Russell's Paradox

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Background

Background

$$\{S|S \notin S\}$$

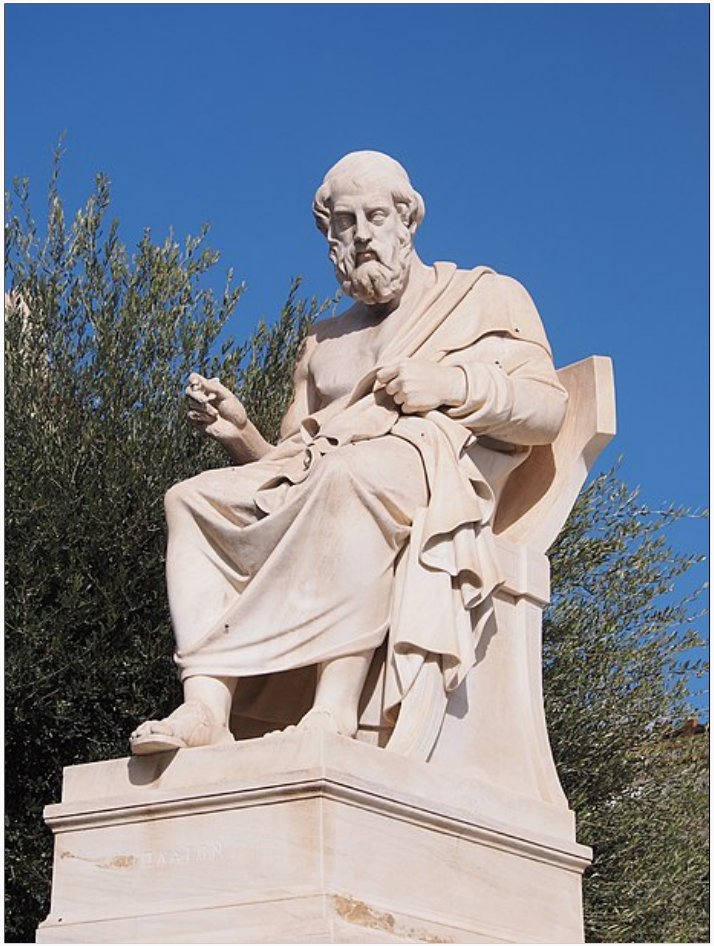
Background

Source: A guy I met at a party once

Background

Disclaimer: The contents of this talk will be approximately true

Historical underpinnings



Historical underpinnings

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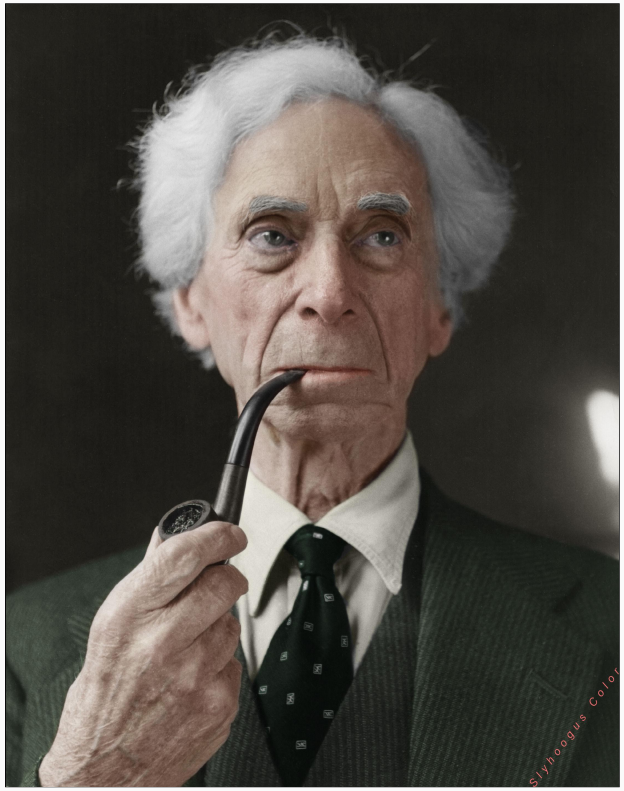


The project

$$0, 1, 2, 3, 4, \ldots$$

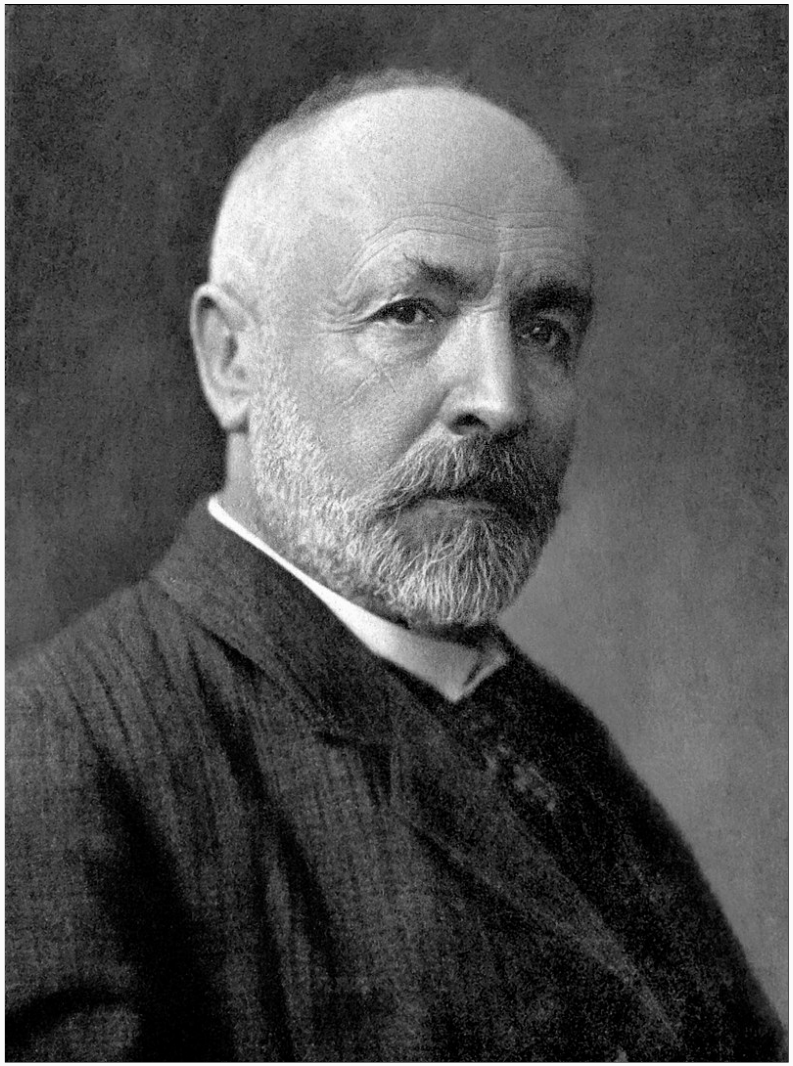
$$x + y = z$$
$$x - y = z$$
$$\ldots$$

The project

$0, 1, 2, 3, 4, \ldots$ $\Longrightarrow$ 🤔

$x + y = z$
$x - y = z$ $\Longrightarrow$ ✅
...

Set theory



Set theory

{🤓, 👩, 👧, 🐶}

{🤓, 😎, 🧐, 👩‍🔬, 👩‍💻, 👩‍🏫}

{monday, tuesday, wednesday, thursday,
friday, saturday, sunday}

{💻, 📱}

Set theory

{🤓, 👩, 👧}

{⚽, 🏀, 🏐}

{tuesday, thursday, saturday}

...

Set theory

$$\{1, 3, 5\}$$

$$\{1, 10, 100\}$$

$$\{\}$$

Set theory

$$\{1, 2, 3, \ldots\}$$

Set theory

$$\{1, 2, 3, \ldots\}$$

$$\{x \mid x > 0\}$$

Set theory

$$\{1, 3, 5, \ldots\}$$

$$\{x \mid x \,\%\, 2 \neq 0\}$$

Set theory

{{🤓, 👩, 👧, 🐶}, {🤓, 😎, 🧐, 👩‍🔬, 👩‍💻, 👩‍💻},
{monday, tuesday, wednesday, thursday, friday, saturday, sunday},
{💻, 📱}, $\{1, 3, 5\}$, $\{1, 10, 100\}$, {}, $\{x \mid x > 0\}$, $\{x \mid x \,\%\, 2 \neq 0\}$

Set theory

$\{\{$🤓, 👩, 👧, 🐶$\}, \{$🤓, 😎, 🧐, 👩‍🔬, 👩‍💻, 👩‍💻$\},$
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$\{$💻, 📱$\}, \{1, 3, 5\}, \{1, 10, 100\}, \{\}, \{x \mid x > 0\}, \{x \mid x \,\%\, 2 \neq 0\}\}$

$$\{\{\}, \{0\}, \{1\}, \{0, 1\}, \ldots\}$$

Set theory

$\{\{$🤓, 👩, 👧, 🐶$\}, \{$🤓, 😎, 🧐, 👩‍🔬, 👩‍💻, 👩‍💻$\},$
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$$\{\{\}, \{0\}, \{1\}, \{0, 1\}, \ldots\}$$

$$\{x \mid x \text{ is a set}\}$$

Set theory

$$\{\{🤓, 👩, 👧, 🐶\}, \{🤓, 😎, 🧐, 👩‍🔬, 👩‍💻, 👩‍💻\},$$
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$$\{\{\}, \{0\}, \{1\}, \{0, 1\}, \ldots\}$$

$$S = \{x \mid x \text{ is a set}\}$$

Set theory

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$$\{\{\}, \{0\}, \{1\}, \{0, 1\}, \ldots\}$$

$$S = \{x \mid x \text{ is a set}\}$$

$$S \in S\,?$$

Set theory

$$S = \{\{\}, \{0\}, \{1\}, \{0, 1\}, \ldots, S, \ldots\}$$

$$S \in S$$

Set theory

$$S = \{\{\}, \{0\}, \{1\}, \{0,1\}, \ldots, S, \ldots\} \qquad S = \{1,3,5\}$$

$$S \in S \qquad S \notin S$$

Set theory

$$S = \{\{\}, \{0\}, \{1\}, \{0, 1\}, \ldots, S, \ldots\} \qquad S = \{1, 3, 5\}$$

$$S \in S \qquad S \notin S$$

$$\{S \mid S \in S\}$$

Set theory

$$S = \{\{\}, \{0\}, \{1\}, \{0,1\}, \ldots, S, \ldots\} \qquad S = \{1,3,5\}$$

$$S \in S \qquad S \notin S$$

$$\{S \mid S \in S\}$$
$$\{S \mid S \notin S\}$$

Set theory

$$T = \{S \mid S \notin S\}$$

Set theory

$$T = \{S \mid S \notin S\}$$

$$T \in T \, ?$$

Set theory

$$T = \{S \mid S \notin S\}$$

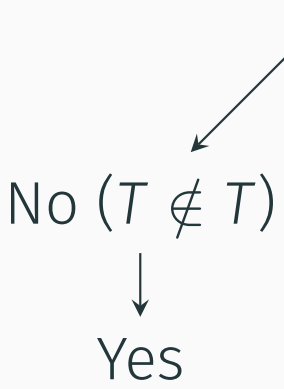
$$T \in T \, ?$$

$$\text{No } (T \notin T)$$

Set theory

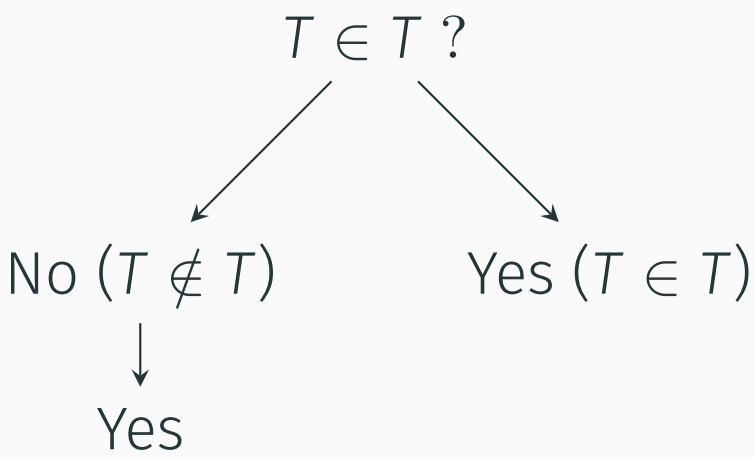
$$T = \{S \mid S \notin S\}$$

$$T \in T\ ?$$



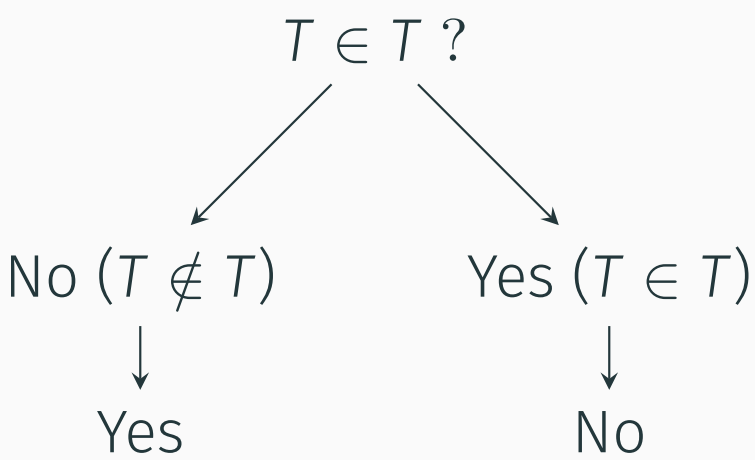
Set theory

$$T = \{S \mid S \notin S\}$$



Set theory

$$T = \{S \mid S \notin S\}$$



Set theory

Set theory

”Hardly anything more unfortunate can befall a scientific writer than to have one of the foundations of his edifice shaken after the work is finished. This was the position I was placed in by a letter of Mr. Bertrand Russell, just when the printing of this volume was nearing its completion.”

Gödels incompleteness theorem

Gödels incompleteness theorem

1. 0 is a natural number
2. If n is a natural number, $S(n)$ is a natural number
3. No numbers have 0 as their successor

$$0$$

$$1 = S(0)$$

$$2 = S(S(0))$$

$$\dots$$

Gödels incompleteness theorem

1. 0 is a natural number
2. If n is a natural number, $S(n)$ is a natural number
3. No numbers have 0 as their successor
4. If $f(0)$ exists and $f(x+1)$ holds whenever $f(x)$ holds, f is a valid function

□

$$0$$

$$1 = S(0)$$

$$2 = S(S(0))$$

$$\cdots$$

$$x + 0 = x$$

$$x + S(y) = S(x + y)$$

□

Gödels incompleteness theorem

□

$$0 := 2$$

$$S(0) := 12965404$$

$$S(S(0)) := 24300749247875100$$

Gödel number

□

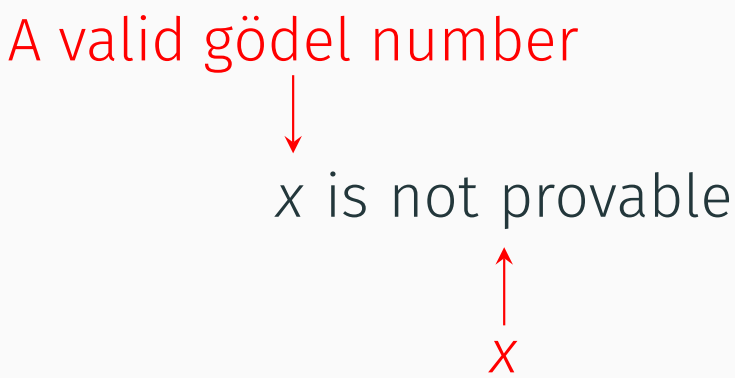
Gödels incompleteness theorem

A valid gödel number

↓

x is not provable

Gödels incompleteness theorem



The halting problem

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Given a program P and an input i, determine if P halts on i

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Given a program P and an input i, determine if P halts on i

```
print('Hello world)
```

Halts

```
while True:
    print('Hello world')
```


Does not halt

The halting problem

In[1]:

```
def decider(program, input):
    """Magical function that decides whether the program
    halts on the input"""
    ...
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    if decider(program, program):
        while True: pass
    else:
        return

f(f)
```

The halting problem

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f(f)
```

Summary

There are limits to what we can do with formal approaches

- These typically break down in the presence of self-references

Are we all strange loops?

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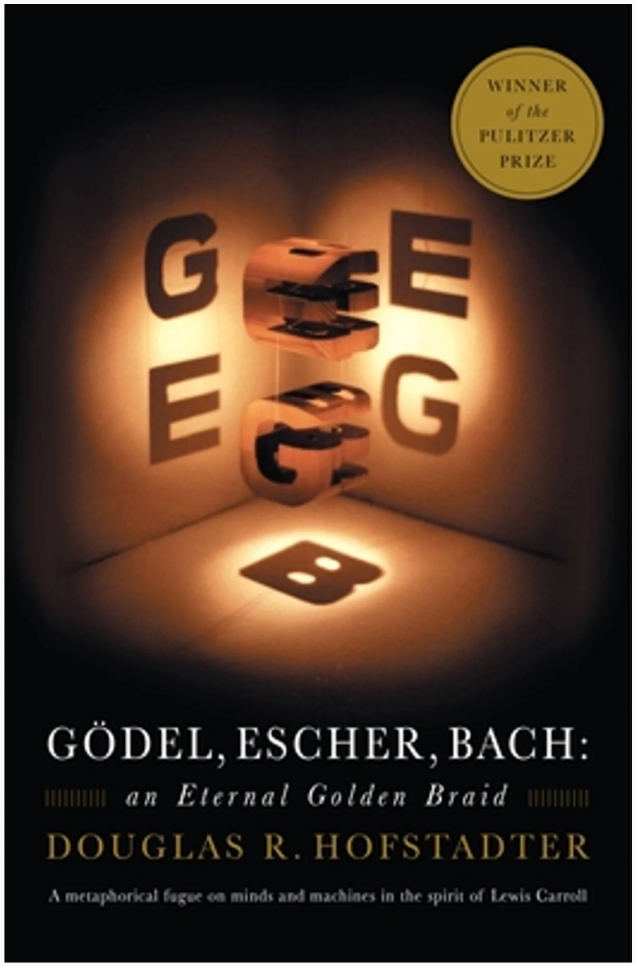
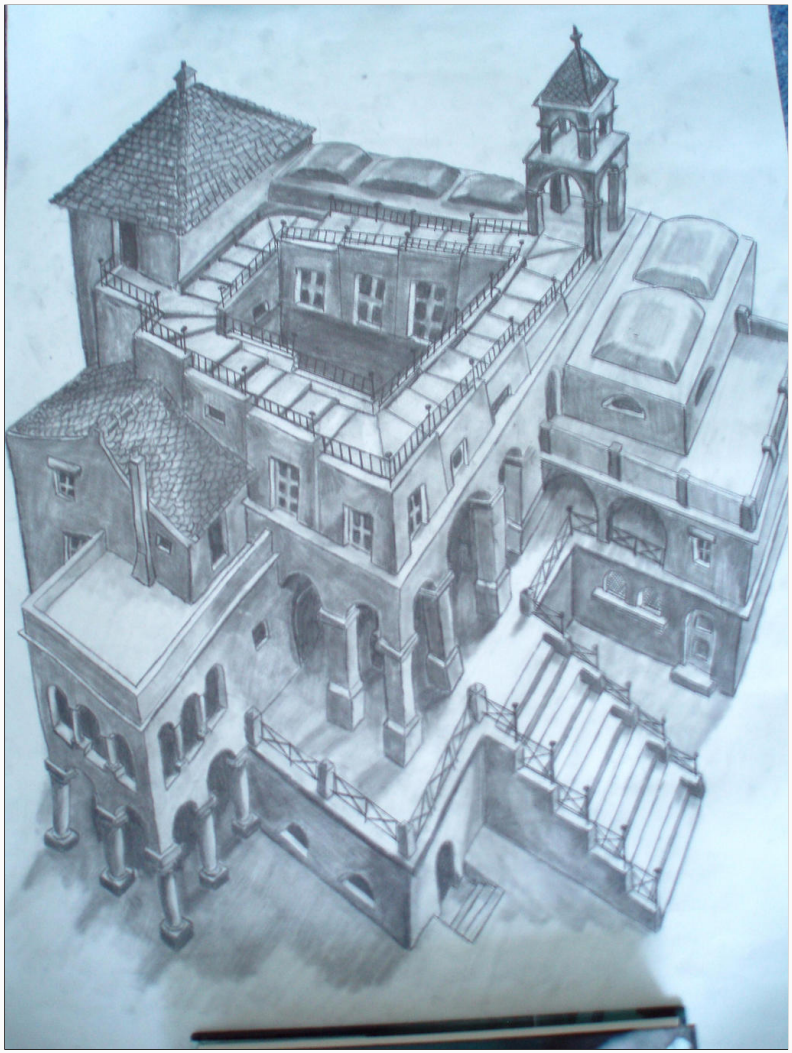


Are we all strange loops?



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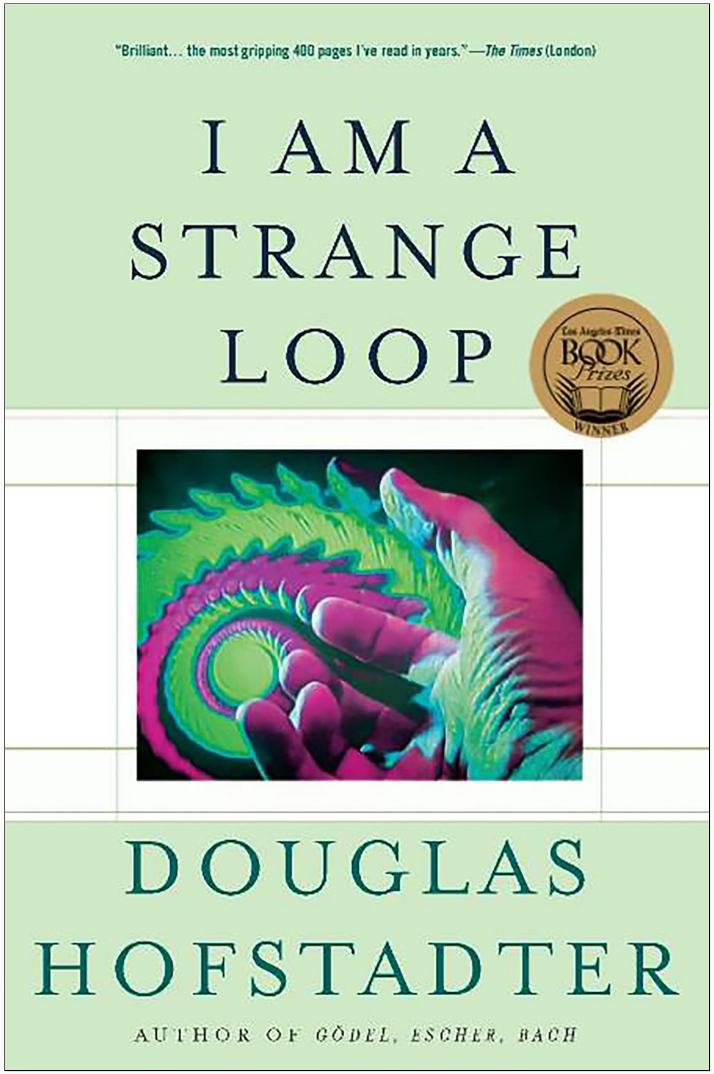

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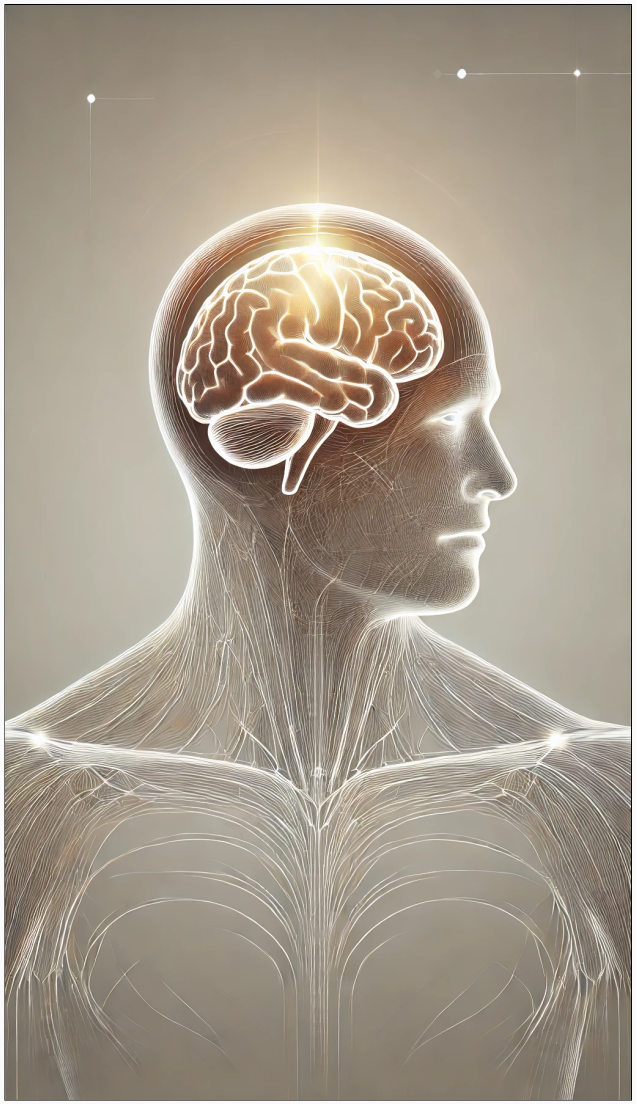

Are we all strange loops?