Pset5 - Econometrics II

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March 2025

1 Question 1 (GMM with known gradient)

Item 1. We want to compute the gradient of the moment condition vector $h(\lambda_0, X_t)$ with respect to the parameter λ_0, where $X_t \sim \text{Exponential}(\lambda_0)$.
Hence, the moment condition vector is defined as:

$$h(\lambda_0, X_t) = \begin{bmatrix} \frac{1}{\lambda_0} - X_t \\ \frac{1}{\lambda_0^2} - \left(X_t - \frac{1}{\lambda_0}\right)^2 \end{bmatrix}$$

Now we want to calculate the Jacobian matrix,

$$H(\lambda_0) = \frac{\partial \mathbb{E}[h(\lambda_0, X_t)]}{\partial \lambda_0}$$

This means we need to take the derivative of each component of $h(\lambda_0, X_t)$ with respect to λ_0, and then take the expectation.

$$h_1(\lambda_0, X_t) = \frac{1}{\lambda_0} - X_t$$

derivative with respect to λ_0:

$$\frac{\partial h_1(\lambda_0, X_t)}{\partial \lambda_0} = -\frac{1}{\lambda_0^2}$$

Now take the expectation (note that this derivative does not depend on X_t):

$$\mathbb{E}\left[\frac{\partial h_1(\lambda_0, X_t)}{\partial \lambda_0}\right] = -\frac{1}{\lambda_0^2}$$

$$h_2(\lambda_0, X_t) = \frac{1}{\lambda_0^2} - \left(X_t - \frac{1}{\lambda_0}\right)^2$$

We compute the derivative term by term such as,
For the first term,

$$\frac{\partial}{\partial \lambda_0}\left(\frac{1}{\lambda_0^2}\right) = -\frac{2}{\lambda_0^3}$$

For the second term,

$$\left(X_t - \frac{1}{\lambda_0}\right)^2$$

Derivative with respect to λ_0,

$$\frac{\partial}{\partial \lambda_0}\left[\left(X_t - \frac{1}{\lambda_0}\right)^2\right] = 2\left(X_t - \frac{1}{\lambda_0}\right)\left(\frac{1}{\lambda_0^2}\right)$$

Therefore, the derivative of $h_2(\lambda_0, X_t)$ is:

$$\frac{\partial h_2(\lambda_0, X_t)}{\partial \lambda_0} = -\frac{2}{\lambda_0^3} - 2\left(X_t - \frac{1}{\lambda_0}\right)\left(\frac{1}{\lambda_0^2}\right)$$

If we take the expectation we have that

$$\mathbb{E}\left[\frac{\partial h_2(\lambda_0, X_t)}{\partial \lambda_0}\right] = -\frac{2}{\lambda_0^3} - 2 \cdot \frac{1}{\lambda_0^2} \cdot \mathbb{E}\left[X_t - \frac{1}{\lambda_0}\right]$$

and making use of the known property of the Exponential distribution we can write,

$$\mathbb{E}[X_t] = \frac{1}{\lambda_0}$$

Thus:

$$\mathbb{E}\left[X_t - \frac{1}{\lambda_0}\right] = 0$$

So the second term dissapears, and we are left with,

$$\mathbb{E}\left[\frac{\partial h_2(\lambda, X_t)}{\partial \lambda}\right] = -\frac{2}{\lambda^3}$$

Now, if we get both results together, we will have,

$$H(\lambda) = \begin{bmatrix} -\frac{1}{\lambda^2} \\ -\frac{2}{\lambda^3} \end{bmatrix}$$

Alternatively, using positive notation from Hamilton (2020) we could factor the negative out and write,

$$H(\lambda) = -\frac{1}{\lambda^2}\begin{bmatrix} 1 \\ \frac{2}{\lambda} \end{bmatrix} = -\frac{1}{\lambda^2}\begin{bmatrix} 1 \\ 2 \cdot \mathbb{E}[X] \end{bmatrix}$$

Item 2.

We estimate the model using the `gmm` function from the `gmm` package in R, and present the results in Table 1. We set the search interval for the Brent method as *lower = 1* and *upper = 7*. The estimate for λ_0 is consistent and sufficiently close to the true parameter value of 5.

Table 1: GMM Simulation for $\lambda_0 = 5$ and 100,000 observations

	Dependent variable:
	GMM estimate
$\theta[1]$	4.995***
	(0.016)
Observations	100,000

Standard errors are reported in parentheses. Significance levels: *p<0.1; **p<0.05; ***p<0.01.

2 Question 2 (Estimating an ARMA(1,2) using GMM

Item 1.
Equation (1) defines the autoregressive component of the ARMA(1,2) model:

$$Y_t = \phi_1 Y_{t-1} + U_t, \tag{1}$$

where the error term U_t follows a moving average process:

$$U_t = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2}, \tag{2}$$

with $\varepsilon_t \sim$ i.i.d. $N(0, 1)$.

The OLS estimation of Equation (1) would require us to use the classical assumption that the regressor Y_{t-1} is uncorrelated with the error term U_t. However, we can not guarantee this assumption in this context.

By construction, U_t depends on past realizations of the white noise ε_t, specifically ε_t, ε_{t-1}, and ε_{t-2}. Since Y_{t-1} itself is a function of past shocks (including ε_{t-1} and ε_{t-2}), there exists correlation between Y_{t-1} and U_t.

This correlation violates the zero conditional mean assumption required for OLS to be consistent. We can show that,

$$\begin{aligned}
\mathbb{E}[Y_{t-1}U_t] &= \mathbb{E}\left[Y_{t-1} \cdot (\varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2})\right] \\
&= \mathbb{E}\left[(\phi_1 Y_{t-2} + \varepsilon_{t-1} + \theta_1\varepsilon_{t-2} + \theta_2\varepsilon_{t-3}) \cdot (\varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2})\right] \\
&= \theta_1\mathbb{E}[\varepsilon_{t-1}^2] + \theta_1\theta_2\mathbb{E}[\varepsilon_{t-2}^2] + \phi_1\theta_2\mathbb{E}[\varepsilon_{t-2}^2] \\
&= \theta_1 + \theta_1\theta_2 + \phi_1\theta_2 > 0,
\end{aligned}$$

Which is the same as,

$$\mathbb{E}[U_t \mid Y_{t-1}] \neq 0. \tag{3}$$

Therefore, OLS estimation of ϕ_1 in Equation (1) would yield biased and inconsistent results.

Item 2.

$$\begin{aligned}
\mathbb{E}[Y_{t-2}U_t] &= \mathbb{E}\left[Y_{t-2} \cdot (\varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2})\right] \\
&= \mathbb{E}\left[(\phi_1 Y_{t-3} + \varepsilon_{t-2} + \theta_1\varepsilon_{t-3} + \theta_2\varepsilon_{t-4}) \cdot (\varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2})\right] \\
&= \theta_2\mathbb{E}[\varepsilon_{t-2}^2] \\
&= \theta_2 \neq 0.
\end{aligned}$$

Item 3.
In the ARMA(1,2) we require instruments that are correlated with Y_{t-1} but uncorrelated with U_t. Thus, to check if Y_{t-3} is a valid instrument, we examine the moment condition,

$$\begin{aligned}
\mathbb{E}[Y_{t-3}U_t] &= \mathbb{E}\left[Y_{t-3} \cdot (\varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2})\right] \\
&= \mathbb{E}[Y_{t-3} \cdot \varepsilon_t] + \theta_1\mathbb{E}[Y_{t-3} \cdot \varepsilon_{t-1}] + \theta_2\mathbb{E}[Y_{t-3} \cdot \varepsilon_{t-2}] \\
&= 0 + \theta_1 \times 0 + \theta_2 \times 0 \\
&= 0,
\end{aligned}$$

and

$$
\begin{aligned}
\mathbb{E}[Y_{t-1}Y_{t-3}] &= \mathbb{E}\left[\left(\phi_1 Y_{t-2} + \varepsilon_{t-1} + \theta_1\varepsilon_{t-2} + \theta_2\varepsilon_{t-3}\right) Y_{t-3}\right] \\
&= \phi_1\mathbb{E}[Y_{t-2}Y_{t-3}] + \theta_2\mathbb{E}[\varepsilon_{t-3}Y_{t-3}] \\
&= \phi_1\mathbb{E}\left[\left(\phi_1 Y_{t-3} + U_{t-2}\right) Y_{t-3}\right] + \theta_2\mathbb{E}[\varepsilon_{t-3}^2] \\
&= \phi_1^2\mathbb{E}[Y_{t-3}^2] + \phi_1\mathbb{E}[U_{t-2}Y_{t-3}] + \theta_2\mathbb{E}[\varepsilon_{t-3}^2] \\
&= \phi_1^2\mathbb{E}[Y_{t-3}^2] + \phi_1\mathbb{E}[U_t Y_{t-1}] + \theta_2\mathbb{E}[\varepsilon_{t-3}^2] \\
&> 0.
\end{aligned}
$$

where the expectations in $\mathbb{E}[Y_{t-3}U_t]$ are zero because Y_{t-3} is a function of shocks ε_{t-3} and earlier, which are independent of ε_t, ε_{t-1}, and ε_{t-2} by assumption (since ε_t is i.i.d.). Therefore, since $\mathbb{E}[Y_{t-3}U_t] = 0$, and $\mathbb{E}[Y_{t-1}Y_{t-3}] \neq 0$, Y_{t-3} is a valid instrument for Y_{t-1}.

Item 4.

Table 2: GMM Estimation of AR(1) Coefficient and Intercept for ARMA(1,2) Process

	Dependent variable: Y_0
Y_1	0.193***
	(0.021)
Constant	−0.002
	(0.004)
Observations	99,997

Note: Standard errors are reported in parentheses. Significance levels: *p<0.1; **p<0.05; ***p<0.01. The estimation was performed using GMM with Y_{t-3} as instrument for Y_{t-1}. Data was generated from an ARMA(1,2) process with $\phi_1 = 0.2$, $\theta_1 = 0.1$, and $\theta_2 = 0.1$, with 100,000 simulated observations.

3 Question 3 (Testing the CAPM model)

Item 1.

```
#item 1

#improting the datasets

bvsp  <- read_csv("data/ABEV3.csv")
bbdc3 <- read_csv("data/BBDC3.csv")
abev3 <- read_csv("data/ABEV3.csv")
itub3 <- read_csv("data/ITUB3.csv")



#creating a function to calculate the daily log
```

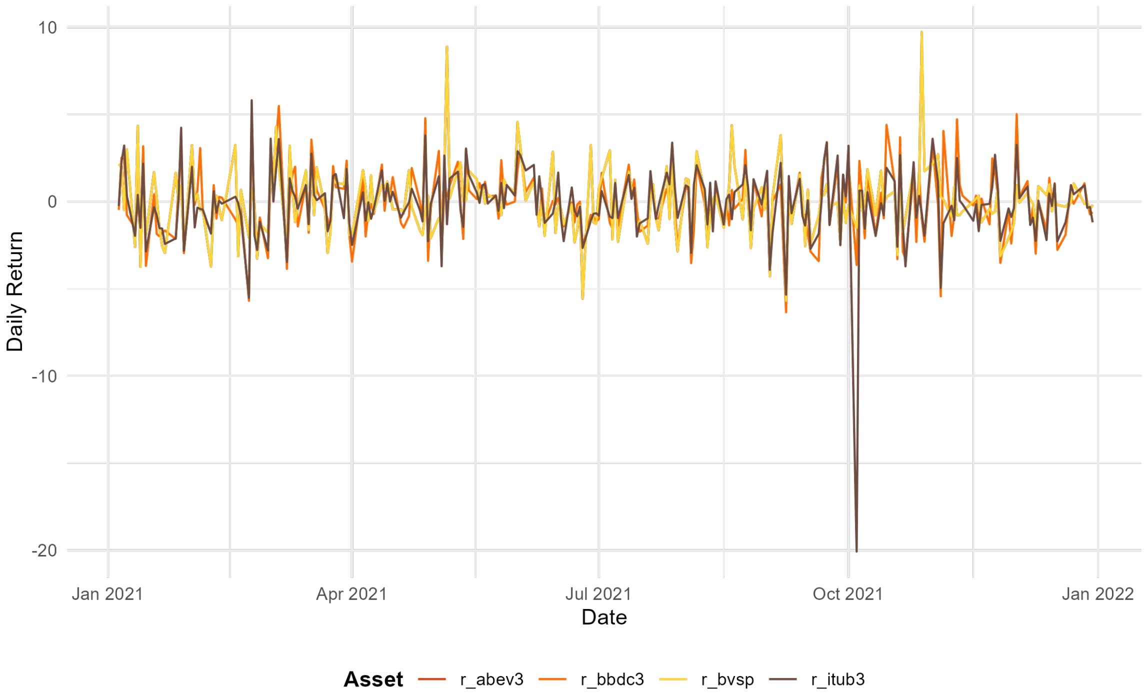
```r
calc_returns <- function(df, ticker) {
  df %>%
    mutate(
      Date = ymd(Date)
    ) %>%
    filter(Date >= as.Date("2021-01-01") & Date <= as.Date("2021-12-31")) %>%
    arrange(Date) %>%
    mutate(
      ret = 100 * (`Adj Close` / lag(`Adj Close`) - 1)  # retorno percentual simples * 1
    ) %>%
    dplyr::select(Date, ret) %>%
    rename(!!ticker := ret)
}

#using the function calc_returns to calculate the returns
bvsp_r  <- calc_returns(bvsp,  "r_bvsp")
bbdc3_r <- calc_returns(bbdc3, "r_bbdc3")
abev3_r <- calc_returns(abev3, "r_abev3")
itub3_r <- calc_returns(itub3, "r_itub3")

#calculating the returns by date

daily_returns_2021 <- bvsp_r %>%
  inner_join(bbdc3_r, by = "Date") %>%
  inner_join(abev3_r, by = "Date") %>%
  inner_join(itub3_r, by = "Date")
```

Figure 1: Daily Returns for 2021



Item 2.

```
#item 2

#importing the selic dataframe
selic <- read_delim("data/selic.csv", delim = ";",
                    locale = locale(decimal_mark = ","),
                    trim_ws = TRUE)

#adjusting it properly
selic <- selic %>%
  mutate(Data = dmy(Data)) %>%
  filter(year(Data) == 2021)

selic <- selic %>%
  mutate(`Fator diário` = as.numeric(`Fator diário`))

selic <- selic %>%
  mutate(Return_Selic = 100 * (`Fator diário` - 1))

selic_daily <- selic %>%
  dplyr::select(Date = Data, Selic_Daily)
```

```
#doing the merge with the main dataframe
daily_returns_selic_2021 <- daily_returns_2021 %>%
  inner_join(selic_daily, by = "Date")
```

Item 3.

```
#item 3

excess_returns <- daily_returns_selic_2021 %>%
  mutate(
    #market excess (IBOV - Selic)
    Market_Excess = bvsp_r - Selic_Daily,

    #returns excess of each stock (Ri - Selic)
    Excess_ABEV3 = abev3_r - Selic_Daily,
    Excess_BBDC3 = bbdc3_r - Selic_Daily,
    Excess_ITUB3 = itub3_r - Selic_Daily
  )
```

Item 4.

Table 3: GMM Estimation of CAPM Parameters

	Dependent Variable:		
	ITUB3	BBDC3	ABEV
	(1)	(2)	(3)
$\hat{\alpha}$	-0.133	-0.101	-0.00005^{***}
	(0.106)	(0.092)	(0.00000)
$\hat{\beta}$	0.464^{***}	0.487^{***}	1.000^{***}
	(0.098)	(0.094)	(0.00000)
Observations	246	246	246
Note:	*p<0.1; **p<0.05; ***p<0.01		

Item 5. The joint hypothesis test examines whether all three intercepts (α_1, α_2, and α_3) in the CAPM regressions are simultaneously equal to zero, as implied by the theoretical CAPM model. The chi-squared test statistic is extremely large ($\chi^2(3) = 2.26 \times 10^{11}$) with a p-value lower than 2.2×10^{-16}. This p-value is far below any conventional significance level we set, leading us to strongly reject the null hypothesis that all three alphas are zero. Therefore, there is clear statistical evidence that at least one of the assets delivers abnormal returns not explained by the CAPM market factor, indicating a failure of the CAPM model in this context.