HILBERT TRANSFORM
&
HILBERT SPECTRUM

Hilbert Transform

Objective: determine a unique, analytic signal from a real signal to calculate instantaneous properties

Analytic signal: $z(t) = f(t) + i\,H\{f(t)\}$

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with amplitude $A(t) = \sqrt{f^2(t) + H\{f(t)\}^2}$

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frequency $\omega(t) = \frac{d\varphi(t)}{dt}$

→ Hilbert Transform provides a unique imaginary component $H\{f(t)\}$

Hilbert Spectrum

Energy-frequency-time distribution

with instantaneous energy: $e = |A(t)|^2$

instantaneous frequency: $\omega(t) = \frac{d\varphi(t)}{dt}$

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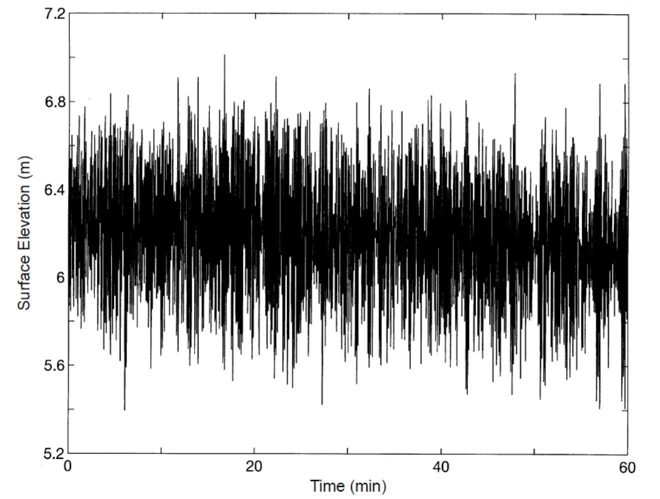

Figure 58. Field wave data measured at a tidal station covering a time span of 1 h. Because of the tides, the mean is not zero.

Huang et al. (1998). *The empirical mode decomposition and the Hilbert spectrum for nonlinear and non-stationary time series analysis*. Proceedings of the Royal Society of London. Series A: mathematical, physical and engineering sciences, 454(1971), 903-995.

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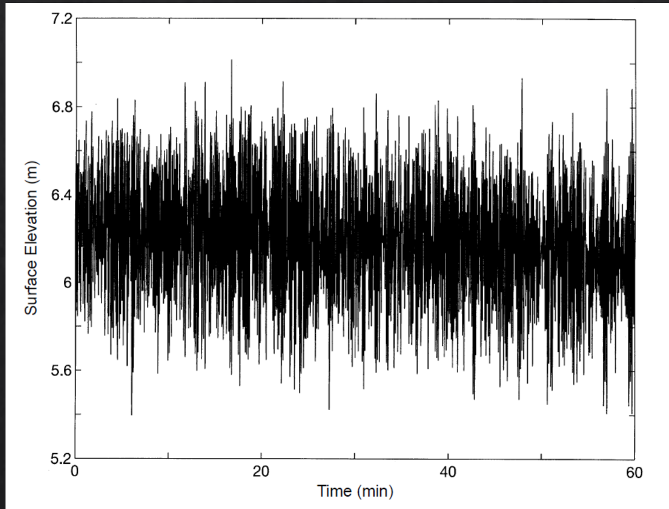

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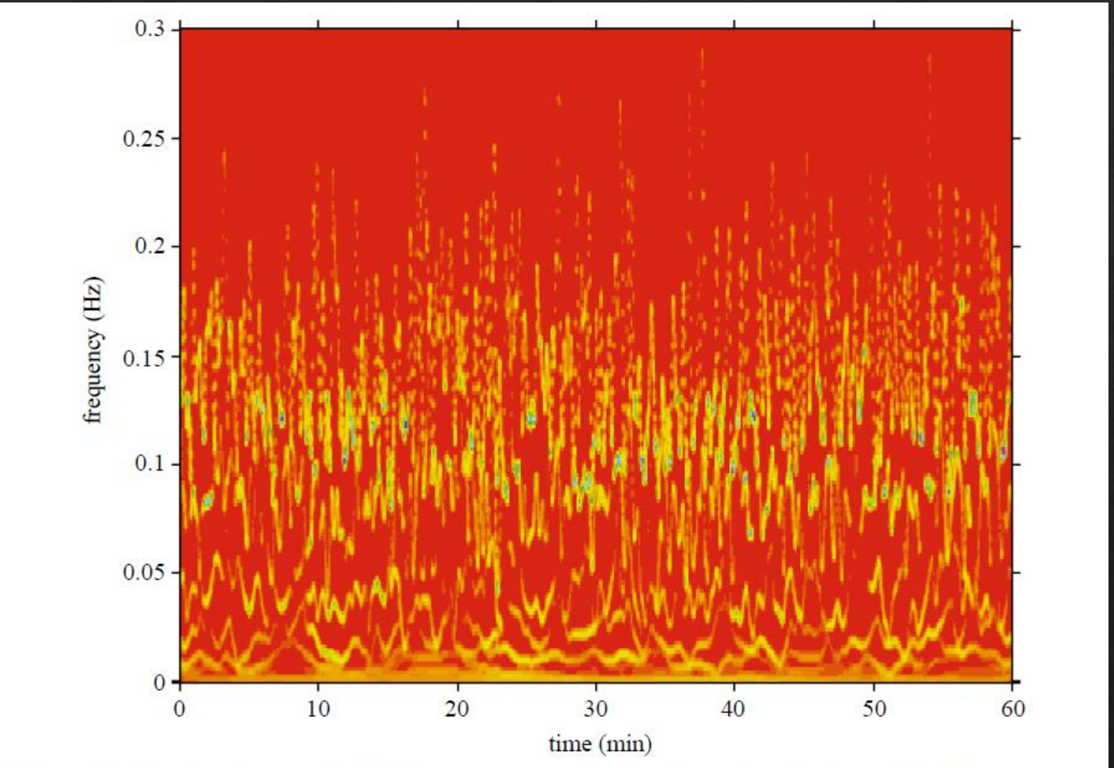

Figure 60. The 9 × 9 smoothed Hilbert spectrum for the data given in figure 58. The spectrum is extremely nodular, an indication that the wave is not stationary.

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$$H\{f(t)\} = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(\tau)}{t-\tau} d\tau$$

with

f: continuous signal

P: Cauchy principal value

t, τ: time variables

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→ In data analysis, we usually only have discrete data points!

Discrete Fourier Transform

Hilbert Transform is based on the Fourier Transform

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$$\mathscr{F}\{f_{k+1}\} = \sum_{t=0}^{n-1} f_{t+1} \cdot e^{-i2\pi tk/n}$$

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$$\left.\begin{array}{rl} f: & \text{measurements} \\ t: & \text{discrete data points} \\ n: & \text{total number of data points} \end{array}\right\} \vec{f} = \begin{pmatrix} f_1 \\ f_2 \\ \cdots \\ f_n \end{pmatrix}, \vec{t} = \begin{pmatrix} 0 \\ 1 \\ \cdots \\ n-1 \end{pmatrix}$$

k: discrete frequencies

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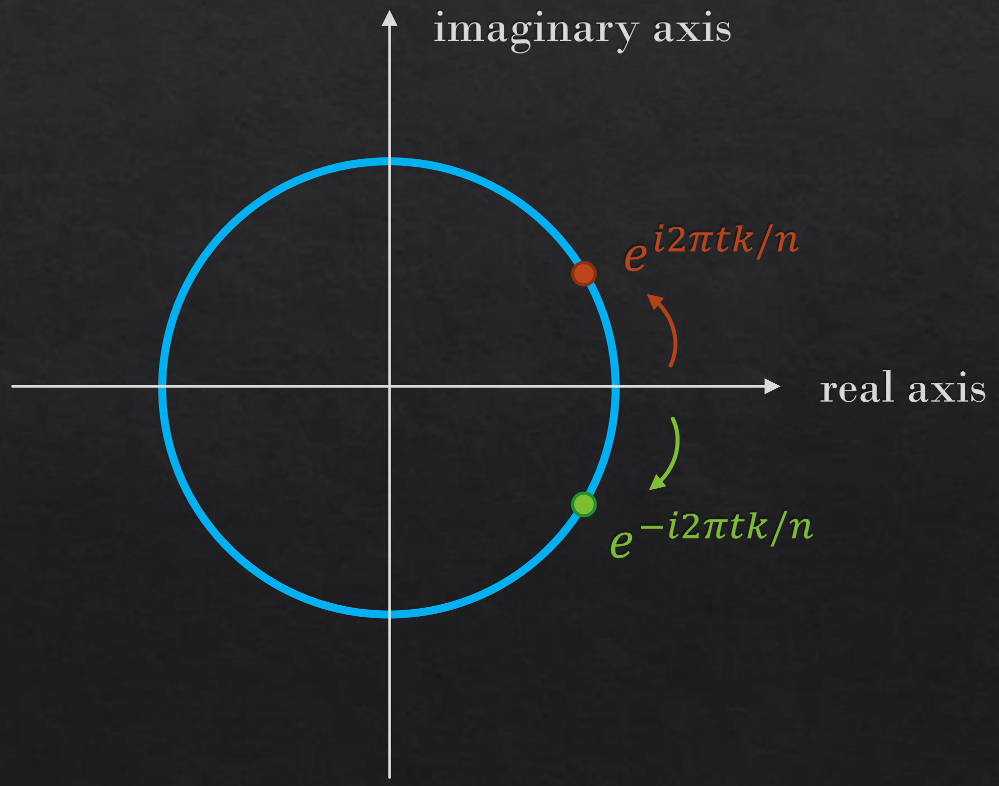

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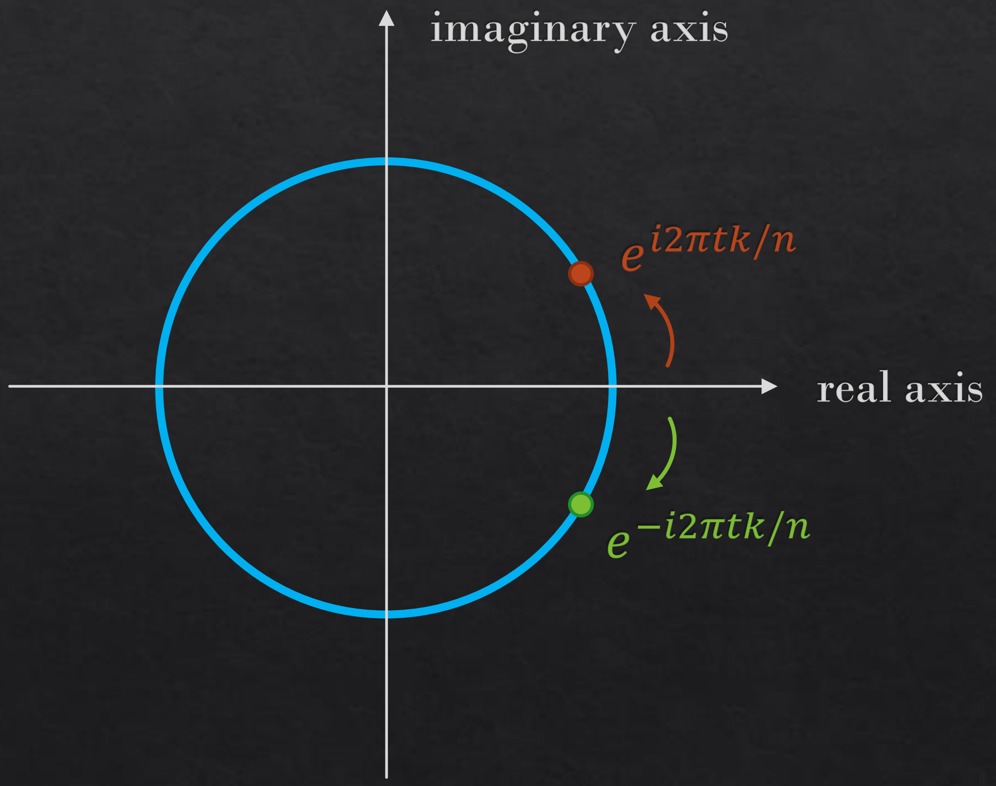

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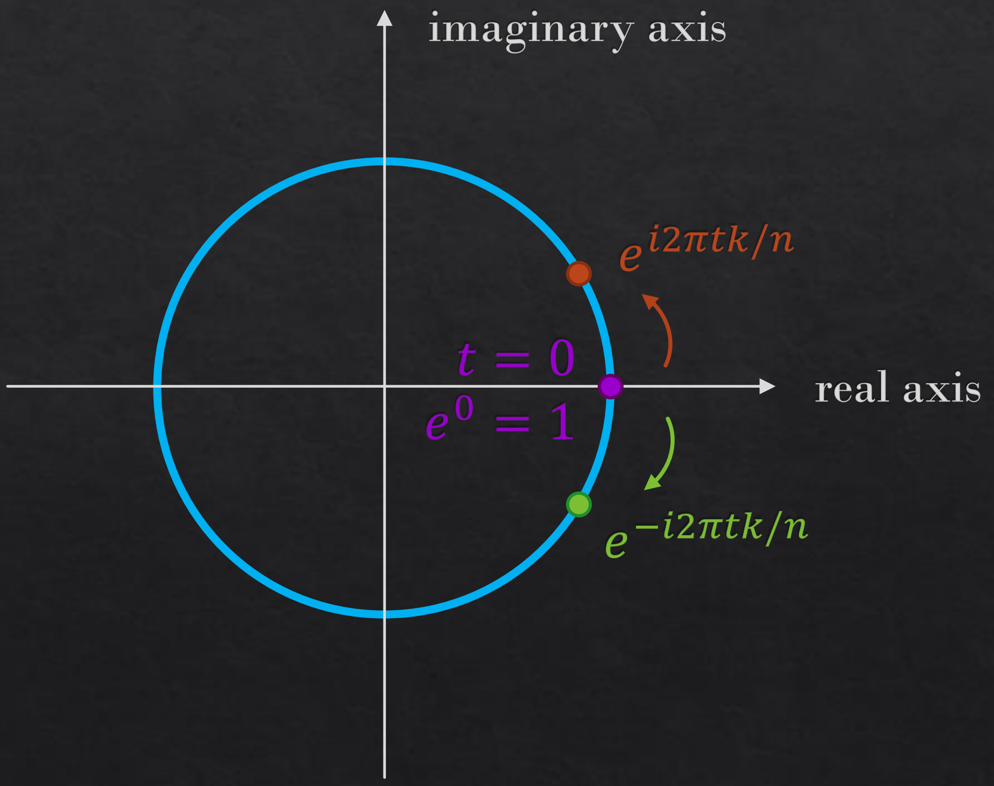

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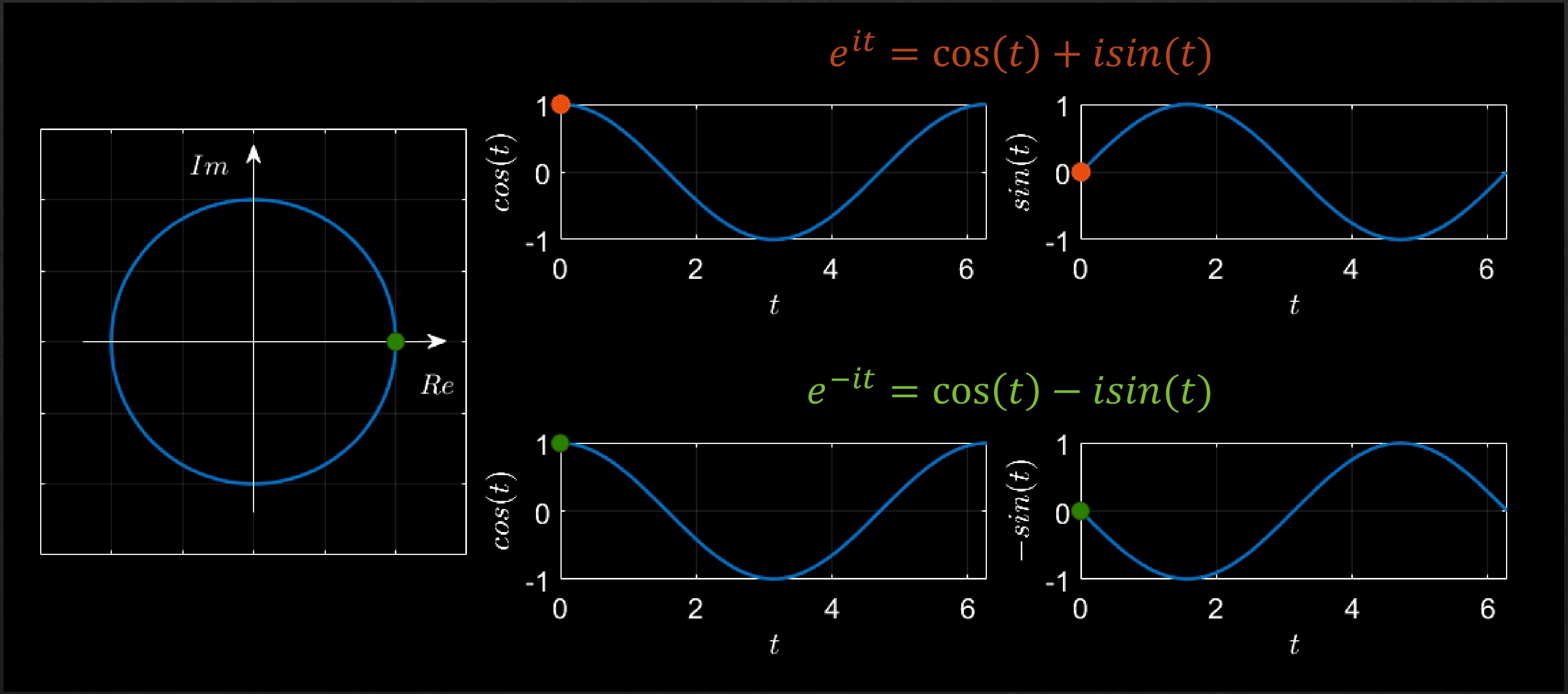

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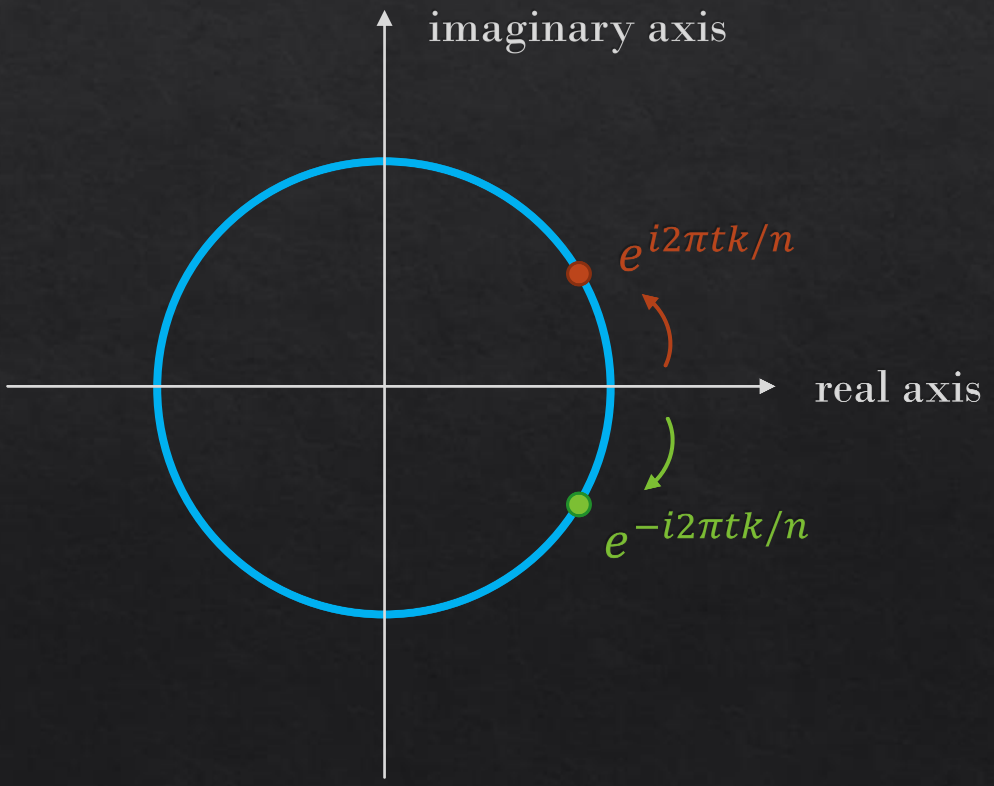

Projection onto complex exponential function → new basis system spanned by sines and cosines ("frequency space")

Euler's formula: $e^{ia} = \cos(a) + i\sin(a)$

Discrete Fourier Transform & its inverse

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Example of DFT

$f(t) = A_1\, sin(2\pi\omega_1 t) + A_2\, sin(2\pi\omega_2 t)$ with $\omega_1 = 5\ Hz, \omega_2 = 20\ Hz, A_1 = 0.7, A_2 = 1$
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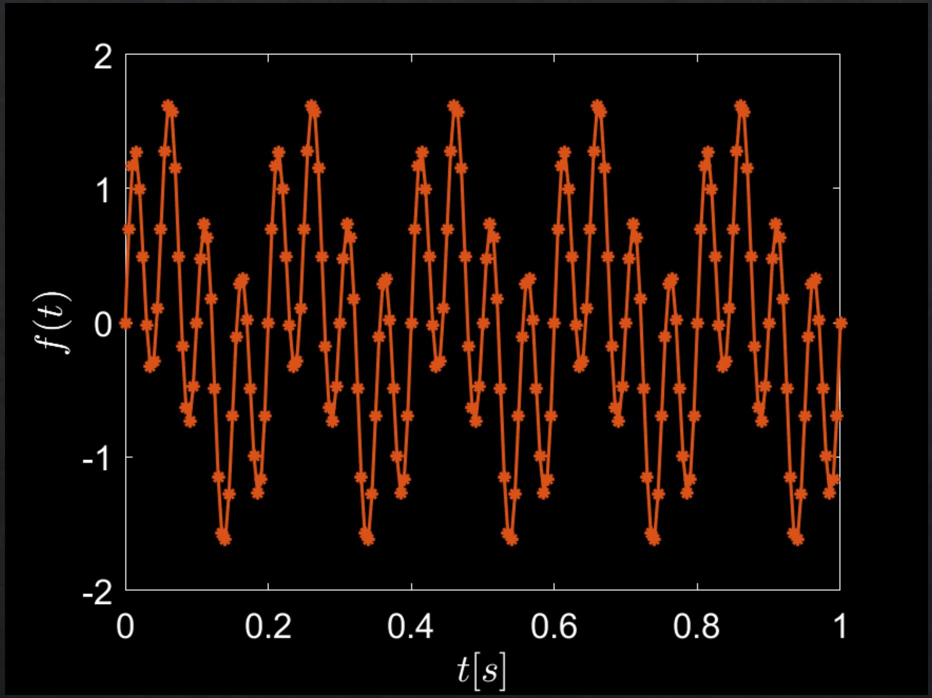

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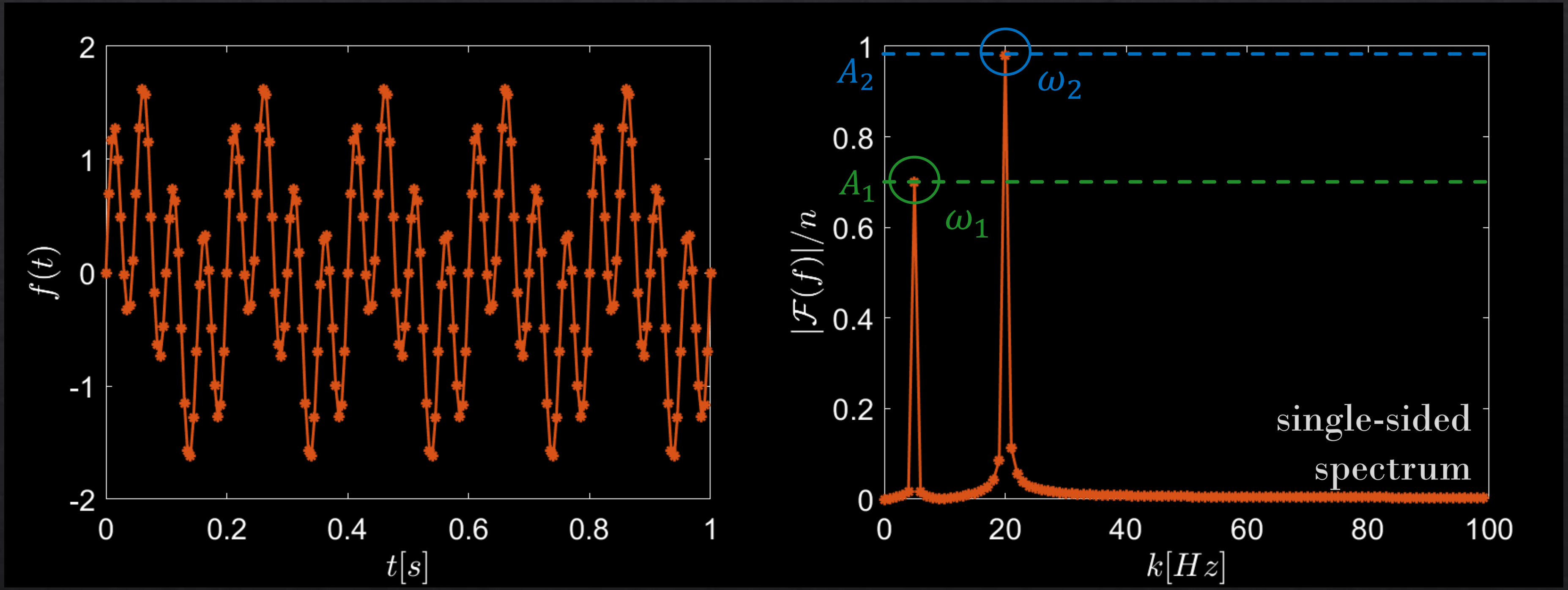

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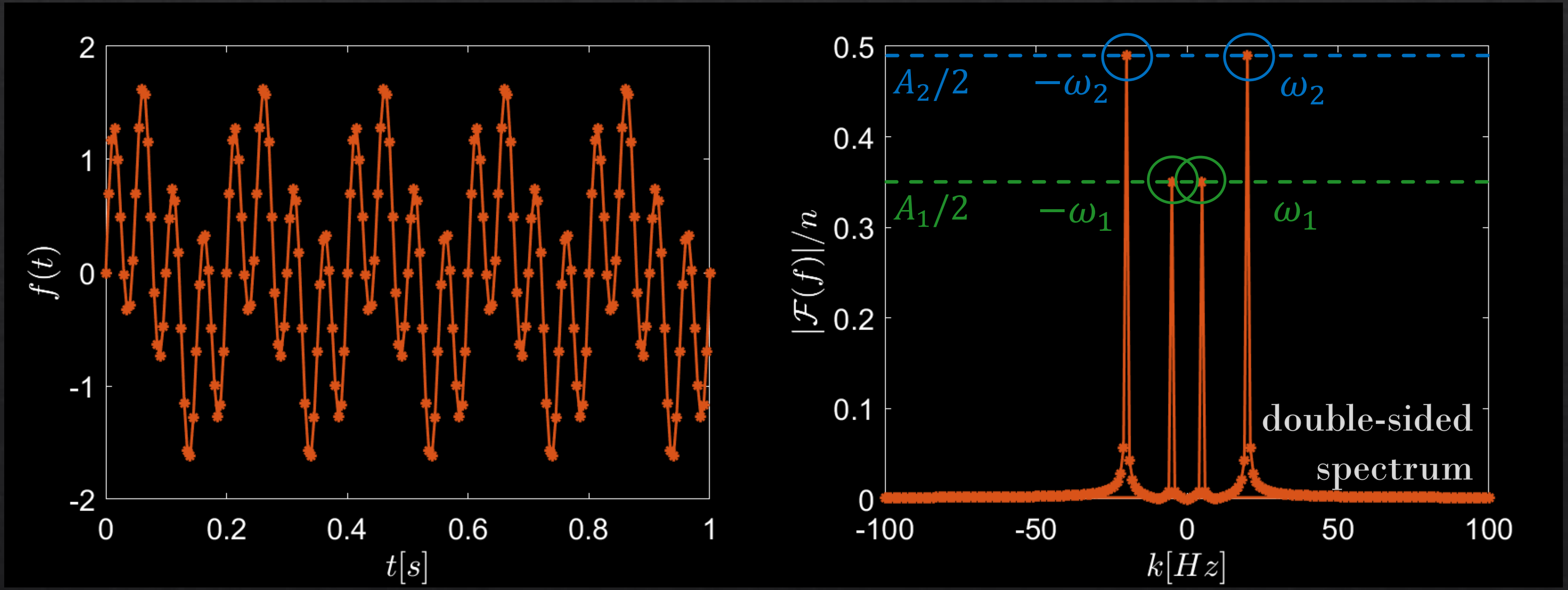

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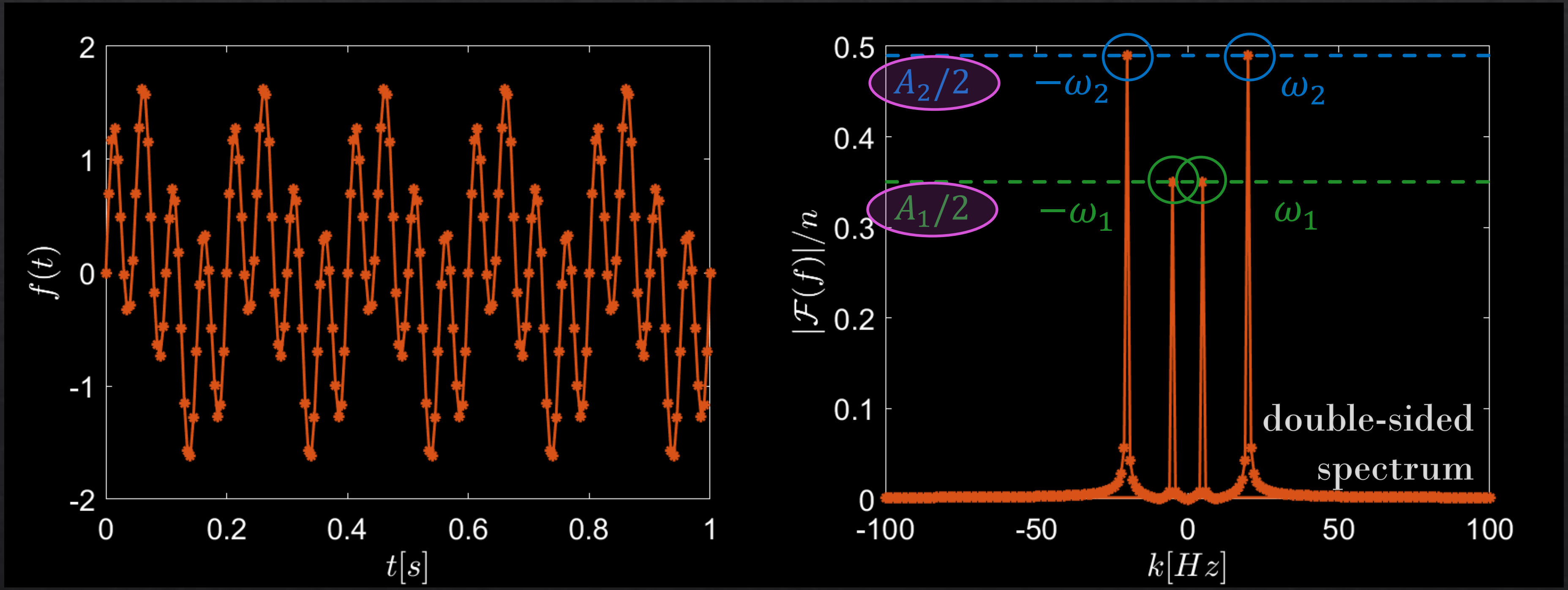

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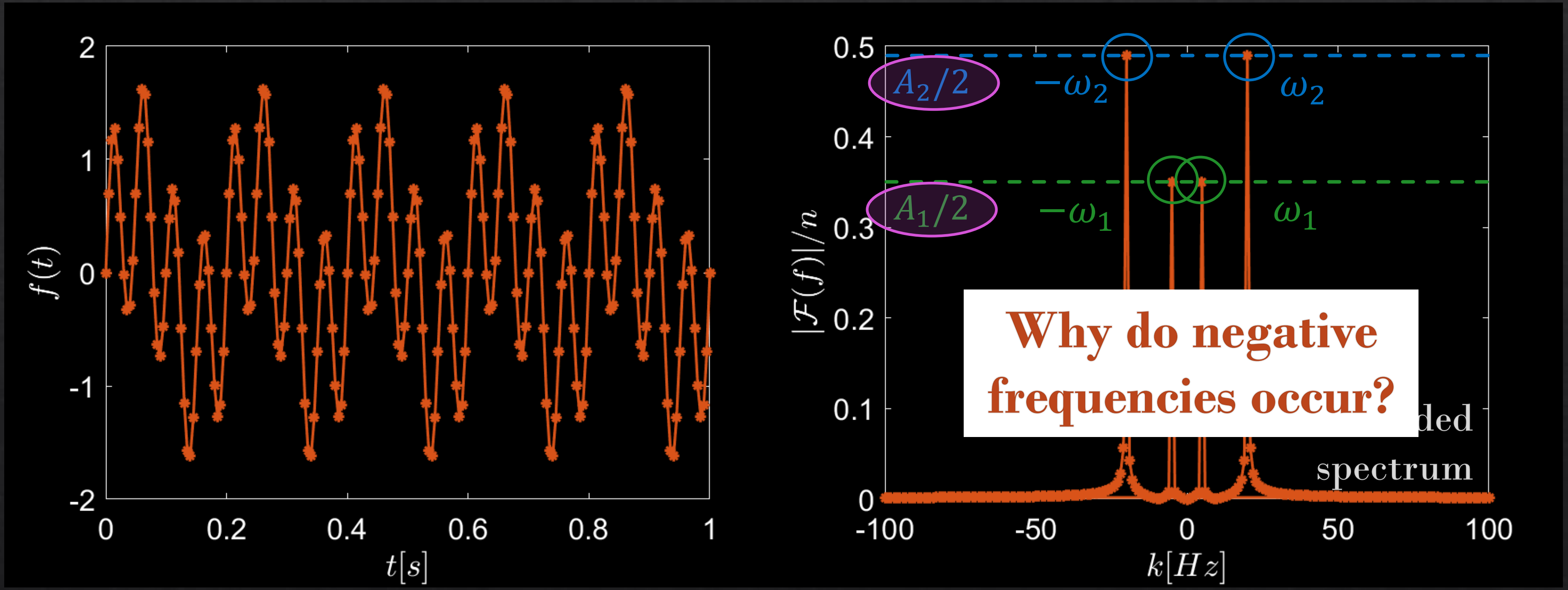

Inverse Discrete Fourier Transform

Why does the Fourier Transform uses negative frequencies at all?

Inverse discrete Fourier Transform to go back into the temporal domain:

only for real-valued data in the temporal domain

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→ Identical amplitude for real and imaginary part

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→ Identical amplitude for real and imaginary part

BUT original data does not contain any imaginary part

→ eliminate sine when transforming back into the original domain

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Euler‘s formula: $e^{ia} = \cos(a) + isin(a)$

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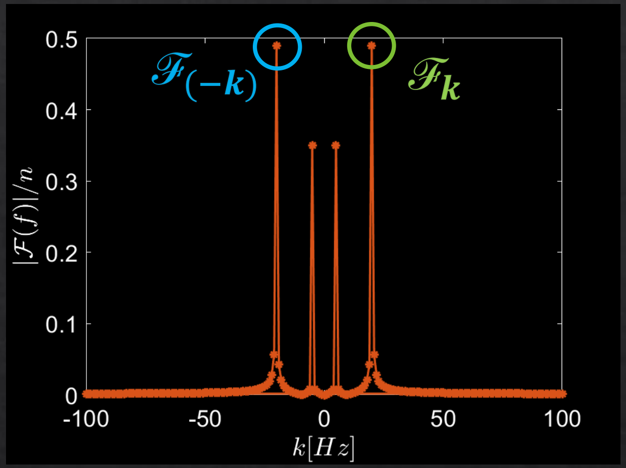

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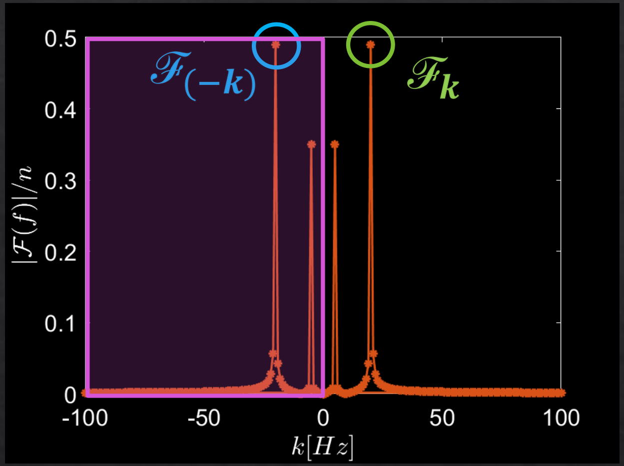

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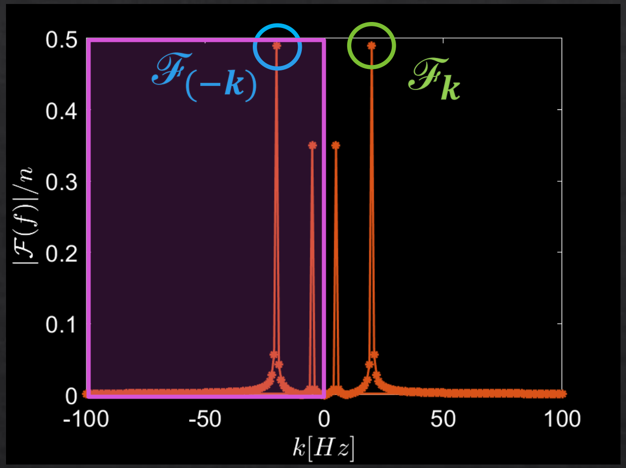

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This only holds for real signals! Otherwise, the information contained in the negative frequencies is not redundant.

Where is the connection to the Hilbert Transform?

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2. Set Fourier coefficients of negative frequencies to zero such that they cannot cancel out the imaginary part related to the positive frequencies during the inverse Fourier Transform

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4. Inverse Fourier Transform to obtain the analytic (complex) signal

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Side note: Steps 2 and 3 are not exactly the same as simply taking the single-sided spectrum; here, you set the coefficients to zero instead of just ignoring them, i.e., the matrix dimensions do not change, which is important for the inverse Fourier Transform

Next time:

Why is it meaningful to perform an EMPIRICAL MODE DECOMPOSITION prior to the Hilbert Transform?

or: on the Hilbert-Huang Transform