Assignment 3 – Math 205 Linear Algebra

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Solutions

1. Does the following set of polynomials constitute a basis for subspace of polynomial of degree less than equal to 2

$$\left\{x^2 + x + 1, x^2 - x + 1, 2x^2, 1\right\}$$

Solution

Setting each vector as a row in a matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Apply Gaussian elimination.

$R_1 + R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$-2R_4 + R_2$

$R_3 + R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Here we get a zero vector which shows that this set is not linearly independent. Hence, this set does not constitute a basis for subspace of polynomials of degree less than equal to 2.

2. Find a basis for the space of polynomials $p(x)$ of degree ≤ 2? Then, find the basis for the subspace with $p(7) = 0$ and sketch the bases.

Solution

Any polynomial of *degree* ≤ 2 can be represented as

$$p(x) = ax^2 + bx + c \tag{1}$$

which can be represented as a linear combination of vectors x^2, x, and 1.

The vectors $\{x^2, x, 1\}$ is a basis for all polynomials with degree ≤ 2.

A linear combination of these bases will form a polynomial of degree 2 (Quadratic) if a_1 is non-zero.

$$p(x) = a_1x^2 + a_2x + a_31$$

If only a_1 is zero, the polynomial is of degree 1 (Linear).

$$p(x) = 0.x^2 + a_2x + a_31$$

$$p(x) = a_2x + a_31$$

If both, a_1 and a_2 are zero, then the polynomial is degree 0(constant)

$$p(x) = 0x^2 + 0x + a_31$$

$$p(x) = a_3$$

For a subspace with $p(7) = 0$,

$$p(7) = a(7)^2 + b(7) + c = 0 \tag{2}$$

$$49a + 7b + c = 0$$

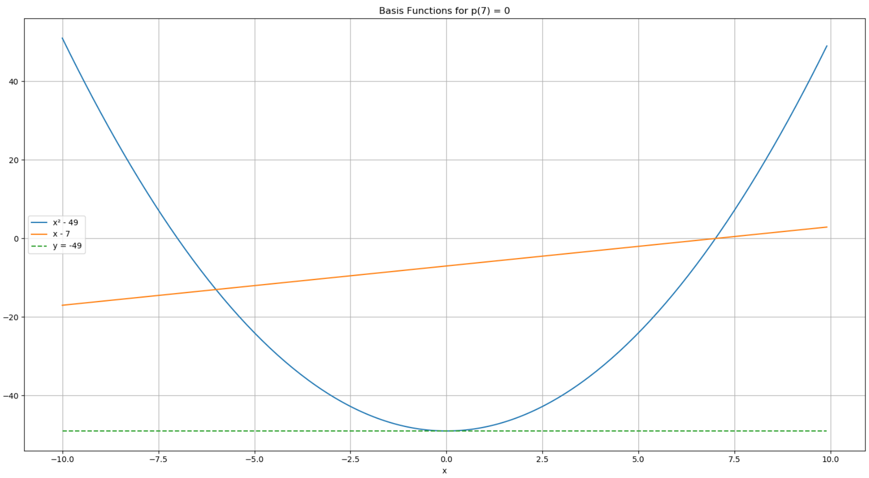
$$c = -49a - 7b$$

$$p(x) = ax^2 + bx - 49a - 7b$$

$$p(x) = ax^2 - 49a + bx - 7b$$

$$p(x) = a(x^2 - 49) + b(x - 7)$$

$$\text{basis} = \{\ (x^2 - 49),\ (x - 7)\ \}$$



Basis Sketch

3. Let P_3 be the real number vector space over $\mathbb{R}$ of cubic polynomials. W is defined as

$$W = \{p(x) \in P_3 \mid p'(-1) = 0 \text{ and } p''(1) = 0\}$$

Determine whether W forms a subspace of V

Solution:

Form of all cubic polynomials:

$$p(x) = ax^3 + bx^2 + cx + d \tag{3}$$

Conditions: $p'(-1) = 0$ and $p''(1) = 0$

$$p'(x) = 3ax^2 + 2bx + c \tag{4}$$

$$p'(-1) = 3a - 2b + c = 0$$

$$c = 2b - 3a$$

$$p''(x) = 6ax + 2b \tag{5}$$

$$p''(1) = 6a + 2b = 0$$

$$b = -3a$$

$$c = 2b - 3a$$

$$c = -6a - 3a$$

$$c = -9a$$

$$a = -\tfrac{1}{9}c$$

$$c = 3b$$

$$b = \tfrac{1}{3}c$$

The form of the cubic polynomials in W then becomes

$$p(x) = -\frac{1}{9}cx^3 + \frac{1}{3}cx^2 + cx + d \tag{6}$$

W contains all polynomials of this form.

Conditions for a valid subspace:

(a) $p(x) + q(x) \in W$

(b) $a.p(x) \in W$ where $a \in \mathbb{R}$

$$p(x) = -\tfrac{1}{9}cx^3 + \tfrac{1}{3}cx^2 + cx + d$$

$$q(x) = -\tfrac{1}{9}ax^3 + \tfrac{1}{3}ax^2 + ax + b$$

$$r(x) = p(x) + q(x)$$

$$\text{Check if } r(x) \in W$$

$$r(x) = -\tfrac{1}{9}cx^3 + \tfrac{1}{3}cx^2 + cx + d + (-\tfrac{1}{9}ax^3 + \tfrac{1}{3}ax^2 + ax + b)$$

$$r(x) = -\tfrac{1}{9}(c+a)x^3 + \tfrac{1}{3}(c+a)x^2 + (c+a)x + b + d$$

$$r'(x) = -\tfrac{1}{3}(c+a)x^2 + \tfrac{2}{3}(c+a)x + c + a$$

$$r'(-1) = -\tfrac{1}{3}(c+a) - \tfrac{2}{3}(c+a) + c + a$$

$$r'(-1) = -\frac{3}{3}(c+a) + c + a$$

$$\mathbf{r'(-1) = 0}$$

$$r''(x) = -\frac{2}{3}(c+a)x + \frac{2}{3}(c+a)$$

$$r''(1) = -\frac{2}{3}(c+a) + \frac{2}{3}(c+a)$$

$$\mathbf{r''(1) = 0}$$

Condition (a) holds True.

$q(x) = a.p(x)$ where $a \in \mathbb{R}$

Check if $q(x) \in W$

$$q(x) = a.p(x) = a(-\frac{1}{9}cx^3 + \frac{1}{3}cx^2 + cx + d)$$

Since a is just a constant, we know

$$q'(x) = a.p'(x)$$

$$q'(-1) = a.p'(-1)$$

$$q'(-1) = a.0 = 0$$

$$q''(x) = a.p''(x)$$

$$q''(1) = a.p''(1)$$

$$q''(1) = a.0 = 0$$

Condition (b) also holds.

W is closed under addition and multiplication, hence W is a subspace of V.

4. Find the bases for the four subspaces associated with **A** and **B**:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix} \quad , B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{bmatrix}$$

Sketch the four subspaces if an arbitrary matrix **C** is

- invertible
- a zero matrix

Solution:

C(A) = span{ $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ }

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-2R_1 + R_2$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 + 2x_2 + 4x_3 = 0$

$x_1 = -2x_2 - 4x_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} -2x_2 - 4x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} -2x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -4x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

N(A) = span{ $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$ }

5. Find the complete solution for the following equations and describe the solution space:

$$\begin{aligned} x + 3y + 3z &= 1 \\ 2x + 6y + 9z &= 5 \\ -x - 3y + 3z &= 5 \end{aligned} \tag{7}$$

Solution

First, we rewrite the above system of equations as an augmented matrix $[\mathbf{A}|\mathbf{b}]$:

$$\left(\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 5 \end{array}\right)$$

Perform the following row operations to obtain the reduced row echelon form $\mathbf{R}$:

$$\begin{gathered} R_3 + R_1 \\ -2R_1 + R_2 \\ -R_2 + R_1 \\ \tfrac{1}{3}R_2 \\ \tfrac{1}{6}R_3 \end{gathered}$$

Which gives us the following augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 3 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

By looking at the left-hand-side of the augmented matrix above, i.e. $\mathbf{R}$, we can deduce that our matrix $\mathbf{A}$ has the rank $r = 2$. $r < m, n$ which means the system is under-determined and hence we will have either ∞ solutions or no solutions.

Our complete solution is found by adding our particular solution and our special solution:

$$\vec{x} = \vec{x}_p + \vec{x}_n \tag{8}$$

First, we find $\vec{x}_n$ by solving $\mathbf{A}\vec{x} = 0$. We only have one free variable, y. Set $y = 1$. This gives us:

$$\begin{gathered} x = -3y \\ y = 1 \\ z = 0 \end{gathered}$$

So, the nullspace of $\mathbf{A}$, $N(\mathbf{A})$, is spanned by the vector $\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$. This means that our special solution is:

$$\vec{x}_n = y \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

Similarly, we find the particular solution by setting $y = 0$:

$$\begin{aligned} x &= -2 \\ y &= 0 \\ z &= 1 \end{aligned}$$

This gives us our particular solution:

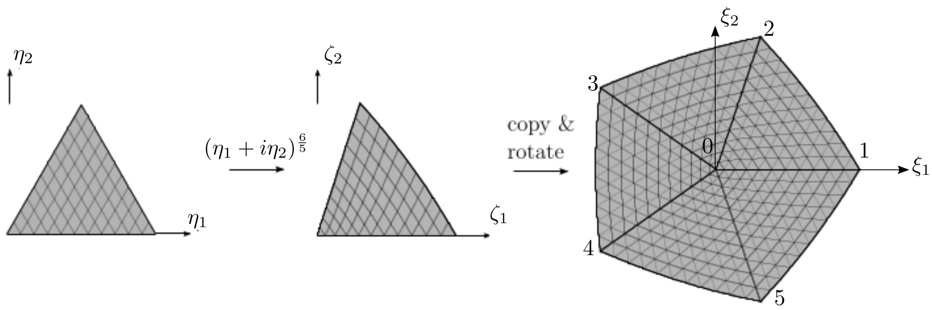
$$\vec{x}_p = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, our complete solution is:

$$\vec{x} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

The solution space is a line in $\mathbb{R}^3$.

6. You are given an equilateral triangle whose sides are of unit length. You are given a complex function $f(\eta_1, \eta_2)$ which transforms the equilateral triangle as shown in the following figure (left and middle). We can then copy and rotate the transformed triangle to form a complete patch as shown in the following figure on the right. Note that we don't have any duplicate points on the final patch



   (a) Calculate the coordinates of the 6 points on the patch.

   Solution

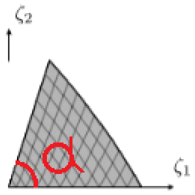
   The transformation in question is a non-linear transformation.

This is evident from that fact that $f(u+v) \neq f(u)+f(v)$. Graphically, this is also evident as the grid-lines of the coordinates are not parallel and evenly spaced after our transformation.

The transformation takes points from the unit square on the real plane to the imaginary plane as given by

$$\vec{\zeta} = f(\eta_1, \eta_2) = (\eta_1 + i\eta_2)^{\frac{6}{5}}$$

To find the coordinates of the patch, we first find the angle α as indicated in the following diagram:



After calculating α, we apply the transformation matrix, given by:

$$\mathbf{R} = \begin{bmatrix} cos(\alpha) & -sin(\alpha) \\ sin(\alpha) & cos(\alpha) \end{bmatrix}$$

This matrix is applied to transformed points on the unit triangle. The transformed points are the original points:[1]

$$(0,0), (0,1), (0.5, \sqrt{0.75})$$

on the unit triangle as given by the transformation $\vec{\zeta}$.

The following code was implemented in **MATLAB**, which explains the procedure for calculating the 6 points:

```
n0 = [0; 0]; %Point 0 of unit triangle
n1 = [1; 0]; %Point 1 of unit triangle
n2 = [0.5; sqrt(0.75)]; %Point 2 of unit triangle
alpha = angle((n2(1,1)+1i*n2(2,1))^(6/5))
%Angle between point 1 and point 2 in radians
P1 = complex_power(n0) %Point 0
```

[1]The point $(\eta_1, \eta_2) = (0.5, \sqrt{0.75})$ is calculated by using the Pythagorean theorem on the unit triangle.

```
P1 = complex_power(n1) %Point 1
P2 = complex_power(n2) %Point 2
P3 = rotation(P2, alpha) %Point 3
P4 = rotation(P3, alpha) %Point 4
P5 = rotation(P4, alpha) %Point 5
function y = complex_power(x) %complex power function
y = (x(1,1)+1i*x(2,1))^(6/5);
y = [real(y); (imag(y))];
end
function y = rotation(x, alpha) %rotation matrix function
R = [cos(alpha), -sin(alpha); sin(alpha), cos(alpha)];
y = R*x;
end
```

Thus, after implementing the code as indicated above, we obtain the following coordinates (ξ_1, ξ_2) for the transformed points:

$$\begin{gathered}\text{P0} = (0,\ 0i)\\ \text{P1} = (1,\ 0i)\\ \text{P2} = (0.3090,\ 0.9511i)\\ \text{P3} = (\text{-}0.8090,\ 0.5878i)\\ \text{P4} = (\text{-}0.8090,\ \text{-}0.5878i)\\ \text{P5} = (0.3090,\ \text{-}0.9511i)\end{gathered}$$

(b) An engineer measured the charge distribution q on the 6 points of the patch as follows:

Point #	0	1	2	3	4	5
q	1	3	1	0	1	2

An electrical engineer is interested in finding out the charge q at $\vec{\xi} = (0.5, 0.5)^T$. He also tells you that the charge inside the patch can be approximated using a bilinear polynomial $(1, \xi_1, \xi_2, \xi_1\xi_2)$. Compute q at $\vec{\xi} = (0.5, 0.5)^T$.

Solution

To obtain the best polynomial approximation for the coordinates provided to us in the table above, we need to calculate the values of the coefficients of the bilinear polynomial described above. The polynomial is:

$$P(\xi_1, \xi_2) = a + b\xi_1 + c\xi_2 + d\xi 1\xi_2$$

The above system is over-determined, as we have only 4 unknowns in the bilinear polynomial but 6 data points to be fitted. To find the solution, we must solve for $\mathbf{A}\vec{x} = \vec{b}$. The matrix $\mathbf{A}$ will be a 6×4 matrix:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0.3090 & 0.9511 & (0.3090 \times 0.9511) \\ 1 & -0.8090 & 0.5878 & (-0.8090 \times 0.5878) \\ 1 & -0.8090 & -0.5878 & (-0.8090 \times -0.5878) \\ 1 & 0.3090 & -0.9511 & (0.3090 \times -0.9511) \end{bmatrix}$$

We also have $\vec{b} = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$.

Solving for $\vec{x}$, we thus obtain the following vector:

$$\vec{x} = \begin{bmatrix} 1.3333 \\ 1.2472 \\ -0.6155 \\ 0.2906 \end{bmatrix}$$

We thus end up with the following bilinear polynomial:

$$P(\xi_1, \xi_2) = 1.3333 + 1.2472\xi_1 - 0.6155\xi_2 + 0.2906\xi 1\xi_2$$

Computing q at $\vec{\xi} = (0.5, 0.5)^T$:

$$P(0.5, 0.5) = 1.3333 + 1.2472(0.5) - 0.6155(0.5) + 0.2906(0.5)^2$$

$$P(0.5, 0.5) = 1.7218$$

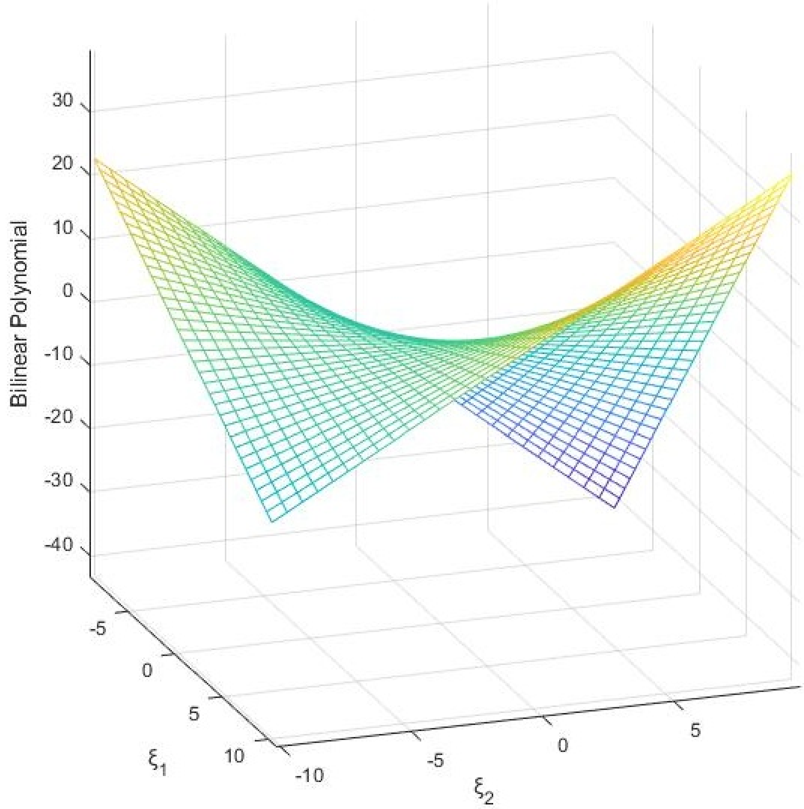
Therefore, the value of q at the point $(0.5, 0.5)$ is 1.7218.

(c) Sketch the charge distribution surface over the $\xi_1 - \xi_2$ patch.

Solution

The figure was plotted in **MATLAB** using the following code:

```
        figure
fmesh(@(x,y) 1.3333 + 1.2472.*x -0.6155.*y + 0.2906.*x.*y,
[-10 10 -10 10])
```



7. We have learned that for underdetermined systems, it is impossible to find unique solutions in the absence of some extra constraints. One way of obtaining the unique solution is to impose a minimum L_2 norm constraint i.e.

$$\text{Solve:} \quad \mathbf{A}\vec{x} = \vec{b} \tag{9}$$

$$\text{such that:} \quad \|\vec{x}\|_2 \ \text{ is minimum} \tag{10}$$

In this case, $\vec{x}_r = \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}\vec{b}$. By way of example explain the relationship between $\vec{x}_p$ and $\vec{x}_n$, and when $\vec{x}_p = \vec{x}_r$. Sketch the four fundamental subspaces

8. Explain, with reason, whether the following statements are true or false

   (a) The complete solution is any linear combination of x_p and x_n

   - This statement is false because x_p can not be any vector whereas x_n can be any constant multiplied by the vector

(b) A system $\mathbf{A}\vec{x} = \vec{b}$ has at most one particular solution

- This statement is false as $\mathbf{A}\vec{x} = \vec{b}$ can have more than one particular solutions. One counterexample: If x_p is one particular solution, and $x_n \in N(\mathbf{A})$, then x_p+ x_n is also a particular solution.

(c) The solution x_p with all free variables set to zero can be the shortest solution (minimum length $\|x\|$)

- Setting free variables to zero, doesn't mean getting the shortest solution Take $A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ The free variable is x_2

  Setting $x_2 = 0$ and $x_1 = 2$
  So, $x_p = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ while length of $x_p = 4$
  Setting $x_2 = 1$ and $x_1 = 1$
  So, $x_p = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ while length of $x_p = \sqrt{2}$

  which shows that setting free variables to zero does not guarantee shortest solution.

(d) If $\mathbf{A}$ is invertible there is no solution x_n in the nullspace

- There is always the zero vector in the nullspace. So, $x_n = 0$ always exists. Thus false.

9. True or false (with a reason or a counterexample)

(a) $\mathbf{A}$ and $\mathbf{A}^T$ have the same number of pivots

- **TRUE** A matrix and its transpose have the same rank as since transposition interchanges rows and columns, and a matrix always has same column and row rank which most essentially means that number of pivot variables which is also seen as the rank of the matrix, must stay the same.

(b) $\mathbf{A}$ and $\mathbf{A}^T$ have the same left nullspace.

- **FALSE** If we take a 1x2 matrix, such as A $= \begin{bmatrix} 1 & 1 \end{bmatrix}$ The left nullspace of A contains vectors in R while the left nullspace of A^T, which is the right nullspace of A, contains vectors in R^2 , so they cannot be the same.

(c) If the row space equals the column space then $\mathbf{A} = \mathbf{A}^T$

- **FALSE**. Counterexample: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ Here, row space and column space equals R^2 as A is invertible, but $A \neq A^T$

(d) If $-\mathbf{A} = \mathbf{A}^T$, then the row space of $\mathbf{A}$ equals the column space

- **TRUE** The null spaces are identical because $Ax = 0 = (-A)x = 0$. The column spaces are identical because any vector v that can be expressed as $v = Ax$ for some x can also be expressed as $v = (A)(x)$. A similar reasoning holds for the two remaining subspaces. The row space of A is equal to the column space of A^T which for this A equals the column space of -A. Since, A and -A have same fundamental subspaces, we can conclude that row space of A equals the column space of A.

10. For the space $\mathbb{R}^4$, let $w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} 3 \\ 3 \\ -1 \\ -1 \end{bmatrix}, y = \begin{bmatrix} 6 \\ 0 \\ 2 \\ 0 \end{bmatrix}$, and let $W = \text{sp}\{w_1, w_2\}$.

(a) Find a basis for W consisting of two orthogonal vectors using Gram-Schmidt process.

(b) Explain the Gram-Schmidt process intuitively.

(c) Express y as the sum of a vector in W and a vector in $W^{\perp}$

Please refer to this link for this question

Appendix A: Code for Q2

```
import matplotlib.pyplot as plt
import numpy as np

x = [i for i in np.arange(-10, 10, 0.1)]

def basis_1(x):
return x**2 - 49
def basis_2(x):
return x - 7

b1 = [basis_1(i) for i in x]
b2 = [basis_2(i) for i in x]

fig = plt.figure()
basis = fig.add_subplot(111)
basis.set_xlabel('x')
basis.set_title("basis for p(7) = 0")

basis.plot(x, b1, label=f'x\N{SUPERSCRIPT TWO} - 49')
basis.plot(x, b2, label='x - 7')
basis.plot(x, [-49 for i in x], linestyle='--', label='y = -49')
basis.set_title('Basis Functions for p(7) = 0')

basis.grid(linestyle='-')
legend = basis.legend()
plt.show()
```

References

[1] Formatting Python strings with super script for Q2.

`https://stackoverflow.com/questions/8651361/how-do-you-print-superscript-in-python/49442772`

[2] Plotting Basis with matplotlib

`https://matplotlib.org/3.1.0/gallery/pyplots/fig_axes_labels_simple.html`