STATISTICS DEPARTMENT
M.S. EXAMINATION

PART I
CLOSED BOOK

Friday, May 16, 2003

9:00 a.m. - 1:00 p.m.

Biella Room (Library, First Floor)

Instructions: Complete *four of the five* problems. Each problem counts 25 points. Unless otherwise noted, points are allocated approximately equally to lettered parts of a problem. Spend your time accordingly.

Begin each problem on a new page. Write the problem number and the page number in the specified locations at the top of each page. Also write your chosen ID code number on every page. Please write only within the black borderlines, leaving at least 1" margins on both sides, top and bottom of each page. Write on one side of the page only.

At the end of this part of the exam you will turn in your answers sheets, but you will keep the question sheets and your scratch paper.

Tables of some distributions are provided. Use them as appropriate.

1. Let $X_{[1]} \le X_{[2]} \le X_{[3]} \le \ldots \le X_{[41]}$ be 41 "ordered" observations from the variable X = number of movies seen per month for a random sample of 41 people. The data are given below. Let θ be the population 70th percentile for this variable.

Note the data and the cumulative binomial probabilities given below.

(a) Explain why $P(X > \theta) = 0.3$.

(b) For θ_0, some specific value of θ, what type of random variable is Y = the number of the 41 observations that are greater than θ_0?

(c) Suppose we test $H_0: \theta = \theta_0$ versus $H_1: \theta \neq \theta_0$, accepting H_0 if $9 \le Y \le 18$, and rejecting H_0 otherwise. What is the value of α = P(Type I error) for this test?

(d) If we test $H_0: \theta = 4.0$ versus $H_1: \theta \neq 4.0$, what is the conclusion?
(Notice that 4.0 is the 23rd observation.)
If we test $H_0: \theta = 6.0$ versus $H_1: \theta \neq 6.0$, what is the conclusion?
(Notice that 6.0 is the 33rd observation.)

(e) Explain why $\{\theta: 4.0 \le \theta < 6.0\}$ is a $100(1-\alpha)\%$ confidence interval for θ.
Use the value of α obtained in part (c).

Data

1.0	1.0	1.5	1.5	1.5	1.5	1.5	1.5	2.0	2.0
2.0	2.0	2.0	3.0	3.0	3.0	3.0	3.0	3.0	3.0
3.0	4.0	4.0	4.0	4.5	4.5	4.5	4.5	5.0	5.0
5.0	5.0	6.0	6.0	6.0	6.0	8.0	10.0	11.0	12.5
15.0									

Cumulative binomial probabilities: n = 41, p = 0.3

k	P(Y<=k)
0	0.00000
1	0.00001
2	0.00008
3	0.00045
4	0.00197
5	0.00680
6	0.01922
7	0.04583
8	0.09429
9	0.17045
10	0.27490
11	0.40105
12	0.53621
13	0.66543
14	0.77619
15	0.86163
16	0.92114
17	0.95864
18	0.98007
19	0.99119
20	0.99643
21	0.99868
22	0.99955
23	0.99986
24	0.99996
25	0.99999
26	1.00000

Part I

Solutions #1 CB

1) Let $X_{[1]} \le X_{[2]} \le X_{[3]} \ldots \le X_{[41]}$ be 41 ordered observations from the variable X = number of movies seen per month for a random sample of 41 people. The data is given below. **Note also the cumulative binomial probabilities given below.** Let θ = the population 70th percentile for this variable.

(a) Explain why $P(X > \theta) = .3$.
(b) For θ_0, some specific value of θ, what type of random variable is Y = the number of the 41 observations that are greater than θ_0.
(c) Suppose we test $H_0: \theta = \theta_0$ versus $H_1: \theta \neq \theta_0$, accepting H_0 if $9 \le Y \le 18$, and rejecting H_0 otherwise. What is the value of $\alpha = P(\text{type I error})$ for this test?
(d) Suppose we test $H_0: \theta = 4.0$ versus $H_1: \theta \neq 4.0$, what is the conclusion (note that 4.0 is the 23rd observation)? Suppose we test $H_0: \theta = 6.0$ versus $H_1: \theta \neq 6.0$, what is the conclusion (note that 6.0 is the 33rd observation)?
(e) Explain why $(4.0 \le \theta < 6.0)$ is a $100(1-\alpha)\%$ confidence interval for θ.

nomovie

1.0	1.0	1.5	1.5	1.5	1.5	1.5	1.5	2.0	2.0
2.0	2.0	2.0	3.0	3.0	3.0	3.0	3.0	3.0	3.0
3.0	4.0	4.0	4.0	4.5	4.5	4.5	4.5	5.0	5.0
5.0	5.0	6.0	6.0	6.0	6.0	8.0	10.0	11.0	12.5
15.0									

Data Display (binomial, n = sample size = 41, p = probability of success = .3, cumulative probabilities)

k	P(Y<=k)
0	0.00000
1	0.00001
2	0.00008
3	0.00045
4	0.00197
5	0.00680
6	0.01922
7	0.04583
8	0.09429
9	0.17045
10	0.27490
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13	0.66543
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18	0.98007
19	0.99119
20	0.99643
21	0.99868
22	0.99955
23	0.99986
24	0.99996
25	0.99999
26	1.00000

(a) since by definition of 70th percentile $P(X \le \theta) = 0.7 \Rightarrow P(X > \theta) = 1 - P(X \le \theta) = 1 - 0.7 = .3 = \alpha$

(b) binomial $n = 41$, $p = .3$.

(c) $(1-\alpha) = P(9 \le Y \le 18) = P(Y \le 18) - P(Y \le 8)$
$= .98007 - .09429 = .886$
Thus $\alpha \approx .114$

(d) $H_0: \theta = 4.0$, $9 \le Y = 17 \le 18$
Thus accept H_0
(For $\theta_0 < 4.0$, $Y > 18$ and we would reject H_0

$H_0: \theta = 6.0$, $Y = 5$ and we would reject H_0; also for $\theta_0 > 6.0$.
For, $4.0 \le \theta_0 < 6.0$, $Y \ge 9$ and $Y \le 18$, and we would accept H_0!

(e) See answer to (d)

2. Consider the following display for data from an incidence study for a disease.

	Disease status		
Risk factor status	*Disease*	*No disease*	*Total*
Exposed	a	b	$a+b$
Not exposed	c	d	$c+d$
Total	$a+c$	$b+d$	n

Where $n = a+b+c+d$. We define the *risk* of the disease in the sample as

$$risk = r = \text{number of cases of disease / number of people at risk} = (a+c)/n$$

which is used to estimate the *population risk*, ϕ. We define the *exposure-specific risks*, for those with the risk factor as $a/(a+b)$ and for those without the risk factor as $c/(c+d)$. We also define the *relative risk* for those with the risk factor, compared to those without the risk factor as

$$\lambda = \frac{a/(a+b)}{c/(c+d)} = \frac{a(c+d)}{c(a+b)} . \tag{1}$$

(a) Let $X = a+c$. What is the distribution of the number X of cases of disease in the sample of size n, assuming constant risk over the risk factor status? Give the formula for the likelihood function of the *risk* ϕ. Show that the maximum likelihood estimate (m.l.e.) of the *risk* ϕ is $\hat{\phi} = (a+c)/n$, based on observing the number of cases of disease in the sample $X = (a+c)$.

(b) What is the large sampling distribution of the m.l.e. $\hat{\phi}$ of ϕ? Give a large sample confidence interval for the *population risk* ϕ.

The derivation of the large sample confidence interval for the *population relative risk* λ is slightly more difficult to derive. The large sample distribution of the *sample relative risk* $\hat{\lambda}$ is skewed and a *log* transformation is used to achieve approximate normality. On the log scale it can be shown that

$$\hat{se}(log(\hat{\lambda})) = \sqrt{\frac{1}{a} - \frac{1}{a+b} + \frac{1}{c} - \frac{1}{c+d}} . \tag{2}$$

Therefore, a 95% confidence interval for $log(\lambda)$ is

$$\log(\hat{\lambda}) \pm 1.96 \hat{se}(\log \hat{\lambda}) \tag{3}$$

with lower and upper confidence limits of

$$L_{log} = \log(\hat{\lambda}) - 1.96\hat{se}(\log\hat{\lambda}) \tag{4}$$

$$U_{log} = \log(\hat{\lambda}) + 1.96\hat{se}(\log\hat{\lambda}) \ . \tag{5}$$

Since we want a 95% confidence interval for λ itself, we can obtain the two limits by raising (L_{log}, U_{log}) to the power of exp. That is

$$L = \exp(L_{log}) \tag{6}$$

$$U = \exp(U_{log}) \tag{7}$$

to give a 95% confidence interval for λ, the *population relative risk*.

(c) Explain why $\hat{\lambda}$, the *sample relative risk* statistic, might have a skewed distribution? In which direction would the distribution be skewed? Why might the log transformation help make the sampling distribution of the *sample relative risk* more normal? What property of m.l.e.s is being used when the confidence interval for $\log(\lambda)$ is transformed using exp? Explain why the results will be valid.

Suppose risk factors for coronary heart disease are being studied in men. The following table gives the smoking status of men entering the study and whether or not a coronary event occurred during the 10 years the study was conducted.

	Coronary event?		
Smoker entering the study	*Yes*	*No*	*Total*
Yes	166	1176	1342
No	50	513	563
Total	216	1689	1905

(d) Compute the estimated *population risk* $\hat{\phi}$ of coronary disease in the study. Compute a 95% confidence interval for the *population risk*, ϕ.

(e) Compute the estimated *relative risk* of coronary disease of smokers to nonsmokers. Compute a 95% confidence interval for the $\log(\lambda)$. Compute a 95% confidence interval for λ. Conduct a hypothesis test of $H_0 : \lambda = 1$ versus $H_1 : \lambda \neq 1$ using the final confidence interval computed in part (d) above. Is there statistically significant evidence that smoking is a risk factor for coronary heart disease?

3. Let $X_1, X_2, \ldots, X_n$ be a random sample from a population with the density function $f(x) = (1/\theta^2)\, x\, e^{-x/\theta}$, for $x > 0$. The parameter θ is unknown.

(a) Identify the distribution family of this population, specifying the value of any known parameter of the population distribution. Derive $E(X_i)$ in terms of θ. State $V(X_i)$ if you know it, otherwise derive it.

(b) Find the method of moments estimators of θ and of $\tau = \tau(\theta) = 1/\theta$.

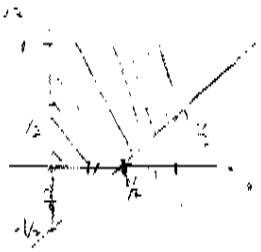
(c) Find the maximum likelihood estimator $\hat{\theta}$ of θ. State the maximum likelihood estimator of τ and the name of the principle by which you found it.

(d) Find the Cramér-Rao bound on unbiased estimators of θ. Can this result be used to determine whether $\hat{\theta}$ is the UMVUE of θ? Why or why not? Can this result be used to find a UMVUE for τ? Why or why not?

(e) Based on $n = 800$ observations from this distribution, suppose that the sample mean is 273.1. Give approximate 95% confidence intervals for θ and τ. Quote appropriate theorem(s) to justify your method.

4. Let $Y_1, Y_2, \cdots$ be independent and identically distributed random variables that have the probability density function

$$f(y) = I_{(0,1)}(y) = \begin{cases} 1 \text{ if } 0 < y < 1 \\ 0 \text{ elsewhere.} \end{cases}$$



5 (a) Find $P\left(Y_1 \le Y_2 + \frac{1}{2}\right)$.

5 (b) For $k = 1, 2, \cdots$, let $X_k = -\ln(Y_k)$.

For $n = 1, 2, \cdots$, let $W_n = \sum_{k=1}^{n} X_k$.

5 (i) Find $P(W_2 \le w)$ for $0 < w < \infty$.

5 (ii) Suppose that $0 < t \le 1$. Find $E\left(t^{W_n}\right)$.

10 (iii) For $0 < \theta < 1$, let N be a random variable that has the probability mass function

$$p(n) = P(N = n) = \theta^{n-1}(1-\theta) \text{ for } n = 1, 2, \cdots.$$

Suppose that N is independent of $X_1, X_2, \cdots$.

Find $E(W_N)$. Also find $E\left(e^{tW_N}\right)$ for $-\infty < t \le 0$.

5. The number of items that arrive at a repair facility by time t is a Poisson process $Y(t)$ for $0 \le t < \infty$. Assume that the items arrive at a rate of λ items per hour.

(a) Let n be a positive integer and suppose that $0 < s < t < \infty$.

5 (i) If $Y(s) = n$, then what is the expected value of $Y(t)$?

5 (ii) If $Y(t) = n$, then what is the variance of $Y(s)$?

(b) Twenty percent of the arriving items require an expensive repair. Assume that the items are independent of each other and that the items independent of $Y(t)$. For $0 \le t < \infty$, let $X(t)$ be the number of items requiring an expensive repair that arrive by time t.

5 (i) For $0 \le t < \infty$, what is the expected value of $X(t)$?

5 (ii) For $0 \le t < \infty$, what is variance of $X(t)$?

5 (iii) For $0 \le t < \infty$ and $k = 0,1,2,\cdots$, what is the probability that $X(t)$ is equal to k?

Upper-tail Areas for the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.00	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641
0.10	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
0.20	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
0.30	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
0.40	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
0.50	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
0.60	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
0.70	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
0.80	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
0.90	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
1.00	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
1.10	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
1.20	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
1.30	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
1.40	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
1.50	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
1.60	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
1.70	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
1.80	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
1.90	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
2.00	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
2.10	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
2.20	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
2.30	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
2.40	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
2.50	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
2.60	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
2.70	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
2.80	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
2.90	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
3.00	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010

z	Area
3.500	.00023263
4.000	.00003167
4.500	.00000340
5.000	.00000029

Percentage Points of the t Distribution

df	a = .1	a = .05	a = .025	a = .01	a = .005	a = .001
1	3.078	6.314	12.706	31.821	63.657	318.309
2	1.886	2.920	4.303	6.965	9.925	22.327
3	1.638	2.353	3.182	4.541	5.841	10.215
4	1.533	2.132	2.776	3.747	4.604	7.173
5	1.476	2.015	2.571	3.365	4.032	5.893
6	1.440	1.943	2.447	3.143	3.707	5.208
7	1.415	1.895	2.365	2.998	3.499	4.785
8	1.397	1.860	2.306	2.896	3.355	4.501
9	1.383	1.833	2.262	2.821	3.250	4.297
10	1.372	1.812	2.228	2.764	3.169	4.144
11	1.363	1.796	2.201	2.718	3.106	4.025
12	1.356	1.782	2.179	2.681	3.055	3.930
13	1.350	1.771	2.160	2.650	3.012	3.852
14	1.345	1.761	2.145	2.624	2.977	3.787
15	1.341	1.753	2.131	2.602	2.947	3.733
16	1.337	1.746	2.120	2.583	2.921	3.686
17	1.333	1.740	2.110	2.567	2.898	3.646
18	1.330	1.734	2.101	2.552	2.878	3.610
19	1.328	1.729	2.093	2.539	2.861	3.579
20	1.325	1.725	2.086	2.528	2.845	3.552
21	1.323	1.721	2.080	2.518	2.831	3.527
22	1.321	1.717	2.074	2.508	2.819	3.505
23	1.319	1.714	2.069	2.500	2.807	3.485
24	1.318	1.711	2.064	2.492	2.797	3.467
25	1.316	1.708	2.060	2.485	2.787	3.450
26	1.315	1.706	2.056	2.479	2.779	3.435
27	1.314	1.703	2.052	2.473	2.771	3.421
28	1.313	1.701	2.048	2.467	2.763	3.408
29	1.311	1.699	2.045	2.462	2.756	3.396
30	1.310	1.697	2.042	2.457	2.750	3.385
40	1.303	1.684	2.021	2.423	2.704	3.307
60	1.296	1.671	2.000	2.390	2.660	3.232
120	1.289	1.658	1.980	2.358	2.617	3.160
240	1.285	1.651	1.970	2.342	2.596	3.125
inf.	1.282	1.645	1.960	2.326	2.576	3.090

Solution - #2 CB

2

a) $X \sim Bin(n, \phi)$

$$L(\phi) = f(x|\phi) = L(\phi) = \binom{n}{x} \phi^x (1-\phi)^{n-x}$$

$$\ell(\phi) = \log \binom{n}{x} + x \log(\phi) + (n-x) \log(1-\phi)$$

$$\ell'(\phi) = \frac{x}{\phi} - \frac{n-x}{1-\phi} = 0$$

$$\frac{x}{\phi} = \frac{n-x}{1-\phi}$$

$$\frac{1}{\phi} - 1 = \frac{1-\phi}{\phi} = \frac{n-x}{x} = \frac{n}{x} - 1$$

$$\hat{\phi} = \frac{x}{n} = \frac{a+c}{n}$$

b) $$\ell''(\phi) = -\frac{x}{\phi^2} - \frac{n-x}{(1-\phi)^2}$$

$$E[-\ell''(\phi)] = E\left[\frac{x}{\phi^2} + \frac{n-x}{(1-\phi)^2}\right]$$

$$= \frac{1}{\phi^2} E[x] + \frac{1}{(1-\phi)^2} E[n-x]$$

$$= \frac{n\phi}{\phi^2} + \frac{n-n\phi}{(1-\phi)^2}$$

$$= \frac{n(1-\phi)^2 + \phi(n-n\phi)}{\phi(1-\phi)^2}$$

$$= \frac{n - 2n\phi + n\phi^2 + n\phi - n\phi^2}{\phi(1-\phi)^2}$$

$$= \frac{n - n\phi}{\phi(1-\phi)^2} = \frac{n(1-\phi)}{\phi(1-\phi)^2} = \frac{n}{\phi(1-\phi)}$$

$$Av(\hat{\phi}) = \frac{1}{E[-\ell''(\phi)]} = \frac{\phi(1-\phi)}{n}$$

$$\hat{\phi} \sim N\left(\phi, \frac{\phi(1-\phi)}{n}\right) \qquad se(\hat{\phi}) = \sqrt{\frac{\phi(1-\phi)}{n}}$$

large sample CI

$$\hat{\phi} \pm z_{\alpha/2}\sqrt{\frac{\hat{\phi}(1-\hat{\phi})}{n}} \qquad \hat{se}(\hat{\phi}) = \sqrt{\frac{\hat{\phi}(1-\hat{\phi})}{n}}$$

or.

$$r \pm z_{\alpha/2}\sqrt{\frac{r(1-r)}{n}}$$

c) Since the relative risk λ can take values from 0 to $+\infty$ and since $\lambda = 1$ when the exposure-specific risks are equal the sampling distribution of λ should be skewed to the right. The log transformation should help by lowering the values of $\lambda > 1$ and spreading out the values below $\lambda = 1$. The invariance property is used and since the transformation e^x is one-to-one it is valid.

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d) $r = \hat{\phi} = (a+c)/n = 216/1905 = .1134$

CI for ϕ

$$r \pm z_{\alpha/2}\sqrt{\frac{r(1-r)}{n}}$$

$$.1134 \pm 1.96\sqrt{\frac{(1134)(.8866)}{1905}}$$

$$.1134 \pm .0142 \qquad (.0992, .1276)$$

c) Smokers $a/(a+b) = 166/1374 = .1237$

non smokers $c/(c+d) = 50/563 = .0888$

relative risk $\hat{\lambda} = \frac{a/(a+b)}{c/(c+d)} = \frac{.1237}{.0888} = 1.3930$

$$\hat{se}(\log(\hat{\lambda})) = \sqrt{\frac{1}{a} - \frac{1}{a+b} + \frac{1}{c} - \frac{1}{c+d}}$$

$$= \sqrt{\frac{1}{166} - \frac{1}{1374} + \frac{1}{50} - \frac{1}{563}}$$

$$= .1533$$

$$L_{\log} = \log(1.3930) - 1.96(.1533) = .0310$$

$$U_{\log} = \log(1.3930) + 1.96(.1532) = .6319$$

$$L = e^{.0310} = 1.0315$$

$$U = e^{.6319} = 1.8813$$

$\therefore (1.0315, 1.8813)$

$H_0: \lambda = 1 \quad H_1: \lambda \neq 1$

Reject H_0 since $\lambda = 1$ is not contained in the CI.

#3 CB

Answers

(a) (Using the notation of Bain and Englehardt) gamma with shape parameter $\kappa = 2$ and unknown scale parameter θ. The derivation (shown in many probability and mathematical statistics texts) of $E(X) = \kappa\theta$ based on the fact that a gamma density for $\kappa = 3$ and θ integrates to 1. That $V(X) = \kappa\theta^2$ can be derived similarly by finding $E(X^2)$, but here it is sufficient just to state the result.

(b) The MME $\overline{X}/2$ of θ is found by setting $\mu = 2\theta = \overline{X}$ and solving for θ. By invariance, the MME of τ is $2/\overline{X}$. (The parameter τ is the rate of the underlying Poisson process and it is often called λ.)

(c) $L(\theta) = \Pi f(X_i|\theta) = \theta^{-2n} \Pi X_i \, e^{-S/\theta}$, where $S = \Sigma X_i$.
Then $\ln L(\theta) = \ell(\theta) = -2n \ln \theta + \Sigma \ln X_i - \theta^{-1} \Sigma X_i$ and $\ell'(\theta) = -2n/\theta + S/\theta^2$.
Solving $\ell'(\theta) = 0$ for θ, we get $\hat{\theta} = \overline{X}/2$ for the MLE (which agrees with the MME). By invariance, the MLE of τ is $2/\overline{X}$.

(d) Fisher's information for a single observation X is

$$I(\theta) = -E[(d^2/d\theta^2) f(X|\theta)] = -E[2\theta^{-2} - 2X\theta^{-3}] = 2\theta^{-2}.$$

So CRLB $= \theta^2/2n$. Because $V(\hat{\theta}) = V(\overline{X}/2) = \theta^2/n =$ CLRB, we know that $\hat{\theta}$ is UMVUE for θ. Because τ is a nonlinear function of θ, we know that the variance of an unbiased estimator of τ cannot achieve its CRLB, so this method won't work.

(e) For large n, the MLE of θ is approximately normal with mean θ and standard deviation $\sigma_n = \theta(2n)^{-1/2}$. This is a standard theorem about the asymptitic properties of MLEs. Considering $n = 800$ as large, we have $\sigma_{800} = \theta/40$ and

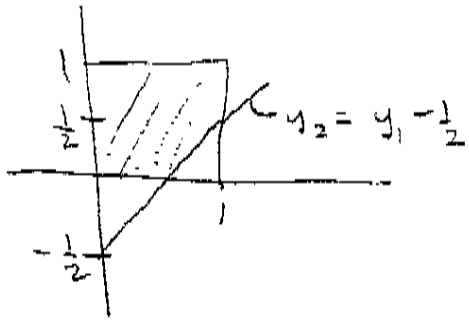
$$P\{\theta - 2\theta/40 < \overline{X}/2 < \theta + 2\theta/40\} = P\{1.9\theta < \overline{X} < 2.1\theta\} = P\{\overline{X}/2.1 < \theta < \overline{X}/1.9\} \approx 0.95$$

(We might have multiplied σ_n by 1.96, but this is only an approximate procedure.) Thus an approximate 95% CI for θ based on $\overline{X} = 273.1$ is (130.0, 143.7). Similarly, a 95% CI for $\tau = 1/\theta$ is $\{1.9/\overline{X} = 0.00696,\ 2.1/\overline{X} = 0.00769\}$. Note that both CIs are based on MLEs and thus on the sufficient statistic $\overline{X}$. [Even though the MLE of τ is not unbiased, it is *asymptotically* unbiased so $2/\overline{X} = 0.00732$ is not a bad point estimate of τ.]

Note to those studying for future MS exams: In (d) you might want to find the constant c that unbiases the MLE of τ; that is, such that $E(2c/\overline{X}) = \tau$. Then find a UMVUE of τ by using theorems of Rao-Blackwell and Lehmann-Scheffé and the ideas of sufficiency, completeness, and standard exponential families. Also, in (e) note that with statistical software you could find CIs based on the exact distribution of $\overline{X}$ (which is what?) rather than using a normal approximation. Note the similarity of this method to the method used to find a CI for the variance (or standard deviation) of a normal population.

Solution (4) CB

1. a.



$P[Y_1 \le Y_2 + \frac{1}{2}] = 1 - \frac{1}{8}$

unshaded area is $\frac{1}{8}$ of the square.

b. (i) Using (ii) W_2 has density $f(w) = \frac{w^{2-1}e^{-w}}{\Gamma(2)} I_{(0,\infty)}(w)$

$P[W_2 \le w]$ or using the fact that $X_k \sim f(x) = e^{-x} I_{(0,\infty)}(x)$

$\int_0^w u e^{-u} du$ $J = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$ $W_1 = X_1$, $W_2 = X_1 + X_2$ so $X_1 = W_1$, $X_2 = W_2 - W_1$

$f_{W_1,W_2}(w_1,w_2) = e^{-w_1} e^{-(w_2-w_1)} I_{(0,\infty)}(w_2) I_{(0,w_2)}(w_1)$

$f_{W_2}(w) = \int_0^w e^{-w_1} e^{-(w-w_1)} dw_1 = \int_0^w e^{-w} dw_1 = w e^{-w} I_{(0,\infty)}(w)$

$-ue^{-u}\Big|_0^w + \int_0^w e^{-u} du$ (ii) $E(t^{W_2}) = [E(t^{X_1})]^2 = [1-\ln(t)]^{-2}$.

$we + 1 - e^{-w}$

$P[X_1 \le x] = P[-\ln(Y_1) \le x]$

$= P[Y_1 \ge e^{-x}]$

$= 1 - e^{-x}$ for $x > 0$. $\therefore f_{X_1}(x) = e^{-x} I_{(0,\infty)}(x)$

$E(t^{X_1}) = \int_0^\infty t^x e^{-x} dx$

$= \int_0^\infty e^{-x[1-\ln(t)]} dx = \frac{1}{1-\ln(t)}$

(iii) Let $s = e^t$. Then $-\infty < t \le 0 \Leftrightarrow 0 < s \le 1$.

Hence $E(e^{tW_n}) = E(s^{W_n}) = [1-\ln(s)]^{-n} = [1-t]^{-n}$

for $n = 1, 2, \ldots$

Hence, $E(e^{tW_N}) = \sum_{n=1}^{\infty} [1-t]^{-n} \theta^{n-1}(1-\theta)$

$= \frac{1-\theta}{1-t} \sum_{n=1}^{\infty} \left[\frac{\theta}{1-t}\right]^{n-1} = \frac{1-\theta}{1-t}\left[\frac{1}{1-\frac{\theta}{1-t}}\right] = \frac{1-\theta}{1-t-\theta} = m(t)$

$m^{(1)}(t) = \frac{1-\theta}{(1-t-\theta)^2}$ Hence $E(W_N) = m^{(1)}(0) = \frac{1}{1-\theta}$.

or $E(W_n) = E(X_1)E(N)$

$= \frac{1}{1-\theta}$,

$N \sim F_2(1-\theta)$ so

$E(N) = \frac{1}{1-\theta}$.

$X_1 \sim \text{Exp}(1)$ so $E(X_1) = 1$

$E(W_1) = E(X_1) = 1$ $N \sim F_2(1-\theta)$

Solution #5 CB

(5) The number of items that arrive at a repair facility by time t is a Poisson process $Y(t)$ for $0 \le t < \infty$. Assume that the items arrive at a rate of λ items per hour.

(a) Let n be a positive integer and suppose that $0 < s < t < \infty$.

(i) If $Y(s) = n$, then what is the expected value of $Y(t)$?

$E(Y(t) \mid Y(s)=n) = E(Y(t)-Y(s)+Y(s) \mid Y(s)=n] = E(Y(t)-Y(s)) + n = \lambda(t-s)+n$

(ii) If $Y(t) = n$, then what is the variance of $Y(s)$?

(b) Twenty percent of the arriving items require an expensive repair. Assume that the items are independent of each other and that the items are independent of $Y(t)$. For $0 \le t < \infty$, let $X(t)$ be the number of items requiring an expensive repair that arrive by time t.

Let $I_i = \begin{cases} 1 & \text{if item } i \text{ requires expensive repair} \\ 0 & \text{otherwise} \end{cases}$

(i) For $0 \le t < \infty$, what is the expected value of $X(t)$?

$E(X(t)) = E\left(\sum_{i=1}^{Y(t)} I_i\right) = E(Y(t))\,E(I_1) = \lambda t (.2) = .2\lambda t$

(ii) For $0 \le t < \infty$, what is variance of $X(t)$?

$Var(X(t)) = Var\left(\sum_{i=1}^{Y(t)} I_i\right) = [E(I_1)]^2 Var(Y(t)) + E(Y(t)) Var(I_1) =$

$(.2)^2 \lambda t + \lambda t (.2)(1-.2) = .2\lambda t.$

(iii) For $0 \le t < \infty$ and $k = 0, 1, 2, \cdots$, what is the probability that $X(t)$ is equal to k?

(a)(ii) $P[Y(s)=k \mid Y(t)=n] = \dfrac{P[Y(t)-Y(s)=n-k,\ Y(s)=k]}{P[Y(t)=n]}$

$= \dfrac{\frac{[\lambda(t-s)]^{n-k} e^{-\lambda(t-s)}}{(n-k)!} \cdot \frac{(\lambda s)^k e^{-\lambda s}}{k!}}{\frac{(\lambda t)^n e^{-\lambda t}}{n!}} = \binom{n}{k} \left(\frac{s}{t}\right)^k \left(1-\frac{s}{t}\right)^{n-k}$

$\therefore Y(s) \mid Y(t)=n \sim \text{Binomial}\left(n, \frac{s}{t}\right) \therefore Var(Y(s) \mid Y(t)=n) = n\frac{s}{t}\left(1-\frac{s}{t}\right).$

(b)(iii) $P[X(t)=k, Y(t)=n]$

$= P[X(t)=k \mid Y(t)=n]\, P[Y(t)=n]$

$= \binom{n}{k} (.2)^k (1-.2)^{n-k} \dfrac{(\lambda t)^n e^{-\lambda t}}{n!}$

$= \dfrac{(.2\lambda t)^k e^{-\lambda t}}{k!} \dfrac{((1-.2)\lambda t)^{n-k}}{(n-k)!}$ for $n = k, k+1, \cdots$

$\therefore P[X(t)=k] = \dfrac{(.2\lambda t)^k e^{-\lambda t}}{k!} \sum_{n=k}^{\infty} \dfrac{((1-.2)\lambda t)^{n-k}}{(n-k)!}$

$= \dfrac{(.2\lambda t)^k e^{-\lambda t}}{k!} e^{(1-.2)\lambda t} = \dfrac{(.2\lambda t)^k e^{-.2\lambda t}}{k!}$ for $k = 0, 1, 2, \cdots$

$\sim \text{Poisson}(.2\lambda t).$

$\therefore$ (i) $E(X(t)) = .2\lambda t$

(ii) $Var(X(t)) = .2\lambda t$

STATISTICS DEPARTMENT
M.S. EXAMINATION

PART II
OPEN BOOK

Tuesday, May 20, 2003

9:00 a.m. - 1:00 p.m.

Statistics Department Computer Lab, SC S152

Instructions: **Complete *four of the five* problems. Each problem counts 25 points. Unless otherwise noted, points are allocated approximately equally to lettered parts of a problem. Spend your time accordingly.**

The web site address for data and program files for this exam is:

http://www.sci.csuhayward.edu/~esuess/msexam/

Begin each problem on a new page. Write the problem number and the page number in the specified locations at the top of each page. Also write your chosen ID code number on every page. Please write only within the black borderlines, leaving at least 1" margins on both sides, top and bottom of each page. Write on one side of the page only.

At the end of this part of the exam you will turn in your answer sheets, but you will keep the question sheets and your scratch paper.

You may use a computer to work any of the problems, but your answers must be handwritten on standard paper provided for the examination. Printers may *not* be used during the exam, and pages printed out by computer may *not* be submitted. As indicated, some problems have data files available on disk.

1. The weights of a random sample of 24 male runners are measured. The sample mean is 60 kilograms (kg). Suppose that the standard deviation is known to be 5 kg.

(a) Describe an appropriate population and sample for this problem. That is, tell the story of this experiment so that it can be analyzed correctly. Name the statistics and parameters mentioned and say whether these are known or unknown. [2 points]

(b) What is the standard error for 60 kg? [2 points]

(c) Is it appropriate to use the Central Limit Theorem here? Why or why not? If it is appropriate, how does the CLT apply? To what does it apply? [4 points]

(d) Give a 95% confidence interval for the mean of the population from which the sample is drawn. [4 points]

(e) Because Americans are less familiar with kilograms, convert to pounds by multiplying by the conversion factor 2.2 pounds per kilogram. What are the new values for the sample mean and standard error of the mean when measured in pounds? [3 points]

(f) In a new sample and with weights measured in pounds, what sample size n would we need to estimate the population mean using a 95%-confidence interval which is centered within a margin of error of 1.5 pounds? [5 points]

(g) Design a test of the hypothesis that the population mean for male runners is under 130 pounds. Use a Type I error of 0.05 and select n so that, if the true mean is 128 pounds, the Type II error is approximately 0.05 as well. [5 points]

2. Two types of laboratory tests (A and B) are used to determine the level of a certain liver enzyme in human blood. It is claimed that both tests accurately measure this enzyme. The question is whether they actually give equivalent results. Suppose that a study to investigate this is conducted at three randomly chosen hospitals. At each hospital blood is drawn from 20 randomly chosen subjects (60 subjects in all). The blood from each subject is divided into two samples, one assayed for the enzyme using each type of test (120 assays in all).

Suppose the enzyme levels obtained are as shown in the table below. For each type of test, the results from the 20 subjects at a hospital are given in the same order across two rows of 10 numbers each. These data are available in the order shown below (but without labels) in the file ENZYME.TXT. They are also displayed in two-column format in ENZYME2.TXT and in a Minitab worksheet ENZYME2.MTW.

TYPE A

Hospital										
1	154	165	149	144	139	160	154	150	146	146
	154	155	143	150	166	157	165	136	149	139
2	152	158	151	157	146	143	143	149	162	152
	136	154	149	150	136	159	155	137	159	157
3	145	135	163	148	152	158	141	159	138	144
	145	158	133	150	147	157	152	152	137	152

TYPE B

Hospital										
1	150	141	160	147	125	144	158	142	135	152
	155	143	165	149	156	153	127	153	146	168
2	128	146	133	135	132	146	128	142	139	156
	132	124	127	128	140	149	133	133	134	129
3	149	152	158	152	167	162	156	159	156	141
	139	165	150	146	155	159	149	157	142	138

(a) Write the most complete ANOVA model supported by these data. Account for all possible interactions. Say whether each main effect is fixed or random, and how many levels there are. If there is nesting, describe it. Also state the assumptions of your model.

(b) Perform the numerical analysis according to your model. Give a table with columns headed Source, DF, SS, MS, F, and P. Discuss any significant effects, explaining their meaning in nontechnical language that could be understood by someone with no background in ANOVA designs.

(c) Perform the appropriate procedures to check assumptions and report your findings.

(d) Would it make any difference in the F-ratios if you changed the model designation of the hospital effect—fixed vs. random? Would this change make any difference in the practical interpretation of your results? Explain.

(e) These are *fake* data. They fail to exhibit an important property that one would expect to see very clearly in *real* data collected according to the story above. Identify what is missing and discuss. (If this were a living room, we would be talking about an elephant lounging on the sofa, not some dust on the coffee table.)

3. Consider a data set containing the cumulative GPA for a random sample of computer science majors at a large university. This data set is located in the text-file GRADES. There are several explanatory variables including High School Mathematics grade (1-10), High School Social Science grade (1-10), High School English grade (1-10), SAT mathematics (1-800), and SAT verbal scores (1-800). Gender is also recorded (m or f). The first few lines of data are as follows:

001	3.32	10	10	10	670	600	m
002	2.26	6	8	5	700	640	m
003	2.35	8	6	8	640	530	m
004	2.08	9	10	7	670	600	m
005	3.38	8	9	8	540	580	m
006	3.29	10	8	8	760	630	m
007	3.21	8	8	7	600	400	m
008	2.00	3	7	6	460	530	m
009	3.18	9	10	8	670	450	m
010	2.34	7	7	6	570	480	m

(a) Read in the file GRADES using a SAS program. (4 pts.)

(b) Ignoring gender, create a model for predicting college GPA containing all 5 other explanatory variables. (4 pts.)

(c) Again ignoring gender, create a smaller model for college GPA containing a subset of the 5 explanatory variables. Describe the method you used to choose this model. Is it is better or worse than the model in (b). (5 pts.)

(d) Discuss the model assumptions using the residuals from (c). Include statistics, hypothesis test(s), and at least one graph that is relevant to model assessment. (6 pts.)

(e) Include gender in the model. Indicate whether the model is improved and whether it is sensible to include an interaction with gender. Discuss why you think this might be true. (6pts.)

4. Consider the number of eggs the Queen Bee lays in a bee hive. Suppose the distribution of the random variable Y = the number of eggs laid by the Queen Bee is $Poisson(\lambda)$. Also suppose the random variable X = number of survivors is of interest to a Biologist.

   A hierarchical model for the number of survivors in terms of the number of eggs laid is defined as:

$$X|Y \sim Binomial(Y, p) \tag{1}$$

   and

$$Y \sim Poisson(\lambda) \tag{2}$$

   where p is the proportion of eggs that result in a surviving bee.

   (a) What is the expected value of Y? What is the variance of Y?

   (b) Early in the spring the Queen Bee may lay several hundred eggs per day. Suppose $\lambda = 300$. Compute the probability that the number of eggs laid is greater than 320, $P(Y > 320)$. Because λ is so large, what distribution could be used to approximate this probability calculation? Using that distribution compute the approximate probability of $P(Y > 320)$.

   (c) What is the expected value of $X|Y = y$? What is the variance of $X|Y = y$?

   (d) Assuming the survival rate of bees is $p = 0.9$ and the number of eggs laid is $Y = 300$, compute the probability that the number of survivors is greater than 280, $P(X > 280|Y = 300)$. Since Y is so large, what distribution could be used to approximate this probability calculation?

   (e) Suggest a different hierarchical model for the number of survivors in terms of the number of eggs laid considering the large sample distributions.

5) Suppose Factor A is fixed with 2 levels, Factor B (nested in A) is random with 3 levels and 3 observations are taken at each of the 6 combinations of A and B.

This model is usually written as $Y_{ijk} = \mu + \alpha_i + \beta_{j(i)} + \varepsilon_{ijk}$ $(i = 1, 2; j, k = 1, 2, 3)$.

(a) What are the usual assumptions for this model?

(b) Let $Y_{1\bullet\bullet} = \sum_{j=1}^{3}\sum_{k=1}^{3} Y_{1jk}$. Show that $Y_{1\bullet\bullet} = 9\mu + 9\alpha_1 + 3(\beta_{1(1)} + \beta_{2(1)} + \beta_{3(1)}) + \varepsilon_{1\bullet\bullet}$, where $\varepsilon_{1\bullet\bullet} = \sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{1jk}$.

(c) Show that $\text{var}(Y_{1\bullet\bullet}) = 9(3\sigma_\beta^2) + 9\sigma^2$, where σ_β^2 is the common variance of $\{\beta_{j(i)}\}$ and σ^2 is the common variance of $\{\varepsilon_{ijk}\}$.

(d) Letting $\bar{Y}_{1\bullet\bullet} = \frac{Y_{1\bullet\bullet}}{9}$, show that $\text{var}(\bar{Y}_{1\bullet\bullet}) = \frac{1}{9}[3\sigma_\beta^2 + \sigma^2]$.

(e) Obtain $E(\bar{Y}_{1\bullet\bullet} - \bar{Y}_{2\bullet\bullet})$ and $\text{var}(\bar{Y}_{1\bullet\bullet} - \bar{Y}_{2\bullet\bullet})$.

(f) Suppose we compare 2 drugs, with 3 randomly selected batches from each drug. We randomly select 3 individuals for each combination of drug and batch; measure Y = improvement for each individual. The data are given below. **The corresponding file is named "improvement" and it is saved as Minitab, Excel and 'dat' files.** For A = drug and B = batch, from the ANOVA table we obtain $E(\text{MS[B(A)]}) = 3\sigma_\beta^2 + \sigma^2$, df[B(A)] = 4, MS[B(A)] = .366. Test H_0: $\alpha_1 = \alpha_2$.

(g) Explain in words, without any technical jargon, what conclusion can be drawn based on the result of the hypothesis test in part (f).

obs.	drug	batch	improvement
1	1	1	1.257
2	1	1	1.415
3	1	1	2.172
4	1	2	2.743
5	1	2	2.250
6	1	2	2.179
7	1	3	1.000
8	1	3	1.657
9	1	3	2.107
10	2	1	6.007
11	2	1	6.457
12	2	1	5.329
13	2	2	5.936
14	2	2	6.493
15	2	2	5.693
16	2	3	6.857
17	2	3	5.550
18	2	3	6.500

Part II

Solution #1

1. The weights of a random sample of 24 male runners are measured. The sample mean is 60 kilograms (kg). Suppose that the standard deviation is known to be 5 kg.

a. Describe an appropriate population and sample for this problem. Name the statistics and parameters mentioned and whether these are known or unknown. (2pts.)

The population is all male runners completing marathons in top fifty, worldwide in 2002. The sample is a randomly selected sample taken from records of all marathon runners who were male. The statistic given is that the sample weight, $\bar{x}$ **is 60 kg. The standard deviation 5 kg is the population parameter σ.**

b. What is the standard error for 60 kilograms? (2pts.)

$\sigma_{\bar{x}} = 5/\sqrt{24}$ **= 1.02**

c. Why or why not is it appropriate to use the Central Limit Theorem here? How and to what does the CLT apply. (3pts.)

We are given σ, therefore μ exists. We have a sample of size 24 that is fairly large. We don't know the distribution of the original measurements. But we can assume that the sample mean is approximately normal using the CLT, so we assume that $\bar{x}$ is approximately normal with unknown mean μ and standard deviation 1.02.

d. Give a 95% confidence interval for the mean of the population from which the sample is drawn. (3pts.)

Applying the CLT, a 95% confidence interval for μ is the interval 60±1.96*1.02 in kilograms.

e. Since Americans are less familiar with kilograms, we wish to convert to pounds by multiplying by the conversion factor 2.2 pounds per kilogram. What are the new values for mean and standard error of the mean measurements in pounds? (3pts.)

$\bar{x}$***2.2=132 pounds with standard error 2.2*** $\sigma_{\bar{x}} = 2.2*5/\sqrt{24}$ **= 2.2*1.02=2.24 pounds**

f. What n would we need in a new sample to estimate the mean using a 95% confidence interval which when measured in pounds is centered within 1.5 pounds of the mean? (3pts.)

1.5=1.96*2.2*5/√n so n must be 207.

g. Design a hypothesis test for the mean, measured in pounds, that asserts the population mean for male runners is under 130 pounds. Use a type I error of .05 and select n so that if the true mean is 128 pounds, the type II error is approximately .05 as well. (4pts.)

The critical value for a .05 test for μ would be 130-1.645*2.2*5/√n. Values above this one would indicate fail to reject Ho and values below would indicate reject Ho. If the true μ is 128, then we would tend to observe values above 130-1.645*2.2*5/√n with probability set at .05. Converting back to a standard normal we have (2-1.645*2.2*5/√n)/ 2.2*5/√n =1.645. Solve for n. n=328.

Answers

(a) Model: $Y_{ijk} = \mu + H_i + S(H)_{j(i)} + \tau_k + (H\tau)_{ik} + e_{ijk}$, where $i = 1,2$, $j = 1, ...,20$, $k = 1,2,3$. Distributions: H_i iid $N(0, \sigma_H^2)$, $S(H)_{j(i)}$ iid $N(0, \sigma_S^2)$, e_{ijk} iid $N(0, \sigma^2)$. Optionally, we use the restricted model here, so that $\Sigma_k (H\tau)_{ik} = 0$, but similar results are obtained without this assumption. Main effects: Hospitals random, Subjects random and nested within Hospitals, and test Types fixed. (Major points off for omitting interaction; fixed/random, crossed/nested errors. Minor points off for skipping other details.)

(b)

Analysis of Variance for Enz

Source	DF	SS	MS	F	P
Type	1	550.41	550.41	0.61	0.518
Hosp	2	1363.55	681.78	7.83	0.001
Type*Hosp	2	1819.12	909.56	12.46	0.000
Subj(Hosp)	57	4960.88	87.03	1.19	0.254
Error	57	4159.98	72.98		
Total	119	12853.93			

Source	Variance component	Error term	Expected Mean Square for Each Term (using restricted model)
1 Type		3	(5) + 20(3) + 60Q[1]
2 Hosp	14.869	4	(5) + 2(4) + 40(2)
3 Type*Hosp	41.829	5	(5) + 20(3)
4 Subj(Hosp)	7.025	5	(5) + 2(4)
5 Error	72.982		(5)

Hospital*Type very highly significant. Also disorderly. We can take no comfort in failure to reject the Type effect, test methods may be implemented differently at different hospitals. (Major points off for incorrect interpretation of main effects ignoring interaction.)

Rows: Hosp Columns: Type

	1	2	All
1	151.05	148.45	149.75
2	150.25	135.70	142.98
3	148.30	152.60	150.45
All	149.87	145.58	147.73

Cell Contents --
Enz:Mean

(c) A normal probability plot of residuals from the model seems satisfactorily close to linear. Also, a plot of residuals in the order shown in the problem reveals no heteroscedastic pattern. One could also do formal tests. (Major points off for checking neither normality nor homoscedasticity; minor penalty for skipping one.)

(d) Yes, the F ratios are different. The interaction and the small number of hospitals keeps Type from having a small P-value (tested against interaction). But if we take the hospitals to be fixed effects, then Type has a very small P-value (tested against error). [Unrestricted model: Random Hospital requires synthetic test, Type not significant; fixed hospital gives same F-ratios as for restricted model.]

Analysis of Variance for Enz

Source	DF	SS	MS	F	P
Type	1	550.41	550.41	7.54	0.008
Hosp	2	1363.55	681.78	7.83	0.001
Type*Hosp	2	1819.12	909.56	12.46	0.000
Subj(Hosp)	57	4960.88	87.03	1.19	0.254
Error	57	4159.98	72.98		
Total	119	12853.93			

Source	Variance component	Error term	Expected Mean Square for Each Term (using restricted model)
1 Type		5	(5) + 60Q[1]
2 Hosp		4	(5) + 2(4) + 40Q[2]
3 Type*Hosp		5	(5) + 20Q[3]
4 Subj(Hosp)	7.025	5	(5) + 2(4)
5 Error	72.982		(5)

However, the disorderly interaction appears in all cases, preventing a clear interpretation of either main effect, so the real-world interpretation is somewhat similar whether hospitals are taken as fixed or random [also restricted or unrestricted].

(e) In *any* experiment with randomly chosen human subjects, one expects the Subject effect to be very highly significant. Otherwise, why use so many people?! Here, it isn't anywhere near significant. (Elephants are large so they can look different from different angles. Equivalently, observe that, for each of the three hospitals, the 20 paired A and B measurements have no significant correlation.) [A related issue: anyone who knows about liver enzymes would be astonished not to find among the 60 randomly chosen subjects a few outliers with A and B assays both very high. Result: both correlation and nonnormal residuals.]

Solution #3

4. Consider a data set containing the cumulative GPA for a random sample of computer science majors at a large university. This data is located in the file GRADES. There are several explanatory variables including High School Mathematics grade, High School Social Science grade, and High School English grade, SAT Mathematics and SAT verbal scores. Gender is also recorded.

a. Read in the file Grades using a SAS program. (3 pts.)

FOR EXAMPLE:

```
data grades;
infile 'c:/temp/CSDATA for regression.txt';
input student gpa HSMATH HSSS HSENG SATMATH SATVERB gender $;
title 'problem a';
/*
001    3.32     10     10     10     670     600     m
002    2.26      6      8      5     700     640     m
003    2.35      8      6      8     640     530     m
004    2.08      9     10      7     670     600     m
005    3.38      8      9      8     540     580     m
006    3.29     10      8      8     760     630     m

*/;
```

b. Ignoring gender, create a model for college GPA containing all 5 explanatory variables. (3 pts.)

Using the following code we get the solution below from SAS:

```
title 'problem b';
proc reg;
model gpa=HSMATH HSSS HSENG SATMATH SATVERB/press r; run;
```

problem b 18:04 Monday, May 19, 2003 1

The REG Procedure
Model: MODEL1
Dependent Variable: gpa

Analysis of Variance

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	5	28.64364	5.72873	11.69	<.0001
Error	218	106.81914	0.49000		
Corrected Total	223	135.46279			

Root MSE	0.70000	R-Square	0.2115
Dependent Mean	2.63522	Adj R-Sq	0.1934

Coeff Var 26.56311

Parameter Estimates

Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Intercept	1	0.32672	0.40000	0.82	0.4149
HSMATH	1	0.14596	0.03926	3.72	0.0003
HSSS	1	0.03591	0.03780	0.95	0.3432
HSENG	1	0.05529	0.03957	1.40	0.1637
SATMATH	1	0.00094359	0.00068566	1.38	0.1702
SATVERB	1	-0.00040785	0.00059189	-0.69	0.4915

Therefore the model is:
gpa=.327+.146*HSMATH+.036*HSSS+.055*HSENG+,00094*SASMATH-.00041*SATVERB

c. Again ignoring gender, create a smaller model for college GPA containing a subset of the 5 exploratory variables. Describe the method you used to choose this model and whether it is better or worse than the model in b. (4 pts.)

Using the code below or some other similar selection method:

```
proc reg;
model gpa=HSMATH HSSS HSENG SATMATH SATVERB/selection=cp rmse adjrsq;
title 'problem c'; run;
proc reg;
model gpa=HSMATH HSENG SATMATH/press r;
output out=resids student=student p=fits;
proc univariate data=resids plot normal;
var student; run;
```

According to the output below, I would choose model 2 and include HSMATH, HSENG, and SATMATH in the model. Although none of these models are very good, this one has the best adjusted r-squared. Model 1 has all coefficients significant while this 3 variable model does not. Only 11 of the 172 studentized residuals are outside ± 2 and none are outside ± 3. Three of the four normal tests reject normal errors, however. Even so, this is a simpler model than the one in part b. The smaller model also has a lower press statistic indicating better prediction.

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The REG Procedure
Model: MODEL1
Dependent Variable: gpa

C(p) Selection Method

Number in Model	C(p)	R-Square	Adjusted R-Square	Root MSE	Variables in Model
2	2.7350	0.2016	0.1943	0.69958	HSMATH HSENG
3	3.2512	0.2069	0.1961	0.69880	HSMATH HSENG SATMATH
2	3.2585	0.1997	0.1924	0.70041	HSMATH HSSS
1	3.7832	0.1905	0.1869	0.70280	HSMATH
3	3.9007	0.2046	0.1937	0.69984	HSMATH HSSS HSENG
3	4.1598	0.2036	0.1928	0.70025	HSMATH HSSS SATMATH
4	4.4748	0.2097	0.1953	0.69916	HSMATH HSSS HSENG SATMATH
3	4.7348	0.2016	0.1907	0.70117	HSMATH HSENG SATVERB
2	4.7775	0.1942	0.1869	0.70281	HSMATH SATMATH
4	4.9023	0.2082	0.1937	0.69984	HSMATH HSENG SATMATH SATVERB
3	5.2570	0.1997	0.1888	0.70199	HSMATH HSSS SATVERB
2	5.6893	0.1909	0.1835	0.70424	HSMATH SATVERB
4	5.8939	0.2046	0.1901	0.70142	HSMATH HSSS HSENG SATVERB
4	5.9527	0.2044	0.1899	0.70152	HSMATH HSSS SATMATH SATVERB
5	6.0000	0.2115	0.1934	0.70000	HSMATH HSSS HSENG SATMATH SATVERB
3	6.7619	0.1942	0.1832	0.70438	HSMATH SATMATH SATVERB
3	17.2321	0.1564	0.1448	0.72074	HSSS HSENG SATMATH
4	17.8214	0.1615	0.1461	0.72020	HSSS HSENG SATMATH SATVERB
2	19.7248	0.1401	0.1323	0.72600	HSSS SATMATH
3	20.9845	0.1428	0.1311	0.72652	HSSS SATMATH SATVERB
2	21.7757	0.1327	0.1248	0.72913	HSENG SATMATH
3	22.7150	0.1365	0.1247	0.72916	HSENG SATMATH SATVERB
2	24.4473	0.1230	0.1151	0.73318	HSSS HSENG
3	26.3825	0.1233	0.1113	0.73474	HSSS HSENG SATVERB
1	26.4555	0.1085	0.1045	0.73755	HSSS
2	28.2181	0.1094	0.1013	0.73886	HSSS SATVERB
1	33.3667	0.0835	0.0794	0.74782	HSENG
2	34.7962	0.0856	0.0773	0.74866	HSENG SATVERB
1	38.9406	0.0634	0.0591	0.75600	SATMATH
2	40.9387	0.0634	0.0549	0.75770	SATMATH SATVERB
1	52.8331	0.0131	0.0087	0.77601	SATVERB

d. Discuss the model assumptions using the residuals from c. Include statistics, hypothesis test(s), and at least one graph that is relevant to model assessment. (5 pts.)

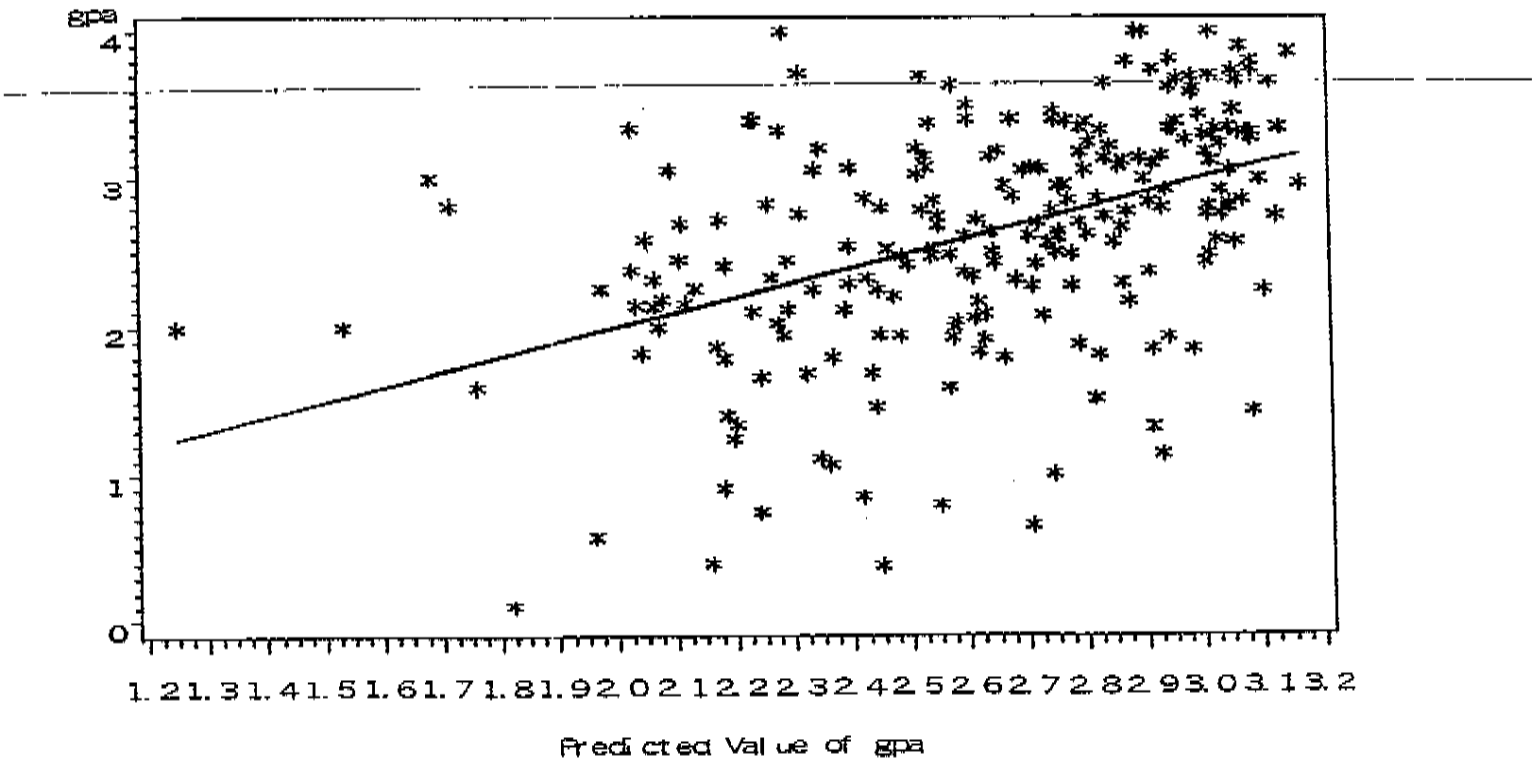
A number of ideas could be used here such as the 4 normal tests, the press statistic for comparison, etc.

Several useful graphs and tests come from the code below:

```
proc univariate data=resids plot normal;
var student;run;

proc gplot data=resids;
symbol1 v=star cv=blue i=none;
symbol2 v=none i=line ci=black;
plot gpa*fits=1 fits*fits=2/overlay;
title 'problem d';
```

Solid line shows fit versus fit. A model with nearly perfect fit would fall much nearer the line. Some important information is missing from this model.



e. Include gender in the model. Indicate whether the model is improved and whether it is sensible to include an interaction with gender and why you think this might be true. (5pts.)

Several models fit approximately the same. Again, I will choose the model with HSENG HSMATH SATMATH plus sex. We have the same problem with lack of normality. We have lowered the Press error so improved predictability somewhat. Many reasonable comparisons could be made--but we are still missing a big part of the picture. Adjusted r-square has increased slightly to .20 and three of the four coefficients are significant. SATMATH is not significant with these other terms in the equation.

One could also consider heteroscedasticity or multicolinearity or including powers or interaction terms. The best interaction term is sex*satmath which increases the adjusted r-squared to .2053 with all three coefficients significant. But there are still problems with the residuals being non-normal by three of the four tests.

When squares of the hs grades are included, hsss becomes important:

C(p) Selection Method

Number in Model	C(p)	R-Square	Adjusted R-Square	Root MSE	Variables in Model
4	1.6853	0.2505	0.2368	0.68089	HSSS sexhseng hsmathsq hssssq
3	1.8439	0.2430	0.2327	0.68273	HSSS hsmathsq hssssq
4	1.9032	0.2497	0.2360	0.68123	HSSS sex hsmathsq hssssq
4	2.0149	0.2494	0.2356	0.68141	HSSS sexsatmath hsmathsq hssssq
5	2.0661	0.2561	0.2391	0.67988	HSSS sex sexhsmath hsmathsq hssssq
5	2.0977	0.2560	0.2389	0.67993	HSSS HSENG sex hsmathsq hssssq
4	2.1007	0.2491	0.2353	0.68154	HSSS sexhsss hsmathsq hssssq
6	2.2298	0.2625	0.2421	0.67852	HSSS HSENG sex sexhsmath hsmathsq hssssq
5	2.2732	0.2554	0.2383	0.68021	HSSS HSENG sexsatmath hsmathsq hssssq

Using the first model here with HSSS and hssssq as well as sexhseng and hsmathsq, the adjusted r-squared is raised to .237. The other difficulties are not removed such as lack of normality. The press statistic is reduced to 106 indicating even more predictability. A possible program follows:

```
data grades;
infile 'c:/temp/CSDATA for regression.txt';
input student gpa HSMATH HSSS HSENG SATMATH SATVERB gender $;
title 'problem a';
/*
001     3.32      10      10      10      670      600      m
002     2.26       6       8       5      700      640      m
003     2.35       8       6       8      640      530      m
004     2.00       5      10       7      670      600      f

223     2.59       5       4       7      630      470      f
224     2.28       3       3       5      550      460      f
*/;
title 'problem b';
proc reg;
model gpa=HSMATH HSSS HSENG SATMATH SATVERB/press r; run;
```

```
    reg;
 el gpa=HSMATH HSSS HSENG SATMATH SATVERB/selection=cp rmse adjrsq;
proc reg;
odel gpa=HSMATH HSENG SATMATH/press r;
utput out=resids student=student p=fits;
itle 'problem c';

roc univariate data=resids plot normal;
ar student;run;
roc gplot data=resids;
ymbol1 v=star cv=blue i=none;
ymbol2 v=none i=line ci=black;
lot gpa*fits=1 fits*fits=2/overlay;
itle 'problem d'; run;

data grades; set grades; sex=0;  if gender='m' then sex=1;
sexhsmath=sex*hsmath; sexhseng=sex*hseng; sexhsss=sex*hsss;
sexsatmath=sex*satmath; sexsatverb=sex*satverb; satmathverb=satmath*satverb;
smathsq=hsmath*hsmath; hsengsq=hseng*hseng; hssssq=hsss*hsss;
run;
proc reg;
model gpa=HSMATH HSSS HSENG SATMATH SATVERB sex/selection=cp rmse adjrsq; run;
proc reg;
model gpa=HSMATH HSENG SATMATH sex/press r;
output out=resids student=student p=fits;
title 'problem e';

p  univariate data=resids plot normal;
   student;run;
  c gplot data=resids;
plot gpa*fits=1 fits*fits=2/overlay; run;

proc reg;
model gpa=HSMATH HSSS HSENG SATMATH SATVERB sex
sexhsmath sexhseng sexhsss sexsatmath sexsatverb/selection=cp rmse adjrsq; run;
proc reg;
model gpa=HSMATH HSENG sexSATMATH/press r;
output out=resids student=student p=fits;
title 'problem e including interaction';

proc univariate data=resids plot normal;
var student;run;
proc gplot data=resids;
plot gpa*fits=1 fits*fits=2/overlay; run;

proc reg;
model gpa=HSMATH HSSS HSENG SATMATH SATVERB sex
sexhsmath sexhseng sexhsss sexsatmath sexsatverb
satmathverb hsmathsq hsengsq hssssq/selection=cp rmse adjrsq; run;
title 'problem e including interaction and squares';
proc reg;
model gpa=HSSS sexhseng hsmathsq hssssq/press r;
output out=resids student=student p=fits;
t  le 'problem e including interaction and squares';

  c univariate data=resids plot normal;
 ar student;run;
proc gplot data=resids;
plot gpa*fits=1 fits*fits=2/overlay;  run;
```

Solution # 4

4.

a. The mean and variance of the Poisson distribution is lambda.

```
mu.y <- lambda
var.y <- lambda
```

b. Using Splus.

```
# exact calculation

lambda <- 300

1 - ppois(320, lambda = 300)
```

Ans: 0.1190045

```
# normal approximation

mu.y <- lambda
var.y <- lambda
sigma.y <- sqrt(lambda)

1 - pnorm(320, mu.y, sigma.y)
```

Ans: 0.1241065

c.

```
mu.xgy <- y*p
var.xgy <- y*p*(1-p)
```

d.

```
# exact calculation

p <- 0.9
y <- 300

1 - pbinom(280,y,p)
```

Ans: 0.01711813

```
# normal approximation

mu.xgy <- y*p
var.xgy <- y*p*(1-p)
sigma.xgy <- sqrt(y*p*(1-p))

1 - pnorm(280,mu.xgy,sigma.xgy)
```

Ans: 0.02714591

e.

$Y \sim N(\mu_y = \lambda, \sigma^2_y = \lambda)$

$X|Y \sim N(\mu_{x|y} = yp, \sigma^2_{x|y} = yp(1-p))$

Solution #5

(5g) Suppose Factor A is fixed with 2 levels, Factor B (nested in A) is random with 3 levels and 3 observations are taken at each of the 6 combinations of A and B.

This model is usually written as $Y_{ijk} = \mu + \alpha_i + \beta_{j(i)} + \varepsilon_{ijk}$ $(i = 1, 2; j, k = 1, 2, 3)$.

(a) What are the usual assumptions for this model?

(b) Let $Y_{1..} = \sum_{j=1}^{3}\sum_{k=1}^{3} Y_{1jk}$. Show that $Y_{1..} = 9\mu + 9\alpha_1 + 3(\beta_{1(1)} + \beta_{2(1)} + \beta_{3(1)}) + \varepsilon_{1..}$, where $\varepsilon_{1..} = \sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{1jk}$.

(c) Show that $\text{var}(Y_{1..}) = 9(3\sigma_\beta^2) + 9\sigma^2$, where σ_β^2 is the common variance of $\{\beta_{j(i)}\}$ and σ^2 is the common variance of $\{\varepsilon_{ijk}\}$.

(d) Letting $\bar{Y}_{1..} = \frac{Y_{1..}}{9}$, show that $\text{var}(\bar{Y}_{1..}) = \frac{1}{9}[3\sigma_\beta^2 + \sigma^2]$.

(e) Obtain $E(\bar{Y}_{1..} - \bar{Y}_{2..})$ and $\text{var}(\bar{Y}_{1..} - \bar{Y}_{2..})$.

(f) Suppose we compare 2 drugs, with 3 randomly selected batches from each drug. We randomly select 3 individuals for each combination of drug and batch and measure Y = improvement for each individual. The data is given below. For A = drug, B = batch, from the ANOVA table, we obtain that $E(MS(B(A))) = 3\sigma_\beta^2 + \sigma^2$, df(B(A)) = 4, MS(B(A)) = .366. Test H_0: $\alpha_1 = \alpha_2$.

(g) In words, without any technical jargon, what conclusion can be made based on the result of the hypothesis test in part (f).

obs.	drug	batch	improvement
1	1	1	1.257
2	1	1	1.415
3	1	1	2.172
4	1	2	2.743
5	1	2	2.250
6	1	2	2.179
7	1	3	1.000
8	1	3	1.657
9	1	3	2.107
10	2	1	6.007
11	2	1	6.457
12	2	1	5.329
13	2	2	5.936
14	2	2	6.493
15	2	2	5.693
16	2	3	6.857
17	2	3	5.550
18	2	3	6.500

Solution (a) $\sum \alpha_i = 0$; $\{\beta_{j(i)}\}$ are normal with mean=0, common variance σ_β^2, mutually independent and independent of $\{\varepsilon_{ijk}\}$; $\{\varepsilon_{ijk}\}$ are normal with mean=0, common variance σ^2 and are mutually independent.

(b) by definition of Y_{ijk}

(c) from (b), assumptions and elementary properties of variance.

(d) from (c) and elementary properties of variance.

(e) $E(\bar{Y}_{1..} - \bar{Y}_{2..}) = (\mu + \alpha_1) - (\mu + \alpha_2) = \alpha_1 - \alpha_2$

$\text{var}(\bar{Y}_{1..} - \bar{Y}_{2..}) = \text{variance}(\bar{Y}_{1..}) + \text{var}(\bar{Y}_{2..})$

$= \frac{2}{9}[3\sigma_\beta^2 + \sigma^2]$

(f) Use $t = (\bar{Y}_{1..} - \bar{Y}_{2..}) \Big/ \sqrt{\frac{2}{9} E(MS(B(A)))} = \underbrace{(1.864 - 6.091)}_{-4.227} \Big/ \sqrt{\frac{2}{9}(.366)} = 14.822$

follows a t with 4 df and is significant for even very small α. Thus conclude that $\alpha_1 \neq \alpha_2$.

g) The improvements for the two drugs (averaged over batches and individuals) is not the same for both drugs.

(Note: if one had the opportunity to look at the ANOVA table, one would not conclude that there is variability from batch to batch, leading to a possibly stronger conclusion)

improvement

Solution (continued)

Results for: improvement-MTW

ANOVA: improvement versus drug, batch

Factor	Type	Levels	Values
drug	fixed	2	1 2
batch(drug)	random	3	1 2 3

Analysis of Variance for improvem

Source	DF	SS	MS	F	P
drug	1	80.400	80.400	219.44	0.000
batch(drug)	4	1.466	0.366	1.38	0.298
Error	12	3.181	0.265		
Total	17	85.046			

Source		Variance component	Error term	Expected Mean Square for Each Term (using unrestricted model)
1	drug		2	(3) + 3(2) + Q[1]
2	batch(drug)	0.03377	3	(3) + 3(2)
3	Error	0.26509		(3)

Results for: improvement.MTW

Data Display

obs.	drug	batch	improvement	
1	1	1	1.257	
2	1	1	1.415	
3	1	1	2.172	
4	1	2	2.743	
5	1	2	2.250	*
6	1	2	2.179	
7	1	3	1.000	
8	1	3	1.657	
9	1	3	2.107	
10	2	1	6.007	
11	2	1	6.457	
12	2	1	5.329	
13	2	2	5.936	
14	2	2	6.493	
15	2	2	5.693	
16	2	3	6.857	
17	2	3	5.550	
18	2	3	6.500	