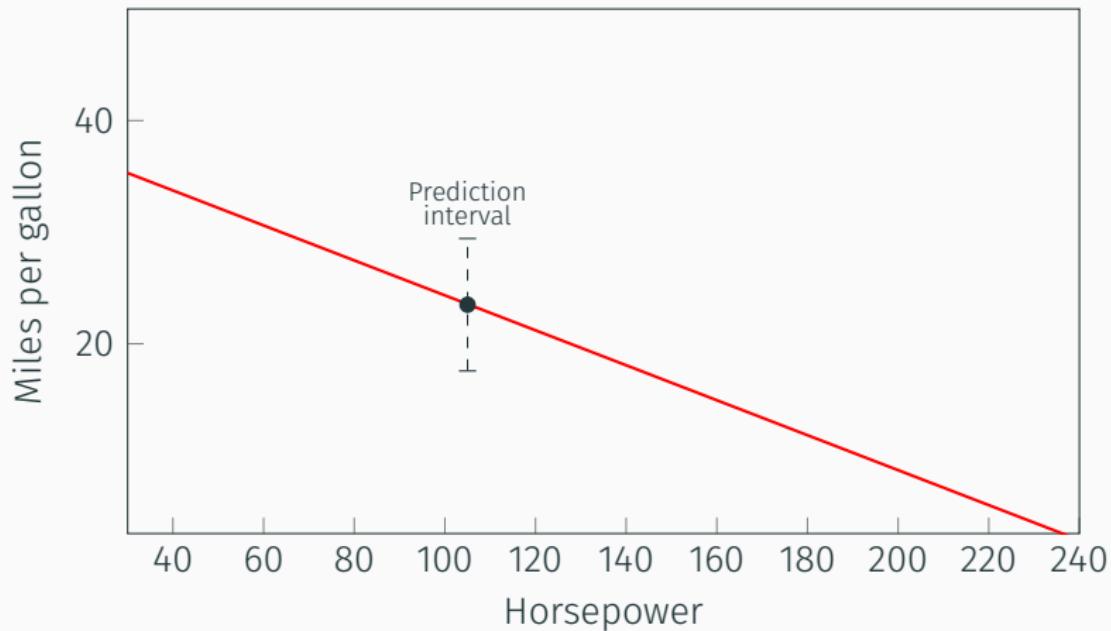
$$\widehat{mpg} = \beta_0 + \beta_1 \times \text{chevrolet} + \beta_2 \times \text{horsepower} \\ + \beta_3 \times \text{chevrolet} \times \text{horsepower}$$



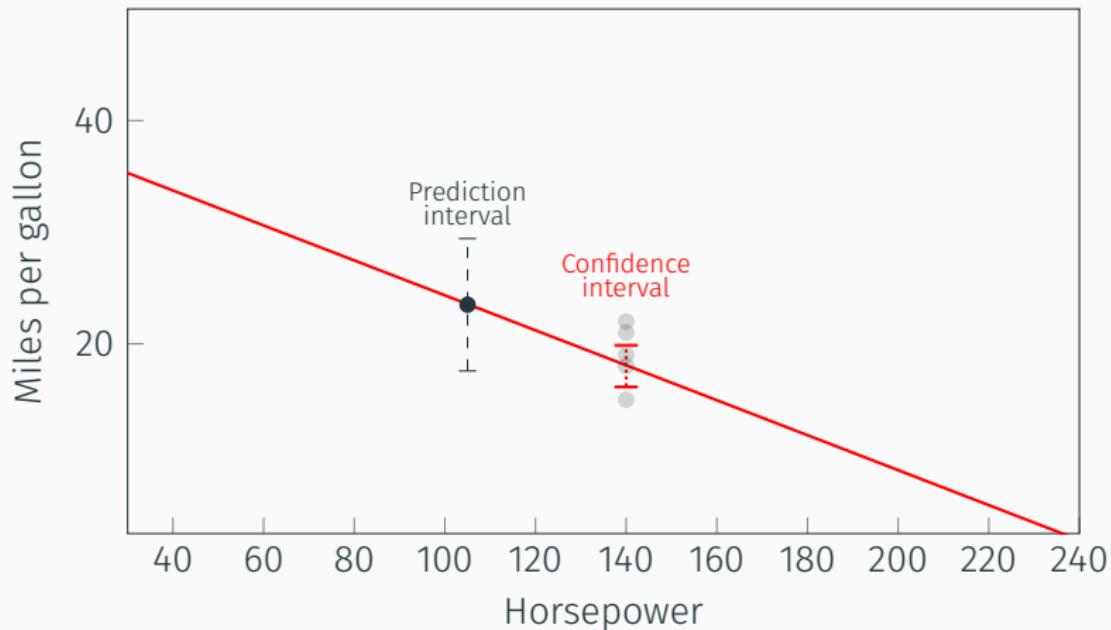
Linear regression (via ordinary least squares)



Linear regression (via ordinary least squares)



Linear regression (via ordinary least squares)



Linear regression (via ordinary least squares)

```
predict(fit, newdata=data.frame(horsepower=105),  
        interval='prediction', level=0.95)
```

	fit	lwr	upr
1	23.535	17.158	29.912

```
predict(fit, newdata=data.frame(horsepower=105),  
        interval='confidence', level=0.95)
```

	fit	lwr	upr
1	23.535	23.023	24.047



Linear regression (via ordinary least squares)

```
In[1]: import statsmodels.api as sm  
  
model = sm.OLS(df['mpg'], sm.add_constant(df[['horsepower']]))

fit = model.fit()
new_input = sm.add_constant(pd.DataFrame({'horsepower': [105, 106]}))
intervals = fit.get_prediction(new_input).summary_frame()
print(intervals)
```

```
Out[1]: mean mean_se mean_ci_lower mean_ci_upper obs_ci_lower obs_ci_upper
0 24.467077 0.251262 23.973079 24.961075 14.809396 34.124758
1 31.096556 0.398740 30.312607 31.880505 21.419710 40.773402
```



Linear regression (via ordinary least squares)



<http://localhost:8888/notebooks/notebooks%2FLinear%20regression.ipynb>



Linear regression (via ordinary least squares)

Linear regression: The true workhorse of machine learning

- Models the relationship between (either singular or multiple) inputs X and (a continuous) output y as a linear function
 - Inputs can be both continuous and categorical
- A strict parametric form limits the expressivity of the model
 - More advanced terms can be explicitly added
 - The strictness allows for further interpretation, such as confidence intervals
 - **Makes the model human interpretable**



K-Nearest Neighbours



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K-Nearest Neighbours

Linear regression:

$$\hat{f}(X) = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \dots + \beta_p X_p$$

K-Nearest Neighbours:

$$\hat{f}(X) = \frac{1}{k} \sum_{x_i \in \mathcal{N}} y_i$$



K-Nearest Neighbours

Linear regression:

$$\hat{f}(X) = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \dots + \beta_p X_p$$

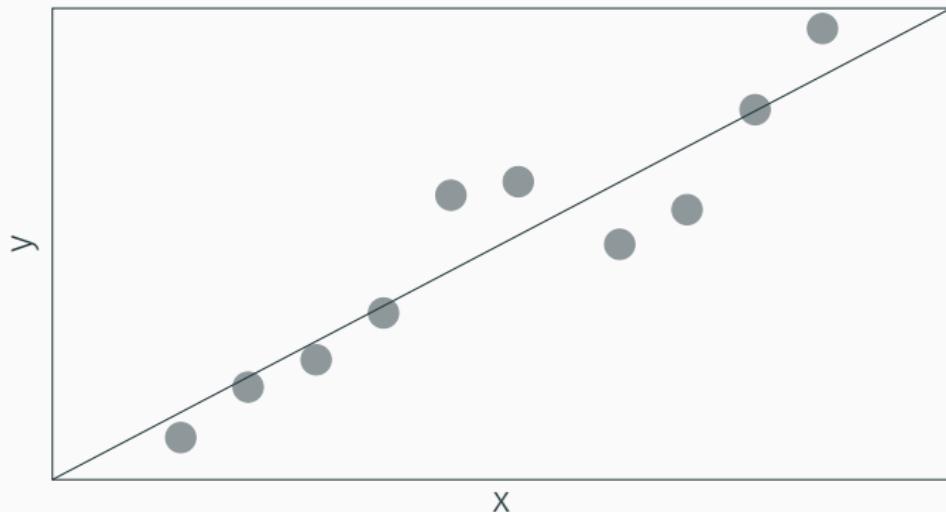
Blackboard!

K-Nearest Neighbours:

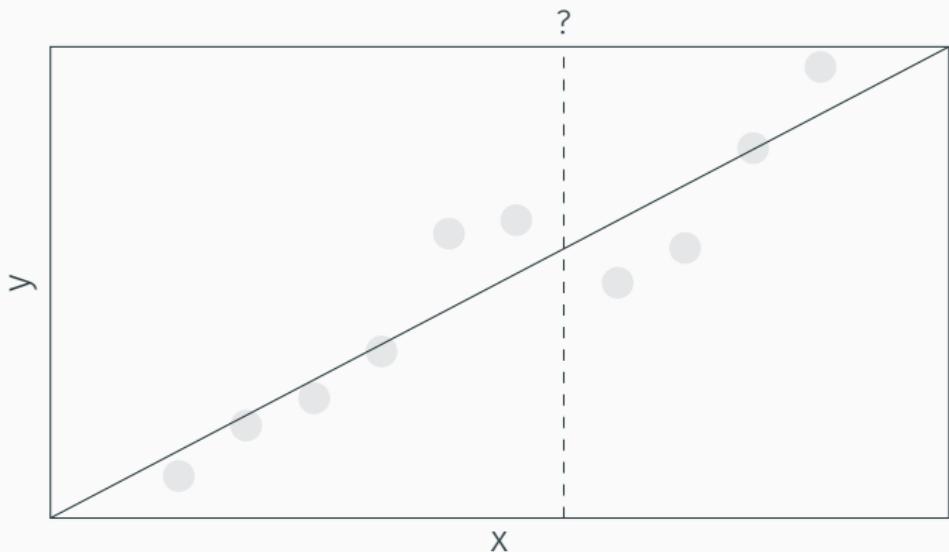
$$\hat{f}(X) = \frac{1}{k} \sum_{x_i \in \mathcal{N}} y_i$$



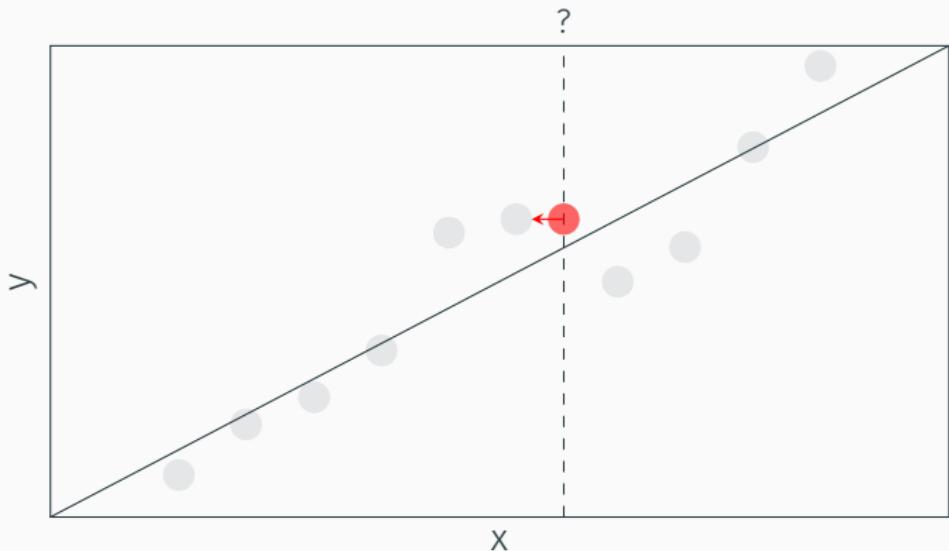
K-Nearest Neighbours



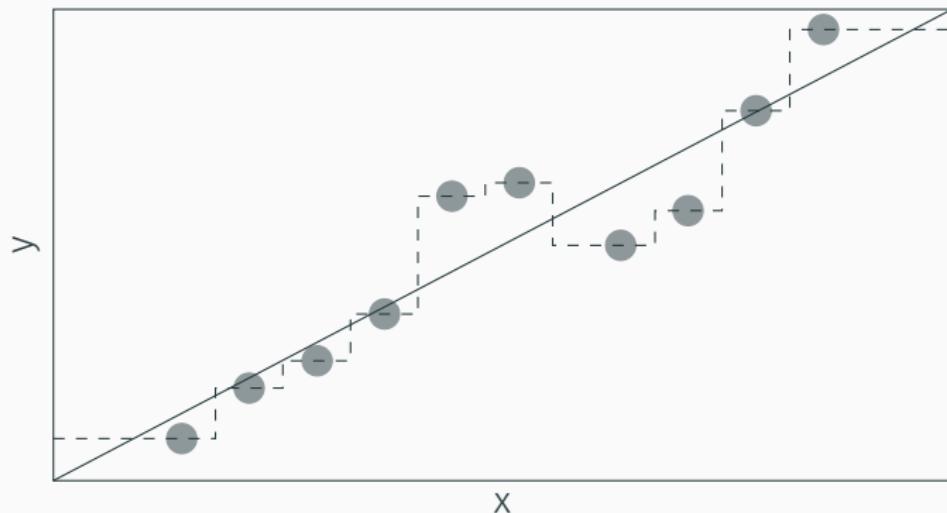
K-Nearest Neighbours



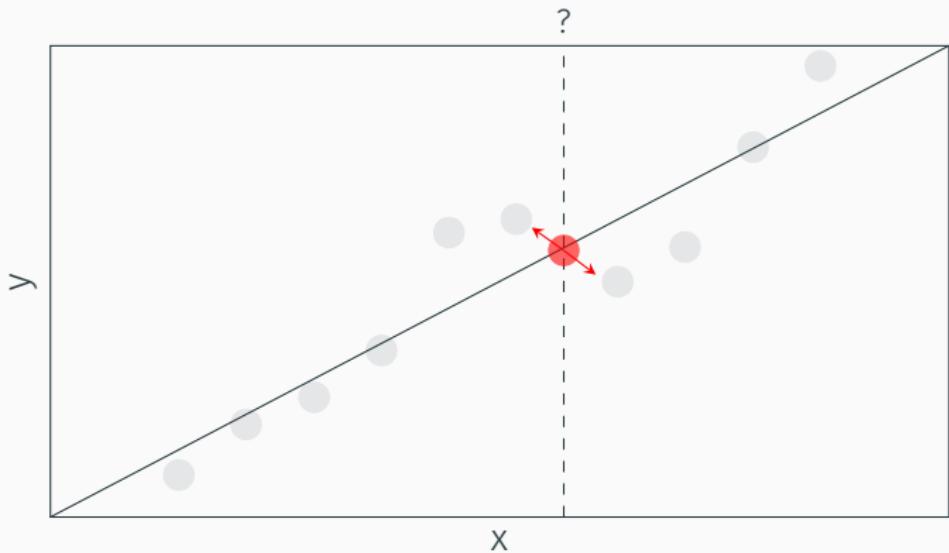
K-Nearest Neighbours



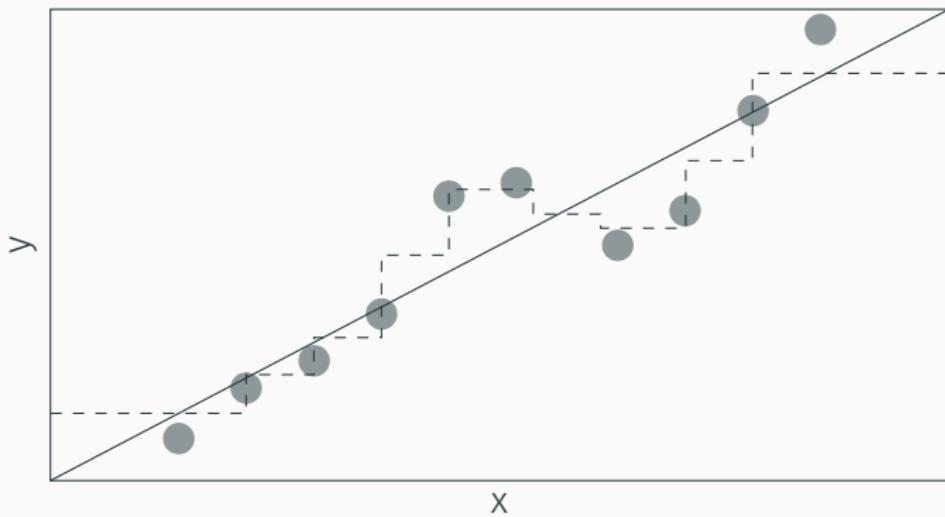
K-Nearest Neighbours



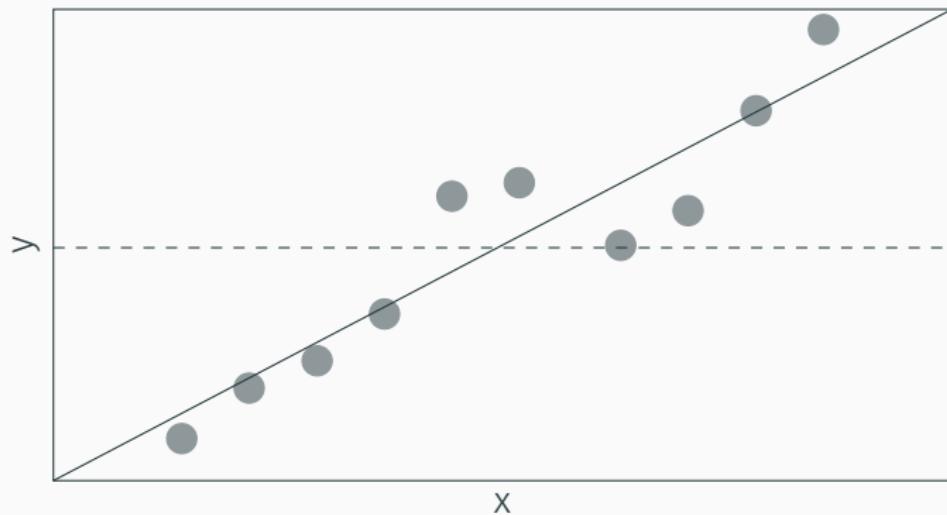
K-Nearest Neighbours



K-Nearest Neighbours



K-Nearest Neighbours



K-Nearest Neighbours

Blackboard exercise:

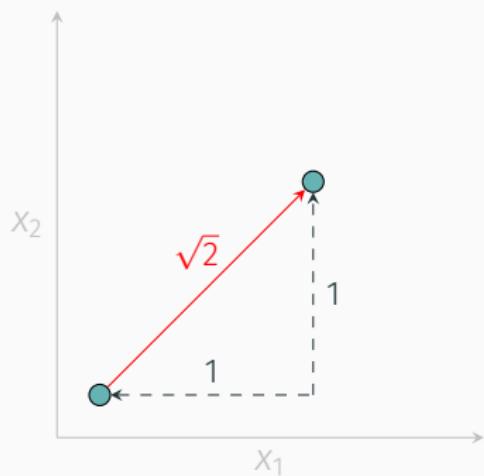
How does the bias-variance trade-off relate to the choice of K ?



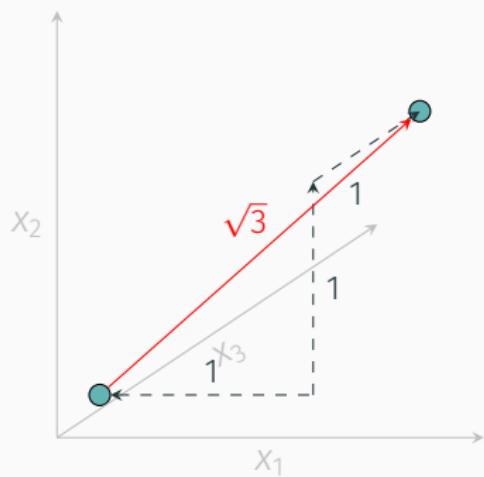
K-Nearest Neighbours



K-Nearest Neighbours



K-Nearest Neighbours



K-Nearest Neighbours

K-Nearest Neighbours: An intuitive model relying on similar datapoints to make a prediction

- No assumptions about the functional relationship between inputs and outputs
- Directly trades off bias and variance through the choice of K
- Does not work well in high dimensions as the neighbourhoods get sparse



Logistic regression



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Logistic regression



mpg	manufacturer	chevrolet
36	Chevrolet	1
15	Ford	0
25	Chevrolet	1
26	Chevrolet	1
17	Ford	0
15	Ford	0
32	Chevrolet	1
14	Ford	0
14	Ford	0
28	Chevrolet	1

$$\widehat{\text{mpg}} = \beta_0 + \beta_1 \times \text{chevrolet}$$



Logistic regression

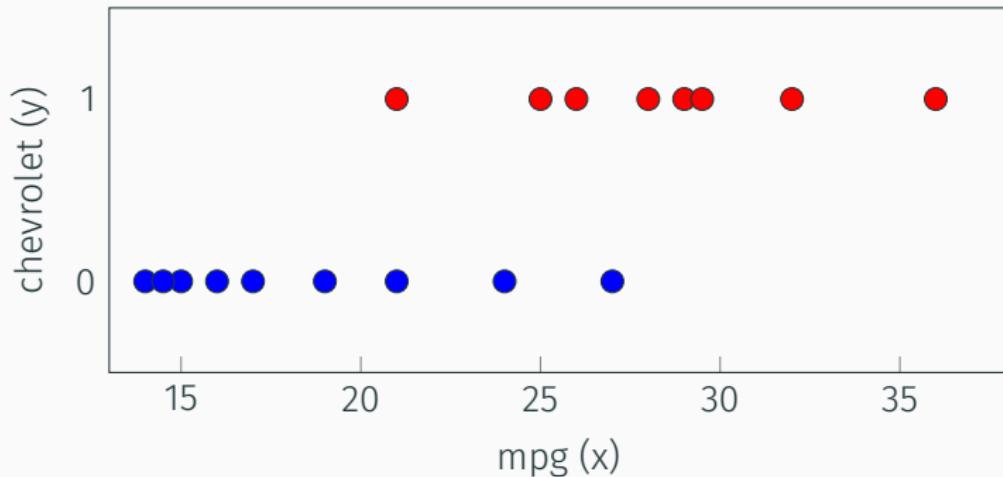


mpg	manufacturer	chevrolet
36	Chevrolet	1
15	Ford	0
25	Chevrolet	1
26	Chevrolet	1
17	Ford	0
15	Ford	0
32	Chevrolet	1
14	Ford	0
14	Ford	0
28	Chevrolet	1

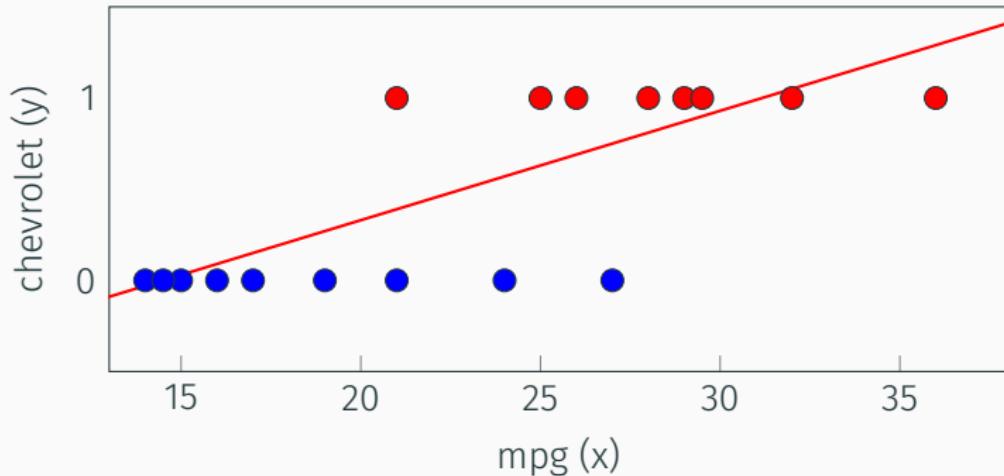
$$\widehat{\text{chevrolet}} = \beta_0 + \beta_1 \times \text{mpg}$$



Logistic regression



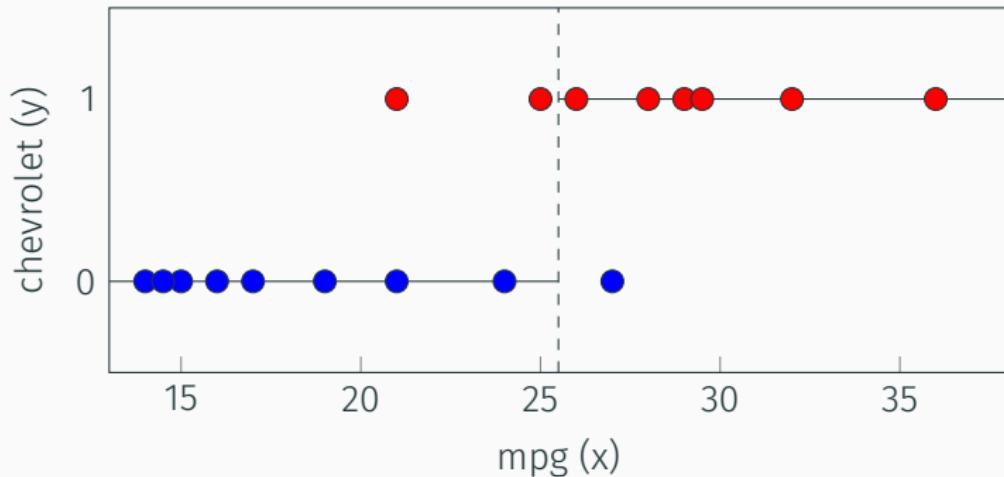
Logistic regression



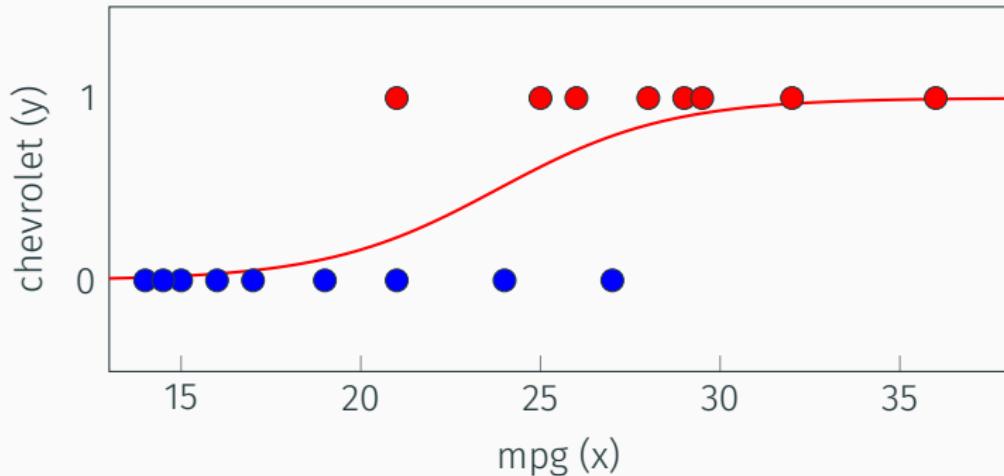
$$\widehat{\text{chevrolet}} = -0.87 + 0.06 \times \text{mpg}$$



Logistic regression



Logistic regression



$$\widehat{\text{chevrolet}} = \frac{e^{-10.22 + 0.42 \times \text{mpg}}}{1 + e^{-10.22 + 0.42 \times \text{mpg}}}$$



Logistic regression



$$\hat{y} = \frac{e^{\beta_0 + \beta_1 x}}{1 + e^{\beta_0 + \beta_1 x}}$$



Logistic regression

□

$$\hat{y} = \frac{e^{\beta_0 + \beta_1 x}}{1 + e^{\beta_0 + \beta_1 x}}$$

$$e^{\beta_0 + \beta_1 x} \rightarrow 0$$

□



Logistic regression

□

$$\hat{y} = \frac{e^{\beta_0 + \beta_1 x}}{1 + e^{\beta_0 + \beta_1 x}}$$

$$e^{\beta_0 + \beta_1 x} \rightarrow 0 \implies \hat{y} = 0$$

□



Logistic regression

□

$$\hat{y} = \frac{e^{\beta_0 + \beta_1 x}}{1 + e^{\beta_0 + \beta_1 x}}$$

$$e^{\beta_0 + \beta_1 x} \rightarrow 0 \implies \hat{y} = 0$$
$$e^{\beta_0 + \beta_1 x} \rightarrow \infty$$

□



Logistic regression

□

$$\hat{y} = \frac{e^{\beta_0 + \beta_1 x}}{1 + e^{\beta_0 + \beta_1 x}}$$

$$e^{\beta_0 + \beta_1 x} \rightarrow 0 \implies \hat{y} = 0$$
$$e^{\beta_0 + \beta_1 x} \rightarrow \infty \implies \hat{y} = 1$$

□



Logistic regression

□

$$\hat{y} = \frac{e^{\beta_0 + \beta_1 x}}{1 + e^{\beta_0 + \beta_1 x}}$$

$$\begin{aligned} e^{\beta_0 + \beta_1 x} \rightarrow 0 &\implies \hat{y} = 0 \\ e^{\beta_0 + \beta_1 x} \rightarrow \infty &\implies \hat{y} = 1 \end{aligned}$$

$$0 \leq \hat{y} \leq 1$$

□



Logistic regression

□

$$\hat{y} = \frac{e^{\beta_0 + \beta_1 x}}{1 + e^{\beta_0 + \beta_1 x}}$$

$$e^{\beta_0 + \beta_1 x} \rightarrow 0 \implies \hat{y} = 0$$
$$e^{\beta_0 + \beta_1 x} \rightarrow \infty \implies \hat{y} = 1$$

$$0 \leq \hat{y} \leq 1 \implies \hat{y} = Pr(Y = 1|X)$$

□



Logistic regression



$$\hat{y} = \frac{e^{\beta_0 + \beta_1 x}}{1 + e^{\beta_0 + \beta_1 x}}$$



Logistic regression

□

$$\hat{y} = \frac{e^{\beta_0 + \beta_1 x}}{1 + e^{\beta_0 + \beta_1 x}}$$

$$\log \left(\frac{p(X)}{1 - p(X)} \right) = \beta_0 + \beta_1 x$$

□



Logistic regression



$$\hat{y} = \frac{e^{\beta_0 + \beta_1 x}}{1 + e^{\beta_0 + \beta_1 x}}$$

$$\log \left(\frac{p(X)}{1 - p(X)} \right) = \beta_0 + \beta_1 x$$

"... counterintuitive and challenging to interpret." - James Jaccard



Logistic regression



$$\hat{y} = \frac{e^{\beta_0 + \beta_1 x}}{1 + e^{\beta_0 + \beta_1 x}}$$



Logistic regression



$$\hat{y} = \frac{e^{\beta_0 + \beta_1 x}}{1 + e^{\beta_0 + \beta_1 x}}$$

More than one class?



Logistic regression



$$\hat{y} = \frac{e^{\beta_0 + \beta_1 x}}{1 + e^{\beta_0 + \beta_1 x}}$$

More than one class?

$$Pr(Y = k | X = x) = \frac{e^{\beta_{0k} + \beta_{1k}x}}{\sum_{l=1}^K e^{\beta_{0l} + \beta_{1l}x}}$$



Logistic regression



$$\hat{y} = \frac{e^{\beta_0 + \beta_1 x}}{1 + e^{\beta_0 + \beta_1 x}}$$

More than one class?

$$Pr(Y = k | X = x) = \frac{e^{\beta_{0k} + \beta_{1k} x}}{\sum_{l=1}^K e^{\beta_{0l} + \beta_{1l} x}}$$



Logistic regression

Logistic regression: Extending linear regression to classification.

- Treats all members of a class (approximately) equally.
- Outputs an understandable quantity: The probability of a sample belonging to the positive class.
- Somewhat interpretable, although not as much as linear regression
- Can trivially be extended to include multiple classes



Logistic regression

<http://localhost:8888/notebooks/notebooks%2FLogistic%20regression.ipynb>



Generative models for classification



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Generative models

Generative models for classification:

- Logistic regression models the probability of a class given predictors: $P(Y|X)$
- Generative models instead consider the probability of the predictors values given the class, $P(X|Y)$
- How would a given class generate the datapoint?



Generative models

Generative models for classification:

- Bayes' theorem: $P(Y|X) = \frac{P(X|Y)P(Y)}{P(X)}$
- $P(Y|X)$ is the posterior probability of the class given the data
- $P(X|Y)$ is the likelihood of the data given the class
- $P(Y)$ is the prior probability of the class
- $P(X)$ is the marginal likelihood of the data



Generative models

Linear Discriminant Analysis

- Assume the predictors are normally distributed within each class separately
 - $P(X|Y = \text{Class1})$
 - $P(X|Y = \text{Class2})$
 - etc
- Class probabilities $P(Y)$ estimated from training data
- Marginal probability of data $P(X)$ can be simplified away
- Posterior probability $P(Y|X)$ can be calculated using Bayes' theorem



Classification metrics



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Classification metrics

The two most common, severe, mistakes made in machine learning studies in psychology and neuroscience (in my opinion) are:

- Poor validation and testing strategies (Next lecture)
- Using incorrect performance measures, most commonly accuracy (Blackboard!)



Assignment 2

https://uiuo.instructure.com/courses/53357/assignments/116394?module_item_id=946345

