2 Slope stability analysis methods

2.1 Introduction

In this chapter, the basic formulation of the two-dimensional (2D) slope stability method will be discussed. Presently, the theory and software for two-dimensional slope stability are rather mature, but there are still some important and new findings which will be discussed in this chapter. Most of the methods discussed in this chapter are available in the program SLOPE 2000 developed by Cheng, an outline of which is given in the Appendix.

2.1.1 Types of stability analysis

There are two different ways for carrying out slope stability analyses. The first approach is the *total stress approach* which corresponds to clayey slopes or slopes with saturated sandy soils under short-term loadings with the pore pressure not dissipated. The second approach corresponds to the *effective stress approach* which applies to long-term stability analyses in which drained conditions prevail. For natural slopes and slopes in residual soils, they should be analysed with the effective stress method, considering the maximum water level under severe rainstorms. This is particularly important for cities such as Hong Kong where intensive rainfall may occur over a long period, and the water table can rise significantly after a rainstorm.

2.1.2 Definition of the factor of safety (FOS)

The factor of safety for slope stability analysis is usually defined as the ratio of the ultimate shear strength divided by the mobilized shear stress at incipient failure. There are several ways in formulating the factor of safety *F*. The most common formulation for *F* assumes the factor of safety to be constant along the slip surface, and it is defined with respect to the force or moment equilibrium:

1 *Moment equilibrium*: generally used for the analysis of rotational landslides. Considering a slip surface, the factor of safety F_m defined with respect to moment is given by:

$$F_m = \frac{M_r}{M_d}, \tag{2.1}$$

where M_r is the sum of the resisting moments and M_d is the sum of the driving moment. For a circular failure surface, the centre of the circle is usually taken as the moment point for convenience. For a non-circular failure surface, an arbitrary point for the moment consideration may be taken in the analysis. It should be noted that for methods which do not satisfy horizontal force equilibrium (e.g. Bishop Method), the factor of safety will depend on the choice of the moment point as 'true' moment equilibrium requires force equilibrium. Actually, the use of the moment equilibrium equation without enforcing the force equilibrium cannot guarantee 'true' moment equilibrium.

2 *Force equilibrium*: generally applied to translational or rotational failures composed of planar or polygonal slip surfaces. The factor of safety F_f defined with respect to force is given by:

$$F_f = \frac{F_r}{F_d}, \tag{2.2}$$

where F_r is the sum of the resisting forces and F_d is the sum of the driving forces.

For 'simplified methods' which cannot fulfil both force and moment equilibrium simultaneously, these two definitions will be slightly different in the values and the meaning, but most design codes do not have a clear requirement on these two factors of safety, and a single factor of safety is specified in many design codes. A slope may actually possess several factors of safety according to different methods of analysis which are covered in the later sections.

A slope is considered as unstable if $F \leq 1.0$. It is however common that many natural stable slopes have factors of safety less than 1.0 according to the commonly adopted design practice, and this phenomenon can be attributed to:

1 application of additional factor of safety on the soil parameters is quite common;
2 the use of a heavy rainfall with a long recurrent period in the analysis;
3 three-dimensional effects are not considered in the analysis;
4 additional stabilization due to the presence of vegetation or soil suction is not considered.

An acceptable factor of safety should be based on the consideration of the recurrent period of heavy rainfall, the consequence of the slope failures, the knowledge about the long-term behaviour of the geological materials and the accuracy of the design model. The requirements adopted in Hong Kong

Table 2.1 Recommended factors of safety *F* (GEO, Hong Kong, 1984)

Risk of economic losses	*Risk of human losses*		
	Negligible	*Average*	*High*
Negligible	1.1	1.2	1.4
Average	1.2	1.3	1.4
High	1.4	1.4	1.5

Table 2.2 Recommended factor of safeties for rehabilitation of failed slopes (GEO, Hong Kong, 1984)

Risk of human losses	*F*
Negligible	>1.1
Average	>1.2
High	>1.3

Note: *F* for recurrent period of 10 years.

are given in Tables 2.1 and 2.2, and these values are found to be satisfactory in Hong Kong. For the slopes at the Three Gorges Project in China, the slopes are very high and steep, and there is a lack of previous experience as well as the long-term behaviour of the geological materials; a higher factor of safety is hence adopted for the design. In this respect, an acceptable factor of safety shall fulfil the basic requirement from the soil mechanics principle as well as the long-term performance of the slope.

The geotechnical engineers should consider the current slope conditions as well as the future changes, such as the possibility of cuts at the slope toe, deforestation, surcharges and excessive infiltration. For very important slopes, there may be a need to monitor the pore pressure and suction by tensiometer and piezometer, and the displacement can be monitored by the inclinometers, GPS or microwave reflection. Use of strain gauges or optical fibres in soil nails to monitor the strain and the nail loads may also be considered if necessary. For larges-scale projects, the use of the classical monitoring method is expensive and time-consuming, and the use of the GPS has become popular in recent years.

2.2 Slope stability analysis – limit equilibrium method

A slope stability problem is a statically indeterminate problem, and there are different methods of analysis available to the engineers. Slope stability analysis can be carried out by the limit equilibrium method (LEM), the limit analysis method, the finite element method (FEM) or the finite difference method. By far, most engineers still use the limit equilibrium method with which they are familiar. For the other methods, they are not commonly adopted in routine design, but they will be discussed in the later sections of this chapter and in Chapter 4.

Presently, most slope stability analyses are carried out by the use of computer software. Some of the early limit equilibrium methods are however simple enough that they can be computed by hand calculation, for example, the infinite slope analysis (Haefeli, 1948) and the $\phi_u = 0$ undrained analysis (Fellenius, 1918). With the advent of computers, more advanced methods have been developed. Most of limit equilibrium methods are based on the techniques of slices which can be vertical, horizontal or inclined. The first slice technique (Fellenius, 1927) was based more on engineering intuition than on a rigorous mechanics principle. There was a rapid development of the slice methods in the 1950s and 1960s by Bishop (1955); Janbu *et al.* (1956); Lowe and Karafiath (1960); Morgenstern and Price (1965); and Spencer (1967). The various 2D slice methods of limit equilibrium analysis have been well surveyed and summarized (Fredlund and Krahn, 1984; Nash, 1987; Morgenstern, 1992; Duncan, 1996). The common features of the methods of slices have been summarized by Zhu *et al.* (2003):

(a) The sliding body over the failure surface is divided into a finite number of slices. The slices are usually cut vertically, but horizontal as well as inclined cuts have also been used by various researchers. In general, the differences between different methods of cutting are not major, and the vertical cut is preferred by most engineers at present.
(b) The strength of the slip surface is mobilized to the same degree to bring the sliding body into a limit state. That means there is only a single factor of safety which is applied throughout the whole failure mass.
(c) Assumptions regarding inter-slice forces are employed to render the problem determinate.
(d) The factor of safety is computed from force and/or moment equilibrium equations.

The classical limit equilibrium analysis considers the ultimate limit state of the system and provides no information on the development of strain which actually occurs. For a natural slope, it is possible that part of the failure mass is heavily stressed so that the residual strength will be mobilized at some locations while the ultimate shear strength may be applied to another part of the failure mass. This type of progressive failure may occur in overconsolidated or fissured clays or materials with a brittle behaviour. The use of the finite element method or the extremum principle by Cheng *et al.* (2007c) can provide an estimation of the progressive failure.

Whitman and Bailey (1967) presented a very interesting and classical review of the limit equilibrium analysis methods, which can be grouped as:

1 *Method of slices*: the unstable soil mass is divided into a series of vertical slices and the slip surface can be circular or polygonal. Methods of analysis which employ circular slip surfaces include: Fellenius (1936); Taylor (1949); and Bishop (1955). Methods of analysis which employ non-circular slip surfaces include: Janbu (1973); Morgenstern and Price (1965); Spencer (1967); and Sarma (1973).

2 *Wedge methods*: the soil mass is divided into wedges with inclined interfaces. This method is commonly used for some earth dam (embankment) designs but is less commonly used for slopes. Methods which employ the wedge method include: Seed and Sultan (1967) and Sarma (1979).

The shear strength mobilized along a slip surface depends on the effective normal stress σ' acting on the failure surface. Frohlich (1953) analysed the influence of the σ' distribution on the slip surface on the calculated F. He suggested an upper and lower bound for the possible F values. When the analysis is based on the lower bound theorem in plasticity, the following criteria apply: equilibrium equations, failure criterion and boundary conditions in terms of stresses. On the other hand, if one applies the upper bound theorem in plasticity, the following alternative criteria apply: compatibility equations and displacement boundary conditions, in which the external work equals the internal energy dissipations.

Hoek and Bray (1977) suggested that the lower bound assumption gives accurate values of the factor of safety. Taylor (1948), using the friction method, also concluded that a solution using the lower bound assumptions leads to accurate F for a homogeneous slope with circular failures. The use of the lower bound method is difficult in most cases, so different assumptions to evaluate the factor of safety have been used classically. Cheng *et al.* (2007c,d) has developed a numerical procedure in Sections 2.8 and 2.9, which is effectively the lower bound method but is applicable to a general type of problem. The upper bound method in locating the critical failure surface will be discussed in Chapter 3.

In the conventional limiting equilibrium method, the shear strength τ_m which can be mobilized along the failure surface is given by:

$$\tau_m = \tau_f / F \tag{2.3}$$

where F is the factor of safety (based on force or moment equilibrium in the final form) with respect to the ultimate shear strength τ_f which is given by the Mohr–Coulomb relation as

$$\tau_f = c' + \sigma'_n \tan\phi' \text{ or } c_u \tag{2.4}$$

where c' is the cohesion, σ'_n is the effective normal stress, ϕ' is the angle of internal friction and c_u is the undrained shear strength.

In the classical stability analysis, F is usually assumed to be constant along the entire failure surface. Therefore, an average value of F is obtained along the slip surface instead of the actual factor of safety which varies along the failure surface if progressive failure is considered. There are some formulations where the factors of safety can vary along the failure surface. These kinds of formulations attempt to model the progressive failure in a simplified way, but the introduction of additional assumptions is not favoured by many engineers. Chugh (1986) presented a procedure for determining a variable factor of safety along the failure surface within the framework of the LEM. Chugh predefined a characteristic shape for the variation of the factor of safety along a failure surface, and this idea actually follows the idea of the variable inter-slice shear force function in the Morgenstern–Price method (1965). The suitability of this variable factor of safety distribution is however questionable, as the local factor of safety should be mainly controlled by the local soil properties. In view of these limitations, most engineers prefer the concept of a single factor of safety for a slope, which is easy for the design of the slope stabilization measures. Law and Lumb (1978) and Sarma and Tan (2006) have also proposed different methods with varying factors of safety along the failure surface. These methods however also suffer from the use of assumptions with no strong theoretical background. Cheng *et al.* (2007c) has developed another stability method based on the extremum principle as discussed in Section 2.8 which can allow for different factors of safety at different locations.

2.2.1 Limit equilibrium formulation of slope stability analysis methods

The limit equilibrium method is the most popular approach in slope stability analysis. This method is well known to be a statically indeterminate problem, and assumptions on the inter-slice shear forces are required to render the problem statically determinate. Based on the assumptions of the internal forces and force and/or moment equilibrium, there are more than ten methods developed for slope stability analysis. The famous methods include those by Fellenius (1936), Bishop (1955), Janbu (1973), Janbu *et al.* (1956), Lowe and Karafiath (1960), Spencer (1967), Morgenstern and Price (1965) and so on.

Since most of the existing methods are very similar in their basic formulations with only minor differences in the assumptions on the inter-slice shear forces, it is possible to group most of the existing methods under a unified formulation. Fredlund and Krahn (1977) and Espinoza and Bourdeau (1994) have proposed a slightly different unified formulation to the more commonly used slope stability analysis methods. In this section, the formulation by Cheng and Zhu (2005) which can degenerate to many existing methods of analysis will be introduced.

Based upon the static equilibrium conditions and the concept of limit equilibrium, the number of equations and unknown variables are summarized in Tables 2.3 and 2.4.

From these tables it is clear that the slope stability problem is statically indeterminate in the order of $6n - 2 - 4n = 2n - 2$. In other words, we have to introduce additional $(2n - 2)$ assumptions to solve the problem. The locations of the base normal forces are usually assumed to be at the middle of the slice, which is a reasonable assumption if the width of the slice is limited. This assumption will reduce unknowns so that there are only $n - 2$ equations to be introduced. The most common additional assumptions are either the location of the inter-slice normal forces or the relation between the inter-slice normal and shear forces. That will further reduce the number of unknowns by $n - 1$ (n slice has only $n - 1$ interfaces), so the problem will become over-specified by 1. Based on different assumptions along the interfaces between slices, there are more than ten existing methods of analysis at present.

The limit equilibrium method can be broadly classified into two main categories: 'simplified' methods and 'rigorous' methods. For the simplified

Table 2.3 Summary of system of equations (n = number of slices)

Equations	*Condition*
n	Moment equilibrium for each slice
$2n$	Force equilibrium in X and Y directions for each slice
n	Mohr–Coulomb failure criterion
$4n$	Total number of equations

Table 2.4 Summary of unknowns

Unknowns	*Description*
1	Safety factor
n	Normal force at the base of slice
n	Location of normal force at base of slice
n	Shear force at base of slice
$n-1$	Inter-slice horizontal force
$n-1$	Inter-slice tangential force
$n-1$	Location of inter-slice force (line of thrust)
$6n-2$	Total number of unknowns

methods, either force or moment equilibrium can be satisfied but not both at the same time. For the rigorous methods, both force and moment equilibrium can be satisfied, but usually the analysis is more tedious and may sometimes experience non-convergence problems. The authors have noticed that many engineers have the wrong concept that methods which can satisfy both the force and moment equilibrium are accurate or even 'exact'. This is actually a wrong concept as all methods of analysis require some assumptions to make the problem statically determinate. The authors have even come across many cases where very strange results can come out from the 'rigorous' methods (which should be eliminated because the internal forces are unacceptable), but the situation is usually better for those 'simplified' methods. In this respect, no method is particularly better than others, though methods which have more careful consideration of the internal forces will usually be better than the simplified methods in most cases. Morgenstern (1992), Cheng as well as many other researchers have found that most of the commonly used methods of analysis give results which are similar to each other. In this respect, there is no strong need to fine tune the 'rigorous' slope stability formulations except for isolated cases, as the inter-slice shear forces have only a small effect on the factor of safety in general.

To begin with the generalized formulation, consider the equilibrium of force and moment for a general case shown in Figure 2.1. The assumptions used in the present unified formulation are:

1. The failure mass is a rigid body.
2. The base normal force acts at the middle of each slice base.
3. The Mohr–Coulomb failure criterion is used.

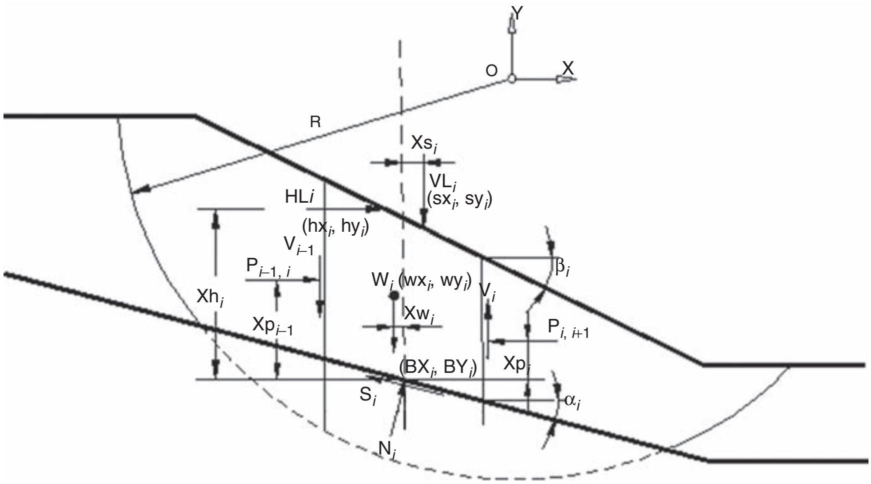

Figure 2.1 Internal forces in a failing mass.

2.2.1.1 Force equilibrium

The horizontal and vertical force equilibrium conditions for slice *i* are given by:

$$N_i \sin\alpha_i - S_i \cos\alpha_i + HL_i = P_{i,i+1} - P_{i-1,i} \tag{2.5}$$

$$W_i + VL_i - N_i \cos\alpha_i - S_i \sin\alpha_i = P_{i,i+1} \tan\phi_{i,i+1} - P_{i-1,i} \tan\phi_{i-1,i} \tag{2.6}$$

The Mohr–Coulomb relation is applied to the base normal force N_i and shear force S_i as

$$S_i = \frac{N_i \tan\phi_i + c_i l_i}{F} \tag{2.7}$$

The boundary conditions to the above three equations are the inter-slice normal forces, which will be 0 for the first and last ends:

$$P_{0,1} = 0; \quad P_{n,n+1} = 0. \tag{2.8}$$

When $i = 1$ (first slice), the base normal force N_1 is given by eqs (2.5)–(2.7) as

$$N_1 = \frac{A_1 \times F + C_i}{H_1 + E_1 \times F}, \quad P_{1,2} = \frac{L_1 + K_1 \times F + M_1}{H_1 + E_1 \times F} \tag{2.9}$$

$P_{1,2}$ is a first order function of the factor of safety *F*. For slice *i* the base normal force is given by

$$N_i = \frac{(\tan\phi_{i-1,i} - \tan\phi_{i,i+1})F \times P_{i-1,i} + A_i \times F + C_i}{H_i + E_i \times F} \tag{2.10}$$

$$P_{i,i+1} = \frac{(J_i \times f_i + G_i \times F)P_{i-1,i} + L_i + K_i \times F + M_i}{H_i + E_i \times F} \tag{2.11}$$

When $i = n$ (last slice), the base normal force is given by

$$N_n = \frac{AA_n \times F + D_n}{J_n + G_n \times F}, \quad P_{n-1,n} = -\frac{L_n + K_n \times F + M_n}{J_n + G_n \times F} \tag{2.12}$$

Eqs (2.11) and (2.12) relate the left and right inter-slice normal forces of a slice, and the subscript $i,i+1$ means the internal force between slice i and $i+1$. Definitions of symbols used in the above equations are:

$$\begin{aligned}
&A_i = W_i + VL_i - HL_i \tan\phi_{i,i+1}, && AA_i = W_i + VL_i - HL_i \tan\phi_{i-1,i}\\
&C_i = (\sin\alpha_i + \cos\alpha_i \tan\phi_{i,i+1})c_i A_i, && D_i = (\sin\alpha_i + \cos\alpha_i \tan\phi_{i-1,i})c_i A_i\\
&E_i = \cos\alpha_i + \tan\phi_{i,i+1}\sin\alpha_i, && G_i = \cos\alpha_i + \tan\phi_{i-1,i}\sin\alpha_i\\
&H_i = (-\sin\alpha_i - \tan\phi_{i,i+1}\cos\alpha_i)f_i, && J_i = (-\sin\alpha_i - \tan\phi_{i-1,i}\cos\alpha_i)f_i\\
&K_i = (W_i + VL_i)\sin\alpha_i + HL_i\cos\alpha_i && V_i = P_{i,i+1}\tan\Phi_{i,i+1}\\
&L_i = (-(W_i + VL_i)\cos\alpha_i - HL_i\sin\alpha_i)f_i, && M_i = (\sin^2\alpha_i - \cos^2\alpha_i)c_i A_i\\
&A_i = W_i + VL_i - HL_i \tan\phi_{i,i+1}, && B_i = W_i + VL_i - HL_i \tan\phi_{i-1,i}
\end{aligned}$$

where

α – base inclination angle, clockwise is taken as positive;
β – ground slope angle, counter-clockwise is taken as positive;
W – weight of slice; VL – external vertical surcharge;
HL – external horizontal load; P – inter-slice normal force;
V – inter-slice shear force; N – base normal force;
S – base shear force; F – factor of safety;
c, f – base cohesion c' and $\tan\phi'$;
l – base length l of slice, $\tan\Phi = \lambda f(x)$;

$\{BX, BY\}$, coordinates of the mid-point of base of each slice; $\{wx, wy\}$, coordinates for the centre of gravity of each slice; $\{sx, sy\}$, coordinates for point of application of vertical load for each slice; $\{hx, hy\}$ coordinates for the point of application of the horizontal load for each slice; Xw, Xs, Xh, Xp are lever arm from middle of base for self weight, vertical load, horizontal load and line of thrust, respectively, where $Xw = BX - wx$; $Xs = BX - sy$; $Xh = BY - hy$.

2.2.1.2 Moment equilibrium equation

Taking moment about any given point O in Figure 2.1, the overall moment equilibrium is given:

$$\sum_{i=1}^{n}[W_i wx_i + VL_i sx_i + HL_i hy_i + (N_i \sin\alpha_i - S_i\cos\alpha_i)BY_i - (N_i\cos_i - S_i\sin\alpha_i)BX_i] = 0 \tag{2.13}$$

It should be noted that most of the 'rigorous' methods adopt the overall moment equilibrium instead of the local moment equilibrium in the formulation, except for the Janbu rigorous method and the extremum method by Cheng *et al.* (2007c) which will be introduced in Section 2.8. The line of thrust can be back-computed from the internal forces after the stability analysis. Since the local moment equilibrium equation is not adopted explicitly, the line of thrust may fall outside the slice which is clearly unacceptable, and it can be considered that the local moment equilibrium cannot be maintained under this case. The effects of the local moment equilibrium are however usually not critical towards the factor of safety, as the effect of the inter-slice shear force is usually small in most cases. The engineers should however check the location of the thrust line as a good practice after performing those 'rigorous' analyses. Sometimes, the local moment equilibrium can be maintained by fine tuning of the inter-slice force function $f(x)$, but there is no systematic way to achieve this except by manual trial and error or the lower bound method by Cheng *et al.* (2007d) as discussed in Section 2.9.

2.2.2 Inter-slice force function

The inter-slice shear force V is assumed to be related to the inter-slice normal force P by the relation $V = \lambda f(x)P$. There is no theoretical basis to determine $f(x)$ for a general problem, as the slope stability problem is statically indeterminate by nature. More detailed discussion about $f(x)$ by the lower bound method will be given in Section 2.9. There are seven types of $f(x)$ commonly in use:

Type 1: $f(x) = 1.0$. This case is equivalent to the Spencer method and is commonly adopted by many engineers. Consider the case of a sandy soil with $c' = 0$. If the Mohr–Coulomb relation is applied to the inter-slice force relation, $V = P \tan\phi'$, then $f(x) = 1.0$ and $\lambda = \tan\phi$. Since there is no strong requirement to apply the Mohr–Coulomb relation for the inter-slice forces, $f(x)$ should be different from 1.0 in general. It will be demonstrated in Section 2.9 that $f(x) = 1.0$ is actually not a realistic relation.

Type 2: $f(x) = \sin(x)$. This is a relatively popular alternative to $f(x) = 1.0$. This function is adopted purely because of its simplicity.

Type 3: $f(x)$ = trapezoidal shape shown in Figure 2.2. Type 3 $f(x)$ will reduce to type 1 as a special case, but it is seldom adopted in practice.

Type 4: $f(x)$ = error function or the Fredlund–Wilson–Fan force function (1986) which is in the form of $f(x) = \Psi \exp(-0.5c^n\eta^n)$, where Ψ, c and n have to be defined by the user. η is a normalized dimensional factor which has a value of -0.5 at left exit end and $=0.5$ at right exit end of the failure surface. η varies linearly with the x-ordinates of the failure surface. This error function is actually based on an elastic finite element stress analysis by Fan *et al.* (1986). Since the stress state in the limit equilibrium analysis is the ultimate condition and is different from the elastic stress analysis by Fan *et al.* (1986), the suitability of this inter-slice force function cannot be

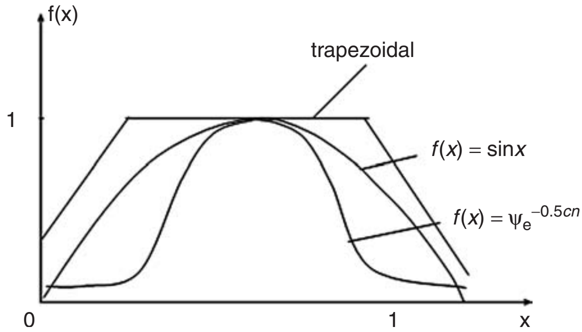

Figure 2.2 Shape of inter-slice shear force function.

justified by the elastic analysis. It is also difficult to define the suitable parameters for a general problem with soil nails, water table and external loads. This function is also not applicable for complicated cases, and a better inter-slice force function will be suggested in Section 2.9.

For the first four types of functions shown above, they are commonly adopted in the Morgenstern–Price and GLE methods, and both the moment and force equilibrium can be satisfied simultaneously. A completely arbitrary inter-slice force function is theoretically possible, but there is no simple way or theoretical background in defining this function except for the extremum principle introduced in Section 2.9, so the arbitrary inter-slice force function is seldom considered in practice.

Type 5: Corps of Engineers inter-slice force function. $f(x)$ is assumed to be constant and is equal to the slope angle defined by the two extreme ends of the failure surface.

Type 6: Lowe–Karafiath inter-slice force function. $f(x)$ is assumed to be the average of the slope angle of the ground profile and the failure surface at the section under consideration.

Type 7: $f(x)$ is defined as the tangent of the base slope angle at the section under consideration, and this assumption is used in the Load factor method in China.

For type 5 to type 7 inter-slice force functions, only force equilibrium is enforced in the formulation. The factors of safety from these methods are however usually very close to those by the 'rigorous' methods, and are usually better than the results by the Janbu simplified method. In fact, the Janbu method is given by the case of $\lambda = 0$ for the Corps of Engineers method, Lowe–Karafiath method and the Load factor method, and results from the Janbu analysis can also be taken as the first approximation in the Morgenstern–Price analysis.

Based on a Mohr circle transformation analysis, Chen and Morgenstern (1983) have established that $\lambda f(x)$ for the two ends of a slip surface which is

Table 2.5 Assumptions used in various methods of analysis (× means not satisfied and √ means satisfied)

Method	*Assumptions*	*Force equilibrium*		*Moment equilibrium*
		X	*Y*	
1 Swedish	$P = V = 0$	×	×	√
2 Bishop simplified	$V = 0$ or $\Phi = 0$	×	√	√
3 Janbu simplified	$V = 0$ or $\Phi = 0$	√	√	×
4 Lowe and Karafiath	$\Phi = (\alpha + \beta)/2$	√	√	×
5 Corps of Engineers	$\Phi = \beta$ or $\Phi_{i-1,i} = \dfrac{\alpha_{i-1} + \alpha_i}{2}$	√	√	×
6 Load transfer	$\Phi = \alpha$	√	√	×
7 Wedge	$\Phi = \phi$	√	√	×
8 Spencer	$\Phi =$ constant	√	√	√
9 Morgernstern–Price and GLE	$\Phi = \lambda f(x)$	√	√	√
10 Janbu rigorous	Line of thrust (Xp)	√	√	√
11 Leshchinsky	Magnitude and distribution of N	√	√	√

the inclination of the resultant inter-slice force should be equal to the ground slope angle. Other than this requirement, there is no simple way to establish $f(x)$ for a general problem. Since the requirement by Chen and Morgenstern (1983) applies only under an infinitesimal condition, it is seldom adopted in practice. Even though there is no simple way to define $f(x)$, Morgenstern (1992), among others, has however pointed out that, for normal problems, F from different methods of analyses are similar so that the assumptions on the internal force distributions are not major issues for practical use except for some particular cases. In views of the difficulty in prescribing a suitable $f(x)$ for a general problem, most engineers will choose $f(x) = 1.0$ which is satisfactory for most cases. Cheng *et al.* (2007d) have however established the upper and lower bounds of the factor of safety and the corresponding $f(x)$ based on the extremum principle which will be discussed in Section 2.9.

2.2.3 Reduction to various methods and discussion

The present unified formulation by Cheng and Zhu (2005) can reduce to most of the commonly used methods of analysis which is shown in Table 2.5. In Table 2.5, the angle of inclination of the inter-slice forces is prescribed for methods 2–9.

The classical Swedish method for undrained analysis (Fellenius analysis) considers only the global moment equilibrium and neglects all the internal forces between slices. For the Swedish method under drained analysis, the left and right inter-slice forces are assumed to be equal and opposite so that the base normal forces become known. The factor of safety can be obtained easily without the need of iteration analysis. The Swedish method is well known to be conservative, and sometimes the results from it can be 20–30 per cent smaller than those from the 'rigorous' methods, hence the Swedish method is seldom adopted in practice. This method is however simple enough to be operated by hand or spreadsheet calculation, and there are no non-convergence problems as iteration is not required.

The Bishop method is one of the most popular slope stability analysis methods and is used worldwide. This method satisfies only the moment equilibrium given by eq. (2.13) but not the horizontal force equilibrium given by eq. (2.5), and it applies only for a circular failure surface. The centre of the circle is taken as the moment point in the moment equilibrium equation given by eq. (2.13). The Bishop method has been used for some non-circular failure surfaces, but Fredlund *et al.* (1992) have demonstrated that the factor of safety will be dependent on the choice of the moment point because there is a net unbalanced horizontal force in the system. The use of the Bishop method to the non-circular failure surface is generally not recommended because of the unbalanced horizontal force problem, and this can be important for problems with loads from earthquake or soil reinforcement. This method is simple for hand calculation and the convergence is fast. It is also virtually free from/convergence problems, and the results from it are very close to those by the 'rigorous' methods. If the circular failure surface is sufficient for the design and analysis, this method can be a very good solution for engineers. When applied to an undrained problem with $\phi = 0$, the Bishop method and the Swedish method will become identical.

For the Janbu simplified method (1956), force equilibrium is completely satisfied while moment equilibrium is not satisfied. This method is also popular worldwide as it is fast in computation with only few convergence problems. This method can be used for a non-circular failure surface which is commonly observed in sandy-type soil. Janbu (1973) later proposed a 'rigorous' formulation which is more tedious in computation. Based on the ratio of the factors of safety from the 'rigorous' and 'simplified' analyses, Janbu proposed a correction factor f_0 given by eq. (2.14) for the Janbu simplified method. When the factor of safety from the simplified method is multiplied with this correction factor, the result will be close to that from the 'rigorous' analysis.

$$
\begin{aligned}
\text{For } c, \phi > 0, \qquad & f_0 = 1 + 0.5\left[\frac{D}{l} - 1.4\left(\frac{D}{l}\right)^2\right] \\
\text{For } c = 0, \qquad & f_0 = 1 + 0.3\left[\frac{D}{l} - 1.4\left(\frac{D}{l}\right)^2\right] \\
\text{For } \phi = 0, \qquad & f_0 = 1 + 0.6\left[\frac{D}{l} - 1.4\left(\frac{D}{l}\right)^2\right]
\end{aligned}
\tag{2.14}
$$

For the correction factor shown above, l is the length joining the left and right exit points while D is the maximum thickness of the failure zone with reference to this line. Since the correction factors by Janbu (1973) are based on limited

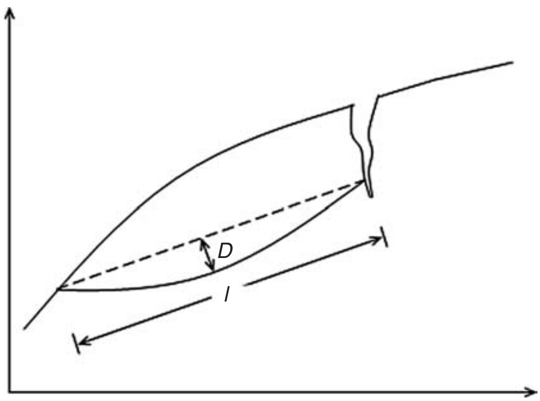

Figure 2.3 Definitions of D and l for the correction factor in the Janbu simplified method.

case studies, the uses of these factors to complicated non-homogeneous slopes are questioned by some engineers. Since the inter-slice shear force can sometimes generate a high factor of safety for some complicated cases which may occur in dam and hydropower projects, the use of the Janbu method is preferred over other methods in these kinds of projects in China.

The Lowe and Karafiath method and the Corps of Engineers method are based on the inter-slice force functions type 5 and type 6. These two methods satisfy force equilibrium but not moment equilibrium. In general, the Lowe and Karafiath method will give results close to that from the 'rigorous' method even though the moment equilibrium is not satisfied. For the Corps of Engineers method, it may lead to a high factor of safety in some cases, and some engineers actually adopt a lower inter-slice force angle to account for this problem (Duncan and Wright, 2005), and this practice is also adopted by some engineers in China. The load transfer and the wedge methods in Table 2.5 satisfy only the force equilibrium. These methods are used in China only and are seldom adopted in other countries.

The Morgenstern–Price method is usually based on the inter-slice force function types 1 to 4, though the use of the arbitrary function is possible and is occasionally used. If the type 1 inter-slice force function is used, this method will reduce to the Spencer method. The Morgenstern–Price method satisfies the force and the global moment equilibrium. Since the local moment equilibrium equation is not used in the formulation, the internal forces of an individual slice may not be acceptable. For example, the line of thrust (centroid of the inter-slice normal force) may fall outside the soil mass from the Morgenstern–Price analysis. The GLE method is basically similar to the Morgenstern–Price method, except that the line of thrust is determined and is closed at the last slice. The acceptability of the line of thrust for any intermediate slice may still be unacceptable from the GLE analysis. In general, the results from these two methods of analysis are very close.

The Janbu rigorous method appears to be appealing in that the local moment equilibrium is used in the intermediate computation. The internal forces will hence be acceptable if the analysis can converge. As suggested by Janbu (1973), the line of thrust ratio is usually taken as one-third of the inter-slice height, which is basically compatible with the classical lateral earth pressure distribution. It should be noted that the equilibrium of the last slice is actually not used in the Janbu rigorous method, so the moment equilibrium from the Janbu rigorous method is not strictly rigorous. A limitation of this method is the relatively poor convergence in the analysis, particularly when the failure surface is highly irregular or there are external loads. This is due to the fact that the line of thrust ratio is pre-determined with no flexibility in the analysis. The constraints in the Janbu rigorous method are more than that in the other methods, hence convergence is usually poorer. If the method is slightly modified by assuming $h_t/h = \lambda f(x)$, where h_t = height of line of thrust above slice base and h = length of the vertical inter-slice, the convergence of this method will be improved. There is however difficulty in defining $f(x)$ for the line of thrust, and hence this approach is seldom considered. Cheng has developed another version of the Janbu rigorous method which is implemented in the program SLOPE 2000.

For the Janbu rigorous (1973) and Leshchinsky (1985) methods, Φ (or λ equivalently) is not known in advance. The relationship between the line of thrust Xp and angle Φ in the Janbu rigorous method can be derived in the following ways:

(a) Taking moment about middle of the slice base in the Janbu rigorous method, the moment equilibrium condition is given by:

$$W_iXw_i + Vl_iXs_i - HL_iXh_i = P_{i,i+1}Xp_i - P_{i-1,i}Xp_{i-1} + \frac{1}{2}(P_{i,i+1}\tan\Phi_{i,i+1} + P_{i-1,i}\tan\Phi_{i-1,i})B_i \tag{2.15}$$

From above, the inter-slice normal force is obtained as:

$$P_{i,i+1} = \frac{Al_i}{2Xp_i + B_i\tan\Phi_{i,i+1}} \tag{2.16}$$

where

$$Al_i = 2W_iXw_i + 2VL_iXs_i - 2VL_iXh_i + 2P_{i-1,i}Xp_{i-1} - B_iP_{i-1,i}\tan\Phi_{i-1,i} \tag{2.17}$$

From eq. (2.9) the inter-slice normal force is also obtained as

$$P_{i,i+1} = \frac{A2_i}{-f_i\sin\alpha_i - f_i\cos\alpha_i\tan\Phi_{i,i+1} + K\cos\alpha_i + K\sin\alpha_i\tan\Phi_{i,i+1}} \tag{2.18}$$

where

$$A2_i = (J_i + G_i \times F)P_{i-1,i} + M_i + L_i + K_iF \tag{2.19}$$

From eqs (2.15) and (2.17), the relation between line of thrust Xp and angle Φ is given by:

$$\tan \Phi_{i,i+1} = -\frac{-2A2_iXp_i - Al_if_i \sin\alpha_i + Al_iF\cos\alpha_i}{-A2_iB_i - Al_if_i\cos\alpha_i + Al_iF\sin\alpha_i} \tag{2.20}$$

(b) For the Leshchinsky method where the distribution of the base normal force N is assumed to be known, Φ can then be determined as:

$$\tan\Phi_{i,i+1} = -\frac{-N_if_i\sin\alpha_i + N_iF\cos\alpha_i - P_{i-1,i}F\tan\Phi_{i-1,i} - c_iA_i\sin\alpha_i - W_iF - VL_iF}{-N_if_i\cos\alpha_i + N_iF\sin\alpha_i + P_{i-1,i}F - c_iA_i\sin\alpha_i + VL_iF} \tag{2.21}$$

Once Φ is obtained from eq. (2.19) or (2.20), the calculation can then proceed as described previously.

2.2.4 Solution of the non-linear factor of safety equation

In eq. (2.11), the inter-slice normal force for slice i, $P_{i,i+1}$, is controlled by the inter-slice normal $P_{i-1,i}$. If we put the equation for inter-slice normal force $P_{1,2}$ (eq. 2.9) from slice 1 into the equation for inter-slice normal force $P_{2,3}$ for slice 2 (eq. 2.11), we will get a second order equation in factor of safety F as

$$P_{2,3} = \frac{(J_2 \times f_2 + G_2 \times F)P_{1,2} + L_2 + K_2 \times F + M_2}{H_2 + E_2 \times F} \tag{2.22}$$

The term $(J_2 \times f_2 + G_2 \times F)P_{1,2}$ is a second order function in F. The numerator on the right hand side of eq. (2.22) is hence a second order function in F. Similarly, if we put the equation $P_{2,3}$ into the equation for $P_{3,4}$, a third order equation in F will be achieved. If we continue this process to the last slice, we will arrive at a polynomial for F and the order of the polynomial is n for $P_{n,n+1}$ which is just 0! Sarma (1987) has also arrived at a similar conclusion for a simplified slope model. The importance of this polynomial under the present formulation is that there are n possible factors of safety for any prescribed Φ. Most of the solutions will be physically unacceptable and are either imaginary numbers or negative solutions. Physically acceptable factors of safety are given by the positive real solutions from this polynomial.

λ and F are the two unknowns in the above equations and they can be determined by several different methods. In most of the commercial programs, the factor of safety is obtained by the use of iteration with an initial trial factor of safety (usually 1.0) which is efficient and effective for most cases. The use of the iteration method is actually equivalent to expressing the complicated factor of safety polynomial in a functional form as:

$$F = f(F) \tag{2.23}$$

Chen and Morgenstern (1983) and Zhu *et al.* (2001) have proposed the use of the Newton–Rhapson technique in the evaluation of the factor of safety F and λ. The gradient type methods are more complicated in the formulation but are fast in solution. Chen and Morgenstern (1983) suggested that the initial trial λ can be chosen as the tangent of the average base angle of the failure surface, and these two values can be determined by the use of the Newton–Rhapson method. Chen and Morgenstern (1983) have also provided the expressions for the derivatives of the moment and shear terms required for the Newton–Rhapson analysis. Zhu *et al.* (2001) admitted that the initial trials of F and λ can greatly affect the efficiency of the computation. In some cases, poor initial trials can even lead to divergence in analysis. Zhu *et al.* proposed a technique to estimate the initial trial value which appears to work fine for smooth failure surfaces. The authors' experience is that, for non-smooth or deep-seated failure surfaces, it is not easy to estimate a good initial trial value, and Zhu *et al.*'s proposal may not work for these cases.

As an alternative, Cheng and Zhu (2005) have proposed that the factor of safety based on the force equilibrium is determined directly from the polynomial as discussed above, and this can avoid the problems that may be encountered using the Newton–Rhapson method or iteration method. The present proposal can be effective under difficult problems while Chen's or Zhu's methods are more efficient for general smooth failure surfaces. The additional advantage of the present proposal is that it can be applied to many slope stability analysis methods if the unified formulation is adopted. To solve for the factor of safety, the following steps can be used:

1. From slice 1 to n, based on an assumed value of λ and $f(x)$ and hence Φ for each interface, the factors of safety can be determined from the polynomial by the Gauss–Newton method with a line search step selection. The internal forces P, V, N and S can be then be determined from eqs (2.5) to (2.11) without using any iteration analysis. The special feature of the present technique is that while determination of inter-slice forces is required for calculating the factor of safety in iterative analysis (for rigorous methods), the factor of safety is determined directly under the present formulation. Since the Bishop analysis does not satisfy horizontal force equilibrium, the present method cannot be applied to the Bishop analysis. This is not important as the Bishop method can be solved easily by the classical iterative algorithm.
2. For those rigorous methods, moment equilibrium has to be checked. Based on the internal forces as determined in step 1 for a specific physically acceptable factor of safety, the moment equilibrium equation (2.13) is then checked. If moment equilibrium is not satisfied with that

specific factor of safety based on the force equilibrium, repeat the step with the next factor of safety in checking the moment equilibrium.

3 If no acceptable factor of safety is found, try the next λ and repeat steps 1 and 2 above. In the actual implementation, the sign of the unbalanced moment from eq. (2.13) is monitored against λ and interpolation is used to accelerate the determination of λ which satisfies the moment equilibrium.

4 For the Janbu rigorous method or the Leshchinsky method, eqs (2.20) and (2.21) have to be used in the above procedures during each step of analysis.

It will be demonstrated in Chapter 3 that there are many cases where iteration analysis may fail to converge but the factors of safety actually exist. On the other hand, using the Gauss–Newton method and the polynomial from by Cheng and Zhu (2005) or the matrix form and the double QR method by Cheng (2003), it is possible to determine the factor of safety without iteration analysis. The root of the polynomial (factor of safety) close to the initial trial can be determined directly by the Gauss–Newton method. For the double QR method, the factor of safety and the internal forces are determined directly without the need of any initial trial at the expense of computer time in solving the matrix equation.

Based on the fact that the inter-slice forces at any section are the same for the slices to the left and to the right of that section, an overall equation can be assembled in a way similar to that in the stiffness method which will result in a matrix equation (Cheng, 2003). The factor of safety equation as given by eq. (2.22) can be cast into a matrix form instead of a polynomial (actually equivalent). The complete solution of all the real positive factors of safety from the matrix can be obtained by the double QR method by Cheng (2003), which is a useful numerical method to calculate all the roots associated with the Hessenberg matrix arising from eq. (2.22). It should be noted that imaginary numbers may satisfy the factor of safety polynomial, so the double QR method instead of the classical QR method is necessary to determine the real positive factors of safety. If a F value from the double QR analysis cannot satisfy the above requirement, the next F value will be computed. Processes 1 to 4 above will continue until all the possible F values are examined. If no factor of safety based on the force equilibrium can satisfy the moment equilibrium, the analysis is assumed to fail in convergence and only imaginary roots will be available.

The advantage of the present method is that the factor of safety and the internal forces with respect to force equilibrium are obtained directly without any iteration analysis. Cheng (2003) has also demonstrated that there can be at most n possible factors of safety (including negative value and imaginary number) from the double QR analysis for a failure mass with n slices. The actual factor of safety can be obtained from the force and moment balance at a particular λ value. The time required for the double QR computation is not excessively long as inter-slice normal and shear forces are not required to be determined in obtaining a factor of safety. In general, if the number of slices used for the analysis is less than 20, the solution time for the double QR method is only 50–100 per cent longer than the iteration method.

Since all the possible factors of safety are examined, this method is the ultimate method in the determination of the factor of safety. If other methods of analysis fail to determine the factor of safety, this method may still work which will be demonstrated in Chapter 3. On the other hand, if no physically acceptable solution is found from the double QR method, the problem under consideration has no solution by nature. More discussion about the use of the double QR method will be given in Section 2.9.

2.2.5 Examples on slope stability analysis

Figure 2.4 is a simple slope given by coordinates (4,0), (5,0), (10,5) and (12,5) while the water table is given by (4,0), (5,0), (10,4) and (12,4). The soil parameters are: unit weight = 19 kNm^{-3}, c' = 5 kPa and ϕ' = 36°. To define a circular failure surface, the coordinates of the centre of rotation and the radius should be defined. Alternatively, a better method is to define the x-ordinates of the left and right exit ends and the radius of the circular arc. The latter approach is better as the left and right exit ends can usually be estimated easily from engineering judgement. In the present example, the x-ordinates of the left and right exit ends are defined as 5.0 and 12.0 m while the radius is defined as 12 m. The soil mass is divided into ten slices for analysis and the details are given below:

Slice	*Weight (kN)*	*Base angle (°)*	*Base length (m)*	*Base pore pressure (kPa)*
1	2.50	16.09	0.650	1.57
2	7.29	19.22	0.662	4.52
3	11.65	22.41	0.676	7.09
4	15.54	25.69	0.694	9.26
5	18.93	29.05	0.715	10.99
6	21.76	32.52	0.741	12.23
7	23.99	36.14	0.774	12.94
8	25.51	39.94	0.815	13.04
9	32.64	45.28	1.421	7.98
10	11.77	52.61	1.647	0.36

The results of analyses for the problem in Figure 2.4 are given in Table 2.6. For the Swedish method or the Ordinary method of slices where only the moment equilibrium is considered while the inter-slice shear force is neglected, the factor of safety from the global moment equilibrium takes the form of:

$$F_{\mathrm{m}} = \frac{\sum \left(c'l + (W\cos\alpha - ul)\tan\phi'\right)}{\sum W\sin\alpha} \tag{2.24}$$

A factor of safety 0.991 is obtained directly from the Swedish method for this example without any iteration. For the Bishop method, which assumes the inter-slice shear force V to be zero, the factor of safety by the global moment equilibrium will reduce to

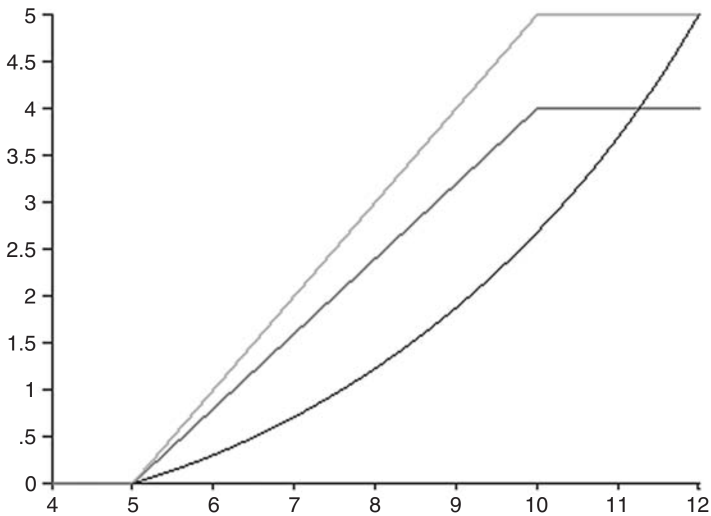

Figure 2.4 Numerical examples for a simple slope.

Table 2.6 Factors of safety for the failure surface shown in Figure 2.4

	Bishop	*Janbu simplified*	*Janbu rigorous*	*Swedish*	*Load factor*	*Sarma*	*Morgenstern–Price*
F	1.023	1.037	1.024	0.991	1.027	1.026	1.028

Note: The correction factor is applied to the Janbu simplified method. The results for the Morgenstern–Price method using $f(x) = 1.0$ and $f(x) = \sin(x)$ are the same. Tolerance in iteration analysis is 0.0005.

$$F_{\mathrm{m}} = \frac{\sum\left(c'b + (W - ul)\tan\phi'\right)\sec\alpha/m_\alpha}{\sum W\sin\alpha} \tag{2.25}$$

where $m_\alpha = \cos\alpha(1 + \tan\alpha\dfrac{\tan\phi'}{F})$

Based on an initial factor of safety 1.0, the successive factors of safety during the Bishop iteration analysis are 1.0150, 1.0201, 1.0219, 1.0225 and 1.0226. For the Janbu simplified method, the factor of safety based on force equilibrium using the iteration analysis takes the form of:

$$F_{\mathrm{f}} = \frac{\sum[c'b + (W - ub)\tan\phi']/n_\alpha}{\sum W\tan\alpha} \quad \text{and} \quad n_\alpha = \cos\alpha\cdot m_\alpha \tag{2.26}$$

The successive factors of safety during the iteration analysis using the Janbu simplified method are 0.9980, 0.9974 and 0.9971. Based on a correction factor of 1.0402, the final factor of safety from the Janbu simplified analysis is 1.0372. If

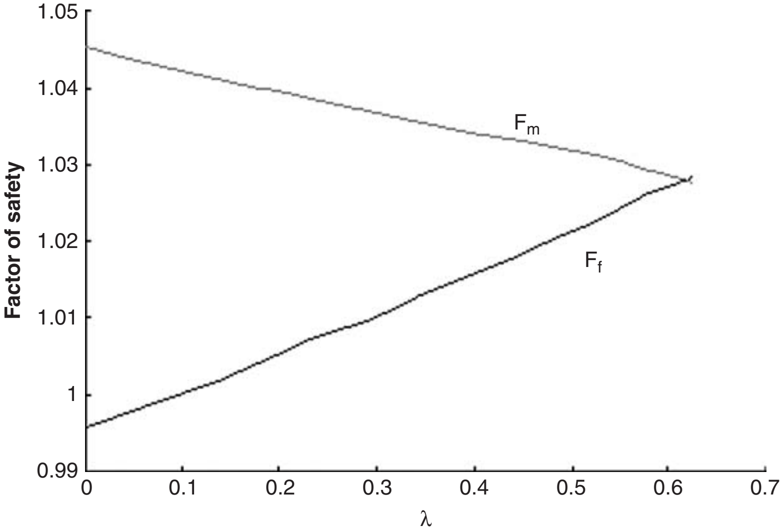

Figure 2.5 Variation of F_f and F_m with respect to λ for the example in Figure 2.4.

the double QR method is used for the Janbu simplified method, a value of 0.9971 is obtained directly from the first positive solution of the Hessenberg matrix without using any iteration analysis. For the Janbu rigorous method, the successive factors of safety based on iteration analysis are 0.9980, 0.9974, 0.9971, 1.0102, 1.0148, 1.0164, 1.0170, 1.0213, 1.0228, 1.0233 and 1.0235. For the Morgenstern–Price method, a factor of safety 1.0282 and the internal forces are obtained directly from the double QR method without any iteration analysis. The variation of F_f and F_m with respect to λ using the iteration analysis for this example is shown in Figure 2.5. It should be noted that F_f is usually more sensitive to λ than F_m in general, and the two lines may not meet for some cases which can be considered as no solution to the problem. There are cases where the lines are very close but actually do not intersect. If a tolerance large enough is defined, then the two lines can be considered as having an intersection and the solution converge. This type of 'false' convergence is experienced by many engineers in Hong Kong. These two lines may be affected by the choice of the moment point, and convergence can sometimes be achieved by adjusting the choice of the moment point. The results shown in Figure 2.5 assume the interslice shear forces to be zero in the first solution step, and this solution procedure appears to be adopted in many commercial programs. Cheng *et al.* (2008a) have however found that the results shown in Figure 2.5 may not be the true result for some special cases, and this will be further discussed in Chapter 3.

From Table 2.6, it is clear that the Swedish method is a very conservative method as first suggested by Whitman and Bailey (1967). Besides, the Janbu simplified method will also give a smaller factor of safety if the correction factor is not used. After the application of the correction factor, Cheng found that the results from the Janbu simplified method are usually close to those 'rigorous' methods. In general, the factors of safety from different methods of analysis are usually close as pointed out by Morgenstern (1992).

2.3 Miscellaneous consideration on slope stability analysis

2.3.1 Acceptability of the failure surfaces and results of analysis

Based on an arbitrary inter-slice force function, the internal forces which satisfy both the force and moment equilibrium may not be kinematically acceptable. The acceptability conditions of the internal forces include:

1 Since the Mohr–Coulomb relation is not used along the vertical interfaces between different slices, it is possible though not common that the inter-slice shear forces and normal forces may violate the Mohr–Coulomb relation.
2 Except for the Janbu rigorous method and the extremum method as discussed in Section 2.8 under which the resultant of the inter-slice normal force must be acceptable, the line of thrust from other 'rigorous' methods which are based on overall moment equilibrium may lie outside the failure mass and is unacceptable.
3 The inter-slice normal forces should not be in tension. For the inter-slice normal forces near to the crest of the slope where the base inclination angles are usually high, if c' is high, it is highly likely that the inter-slice normal forces will be in tension to maintain the equilibrium. This situation can be eliminated by the use of a tension crack. Alternatively, the factor of safety with tensile inter-slice normal forces for the last few slices may be accepted, as the factor of safety is usually not sensitive to these tensile forces. On the other hand, tensile inter-slice normal forces near the slope toe are usually associated with special shape failure surfaces with kinks, steep upward slope at the slope toe or an unreasonably high/low factor of safety. The factors of safety associated with these special failure surfaces need special care in the assessment and should be rejected if the internal forces are unacceptable. Such failure surfaces should also be eliminated during the location of the critical failure surfaces.
4 The base normal forces may be negative near the toe and crest of the slope. For negative base normal forces near the crest of the slope, the situation is similar to the tensile inter-slice normal forces and may be tolerable. For negative base normal forces near the toe of the slope which is physically unacceptable, it is usually associated with deep-seated failure with a high upward base inclination. Since a very steep exit angle is not likely to occur, it is possible to limit the exit angle during the automatic location of the critical failure surface.

If the above criteria are strictly enforced to all slices of a failure surface, many slip surfaces will fail to converge. One of the reasons is the effect of the last slice when the base angle is large. Based on the force equilibrium, the tensile