HILBERT TRANSFORM
&
EMPIRICAL MODE DECOMPOSITION

Hilbert Transform

<u>Objective:</u> determine a unique, analytic signal from a real signal to calculate instantaneous properties

Analytic signal: $z(t) = \underbrace{f(t)}_{\text{real part}} + i\,\underbrace{H\{f(t)\}}_{\text{imaginary part}}$

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Analytic signal: $z(t) = f(t) + i\, H\{f(t)\} = A(t) \cdot e^{i\varphi(t)}$

real part imaginary part

with amplitude $A(t) = \sqrt{f^2(t) + H\{f(t)\}^2}$

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energy $e = |A(t)|^2$

$\boxed{\text{Hilbert spectrum}}$

phase $\varphi(t) = arctan\left(\frac{H\{f(t)\}}{f(t)}\right)$

frequency $\omega(t) = \frac{d\varphi(t)}{dt}$

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- Based on the discrete Fourier Transform
- Fourier transformed data contains the same information as the original, real-valued data in the temporal domain but it is complex

  → Preserve the complex nature during the back transformation

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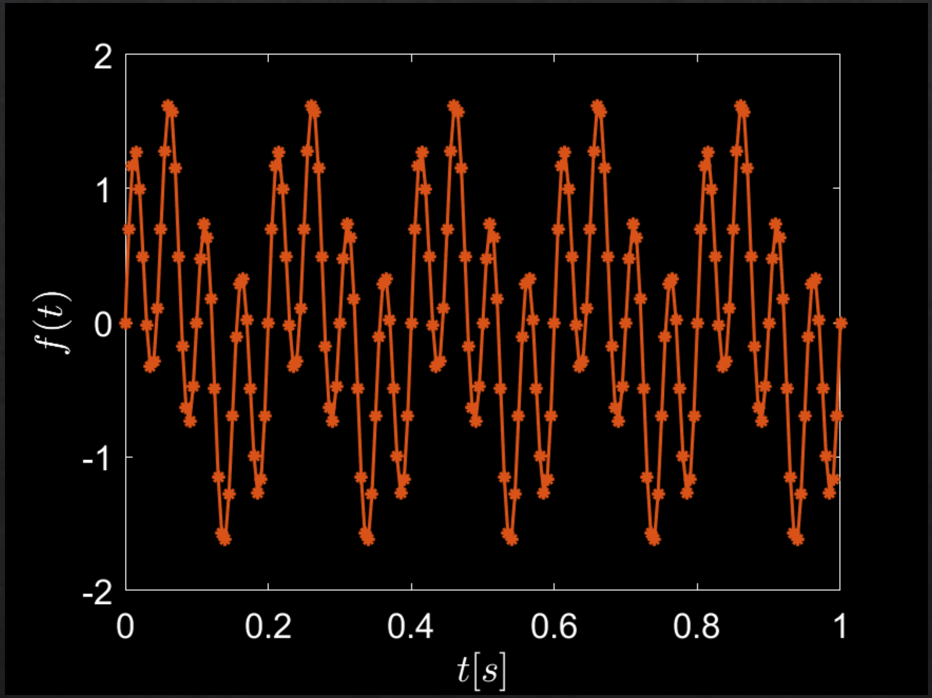
3. Double the amplitude related to the positive frequencies for energy conservation
4. Inverse Fourier Transform to obtain the analytic (complex) signal

Why should I use the Hilbert-Huang Transform?

Why is it meaningful to perform an
EMPIRICAL MODE DECOMPOSITION
prior to the Hilbert Transform?

Example

$f(t) = A_1\, sin(2\pi\omega_1 t) + A_2\, sin(2\pi\omega_2 t)$ with $\omega_1 = 5\ Hz, \omega_2 = 20\ Hz, A_1 = 0.7, A_2 = 1$ sampled at $200\ Hz$



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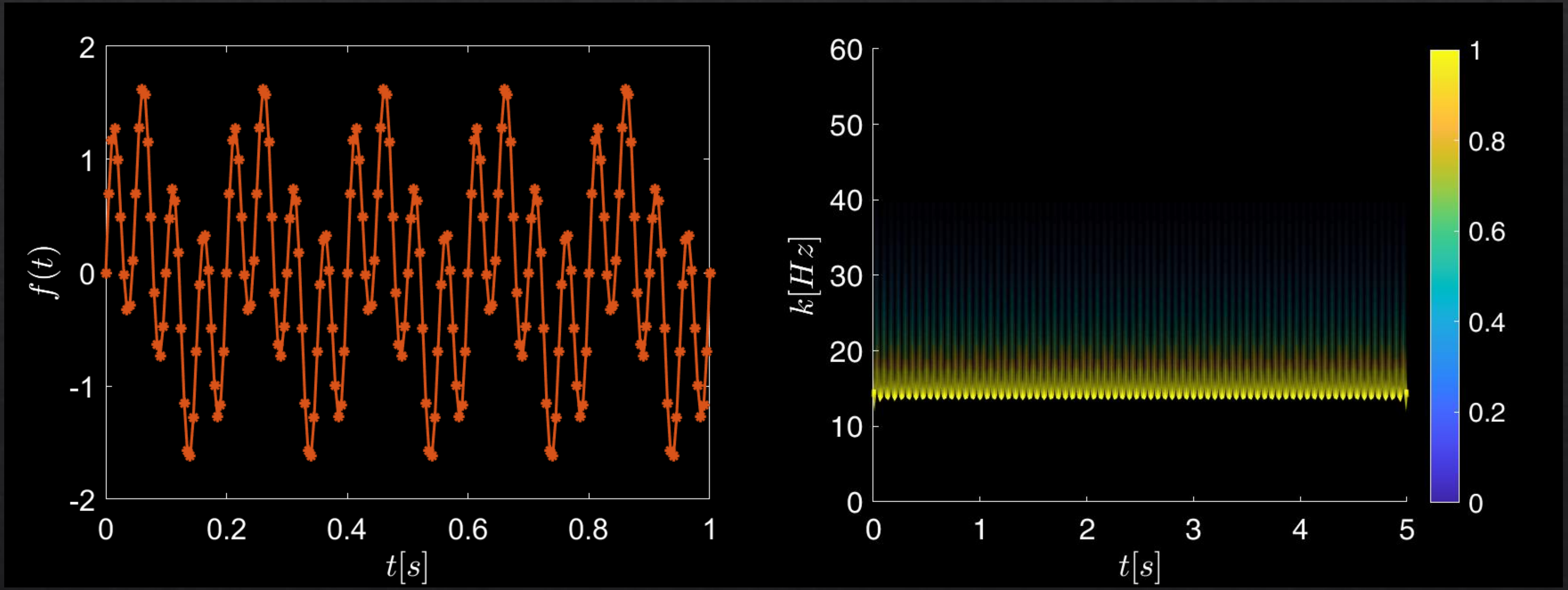

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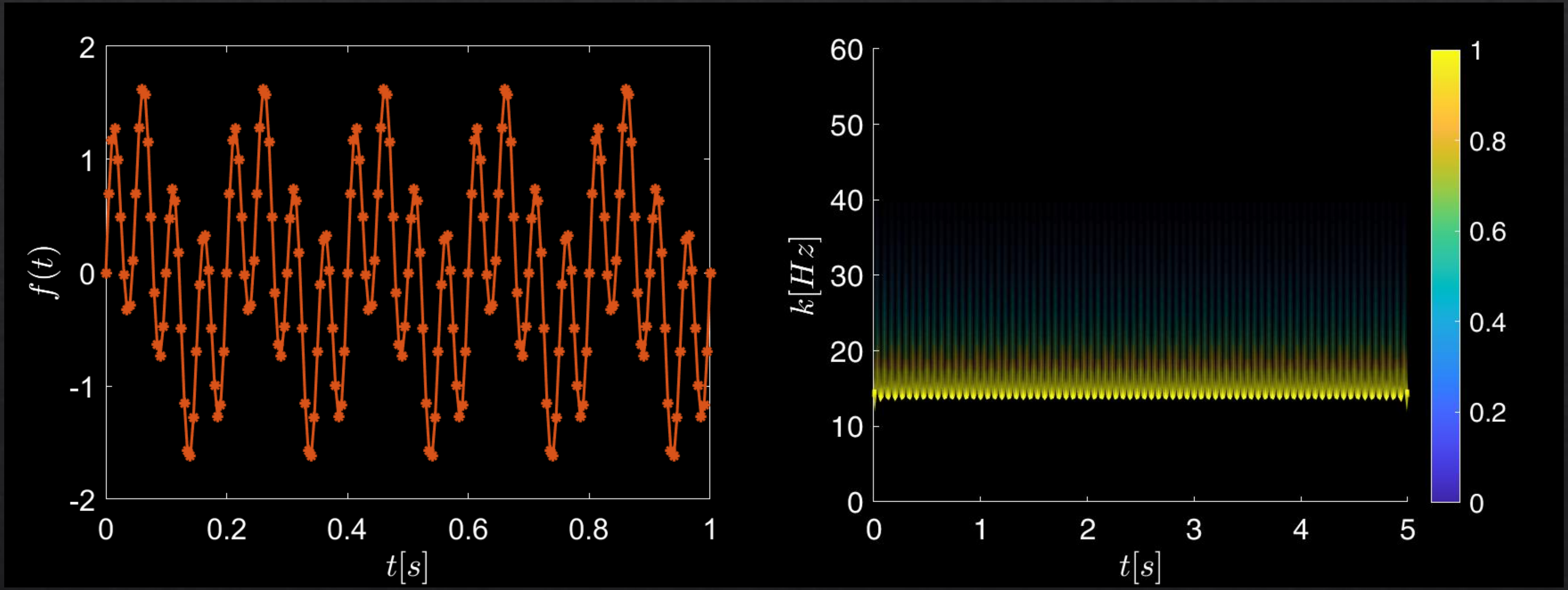

→ Hilbert spectrum does not reveal the correct frequency content

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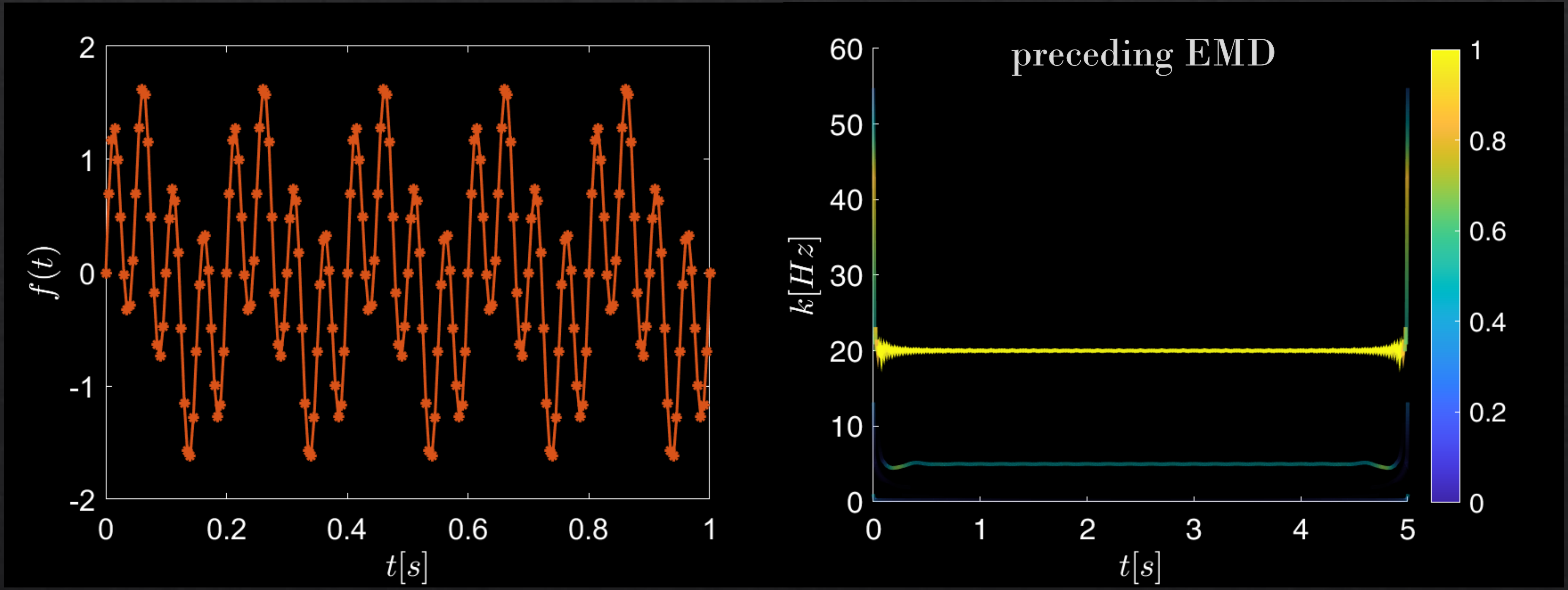

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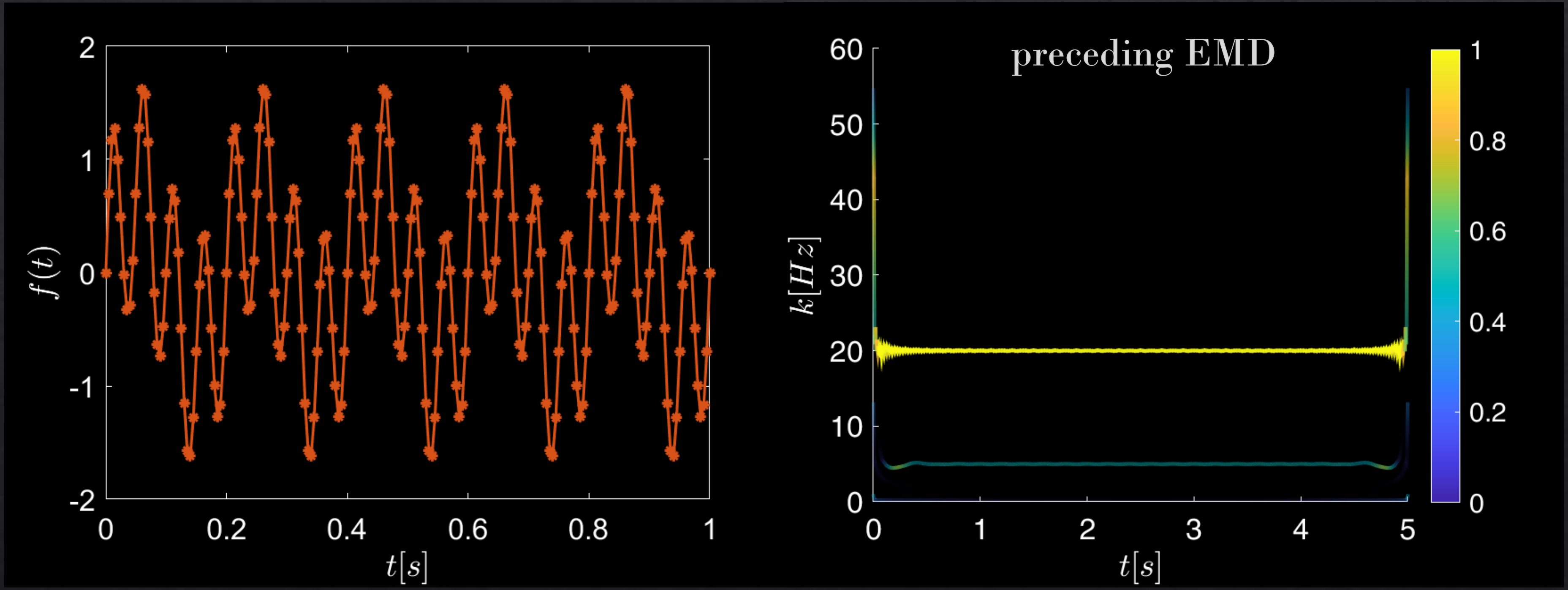

→ Hilbert spectrum with preceding EMD clearly reveals the contained frequencies and amplitudes

Instantaneous frequency

$$\boxed{\omega(t) = \frac{d\varphi(t)}{dt}}$$

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Conditions for narrow bandwidth:

- Equal number of zero crossings and extrema (global condition)
- Local symmetry with respect to zero level (local condition)

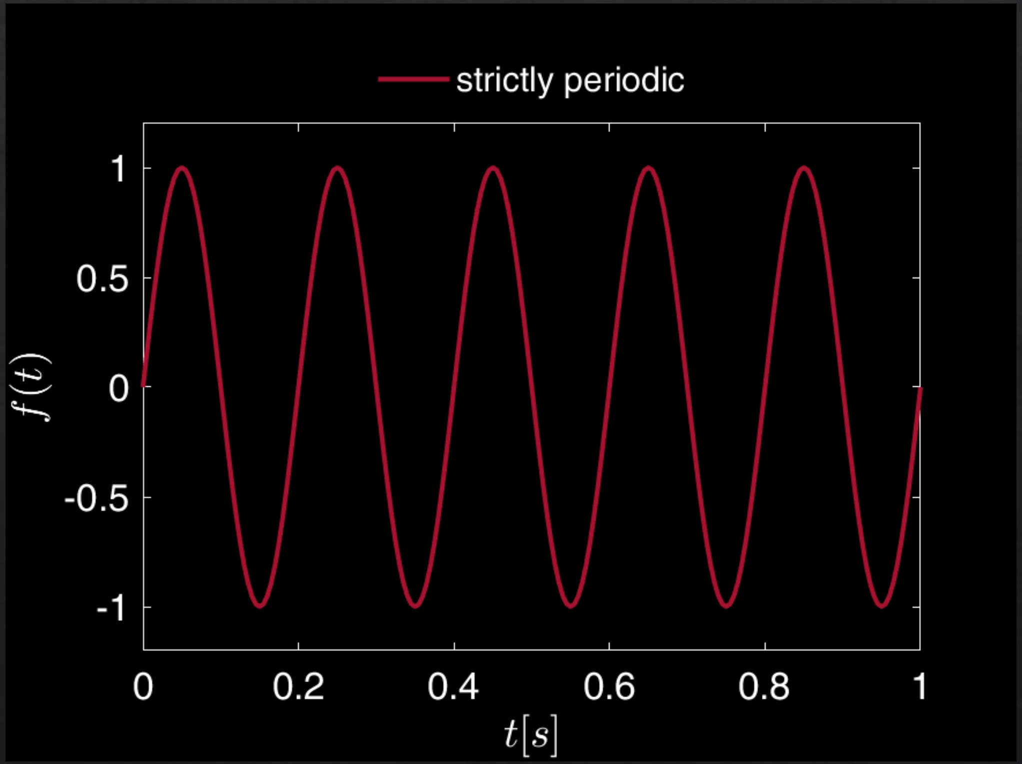
→ Definition of **Intrinsic Mode Functions** (IMFs) obtained by EMD

→ Determine Hilbert Spectrum of each IMF and combine afterwards

Limitations of the FT

Data must be linear and stationary or strictly periodic.

Fourier Transform uses uniform, trigonometric functions (sine, cosine)



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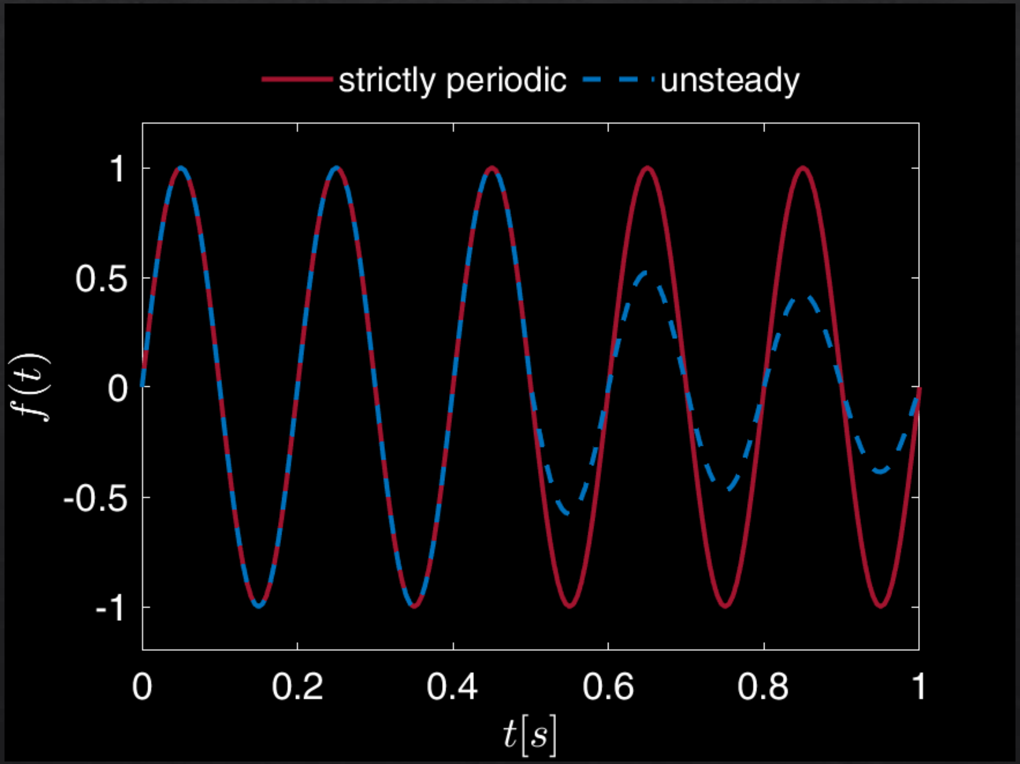

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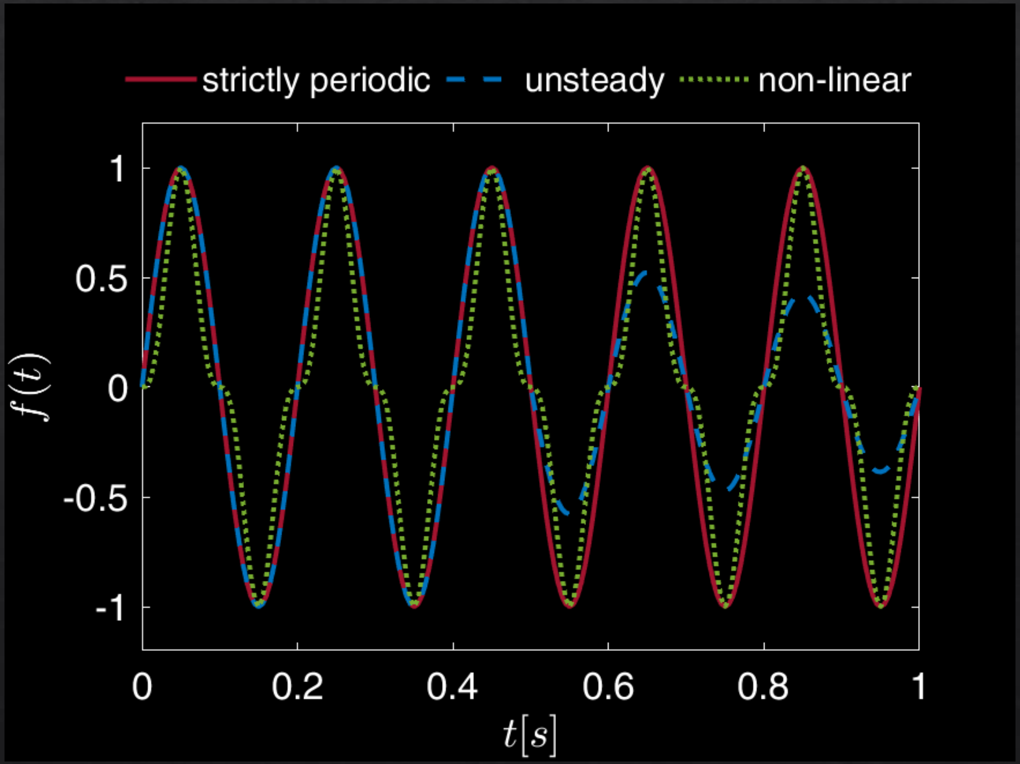

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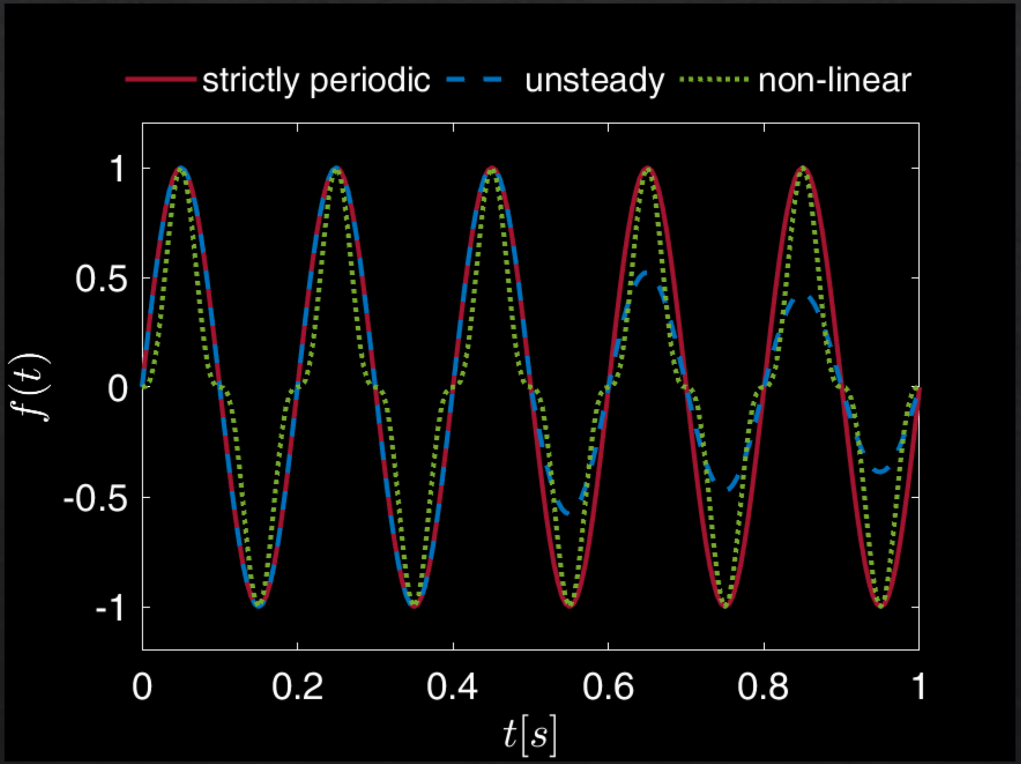
→ Energy spreading across spurious harmonics

→ Non-physical representation of data



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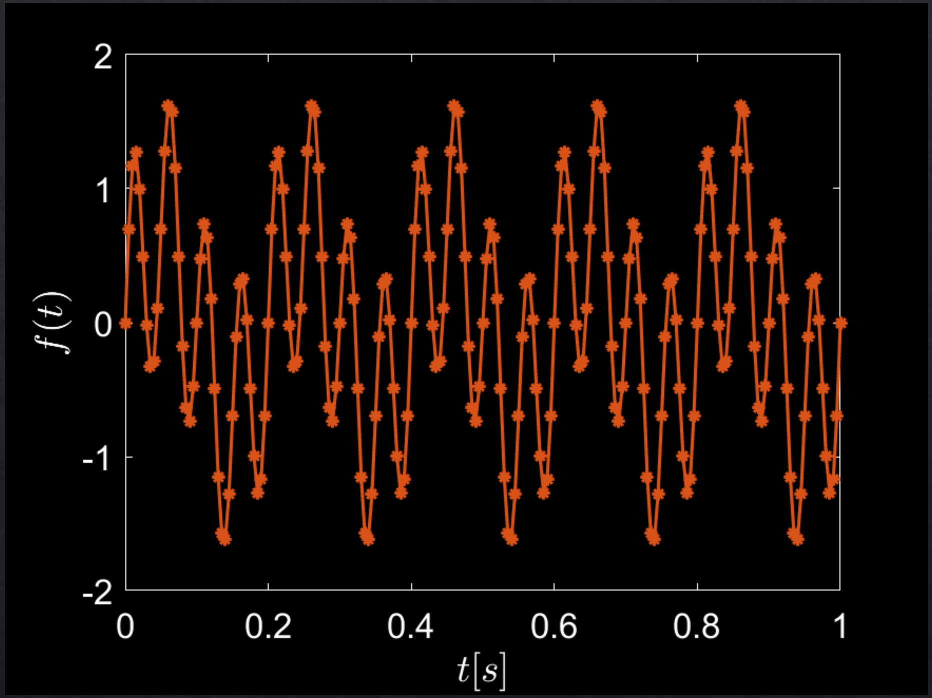

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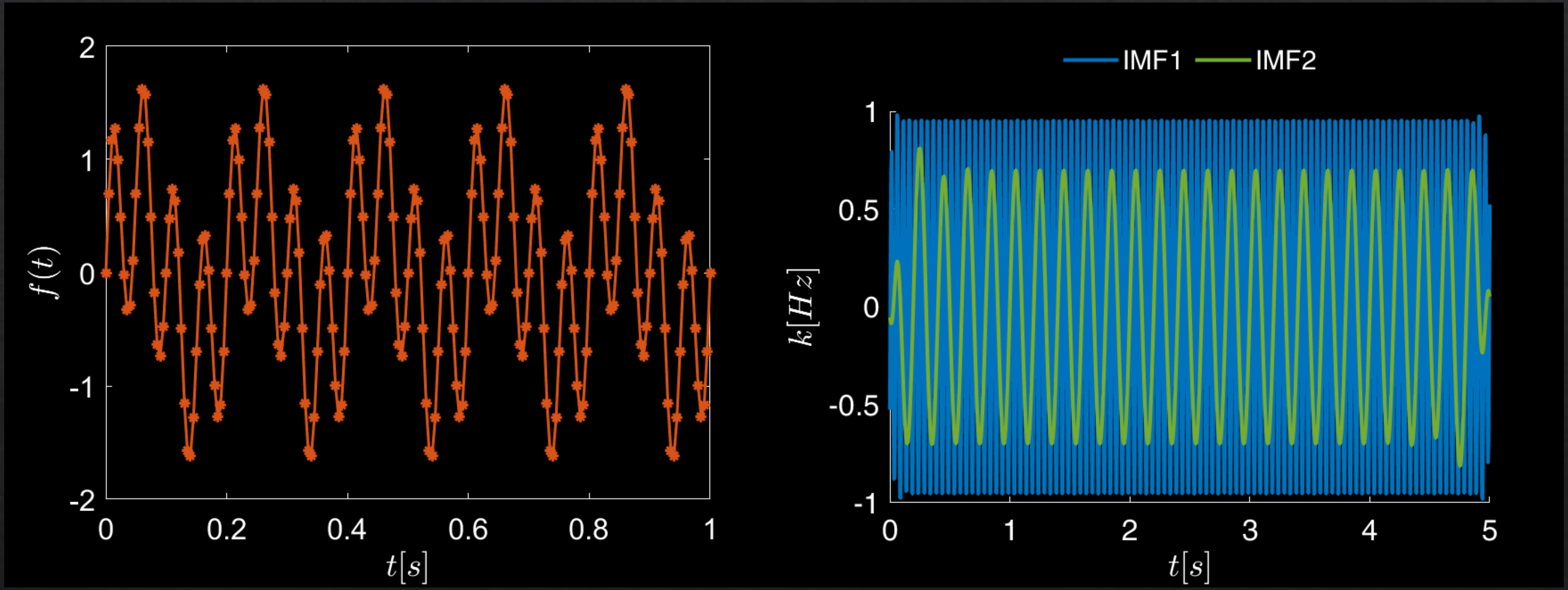

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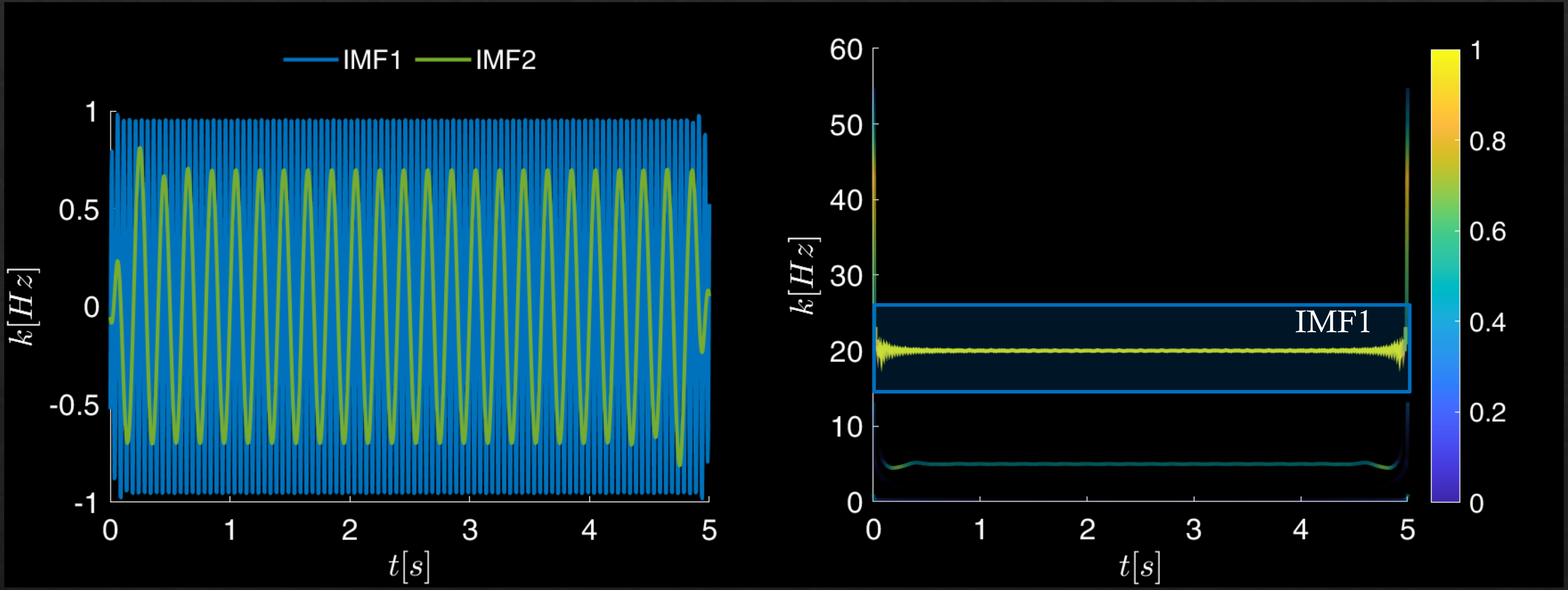

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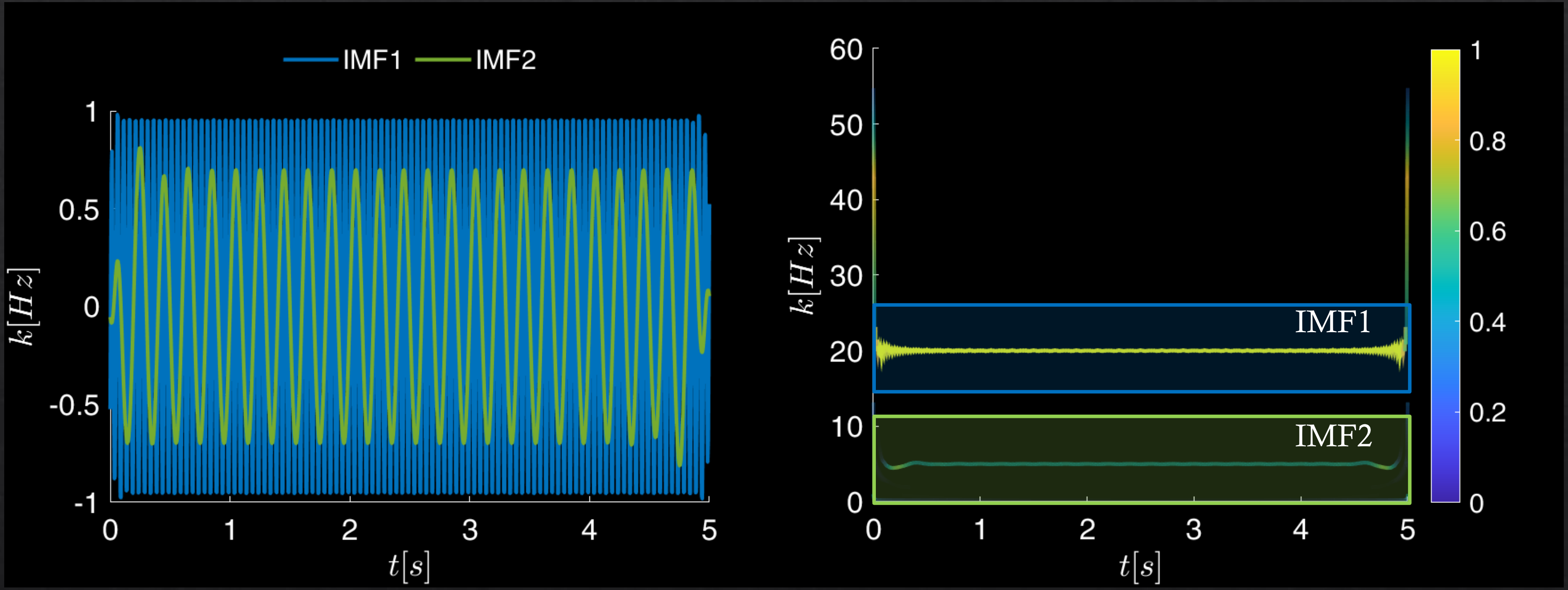

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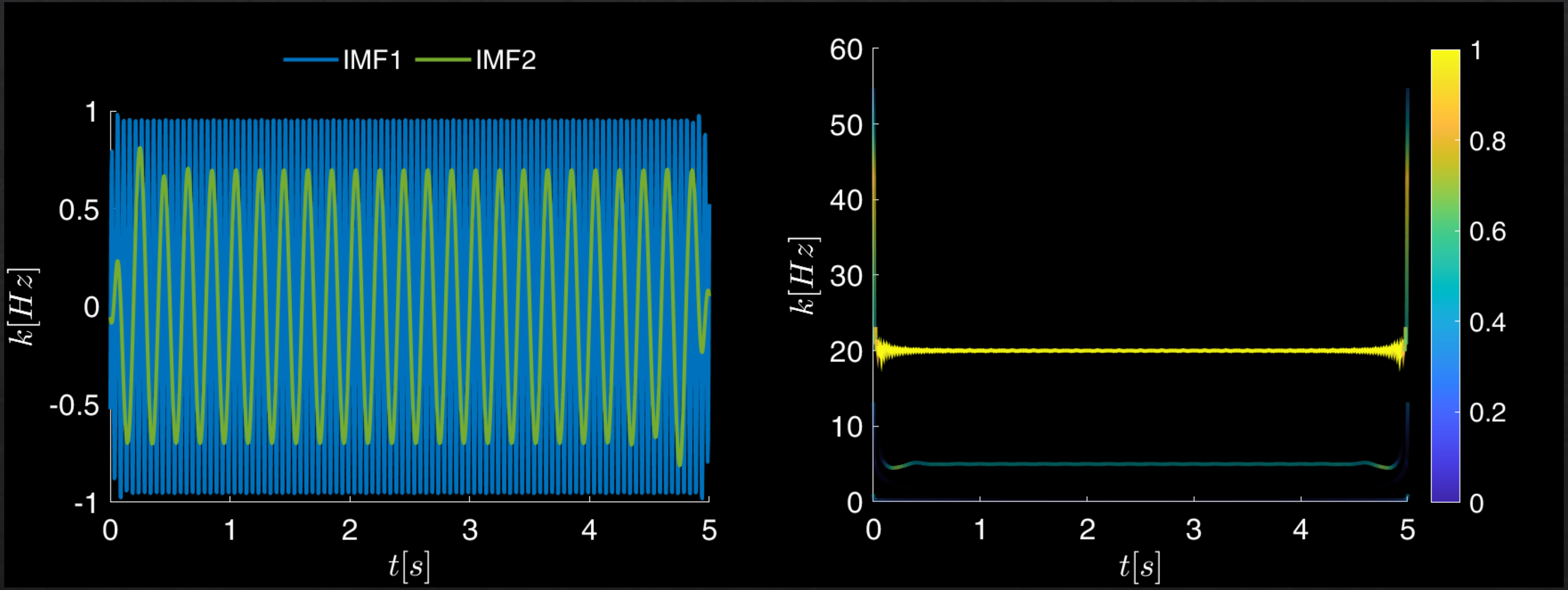

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Example: whale song

Audio data from a Pacific blue whale, sampled at 4 kHz
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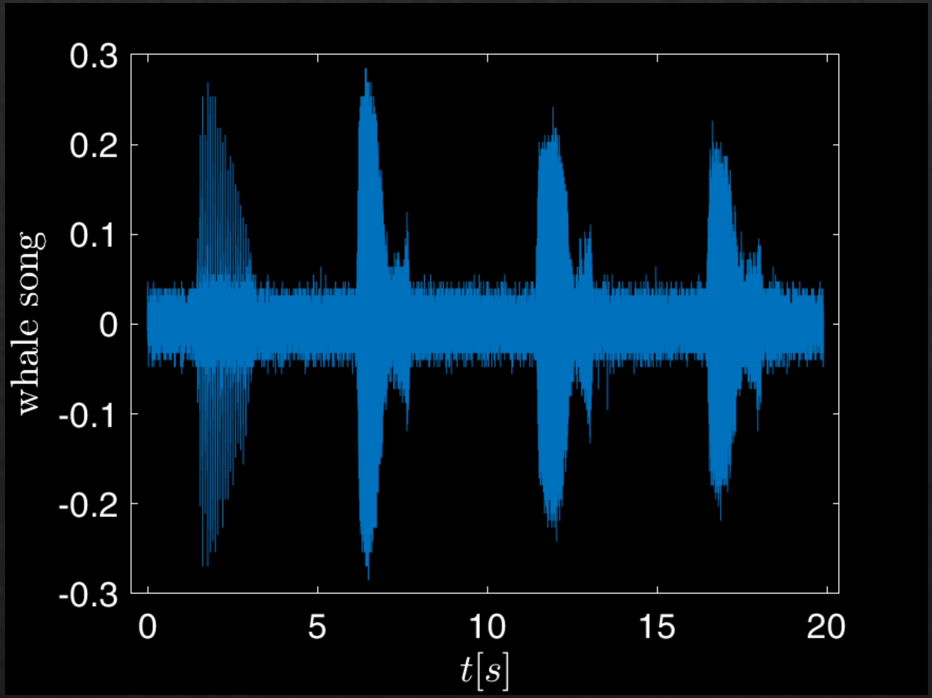

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