

# Mineria de Datos

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## 4.8

Using (4.57) and (4.58), derive the result (4.65) for the posterior class probability in the two-class generative model with Gaussian densities, and verify the results (4.66) and (4.67) for the parameters  $w$  and  $w_0$ .

$$p(c_1|x) = \frac{p(x|c_1)p(c_1)}{p(x|c_1)p(c_1) + p(x|c_2)p(c_2)} = \frac{1}{1 + \exp(-a)} = \sigma(a)$$

$$a = \ln \frac{p(x|c_1)p(c_1)}{p(x|c_2)p(c_2)}$$

$$p(c_1|x) = \sigma(w^T x + w_0)$$

$$w = \Sigma^{-1}(\mu_1 - \mu_2)$$

$$w_0 = -\frac{1}{2}\mu_1^T \Sigma^{-1} \mu_1 + \frac{1}{2}\mu_2^T \Sigma^{-1} \mu_2 + \ln \frac{c_1}{c_2}$$

Entonces

$$\ln \frac{p(x|c_1)p(c_1)}{p(x|c_2)p(c_2)} = w^T x + w_0$$

Teniendo en cuenta que:

$$p(x|c_k) = \frac{1}{(2\pi)^{D/2}} \frac{1}{|\Sigma|^{1/2}} \exp \left\{ -\frac{1}{2} (x - \mu_k)^T \Sigma^{-1} (x - \mu_k) \right\}$$

$$\ln \frac{p(x|c_1)p(c_1)}{p(x|c_2)p(c_2)} =$$

$$\ln \left\{ \frac{\frac{1}{(2\pi)^{D/2}} \frac{1}{|\Sigma|^{1/2}} \exp \left\{ -\frac{1}{2} (x - \mu_1)^T \Sigma^{-1} (x - \mu_1) \right\}}{\frac{1}{(2\pi)^{D/2}} \frac{1}{|\Sigma|^{1/2}} \exp \left\{ -\frac{1}{2} (x - \mu_2)^T \Sigma^{-1} (x - \mu_2) \right\}} \right\} + \ln \frac{p(c_1)}{p(c_2)} =$$

$$\ln \left\{ \exp \left\{ -\frac{1}{2} (x - \mu_1)^T \Sigma^{-1} (x - \mu_1) + \frac{1}{2} (x - \mu_2)^T \Sigma^{-1} (x - \mu_2) \right\} \right\} + \ln \frac{p(c_1)}{p(c_2)} =$$

$$-\frac{1}{2} \{ x^T \Sigma^{-1} (x - \mu_1) - \mu_1^T \Sigma^{-1} (x - \mu_1) - x^T \Sigma^{-1} (x - \mu_2) + \mu_2^T \Sigma^{-1} (x - \mu_2) \} + \ln \frac{p(c_1)}{p(c_2)} =$$

$$-\frac{1}{2} \{ x^T \Sigma^{-1} x - x^T \Sigma^{-1} \mu_1 - \mu_1^T \Sigma^{-1} x + \mu_1^T \Sigma^{-1} \mu_1 - x^T \Sigma^{-1} x + x^T \Sigma^{-1} \mu_2 + \mu_2^T \Sigma^{-1} x - \mu_2^T \Sigma^{-1} \mu_2 \} + \ln \frac{p(c_1)}{p(c_2)} =$$

$$-\frac{1}{2} \{ -x^T \Sigma^{-1} (\mu_1 - \mu_2) - (\mu_1 - \mu_2)^T \Sigma^{-1} x + \mu_1^T \Sigma^{-1} \mu_1 - \mu_2^T \Sigma^{-1} \mu_2 \} + \ln \frac{p(c_1)}{p(c_2)} =$$

$$-\frac{1}{2} \{ -2(\mu_1 - \mu_2)^T \Sigma^{-1} x + \mu_1^T \Sigma^{-1} \mu_1 - \mu_2^T \Sigma^{-1} \mu_2 \} + \ln \frac{p(c_1)}{p(c_2)} = (\mu_1 - \mu_2)^T \Sigma^{-1} x - \frac{1}{2} \mu_1^T \Sigma^{-1} \mu_1 + \frac{1}{2} \mu_2^T \Sigma^{-1} \mu_2 +$$

$$\ln \frac{p(c_1)}{p(c_2)} =$$

$$\underbrace{(\Sigma^{-1}(\mu_1 - \mu_2))^T x}_{w^T} - \underbrace{\frac{1}{2}\mu_1^T \Sigma^{-1} \mu_1 + \frac{1}{2}\mu_2^T \Sigma^{-1} \mu_2 + \ln \frac{p(c_1)}{p(c_2)}}_{w_0} = w^T x + w_0$$

#### 4.10

Consider the classification model of Exercise 4.9 and now suppose that the class-conditional densities are given by Gaussian distributions with a shared covariance matrix, so that

$$p(\phi_n | C_k) = \mathcal{N}(\phi_n | \mu, \Sigma)$$

Show that the maximum likelihood solution for the mean of the Gaussian distribution for class  $C_k$  is given by

$$\mu_k = \frac{1}{N_k} \sum_{n=1}^N t_{n,k} \phi_n$$

which represents the mean of those feature vectors assigned to class  $C_k$ . Similarly, show that the maximum likelihood solution for the shared covariance matrix is given by

$$\Sigma = \sum_{k=1}^K \frac{N_k}{N} S_k$$

where

$$S_k = \frac{1}{N_k} \sum_{n=1}^N t_{n,k} (\phi_n - \mu_k)(\phi_n - \mu_k)^T$$

Thus  $\Sigma$  is given by a weighted average of the covariances of the data associated with each class, in which the weighting coefficients are given by the prior probabilities of the classes.

$$p(\{\phi_n, t_n\} | \{\pi_k\}) = \prod_{n=1}^N \prod_{k=1}^K \{p(\phi_n | C_k) p(C_k)\}^{t_{n,k}}$$

$$\begin{aligned} p(C_k) &= \pi_k \\ p(\phi_n | C_k) &= \mathcal{N}(\phi_n | \mu, \Sigma) \end{aligned}$$

$$\ln p(\{\phi_n, t_n\} | \{\pi_k\}) = \ln \prod_{n=1}^N \prod_{k=1}^K \{\mathcal{N}(\phi_n | \mu, \Sigma) \pi_k\}^{t_{n,k}}$$

$$\begin{aligned} \ln \prod_{m=1}^M a_m &= \ln \sum_{m=1}^M a_m \\ \ln a^n &= n \ln a \end{aligned}$$

$$\ln p(\{\phi_n, t_n\} | \{\pi_k\}) = \sum_{n=1}^N \sum_{k=1}^K t_{n,k} \ln \{\mathcal{N}(\phi_n | \mu, \Sigma) \pi_k\}$$

$$\ln \mathcal{N}(\phi_n | \mu, \Sigma) = \ln (2\pi |\Sigma|)^{-1/2} - \frac{1}{2} (\phi_n - \mu_k)^T \Sigma^{-1} (\phi_n - \mu_k)$$

$$\ln p(\{\phi_n, t_n\} | \{\pi_k\}) = \sum_{n=1}^N \sum_{k=1}^K t_{n,k} \left\{ -\frac{1}{2} \ln(|\Sigma|) - \frac{1}{2} (\phi_n - \mu_k)^T \Sigma^{-1} (\phi_n - \mu_k) - \frac{1}{2} \ln(2\pi) + \ln \pi_k \right\}$$

$$\frac{\partial}{\partial \mu_m} \left\{ \sum_{m=1}^m F(x_m) \right\} = \frac{\partial F(x_m)}{\partial \mu_m}$$

$$\frac{\partial}{\partial \mu_k} \{ \ln p(\{\phi_n, t_n\} | \{\pi_k\}) \} = \frac{\partial}{\partial \mu_k} \left\{ \sum_{n=1}^N t_{n,k} \left\{ -\frac{1}{2} (\phi_n - \mu_k)^T \Sigma^{-1} (\phi_n - \mu_k) \right\} \right\} = 0$$

$$\frac{\partial}{\partial s} (x-s)^T W (x-s) = 2W (x-s)$$

$$\sum_{n=1}^N t_{n,k} \{ -\Sigma^{-1} (\phi_n - \mu_k) \} = 0 \implies -\sum_{n=1}^N t_{n,k} \Sigma^{-1} \phi_n - t_{n,k} \Sigma^{-1} \mu_k = 0$$

$$\sum a-b=\sum a-\sum b$$

$$\sum_{n=1}^N t_{n,k} \Sigma^{-1} \mu_k - \sum_{n=1}^N t_{n,k} \Sigma^{-1} \phi_n = 0 \implies \sum_{n=1}^N t_{n,k} \Sigma^{-1} \mu_k = \sum_{n=1}^N t_{n,k} \Sigma^{-1} \phi_n$$

$$\Sigma \cdot \Sigma^{-1} = I$$

$$\Sigma^{-1} \mu_k \sum_{n=1}^N t_{n,k} = \Sigma^{-1} \sum_{n=1}^N t_{n,k} \phi_n \implies \Sigma \left\{ \Sigma^{-1} \mu_k \sum_{n=1}^N t_{n,k} \right\} = \Sigma \left\{ \Sigma^{-1} \sum_{n=1}^N t_{n,k} \phi_n \right\}$$

$$N_k = \sum_{n=1}^N t_{n,k}$$

$$\mu_k N_k = \sum_{n=1}^N t_{n,k} \phi_n \implies \mu_k = \frac{1}{N_k} \sum_{n=1}^N t_{n,k} \phi_n$$

$$\frac{\partial}{\partial \Sigma} \{ \ln p(\{\phi_n, t_n\} | \{\pi_k\}) \} = \frac{\partial}{\partial \Sigma} \left\{ \sum_{n=1}^N \sum_{k=1}^K t_{n,k} \left\{ -\frac{1}{2} \ln(|\Sigma|) - \frac{1}{2} (\phi_n - \mu_k)^T \Sigma^{-1} (\phi_n - \mu_k) \right\} \right\}$$

$$\sum a+b=\sum a+\sum b\\ D\left(a+b\right)=Da+Db$$

$$-\frac{1}{2} \frac{\partial}{\partial \Sigma} \left\{ \sum_{n=1}^N \sum_{k=1}^K t_{n,k} \ln(|\Sigma|) \right\} - \frac{1}{2} \frac{\partial}{\partial \Sigma} \left\{ \sum_{n=1}^N \sum_{k=1}^K t_{n,k} (\phi_n - \mu_k)^T \Sigma^{-1} (\phi_n - \mu_k) \right\} = 0$$

$$S_k = \frac{1}{N_k} \sum_{n=1}^N t_{n,k} (\phi_n - \mu_k) (\phi_n - \mu_k)^T$$

$$-\frac{1}{2} \left\{ \sum_{n=1}^N \sum_{k=1}^K t_{n,k} \frac{\partial \ln(|\Sigma|)}{\partial \Sigma} \right\} = \frac{1}{2} \left\{ \sum_{k=1}^K \frac{\partial}{\partial \Sigma} \underbrace{\sum_{n=1}^N t_{n,k} (\phi_n - \mu_k)^T \Sigma^{-1} (\phi_n - \mu_k)}_{N_k \text{ } Tr(\Sigma^{-1} S_k)} \right\}$$

$$\frac{\partial l(n(|A|))}{\partial \Sigma} = (A^{-1})^T$$

$$-\sum_{n=1}^N \sum_{k=1}^K t_{n,k} \underbrace{(\Sigma^{-1})^T}_{\Sigma^{-1}} = \sum_{k=1}^K \frac{\partial \{N_k \text{Tr}(\Sigma^{-1} S_k)\}}{\partial \Sigma}$$

$$\frac{\partial \{ \text{Tr}(AX^{-1}B) \}}{\partial X} = (X^{-1}B \ A \ X^{-1})^T$$

$$-\sum_{n=1}^N \sum_{k=1}^K t_{n,k} \Sigma^{-1} = -\sum_{k=1}^K N_k (\Sigma^{-1} S_k \Sigma^{-1})^T$$

$$(A \ B \ C)^T = C^T B^T A^T \ N = \sum_{n=1}^N \sum_{k=1}^K t_{n,k}$$

$$\Sigma^{-1} \underbrace{\sum_{n=1}^N \sum_{k=1}^K t_{n,k}}_N = \sum_{k=1}^K N_k (\Sigma^{-1} S_k \Sigma^{-1})^T$$

Multiplicando por  $\Sigma$  tanto por la izquierda como por la derecha ambos miembros de la ecuación

$$\Sigma (\Sigma^{-1} N) \Sigma = \Sigma \left( \sum_{k=1}^K N_k \Sigma^{-1} S_k \Sigma^{-1} \right) \Sigma \ N \ \Sigma = \sum_{k=1}^K N_k S_k \implies \Sigma = \sum_{k=1}^K \frac{N_k}{N} S_k$$