Pset3 - Econometrics II

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1 Question 1 (General Time Series Regression - 90 points)

In this question, we will use the dataset `data_brazil.csv`. It contains four variables: *date*, which refers to the year (1901–2021); *real gdp growth pct*, which is the annual real GDP growth measured in percentage; *exchange rate real dolar annual average*, which corresponds to the annual average exchange rate between the Brazilian Real and the US Dollar measured in R$/US$; and *ipc fipe pct*, which represents annual inflation measured by the IPC-FIPE, also in percentage. Our goal is to forecast GDP growth in 2020. To do so, we will use data from the period 1942 to 2019.

1. Model 1: Run an ADL(2,1) using GDP growth as your dependent variable and Exchange Rate as your predictor. (a) Report the estimated coefficient and their standard errors. (b) Predict GDP growth in 2020 using model 1.

2. Model 2: Run an ADL(2,2) using GDP growth as your dependent variable and inflation as your predictor. (a) Report the estimated coefficient and their standard errors. (b) Predict GDP growth in 2020 using model 2.

3. Model 3: Run an general time series regression model using GDP growth as your dependent variable and two lags of GDP growth, Exchange Rate and Inflation as your predictors. (a) Report the estimated coefficient and their standard errors. (b) Predict GDP growth in 2020 using model 3.

4. Model 4: Run an ARMA(2,0) using GDP growth as your dependent variable. (a) Report the estimated coefficient and their standard errors. (b) Predict GDP growth in 2020 using model 4.

5. Which model generate the prediction that is closest to the realized value?

In Table 1, we present the results for items 1–4. Along the next comments we also discuss question 5. The findings indicate that the ADL(2,2) and ARMA(2,0) models produce quite similar estimates for the lags of the GDP variable. In contrast, the ADL(2,1) and the General model yield lower values for the same coefficients. Moreover, all these estimates are statistically significant. This same pattern holds for the second lag of the GDP variable. The estimates for the inflation variable are not statistically significant in any of the models. Lastly, we observe different estimates for the exchange rate coefficient. Since its statistical significance is only at the 10% level, we argue that caution should be exercised when interpreting the real contribution of this variable to the outcome.

Table 1: Results for GDP forecast using DL and AR models

	ADL(2,1)	ADL(2,2)	General	ARMA(2,0)
Intercept	2.49***	1.78***	3.07***	1.62***
	(0.883)	(0.746)	(1.064)	(0.665)
GDP_{t-1}	0.32***	0.37***	0.29**	0.37***
	(0.114)	(0.111)	(0.118)	(0.109)
GDP_{t-2}	0.23**	0.26**	0.20*	0.27***
	(0.110)	(0.110)	(0.115)	(0.107)
EXR_{t-1}	-0.59*		-0.77*	
	(0.401)		(0.439)	
INFL_{t-1}		0.00	-0.00	
		(0.001)	(0.001)	
INFL_{t-2}		-0.00		
		(0.001)		
AIC	406.76	410.50	407.73	407.02
BIC	418.41	424.48	421.72	416.34
GDP_{2020} Forecast	0.96	2.70	0.75	2.57
Forecast Lower Bound	-1.46	1.48	-1.71	1.49
Forecast Upper Bound	3.38	3.91	3.21	3.65

Note: $^{*}p < 0.10$, $^{**}p < 0.05$, $^{***}p < 0.01$. Standard errors are in parentheses. The variable EXR_t referes refers to real annual average exchange rate, and the variable $INFL_t$ stands for inflation. We used the funcion $lm()$ form the papckage `lmtest` to estimate a OLS for all models.

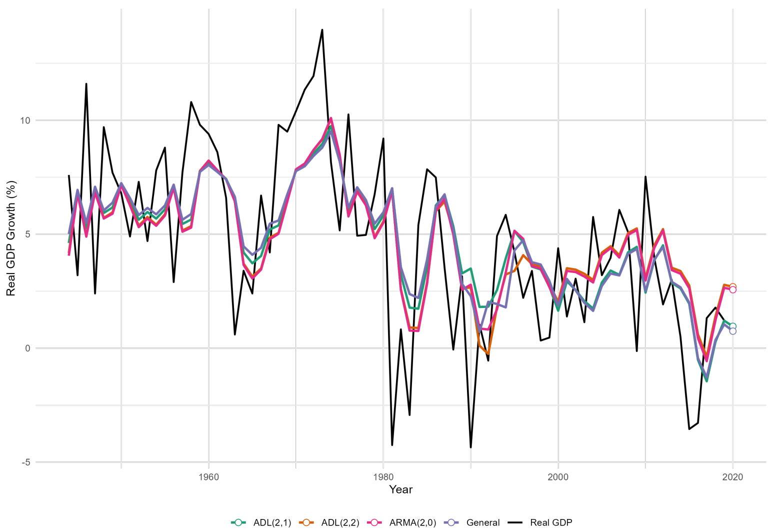

Figure 1: Comparasion of forecast results using DL and AR settings

The graphical interpretation in Figures 2 and 3 allows us to infer that the best approximations for the actual GDP realization in 2020 are given by the General model (0.75) and the ADL(2,1) model (0.96), although they have wider confidence intervals for this estimate.

However, when considering the information criteria and coefficient values, we argue that the ARMA(2,0) model is the best. Ultimately, we favor the General model as the most appropriate.

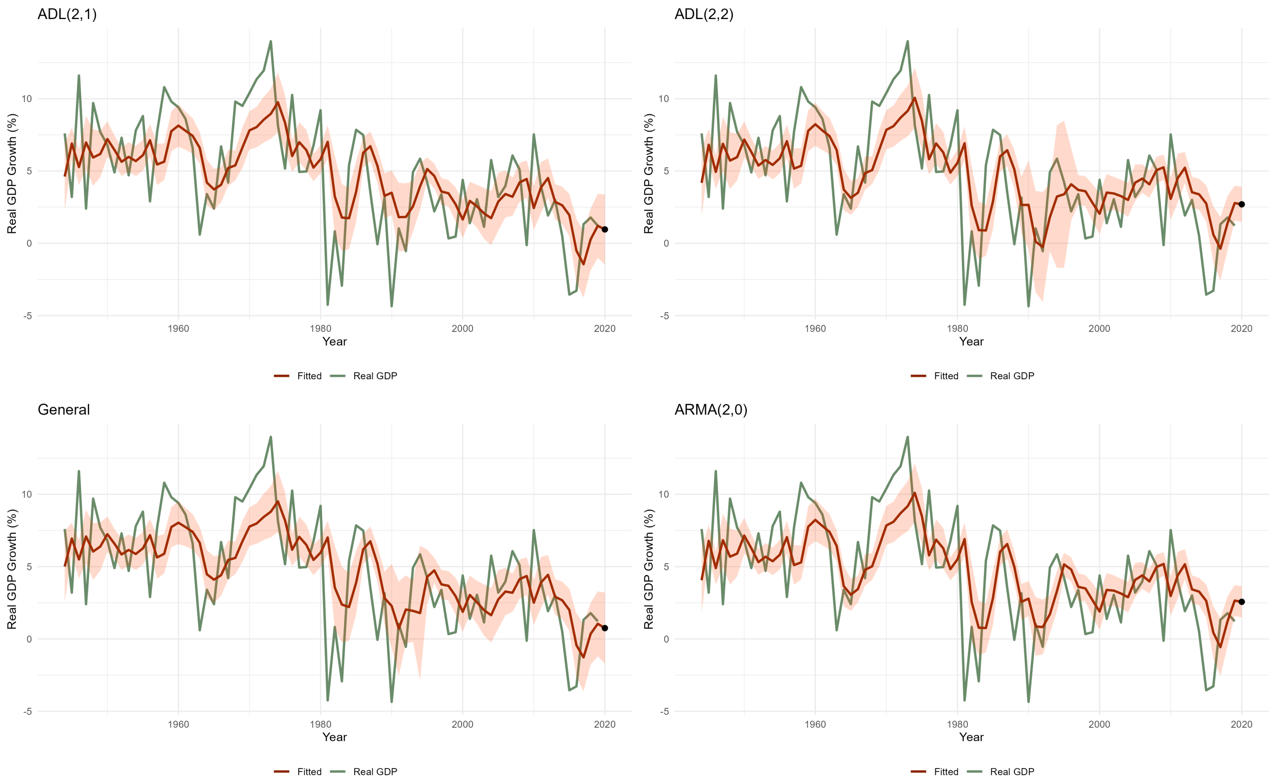

Figure 2: Forecast results using DL and AR settings

2 Question 2 (Expected Value of a VAR (p) Process)

Let $\{Y_t\}$ be a $n \times 1$ process wich is covariance stationary and follows a $VAR(p)$ model. Find its mean $\mu := \mathrm{E}[Y_t]$.

Folowing the definition in Hamilton (2020), let $\{Y_t\}$ be a *pth-order vector autoregression* process wich is covariance stationary, where c is a $n \times 1$ vector of constants, ϕ_j is a $n \times n$ matrix of coefficients for each $j \in \{1, ..., p\}$, and ε_t is a generalization of a white noise. Thus, this process follows a $VAR(p)$ model such that,

$$\mathbf{y}_t = c + \Phi_1 \mathbf{y}_{t-1} + \Phi_2 \mathbf{y}_{t-2} + \cdots + \Phi_p \mathbf{y}_{t-p} + \varepsilon_t$$

where $\mathrm{E}(\varepsilon_t) = 0$ and

$$\mathrm{E}(\varepsilon_t \varepsilon_\tau) = \begin{cases} \sigma^2 & \text{if } t = \tau \\ 0 & \text{otherwise} \end{cases}$$

Proof. We apply the expectation on both sides of the model,

$$\mathrm{E}[\mathbf{y}_t] = \mathrm{E}\left[c + \Phi_1 \mathbf{y}_{t-1} + \cdots + \Phi_p \mathbf{y}_{t-p} + \varepsilon_t\right]$$

Leveraging the linearity of expectation and the assumption of stationarity, we get:

$$\mu = c + \Phi_1 \mu + \Phi_2 \mu + \cdots + \Phi_p \mu$$

Factoring out μ,

$$\mu = \left(\Phi_1 + \Phi_2 + \cdots + \Phi_p\right) \mu + c$$

Bringing terms to one side,

$$\left[I - \sum_{j=1}^{p} \Phi_j\right] \mu = c$$

Therefore,

$$\mu = \left[I - \sum_{j=1}^{p} \Phi_j\right]^{-1} c$$

□

3 Question 3 (The $j-$th Autocovariance Matrix)

Proof that $\Gamma_j' = \Gamma_{-j}$.

Let $\{Y_t\}$ be an $n \times 1$ vector stochastic process that is covariance-stationary and follows a $VAR(p)$ model with mean μ. Following Hamilton (2020), we define its j-th order autocovariance matrix as,

$$\Gamma_j \coloneqq \mathbb{E}[(Y_t - \mu)(Y_{t-j} - \mu)']$$

We claim that $\Gamma_j' = \Gamma_{-j}$ under the assumption of covariance stationarity, which implies that the second moments of the process depend only on the time lag j, and not on the specific date t.

Proof. First apply the property of transposition inside the expectation to get

$$\Gamma_{-j} = (\mathbb{E}[(Y_{t+j} - \mu)(Y_t - \mu)'])'$$

Since expectation and transposition commute, and by renaming variables (or noting symmetry of the lag), we can write,

$$
\begin{aligned}
\Gamma_{-j} &= (\mathbb{E}[(Y_t-\mu)(Y_{t-j}-\mu)'])' \\
&= \Gamma_j'
\end{aligned}
$$

Therefore, we conclude that:

$$
\Gamma_j' = \Gamma_{-j}
$$

$\square$

4 Question 4 (Vector MA (q) Process)

Prove that any vector MA (q) process is covariance-stationary.

Following the definition in Hamilton (2020), let $\{Y_t\}$ be a moving average process of order q taing the form,

$$
Y_t = \boldsymbol{\mu} + \boldsymbol{\varepsilon}_t + \boldsymbol{\Theta}_1\boldsymbol{\varepsilon}_{t-1} + \boldsymbol{\Theta}_2\boldsymbol{\varepsilon}_{t-2} + \cdots + \boldsymbol{\Theta}_q\boldsymbol{\varepsilon}_{t-q},
$$

where $\boldsymbol{\varepsilon}_t$ is a vector white noise process satisfying the standard properties of zero mean and covariance $\boldsymbol{\Omega}$, and $\boldsymbol{\Theta}_j$ denotes an $(n \times n)$ matrix of MA coefficients for $j = 1, 2, \ldots, q$. The mean of Y_t is $\boldsymbol{\mu}$, and the variance is

$$
\begin{aligned}
\boldsymbol{\Gamma}_0 &= \mathbb{E}[(Y_t-\boldsymbol{\mu})(Y_t-\boldsymbol{\mu})'] \\
&= \mathbb{E}[\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'] + \boldsymbol{\Theta}_1\mathbb{E}[\boldsymbol{\varepsilon}_{t-1}\boldsymbol{\varepsilon}_{t-1}']\boldsymbol{\Theta}_1' + \boldsymbol{\Theta}_2\mathbb{E}[\boldsymbol{\varepsilon}_{t-2}\boldsymbol{\varepsilon}_{t-2}']\boldsymbol{\Theta}_2' + \cdots + \boldsymbol{\Theta}_q\mathbb{E}[\boldsymbol{\varepsilon}_{t-q}\boldsymbol{\varepsilon}_{t-q}']\boldsymbol{\Theta}_q' \\
&= \boldsymbol{\Omega} + \boldsymbol{\Theta}_1\boldsymbol{\Omega}\boldsymbol{\Theta}_1' + \boldsymbol{\Theta}_2\boldsymbol{\Omega}\boldsymbol{\Theta}_2' + \cdots + \boldsymbol{\Theta}_q\boldsymbol{\Omega}\boldsymbol{\Theta}_q'.
\end{aligned}
$$

For a vector MA(q) process, the autocovariances are given by:

$$
\boldsymbol{\Gamma}_j = \begin{cases}
\boldsymbol{\Theta}_j\boldsymbol{\Omega} + \boldsymbol{\Theta}_{j+1}\boldsymbol{\Omega}\boldsymbol{\Theta}_1' + \boldsymbol{\Theta}_{j+2}\boldsymbol{\Omega}\boldsymbol{\Theta}_2' + \cdots + \boldsymbol{\Theta}_q\boldsymbol{\Omega}\boldsymbol{\Theta}_{q-j}' & \text{for } j = 1, 2, \ldots, q, \\
\boldsymbol{\Omega}\boldsymbol{\Theta}_{-j}' + \boldsymbol{\Theta}_1\boldsymbol{\Omega}\boldsymbol{\Theta}_{-j+1}' + \boldsymbol{\Theta}_2\boldsymbol{\Omega}\boldsymbol{\Theta}_{-j+2}' + \cdots + \boldsymbol{\Theta}_{q+j}\boldsymbol{\Omega}\boldsymbol{\Theta}_q' & \text{for } j = -1, -2, \ldots, -q, \\
\mathbf{0} & \text{for } |j| > q,
\end{cases}
$$

Proof. We are interested in showt that $\{Y_t\}$ is covariance-stationary. Then let

$$
\boldsymbol{\Gamma}_j = \mathbb{E}\left[(Y_t-\boldsymbol{\mu})(Y_{t-j}-\boldsymbol{\mu})'\right] = \mathbb{E}[Y_tY_{t-j}'].
$$

Notice that we can expand Y_t and Y_{t-j} using their definitions to get,

$$
\begin{aligned}
Y_t &= \boldsymbol{\varepsilon}_t + \boldsymbol{\Theta}_1\boldsymbol{\varepsilon}_{t-1} + \cdots + \boldsymbol{\Theta}_q\boldsymbol{\varepsilon}_{t-q}, \\
Y_{t-j} &= \boldsymbol{\varepsilon}_{t-j} + \boldsymbol{\Theta}_1\boldsymbol{\varepsilon}_{t-j-1} + \cdots + \boldsymbol{\Theta}_q\boldsymbol{\varepsilon}_{t-j-q}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\boldsymbol{\Gamma}_j &= \mathbb{E}\left[\left(\sum_{i=0}^{q} \boldsymbol{\Theta}_i \boldsymbol{\varepsilon}_{t-i}\right)\left(\sum_{k=0}^{q} \boldsymbol{\Theta}_k \boldsymbol{\varepsilon}_{t-j-k}\right)'\right] \quad (\text{with } \boldsymbol{\Theta}_0 = \boldsymbol{I}_n) \\
&= \sum_{i=0}^{q}\sum_{k=0}^{q} \boldsymbol{\Theta}_i \, \mathbb{E}\left[\boldsymbol{\varepsilon}_{t-i}\boldsymbol{\varepsilon}'_{t-j-k}\right] \boldsymbol{\Theta}'_k.
\end{aligned}
$$

Now conisider the case for $j = 0$,

$$
\begin{aligned}
\Gamma_0 &= \mathbb{E}\left[(\boldsymbol{\varepsilon}_t + \boldsymbol{\Theta}_1\boldsymbol{\varepsilon}_{t-1} + \cdots + \boldsymbol{\Theta}_q\boldsymbol{\varepsilon}_{t-q})(\boldsymbol{\varepsilon}_t + \boldsymbol{\Theta}_1\boldsymbol{\varepsilon}_{t-1} + \cdots + \boldsymbol{\Theta}_q\boldsymbol{\varepsilon}_{t-q})'\right] \\
&= \mathbb{E}[\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}'_t] + \mathbb{E}[\boldsymbol{\Theta}_1\boldsymbol{\varepsilon}_{t-1}\boldsymbol{\varepsilon}'_{t-1}\boldsymbol{\Theta}'_1] + \cdots + \mathbb{E}[\boldsymbol{\Theta}_q\boldsymbol{\varepsilon}_{t-q}\boldsymbol{\varepsilon}'_{t-q}\boldsymbol{\Theta}'_q] \\
&= \boldsymbol{\Omega} + \sum_{i=1}^{q} \boldsymbol{\Theta}_i\boldsymbol{\Omega}\boldsymbol{\Theta}'_i,
\end{aligned}
$$

where we used the fact that $\{\boldsymbol{\varepsilon}_t\}$ is a white noise process, so that

$$
\mathbb{E}[\boldsymbol{\varepsilon}_{t-i}\boldsymbol{\varepsilon}'_{t-j-k}] = \begin{cases} \boldsymbol{\Omega}, & \text{if } i = j + k, \\ \mathbf{0}, & \text{otherwise.} \end{cases}
$$

Moreover, in case $j \in \{1, \ldots, q\}$, it is easy to see that,

$$
\begin{aligned}
\Gamma_j &= \mathbb{E}\left[(\boldsymbol{\varepsilon}_t + \boldsymbol{\Theta}_1\boldsymbol{\varepsilon}_{t-1} + \cdots + \boldsymbol{\Theta}_q\boldsymbol{\varepsilon}_{t-q})(\boldsymbol{\varepsilon}_{t-j} + \boldsymbol{\Theta}_1\boldsymbol{\varepsilon}_{t-j-1} + \cdots + \boldsymbol{\Theta}_q\boldsymbol{\varepsilon}_{t-j-q})'\right] \\
&= \boldsymbol{\Theta}_j\boldsymbol{\Omega} + \boldsymbol{\Theta}_{j+1}\boldsymbol{\Omega}\boldsymbol{\Theta}'_1 + \cdots + \boldsymbol{\Theta}_q\boldsymbol{\Omega}\boldsymbol{\Theta}'_{q-j} \\
&= \boldsymbol{\Theta}_j\boldsymbol{\Omega} + \sum_{i=j+1}^{q} \boldsymbol{\Theta}_i\boldsymbol{\Omega}\boldsymbol{\Theta}'_{i-j}.
\end{aligned}
$$

And last, in case $j \in \{-q, \ldots, -1\}$, let $h = -j$, then we get,

$$
\begin{aligned}
\Gamma_j &= \mathbb{E}\left[(\boldsymbol{\varepsilon}_t + \boldsymbol{\Theta}_1\boldsymbol{\varepsilon}_{t-1} + \cdots + \boldsymbol{\Theta}_q\boldsymbol{\varepsilon}_{t-q})(\boldsymbol{\varepsilon}_{t+h} + \boldsymbol{\Theta}_1\boldsymbol{\varepsilon}_{t+h-1} + \cdots + \boldsymbol{\Theta}_q\boldsymbol{\varepsilon}_{t+h-q})'\right] \\
&= \boldsymbol{\Omega}\boldsymbol{\Theta}'_h + \boldsymbol{\Theta}_1\boldsymbol{\Omega}\boldsymbol{\Theta}'_{h+1} + \cdots + \boldsymbol{\Theta}_{q-h}\boldsymbol{\Omega}\boldsymbol{\Theta}'_q \\
&= \boldsymbol{\Omega}\boldsymbol{\Theta}'_{-j} + \sum_{i=1}^{q+j} \boldsymbol{\Theta}_i\boldsymbol{\Omega}\boldsymbol{\Theta}'_{j+i}.
\end{aligned}
$$

Note that in general $\Gamma_j \neq \Gamma_{-j}$, since the matrices $\boldsymbol{\Theta}_i$ may be asymmetric or unequal.

Hence, in case $|j| > q$, there are no overlapping lags between Y_t and Y_{t-j}, so all cross terms dissapear and we get,

$$\mathbf{\Gamma}_j = \mathbf{0}.$$

This way we conclude that the process $\{Y_t\}$ is **covariance stationary**. $\square$

5 Question 5 (A Type of LLN)

Proposition 1 *Let $\{Y_t\}$ be a covariance-stationary process with moments given by*

$$\begin{aligned}\mathbb{E}[Y_t] &= \mu, \\ \mathbb{E}\left[(Y_t - \mu)(Y_{t-j} - \mu)'\right] &= \Gamma_j\end{aligned}$$

and with absolutely summable autocovariances (i.e., $\left(\sum_{\nu=-\infty}^{+\infty} \Gamma_\nu\right) \in \mathbb{R}$). Assume we have a sample of size T drawn from $\{Y_t\}$. Then, the sample mean $\bar{Y}_T := \dfrac{1}{T} \sum_{t=1}^{T} Y_t$ satisfies

1. $\bar{Y}_T \xrightarrow{p} \mu$
2. $\lim_{T\to+\infty} \left\{T \cdot \mathbb{E}\left[(\bar{Y}_T - \mu)(\bar{Y}_T - \mu)'\right]\right\} = \sum_{\nu=-\infty}^{+\infty} \Gamma_\nu$.

Proof. We shall start from the second point, as in Hamilton (p. 280). First, note that if we open the terms that make $\bar{Y}_T$, we can write the autocovariance matrix as:

$$\begin{aligned}\mathbb{E}\left[(\bar{Y}_T - \mu)(\bar{Y}_T - \mu)'\right] = \frac{1}{T^2}\mathbb{E}\Big\{&(Y_1 - \mu + Y_2 - \mu + \cdots + Y_T - \mu) \\ &\cdot (Y_1 - \mu + Y_2 - \mu + \cdots + Y_T - \mu)'\Big\}.\end{aligned}$$

Expanding this expression results in a double sum over all cross-products,

$$\mathbb{E}\left[(\bar{Y}_T - \mu)(\bar{Y}_T - \mu)'\right] = \frac{1}{T^2} \sum_{t=1}^{T} \sum_{s=1}^{T} \mathbb{E}\left[(Y_t - \mu)(Y_s - \mu)'\right].$$

Since the process is covariance-stationary, we have that,

$$\mathbb{E}\left[(Y_t - \mu)(Y_s - \mu)'\right] = \Gamma_{t-s},$$

and so,

$$\mathbb{E}\left[(\bar{Y}_T - \mu)(\bar{Y}_T - \mu)'\right] = \frac{1}{T^2} \sum_{t=1}^{T} \sum_{s=1}^{T} \Gamma_{t-s}.$$

We can rearrange the sum by grouping terms with the same lag $v = t - s$, which leads to,

$$\mathbb{E}\left[(\bar{Y}_T - \mu)(\bar{Y}_T - \mu)'\right] = \frac{1}{T^2} \sum_{v=-(T-1)}^{T-1} (T - |v|)\Gamma_v.$$

Now, multiplying both sides by T, we obtain,

$$\begin{aligned} T \cdot \mathbb{E}\left[(\bar{Y}_T - \mu)(\bar{Y}_T - \mu)'\right] &= \Gamma_0 + \frac{T-1}{T}\Gamma_1 + \frac{T-2}{T}\Gamma_2 + \cdots + \frac{1}{T}\Gamma_{T-1} \\ &\quad + \frac{T-1}{T}\Gamma_{-1} + \frac{T-2}{T}\Gamma_{-2} + \cdots + \frac{1}{T}\Gamma_{-(T-1)} \\ &= \sum_{v=-(T-1)}^{T-1} \left(1 - \frac{|v|}{T}\right)\Gamma_v. \end{aligned} \tag{1}$$

Observe that as $T \to \infty$, we have $\frac{|v|}{T} \to 0$ for any fixed v. Then, we find that,

$$\lim_{T\to\infty} T \cdot \mathbb{E}\left[(\bar{Y}_T - \mu)(\bar{Y}_T - \mu)'\right] = \sum_{v=-\infty}^{+\infty} \Gamma_v.$$

Notably, consider the following matrix,

$$S := \sum_{v=-\infty}^{+\infty} \Gamma_v - T \cdot \mathbb{E}\left[(\bar{Y}_T - \mu)(\bar{Y}_T - \mu)'\right],$$

which converges to zero as $T \to \infty$.
Here, note that all autocovariances of order higher than T are not in Equation (1) Then,

$$S = \sum_{|v|\geq T} \Gamma_v + \sum_{v=-(T-1)}^{T-1} \frac{|v|}{T}\Gamma_v.$$

Let $\gamma_{ij}^{(v)}$ denote the row i, column j element of Γ_v. We can express the row i, column j element of S as,

$$\sum_{|v|\geq T} \gamma_{ij}^{(v)} + \sum_{v=-(T-1)}^{T-1} \frac{|v|}{T}\gamma_{ij}^{(v)}.$$

By assumption, the autocovariances are absolutely summable, i.e.,

$$\sum_{v=-\infty}^{+\infty} \|\Gamma_v\| < \infty.$$

This implies that for a sufficiently large lag, the terms $\gamma_{ij}^{(v)}$ become arbitrarily small. Formally, for any $\varepsilon > 0$, there might exist $q \in \mathbb{N}$ such that,

$$\sum_{|v|\geq q} \left|\gamma_{ij}^{(v)}\right| < \frac{\varepsilon}{2}.$$

Now, for $T > q$, we have,

$$\left| \sum_{|v| \geq T} \gamma_{ij}^{(v)} + \sum_{v=-(T-1)}^{T-1} \frac{|v|}{T} \gamma_{ij}^{(v)} \right| < \frac{\varepsilon}{2} + \sum_{v=-(T-1)}^{T-1} \frac{|v|}{T} \left| \gamma_{ij}^{(v)} \right|$$
$$\leq \frac{\varepsilon}{2} + \frac{1}{T} \sum_{v=-(T-1)}^{T-1} |v| \cdot |\gamma_{ij}^{(v)}| \xrightarrow{T \to \infty} \frac{\varepsilon}{2}.$$

Therefore, each element of S converges to zero as $T \to \infty$, so the matrix S converges to the zero matrix. That is,

$$\lim_{T \to \infty} S = 0 \iff \sum_{v=-\infty}^{\infty} \Gamma_v = \lim_{T \to \infty} T \cdot \mathbb{E}\left[(\bar{Y}_T - \mu)(\bar{Y}_T - \mu)' \right],$$

which proves the second point.

Now, observe that:

$$\lim_{T \to \infty} \mathbb{E}\left[(\bar{Y}_T - \mu)(\bar{Y}_T - \mu)' \right] = 0 \quad \Rightarrow \quad \lim_{T \to \infty} \mathbb{E}[(\bar{Y}_{T,i} - \mu_i)^2] = 0.$$

This implies:

$$\bar{Y}_{T,i} \xrightarrow{p} \mu_i, \quad \text{for all } i = 1, \ldots, n,$$

and therefore:

$$\bar{Y}_T \xrightarrow{p} \mu,$$

which proves the first point.

□

6 Question 6 $VAR(p)$ Model

Your goal is to forecast GDP growth in 2020. To do so, we will use data from 1942 to 2019.

1. First, we will use a $VAR(p)$ to forecast GDP growth in 2020.
(a) Model A: Set p = 1 and estimate a reduced-form VAR(1) model.

i) Report the estimated coefficient and their standard errors.

ii) Predict GDP growth in 2020 using model A.

Table 2: Results for real GDP forecast and reduced forms using VAR(p) models with $p = 1$

	Dependent Variables		
	GDP	EXR	INFL
AIC	415.417	-20.310	1100.027
BIC	427.136	-8.591	1111.746
GDP_{2020} Forecast	0.234	-	-
Forecast Lower Bound	-6.541	-	-
Forecast Upper Bound	7.009	-	-
Intercept	4.478***	0.011	182.530*
	(0.874)	(0.052)	(74.463)
EXR_{t-1}	-1.164**	1.041**	-61.510
	(0.411)	(0.024)	(35.044)
$INFL_{t-1}$	-0.002	0.000	0.632**
	(0.001)	(0.000)	(0.087)
GDP_{t-1}	0.292**	-0.001	-17.725
	(0.111)	(0.007)	(9.499)

Note: $^{*}p < 0.10$, $^{**}p < 0.05$, $^{***}p < 0.01$. Standard errors are in parentheses. Variables are lagged by number of periods indicated in the subscript. EXR refers to the real annual average exchange rate. INFL refers to the inflation index. Models estimated by VAR with $p = 1$ lags.

(b) Model B: Set p = 2 and estimate a reduced-form $VAR(2)$ model.

i) Report the estimated coefficient and their standard errors.

ii) Predict GDP growth in 2020 using model B.

Table 3: Results for real GDP forecast and reduced forms using VAR(p) models with $p = 2$

	Dependent Variables		
	GDP	EXR	INFL
AIC	410.966	-19.424	1086.988
BIC	429.612	-0.778	1105.634
GDP_{2020} Forecast	0.725	-	-
Forecast Lower Bound	-5.966	-	-
Forecast Upper Bound	7.416	-	-
Intercept	3.200***	0.000	288.045*
	(1.085)	(0.064)	(92.642)
EXR_{t-1}	-1.497*	1.334***	-194.682
	(2.051)	(0.121)	(175.155)
EXR_{t-2}	0.751	-0.309*	112.603
	(2.122)	(0.125)	(181.280)
$INFL_{t-1}$	0.000	0.000	0.707***
	(0.001)	(0.000)	(0.120)
$INFL_{t-2}$	-0.001	0.000	-0.173
	(0.001)	(0.000)	(0.117)
GDP_{t-1}	0.282**	0.001	-18.671
	(0.121)	(0.007)	(10.306)
GDP_{t-2}	0.195	0.000	-13.218
	(0.117)	(0.007)	(9.982)

Note: $^{*}p < 0.10$, $^{**}p < 0.05$, $^{***}p < 0.01$. Standard errors are in parentheses. Variables are lagged by number of periods indicated in the subscript. EXR refers to the real annual average exchange rate. INFL refers to the inflation index. Models estimated by VAR with $p = 2$ lags.

(c) Model C: Set p = 3 and estimate a reduced-form $VAR(3)$ model.

i) Report the estimated coefficient and their standard errors.

ii) Predict GDP growth in 2020 using model C.

Table 4: Results for real GDP forecast and reduced forms using VAR(p) models with $p = 3$

	Dependent Variables		
	GDP	EXR	INFL
AIC	411.836	-15.023	1058.493
BIC	437.328	10.470	1083.985
GDP_{2020} Forecast	0.799	-	-
Forecast Lower Bound	-6.052	-	-
Forecast Upper Bound	7.650	-	-
Intercept	3.269**	0.058	217.662**
	(1.292)	(0.075)	(96.271)
EXR_{t-1}	-1.557	1.342***	-97.002
	(2.181)	(0.127)	(162.545)
EXR_{t-2}	0.941	-0.341*	-199.978
	(3.544)	(0.206)	(264.062)
EXR_{t-3}	-0.141	0.008	239.065
	(2.294)	(0.133)	(170.953)
$INFL_{t-1}$	-0.001	0.000	0.800***
	(0.001)	(0.000)	(0.111)
$INFL_{t-2}$	-0.001	0.000	-0.568***
	(0.002)	(0.000)	(0.136)
$INFL_{t-3}$	0.000	0.000	0.495***
	(0.001)	(0.000)	(0.108)
GDP_{t-1}	0.269**	0.003	-12.038
	(0.128)	(0.007)	(9.569)
GDP_{t-2}	0.235*	0.001	-15.233
	(0.131)	(0.008)	(9.755)
GDP_{t-3}	-0.043	-0.012*	1.963
	(0.123)	(0.007)	(9.176)

Note: $^{*}p < 0.10$, $^{**}p < 0.05$, $^{***}p < 0.01$. Standard errors are in parentheses. Variables are lagged by number of periods indicated in the subscript. EXR refers to the real annual average exchange rate. INFL refers to the inflation index. Models estimated by VAR with $p = 3$ lags.

(d) Which model generate the prediction that is closest to the realized value?

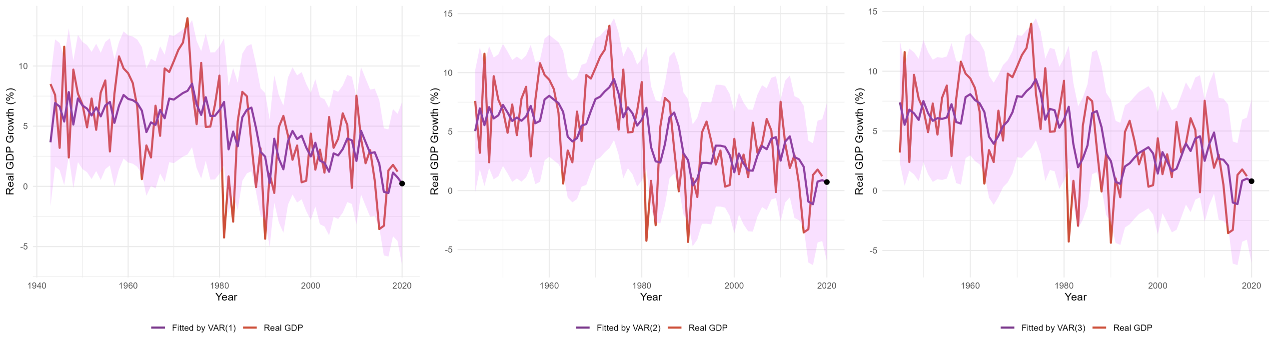

Figure 3: $VAR(p)$ estimations with GDP as outcome variable

In Tables 2, 3, 4 we present the results of our estimations as well as the forecast for the year 2020 for the variable GDP. Moreover, in Figure 3 the graphical results of the fitted values for each $VAR(p)$, such that $p \in \{1, 2, 3\}$, are displayed. Based on the estimated results and the graphical representation, the most precise forecast is the one from $VAR(1)$, with a forecasted value of 0.23. However, if we look at the best model in terms of Information Criteria, comparing the AIC[1] of each one, we find that the "best" model is $VAR(2)$. In the end, we argue that, since the lagged regressors in $VAR(1)$ are all significant, it is the most appropriate model to forecast the variable GDP_t in 2020.

2. Now, we will focus on a $VAR(2)$ model and focus on structural IRFs.

(a) Choose the order of your variables and justify your exclusion restrictions.

In our dataset, we were given three variables: the annual real GDP growth, denoted as GDP_t, the real exchange rate averaged by year, EXR_t, and the annual inflation, $INFL_t$. The first variable we include in the vector is $INFL_t$. As widely discussed in macroeconomics, especially in the context of price behavior, inflation typically does not respond contemporaneously to shocks in GDP or the exchange rate. This is particularly true in Brazil, where inflation dynamics are influenced by institutional and market rigidities. Therefore, we consider this variable sufficiently exogenous to allow inference about the effects of current shocks in GDP_t and EXR_t.

The second variable in the ordering is EXR_t, since it is closely linked to inflation dynamics but is relatively less affected by contemporaneous GDP shocks. The exchange rate is a key channel through which price levels are influenced, especially in an economy with high exposure to international trade and capital flows.

Lastly, we place GDP_t as the third variable. This is because it is evidently impacted by both contemporaneous shocks to EXR_t and $INFL_t$. The exchange rate affects GDP in both directions: on one hand, a more favorable exchange rate can boost exports, which are a significant component of the Brazilian economy, thereby stimulating GDP. On the other hand, higher inflation tends to hinder GDP growth, as it often prompts the central bank to raise interest rates, which dampens economic activity.

(b) Estimate and plot all nine structural impulse response functions and their 90%-confidence intervals based on 1,000 bootstrap repetitions.

[1] AIC is the most recommended when making forecasts

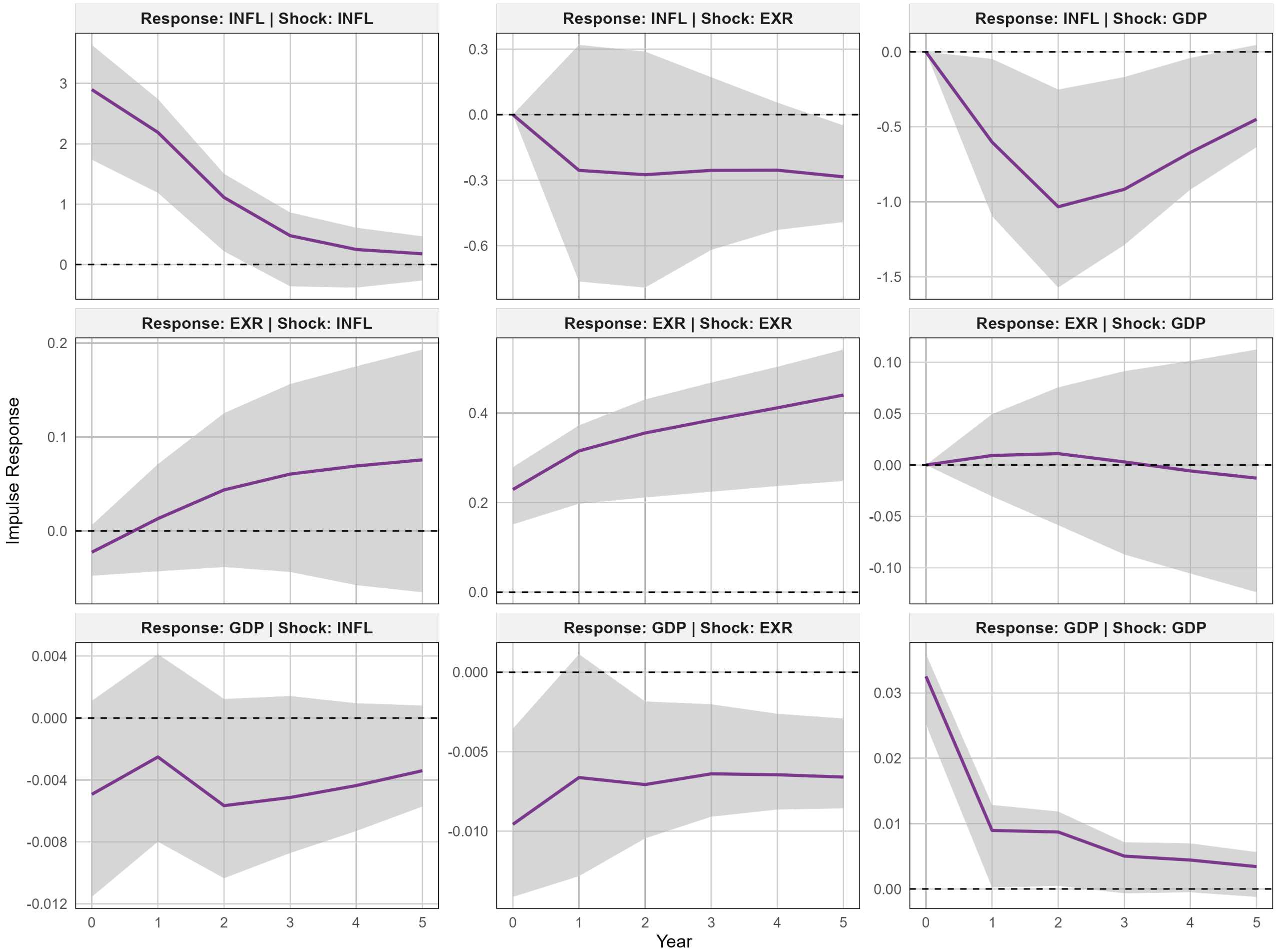

Figure 4: $VAR(2)$ SIRFs

In Figure 4 we present the result of the Structural Impulse-Response Functions (SIRFs). We used the Cholesky Decomposition, therefore the SIRFs should be interpreted as the "effect" of one standard deviation of the Impulse variable on the Response of each variable. Each row of the figure represents the responses of a given variable to shocks in all three variables. For example, the first row shows the response of inflation to shocks in inflation, exchange rate, and GDP. This ordering follows the structure we defined for tracing the transmission of shocks.

(c) Do you believe your results are credible? Justify your answer. (20 points)

When we look at the graphical results in Figure 4 in the second line, we observe that the exchange rate responds slightly positively to price shocks, which is broadly consistent with the macroeconomic literature. This are very related with the common view that higher inflation often leads to currency depreciation, especially in emerging economies like Brazil.

However, the impulse response of inflation appears to be relatively noisy and less stable. This may be due to the presence of historical episodes of very high inflation in Brazil, particularly during the late 20th century, which likely distort the estimation of the average response of inflation to shocks. As a result, part of the estimated responses may not be fully representative of the current inflation dynamics under a more stable monetary regime.

In the long run, the effects of shocks to EXR seem to dissipate, as the impulse responses converge to zero and the confidence intervals include the null value. We observe this same pattern in all variables. This behavior supports the idea that the system is relatively stationary and that shocks have only temporary effects.

On the other hand, a puzzling result arises from the lack of a suficiently large reaction of the exchange rate to GDP shocks. This is somewhat counterintuitive, as in an open economy we would expect that changes in output affect the trade balance and capital flows, thereby influencing the exchange rate. This inconsistency suggests that our model might be omitting important structural variables, such as interest rates, fiscal indicators (e.g., public debt or tax revenues), or external variables (e.g., commodity prices or global financial conditions). Moreover, GDP appears to be more responsive to shocks in the exchange rate, as most of the confidence interval does not include zero. This result is quite reasonable, especially in the context of the Brazilian economy.

Therefore, while some of our results are in line with economic theory and seem credible, others appear questionable and may reflect model limitations. Improving the specification by including additional relevant macroeconomic variables and testing alternative identification schemes could help increase the credibility and robustness of the results.

Statement Regarding Artificial Intelligence Usage

This project made use of ChatGPT 4.0 for coding assistance and language correction.

References

Hamilton, James D (2020). *Time series analysis*. Princeton university press.