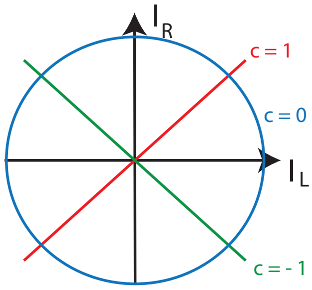
TD8 – Learning II Unsupervised Learning

1 Modeling inputs

① *Distributions of inputs*
Different inputs correspond to points in the plane (I_L, I_R). The correlation sets the direction along which inputs align.

- If $c = 0$, then inputs are not correlated. Thus, they can span the full unit disk.
- If $c = 1$, then inputs are positively correlated, such that high values of I_L are associated to high values of I_R. Thus, inputs lie in an ellipse along the identity line.
- If $c = -1$, then inputs are negatively correlated, such that positive values of I_L are associated to negative values of I_R. Thus, inputs lie in an ellipse along the line $y = -x$.



② *Correlation for visual inputs*
Both eyes receive light from the same visual scene, with slightly different viewpoints. Thus, left and right inputs are correlated, which implies a positive correlation $c \geq 0$.

③ *Variances and Covariance*
According to Koenig-Huygens formula, for two random variables X, Y :

$$\mathbb{C}\text{ov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) \qquad \mathbb{V}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

The means of the variables I_L and I_R are null by assumption, such that :

$$\mathbb{C}\text{ov}(I_L, I_R) = \mathbb{E}(I_L I_R) \qquad \mathbb{V}(I_L) = \mathbb{E}(I_L^2) \qquad \mathbb{V}(I_R) = \mathbb{E}(I_R^2)$$

④ *Ordering variances and covariances*
The relation between the variance and the covariance can be obtained by developing the square of the sum and the difference of the random variable I_R and I_L. As cubes are positive, so are the results :

$$\langle (I_L - I_R)^2 \rangle = \langle I_L^2 \rangle - 2\langle I_L I_R \rangle + \langle I_R^2 \rangle = 2(v - c) > 0 \implies v > c$$
$$\langle (I_L + I_R)^2 \rangle = \langle I_L^2 \rangle + 2\langle I_L I_R \rangle + \langle I_R^2 \rangle = 2(v + c) > 0 \implies -v < c$$

Therefore :

$$-v \leq c \leq v$$

⑤ *Axes of maximal correlation and anti-correlation*

- The axis reflecting perfect correlation is along the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and has a unit norm : $\vec{e}_1 = \dfrac{\vec{e}_L + \vec{e}_R}{\sqrt{2}}$.
- The axis reflecting perfect anti-correlation is along the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and has a unit norm : $\vec{e}_2 = \dfrac{\vec{e}_L - \vec{e}_R}{\sqrt{2}}$.
- Equivalently, the vectors $\vec{e}_L$, $\vec{e}_R$ can be expressed in the basis $\vec{e}_1$, $\vec{e}_2$:

$$\begin{cases} \sqrt{2}\vec{e}_1 = \vec{e}_L + \vec{e}_R \\ \sqrt{2}\vec{e}_2 = \vec{e}_L - \vec{e}_R \end{cases} \implies \begin{cases} 2\vec{e}_L = \sqrt{2}(\vec{e}_1 + \vec{e}_2) \\ 2\vec{e}_R = \sqrt{2}(\vec{e}_1 - \vec{e}_2) \end{cases} \implies \begin{cases} \vec{e}_L = \dfrac{\vec{e}_1 + \vec{e}_2}{\sqrt{2}} \\ \vec{e}_R = \dfrac{\vec{e}_1 - \vec{e}_2}{\sqrt{2}} \end{cases}$$

⑥ *Coordinates in the new basis*

- For any vector $\vec{I} = I_L \vec{e}_L + I_R \vec{e}_R$, the corresponding coordinates in the new basis $\vec{I} = I_1 \vec{e}_1 + I_2 \vec{e}_2$ are :

$$I_1 = \frac{I_L + I_R}{\sqrt{2}} \qquad I_2 = \frac{I_L - I_R}{\sqrt{2}}$$

Method – Expressing the coordinates of a vector in a new basis

Method ① Replacing the expressions of the vectors $\vec{e}_L$, $\vec{e}_R$ by $\vec{e}_1$, $\vec{e}_2$ and identifying.

$$\vec{I} = I_L \vec{e}_L + I_R \vec{e}_R = I_L \frac{\vec{e}_1 + \vec{e}_2}{\sqrt{2}} + I_R \frac{\vec{e}_1 - \vec{e}_2}{\sqrt{2}} = \underbrace{\frac{I_L + I_R}{\sqrt{2}}}_{I_1} \vec{e}_1 + \underbrace{\frac{I_L - I_R}{\sqrt{2}}}_{I_2} \vec{e}_2$$

Method ② Projecting the vector $\vec{I} = I_L \vec{e}_L + I_R \vec{e}_R$ onto the basis vectors $\vec{e}_1$ and $\vec{e}_2$ through the scalar product (if the basis vectors have unit norm and are orthogonal).

$$I_1 = \|\mathrm{Proj}_{\vec{e}_1}(\vec{I})\| = (I_L \vec{e}_L + I_R \vec{e}_R) \cdot \vec{e}_1 = I_L \vec{e}_L \cdot \vec{e}_1 + I_R \vec{e}_R \cdot \vec{e}_1 = I_L \times \frac{1}{\sqrt{2}} + I_R \times \frac{1}{\sqrt{2}} = \frac{I_L + I_R}{\sqrt{2}}$$
$$I_2 = \|\mathrm{Proj}_{\vec{e}_2}(\vec{I})\| = (I_L \vec{e}_L + I_R \vec{e}_R) \cdot \vec{e}_2 = I_L \vec{e}_L \cdot \vec{e}_2 + I_R \vec{e}_R \cdot \vec{e}_2 = I_L \times \frac{1}{\sqrt{2}} + I_R \times \left(-\frac{1}{\sqrt{2}}\right) = \frac{I_L - I_R}{\sqrt{2}}$$

⑦ *Correlations in the new basis*

The coefficients I_1 and I_2 can be expressed as a function of I_R and I_L :

$$\mathbb{E}(I_1^2) = \mathbb{E}\left(\frac{(I_L + I_R)^2}{\sqrt{2}^2}\right) = \frac{\mathbb{E}(I_L^2) + \mathbb{E}(I_R^2) + 2\mathbb{E}(I_L I_R)}{2} = \frac{v + v + 2c}{2} = v + c$$
$$\mathbb{E}(I_2^2) = \mathbb{E}\left(\frac{(I_L - I_R)^2}{\sqrt{2}^2}\right) = \frac{\mathbb{E}(I_L^2) + \mathbb{E}(I_R^2) - 2\mathbb{E}(I_L I_R)}{2} = \frac{v + v - 2c}{2} = v - c$$
$$\mathbb{E}(I_1 I_2) = \mathbb{E}\left(\frac{(I_L - I_R)(I_L + I_R)}{\sqrt{2}\sqrt{2}}\right) = \frac{\mathbb{E}(I_L^2) - \mathbb{E}(I_R^2)}{2} = \frac{v - v}{2} = 0$$

2 Hebbian learning algorithm

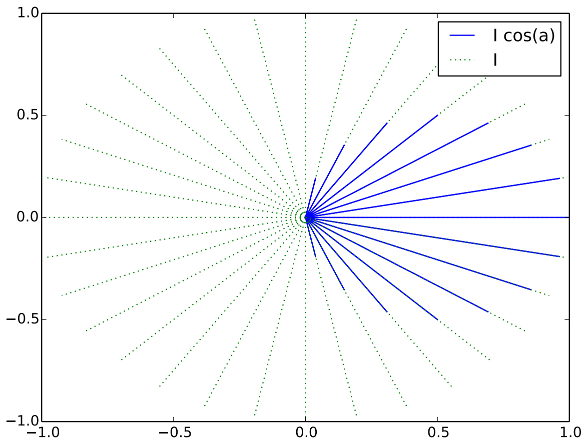
2.1 Standard Hebbian learning

⑧ *Update of the weight vector*

• With the Hebbian learning rule, $\vec{W}(t+1) - \vec{W}(t) = \epsilon V(t)\vec{I}(t)$, such that the update vector is aligned in the direction of the input $\vec{I}(t)$, with a magnitude $\epsilon V(t)\|\vec{I}\| = \epsilon V(t)$ (under the assumption $\|\vec{I}\| = 1$).
Moreover, the activity of the neuron is exactly the scalar product between $\vec{W(t)}$ and $\vec{I}(t)$: $V(t) = \vec{W}(t)\cdot\vec{I}(t)$. This scalar product can be interpreted with the cosine of the angle between both vectors : $V(t) = \|\vec{W}\|.\|\vec{I}\|\cos(\vec{W},\vec{I}) = \|\vec{W}\|\cos(\alpha)$.
Therefore, the update vector can be expressed as a function of α :

$$\vec{W}(t+1) - \vec{W}(t) = \epsilon\|\vec{W}\|\cos(\alpha)\vec{I}$$

• Whatever the angle α, the norm of the weight vector $\|\vec{W}\|$ increases at each update. Indeed :
  - If $\vec{W}, \vec{I}$ are oriented in 'similar direction', then $\alpha \in [-\pi/2, \pi/2]$ and $\cos(\vec{W},\vec{I}) \geq 0$. The update vector is in the direction of $\vec{I}$, and consequently in the direction of $\vec{W}$ too.
  - If $\vec{W}, \vec{I}$ are oriented in 'opposite direction', then $\alpha \in [\pi/2, 3\pi/2]$ and $\cos(\vec{W},\vec{I}) \leq 0$. The update vector is in the opposite direction of $\vec{I}$, and consequently still in the direction of $\vec{W}$.



⑨ *Update of along main axes*

If $\vec{W}$ is along one of the axes $\vec{e}_1$, $\vec{e}_2$, then through averaging, the update vector $\frac{\mathrm{d}\vec{W}}{\mathrm{d}t}$ will be parallel to $\vec{W}$.
Indeed, for instance with $\vec{W} = w\vec{e_1}$:

$$\begin{aligned}
\langle\Delta\vec{W}\rangle &= \langle V(t)\vec{I}(t)\rangle \\
&= \left\langle \vec{W}(t)\cdot\vec{I}(t)\times\vec{I}(t)\right\rangle \\
&= \left\langle w\,\underbrace{\vec{e}_1\cdot\vec{I}(t)}_{I_1}\times\underbrace{\vec{I}(t)}_{I_1\vec{e}_1+I_2\vec{e}_2}\right\rangle \\
&= w\,\langle I_1\times(I_1\vec{e}_1+I_2\vec{e}_2)\rangle \\
&= w\left(\langle I_1^2\rangle\vec{e}_1+\langle I_1I_2\rangle\vec{e}_2\right) \\
&= w\,((v+c)\times\vec{e}_1+0\times\vec{e}_2) \quad \text{question ⑦} \\
&= w(v+c)\vec{e}_1 \\
&= (v+c)\vec{W}
\end{aligned}$$

Therefore, the axes $\vec{e}_1$, $\vec{e}_2$ are the *eigenvectors* of the dynamics :
- $\vec{e}_1$ is associated to the largest eigenvalue $v+c$,
- $\vec{e}_2$ is associated to the lowest eigenvalue $v-c$.

10 *Evolution of $\vec{W}$*

Method 1 – In the basis of eigenvectors $\vec{e}_1, \vec{e}_2$

The dynamics can be expressed directly in the eigenvectors basis $\vec{e}_1, \vec{e}_2$ (question 9), writing $\vec{W} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$:

$$\begin{aligned}
\langle \Delta \vec{W} \rangle = \langle V(t) \vec{I}(t) \rangle &= \left\langle \begin{bmatrix} w_1 I_1^2 + w_2 I_1 I_2 \\ w_2 I_2^2 + w_1 I_1 I_2 \end{bmatrix} \right\rangle \\
&= \begin{bmatrix} w_1(v+c) + 0 \\ 0 + w_2(v-c) \end{bmatrix} \qquad \text{(question 9)} \\
&= \underbrace{\begin{pmatrix} v+c & 0 \\ 0 & v-c \end{pmatrix}}_{\mathbf{D}} \vec{W}
\end{aligned}$$

The system is already diagonalized, such that the evolution of the weights are governed by simple geometric sequences :

$$\begin{aligned}
\langle \Delta \vec{w_1} \rangle &= (v+c) w_1 \\
\langle \Delta \vec{w_2} \rangle &= (v-c) w_2
\end{aligned}$$

Method 2 – In the initial basis $\vec{e}_L, \vec{e}_R$

The evolution of the weight vector can be expressed by a linear transformation in the basis $\vec{e}_L, \vec{e}_R$, writing $\vec{W} = \begin{bmatrix} w_L \\ w_R \end{bmatrix}$:

$$\begin{aligned}
\langle \Delta \vec{W} \rangle = \langle V(t) \vec{I}(t) \rangle &= \left\langle \vec{W}(t) \cdot \vec{I}(t) \times \vec{I}(t) \right\rangle \\
&= \left\langle (w_L I_L + w_R I_R) \begin{bmatrix} I_L \\ I_R \end{bmatrix} \right\rangle \\
&= \left\langle \begin{bmatrix} w_L I_L^2 + w_R I_L I_R \\ w_R I_R^2 + w_L I_L I_R \end{bmatrix} \right\rangle \\
&= \begin{bmatrix} w_L \langle I_L^2 \rangle + w_R \langle I_L I_R \rangle \\ w_R \langle I_R^2 \rangle + w_L \langle I_L I_R \rangle \end{bmatrix} \\
&= \begin{bmatrix} w_L v + w_R c \\ w_R v + w_L c \end{bmatrix} \\
&= \underbrace{\begin{pmatrix} v & c \\ c & v \end{pmatrix}}_{\mathbf{A}} \vec{W}
\end{aligned}$$

Projecting on each axis, this relation is equivalent to :

$$\begin{aligned}
\langle \Delta \vec{w_1} \rangle &= w_L v + w_R c \\
\langle \Delta \vec{w_2} \rangle &= w_R v + w_L c
\end{aligned}$$

Those equations are coupled. To solve them, the evolution of $\vec{W}$ can be expressed in the orthogonal axes defined by the eigenvectors of the matrix $\mathbf{A}$ (which are $\vec{e}_1$, $\vec{e}_2$ according to question 9, but in this method it is assumed that they have not been identified previously). The eigenvalues are found by the roots of the characteristic polynomial :

$$\begin{aligned}
\det[\mathbf{A} - \lambda \, \mathrm{Id}] &= 0 \\
(v - \lambda)^2 - c^2 &= 0 \\
v - \lambda &= \pm c
\end{aligned}$$

The two solutions are $\lambda_1 = v + c$ and $\lambda_2 = v - c$. The normalised eigenvector $\vec{e_1}$ associated to the first eigenvalue satisfies :

$$A \vec{e_1} = (v+c) \vec{e_1}$$

Writing $\vec{e_1} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and normalizing at the end :

$$\begin{aligned}
\begin{pmatrix} v & c \\ c & v \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} v x_1 + c y_1 \\ c x_1 + v y_1 \end{pmatrix} &= \begin{pmatrix} (v+c) x_1 \\ (v+c) y_1 \end{pmatrix} \\
\Rightarrow x_1 = y_1 \Rightarrow \vec{e_1} &= \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}
\end{aligned}$$

The same development for $\vec{e_2}$ leads to :

$$\begin{pmatrix} v & c \\ c & v \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} vx_2 + cy_2 \\ cx_2 + vy_2 \end{pmatrix} = \begin{pmatrix} (v-c)x_2 \\ (v-c)y_2 \end{pmatrix}$$
$$\Rightarrow x_2 = -y_2 \Rightarrow \vec{e_2} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

2.2 Improvements of Hebbian learning

11 *Evolution of $\vec{W}$ with homeostasis*

• Is it *not* possible to obtain a *linear* differential equation for the evolution of $\vec{W}$, because the second term includes components of $\vec{W}$ to the power three :

$$\begin{aligned}
\langle V(t)^2\rangle &= \langle(\vec{W}\cdot\vec{I})^2\rangle \\
&= \langle(w_1 I_1 + w_2 I_2)^2\rangle \\
&= w_1^2\langle I_1^2\rangle + w_2^2\langle I_2^2\rangle + w_1 w_2\langle I_1 I_2\rangle \\
&= w_1^2(v+c) + w_2^2(v-c) + 0 \\
\langle V(t)^2\rangle\vec{W} &= (w_1^2(v+c) + w_2^2(v-c))\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \\
&= \begin{bmatrix} w_1^3(v+c) + w_1 w_2^2(v-c) \\ w_1^2 w_2(v+c) + w_2^3(v-c) \end{bmatrix}
\end{aligned}$$

• The evolution of the components of $\vec{W}$ obeys the following differential equations :

Note : terms in red stem from the standard hebbian rule (question 10) and terms in blue correspond to the homeostatic term :

$$\begin{cases} \langle\Delta\vec{w_1}\rangle = (v+c)w_1 - (w_1^3(v+c) + w_1 w_2^2(v-c)) \\ \langle\Delta\vec{w_2}\rangle = (v-c)w_2 - (w_1^2 w_2(v+c) + w_2^3(v-c)) \end{cases} \iff \begin{cases} \langle\Delta\vec{w_1}\rangle = w_1(v+c-w_1^2(v+c) - w_2^2(v-c)) \\ \langle\Delta\vec{w_2}\rangle = w_2(v-c-w_2^2(v-c) - w_1^2(v+c)) \end{cases}$$

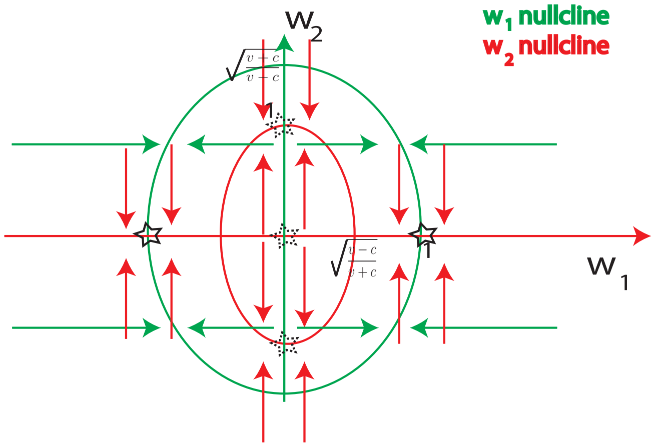
12 *Nullclines and equilibria*

• The differential equations can be rewritten under the form of the equation of ellipses :

$$\langle\Delta\vec{w_1}\rangle = -(v+c)\,w_1\left(w_1^2 + \frac{v-c}{v+c}w_2^2 - 1\right)$$
$$\langle\Delta\vec{w_2}\rangle = -(v-c)\,w_2\left(\frac{v+c}{v-c}w_1^2 + w_2^2 - 1\right)$$

• The nullclines correspond to the points where $\frac{\mathrm{d}W_1}{\mathrm{d}t}$ and $\frac{\mathrm{d}W_2}{\mathrm{d}t}$ cancel respectively.

- The w_1-nullcline contains the line $w_1 = 0$ and the ellipse $w_1^2 + \frac{v-c}{v+c}w_2^2 = 1$, which sets $a_1 = 1,\ b_1 = \sqrt{\frac{v+c}{v-c}} > 1$.
- The w_2-nullcline contains the line $w_2 = 0$ and the ellipse $\frac{v+c}{v-c}w_1^2 + w_2^2 = 1$, which sets $a_2 = \sqrt{\frac{v-c}{v+c}} < 1,\ b_2 = 1$.



The arrows indicate the direction of evolution of the system in any part of the (W_1, W_2) space, which can be determined at any point depending on the sign of the derivatives of w_1 and w_2. A positive derivative entails a right arrow for w_1 and an up arrow for w_2, and conversely.

- The equilibria are obtained at the intersections the w_1 and w_2-nullclines. The only stable intersection points are :

$$\begin{cases} w_2 = 0 \\ w_1 = \pm 1 \end{cases}$$

(13) *Interpretation of the homeostatic learning rule*
In both cases, at the equilibrium, the component w_2 is null. For instance, if initially the weights are positive, $w_1(t = 0) > 0$, then the system converges to $w_1 = 1, w_2 = 0$.
This means the homeostatic learning rule keeps the projection onto the principal component, that is the eigenvector associated to the largest eigenvalue. The projection on the second eigenvector is discarded.

(14) *Evolution of $\vec{W}$ in competitive hebbian learning*
The second term of the learning rule can be expressed in the basis $(\vec{e}_1, \vec{e}_2)$, through the following relation (question (6)) :

$$\begin{bmatrix} \frac{I_L+I_R}{2} \\ \frac{I_L+I_R}{2} \end{bmatrix} = \frac{I_L + I_R}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{\sqrt{2} I_1}{2} \sqrt{2} \vec{e_1} = I_1 \vec{e_1}$$

Thereby :

$$\begin{aligned} \left\langle V(t) \begin{bmatrix} \frac{I_L+I_R}{2} \\ \frac{I_L+I_R}{2} \end{bmatrix} \right\rangle &= \langle (w_1 I_1 + w_2 I_2) I_1 \vec{e}_1 \rangle \\ &= (w_1 \langle I_1^2 \rangle + w_2 \langle I_2 I_1 \rangle) \vec{e}_1 \\ &= w_1 (v + c) \vec{e}_1 \end{aligned}$$

- The evolution of the components of $\vec{W}$ obeys the following differential equations :

Note : terms in red stem from the standard hebbian rule (question (10)) and terms in blue correspond to the competitive term :

$$\begin{cases} \langle \Delta \vec{w_1} \rangle = (v + c) w_1 - (v + c) w_1 \\ \langle \Delta \vec{w_2} \rangle = (v - c) w_2 \end{cases} \iff \begin{cases} \langle \Delta \vec{w_1} \rangle = 0 \\ \langle \Delta \vec{w_2} \rangle = (v - c) w_2 \end{cases} \iff \langle \Delta \vec{W} \rangle = \begin{pmatrix} 0 & 0 \\ 0 & v - c \end{pmatrix} \vec{W}$$

The component w_1 does not change and the component w_2 grows exponentially, towards $\pm\infty$ depending on the sign of w_2.

(15) *Positive weights*
Returning in the initial basis, $w_L + w_R$ is constant, and $w_L - w_R$ is growing exponentially.
Enforcing both to be positive, their sum is fixed and their difference goes exponentially to $+\infty$ if initially $w_L > w_R$ or to $-\infty$ in the other case.
This means the weight with the highest initial value 'wins' the competition, while the other becomes null.