TD7 – Learning I Unsupervised Learning

Practical Information

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TD Material
https://github.com/esther-poniatowski/2223_UlmM2_ThNeuro

Goals of the TD

This series of TDs aim to study three types of learning algorithms : supervised, unsupervised and reinforcement learning.

This TD presents **unsupervised learning** through the paradigm of **Hebbian learning**, which is a biologically plausible mechanism in neurons.

Part 1 Modeling of a binocular neuron receiving two inputs.

Part 2 Hebbian learning & Variant algorithms.

1 Modeling a binocular neuron

Model of a binocular neuron

The model consists of a binocular neuron sensitive to two visual inputs, one from the left eye I_L and another one from the right eye I_R.

Each input is drawn from a random distribution of mean 0 and variance v.

The two inputs are correlated according to : $\mathbb{C}\mathrm{ov}(I_L, I_R) = c$.

① For $v = 1$ and $c \in \{-1, 0, 1\}$, sketch a distribution in the plane (I_L, I_R) where each input varies between -1 and 1.

② Justify why, for visual inputs, the correlation between left and right inputs should be modeled as $c \geq 0$.

③ Justify that $\mathbb{C}\mathrm{ov}(I_L, I_R) = \mathbb{E}(I_L I_R)$, $\mathbb{V}(I_L) = \mathbb{E}(I_L^2)$ and $\mathbb{V}(I_R) = \mathbb{E}(I_R^2)$.

④ Show that $-v \leq c \leq v$.

⑤ Let $\vec{e}_1, \vec{e}_2$ be two basis vectors (with unit norm) aligned with the axes of perfect correlation and perfect anti-correlation. Express those basis vectors as combinations of the canonical basis vectors $\vec{e}_L, \vec{e}_R$.

Each input can be decomposed in the canonical basis : $\vec{I} = \begin{bmatrix} I_L \\ I_R \end{bmatrix}$, or equivalently in the new basis : $\vec{I} = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$.
The coefficients I_L, I_R, I_1 and I_2 correspond to random variables related between each other.

⑥ For any vector $\vec{I} = I_L \vec{e}_L + I_R \vec{e}_R$, express its coordinates in the new basis : $\vec{I} = I_1 \vec{e}_1 + I_2 \vec{e}_2$.

⑦ Compute the correlations $\mathbb{E}(I_1^2)$, $\mathbb{E}(I_2^2)$, $\mathbb{E}(I_1 I_2)$. Comment on the interest of the change of basis to study the situation.

2 Hebbian learning algorithm

Unsupervised learning rules

In the model, the activity of the binocular neuron is a linear combination of the inputs its receives at any time :

$$V(t) = \vec{W}(t) \cdot \vec{I}(t) \tag{1}$$

The weights W represent the synaptic strengths between the sensory/retina neurons and the binocular neuron.

In unsupervised learning, the synaptic weights evolve depending only on the neuron's activity itself. The *learning rule* specifies the update of the weights' vector $\vec{W}$ every time an input $\vec{I}(t)$ is presented, under the form :

$$\vec{W}(t+1) = \vec{W}(t) + f(V(t), \vec{I}(t)) \tag{2}$$

The output of the function $f(V(t), \vec{I}(t))$ represents the *'update'* vector $\langle \Delta\vec{W} \rangle = \langle \vec{W}(t+1) - \vec{W}(t+1) \rangle$ (of same dimension as $\vec{W}$). The latter depends on the input presented at time t and the resulting activity elicited by this input.

Several learning rules exist, implementing different choices for the update function f. Most of them are variants of the standard Hebbian learning rule presented below.

Notations for the following questions :

- $\vec{I} = \begin{bmatrix} I_L \\ I_R \end{bmatrix}$ still denotes two-dimensional inputs to the neuron under consideration.
- $\langle \cdot \rangle = \mathbb{E}(\cdot)$ denotes the average taken over the distribution of the inputs $\vec{I}$.
- $\langle \Delta\vec{W} \rangle = \langle \vec{W}(t+1) - \vec{W}(t+1) \rangle$ denotes the mean update vector over one learning step.
- $\alpha(t)$ denotes the angle between $\vec{I}(t)$ and $\vec{W}(t)$.

2.1 Standard Hebbian learning

Hebbian learning algorithm

According to the Hebbian learning rule, every time an input $\vec{I}(t)$ is presented, the neuron weights are updated according to :

$$\vec{W}(t+1) = \vec{W}(t) + \epsilon V(t)\vec{I}(t) \tag{3}$$

In other words, the update function is given by :

$$\langle \Delta\vec{W} \rangle = \epsilon \langle V(t)\vec{I}(t) \rangle \tag{4}$$

8 Assuming $||\vec{I}|| = 1$, sketch the update of the vector $\vec{W}$ in the plane (w_1, w_2), for different values of α. Comment on the evolution of $||\vec{W}||$.

To simplify, the learning rate is set to $\epsilon = 1$ in all the following questions.

9 In the case in which $\vec{W}$ is initially along the direction of one main axis of the distribution, $\vec{e}_1$ or $\vec{e}_2$, determine the corresponding direction of the update $\Delta\vec{W}$. Along which of these two directions would the update vector have the largest magnitude ?

10 Obtain a linear differential equation for the evolution of the weight vector $\vec{W}$. Determine the eigenvectors and associated eigenvalues of the dynamics. Comment.

2.2 Improvements of Hebbian learning

Hebbian learning with homeostasis

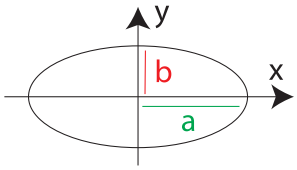
In order to prevent the weights from growing exponentially, it is possible to add a "homeostatic" term to the dynamics, such that :

$$\langle \Delta \vec{W} \rangle = \langle V(t)\vec{I}(t) \rangle - \langle V(t)^2 \rangle \vec{W}(t) \tag{5}$$

(11) Is it possible to obtain a *linear* differential equation for the evolution of $\vec{W}$? Obtain a differential equation on the components of $\vec{W}$ in the basis $(\vec{e_1}, \vec{e_2})$.

The equation of an ellipse, given by :

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{6}$$



(12) Draw the nullclines in the space (I_L, I_R). Identify the equilibrium points for $\vec{W}$, and determine their stability.

(13) Comment on the outcome of the homeostatic learning rule.

Competitive Hebbian learning

Competitive Hebbian learning consists in adding a term to the dynamics so as to introduce competition between the left and right inputs. In the basis $(\vec{e_L}, \vec{e_R})$, the dynamics are now given by :

$$\langle \Delta \vec{W} \rangle = \langle V(t)\vec{I}(t) \rangle - \left\langle V(t) \begin{bmatrix} \frac{I_L + I_R}{2} \\ \frac{I_L + I_R}{2} \end{bmatrix} \right\rangle \tag{7}$$

(14) Obtain a linear differential equation on $\vec{W}$ in the basis $(\vec{e_1}, \vec{e_2})$. Comment on the dynamics.

(15) If the the weights are forced to remain positive, comment on the outcome of the competitive hebbian learning rule.