Mineria de Datos

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4.8

Using (4.57) and (4.58), derive the result (4.65) for the posterior class probability in the two-class generative model with Gaussian densities, and verify the results (4.66) and (4.67) for the parameters w and w0.

$$p\left(c_1|x\right)=\frac{p\left(x|c_1\right)p\left(c_1\right)}{p\left(x|c_1\right)p\left(c_1\right)+p\left(x|c_2\right)p\left(c_2\right)}=\frac{1}{1+\exp\left(-a\right)}=\sigma\left(a\right)$$
$$a=\ln\frac{p\left(x|c_1\right)p\left(c_1\right)}{p\left(x|c_2\right)p\left(c_2\right)}$$
$$p\left(c_1|x\right)=\sigma\left(w^Tx+w_0\right)$$
$$w=\Sigma^{-1}\left(\mu_1-\mu_2\right)$$
$$w_0=-\frac{1}{2}\mu_1^T\Sigma^{-1}\mu_1+\frac{1}{2}\mu_2^T\Sigma^{-1}\mu_2+\ln\frac{c_1}{c_2}$$

Entonces
$$\ln\frac{p\left(x|c_1\right)p\left(c_1\right)}{p\left(x|c_2\right)p\left(c_2\right)}=w^Tx+w_0$$
Teniendo en cuenta que:
$$p\left(x|c_k\right)=\frac{1}{\left(2\pi\right)^{D/2}}\frac{1}{\left|\Sigma\right|^{1/2}}\exp\left\{-\frac{1}{2}\left(x-\mu_k\right)^T\Sigma^{-1}\left(x-\mu_k\right)\right\}$$
$$\ln\frac{p\left(x|c_1\right)p\left(c_1\right)}{p\left(x|c_2\right)p\left(c_2\right)}=$$
$$\ln\left\{\frac{\frac{1}{\left(2\pi\right)^{D/2}}\frac{1}{\left|\Sigma\right|^{1/2}}\exp\left\{-\frac{1}{2}\left(x-\mu_1\right)^T\Sigma^{-1}\left(x-\mu_1\right)\right\}}{\frac{1}{\left(2\pi\right)^{D/2}}\frac{1}{\left|\Sigma\right|^{1/2}}\exp\left\{-\frac{1}{2}\left(x-\mu_2\right)^T\Sigma^{-1}\left(x-\mu_2\right)\right\}}\right\}+\ln\frac{p\left(c_1\right)}{p\left(c_2\right)}=$$
$$\ln\left\{\exp\left\{-\frac{1}{2}\left(x-\mu_1\right)^T\Sigma^{-1}\left(x-\mu_1\right)+\frac{1}{2}\left(x-\mu_2\right)^T\Sigma^{-1}\left(x-\mu_2\right)\right\}\right\}+\ln\frac{p\left(c_1\right)}{p\left(c_2\right)}=$$
$$-\frac{1}{2}\left\{x^T\Sigma^{-1}\left(x-\mu_1\right)-\mu_1{}^T\Sigma^{-1}\left(x-\mu_1\right)-x^T\Sigma^{-1}\left(x-\mu_2\right)+\mu_2{}^T\Sigma^{-1}\left(x-\mu_2\right)\right\}+\ln\frac{p\left(c_1\right)}{p\left(c_2\right)}=$$
$$-\frac{1}{2}\left\{x^T\Sigma^{-1}x-x^T\Sigma^{-1}\mu_1-\mu_1{}^T\Sigma^{-1}x+\mu_1{}^T\Sigma^{-1}\mu_1-x^T\Sigma^{-1}x+x^T\Sigma^{-1}\mu_2+\mu_2{}^T\Sigma^{-1}x-\mu_2{}^T\Sigma^{-1}\mu_2\right\}+\ln\frac{p\left(c_1\right)}{p\left(c_2\right)}=$$
$$-\frac{1}{2}\left\{-x^T\Sigma^{-1}\left(\mu_1-\mu_2\right)-\left(\mu_1-\mu_2\right)^T\Sigma^{-1}x+\mu_1{}^T\Sigma^{-1}\mu_1-\mu_2{}^T\Sigma^{-1}\mu_2\right\}+\ln\frac{p\left(c_1\right)}{p\left(c_2\right)}=$$
$$-\frac{1}{2}\left\{-2\left(\mu_1-\mu_2\right)^T\Sigma^{-1}x+\mu_1{}^T\Sigma^{-1}\mu_1-\mu_2{}^T\Sigma^{-1}\mu_2\right\}+\ln\frac{p\left(c_1\right)}{p\left(c_2\right)}=\left(\mu_1-\mu_2\right)^T\Sigma^{-1}x-\frac{1}{2}\mu_1{}^T\Sigma^{-1}\mu_1+\frac{1}{2}\mu_2{}^T\Sigma^{-1}\mu_2+$$
$$\ln\frac{p\left(c_1\right)}{p\left(c_2\right)}=$$

$$\underbrace{\left(\Sigma^{-1}\left(\mu_1-\mu_2\right)\right)^T}_{w^T} x \underbrace{-\frac{1}{2}\mu_1{}^T\Sigma^{-1}\mu_1+\frac{1}{2}\mu_2{}^T\Sigma^{-1}\mu_2+\ln\frac{p\left(c_1\right)}{p\left(c_2\right)}}_{w_0} =$$

$$w^T x + w_0$$

4.10

Consider the classification model of Exercise 4.9 and now suppose that the class-conditional densities are given by Gaussian distributions with a shared covariance matrix, so that

$$p\left(\phi_n|C_k\right)=\mathcal{N}\left(\phi_n|\mu,\Sigma\right)$$

Show that the maximum likelihood solution for the mean of the Gaussian distribution for class Ck is given by

$$\mu_k = \frac{1}{N_k}\sum_{n=1}^{N} t_{n,k}\phi_n$$

which represents the mean of those feature vectors assigned to class Ck. Similarly, show that the maximum likelihood solution for the shared covariance matrix is given by

$$\Sigma=\sum_{k=1}^{K}\frac{N_k}{N}\ S_k$$

where

$$S_k=\frac{1}{N_k}\sum_{n=1}^{N}\ t_{n,k}\left(\phi_n-\mu_k\right)\left(\phi_n-\mu_k\right)^T$$

Thus Σ is given by a weighted average of the covariances of the data associated with each class, in which the weighting coefficients are given by the prior probabilities of the classes.

$$p\left(\left\{\phi_n,t_n\right\}|\left\{\pi_k\right\}\right)=\prod_{n=1}^{N}\prod_{k=1}^{K}\left\{p\left(\phi_n|C_k\right)p\left(C_k\right)\right\}^{t_{n,k}}$$

$$p\left(C_k\right)=\pi_k$$
$$p\left(\phi_n|C_k\right)=\mathcal{N}\left(\phi_n|\mu,\Sigma\right)$$

$$\ln p\left(\left\{\phi_n,t_n\right\}|\left\{\pi_k\right\}\right)=\ln\prod_{n=1}^{N}\prod_{k=1}^{K}\left\{\mathcal{N}\left(\phi_n|\mu,\Sigma\right)\ \pi_k\right\}^{t_{n,k}}$$

$$\ln\prod_{m=1}^{M}a_m=\ln\sum_{m=1}^{M}a_m$$
$$\ln a^n = n\ln a$$

$$\ln p\left(\left\{\phi_n,t_n\right\}|\left\{\pi_k\right\}\right)=\sum_{n=1}^{N}\sum_{k=1}^{K}t_{n,k}\ln\left\{\mathcal{N}\left(\phi_n|\mu,\Sigma\right)\ \pi_k\right\}$$

$$\ln\mathcal{N}\left(\phi_n|\mu,\Sigma\right)=\ln\left(2\pi\left|\Sigma\right|\right)^{-1/2}-\frac{1}{2}\left(\phi_n-\mu_k\right)^T\Sigma^{-1}\left(\phi_n-\mu_k\right)$$

$$\ln p\left(\left\{\phi_n,t_n\right\}|\left\{\pi_k\right\}\right)=\sum_{n=1}^{N}\sum_{k=1}^{K}t_{n,k}\left\{-\frac{1}{2}\ln\left(\left|\Sigma\right|\right)-\frac{1}{2}\left(\phi_n-\mu_k\right)^T\Sigma^{-1}\left(\phi_n-\mu_k\right)-\frac{1}{2}\ln\left(2\pi\right)+\ln\pi_k\right\}$$

$$\frac{\partial}{\partial \mu_m}\left\{\sum_{m=1}^{m} F\left(x_m\right)\right\}=\frac{\partial F\left(x_m\right)}{\partial \mu_m}$$

$$\frac{\partial}{\partial \mu_k}\left\{\ln p\left(\left\{\phi_n, t_n\right\} \mid \left\{\pi_k\right\}\right)\right\}=\frac{\partial}{\partial \mu_k}\left\{\sum_{n=1}^{N} t_{n,k}\left\{-\frac{1}{2}\left(\phi_n-\mu_k\right)^T \Sigma^{-1}\left(\phi_n-\mu_k\right)\right\}\right\}=0$$

$$\frac{\partial}{\partial s}(x-s)^T W(x-s)=2W(x-s)$$

$$\sum_{n=1}^{N} t_{n,k}\ \left\{-\Sigma^{-1}\left(\phi_n-\mu_k\right)\right\}=0 \quad \Longrightarrow \quad -\sum_{n=1}^{N}\ t_{n,k}\Sigma^{-1}\phi_n-t_{n,k}\Sigma^{-1}\mu_k=0$$

$$\sum a-b=\sum a-\sum b$$

$$\sum_{n=1}^{N}\ t_{n,k}\Sigma^{-1}\mu_k-\sum_{n=1}^{N}\ t_{n,k}\Sigma^{-1}\phi_n=0 \quad \Longrightarrow \quad \sum_{n=1}^{N}\ t_{n,k}\Sigma^{-1}\mu_k=\sum_{n=1}^{N}\ t_{n,k}\Sigma^{-1}\phi_n$$

$$\Sigma \cdot \Sigma^{-1}=I$$

$$\Sigma^{-1}\mu_k\sum_{n=1}^{N}\ t_{n,k}=\Sigma^{-1}\sum_{n=1}^{N}\ t_{n,k}\phi_n \implies \Sigma\left\{\Sigma^{-1}\mu_k\sum_{n=1}^{N}\ t_{n,k}\right\}=\Sigma\left\{\Sigma^{-1}\sum_{n=1}^{N}\ t_{n,k}\phi_n\right\}$$

$$N_k=\sum_{n=1}^{N}\ t_{n,k}$$

$$\mu_k\ N_k=\sum_{n=1}^{N}\ t_{n,k}\phi_n \quad \Longrightarrow \quad \mu_k\ =\frac{1}{N_k}\ \sum_{n=1}^{N}\ t_{n,k}\phi_n$$

$$\frac{\partial}{\partial \Sigma}\left\{\ln p\left(\left\{\phi_n, t_n\right\} \mid \left\{\pi_k\right\}\right)\right\}=\frac{\partial}{\partial \Sigma}\left\{\sum_{n=1}^{N}\sum_{k=1}^{K} t_{n,k}\ \left\{-\frac{1}{2}\ln\left(|\Sigma|\right)-\frac{1}{2}\left(\phi_n-\mu_k\right)^T \Sigma^{-1}\left(\phi_n-\mu_k\right)\right\}\right\}$$

$$\sum a+b=\sum a+\sum b$$
$$D(a+b)=Da+Db$$

$$-\frac{1}{2}\frac{\partial}{\partial \Sigma}\left\{\sum_{n=1}^{N}\sum_{k=1}^{K} t_{n,k}\ \ln\left(|\Sigma|\right)\right\}-\frac{1}{2}\frac{\partial}{\partial \Sigma}\left\{\sum_{n=1}^{N}\sum_{k=1}^{K} t_{n,k}\ \left(\phi_n-\mu_k\right)^T \Sigma^{-1}\left(\phi_n-\mu_k\right)\right\}=0$$

$$S_k=\frac{1}{N_k}\sum_{n=1}^{N}\ t_{n,k}\left(\phi_n-\mu_k\right)\left(\phi_n-\mu_k\right)^T$$

$$-\frac{1}{2}\left\{\sum_{n=1}^{N}\sum_{k=1}^{K} t_{n,k}\frac{\partial \ln\left(|\Sigma|\right)}{\partial \Sigma}\right\}=\frac{1}{2}\left\{\sum_{k=1}^{K}\frac{\partial}{\partial \Sigma}\underbrace{\sum_{n=1}^{N} t_{n,k}\left(\phi_n-\mu_k\right)^T \Sigma^{-1}\left(\phi_n-\mu_k\right)}_{N_k\ Tr\left(\Sigma^{-1}S_k\right)}\right\}$$

$$\frac{\partial l\ n(|A|)}{\partial \Sigma} = \left(A^{-1}\right)^T$$

$$-\sum_{n=1}^{N}\sum_{k=1}^{K} t_{n,k} \underbrace{\left(\Sigma^{-1}\right)^T}_{\Sigma^{-1}} = \sum_{k=1}^{K} \frac{\partial \left\{ N_k\ Tr\left(\Sigma^{-1} S_k\right)\right\}}{\partial \Sigma}$$

$$\frac{\partial \left\{\ Tr\left(AX^{-1}B\right)\right\}}{\partial X} = \left(X^{-1}B\ A\ X^{-1}\right)^T$$

$$-\sum_{n=1}^{N}\sum_{k=1}^{K} t_{n,k}\Sigma^{-1} = -\sum_{k=1}^{K} N_k \left(\Sigma^{-1} S_k \Sigma^{-1}\right)^T$$

$$(A\ B\ C)^T = C^T B^T A^T \ N = \sum_{n=1}^{N}\sum_{k=1}^{K} t_{n,k}$$

$$\Sigma^{-1} \underbrace{\sum_{n=1}^{N}\sum_{k=1}^{K} t_{n,k}}_{N} = \sum_{k=1}^{K} N_k \left(\Sigma^{-1} S_k \Sigma^{-1}\right)^T$$

Multiplicando por Σ tanto por la izquierda como por la derecha ambos miembros de la ecuación

$$\Sigma\left(\Sigma^{-1}N\right)\Sigma = \Sigma\left(\sum_{k=1}^{K} N_k \Sigma^{-1} S_k \Sigma^{-1}\right)\Sigma \ N\ \Sigma = \sum_{k=1}^{K} N_k S_k \Longrightarrow \quad \Sigma = \sum_{k=1}^{K} \frac{N_k}{N}\ S_k$$