Assignment 2
Math 205 Linear Algebra

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1 Answers

1. We have already indirectly talked about the notions of **span, independent vectors, basis** and **dimension**, but now you have to put the intuition we built in proper mathematical terms. The **span** has already gotten a crucial amount of attention, so we just add the formal definition here:

   For $\vec{v}_1, ..., \vec{v}_k$ vectors in $\mathbb{R}^n$, the *span* of $\vec{v}_1, ..., \vec{v}_k$ is the set of all linear combinations of those vectors, i.e.

   $$\text{Span}\{\vec{v}_1, ..., \vec{v}_k\} = \{x_1\vec{v}_1 + ... + x_k\vec{v}_k | x_1, ...x_k \in \mathbb{R}\}$$

   We also say that $\text{Span}\{\vec{v}_1, ..., \vec{v}_k\}$ is the subset *spanned by* or *generated by* the vectors $\vec{v}_1, ..., \vec{v}_k$.
   Conversely we say that a set of vectors *spans* a space if their linear combination fills the space, i.e. each vector in the space can be represented by a linear combination of those vectors.

   In order to understand the other notions read the following chapters `https://textbooks.math.gatech.edu/ila/linear-independence.html` and `https://textbooks.math.gatech.edu/ila/dimension.html` and summarize those three other concepts before answering the following questions (including explanation):

   a) Are the vectors $\left\{ \begin{bmatrix} 1 \\ 4 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -14 \\ 13 \\ 7 \\ -19 \end{bmatrix} \right\}$ linearly dependent?

b) Can the columns of a wide matrix, $n > m$ be linearly independent?

c) Can the columns of a tall matrix, $n < m$ be linearly independent?

d) If the columns of $\mathbf{A}$ are linearly independent, how many solutions are there to the system $\mathbf{A}\vec{x} = 0$?

e) To determine whether a set of n vectors from $\mathbb{R}^n$ is independent, we can form a matrix $\mathbf{A}$ whose columns are the vectors in the set and then put that matrix in reduced row echelon form. If the vectors are linearly independent, what will we see in the reduced row echelon form?

f) If your vector space V has dimension m, do any m vectors in V form a basis of V?

2. Suppose $\mathbf{A} = \begin{bmatrix} 3 & 6 & 9 \\ 6 & 9 & 12 \\ 9 & 12 & 15 \end{bmatrix}$

- Find $N(\mathbf{A})$
- $\mathbf{B} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. Find $N(\mathbf{B})$ and $N(\mathbf{B}^2)$
- What can you say about the relationship between $N(\mathbf{C})$ and $N(\mathbf{C}^2)$. Assume for this part that $\mathbf{C}$ can be any arbitrary square matrix

3. A linear system $\mathbf{A}\vec{x} = \vec{b}$ has special solution of the form

$$\vec{x}_n = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

(a) What are the basis for $N(\mathbf{A})$

(b) What is the dimension of $N(\mathbf{A})$

(c) Write the reduced row echelon form for $\mathbf{A}$

(d) Describe intuitively the geometry of the solution space

4. Determine with reason which of the following are subspaces of 3×3 matrix $\mathbf{M}$

(a) all 3×3 matrices $\mathbf{A}$ such that $\mathbf{A}^T = -\mathbf{A}$

(b) all 3×3 matrices such that the linear system $\mathbf{A}\vec{x} = \vec{0}$ has only trivial solution

5. For which right sides are these systems solvable? Give your reasons

For all $\vec{b}$ in the $C(\mathbf{A})$, there exists a solution for the system

(a)

$$\begin{bmatrix} 5 & 7 & 5 \\ 10 & 14 & 15 \\ 20 & 28 & 25 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Applying row operations on $\mathbf{A}$:

$$-2R_1 + R_2$$

$$-4R_1 + R_3$$

$$-R_3 + R_1$$

$$-R_3 + R_2$$

$$\tfrac{1}{5}R_1$$

$$\tfrac{1}{5}R_3$$

$$R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & \frac{7}{5} & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

$\vec{a}_1$ and $\vec{a}_3$ are linearly independent and their span is the $C(\mathbf{A})$ hence

$\forall \vec{b} \in C(\mathbf{A})$ the system is solvable.

(b)

$$\begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Applying row operations on $\mathbf{A}$

$$R_1 + R_3$$

$$-2R_1 + R_2$$

$$-4R_2 + R_1$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$\forall \vec{b}$ where $\vec{b}$ is the linear combination of the two columns of $\mathbf{A}$, the system is solvable.

So speaking in terms of the cartesian coordinate system, all vectors lying on the xy plane, with z component zero.

(c)

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Since all three vectors of this 3×3 matrix are independent, the system is solvable for all vectors $\vec{b}$ in $\mathbb{R}^3$

(d)

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

In this part, $\mathbf{A}$ has only 2 independent vectors, hence similar to part b, the system is solvable for all vectors $\vec{b}$ which are the linear combinations of $\vec{a}_1$ and $\vec{a}_2$.

6. Given that $\mathbf{A}$ is an arbitrary 4×3 matrix, if we add an extra column $\vec{a}_4$ to a matrix $\mathbf{A}$. then the column space gets larger unless __________________. Give an example where the column space gets larger and an example where it doesn't.

   What should be the condition on $\vec{b}$ for $\mathbf{A}\vec{x} = \vec{b}$ to have a solution if the $C(\mathbf{A})$ doesn't get larger?

7. We begin by making a matrix of all of these vectors.

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \vec{v}_5 & \vec{v}_6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & -1 \end{bmatrix}$$

We then perform row operations on this matrix:

$$R_1 + R_2$$

$$R_2 + R_3$$

$$R_3 + R_4$$

$$\frac{1}{2}R_4$$

The Matrix then becomes

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Out of these we can see $\vec{v}_1, \vec{v}_4, \vec{v}_6$, and $\vec{v}_5$ to be linearly independent.

So we have atmost 4 linearly independent vectors.

8. Find the bases for the $C(\cdot)$ and $N(\cdot)$ associated with **A** and **B**:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix} \qquad , B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{bmatrix}$$

9. If **V** is the subspace spanned by (1, 1, 1) and (2, 1, 0), find a matrix **A** that has **V** as its column space and a matrix **B** that has **V** as its nullspace.

10. Find the complete solution for the following equations and describe the solution space:

$$\begin{aligned} x + 3y + 3z &= 0 \\ 2x + 6y + 9z &= 0 \\ -x - 3y + 3z &= 0. \end{aligned} \tag{2}$$