Assignment 2 – Math 205 Linear Algebra

October 3, 2019

1 Instructions

This assignment will contribute 6.67% towards the final marks. The deadline to submit the assignment is 18:30 hours October 11, 2019. The assignment must be submitted online via LMS. You should include both the tex and pdf files, and any code files you may have.

1.1 Marking scheme

Q1 is worth 17 points and the remaining questions are worth 7 points each. Assignment will be marked out of 100.

- 20 marks will be for the presentation.
- 80 for the assignment

A good presentation would contain, but not limited to the following:

- Document typed in LaTeX. The basic template for LaTeXand the associated makefile is available to download from the course website on LMS. You are welcome to use your own LaTeXtemplate. Note: **You are required to type five questions in LaTeX. Question 1 must be typed in LaTeX. You can decide which four of the remaining nine questions you would like to type in LaTeX. The rest of the questions can be submitted by scanning your handwritten solutions**. *Please make sure that you submit your pdf and tex files (and any other related file), and submit a single zip file on LMS*
- A document free of typing errors
- Using figures/diagrams/set diagam, where possible, to explain your answers. As the cliché goes – a picture is worth a thousand words

- Concise and thorough answers. A long report doesn't necessarily mean a good report
- Comments in the code
- List of references

1.2 Late submission policy

Late submissions, unless approved beforehand, will be penalised according to the following.

# hours past the deadline	Percentage penalty on assignment
< 1 hour	5%
1-2 hours	15%
2-3 hours	30%
3-4 hours	45%
>4 hours	0% (not accepted)

1.3 Plagiarism and collusion

There is a zero tolerance policy towards plagiarism and/or collusion. If a student(s) is found to have plagiarised and/or colluded, or the work submitted is not their own, (s)he will be given a **zero** in this assignment. Furthermore, they will most likely be reported to the academic code of conduct committee which would affect your academic standing in the university. If you are unsure whether you are plagiarising, please ask.

Please note that even if you understand everything, copying someone else's work is still plagiarism.

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In the event that something is not clear from the question, you are strongly encouraged to use the discussion forum on workplace. **Individual enquiries to instructors and TA will not be entertained**

2 Questions

1. We have already indirectly talked about the notions of **span, independent vectors, basis** and **dimension**, but now you have to put the intuition we built in proper mathematical terms. The **span** has already gotten a crucial amount of attention, so we just add the formal definition here:

For $\vec{v}_1, ..., \vec{v}_k$ vectors in $\mathbb{R}^n$, the *span* of $\vec{v}_1, ..., \vec{v}_k$ is the set of all linear combinations of those vectors, i.e.

$$\text{Span}\{\vec{v}_1, ..., \vec{v}_k\} = \{x_1\vec{v}_1 + ... + x_k\vec{v}_k | x_1, ...x_k \in \mathbb{R}\}$$

We also say that $\text{Span}\{\vec{v}_1, ..., \vec{v}_k\}$ is the subset *spanned by* or *generated by* the vectors $\vec{v}_1, ..., \vec{v}_k$.
Conversely we say that a set of vectors *spans* a space if their linear combination fills the space, i.e. each vector in the space can be represented by a linear combination of those vectors.

In order to understand the other notions read the following chapters `https://textbooks.math.gatech.edu/ila/linear-independence.html` and `https://textbooks.math.gatech.edu/ila/dimension.html` and summarize those three other concepts before answering the following questions (including explanation):

a) Are the vectors $\left\{ \begin{bmatrix} 1 \\ 4 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -14 \\ 13 \\ 7 \\ -19 \end{bmatrix} \right\}$ linearly dependent?

b) Can the columns of a wide matrix, $n > m$ be linearly independent?

c) Can the columns of a tall matrix, $n < m$ be linearly independent?

d) If the columns of $\mathbf{A}$ are linearly independent, how many solutions are there to the system $\mathbf{A}\vec{x} = 0$?

e) To determine whether a set of n vectors from $\mathbb{R}^n$ is independent, we can form a matrix $\mathbf{A}$ whose columns are the vectors in the set and then put that matrix in reduced row echelon form. If the vectors are linearly independent, what will we see in the reduced row echelon form?

f) If your vector space V has dimension m, do any m vectors in V form a basis of V?

2. Suppose $\mathbf{A} = \begin{bmatrix} 3 & 6 & 9 \\ 6 & 9 & 12 \\ 9 & 12 & 15 \end{bmatrix}$

- Find $N(\mathbf{A})$

- $\mathbf{B} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. Find $N(\mathbf{B})$ and $N(\mathbf{B}^2)$
- What can you say about the relationship between $N(\mathbf{C})$ and $N(\mathbf{C}^2)$. Assume for this part that $\mathbf{C}$ can be any arbitrary square matrix

3. A linear system $\mathbf{A}\vec{x} = \vec{b}$ has special solution of the form

$$\vec{x}_n = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

   (a) What are the basis for $N(\mathbf{A})$

   (b) What is the dimension of $N(\mathbf{A})$

   (c) Write the reduced row echelon form for $\mathbf{A}$

   (d) Describe intuitively the geometry of the solution space

4. Determine with reason which of the following are subspaces of 3×3 matrix $\mathbf{M}$

   (a) all 3×3 matrices $\mathbf{A}$ such that $\mathbf{A}^T = -\mathbf{A}$

   (b) all 3×3 matrices such that the linear system $\mathbf{A}\vec{x} = \vec{0}$ has only trivial solution

5. For which right sides are these systems solvable? Give your reasons

   (a)

$$\begin{bmatrix} 5 & 7 & 5 \\ 10 & 14 & 15 \\ 20 & 28 & 25 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

   (b)

$$\begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

   (c)

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

(d)

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

6. Given that $\mathbf{A}$ is an arbitrary 4×3 matrix, if we add an extra column $\vec{a}_4$ to a matrix $\mathbf{A}$. then the column space gets larger unless ____________________. Give an example where the column space gets larger and an example where it doesn't.

   What should be the condition on $\vec{b}$ for $\mathbf{A}\vec{x} = \vec{b}$ to have a solution if the $C(\mathbf{A})$ doesn't get larger?

7. Find the largest possible number of independent vectors among

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} v_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} v_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} v_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

8. Find the bases for the $C(\cdot)$ and $N(\cdot)$ associated with $\mathbf{A}$ and $\mathbf{B}$:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix} \qquad , B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{bmatrix}$$

9. If $\mathbf{V}$ is the subspace spanned by (1, 1, 1) and (2, 1, 0), find a matrix $\mathbf{A}$ that has $\mathbf{V}$ as its column space and a matrix $\mathbf{B}$ that has $\mathbf{V}$ as its nullspace.

10. Find the complete solution for the following equations and describe the solution space:

$$\begin{aligned} x + 3y + 3z &= 0 \\ 2x + 6y + 9z &= 0 \\ -x - 3y + 3z &= 0. \end{aligned} \tag{2}$$