Introduction to Mathematical Modeling

Difference Equations, Differential Equations, & Linear Algebra

(The First Course of a Two-Semester Sequence)

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Chapter 0

To the Student and the Instructor

This document contains lecture notes, classroom activities, examples, and challenge problems specifically designed for a first semester of differential equations and linear algebra taught with a focus on mathematical modelingThe content herein is written and maintained by Dr. Eric Sullivan of Carroll College. Problems were either created by Dr. Sullivan, the Carroll Mathematics Department faculty, part of NSF Project Mathquest, part of the Active Calculus text, or come from other sources and are either cited directly or cited in the LaTeX source code for the document (and are hence purposefully invisible to the student).

0.1 An Inquiry Based Approach

Problem 0.1 (Setting The Stage).

- Get in groups of size 3-4.
- Group members should introduce themselves.
- For each of the questions that follow I will ask you to:
  1. **Think** about a possible answer on your own
  2. **Discuss** your answers with the rest of the group
  3. **Share** a summary of each group's discussion

Questions:

Question #1: What are the goals of a university education?

Question #2: How does a person learn something new?

Question #3: What do you reasonably expect to remember from your courses in 20 years?

Question #4: What is the value of making mistakes in the learning process?

Question #5: How do we create a safe environment where risk taking is encouraged and productive failure is valued?

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(The previous problem is inspired by Dana Ernst's first day activity in IBL activity titled: Setting the Stage.)

> "Any creative endeavor is built in the ash heap of failure."
> –Michael Starbird

This material is written with an Inquiry-Based Learning (IBL) flavor. In that sense, this document could be used as a stand-alone set of materials for the course but these notes are not a *traditional textbook* containing all of the expected theorems, proofs, examples, and exposition. The students are encouraged to work through problems and homework, present their findings, and work together when appropriate. You will find that this document contains collections of problems with only minimal interweaving exposition. It is expected that you do every one of the problems and then use other more traditional texts as a backup when you are stuck. Let me say that again: this is not the only set of material for the course. Your brain, your peers, and the books linked in the next section are your best resources when you are stuck.

To learn more about IBL go to http://www.inquirybasedlearning.org/about/. The long and short of it is that the students in the class are the ones that are doing the work; building models, proving theorems, writing code, working problems, leading discussions, and pushing the pace. The instructor acts as a guide who only steps in to redirect conversations or to provide necessary insight. If you are a student using this material you have the following jobs:

1. Fight! You will have to fight hard to work through this material. The fight is exactly what we're after since it is ultimately what leads to innovative thinking.

2. Screw Up! More accurately, don't be afraid to screw up. You should write code, work problems, and prove theorems then be completely unafraid to scrap what you've done and redo it from scratch. Learning this material is most definitely a non-linear path.* Embrace this!

3. Collaborate! You should collaborate with your peers with the following caveats: (a) When you are done collaborating you should go your separate ways. When you write your solution you should have no written (or digital) record of your collaboration. (b) The internet is not a collaborator. Use of the internet to help solve these problems robs you of the most important part of this class; the chance for original though.

4. Enjoy! Part of the fun of IBL is that you get to experience what it is like to think like a true mathematician / scientist. It takes hard work but ultimately this should be fun!

*Pun intended: our goal, after all, is really to understand that linear algebra is the glue that holds mathematics together.

0.2 Online Texts and Other Resources

If you are looking for online textbooks for linear algebra and differential equations I can point you to a few. Some of the following online resources may be a good place to help you when you're stuck but they will definitely say things a bit differently. Use these resources wisely.

- The book *Differential Equations with Linear Algebra, An inquiry based approach to learning* is a nice collection of notes covering much of the material that we cover in our class. The order is a bit different but the notes are well done.
  content.byui.edu/file/664390b8-e9cc-43a4-9f3c-70362f8b9735/1/316-IBL%20(2013Spring).pdf

- The ODE Project by Thomas Juson is a nice online text that covers many (but not all) of the topics that we cover in differential equations.
  faculty.sfasu.edu/judsontw/ode/html/odeproject.html

- Elementary Differential Equations by William Trench. This book contains everything(!) you would ever want to look up for ordinary differential equations. It is a great resource to look up ODE techniques.
  ramanujan.math.trinity.edu/wtrench/texts/TRENCH_DIFF_EQNS_I.PDF

- A First Course in Linear Algebra by Robert Beezer. This book is very thorough and covers everything that we do in linear algebra and much more.
  linear.ups.edu/fcla/index.html

- Linear Algebra Workbook by TJ Hitchman. This is a workbook for Dr. Hitchman's class at U. Northern Iowa. Even though it is only a "workbook" it contains some nice explanations and it has embedded executable code for some problems.
  theronhitchman.github.io/linear-algebra/course-materials/workbook/LinAlgWorkbook.html

0.3 To the Instructor

If you are an instructor wishing to use these materials then I only ask that you adhere to the Creative Commons license. You are welcome to use, distribute, and remix these materials for your own purposes. Thanks for considering my materials for your course!

My typical use of these materials are to let the students tackle problems in small groups during class time and to intervene when more explanation appears to be necessary or if the students appear to be missing the deeper connections behind problems. The course that I have in mind for these materials is a first semester of differential equations and linear algebra taught from the standpoint of mathematical modeling. As such, this is not a complete collection of materials for either differential equations or linear algebra in isolation. We discuss matrix operations, Gaussian elimination, the eigenvalue problem, first order linear homogeneous and non-homogeneous differential equations, and second order homogeneous differential equations. In the second course we will expand upon these ideas and include more advanced topics.

Many of the theorems in the text come without a proof. If the theorem is followed by the statement “prove the previous theorem” then I expect the students to have the skill to prove that theorem and to do so with the help of their small group. However, this course is not intended to be a proof-based mathematics course so several theorems are stated without rigorous proof. If you are looking for a proof-based linear algebra or differential equations course then I believe that these notes will not suffice. I have, however, tried to give thought provoking problems throughout so that the students can engage with the material at a level higher than just the mechanics of differential equations and linear algebra. There are also several routine exercises throughout the notes that will allow students to practice mechanical skills.

There is a toggle switch in the LaTeX code that allows you to turn on and off the solutions to problems. The line of code

```
\def\ShowSoln{0}
```

is a switch that, when set to 0, turns the solutions off and when set to 1 turns the solutions on. Just re-compile (`pdflatex`) the document to display the solutions. I typically do not show the solutions to the students while they're learning the material.

Chapter 1

Fundamental Notions from Calculus

Welcome to the mathematical modeling class. We'll start the class with a brief review of the most basic ideas and notions from calculus. If you have never taken a Calculus course before then you should consider first taking a formal Calculus course before tackling this class. We only need a few of the main ideas for this class so what you'll find in this chapter is necessarily brief.

1.1 Sections from Active Calculus

The Active Calculus Textbook is a wonderful online resource that stands in place of this chapter. We will only discuss a few select sections and you can find the relevant links below. If you need further reading to brush up on Calculus you should use the Active Calculus text.

1. Active Calculus Section 1.1: How do we measure velocity?
2. Active Calculus Section 1.2: The notion of a limit
3. Active Calculus Section 1.4 The Derivative Function
4. Active Calculus Section 2.1: Elementary derivative rules
5. Active Calculus Section 2.2: The sine and cosine functions
6. Active Calculus Section 2.3: The product and quotient rules
7. Active Calculus Section 2.4: The chain rule
8. Active Calculus Section 4.3: The definite integral
9. Active Calculus Section 4.4: The Fundamental Theorem of Calculus

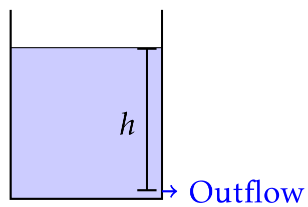
1.2 Lab Activities for Calculus and Modeling

What remains in this chapter of these notes are several laboratory exercises meant to support your review of Calculus and to introduce you to the basic notions of mathematical modeling.

Lab Exploration 1.1. Water is being drained from a hole near the bottom of a cylindrical tank. According to Torrecelli's Law it can be shown that the rate at which the height h of the water changes is proportional to the square root of the height. This can be written with average rates of change as:

$$\text{average rate of change of the height} = \frac{\Delta h}{\Delta t} \approx K \cdot \sqrt{h} \tag{1.1}$$

where K is a constant that depends on gravity as well as the size of the hole and the shape of the tank.



(a) Using the video demonstration of Torrecelli's Law found here: https://www.youtube.com/watch?v=gsNdsuQ1ZCo&app=desktop and, using the pause button to your advantage, create a table of time vs. the height of the water in the container. Use as many data points as you feel necessary.

Time (sec)	Height (cm)
⋮	⋮

(b) Use your data to estimate the value of K in the experiment shown in the video. Be sure to include a discussion of units of K and discuss which parts of this experiment are being described by K?
(Hint: Given equation (1.1), what should a plot of $\sqrt{h}$ (on the x axis) vs $\frac{\Delta h}{\Delta t}$ (on the y axis) look like? How would you find K from this plot?)

(c) Two more experiments were performed with different cylinders and different sized drain holes. Find the values of K for each of these experiments, and from the data make comparisons between the sizes of the cylinders and the sizes of the holes for the three experiments.

Experiment #1			Experiment #2	
Time (sec)	Height (cm)		Time (sec)	Height (cm)
0	35		0	13
8	30		0.58	12
16	25		1.35	11
25	20		1.95	10
35.5	15		2.85	9
48.7	10		3.65	8
66.8	5		4.55	7
			5.55	6
			6.55	5
			7.55	4
			8.55	3
			10.45	2
			13.45	1

(d) Discussion: What would happen if this experiment were run on a different, non-cylindrical, tank? Provide a detail

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Lab Exploration 1.2. A farmer with large land holdings has historically grown a wide variety of crops. With the price of ethanol fuel rising, he decides that it would be prudent to devote more and more of his acreage to producing corn. As he grows more and more corn, he learns efficiencies that increase his yield per acre. In the present year, he used 7000 acres of his land to grow corn, and that land had an average yield of 170 bushels per acre. At the current time, he plans to increase his number of acres devoted to growing corn at a rate of 600 acres/year, and he expects that right now his average yield is increasing at a rate of 8 bushels per acre per year. Use this information to answer the following questions.

(a) Say that the present year is $t = 0$, that $A(t)$ denotes the number of acres the farmer devotes to growing corn in year t, $Y(t)$ represents the average yield in year t (measured in bushels per acre), and $C(t)$ is the total number of bushels of corn the farmer produces. What is the formula for $C(t)$ in terms of $A(t)$ and $Y(t)$? Why?

(a) What is the value of $C(0)$? What does it measure?

(a) Write an expression for $C'(t)$ in terms of $A(t)$, $A'(t)$, $Y(t)$, and $Y'(t)$. Explain your thinking.

(a) What is the value of $C'(0)$? What does it measure?

(a) Based on the given information and your work above, estimate the value of $C(1)$.

(a) Assume that the annual yield decreases every year by 8 bushels per acre. Write expressions for $C(t)$ and $C'(t)$, find the approximate time and number of bushels when the total number of bushels is maximized, and discuss how the maximum value would change if the farmer were able to control the rate at which the yield decreased. Present your solution with thorough discussion and appropriate plots.

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Lab Exploration 1.3. Let $f(v)$ be the gas consumption (in liters/km) of a car going at velocity v (in km/hour). In other words, $f(v)$ tells you how many liters of gas the car uses to go one kilometer if it is traveling at v kilometers per hour. In addition, suppose that $f(80) = 0.05$ and $f'(80) = 0.0004$.

(a) Let $g(v)$ be the distance the same car goes on one liter of gas at velocity v. What is the relationship between $f(v)$ and $g(v)$? Hence find $g(80)$ and $g'(80)$.

(b) Let $h(v)$ be the gas consumption in liters per hour of a car going at velocity v. In other words, $h(v)$ tells you how many liters of gas the car uses in one hour if it is going at velocity v. What is the algebraic relationship between $h(v)$ and $f(v)$? Hence find $h(80)$ and $h'(80)$.

(c) How would you explain the practical meaning of these function and derivative values to a driver who knows no calculus? Include units on each of the function and derivative values you discuss in your response.

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Lab Exploration 1.4. The velocity (m/s) of an object dropped from a helicopter 1000 meters high is given in the table below. Use the velocity data to approximate the acceleration (m/s^2) and position (m) data for the object at each of the given times. Unfortunately the motion sensor broke 3 seconds into the experiment. Extrapolate from the data to approximate the velocity and acceleration when the object hit the ground. Obviously there is drag on this object. Make an argument to estimate when an object with 10% more drag would hit the ground.

In your writeup of this problem:

- Be sure to give enough detail about the decisions and assumptions that you made.
- Be sure to approximate the time that the object hit the ground in each instance.
- Be sure to thoroughly explain where calculus was (and wasn't) used in your solution.

Time (sec)	Velocity (meters/sec)
0.0	0
0.2	−1.96
0.4	−3.89
0.6	−5.70
0.8	−7.33
1.0	−8.76
1.2	−9.95
1.4	−10.92
1.6	−11.69
1.8	−12.29
2.0	−12.74
2.2	−13.08
2.4	−13.33
2.6	−13.52
2.8	−13.65
3.0	−13.75

▲

Lab Exploration 1.5. Consider the following letter:

11 Patinkin Way
First National Park of Drachma

Mathematics Students
Carroll College
Helena, MT, USA

Dear Calculus Students:

Things have finally quieted down around Drachma since the Prince was kicked out of office. The good news is that I've managed to find a government jobs as the head of the First National Park of Drachma. The bad news is that most of the Park consists of a Fire Swamp (google *fire swamp* if you need to). When I went looking for help with our long range planning, your enterprising and resourceful professor naturally referred me to you.

We have two species that have me worried about the future of the Park: the indigenous ROUS (rodents of unusual size) and the brown tree snake which entered the Park about 50 years ago as a stowaway on the Dread Pirate Roberts' ship. Fortunately, ROUS's eat brown tree snakes. Unfortunately, brown tree snakes reproduce very rapidly.

My predecessor at the Park was a meticulous census taker, so I have records of approximate populations for each species for a 30 year period (see Table 1.1).

It looks like the populations are following some sort of pattern, but I'm not sure what it is. My real problem is that when either population gets very large, I will need additional employees to make sure that both species stay within the park and don't escape in the neighboring farmland. This is where I need your expert help. Specifically, I need a

Year	Brown Tree Snakes	ROUS's
1982	15,300	415
1984	9,890	910
1986	2,860	950
1988	3,340	525
1990	9,340	250
1992	12,290	460
1994	9,050	830
1996	4,840	855
1998	5,130	545
2000	8,720	340
2002	10,490	500
2004	8,550	770
2006	6,030	790
2008	6,200	560
2010	8,350	410
2012	9,410	525

Table 1.1. Populations by year since 1982.

prediction for how large the populations will be in each of the next 20 years. I also need an estimate of the rate at which each of the populations are changing with time*. When the rates of change of the populations are largest the local witch doctors head into the woods to raid the snake nests and ROUS borrows for potion ingredients.

I believe that the populations are fluctuating less and less, and may eventually stabilize. I would like your expert opinion on whether or not the populations do stabilize, and if they do, I need to know how long it will take and what the eventual populations will be.

Once the populations stop fluctuating so drastically, we will be able to dramatically improve access to the Park by offering summer camps, establishing permanent camp grounds, and perhaps even adding a log ride. There are still some flame-retardant issues to be worked out and the 6-fingered man is terribly afraid of the log ride idea. This should all be possible when the ROUS population is fluctuating by less than 75 per year and the brown tree snake population is fluctuating by less than 500 per year. I need your expert recommendation on when this will occur.

I have a meeting with the Budget Advisory Committee in 8 days to propose our budget for the next two decades, so I would greatly appreciate your group's report in 1 week.

Gratefully yours,
Amigo Flamboya

*I'm thinking that a plot with time on the x-axis and average rate of change on the y-axis might be very informative

Notes from your professor:

- Work in groups of 2 or 3, but don't be afraid to bounce ideas off of other groups.
- To see the general trend of the populations, I would suggest plotting the points for each population separately (maybe in Excel), with time on the horizontal axis and population on the vertical axis. It may make things a bit easier if you let $t = 0$ be 1982.
- Hint: Once you plot the populations, what two major types of functions do you see controlling the behavior? You can fit the function by estimating things like the period (or frequency), the equilibrium value, and the function that controls the amplitude.
- Be sure to respond to Amigo Flamboya with an appropriate technical report. He understands mathematics quite well so don't be afraid to include all necessary detail (explanations, plots, functions, etc) in your report. Make sure that you answer every question.

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Chapter 2

An Introduction to Linear Algebra

2.1 Why Linear Algebra?

**Of all the mathematical tools that an applied scientist has,
linear algebra is the most important.**

When a student first encounters linear algebra it may seem like a stretch to call it the *most* important mathematical tool. That is, until the student relizes that almost every mathematical operator (such as differentiation, integration, reflection, rotation) and mathematical process (such as finding a best fit line, solving a system of equations, or even revealing the frequencies of a sound wave) are all based on the concepts from linear algebra.

In this chapter we will take a brief tour of several of the large ideas from linear algebra to give the reader a flavor of the richness and depth of the topic. In this section we will present the *three main problems* from linear algebra to highlight the importance and impact of the field. Before launching into these problems, the following problem will give you an introduction to the organizational structure, called a *matrix*, as well as the basic concept of matrix multiplication.

Problem 2.1. Advertisements tend to change people's opinions about political issues. Suppose that on a certain political issue there are 3 different popular opinions (A, B, and C). A psychologist wants to study the shifts in people's opinions after viewing advertisements and hence gathers the data listed in the table below.

Previous Opinion	New Opinion After Viewing Advertisement	Percent Making This Switch
A	A	50%
A	B	20%
A	C	30%
B	A	10%
B	B	70%
B	C	20%
C	A	5%
C	B	5%
C	C	90%

(a) Create a visual representation of the psychologist's data*.

(b) Create a tabular representation of the psychologist's data.

	From A	From B	From C
To A			
To B			
To C			

(c) If there are currently 1200 people in a population with opinion A, 500 people with opinion B, and 800 with opinion C, then what would the psychologist's data predict about the numbers of people with each opinion after viewing the advertisements?

(d) The psychologist did the study twice, but in an unfortunate instance with a hot latte she lost her record of the numbers of people in each category before watching the advertisements. She knows that the end result was 1000 people with opinion A, 800 people with opinion B, and 700 people with opinion C. How many people were originally in each category?

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Systems of Linear Equations: $Ax = b$

Most people are familiar with systems of linear equations arising from problems in algebra, business, calculus, and a plethora of other fields. What is often overlooked in lower-level mathematics classes is that there is an immense amount of structure embedded inside a system of equations just waiting to be exploited.

Problem 2.2. Consider a long metal rod that is being heated at one end and held at a constant temperature at the other end. After some time the temperature profile throughout the rod will reach a steady state (independent of time). One way to estimate the steady state temperature of the rod is to partition the rod into several equally spaced points (see Figure 2.1) and then to observe that the temperature at a point is the average of the temperature of the point to the right and the point to the left.

(a) Write a system of equations associated with the steady state temperature profile shown in Figure 2.1.

(b) Solve the system of equations from part (a) to find the steady state temperatures on the rod. You should either use the substitution method or the elimination method (which do you think will be more efficient? more organized?).

(c) Now let's go back to the system of equations in part (a) and build a more organized description of the problem. Rearrange the equations so that all of the variables are on the left-hand side and all of the corresponding variables are aligned vertically.

*The type of visual representation that most people use here is called a *graph* in mathematics.

This organization lends itself nicely to a *matrix equation*, where all of the coefficients are gathered into one matrix, the variables into a column vector, and the right-hand sides of the equations in another column vector. Write the system of equations from part (a) as a matrix equation. Explain how this structure lends itself nicely to the elimination method.

$$\begin{pmatrix} \text{matrix} \\ \text{of} \\ \text{coefficients} \end{pmatrix} \cdot \begin{pmatrix} \text{column} \\ \text{of} \\ \text{variables} \end{pmatrix} = \begin{pmatrix} \text{right} \\ \text{hand} \\ \text{side} \end{pmatrix}$$

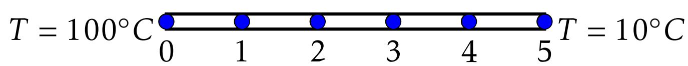

Figure 2.1. A metal rod partitioned into several discrete points.

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Fundamental Matrix Behavior: $Ax = \lambda x$

The second fundamental problem of linear algebra is to decompose a complicated system of equations to a collection of elements that are easily visualized. This is known as the eigenvector-eigenvalue problem. In the following activity we will set up a problem that, without the tools of linear algebra, are very difficult to answer. We will return to this example in future sections.

Problem 2.3. The female owls in a certain population can be classified as *juvenile*, *subadult*, and *adult*. In a given year, the number of new juvenile females in year $k+1$ is 0.33 times the number of adult females in year k, 18% of last year's juveniles become subadults, 71% of last year's subadults become adults, and only 94% of last year's adults survive.

(a) Write a discrete dynamical system for the populations of juveniles j_k, subadults s_k, and adults a_k where k is the year.

(b) Organize the discrete dyamical system into a matrix equation

$$\begin{pmatrix} \text{column} \\ \text{of} \\ \text{variables} \\ (\text{time} = k+1) \end{pmatrix} = \begin{pmatrix} \text{matrix} \\ \text{of} \\ \text{coefficients} \\ (\text{time} = k) \end{pmatrix} \cdot \begin{pmatrix} \text{column} \\ \text{of} \\ \text{variables} \\ (\text{time} = k) \end{pmatrix}$$

(c) The owl population will change over time, but it is very important to determine ahead of time if the owl popluation is in danger of going extinct. Use a spreadsheet program to predict the future of the owl population. The initial populations are not known.

(d) What is the long term behavior of the female owl population? If you used your spreadsheet model from part (c) to make this determination, then how do you know that your answer doesn't depend on your chosen intitial conditions?

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Least Squares: $A^T Ax = A^T b$

The final of the three big problems from linear algebra is that of least squares curve fitting. Instead of *least squares*, often times this is referred to as the *best fit line*. Of course, the word *best* is relative to how you measure the error. In the following activity you'll set up a best fit line problem with linear algebra. The techniques to solve the problem will not be covered in this text; this problem is presented here for completeness of the introduction.

Problem 2.4. Research on study time vs. exam scores yields the following data points

Time Studying	Exam Score (%)
0	75
25	70
30	92
45	88
15	90
30	70

(a) We would like to find an linear equation of the form $y = ax + b$ where x is the time spent studying and y is the exam score. Write 6 equations where the two unknowns are the parameters a and b.

(b) Organize your 6 equations into a matrix equation. The solution to this matrix equation is beyond the scope of this chapter, but if we could *solve*[†] this problem then we would know the slope and y-intercept that minimize the error between the predictor line and the data. This is a wide reaching topic that will unfortunately have to wait for a different course.

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2.2 Matrix Operations and Gaussian Elimination

One of the first natural questions to ask when first encountering matrices is whether the regular operations of addition, subtraction, multiplication, division, and exponentiation make sense. In the cases of addition and subtraction the answer is simple: Yes! Addition and subtraction work in the simplest most natural way with matrices. The other operations, on the other hand, need a bit more care but their definitions are robust and immensely useful.

[†]Even the word *solve* here is arbitrary since there are more equations than there are unknowns!

Problem 2.5. Consider the matrices

$$A = \begin{pmatrix} 1 & 7 & -3 \\ 2 & -3 & 5 \\ 2 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 3 & -7 & 0 \\ 0 & 0 & -2 \\ 2 & 4 & 5 \end{pmatrix}$$

(a) Calculate $A + B$.

(b) Calculate $A - B$.

(c) Calculate $2A$.

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Matrix Arithmetic

In this subsection we'll take a brief glimpse at each of the most fundamental matrix operations as well as some of the foundational definitions for linear algebra.

Definition 2.6 (Matrix Arithmetic)**.** Below are several definitions associated with matrices.

Size of a Matrix: If A is a matrix with m rows and n columns then we say that A has **size** (or dimensions) $m \times n$.

Equality: Two matrices are **equal** if their corresponding entries are equal. Matrices can only be equal if the sizes are equal.

Addition and Subtraction: Matrix **addition and subtraction** and done by regular addition and subtraction on the corresponding entries. Matrix addition and subtraction can only be performed on matrices of the same size.

Scalar Multiplication: If A is a matrix then cA is a **scalar multiple** of the matrix. Multiplying a matrix by a scalar multiplies every entry by the scalar.

Transposition: If A is a matrix then A^T is the **transpose** of the matrix found by interchanging the rows and columns of A. If A is $m \times n$ then A^T is $n \times m$.

The basic operations of addition, subtraction, scalar multiplication, and trasposition all follow our natural intuition, and we'll get a chance to play with them in the homework. The operation of multiplication, on the other hand, takes a bit more care to define.

Before giving a full description of matrix multiplication let us define some very common notation for matrices. The size of the matrix is stated by the number of rows then the number of columns. This lends itself to a system of double indices for keeping track of the entries in a matrix. For the matrix A of size $m \times n$ we denote the individual entry in row i and column j as a_{ij}. In the entire matrix, this becomes

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

So, for example, a_{37} is the entry in row 3 column 7. More concretely, in the matrix, $A = \begin{pmatrix} 7 & 4 & -2 \\ 3 & 1 & 4 \end{pmatrix}$, the size is 2×3 and the entry in row 2 column 1 is $a_{21} = 3$.

Definition 2.7. If A is an $m \times n$ matrix and B is an $n \times p$ matrix then the **product** of A and B is $C = AB$.

- The size of AB is $m \times p$. The number of columns in A must be the same as the number of rows of B.
- The entry in row i and column j of $C = AB$ is

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

It is very important to note that in general $AB \neq BA$.

Example 2.8. Consider the matrices A and B defined as

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 5 & -2 \\ -1 & 0 \\ 1 & 3 \end{pmatrix}.$$

Find AB and BA if they exist.
Solution: First note that A is a 2×3 matrix and B is a 3×2 matrix. Hence, AB will be 2×2 and BA will be 3×3. [a]

$$\begin{aligned} AB &= \begin{pmatrix} 1 & 2 & -3 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & -2 \\ -1 & 0 \\ 1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 \cdot 5 + 2 \cdot (-1) + (-3) \cdot 1 & 1 \cdot (-2) + 2 \cdot 0 + (-3) \cdot 3 \\ 2 \cdot 5 + 0 \cdot (-1) + \cdot 1 \cdot 1 & 2 \cdot (-2) + 0 \cdot 0 + 1 \cdot 3 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -11 \\ 11 & -1 \end{pmatrix} \end{aligned}$$

To be very clear about the process, the -11 in row 1 column 2 of the answer came from multiplying the corresponding entries of row 1 of matrix A by column 2 of matrix B and finding the sum of the products.

$$
\begin{aligned}
BA &= \begin{pmatrix} 5 & -2 \\ -1 & 0 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & -3 \\ 2 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 5\cdot 1 + (-2)\cdot 2 & 5\cdot 2 + (-2)\cdot 0 & 5\cdot(-3) + (-2)\cdot 1 \\ (-1)\cdot 1 + 0\cdot 2 & (-1)\cdot 2 + 0\cdot 0 & (-1)\cdot(-3) + 0\cdot 1 \\ 1\cdot 1 + 3\cdot 2 & 1\cdot 2 + 3\cdot 0 & 1\cdot(-3) + 3\cdot 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 10 & -17 \\ -1 & -2 & 3 \\ 7 & 2 & 0 \end{pmatrix}
\end{aligned}
$$

[a] If you read this example without picking up your pencil and trying the example then you may want to pause and rethink your decisions. Mathematics is not a spectator's sport!

Problem 2.9. Consider the matrices

$$
A = \begin{pmatrix} 2 & -1 & 4 \\ 3 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 0 & -3 \\ 4 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -1 \\ 3 & 2 \\ -2 & 1 \end{pmatrix}, \quad \text{and} \quad \mathsf{x} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}.
$$

(a) Determine which products are possible: AB, AC, $A\mathsf{x}$, BA, CA, $\mathsf{x}A$, BC, $B\mathsf{x}$, CB, $C\mathsf{x}$. For each of the products that is possible, find the size of the result.

(b) Write the product AB and the product BA. Does $AB = BA$?

▲

2.3 Gaussian Elimination: A First Look At Solving Systems

A truly beautiful application of matrices, and the first real application of linear algebra, is the technique of solving systems of linear equations. The technique that we'll describe in the next several activities and examples is used to solve systems of equations in a very organized fashion. Most students are familiar with the *elimination method* from high school algebra, and the technique of *Gaussian Elimination* described herein is simply a more organized way to perform the exact same technique. You'll find that systems of linear equations arise naturally in all sorts of applications so we include this as one of the essential tools for mathematical modeling.

Example 2.10. Consider the following system of equations.

$$
\begin{aligned}
-x_1 + x_2 - x_3 &= 1 \\
3x_2 + 2x_3 &= -8 \\
x_3 &= 2
\end{aligned} \tag{2.1}
$$

Solve the system algebraically and reorganize the system using the powerful and beautiful structure of matrices [a]

Solution: Any technique for solving systems will suffice. This particular system is set up to reveal a solution quickly. Indeed, it is obvious that $x_3 = 2$. Using this fact, the second equation can be rewritten and solved as

$$3x_2 + 2(2) = -8 \quad \implies \quad 3x_2 = -12 \quad \implies \quad x_2 = -4.$$

Now that we have both x_2 and x_3 the first equation can be rewritten as solved as

$$-x_1 + (-4) - (2) = 1 \quad \implies \quad -x_1 = 7 \quad \implies \quad x_1 = -7$$

Now we'll leverage the organizational power of matrices:
The system of equations can be written in matrix form as

$$\begin{pmatrix} -1 & 1 & -1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -8 \\ 2 \end{pmatrix} \tag{2.2}$$

Multiplying the 3×3 matrix on the left-hand side by the 3×1 vector $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ reveals that the matrix equation in (2.2) is indeed the same as the system of equations in (2.1). This important observation illustrated that we can take any system of linear equations and write it in such a way.

The matrix equation can be further reorganized into an *augmented system*:

$$\left(\begin{array}{ccc|c} -1 & 1 & -1 & 1 \\ 0 & 3 & 2 & -8 \\ 0 & 0 & 1 & 2 \end{array} \right)$$

This is simply an organizational technique and we use it because the names of the variables are arbitrary and irrelevant to the solution. The first line of the augmented system can be read as $-x_1 + x_2 - x_3 = 1$ where the variables are inferred and only inserted when necessary. The last line of the augmented system can be read as $0x_1 + 0x_2 + x_3 = 2$, so this clearly reveals that $x_3 = 2$.

[a] If you haven't notice, the author loves matrices!

Problem 2.11. In this activity we wish to solve the system of equations.

$$\begin{aligned} -x_1 + x_2 - x_3 &= -6 \\ x_1 + x_3 &= 15 \\ 2x_1 - x_2 + x_3 &= 9 \end{aligned}$$

We will do so in a very structured and organized fashion to illustrate the *Gaussian Elimination* technique for solving systems.

(a) First write the system as a matrix equation.

$$\begin{pmatrix} \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \_\_\_ \\ \_\_\_ \\ \_\_\_ \end{pmatrix}$$

(b) Now write the system as an *augmented system*

$$\left(\begin{array}{ccc|c} \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ \end{array} \right)$$

(c) Using the operations:

- multiply one row by a scalar quantity
- add a multiple of one row to another row
- interchange two rows

we wish to transform the augmented system you wrote in part (b) to something of the form

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \star \\ 0 & 1 & 0 & \star \\ 0 & 0 & 1 & \star \end{array} \right)$$

Discuss with your partners why the above operations are mathematically valid.

(d) Work with your partners to discuss the first and most logical operation to do that will move you toward that direction.

(e) Use the operations outlined in part (c) to solve the system. Pay particular attention to the order in which you perform the row reduction.

▲

Definition 2.12. The **Gaussian Elimination** technique (also called **row reduction**) is an algorithm used to perform the elimination method on a system of linear equations of virtually any size.

1. Write the system of equations in augmented form.

2. Perform row operations to get to a triangular system of equations. The row operations allowed are:

    - Multiply a row by any nonzero number.
    - Add a multiple of one row to another.
    - Interchange two rows.

3. Once the system is written in triangular form, either back substitute to solve

the system or continue performing row operations to arrive at the form

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & \cdots & 0 & \star \\ 0 & 1 & 0 & \cdots & 0 & \star \\ 0 & 0 & \ddots & \cdots & 0 & \star \\ \vdots & \vdots & \cdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \star \end{array}\right)$$

At which point, read the answer from the augmented form.

Next we will show a fully worked example of Gaussian Elimination in action to give some hints to the thought process that goes on behind the scenes.

Example 2.13. Solve the system of equations

$$\begin{aligned} x_1 + 0x_2 + 3x_3 + 2x_4 &= -20 \\ 0x_1 + x_2 - 4x_3 - 4x_4 &= 32 \\ 2x_1 - 3x_2 + 16x_3 + 16x_4 &= -120 \\ 0x_1 - x_2 + 4x_3 + 9x_4 &= -27 \end{aligned}$$

Solution: If we first write this as an augmented matrix we get

$$\left(\begin{array}{cccc|c} 1 & 0 & 3 & 2 & -20 \\ 0 & 1 & -4 & -4 & 32 \\ 2 & -3 & 16 & 16 & -120 \\ 0 & -1 & 4 & 9 & -27 \end{array}\right).$$

Next we start performing row operations with the goal of creating a triangular system. The observant reader will notice that, while this system of equations can be solved with any (non-graphical) technique from high school algebra, the Gaussian Elimination technique is far more organized.

Add (-2) times row 1 to row 3. Put the answer in row 3.

$$\xrightarrow{-2R_1+R_3} \left(\begin{array}{cccc|c} 1 & 0 & 3 & 2 & -20 \\ 0 & 1 & -4 & -4 & 32 \\ 0 & -3 & 10 & 12 & -80 \\ 0 & -1 & 4 & 9 & -27 \end{array}\right)$$

Add (3) times row 2 to row 3. Put the answer in row 3.

$$\xrightarrow{3R_2+R_3} \left(\begin{array}{cccc|c} 1 & 0 & 3 & 2 & -20 \\ 0 & 1 & -4 & -4 & 32 \\ 0 & 0 & -2 & 0 & 16 \\ 0 & -1 & 4 & 9 & -27 \end{array}\right)$$

Add (1) times row 2 to row 4. Put the answer in row 4.

$$\xrightarrow{(1)R_2+R_4}\left(\begin{array}{cccc|c} 1 & 0 & 3 & 2 & -20 \\ 0 & 1 & -4 & -4 & 32 \\ 0 & 0 & -2 & 0 & 16 \\ 0 & 0 & 0 & 5 & 5 \end{array}\right)$$

Divide row 3 by (-2) and put the answer in row 3.

$$\xrightarrow{R_3/(-2)}\left(\begin{array}{cccc|c} 1 & 0 & 3 & 2 & -20 \\ 0 & 1 & -4 & -4 & 32 \\ 0 & 0 & 1 & 0 & -8 \\ 0 & 0 & 0 & 5 & 5 \end{array}\right)$$

Divide row 4 by 5 to arrive at a triangular form.

$$\xrightarrow{R_4/5}\left(\begin{array}{cccc|c} 1 & 0 & 3 & 2 & -20 \\ 0 & 1 & -4 & -4 & 32 \\ 0 & 0 & 1 & 0 & -8 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

Now that this is in triangular form you can back substitute or simply continue performing row operations. We will choose to perform the row operations to determine the solution.

$$\xrightarrow{4R_3+R_2,4R_4+R_2}\left(\begin{array}{cccc|c} 1 & 0 & 3 & 2 & -20 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -8 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)\xrightarrow{(-3)R_3+R_1,(-2)R_4+R_1}\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -8 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

After all of this simplification, the final solution is

$$\boxed{x_1 = 2,\ x_2 = 4,\ x_3 = -8,\ x_4 = 1}$$

The process of performing Gaussian Elimination may take a lot of paper, but once you get the hang of the process it is far more organized than any other technique for solving systems of linear equations.

Technique 2.14 (Practical Tips for Gaussian Elimination)**.** You should use the following tips for doing Gaussian Elimination.

- First try to get a 1 in the upper left-hand corner of the augmented matrix.
- Next, use the new first row to eliminate all of the non-zero entries in the first column. By the time you're done with this you should have a column with a 1

on top and zeros below.

- Next get a 1 in row 2 column 2.
- Use your new second row to eliminate all of the non-zero entries in the second column.
- Proceed in a similar fashion until you have reached the final row.

Problem 2.15. Write the following system in augmented form and use Gaussian Elimination to solve for x_1, x_2, and x_3.

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

▲

2.4 Systems of Linear Equations

In this section we further explore the notion of solving a linear system of equations. To begin out study consider the following Preview Activity.

Problem 2.16. Solve each of the three systems of two equations and two unknowns. One of the systems has infinitely many solutions, one of the systems has exactly one solution, and one of the systems has no solutions. In each case, use augmented matrices and Gaussian Elimination to solve the system.

(a) Solve the system

$$\begin{aligned} x_1 - 2x_2 &= 4 \\ -2x_1 + 4x_2 &= 5 \end{aligned}$$

(b) Solve the system

$$\begin{aligned} x_1 - 2x_2 &= 4 \\ -2x_1 + 4x_2 &= -8 \end{aligned}$$

(c) Solve the system

$$\begin{aligned} x_1 - 2x_2 &= 4 \\ 2x_1 + 4x_2 &= 5 \end{aligned}$$

▲

Systems, Matrix Equations, and Vector Equations

A system of linear equations can always be written in several different ways. The most familiar of which is the collection of equations themselves. The three other ways to write a system of equations are the **matrix form**, the **vector form**, and the **augmented matrix form**.

Consider the system of m linear equations with n unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n &= b_3 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{2.3}$$

Definition 2.17. The **matrix form** of the system of equations (2.3) is

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{pmatrix}$$

In symbols, this is denoted $A\mathsf{x} = \mathsf{b}$.

Definition 2.18. The **vector form** of the system of equations (2.3) is

$$x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \\ \vdots \\ a_{m2} \end{pmatrix} + x_3 \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \\ \vdots \\ a_{m3} \end{pmatrix} + \cdots x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ a_{3n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{pmatrix}$$

In symbols, this is denoted $x_1\mathsf{a}_1 + x_2\mathsf{a}_2 + x_3\mathsf{a}_3 + \cdots + x_n\mathsf{a}_n = \mathsf{b}$.

Definition 2.19. The **augmented form** of the system of equation (2.3) is

$$\left(\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{array}\right)$$

In symbols this is denoted $(A|\mathbb{b})$.

Solution Sets to Systems of Equations

It is not guaranteed that a system of equations will have a solution. Moreover, if there is a solution it is not guaranteed that the solution will be unique. Theorem 2.20 gives the conditions for which a system of linear equations will have no solutions, infinitely many solutions, or exactly one solution.

Theorem 2.20.

1. A system of linear equations has no solutions if after it is row reduced it has a row of the form

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & | & \star \end{pmatrix}$$

where the number $\star$ is nonzero. This row in the reduced matrix is equivalent to saying that $0 = \star$; which is never true.

2. A system of linear equations has one unique solution if at the end of the row reduction one can determine every variable.

3. A system of linear equations has infinitely many solutions if at the end of the row reduction there are variables that you cannot determine uniquely.

Problem 2.21. (a) Consider the following augmented matrices in reduced row echelon form. Determine the number of solutions. If there is one unique solution then find it.

(i)

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -5 \end{array}\right)$$

(ii)

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & -5 \end{array}\right)$$

(iii)

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

(b) Determine the value of h such that the matrix is the augmented matrix of a linear system with infinitely many solutions

$$\left(\begin{array}{cc|c} 3 & -4 & 4 \\ 9 & h & 12 \end{array}\right)$$

▲

In Activity 2.21, problem (a) part (iii) has infinitely many solutions. In order to find a complete description of those solutions we can rewrite the problem in terms of the variables to get

$$\begin{aligned} x_1 + x_3 &= 3 \\ x_2 &= 2. \end{aligned}$$

It is obvious from this description that x_2 is fixed at 2, but the values of x_1 and x_3 depend on each other. In cases like this we choose one variable to be a *parameter* and express the other variable in terms of that parameter. In this case, we let $x_3 = t$ and we can write the first equation as

$$x_1 = 3 - t$$

Since t can take on any real value we finally write the solution as

$$x_1 = 3 - t \quad \text{and} \quad x_2 = 2 \quad \text{where} \quad x_3 = t \quad \text{and} \quad -\infty < t < \infty.$$

Written as a vector, the solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} t.$$

In two and three dimensions there are nice geometric interpretations for these types of solutions.

Example 2.22. Consider the following three systems of equations and their row reduced forms. Describe their solution sets geometrically.

$$\text{System \#1:} \left(\begin{array}{cc|c} 1 & -1 & 3 \\ 2 & 1 & 0 \end{array}\right) \to \cdots \to \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -2 \end{array}\right)$$

$$\text{System \#2:} \left(\begin{array}{cc|c} 1 & -1 & 3 \\ -1 & 1 & 0 \end{array}\right) \to \cdots \to \left(\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 0 & 3 \end{array}\right)$$

$$\text{System \#3:} \left(\begin{array}{cc|c} 1 & -1 & 3 \\ -1 & 1 & -3 \end{array}\right) \to \cdots \to \left(\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 0 & 0 \end{array}\right)$$

Solution: Figure 2.2 shows the graphical interpretation for each system. Clearly if there is a unique solution then there is one unique point where the lines cross. If there are no solutions then the lines are parallel. In the case where there are infinitely many solutions (system #3) we see that we can write $x_2 = x_1 - 3$. Letting $x_1 = t$ we have $x_2 = t - 3$. This is clearly the line with y-intercept -3 and slope 1.

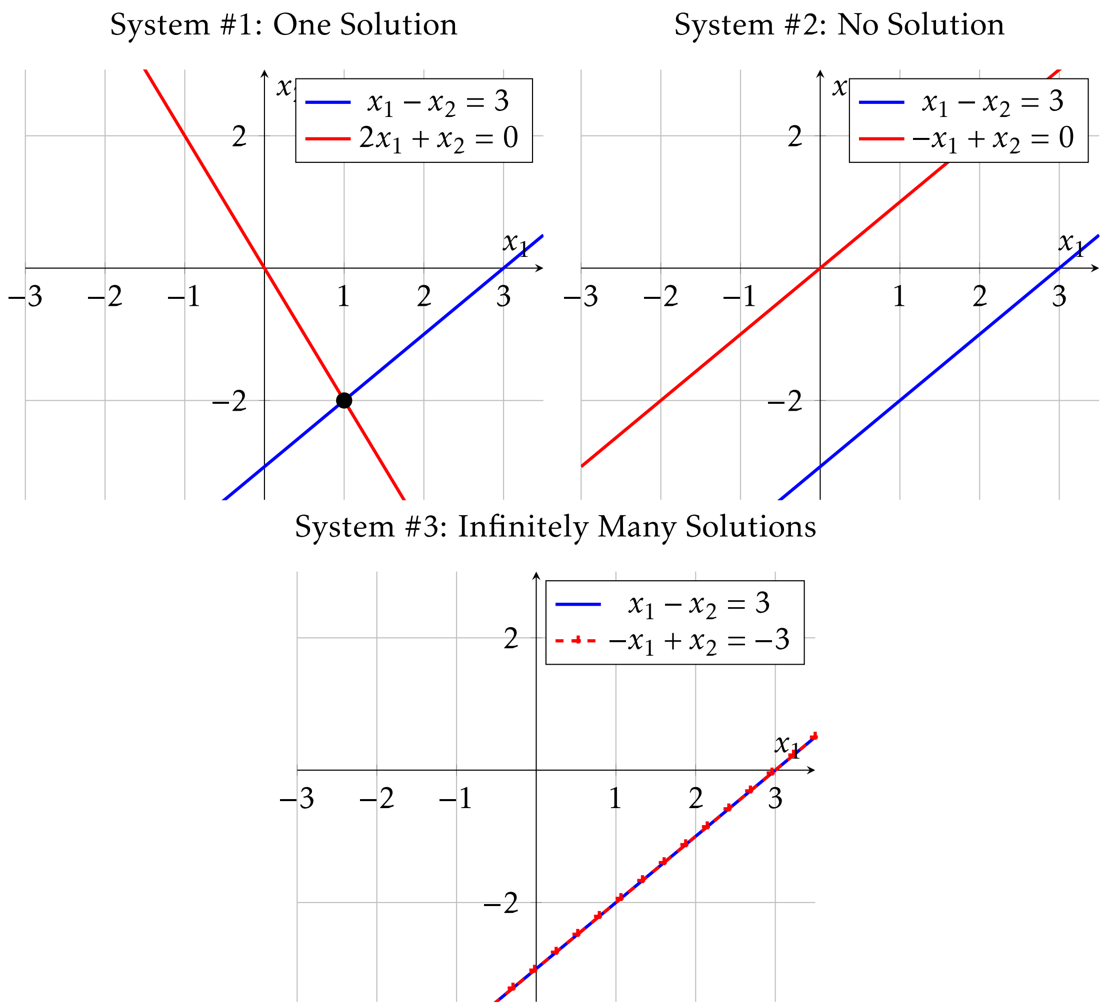

Figure 2.2. Three possible solution sets in two spatial dimensions

Example 2.23. Solve the system of equations. If there are infinitely many solution, express them as a parameterization.

$$-4x_1 + x_2 = 0$$
$$-12x_1 + 3x_2 = 0$$

Solution: First we write the system of equations as an augmented system. Then we row reduce as much as possible.

$$\left(\begin{array}{cc|c} -4 & 1 & 0 \\ -12 & 3 & 0 \end{array}\right) \overset{(-3)R_1+R_2}{\longrightarrow} \left(\begin{array}{cc|c} -4 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

In this particularly simple example, this means that $-4x_1 + 1x_2 = 0$. Written another way, $x_2 = 4x_1$.

If we write $x_1 = t$ then the solution is $x_1 = t, x_2 = 4t$ for $-\infty < t < \infty$. If, on the other hand, we write $x_2 = t$ then the solution is $x_2 = t, x_1 = t/4$ for $-\infty < t < \infty$.

In vector form, this solution can be written as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} t$$

or

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ 1 \end{pmatrix} t.$$

Example 2.24. Solve the system of equations $A\mathsf{x} = \mathsf{b}$ where

$$A = \begin{pmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{pmatrix} \quad \text{and} \quad \mathsf{b} = \begin{pmatrix} 7 \\ -1 \\ -4 \end{pmatrix}.$$

Solution: Writing the augmented matrix $(A|\mathsf{b})$ and doing several steps of row reduction gives

$$\left(\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This implies that $x_2 = 2$ and $x_1 = -1 + \frac{4}{3}t$ for some parameter t.

Written in vector form, the solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} \frac{4}{3} \\ 0 \\ 1 \end{pmatrix}.$$

Example 2.25. Solve the system of equations

$$4x_1 + x_2 + 5x_3 + 7x_4 = 0$$
$$8x_1 + x_2 - 5x_3 + 4x_4 = 0$$

Solution: Written in augmented form and row reduced we see that

$$\left(\begin{array}{cccc|c} 4 & 1 & 5 & 7 & 0 \\ 8 & 1 & -5 & 4 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 4 & 1 & 5 & 7 & 0 \\ 0 & -1 & -15 & -18 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 4 & 0 & -10 & -11 & 0 \\ 0 & 1 & 15 & 18 & 0 \end{array}\right)$$

In this case there are two variables that cannot be solve for: x_3 and x_4. These are now both parameters. If $x_3 = s$ and $x_4 = t$ then

$$x_1 = \frac{10s + 11t}{4} \quad \text{and} \quad x_2 = -15s - 18t \quad \text{where} \quad x_3 = s \quad \text{and} \quad x_4 = t.$$

In vector form, this solution can be written as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 10/4 \\ -15 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} 11/4 \\ -18 \\ 0 \\ 1 \end{pmatrix} t.$$

Problem 2.26. Each of the following systems has been row reduced as much as possible. If there is a unique solution to the system then find it. If there are infinitely many solutions then express them as a vector equation. If there are no solutions then indicate the reason.

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \tag{2.4}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 3 \end{array}\right) \tag{2.5}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right) \tag{2.6}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right) \tag{2.7}$$

▲

2.5 Linear Combinations

One of the most beautiful parts of linear algebra is the richness of the structure of matrices. As we showed earlier, every system of linear equations can be written several different ways (as a system, as a matrix equation, as a vector equation, or as an augmented system). In this subsection we'll look in particular at the vector equation. Hiding below a vector equation is one of the most fundamental ideas behind all of linear algebra: the linear combination.

Definition 2.27. Let $\mathsf{v}_1, \mathsf{v}_2, \ldots, \mathsf{v}_p$ be vectors in n-dimensional space and let $c_1, c_2, \ldots, c_p$ be scalar quantities. The vector u defined by

$$\mathsf{u} = c_1\mathsf{v}_1 + c_2\mathsf{v}_2 + \cdots c_p\mathsf{v}_p$$

is called a **linear combination** of the vectors $\mathsf{v}_1, \mathsf{v}_2, \ldots, \mathsf{v}_p$ with weights $c_1, c_2, \ldots, c_p$.

In the system of equations

$$2x_1 + 3x_2 = 5 \tag{2.8}$$
$$4x_1 - 6x_2 = 6, \tag{2.9}$$

we can rephrase the underlying question as: *find the weights which solve the vector equation*

$$x_1\begin{pmatrix}2\\4\end{pmatrix} + x_2\begin{pmatrix}3\\-6\end{pmatrix} = \begin{pmatrix}5\\6\end{pmatrix}. \tag{2.10}$$

Notice that this is simply stating that a system of equations is nothing more than a linear combination with unknown weights!

There is also a nice graphical interpretation of linear combination (2.10): How many $\begin{pmatrix}2\\4\end{pmatrix}$ plus how many $\begin{pmatrix}3\\-6\end{pmatrix}$ do we need to create $\begin{pmatrix}5\\6\end{pmatrix}$? Solving for x_1 and x_2 (using Gaussian elimination) we find that $x_1 = 2$ and $x_2 = 1/3$. Hence, $\begin{pmatrix}5\\6\end{pmatrix} = 2\begin{pmatrix}2\\4\end{pmatrix} + (1/3)\begin{pmatrix}3\\-6\end{pmatrix}$ as seen in Figure 2.3.

Problem 2.28. Open the GeoGebra applet http://tube.geogebra.org/student/m1254137 in a browser window.

(a) Move the vectors u and v to

$$\mathsf{u} = \begin{pmatrix}1\\2\end{pmatrix} \quad \text{and} \quad \mathsf{v} = \begin{pmatrix}2\\-1\end{pmatrix}.$$

We would like to determine all of the combinations possible forming the vector

$$\mathsf{w} = c_1\mathsf{u} + c_2\mathsf{v}.$$

Use the sliders in the GeoGebra applet to answer the following questions:

(i) Describe all of the possible vectors w if $c_1 = 0$.

(ii) Describe all of the possible vectors w if $c_2 = 0$.

(iii) Which vector results if $c_1 = c_2 = 0$?

(iv) Is it possible to find c_1 and c_2 such that $\mathsf{w} = \begin{pmatrix}-6\\0.5\end{pmatrix}$? If so, what are c_1 and c_2. If not, why not?

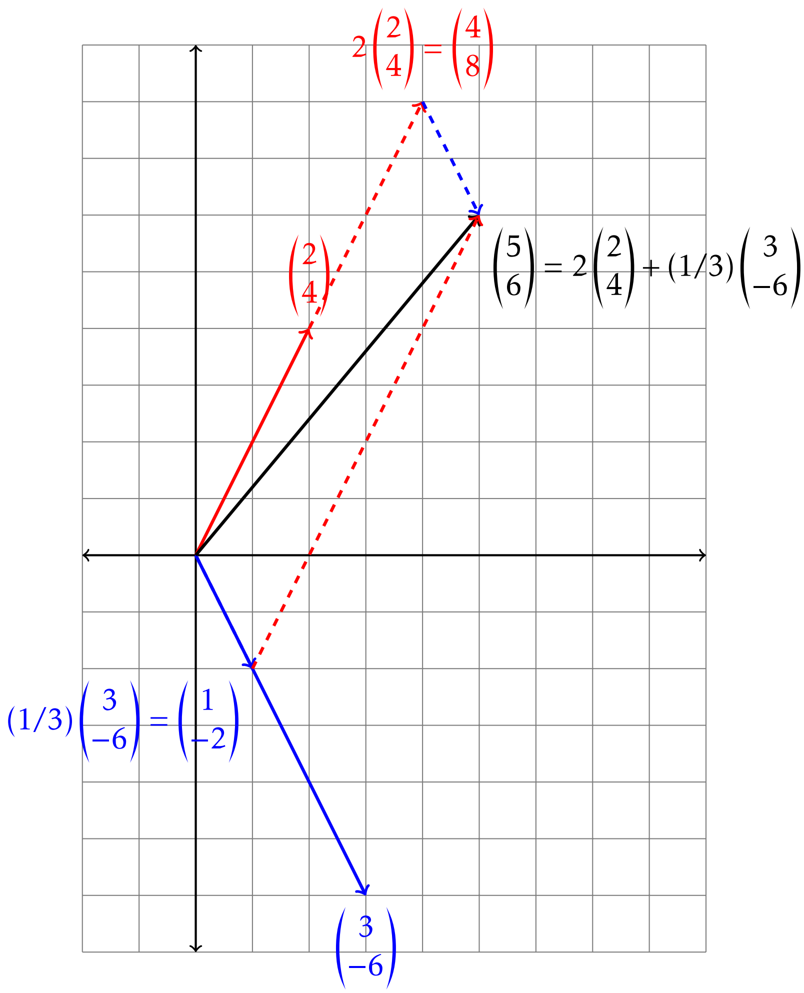

Figure 2.3. A graphical example of a linear combination.

(b) Move u and v so they are parallel. Form the vector $\mathsf{w} = c_1\mathsf{u} + c_2\mathsf{v}$ and describe all of the possible values of the vector w.

(c) Find c_1 and c_2 both algebraically and graphically so that

$$\begin{pmatrix}1\\-12\end{pmatrix} = c_1\begin{pmatrix}1\\-2\end{pmatrix} + c_1\begin{pmatrix}0\\2\end{pmatrix}.$$

(d) Express the vector $\mathsf{w} = \begin{pmatrix}5\\-4\end{pmatrix}$ as a linear comination of $\mathsf{u} = \begin{pmatrix}-2\\-1\end{pmatrix}$ and $\mathsf{v} = \begin{pmatrix}1\\-6\end{pmatrix}$.

▲

2.6 Inverses and Determinants

Division is always a bit of a touchy subject. In the real numbers division is well defined except when the denominator is zero. The same story is true in the rational numbers: a fraction divided by a fraction is another fraction so long as the divisor is not zero. What if we wanted to stay only in the integers? Can we divide two integers and get another integer? Of course you can always divide by 1, but in most other cases division will move you into the rational numbers. Hence, division on the integers doesn't really make sense.‡

Similarly, if we try to define division on matrices we run into trouble. What does it mean to *divide by a matrix*? In general, that phrase is meaningless! Let's expand our view a bit.

When considering the operation of addition, we call 0 the **additive identity** and we call $(-a)$ the *additive inverse* of a since $a+(-a)=0$. When considering multiplication, we call 1 the **multiplicative identity** and $1/a$ is the *multiplicative inverse* of a (when $a \neq 0$) since $a \cdot \frac{1}{a} = 1$.

Problem 2.29. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 \\ -4 & 3 \end{pmatrix}.$$

(a) Find a matrix B such that $A+B=A$ and $B+A=A$.

(b) Find a matrix I such that $I \cdot A = A$ and $A \cdot I = A$.

(c) Find a matrix C such that $C \cdot A = I$ and $A \cdot C = I$.

▲

One way to tackle the third part of the preceding problem is let let C be a matrix filled with unknowns and then to build the associated system of equations. More specifically, if we let

$$C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and then observe that the equation $AC = I$ becomes

$$\begin{pmatrix} 1 & 2 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

After multiplying the left-hand side we get the equation

$$\begin{pmatrix} a+2c & b+2d \\ -4a+3c & -4b+3d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

‡The mathematician would say that the integers are not closed under division.

This results in a system of four equations with four unknowns:

$$\begin{cases} a+2c &= 1 \\ b+2d &= 0 \\ -4a+3c &= 0 \\ -4b+3d &= 1 \end{cases}$$

which can be solved using Gaussian Elimination (row reduction):

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 \\ -4 & 0 & 3 & 0 & 0 \\ 0 & -4 & 0 & 3 & 1 \end{array}\right) \to \cdots \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3/11 \\ 0 & 1 & 0 & 0 & -2/11 \\ 0 & 0 & 1 & 0 & 4/11 \\ 0 & 0 & 0 & 1 & 1/11 \end{array}\right).$$

Hence,

$$C = \frac{1}{11}\begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix}.$$

You should fill in all of the missing row reduction to verify this answer. Also, to check this answer you should multiply AC and CA to be sure that you get the identity matrix with both multiplications.

Definition 2.30. In matrices we define the following:

- The **additive identity** of an $m \times n$ matrix is

$$0_{m\times n} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

- The **additive inverse** of an $m \times n$ matrix A is $(-A)$ since $A + (-A) = (-A) + A = 0$.
- The **multiplicative identity** of an $n \times n$ matrix A is the matrix

$$I = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

- The **multiplicative inverse** of an $n \times n$ matrix A is an $n \times n$ matrix C such that $AC = CA = I$.

The zero matrix act's like the zero integer; adding the zero matrix doesn't change the sum. Similarly, the identity matrix (with ones down the main diagonal and zeros elsewhere) acts like the integer 1; when multiplying by the identity matrix the product doesn't change.

2.6.1 Inverses

The following activity defines the matrix inverse for a 2×2 matrix. The reader should carefully work this problem and take careful note of the result since it shows the general process for finding inverses.

Problem 2.31. Consider the matrix $A = \begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix}$. We would like to find the inverse of A so that when we multiply the inverse by A we get the identity I.

(a) Let's first try a na ive *inverse*. Let

$$B = \begin{pmatrix} 1/2 & 1/3 \\ 1/2 & 1/4 \end{pmatrix}.$$

Find the products AB and BA and verify that $AB \neq I$ and $BA \neq I$.

We really want to find the matrix B such that $AB = BA = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The following parts of this activity will guide you toward that goal.

(b) Create an augmented matrix $(A|I)$.

(c) Use elementary row operations to reduce the matrix in part (a) to an augmented matrix of the form $(I|\star)$.

(d) Check that the matrix on the right-hand side of your answer in part (b) is actually the inverse of A.

(e) Now consider the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Repeat parts (a) and (b) to find the inverse of a general 2×2 matrix. You should have a factor of $\frac{1}{ad-cb}$ in your matrix. The denominator of this fraction is called the *determinant* of the matrix A.

▲

The previous problem illustrated the method for finding the inverse of a matrix:

Technique 2.32 (Process for finding A^{-1} if it exists)**.** The following is the technique for find the inverse of a matrix (if it exists).

1. augment the matrix with the identity, then

2. row reduce to get the identity on the left-hand side of the augmented matrix.

Example 2.33. Let's find the inverse of the matrix from the preview activity using this method instead. Let $A = \begin{pmatrix} 1 & 2 \\ -4 & 3 \end{pmatrix}$. We want to find A^{-1} such that $AA^{-1} = I$ and $A^{-1}A = I$.

Solution: Using the equation $AA^{-1} = I$ and knowing that we are seeking A^{-1} we can write the augmented system $(A|I)$ and row reduce until we get $\left(I|A^{-1}\right)$:

$$
\begin{aligned}
(A|I) &= \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -4 & 3 & 0 & 1 \end{array}\right) \\
&\to \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 11 & 4 & 1 \end{array}\right) \\
&\to \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 4/11 & 1/11 \end{array}\right) \\
&\to \left(\begin{array}{cc|cc} 1 & 0 & 3/11 & -2/11 \\ 0 & 1 & 4/11 & 1/11 \end{array}\right) = \left(I|A^{-1}\right)
\end{aligned}
$$

Hence, the inverse of A is $A^{-1} = \frac{1}{11}\begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix}$

Example 2.34. In part (e) of the previous activity you also worked to find the inverse for the generic 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If you did all of your work correctly you will have found that

$$A^{-1} = \frac{1}{ad - bc} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Let's use this formula to verify (for a third time) the inverse of the matrix from the preview.

Solution: Since $A = \begin{pmatrix} 1 & 2 \\ -4 & 3 \end{pmatrix}$ we can apply the 2×2 inverse formula to get

$$A^{-1} = \frac{1}{(1)(3) - (2)(-4)} \cdot \begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix}$$

The reader should be cautious here. The formula that you derived for 2×2 matrices only makes sense for that size. The only true method for finding the inverse (with the tools we have) is to augment your matrix with the identity and to row reduce.

One other trouble comes when it is impossible to get the identity matrix to appear on the right. When this happens it is an indication that the matrix does not have a multiplicative inverse. In the following activity you will practice this technique on a few matrices.

Problem 2.35. Find the inverse for each of the following matrices if it exists. If it does not exist then determine why not.

(a) $\begin{pmatrix} 1 & -3 \\ 4 & -9 \end{pmatrix}$

(b) $\begin{pmatrix} 3 & 6 \\ 4 & 7 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 & -1 \\ -4 & -7 & 3 \\ -2 & -6 & 4 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

▲

2.6.2 Determinants

Finding a matrix inverse is often a tedious task. As it turns out, there is a very handy number associated with a square matrix that one can use to determine if a matrix is invertible[§]. As we saw previously, if the quantity $ad - bc$ is zero in the 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the matrix cannot have an inverse. This is not a peculiarity of 2×2 matrices! The value $ad - bd$ is called the determinant of the 2×2 matrix.

What we need is a way to define the **determinant** of a square matrix of any size. In order to do so it is easiest to observe how the formula works by following a pattern rather than reading the technical definition. The next problem will walk you through the process.

Problem 2.36. Follow these instructions to see how to find a determinant of a square matrix.

(a) Find the determinant of the 2×2 matrix $\begin{pmatrix} 4 & -1 \\ -2 & 1 \end{pmatrix}$ using the fact that

$$\det(A) = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

(b) Now consider the matrix

$$A = \begin{pmatrix} 1 & 5 & 3 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{pmatrix}.$$

(i) Cross out row 1 and column 1. Call the resulting 2×2 matrix A_{11}.

(ii) Cross out row 1 and column 2. Call the resulting 2×2 matrix A_{12}.

(iii) Cross out row 1 and column 3. Call the resulting 2×2 matrix A_{13}.

[§]This is actually one of many tests that you can be used to determine if a matrix is invertible. A discussion of such techniques will wait until you know a bit more linear algebra.

(iv) The determinant of A is

$$\det(A) = 1 \cdot \det(A_{11}) - 5 \cdot \det(A_{12}) + 3 \cdot \det(A_{13}).$$

Perform this computation.

(c) Find the determinant of the matrix

$$A = \begin{pmatrix} 2 & 3 & -4 \\ 4 & 0 & 5 \\ 5 & 1 & 6 \end{pmatrix}.$$

(d) Fill in the blanks below to set up the determinant calculation for the following 4×4 matrix.

$$A = \begin{pmatrix} 2 & 3 & 5 & 7 \\ 0 & 3 & 2 & 9 \\ 3 & -2 & 6 & 1 \\ 2 & 0 & 0 & 4 \end{pmatrix}$$

$$\det(A) = 2 \cdot \begin{vmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{vmatrix} - 3 \cdot \begin{vmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{vmatrix} + 5 \cdot \begin{vmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{vmatrix} - 7 \cdot \begin{vmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{vmatrix}$$

(e) In your notes, write a few sentences describing the process for finding determinants of square matrices.

▲

Example 2.37. In this example we will work through the determinant of the 3×3 matrix

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & -5 \\ 0 & 0 & 3 \end{pmatrix}$$

Solution: Let's expand along the first row:

$$\begin{aligned} \det(A) &= 1 \cdot \begin{vmatrix} 2 & -5 \\ 0 & 3 \end{vmatrix} - 0 \cdot \begin{vmatrix} 0 & -5 \\ 0 & 3 \end{vmatrix} + 3 \cdot \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} \\ &= 1 \cdot ((2)(3) - (0)(-5)) - 0 \cdot ((0)(3) - (0)(-5)) + 3 \cdot ((0)(0) - (0)(2)) \\ &= 1 \cdot 6 - 0 \cdot 0 + 3 \cdot 0 \\ &= 6. \end{aligned}$$

Also notice in this example that the entire lower triangle of the matrix is filled with zeros. When this is the case you may observe the nice pattern that the determinant is actually just the product of the entries on the main diagonal (you should prove that this is true). Hence, in this problem we know that $\det(A) = 1 \cdot 2 \cdot 3 = 6$. Be careful! If you don't have an entire triangle of zeros then this little *trick* will not work.

Problem 2.38. Find the determinant of the following matrices. Is there anything special that you can say about these matrices? Do you notice any ways to make the determinant computation faster on these matrices?

$$A = \begin{pmatrix} 1 & 3 \\ 6 & 2 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$$

$$C = \begin{pmatrix} 2 & 3 & 2 \\ 4 & 7 & 3 \\ 1 & 0 & 5 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 3 & 2 \\ 0 & 0 & 3 \\ 1 & 0 & 5 \end{pmatrix}$$

$$E = \begin{pmatrix} 2 & 3 & 2 \\ 0 & 7 & 3 \\ 0 & 0 & 5 \end{pmatrix}$$

▲

Problem 2.39. Find the value of k so that the matrix A is not invertible.

$$A = \begin{pmatrix} 2 & 4 \\ 3 & k \end{pmatrix}$$

▲

Problem 2.40. Given the matrix

$$B = \begin{pmatrix} 2-x & 1 \\ 4 & 2-x \end{pmatrix}$$

find all of the values of x that are solutions to the equation $\det(B) = 0$. ▲

The following Theorem states several properties of determinants. While these are all very useful in their own right, we will not take the time here to expand upon their proofs.

Theorem 2.41. Important properties of determinants:

1. A square matrix A is invertible if and only if $\det(A) \neq 0$.
2. The determinant of the identity matrix is $\det(I) = 1$.
3. If A is a triangular matrix, then $\det(A)$ is the product of the entries on the main diagonal of A.
4. If A and B are both square matrices of the same size, then $\det(AB) = \det(A)\det(B)$.

5. If A is a square matrix, then $\det(A^T) = \det(A)$.
6. If A is an $n \times n$ matrix and c is a real number, then $\det(cA) = c^n \det(A)$.
7. If A is an invertible square matrix then from parts (2) and (4) above, $\det(A^{-1}) = \frac{1}{\det(A)}$.

In Theorem 2.41, every property listed has significant impact on the computation of determinants and inverses. The final activity in this section illustrated the time savings and possible pitfalls that can occur.

Problem 2.42. In this activity we'll explore a few properties of the determinant.

(a) How do $\det(A)$ and $\det(A^{-1})$ relate to each other? Hint: we know that $AA^{-1} = I$.

(b) Determine if the matrix B is invertible without trying to calculate the inverse.

$$B = \begin{pmatrix} 2 & 4 & -3 \\ 0 & 0 & 7 \\ 0 & 0 & 3 \end{pmatrix}$$

(c) If the entries in a 10×10 matrix A are known to within a 5% error, what is the maximum error in the determinant computation?
Hint: Let $c = 1.05$. What is $\det(cA)$?

▲

The last problem in the previous problem should serve as a warning! When doing a determinant computation on a computer there is natural roundoff error due to the fact that any number stored in a computer can only be stored with finite bits. Hence, every determinant computation on a large matrix should be immediately suspect! For this reason, it is often preferred to avoid the use of the determinant if at all possible when dealing with large matrices.

2.7 Technology For Linear Algebra

There are many technological tools designed to efficiently perform linear algebra operations. At this stage in many student's academic careers they are very familiar with the TI calculators (TI-89, TI-Nspire, TI-Voyage 200, etc). The computer software MatLab is also very prevalent in many colleges and industries as well as Maple, Mathematica, Sage, MS Excel, and many many others. The following is a very brief guide to using the TI Calculator and MatLab for common operations. If the reader wants more information about how to use these tools they are only an internet search away.

Entering matrices: Consider the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

MatLab: A matrix is written in square brackets with columns separated by commas and rows separated by semicolons.

```
A = [1 , 2 , 3 ; 4 , 5 , 6 ; 7 , 8 , 9]
```

TI Calculator: A matrix is written with curly brackets around the entire matrix and curly brackets around each row. The rows and entries are separated by commas.

```
{{1,2,3},{4,5,6},{7,8,9}} -> A
```

The right arrow is the `STO` button. Storing the matrix as a letter will allow you to do computations with the matrix.

Arithmetic: Addition, subtraction, and multiplication all work as expected in both pieces of software.

Transpose: The transpose switches the rows and columns.

MatLab: The apostrophe is the transpose operator in MatLab. For example, $B = A^T$ in MatLab is

```
B = [1 , 2 , 3 ; 4 , 5 , 6 ; 7 , 8 , 9]'
```

TI Calculator: After entering your matrix and storing it as a variable, the additional matrix operations are found under the `MATH` – `Matrix` menu. The `MATH` menu is `2ND` – `5`.

Reduced Row Echelon Form: Both pieces of software will perform elementary row operations with the `rref` command.

Solving Systems in MatLab: The `rref` command will give the reduced form of the matrix, but to solve a system, MatLab has a very powerful system solver: the `\`. As an example, consider the matrix equation

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ 7 \end{pmatrix} \quad \text{symbolically:} \quad A\mathsf{x} = \mathsf{b}$$

```
A = [1 , 2 , 3 ; 4 , 5 , 6 ; 7 , 8 , -2];
b = [-3 ; 2 ; 7];
x = A\b
```

2.8 Span, Linear Independence and Basis

The importance of linear algebra cannot be under stated. Indeed, most every mathematician will agree that linear algebra is the most important mathematical subject that a mathematical scientist can possibly learn. Let's say that again

> **Linear algebra is the most important mathematical subject a student can learn!**

As such, this section is dedicated to a few of the fundamental theoretical ideas behind linear algebra. The presentation in this section is restricted to matrices and vectors but we'll soon see that the ideas presented herein are applicable in a very wide variety of areas. Let's start with a short exploration.

Problem 2.43 (The Magic Carpet Ride 1)**.** You are a young traveler leaving home for the first time. Your parents want to help you on your journey, so just before your departure they give you two gifts. Specifically, they give you two forms of transportation: a hover board and a magic carpet. Your parents inform you that both the hover board and the magic carpet have restrictions in how they operate:

- If you traveled "forward" on the hover board for one hour it wold move along a diagonal path that would result in a displacement of 3 miles East and 1 mile North of the starting location. Mathematically, the hover board's motion is restricted to the vector $\mathsf{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$
- If you traveled "forward" on the magic carpet for one hour it wold move along a diagonal path that would result in a displacement of 1 mile East and 2 miles North of the starting location. Mathematically, the magic carpet's motion is restricted to the vector $\mathsf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Your Uncle Euler suggests that your first adventure should be to go visit the wise man, Old Man Gauss. Uncle Euler tells you that Old Man Gauss lives in a cabin that is 107 miles East and 64 miles North of your home. Can you use the hover board and the magic carpet to get to Old Man Gauss' cabin? Be able to defend your answer. ▲

Problem 2.44 (Magic Carpet Ride 2)**.** Old Man Gauss wants to move to a cabin in a different location. You are not sure whether Gauss is just trying to test your wits at finding him or if he actually wants to hide somewhere that you can't visit him.

Are there some locations that he can hide and you cannot reach him with using the hover board and the magic carpet? Describe the places that you can reach using a combination of the hover board and the magic carpet and those you cannot. Be able to support your answers. ▲

Problem 2.45 (Magic Carpet Ride 3)**.** Suppose now that you get a third mode of transporation: a jet pack!. In this new scenario assume that your three modes of transportation work as follows:

- The hover board's motion is restricted to the vector $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.
- The magic carpet's motion is restricted to the vector $v_2 = \begin{pmatrix} 4 \\ 1 \\ 6 \end{pmatrix}$.
- The jet pack's motion is restricted to the vector $v_3 = \begin{pmatrix} 6 \\ 3 \\ 8 \end{pmatrix}$.

You are allowed to use each mode of transportation **EXACTLY ONCE** (in the forward or backward direction) for a fixed amount of time (c_1 on v_1, c_2 on v_2, and c_3 on v_3). Find the amounts of time on each mode of transportation (c_1, c_2, and c_3 respectively) need t to go on a journey that starts and ends at home $(0,0,0)$ OR explain why it is not possible to do so. ▲

Problem 2.46 (Magic Carpet Ride 4)**.** Modify the jet pack's restriction so that it is not possible to ride each mode of transportation exactly once and end up back at home. ▲

2.8.1 Span

Now let's formalize a few of the ideas that we just ran into.

Definition 2.47. The **span** of a collection of vectors $\{u_1, u_2, \dots, u_n\}$ is the set

$$\{c_1 u_1 + c_2 u_2 + \cdots c_n u_n : c_j \in \mathbb{R}\}$$

This is the set of all linear combinations of the vectors $u_1, \dots, u_n$.

Problem 2.48. In this activity we will explore the concept of span. The formal definition will be delayed slightly. Instead we will use technology in this activity to build intuition. Open the GeoGebra applet http://tube.geogebra.org/student/m1254137 in a browser window and use the applet to help answer the following questions.

Loosly speaking, the **span** of a set of vectors is the collection of all vectors that can be formed by taking linear combinations.

(a) If $u = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $v = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ then the span of u and v is the set of all vectors of the form $c_1 u + c_2 v$. Use the applet to geometrically describe span($\{u, v\}$).

(b) Describe the span of $u = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $v = \begin{pmatrix} -3 \\ -6 \end{pmatrix}$ geometrically.

(c) Is the zero vector always part of the span of a collection of vectors? Explain?

(d) What must be true (geometrically) about two 2-dimensional vectors if their span is the entire two-dimensional plane?

(e) What must be true (geometrically) about two 2-dimensional vectors if their span is a line in the two-dimensional plane?

▲

Now we'll officially define the span of a collection of vectors. In order to do that we'll define a few symbols that are very useful in mathematics.

Definition 2.49. The space $\mathbb{R}^n$ is the set of order n-tuples with real entries. For example, $\mathbb{R}^2$ is the set of all ordered pairs $\mathbb{R}^2 = \{(x, y)$ such that x and y are real numbers$\}$ that make up 2D space (you're probably used to this set from prior math classes). The set $\mathbb{R}^3$ is the set of ordered triples $\mathbb{R}^3 = \{(x, y, z)$ such that x, y and z are real numbers$\}$ that make up 3D space.

Example 2.50. What is the span of the vectors $\mathsf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\mathsf{v}_2 = \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}$ in $\mathbb{R}^3$? In other words, what is

$$\text{span}\left(\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}\right)?$$

Is the vector $\mathsf{u} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ in the span of v_1 and v_2?

Solution: The span is the collection of all linear combinations of the two vectors. Therefore, any vector w that is in the span is a linear combination of v_1 and v_2:

$$\mathsf{w} = c_1 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}.$$

Since there are only two vectors in this three-dimensional space the span will be a plane in $\mathbb{R}^3$ (shown in Figure 2.4). To test if u is in span$(\{\mathsf{v}_1, \mathsf{v}_2\})$ we need to see if there are constants c_1 and c_2 such that $c_1\mathsf{v}_1 + c_2\mathsf{v}_2 = \mathsf{u}$. Setting this up as a system of equations we see that we want to solve

$$c_1 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

for c_1 and c_2. This vector equation can be rewritten as an augmented system and row

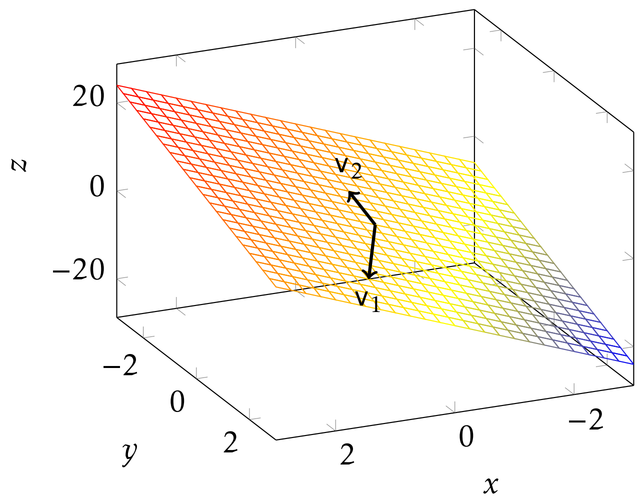

Figure 2.4. The span of v_1 and v_2 in $\mathbb{R}^3$.

reduced as follows:

$$\left(\begin{array}{cc|c} 1 & 0 & 2 \\ 2 & -1 & -1 \\ -1 & 3 & 3 \end{array}\right) \to \cdots \to \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & -10 \end{array}\right).$$

Given that the system of equations does not have a unique solution we see that u is not in the span of v_1 and v_2.

Problem 2.51. In each of the following, determine if w is in the span of the other vectors. If it is then express w as a linear combination of the other vectors.

(a) $v_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, and $w = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$. (Use the applet http://tube.geogebra.org/student/m1254137 to visualize this problem)

(b) $v_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, $v_3 = \begin{pmatrix} 5 \\ -6 \\ 8 \end{pmatrix}$, and $w = \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix}$.

▲

2.8.2 Linear Independence, Linear Dependence, and Basis

The span of the vectors $u = (1,0)^T$ and $v = (0,1)^T$ is the entire two dimensional plane $\mathbb{R}^2$. This means that if we have a vector in $\mathbb{R}^2$, say $x = (3,5)^T$ then we know that it can be written as a linear combination of the vectors u and v:

$$\begin{pmatrix} 3 \\ 5 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 5\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The vector $x = (3,5)^T$ can be built from the vectors u and v so, in some sense, x *depends* on u and v. We would say that the set of vectors $\{u, v, x\}$ is *linearly dependent* since one of the vectors in the set can be built from linear combinations of the others. The set $\{u, v\}$ is called *linearly independent* since none of the vectors in the set can be built from linear combinations of the other vectors in the set.

Definition 2.52. A collection of vectors $\{v_1, v_2, \dots, v_p\}$ in $\mathbb{R}^n$ is called **linearly independent** if the vector equation

$$c_1 v_1 + c_2 v_2 + \cdots c_p v_p = 0$$

has *only* the trivial solution $c_1 = 0, c_2 = 0, \dots, c_p = 0$.

Definition 2.53. A collection of vectors $\{v_1, v_2, \dots, v_p\}$ is called **linearly dependent** if it is not linearly independent.

Example 2.54. Determine whether or not the vectors v_1, v_2, and v_3 are linearly independent or dependent.

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad v_2 = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} \quad v_3 = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$$

Solution: To determine if v_1, v_2, and v_3 are linearly independent or not we need to consider the equation

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0.$$

If the solution to this equation is that $c_1 = c_2 = c_3 = 0$ then the vectors will be linearly independent. Otherwise they will be linearly dependent. This can be viewed as a question about the solutions to the following system of equations (which we will write in several different ways):

$$c_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which is the same as

$$\begin{cases} c_1 + 3c_2 + 4c_3 & = 0 \\ 2c_1 - 2c_2 + c_3 & = 0 \\ 3c_1 + c_3 & = 0 \end{cases}$$

which is the same as the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 2 & -2 & 1 & 0 \\ 3 & 0 & 1 & 0 \end{array}\right)$$

Row reducing the augmented form gives us

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right).$$

Hence, we can see that the solution to the system of equations is $c_1 = c_2 = c_3 = 0$. This implies that $\mathsf{v}_1, \mathsf{v}_2$, and v_3 are linearly independent.

Now let's look at one more example.

Example 2.55. Determine whether or not the vectors v_1, v_2, and v_3 are linearly independent or dependent.

$$\mathsf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \mathsf{v}_2 = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} \quad \mathsf{v}_3 = \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$$

Solution: We'll start by writing the system of equations $c_1\mathsf{v}_1 + c_2\mathsf{v}_2 + c_3\mathsf{v}_3 = 0$ in multiple ways just as we did before.

$$c_1\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + c_2\begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ or } \begin{cases} c_1 + 3c_2 + 4c_3 & = 0 \\ 2c_1 - 2c_2 & = 0 \\ 3c_1 + 3c_3 & = 0 \end{cases} \text{ or } \left(\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 2 & -2 & 0 & 0 \\ 3 & 0 & 3 & 0 \end{array}\right)$$

Row reducing the augmented form gives us

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Now we can see that there are infinitely many solutions to the equation $c_1\mathsf{v}_1 + c_2\mathsf{v}_2 + c_3\mathsf{v}_3 = 0$ and so the vectors $\mathsf{v}_1, \mathsf{v}_2$, and v_3 are linearly dependent.

Moreover, we can see that $c_1 + c_3 = 0$ and $c_2 + c_3 = 0$. We can rewrite these equations as $c_1 = -c_3$ and $c_2 = -c_3$, and therefore c_1, c_2 and c_3 must satisfy the equation

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} t, \quad \text{for} \quad t \in \mathbb{R}.$$

The fact that there are infinitely many solutions to the system of equations further illustrates the idea that the vectors $\mathsf{v}_1, \mathsf{v}_2$ and v_3 are linearly dependent: the coefficients of c_1 and c_2 depend on the value for c_3.

Problem 2.56. Use the definitions of linearly independent and linearly dependent to answer the following questions.

(a) Are the vectors $\mathsf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathsf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ linearly independent or dependent? Explain.

(b) Are the vectors $\mathsf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathsf{v}_2 = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$ linearly independent or dependent? Explain.

(c) Are the vectors $\mathsf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \mathsf{v}_2 = \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}, \mathsf{v}_3 = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$, and $\mathsf{v}_4 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ linearly independent or dependent? Explain.

(d) If nonzero vectors v_1 and v_2 in $\mathbb{R}^2$ only span a line in $\mathbb{R}^2$ are they linearly independent or linearly dependent?

(e) If nonzero vectors v_1, v_2 and v_3 in $\mathbb{R}^3$ span all of $\mathbb{R}^3$ are they linearly independent or linearly dependent?

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Loosely speaking, linear independence means that no vector in the collection can be *built* from the other vectors in the collection. Linearly dependence, on the other hand, means that at least one vector can be built from the other vectors in the collection. The real beauty of linearly independent sets is that they can serve as very simple descriptors of a much larger set. For example, the vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are linearly independent (as seen in the previous activity) and the span of these two simple vectors is all of $\mathbb{R}^2$. These two facts together mean that $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are the simplest building blocks for the entire xy-plane; there is no unneeded information and there is enough to build every point. In some sense, these two vectors are the DNA of the xy-plane.

There are many other linearly independent spanning sets for $\mathbb{R}^2$ but the amazing fact is that they all have exactly 2 vectors in them. Similarly, all of the linearly independent spanning sets of $\mathbb{R}^3$ will all contain exactly 3 vectors. In fact, this is where our intuitive notion of dimensionality comes from; the number of vectors in the linearly independent spanning set.

Definition 2.57. If the set of vectors $\mathcal{B} = \{\mathsf{b}_1, \mathsf{b}_2, \dots, \mathsf{b}_p\}$ spans a space V then $\mathcal{B}$ is said to be a **basis** of V if it is linearly independent. In other words, a basis is a linearly independent spanning set.

Example 2.58. Show that $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^2$.

Solution: We first need to show that $\mathcal{B}$ is linearly independent. This means that we need to show that the equation $c_1\mathbf{b}_1 + c_2\mathbf{b}_2 = 0$ has only the trivial solution $c_1 = c_2 = 0$. Indeed,

$$\begin{aligned}
&c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
&\implies \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
&\implies \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & 1 & 0 \end{array}\right) \to \cdots \to \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \\
&\implies c_1 = c_2 = 0.
\end{aligned}$$

Hence, $\mathcal{B}$ is a linearly independent set.

To show that $\mathcal{B}$ spans $\mathbb{R}^2$ we need to show that any vector $\begin{pmatrix} x \\ y \end{pmatrix}$ can be *built* as a linear combination of the vectors in $\mathcal{B}$. Indeed,

$$\begin{aligned}
&c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \\
&\implies \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \\
&\implies \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & 1 & 0 \end{array}\right) \to \cdots \to \left(\begin{array}{cc|c} 1 & 0 & \frac{x+y}{3} \\ 0 & 1 & \frac{-2x+y}{3} \end{array}\right) \\
&\implies c_1 = \frac{x+y}{3} \quad \text{and} \quad c_2 = \frac{-2x+y}{3}.
\end{aligned}$$

Hence, given **any** point in the xy-plane we can find c_1 and c_2 that will build the point as a linear combination of vectors in $\mathcal{B}$. Therefore, $\mathcal{B}$ is a basis for $\mathbb{R}^2$.

Example 2.59. Is $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right\}$ a basis for $\mathbb{R}^2$?

Solution: No. The reader should verify that the two vectors in $\mathcal{B}$ are linearly dependent so they cannot possible span all of $\mathbb{R}^2$.

Problem 2.60. (a) Find a basis for the set of vectors in $\mathbb{R}^2$ on the line $y = 3x$.

(b) Find a basis for the space spanned by the vectors $\mathsf{v}_1, \dots, \mathsf{v}_5$.

$$\mathsf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 3 \end{pmatrix}, \quad \mathsf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathsf{v}_3 = \begin{pmatrix} 2 \\ -2 \\ -8 \\ 0 \end{pmatrix}, \quad \mathsf{v}_4 = \begin{pmatrix} 2 \\ -1 \\ 10 \\ 3 \end{pmatrix}, \quad \mathsf{v}_5 = \begin{pmatrix} 3 \\ -1 \\ -6 \\ 9 \end{pmatrix}.$$

(Use technology to complete this exercise to save a bit of time)

(c) Let $\mathsf{v}_1 = \begin{pmatrix} 4 \\ -3 \\ 7 \end{pmatrix}$, $\mathsf{v}_2 = \begin{pmatrix} 1 \\ 9 \\ -2 \end{pmatrix}$, and $\mathsf{v}_3 = \begin{pmatrix} 7 \\ 11 \\ 6 \end{pmatrix}$, and also let $H = \text{span}(\{\mathsf{v}_1, \mathsf{v}_2, \mathsf{v}_3\})$. It can be verified that $4\mathsf{v}_1 + 5\mathsf{v}_2 - 3\mathsf{v}_3 = 0$. Use this information to find a basis for H. (There is more than one answer)

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Problem 2.61. Mark each statement as True or False. Justify each answer.

(a) A single vector by itself is linearly dependent.

(b) If $H = \text{span}(\{\mathsf{b}_1, \dots, \mathsf{b}_p\})$ then $\{\mathsf{b}_1, \dots, \mathsf{b}_p\}$ is a basis for H.

(c) The columns of an invertible $n \times n$ matrix form a basis for $\mathbb{R}^n$.

(d) A basis is a spanning set that is as large as possible.

(e) Any linearly independent set is a space V is a basis for V.

(f) If a finite set S of nonzero vectors spans a space V, then some subset of S is a basis for V.

(g) A basis is a linearly independent set that is as large as possible.

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2.8.3 The Column and Null Spaces of a Matrix

For any matrix there are several fundamental spaces that fully describe the actions that a matrix can have on a vector. The two that we focus on here are the column space and the null space.

Definition 2.62. The **column space** of a matrix A is the space that is spanned by the columns of the matrix.

Definition 2.63. The **null space** of an $m \times n$ matrix A, written $\text{Nul}(A)$, is the set of all solutions of the homogeneous equation $A\mathsf{x} = 0$.

Example 2.64. Find the null and column spaces of the matrix

$$A = \begin{pmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{pmatrix}$$

and determine if $\mathsf{u} = \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix}$ belongs to the null space of A.

Solution: The column space of A is the space spanned by the columns of A. We need to determine if the three columns are linearly independent. Row reduction shows us that

$$\begin{pmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{pmatrix} \to \cdots \to \begin{pmatrix} 1 & 0 & \frac{5}{2} \\ 0 & 1 & \frac{3}{2} \end{pmatrix}$$

so the third column vector depends on the other two. Hence, there are two linearly independent vectors that form a basis for the column space. Since each vector is in $\mathbb{R}^2$ we see that the column space is all of $\mathbb{R}^2$.

The null space of A is the set of solutions to $A\mathsf{x} = 0$. We have already row reduced the homogeneous system so we know that $x_1 = -5/2x_3$ and $x_2 = -3/2x_3$. Hence, if $x_3 = t$ then

$$\mathsf{x} = \begin{pmatrix} -5/2t \\ -3/2t \\ t \end{pmatrix} = \begin{pmatrix} -5/2 \\ -3/2 \\ 1 \end{pmatrix} t$$

where t is any real number.

Clearly, if we take $t = -2$ then $\mathsf{x} = \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix}$ showing that u is in the null space of A.

Another (simpler) way to see that u is in the null space of A is to observe that

$$A\mathsf{u} = \begin{pmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 5-9+4 \\ -25+27-2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Problem 2.65. Find the null and column spaces for the matrix

$$A = \begin{pmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{pmatrix}.$$

The matrix A is 3×4. Be sure to specify where the null and column spaces

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2.9 Lab Activities for Linear Algebra

In 1973, Wassily Leontief was awarded the Nobel prize for his work in economics. Part of his work was to apply the basic concepts of linear algebra to model supply and demand within simple economies. His theory, called the Leontief Input Output model, serves as a simplified model to predict production.

Lab Exploration 2.66. Suppose that an economy is divided into three sectors: manufacturing, agriculture, and services. Let x be a 3×1 vector (called the *production vector*) that lists the outputs of each sector for one year. At the same time, let d be a 3×1 vector (called the *demand vector*) that lists any external demands on the goods and services from the non productive part of the economy (consumer demand, government consumption, exports, etc.)

As the three sectors produce goods to meet consumer demand, the producers themselves create additional *intermediate demand* for goods they need as inputs for their own production. For example, the agriculture sector will use equipment made by the manufacturing sector. The basic question Leontief asked is "is there a production level x such that the amounts produced will exactly balance the total demand for the production?" In other words, find x so that

$$\underbrace{\text{amount produced}}_{\mathsf{x}} = \underbrace{\text{intermediate demand}}_{C\mathsf{x}} + \underbrace{\text{external demand}}_{\mathsf{d}}.$$

Here we take C to be a matrix defining unit consumptions of the various sectors (see the table).

	Inputs per Unit of Output		
Purchased From:	**Manufacturing**	**Agriculture**	**Services**
Manufacturing	0.50	0.40	0.20
Agriculture	0.20	0.30	0.10
Services	0.10	0.10	0.30

In the table, the columns read as follows:

- To produce 1 unit, manufacturing needs 0.50 units from other parts of manufacturing, 0.20 units from agriculture, and 0.10 units from services.
- To produce 1 unit, agriculture needs 0.40 units from manufacturing, 0.30 units from other parts of agriculture, and 0.10 units from services.
- To produce 1 unit, services need 0.20 units from manufacturing, 0.10 units from agriculture, and 0.30 units from other services.

Suppose that the demand is 50 units for manufacturing, 30 units for agriculture, and 20 units for services.

(a) If the production is

$$\mathsf{x} = \begin{pmatrix} 140 \\ 20 \\ 50 \end{pmatrix}$$

then what is the sum of the intermediate demand and the external demand? Did the economy under or over produce in each of the sectors?

(b) We want to find the production x that gives a perfectly balance economy: $\mathsf{x} = C\mathsf{x}+\mathsf{d}$. How do we solve this matrix equation?

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2.9.1 Input-Output Economies

In the previous problem we explored situations where a simple economy over or under produces. Theoretically, a perfectly functioning economy has no surplus or shortage. We would like to explore Leontief's original question and determine if there is a perfect production level that will yield no surplus or shortage.

The matrix equation associated with the Leontif Input Output model is

$$\mathsf{x} = C\mathsf{x} + \mathsf{d}. \tag{2.11}$$

The trouble is that the production level, x, shows up on both sides of the matrix equation. To overcome this fact we do a few simple steps of matrix arithmetic to equation (2.11):

$$\mathsf{x} = C\mathsf{x} + \mathsf{d} \quad \Longrightarrow \quad \mathsf{x} - C\mathsf{x} = \mathsf{d} \quad \Longrightarrow \quad (I - C)\mathsf{x} = \mathsf{d} \quad \Longrightarrow \quad \mathsf{x} = (I - C)^{-1}\mathsf{d}. \tag{2.12}$$

The right-hand equation of (2.12) shows that there is a ready-made formula that allows us to solve for the production level.

Example 2.67. Solve for the optimal production level using the internal and external demands from Problem 2.66.

Solution: We want to find the production level x given that the consumption matrix C and the demand vector d are

$$C = \begin{pmatrix} 0.5 & 0.4 & 0.2 \\ 0.2 & 0.3 & 0.1 \\ 0.1 & 0.1 & 0.3 \end{pmatrix} \quad \text{and} \quad \mathsf{d} = \begin{pmatrix} 50 \\ 30 \\ 20 \end{pmatrix}.$$

From equation (2.12) we need to solve the equation $(I - C)\mathsf{x} = \mathsf{d}$ by evaluating $\mathsf{x} = (I - C)^{-1}\mathsf{d}$.

It is impractical to actually compute the inverse[a], so instead we can create an augmented matrix $(I - C\,|\,\mathsf{d})$ and use row reduction.

$$(I - C\,|\,\mathsf{d}) = \left(\begin{array}{ccc|c} 0.5 & -0.4 & -0.2 & 50 \\ -0.2 & 0.7 & -0.1 & 30 \\ -0.1 & -0.1 & 0.7 & 20 \end{array}\right) \to \cdots \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 225.9 \\ 0 & 1 & 0 & 118.5 \\ 0 & 0 & 1 & 77.8 \end{array}\right).$$

Hence, in this economy, manufacturing must produce approximately 226 units, agriculture must produce approximately 119 units, and services must produce approximately 78 units. This will lead to a balanced economy where this is no surplus or shortage.

[a] In the vast majority of linear algebra applications the inverse is never computed.

Lab Exploration 2.68. The consumption matrix C below is based on input-output data for the U.S. economy in 1958, with data for 81 sectors grouped into 7 larger sectors:

1. nonmetal household and personal products,
2. final metal products (such as motor vehicles),
3. basic metal products and mining,
4. basic nonmetal products and agriculture,
5. energy,
6. services, and
7. entertainment and miscellaneous products.

$$C = \begin{pmatrix} 0.1588 & 0.0064 & 0.0025 & 0.0304 & 0.0014 & 0.0083 & 0.1594 \\ 0.0057 & 0.2645 & 0.0436 & 0.0099 & 0.0083 & 0.0201 & 0.3414 \\ 0.0264 & 0.1506 & 0.3557 & 0.0139 & 0.0142 & 0.0070 & 0.0236 \\ 0.3299 & 0.0565 & 0.0495 & 0.3636 & 0.0204 & 0.0483 & 0.0649 \\ 0.0089 & 0.0081 & 0.0333 & 0.0295 & 0.3412 & 0.0237 & 0.0020 \\ 0.1190 & 0.0901 & 0.0996 & 0.1260 & 0.1722 & 0.2368 & 0.3369 \\ 0.0063 & 0.0126 & 0.0196 & 0.0098 & 0.0064 & 0.0132 & 0.0012 \end{pmatrix}, \quad \mathsf{d} = \begin{pmatrix} 74000 \\ 56000 \\ 10500 \\ 25000 \\ 17500 \\ 196000 \\ 5000 \end{pmatrix}$$

(a) Use technology to find the production levels needed to satisfy the final demand d.

(b) The demand in 1964 was

$$\mathsf{d} = \begin{pmatrix} 99640 & 75548 & 14444 & 33501 & 23527 & 263985 & 6526 \end{pmatrix}^T$$

Find the production levels needed to satisfy this demand.

(c) In the six years between 1958 and 1964 the demand changed drastically in several categories leading to changes in the production levels. Use the data in this problem to extrapolate to current demands and production levels. Give several reasons why your extrapolation will ultimately give poor estimates.

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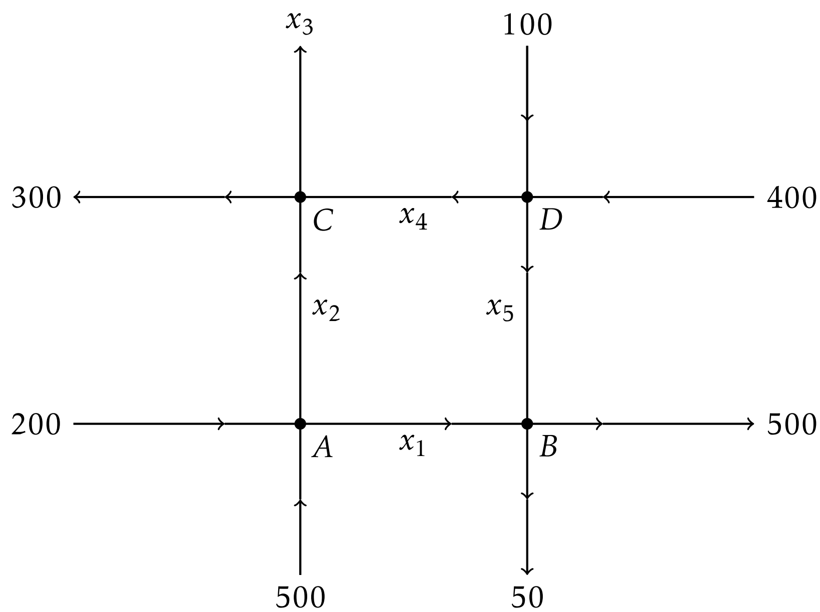
2.9.2 Traffic Networks

A traffic network can be viewed as a collection of lines (streets) and intersections. When modeling a large street network, the influx from certain suburbs and other cities is generally well known. To estimate the number of cars on given streets, traffic engineers often put counters at the intersections. This allow them to calculate flow given that

$$\text{flow into an intersection} = \text{flow out of an intersection.}$$

This simple *conservation law* creates a system of equations for all of the streets that can be solved to find the flow on each individual street.

Lab Exploration 2.69. Consider the collection of one-way streets in a downtown area.



(a) Fill in the table for each intersection. The first line has been done to get you started.

Intersection	Flow In	Flow Out
A	$200 + 500$	$x_1 + x_2$
B		
C		
D		

(b) Write the equation

$$\text{total flow into the network} = \text{total flow out of the network.}$$

This equation describes the fact that the network is not closed but the number of cars in the network must be conserved (just like conservation of mass or momentum from physics).

(c) Use parts (a) and (b) to write a system of equations. Solve the system to determine the traffic flow at $x_1, x_2, \ldots, x_5$.

(d) **Sensitivity Analysis:** There are 7 different given values in this network. What is the percent change in each of the variables $x_1, \ldots, x_5$ if each of the given values were to increase or decrease by 10%? Use a table like the one below to help organize the results, and use your results to write a short report to the traffic division.
Hint: Test each value one at a time.

		Percent Change				
Old Value	New Value	x_1	x_2	x_3	x_4	x_5
100	110					
100	90					
400	440					
400	360					
500	550					
500	450					
50	55					
50	45					
500	550					
500	450					
200	220					
200	180					
300	330					
300	270					

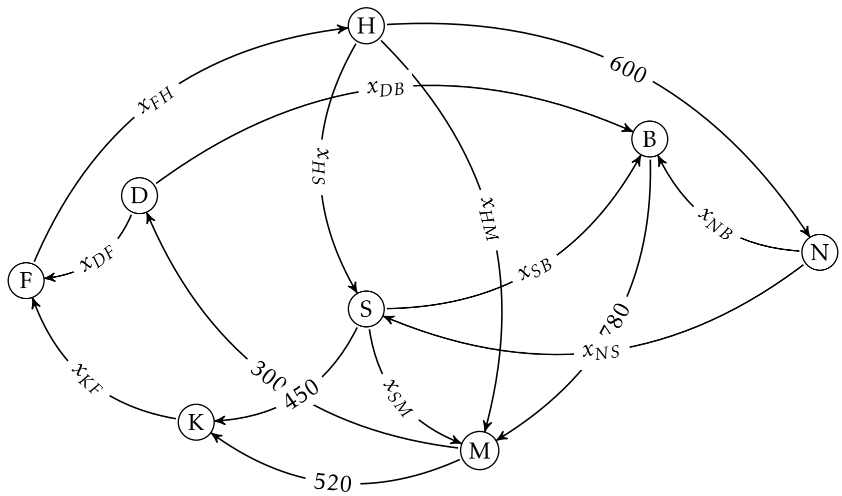
▲

Lab Exploration 2.70. Your boss at the trucking company wants you to solve the following traffic flow problem. In this problem, the lettered nodes are distribution centers for your trucking company and the numbers and variables along the arcs are the truck flows in a given year. Remember that for each distribution center

yearly flow into the distribution center = yearly flow out of the distribution center.

For example, at node H we must have $x_{FH} = x_{HS} + x_{HM} + 600$.
Hint: write an equation for each node and then rearragne into an augmented system of equations.

Your goal is to find the roads with the maximum truck volume and to perform a sensitivity analysis on the known information in the problem (i.e. if each known value were to change by as much as 10%, how much would the maximum road volume change). Give advice to your boss based on your analysis keeping in mind that the highway department would ultimately like to minimize the maximum truck volume on the roads.



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2.9.3 Balancing Chemical Equations

The final application that we will discuss here is that of balancing chemical equations.

Consider the chemical reaction: Nitrogen Dioxide plus water yields Nitrous acid and Nitric acid.

$$aNO_2 + bH_2O \rightarrow cHNO_2 + dHNO_3$$

The coefficients $a, b, c,$ and d are unknown positive integers. The reaction must be balanced; that is, the number of atoms of each element must the same before and after the reaction. Because the number of atoms must remain the same we end up with the following system of equations:

$$\begin{array}{ll} \text{Oxygen:} & 2a + b = 2c + 3d \\ \text{Nitrogen:} & a = c + d \\ \text{Hydrogen:} & 2b = c + d \end{array}$$

Rearranging this into an augmented system gives the homogeneous system

$$\left(\begin{array}{cccc|c} 2 & 1 & -2 & -3 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 2 & -1 & -1 & 0 \end{array}\right)$$

Performing row reductions on this system gives

$$\left(\begin{array}{cccc|c} 2 & 1 & -2 & -3 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 2 & -1 & -1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 2 & -1 & -1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right) \implies \begin{array}{l} a = 2d \\ b = d \\ c = d \end{array}$$

In this chemical equation we see that d, the coefficient of Nitric acid, is a free variable. Mathematically this means that we can freely choose d to be any value we like, but scientifically we choose the smallest positive integer: $d = 1$. Hence

$$a = 2, \quad b = 1, \quad c = 1, \quad \text{and} \quad d = 1$$

so the balanced chemical equation becomes

$$2NO_2 + H_2O \to HNO_2 + HNO_3$$

Lab Exploration 2.71. Find the lower case values that balance each of the chemical equations.

(a) Aluminum oxide and carbon react to create elemental aluminum and carbon dioxide:

$$aAl_2O_3 + bC \to cAl + dCO_2.$$

(b) Limestone, $CaCO_3$, netralizes the acid, H_3O, in acid rain by following the equation

$$aH_3O + bCaCO_3 \to cH_2O + dCa + eCO_2.$$

(c) Boron sulfide reacts violently with water for form boric acid and hydrogen sulfide gas (the smell of rotten eggs).

$$aB_2S_3 + bH_2O \to cH_3BO_3 + dH_2S$$

▲

Chapter 3

First Order Models

3.1 Birth, Death, and Immigration Exploration

Supplies

1 partner, 1 cup, 1 handfull or bag of M&M's, and 1 computer with access to Moodle and Excel.

Scenarios

The M&M's in this activity represent a very fragile species with a high death rate. To determine which M&M's survive and which die in a given year we will shake the M&M's and pour them on the table. The M&M's that land with the M up unfortunately die. The remaining M&M's survive for another year.

Scenario #1 (Deaths) Count the number of M&M's in your cup and track the number of M&M's remaining until the species goes extinct. Add your data to the Google Sheet linked from Moodle.

Scenario #2 (Deaths and Immigration): In this scenario, every M&M that lands M-up will die, but at the end of the year there are 5 new M&M's added to the population due to immigration. Track the number of M&M's remaining until you believe that the population has reached a steady state. Add your data to the second tab of the Google Sheet linked from Moodle.

Scenario #3 (Births, Deaths, and Immigration): In this final scenario, every M&M that lands M-up will die, and at the end of the year there are 5 new M&M's added to the population due to immigration. After the immigration occurs, 10% of the remaining M&M's give birth to new M&M's. Add these births to the population. Track the number of M&M's (after the death, immigration, and the births) until you think the population has reached a steady state. input your data into the Google Sheet, and move to the next year.

Mathematical Modeling

Our goal is to find a function that models each scenario. Copy your data sets from the Google Sheet to your own machines and try to determine the type of function that best models this scenario. When you write your functions, use the variable n for the time.

Remember, we are trying to find a mathematical function that best models the scenario. That does not necessarily mean that you should be doing any data fitting! Instead, you should try two methods:

1. Can you find an algebraic function that appears to fit your data?

$$M(n) = \underline{\hspace{6cm}}$$

2. Or, can you write an expression for how the population changes? In other words,

$$\text{number of } \underline{\text{new}} \text{ M\&M's at year } n = \underline{\hspace{6cm}}$$

3.2 Models for Birth, Death, and Immigration

Problem 3.1. In the birth, death, and immigration exploration you should have generated data with your class for the three scenarios. Go back to the Google Sheet with all of the data, copy your group's data to Excel and make the following plots:

- A plot with population on the horizontal axis and average rate of change of the population for the birth model on the vertical axis. You will need to calculate the average rate of change in Excel first.
- A plot with population on the horizontal axis and average rate of change of the population for the birth & death model on the vertical axis. You will need to calculate the average rate of change in Excel first.
- A plot with population on the horizontal axis and average rate of change of the population for the birth, death, & death model on the vertical axis. You will need to calculate the average rate of change in Excel first.

▲

Problem 3.2. All of the plots that you made in the previous problem *should* appear to follow a linear trend. Use Excel to fit a line to each of these plots. Use your results from Excel to fill in the following blanks.

$$\text{Birth Model:} \quad \frac{\Delta P}{\Delta t} = \underline{\hspace{6cm}}$$

$$\text{Birth \& Death Model:} \quad \frac{\Delta P}{\Delta t} = \underline{\hspace{6cm}}$$

$$\text{Birth, Death, \& Immigration Model:} \quad \frac{\Delta P}{\Delta t} = \underline{\hspace{6cm}}$$

(a) Explain why the slope of each of these linear functions is negative.

(b) Explain what the "y-intercept" means in the context of each model.

▲

In the birth model you should have found a model similar to

$$\frac{\Delta P}{\Delta t} = -0.5P,$$

but notice that in our exploration of the problem with the M&M's we were taking Δt to be 1. Hence we can rewrite our model more simply as

$$\Delta P = -0.5P.$$

When we're dealing with discrete time steps as in this problem we often use the notation P_n to talk about the population at time $t = n$ and P_{n+1} to talk about the population at time $t = n+1$. With this new notation the model becomes

$$P_{n+1} - P_n = -0.5P_n,$$

and this is known as a **difference equation** since the left-hand side of the equation gives the difference between two time steps.

Problem 3.3. Write a difference equation model for the Birth & Death model and for the Birth, Death, & Immigration model.

$$\begin{aligned} \text{Birth Model:} \quad & P_{n+1} - P_n = -0.5P_n \\ \text{Birth \& Death Model:} \quad & P_{n+1} - P_n = \underline{\hspace{5cm}} \\ \text{Birth, Death, \& Immigration Model:} \quad & P_{n+1} - P_n = \underline{\hspace{5cm}} \end{aligned}$$

▲

In a difference equation model it is often convenient to solve for P_{n+1} and rewrite as $P_{n+1} = P_n - 0.5P_n$ (in the birth model). This clearly simplifies to $P_{n+1} = 0.5P_n$, and furthermore this allows for an easy implementation into a spreadsheet program.

Problem 3.4. Using a spreadsheet program like Excel, implement the difference equation models to built a table and graph that predicts the population at any discrete time. ▲

Problem 3.5. In the M&M exploration we were treating time as if it occurred only in discrete steps. What is wrong with this assumption? Specifically, if the M&M's are supposed to represent a population then what is wrong with using discrete time in our model? ▲

Let's return to the models in Problem 3.2 and now consider that time really shouldn't be taken in discrete steps. Instead, time really happens on a continuous scale and, as such, taking $\Delta t = 1$ doesn't really make sense. If we take $\Delta t \to 0$ the average rate of change $\Delta P/\Delta t$ becomes the derivative $\frac{dP}{dt}$. For example, in the birth model $\frac{\Delta P}{\Delta t} = -0.5P$ becomes $\frac{dP}{dt} = -0.5P$ as $\Delta t \to 0$. This new model is called a **differential equation**.

Problem 3.6. Write a differential equation modeling the Birth & Death and the Birth, Death, & Immigration scenarios.

$$\begin{aligned} \text{Birth Model:} \quad & \frac{dP}{dt} = -0.5P \\ \text{Birth \& Death Model:} \quad & \frac{dP}{dt} = \underline{\hspace{5cm}} \\ \text{Birth, Death, \& Immigration Model:} \quad & \frac{dP}{dt} = \underline{\hspace{5cm}} \end{aligned}$$

▲

The differential equation models are most likely more realistic if we thinking of the M&M exploration as modeling population. Populations just simply don't change all at once; they change continuously in time.

For the birth model

$$\frac{dP}{dt} = -0.5P$$

we can still use Excel to create a visual model of the solution to the differential equation. Recall that

$$\frac{dP}{dt} \approx \frac{\Delta P}{\Delta t}$$

and if we take Δt to be *really small* then the approximation isn't *too bad*. Hence, the differential equation $\frac{dP}{dt} = -0.5P$ can be approximated by

$$\frac{\Delta P}{\Delta t} \approx -0.5P,$$

and using the notation of difference equations we can rewrite as

$$\frac{P_{n+1} - P_n}{\Delta t} \approx -0.5P_n.$$

Now if we solve for P_{n+1} we can get an equation that can be implemented in a spreadsheet program:

$$P_{n+1} \approx P_n - 0.5\Delta t P_n.$$

Problem 3.7. Use a spreadsheet program to get a graphical approximation to the Birth model

$$\frac{dP}{dt} = -0.5P.$$

Compare your results to the data and to the solution to the difference equation that you built in Problem 3.4. ▲

Problem 3.8. Now use a spreadsheet program to build approximations to the differential equations for the Birth & Death and Birth, Death, & Immigration models that you built in Problem 3.6. Compare your graphical solutions to the graphical solutions of the discrete time models and to the data. ▲

3.3 Difference Equations and Differential Equations

In Calculus, change is measured over infinitesimally small increments. Unfortunately, this is not always appropriate. For example, if a bank compounds the interest on a savings account every month then it may not make sense to examine the value of the account during the middle of the month; it will be the same as after the interest was last computed. On the other hand, if the temperature of a cup of coffee is being modeled over time then it makes sense to talk about the temperature at any given instant in time. Time in the bank account example is taken in discrete steps (one per month), while time in the coffee example is continuous.

Definition 3.9. A **discrete time model** is a mathematical model where time is measured in individual steps that can be enumerated. The model either does not make sense at intermediate steps or is considered constant.

Definition 3.10. A **continuous time model** is a mathematical model where time can take any non-negative real number quantity. In continuous time models it makes sense to discuss the meaning of the model at any non-negative value of time.

Problem 3.11. In the following scenarios decide whether you should use a discrete time model or a continuous time model.

(a) Water draining from a tank drains at a rate proportional to the square root of the height of the water.

(b) The rate of change of the money in an interest-bearing savings account is proportional to the amount of money that is in the account.

(c) The Department of Fish, Wildlife, and Parks proposes a restocking plan where they restock the mountain lakes in Montana once per year.

(d) The displacement of a spring-mass oscillator is proportional to the acceleration of the mass.

▲

As we've already seen in the M&M exploration, it was easier to model the change in the population that it was to *guess* the algebraic function that modeled the populations. The two primary types of mathematical models that model *change* are **difference equations** and **differential equations.**

Definition 3.12. A **difference equation** is a discrete-time model that relates discrete rates of change.

$$\text{(new value)} - \text{(old value)} = \text{(change in value)}. \tag{3.1}$$

Definition 3.13. A **differential equation** is a continuous-time model that relates instantaneous rates of change.

Problem 3.14. When money is placed into a bank savings account, the bank pays interest into the account once a month, thereby increasing the value of the account, provided no money is withdrawn!

Consider a bank account that has been opened with an initial deposit of $\$4,500$ and which receives monthly interest equal to 1.1% of the account value. The account is to be held for two years before any withdrawals are made.

(a) should we be modeling this situation with discrete time or continuous time?

(b) For the given initial deposit and interest rate, how much money would the bank add to the savings account after one month?

(c) How much money would the savings account be worth after 1 month?

(d) What is the value of the savings account after one and a half months?

(e) How much money would there be in the savings account at the beginning of the second month?

(f) How much interest does the bank add to the account at the end of the second month?

(g) How much money is in the savings account after two months?

(h) In words, describe how the money in your savings account changes from one month to the next.

(i) Use everything that you just did to write a difference equation modeling the amount of money in the bank account.

$$A_{n+1} - A_n = \underline{\hspace{5cm}}$$

(j) Use a spreadsheet program to build a numerical and graphical solution to the difference equation.

▲

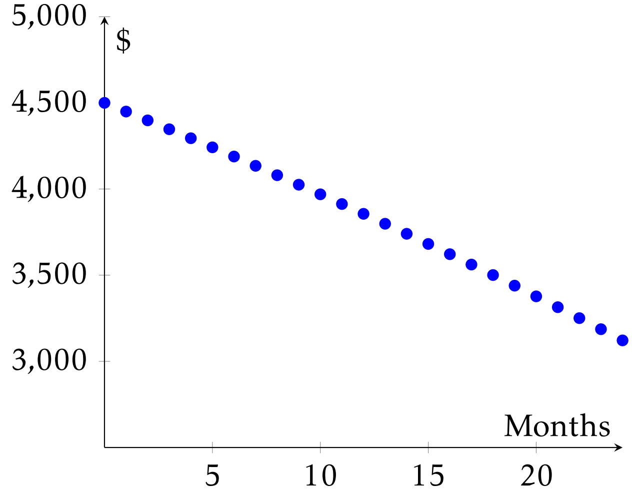
Example 3.15. Consider the savings account situation from Problem 3.14 but include a deduction of $\$100$ per month. Find a difference equation and find a numerical solution to model this situation over a two-year horizon.

The difference equation is

$$\underbrace{\text{new amount in account}}_{a_{n+1}} - \underbrace{\text{old amount in account}}_{a_n} = \underbrace{\text{interest}}_{0.011a_n} - \underbrace{\text{withdrawal}}_{100}$$

More simply: $a_{n+1} = a_n + 0.011a_n - 100$. Or, after some algebra, $a_{n+1} = 1.011a_n - 100$. A numerical solution is simply a table of values (see this video) but is much simpler to visualize as a plot (see this video and this video).

Month	Amount
0	\$4500.00
1	\$4449.50
2	\$4398.44
3	\$4346.83
4	\$4294.64
5	\$4241.88
6	\$4188.54
7	\$4134.62
8	\$4080.10
9	\$4024.98
10	\$3969.25
11	\$3912.92
12	\$3855.96
13	\$3798.37
14	\$3740.16
15	\$3681.30
16	\$3621.79
17	\$3561.63
18	\$3500.81
19	\$3439.32
20	\$3377.15
21	\$3314.30
22	\$3250.76
23	\$3186.52
24	\$3121.57



Example 3.16. A student has a college loan of $\$A$ with an interest rate of $r\%$ APR compounded monthly. She pays $\$p$ per month. Model the amount that the student owes on the loan with a discrete dynamical system, examine the behavior of the system based on different values of the parameter, and give a graph of several numerical solutions.

This situation calls for a discrete dynamical system because the interest and payments are made in discrete time steps. Let a_n be a sequence that represents the amount owed on the loan after n months. The initial loan amount, $\$A$, is the ini-

tial condition, so
$$a_0 = A.$$
The interest is compounded monthly, so each month the new interest is $r/12\%$ of the amount owed in the previous month.
$$\text{new interest} = \frac{r}{12}a_n.$$
The payments are subtracted from the amount owed. Hence, the mathematical model for the amount owed on the loan is
$$\underbrace{a_{n+1} - a_n}_{\text{change in monthly balance}} = \underbrace{\frac{r}{12}a_n}_{\text{amount accumulated from interest}} \underbrace{-p}_{\text{payment}}, \quad \text{where} \quad \underbrace{a_0 = A}_{\text{initial condition}}$$
More simply,
$$a_{n+1} = a_n + \frac{r}{12}a_n - p \quad \text{where} \quad a_0 = A.$$
Even simpler yet (after a little algebra)
$$a_{n+1} = \left(1 + \frac{r}{12}\right)a_n - p.$$
This is a mathematical model for any such situation, and there are many unknown parameters. The fact that you don't know the parameters is what makes the problem interesting!

To discuss the parameters one must think like a scientist; vary one parameter at a time and try to cover all of the possible bases. Let us first consider a loan of $A = \$5000$ at 15% APR. Figure 3.1 shows numerical solutions for several different payments. Clearly if the payment is larger then the amount of time to pay off the loan decreases. Next let us discuss a different situation. Assume that the initial loan is still \$5000 but now assume that the payments are fixed at \$200 and we don't know the rate. Figure 3.2 shows several numerical solutions to the discrete dynamical system at various rates.

Again, there aren't any real surprises here; as the rate goes down the time to pay off the loan goes down. In order to fully explore the parameter space of a DDS it is easy to implement the model in Excel and to use absolute cell references for the parameters.

Problem 3.17. For each of the following situations, create a mathematical model using a difference equation or a differential equation.

(a) The population of a town grows at an annual rate of 1.25%.

(b) A radioactive sample losses 5.6% of its mass every day.

(c) You have a bank account that earns 4% interest every year. At the same time, you withdraw money continually from the account at the rate of \$1000 per year.

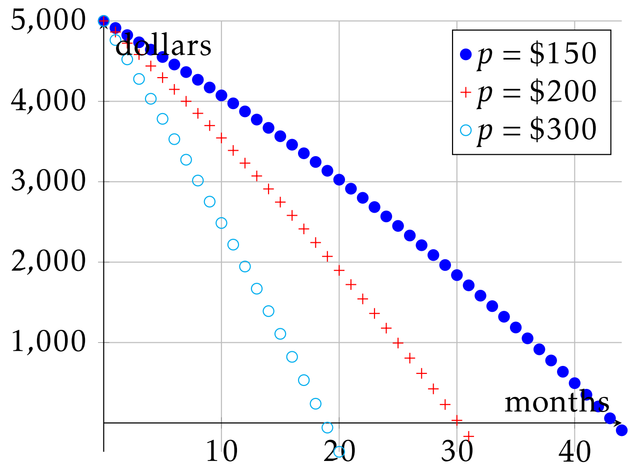


Figure 3.1. Numerical solutions to $a_{n+1} = a_n + \frac{r}{12}a_n - p$ with $a_0 = 5000$ and $r = 0.1$ and different values of p

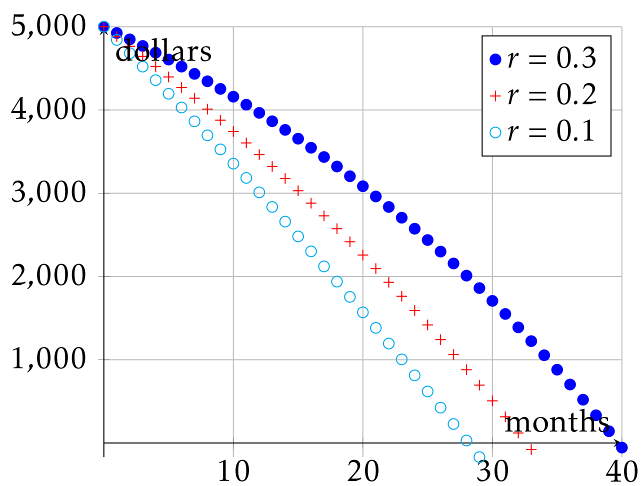


Figure 3.2. Numerical solutions to $a_{n+1} = a_n + \frac{r}{12}a_n - p$ with $a_0 = 5000$ and $p = \$200$ and different values of r

(d) A cup of hot chocolate is sitting in a 70° room. The temperature of the hot chocolate cools by 10% of the difference between the hot chocolate's temperature and the room temperature every minute.

(e) A can of cold soda is sitting in a 70° room. The temperature of the soda warms at the rate of 10% of the difference between the soda's temperature and the room's temperature every minute.

(f) Suppose that there are 400 college students living in a dormitory and that one or more students has a severe case of the flu. Assume some interaction between those infected and those not infected is required to pass the flue. Assume that a student is either infected or susceptible (no immunities). Further assume the number of new infected students each hour is proportional to the product of the infected and the susceptible students. Create a mathematical model for the number of infected students over the course of a day.

Hint: If there are 400 students in the dorm, then the number of susceptible students is 400 minus the number of infected students.

(g) A town has a large reservoir of water that currently contains 450,000 gallons. Each week during the spring W gallons are used by the town, while M gallons flow in from snow melt. If r% of the water evaporates each week, formulate a mathematical model for the amount of water in the reservoir and explore the parameter space with the help of Excel.

Estimates of parameters are as follows: Last year the average water usage during the spring was $W = 50,000$ gallons and the average intake from snow melt was $M = 62,500$ gallons. During the spring the water is estimated to evaporate at approximately 1%.

▲

3.4 Stability and Equilibria

Problem 3.18. Consider the differential equation

$$\frac{dA}{dt} = -0.5A + 0.1.$$

This model comes from a drug dosing problem where your body removes 50% of the drug every hour and an IV drip administers 0.1mg of drug each hour. The value of A is the number of milligrams of drug in the body at time t. If done correctly, the amount of drug in your system will stabilize at a constant level. What is that level? Be sure to fully explain your thinking. ▲

Problem 3.19. Consider the difference equation

$$A_{n+1} - A_n = -0.2A_n + 100.$$

This model comes from a car sales problem where 20% of the cars on the lot are sold in a given month and 100 new cars are delivered each month. If you've done things correctly as the manager of the lot, the number of cars will stabilize at a certain amount. How many cars is this? Be sure to fully explain your thinking. ▲

Problem 3.20. For each of the two previous problems make a plot with "change" on the vertical axis and "amount" on the horizontal axis. Each of these should be a linear function just like what we saw in the birth, death, and immigration models. What is the meaning of the "x intercept" on each of these plots? ▲

Problem 3.21. return to the birth, death and immigration models. You already made a plot with "change" on the vertical axis and "population" on the horizontal axis. Find the point where the linear functions intersect the horizontal axis and interpret these points in the context of the problem. ▲

Definition 3.22 (Equilibrium Point)**.** An **equilibrium point** in a difference or differential equation is the point where the change is 0.

Example 3.23. Consider the differential equation $\frac{dy}{dt} = -\frac{1}{2}(y-4)$. If we set the change to zero then we find that

$$0 = -\frac{1}{2}(y_{eq} - 4) \implies y_{eq} = 4.$$

The equilibrium is at $y = 4$. Furthermore, if we have a y value a bit less than 4 we see that dy/dt is positive so y will be going up. Also, if we have a y value that is a bit more than 4 then dy/dt is negative so y will be going down (see Figure 3.3). We call this type of equilibrium point **stable**.

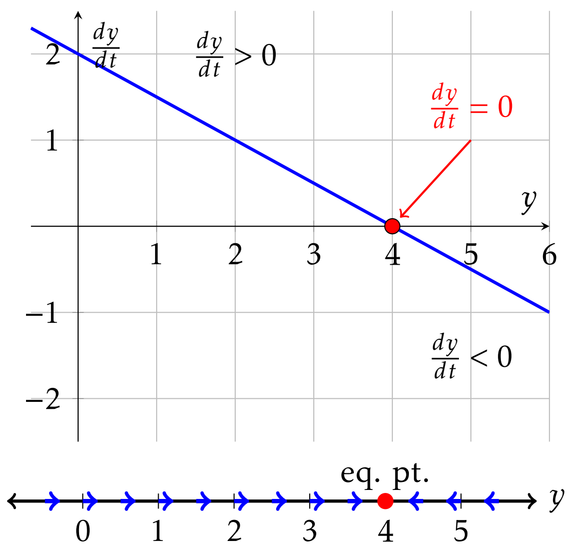


Figure 3.3. A plot of $\frac{dy}{dt}$ vs y (top) and a phase line diagram (bottom) for $\frac{dy}{dt} = -\frac{1}{2}(y-4)$

Example 3.24. Consider the differential equation $\frac{dy}{dt} = \frac{1}{2}(y-4)$. If we set the change to zero then we find that

$$0 = \frac{1}{2}(y_{eq} - 4) \implies y_{eq} = 4.$$

The equilibrium is at $y = 4$. Furthermore, if we have a y value a bit less than 4 we see that dy/dt is negative so y will be going down. Also, if we have a y value that is a bit more than 4 then dy/dt is positive so y will be going up (see Figure 3.4). We call this type of equilibrium point **unstable**.

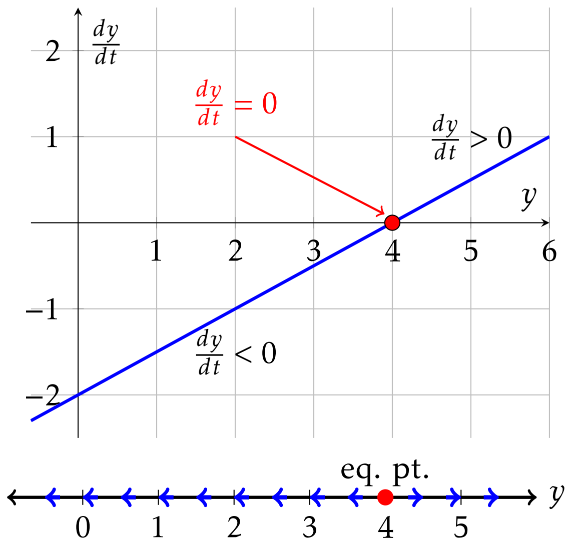


Figure 3.4. A plot of $\frac{dy}{dt}$ vs y (top) and a phase line diagram (bottom) for $\frac{dy}{dt} = -\frac{1}{2}(y-4)$

Problem 3.25. Find and classify the equilibrium points for each of the following difference or differential equations.

(a) $\dfrac{dy}{dt} = y(1-y)$

(b) $A_{n+1} - A_n = 2A_n + 6$

(c) $x_{n+1} - x_n = -3x_n - 2$

(d) $\dfrac{dP}{dt} = -0.5P + 2$

▲

Problem 3.26. Find and classify the equilibrium point for the difference equations modeling the three birth, death, and immigration problems. ▲

3.5 Euler's Method

For a complete discussion of Euler's method see the Active Calculus book Section 7.3.

3.6 Solution Techniques for First Order Models (FINISH)

We now turn to finding analytic solutions to the difference equations and differential equation that we've encountered thus far. We will solve the difference and differential equations in parallel as the techniques are almost identical; the only primary difference is that in a differential equation we can leverage the ideas of calculus to solve the differential equation.

We wish to solve $a_{n+1} = ra_n$ with initial condition a_0. We have seen this type of discrete dynamical system before, and the solution technique comes from observing the following pattern in the sequence:

n	a_n
0	a_0
1	$a_1 = ra_0$
2	$a_2 = ra_1 = r(ra_0) = r^2 a_0$
3	$a_3 = ra_2 = r(r^2 a_0) = r^3 a_0$
4	$a_4 = ra_3 = r(r^3 a_0) = r^4 a_0$
$\vdots$	$\vdots$
k	$a_k = r^k a_0$

Hence, we have just provided sufficient evidence to believe following theorem. *

Theorem 3.27. If $a_{n+1} = ra_n$ with initial condition a_0 then the analytic solution to the DDS is

$$a_k = a_0 \cdot r^k. \tag{3.2}$$

Problem 3.28. Find the solution to the difference equation

$$P_{n+1} - P_n = -0.5P_n$$

with initial condition $P_0 = 50$. ▲

Now let's solve almost the exact same differential equation: $\frac{dy}{dt} = ky$.

Problem 3.29. Which function(s) from calculus has the property that if you take the derivative you get the exact same function back? ▲

3.6.1 Separation of Variables

Problem 3.30. Based on your answer to the previous problem, what is the solution to the differential equation $\frac{dy}{dt} = ky$? ▲

*We hesitate to call this a proof since technically the proof would use the principle of mathematical induction.

You should notice that in both the difference equation, $a_{n+1} - a_n = ka_n$, and the differential equation, $\frac{dy}{dt} = ky$, the solution turns out to be exponential. The only big difference is the base of the exponential.

Problem 3.31. Now follow these steps to develop another technique for solving the differential equation $\frac{dy}{dt} = ky$.

(a) Divide both sides of the equation by y and multiply both sides by dt.

(b) You should now have an equation where the only variable on the left-hand side is y and the only variable on the right-hand side is t. Integrate both sides.

(c) Solve for y. You should find the same solution that we *guessed* before, but this time we did it without any guesswork.

▲

Problem 3.32. Next we'll consider a differential equation of the form $\frac{dy}{dt} = y + 1$. Follow the exact same steps as in the previous problem to solve for y. ▲

For more problems related to separation of variables see Active Calculus Section 7.3.

3.6.2 Undetermined Coefficients (FINISH)

FINISH

3.7 First Order Modeling Problems (FINISH)

In this section I intend to show a few modeling problems with fully worked solutions to the resulting difference or differential equation

3.8 Lab Exercises and Explorations for Difference and Differential Equations

Problem 3.33. Watch the following YouTube videos

- This video describes how to use Excel for calculating a sequence from a discrete dynamical system.
  https://youtu.be/kVx7bZTP9L4
- This video shows how to graph a discrete dynamical system in Excel.
  https://youtu.be/k3bwXAeQsKo
- This video shows a bit more about plotting multiple simumlations of a discrete dynamical system with Excel. https://youtu.be/g6OGCmvxCHQ

▲

Now consider the following problems related to discrete dynamical systems. For each problem you need to fully explore the situation by examining all of the possible parameters and hypothetical situations. A good mathematical modeler will consider every possible case and then present their arguments in a thorough, logical, and complete way.

Lab Exploration 3.34. A classmate in MA141 has a credit card that charges 1.25% interest per month. Suppose he can only afford to pay \$75 per month, and he does not make any other charges on this credit card. Your classmate has already racked up \$1,200 on the card and he seems pretty freaked out about this debt! After all, it is very early in the school year!

Model the amount owed on the credit card using a discrete dynamical system (DDS). Use your DDS model to give your classmate advice about how to handle credit cards. Use appropriate mathematics and graphics to make your arguments. Scare tactics are absolutely allowed (and encouraged)!

Some good points might be (<u>but are certainly not limited to</u>):

- If he makes the \$75 payments each month, when will he pay off the card and what will the last payment be? Furthermore, how much will he have paid for this debt?
- If the initial debt was doubled what would happen?
- Is there an initial debt where his \$75 payments will only pay the interest (this is called an equilibrium state)? What happens if he has more on the card that this amount? What about less? Would you classify this equilibrium as *stable* or *unstable*?
- What happens if he changes his payments?
- . . . there are many other scenarios that you <u>should</u> explore. Write up your arguments in a way that will better educate your classmate. Be sure to include all relevant mathematics and graphics with thorough explanations.

▲

Lab Exploration 3.35. Presume that there is a population of about 150 moose in the Bob Marshall Wilderness in northwest Montana (often called "The Bob" . . . if you've never been up there, you absolutely must go!). The Fish, Wildlife, and Parks department in Montana estimates that in each year about 25 moose permanently migrate out of The Bob to other places like Glacier National Park, Canada, or further west into the Flathead Wilderness. They also estimate that about 0.5% of the population die off due to old age and other natural causes. Finally, they estimate that new moose calves are born equal to about 9% of the total population in The Bob.

Write a discrete dynamical system for the yearly moose population, model it in Excel, and then consider the fact that every parameter given in the paragraph above is subject to variability. Perform a sensitivity analysis on your model by allowing the parameters to vary by as much as 10%, and use your results to write an informative paragraph to the Fish, Wildlife, and Parks department of Montana about the moose population in The Bob.

▲

Lab Exploration 3.36. How long does it take an ant to build a tunnel? That seems like a reasonable question. If you ever had an ant colony purchased by a well-meaning aunt for you in grade school you may have watched ants building industriously and you just might have an idea on this. To answer the question we might need some narrowing of scope, some simplification, and certainly some identification of terms and variables before we can get a nice answer. Let us identify some variables and then together make some assumptions which will lead to a mathematical model.

- Let x be the length of the tunnel in feet that an ant builds.
- Let $T(x)$ be the time in hours it takes the ant to build the tunnel of length x.
- We can get some idea of our situation by making a sketch.

1. If we are going to create a model for the time T as a function of the length of the tunnel x should we use a difference equation or a differential equation? Explain.
2. Maybe we can circumvent building a difference or differential equation by simply writing down an algebraic equation for $T(x)$.
   (a) Write down several candidate functions for $T(x)$ and give one or two statements in each's defense and one or two statements against each.
   (b) You may not have gotten very far with part (a), so how about we try some graphical intuition. Make several sketches of $T(x)$ (tunnel length (x) on the x-axis and total time (T) on the y-axis). Give one or two statements in each's defense and one or two statements against each.
3. Hopefully you see that attempting to jump right on top of $T(x)$ can be hard. So, instead of going after $T(x)$ directly let us examine
   (a) List some assumptions which reflect the reality of such a situation and might make the model simple in a first attempt.
   (b) Modeling *change* is often times much simpler than trying to create an algebraic model from scratch. For the present tunnel-building situation, $T(x+h) - T(x)$ models the amount of time it might take an ant to *extend* a tunnel from distance x to distance $x+h$.

   Below are several possible mathematical models for equation $T(x+h) - T(x)$. Defend or reject each and offer your reasons. Perhaps modify one or two and make it better. When trying to reject a model consider some trivial cases and see if it makes sense, e.g., $h = 0$ or $x = 0$ or either h or x very large.

   i) $T(x+h) - T(x) = x + h$.
   ii) $T(x+h) - T(x) = x - h$.
   iii) $T(x+h) - T(x) = x^h$.
   iv) $T(x+h) - T(x) = x \cdot h$.
   v) $T(x+h) - T(x) = h^x$.

vi) $T(x+h)-T(x)=c$.

4. At this point you are ready to write your own equation.

$$T(x+h)-T(x) = \underline{\hspace{6cm}}$$

   (a) List the variables and parameters with all of their units. Also list any assumption on which your equation depends.
   (b) Convert your model difference equation to a differential equation with appropriate initial conditions.
   (c) Solve the differential equation you create in (d) for $T(x)$. Hint: What initial condition $T(0)$ will you use?
   (d) Use your solution from (c) to determine how much longer it takes to build a tunnel which is twice as long as an original tunnel of length L. What would some of your original function models you set forth in 2(a) have told you here?

5. Suppose we had two ants digging from either side of our sand hill along the same straight line. How would this alter the total time for digging the tunnel?

6. Of course, we can apply these same principles of our model to real tunnel building for engineers. If we were considering #5 as related to engineering construction of a long tunnel of length L, outline some of the issues we should be aware of when having two crews (one from each end of the tunnel) working on the tunnel.

▲

Lab Exploration 3.37. Consider a situation in which we are studying Helicobacter Pyloria; an antiobiotic resistant organism that gives people an upset stomach. Assume that there are 50 Helicobacter Pyloria initially in a Petri dish. We lose 35% of the population each hour due to "forces of death," but through a one way hatch, 2 microorganisms per hour can enter our Petri dish in the first hour, 4 microorganisms per hour can enter our Petri dish in the second hour, 6 microorganisms per hour can enter our Petri dish in the third hour, 8 in the fourth hour, etc.

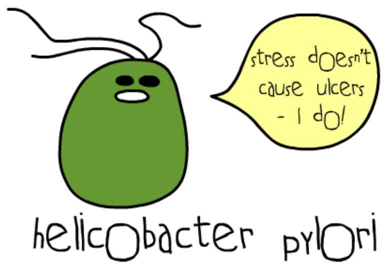
1. Model this situation with (a) a discrete difference equation model **and** (b) a continuous differential equation model.

2. State all of your assumptions used in the model building process.

3. Is there an equilibrium for your model? If so, what it is? If not, why not?

4. Solve each of your models numerically. Use Excel for both the difference and differential equation models. You will need to use Euler's method for the differential equation model (choose a small time step!). Create the plots for the first 12 hours of the experiment.

5. Compare your models and comment on differences and similarities.



6. Your models should take the form

$$\begin{aligned}
&\text{Difference Equation:} && a_{n+1} - a_n = r \cdot a_n + m \cdot n + b \\
&\text{Difference Equation (after simplifying):} && a_{n+1} = (1+r) \cdot a_n + m \cdot n + b \\
&\text{Differential Equation:} && \frac{dy}{dt} = r \cdot y + m \cdot t + b
\end{aligned}$$

   What are the values of r, m, and b for this modeling scenario?

7. We would like to find an analytic solution for each of these models, but we haven't encountered these types of difference or differential equations yet. One technique is to *guess* the form of the solution and then use the difference equation or differential equation along with the intial conditions to find the coefficients.

   The guesses for this model are:

   - Difference Equation:
   $$a_n = C_1(1+r)^n + C_2 t + C_3$$
   - Differential Equation:
   $$y(t) = C_1 e^{rt} + C_2 t + C_3$$

   Work with your team to find C_1, C_2, and C_3 for each of the two models.

8. Finally, plot the analytic solutions along side your numerical solutions.

▲

Lab Exploration 3.38. A lake in northern Montana is dominated by Arctic Grayling (henceforth called "species A") but the Department of Fish, Wildlife, and Parks is planning to slowly introduce Bull Trout ("species B"). The lake is popular with sport fishermen who remove both species of fish from the lake regularly.

The Department of Fish, Wildlife, and Parks has carefully estimated the number of fish taken by sport fishing each week, and they have decided to keep the fish population as constant as possible by replacing the fish lost by an equal number of Arctic Grayling and Bull Trout. For example, if there are $N = 50$ fish in the lake at the beginning of the week and fishermen remove $M = 10$ fish during that week, then the fish and wildlife people will restock the lake with 5 Arctic Grayling and 5 Bull Trout. Hence the population of the lake will remain $N = 50$ fish at the end of each week, assuming no new fish are born.

Both fish species swim freely throughout the lake and both are targeted by similar bait used by sport fisherman.

In your lake you will use $N =$ ____________ and $M =$ ____________.

In summary:

- The week starts with $N =$ ____________ fish.
- The fish swim freely around the lake.
- $M =$ ____________ fish are removed from the lake at random during the week.
- $M =$ ____________ fish are restocked at the end of the week. $M/2 =$ ____________ of those fish are Arctic Grayling and $M/2 =$ ____________ of those fish are Bull Trout.

1. **Conjecture:**
   (a) What do you think will happen to the populations of species A and B over a long period of time?
   (b) Is it possible that species A will be eliminated from the lake with the restocking plan? Explain.

2. **Simulate:**
   (a) Use pennies to represent your N fish and decide with your partner(s) which coin face represents which species. Start your lake with 100% species A.
   (b) Decide with your partner(s) how to simulate the swimming of fish, the fishermen, and the Department of Fish, Wildlife, and Parks' restocking plan. Simulate roughly 15 weeks of the fish population representing species A and B with coins. Be sure to let the fish swim thoroughly around the lake and keep track of the proportions made up by species A and B.

Week #	**Number in population**		**Proportion of population**	
	species A	species B	species A	species B
0			1	0
1				
2				
3				
4				
⋮	⋮	⋮	⋮	⋮

3. **Model:**
   (a) Propose a verbal model for the rate of change of species B in the lake.

   rate at which species B changes = ________________________

   (b) Explicitly state any assumptions that you are using in your verbal model.

(c) Introduce mathematical notation for your proposed model and write your verbal model mathematically. Be sure to include any necessary condition(s).

model: ________________________________

condition(s): ________________________________

4. **Analyze:**

(a) According to your model, what is the long term effect on the fish population in the lake? Use your model to justify your answer algebraically and graphically.

(b) Solve your mathematical model (either numerically or analytically) and compare with your data.

(c) (extension) Suppose now that the Department of Fish, Wildlife, and Parks does not attempt to keep the population in the lake constant. That is, suppose that fishing reduces the population by M_1 fish each week and the Department of Fish, Wildlife, and Parks restocks M_2 fish each week. Fully explore this scenario.

▲

Lab Exploration 3.39.

Niedjatu Elpmeyout
MT Environmental Law Partners
101 Park St.
Helena, MT 59625

O.D.E. Consulting
1601 N. Benton Ave.
Helena, MT 59625

Dear sir or madam:

I have been assigned a case here at my law offices defending a client who got himself into a quite a sticky situation (or rather, a slippery one). My firm would like to secure your services to help us understand the physical aspects and data surrounding the event. In order to protect our client's anonymity, we will request your discretion in sharing this information with the press.

Our client allegedly caused an oil spill over some open water while transporting some cargo. There seems to be some dispute with respect to the amount of oil spilled, and the EPA (those tree-huggers!) has assigned massive fines, which we dispute. While we concede that there was a small amount of oil spilled, we contend that the amount is really not nearly as much as they claim. In fact, our client actually improved the local economy by hiring local workers to assist with containing and cleaning the oil. They should be thanking our client, really. But I digress.

Here's where we need your help. We know that as soon as the resulting oil slick was detected, the Coast Guard wanted to document the size of the oil slick. From time to time, but irregularly, a helicopter was dispatched to photograph the oil slick. On each trip, it arrived over the slick, the pilot took a picture, waited 10 minutes, took another, and then headed home. On each of seven trips the size (in area) of the slick was measured from both photographs, as below.

Area of oil slick (in miles):

Initial Obs.	10 min. later
1.047	1.139
2.005	2.087
3.348	3.413
5.719	5.765
7.273	7.304
8.410	8.426
9.117	9.127

We would like to request the following information from you.

- Build a model for the growth of the oil slick at time t.
- Predict the size of the oil slick, say at $t = 10$, $t = 20$, and $t = 120$ minutes from the start of the oil spill.
- Plot your model of the size of the oil slick as a function of time.
- Find the time at which the oil slick was 8 square miles.
- Determine the time of each of the observations.

Please help us help our client (who, despite what you might have heard in the news, was definitely *not* under the influence of an illegal substance–not at the time of the incident, anyway). We will have to present your argument in court, so please fully explain your work in a clear and concise fashion.

Your company was suggested by one Professor Sullivan of Carroll College, whose services we have used before. He has promised to be available to you, but cannot himself commit to this work because he is teaching some talented and motivated students techniques in mathematical modeling this semester.

Looking forward to seeing your results soon.

Sincerely,

Niedjatu Elpmeyout
MT Environmental Law Partners ▲

Lab Exploration 3.40. A beaker of warm water is placed in a room with an ambient temperature of 72°F. The data for this experiment can be found in the `Newton.xlsx` Excel file on Moodle.

1. Below are 5 proposed differential equation models for the temperature of the water in the beaker.

$$\frac{dT}{dt} = a \tag{3.3}$$
$$\frac{dT}{dt} = a + bt \tag{3.4}$$
$$\frac{dT}{dt} = \frac{A}{B + Ct} \tag{3.5}$$
$$\frac{dT}{dt} = -k(T - T_{env}) \tag{3.6}$$
$$\frac{dT}{dt} = -kT \tag{3.7}$$
$$\frac{dT}{dt} = Ae^{-kt} \tag{3.8}$$

   Spend a few minutes critiquing each of these models. For each model that seems unreasonable, be sure to give a brief explanation.

2. Choose the most appropriate model from the above list (only 1 of them is the *right* one!) and do the following:
   (a) Find any equilibrium points and determine their stability
   (b) solve the differential equation using an appropriate technique. Your answer will have some unknown parameters.

3. Use the Solver in Excel to find the value(s) of the parameter(s) in your model so that your model best fits the data.

4. How would the data (and your solution) change if the beaker had been insulated?

▲

Chapter 4

Second Order Models

4.1 $F = ma$ (Finish)

4.2 Modeling Mass Spring Oscillators (Finish)

4.3 Solution Techniques for Second Order Differential Equations (Finish)

Chapter 5

Systems of Difference and Differential Equations

5.1 Rumor Spreading System (Finish)

5.2 Linear Systems (Finish)

5.3 The Eigenvalue / Eigenvector Problem (Finish)

5.4 Analysis of Linear Systems (Finish)

Appendices

Appendix A

MATLAB Basics

In this appendix we'll go through a few of the basics in MATLAB. This is by no means meant to be an all-encompassing resource for MATLAB programming. A few more thorough resources for MATLAB are listed here.

- https://www.mathworks.com/help/pdf_doc/matlab/matlab_prog.pdf
- https://www.mathworks.com/products/matlab/examples.html
- https://en.wikibooks.org/wiki/MATLAB_Programming
- http://gribblelab.org/scicomp/scicomp.pdf (this is a personal favorite)

In this appendix we'll give examples of some of the more common coding practices that the reader will run into while working through the exercises and problems in these notes.

A.1 Vectors and Matrices

Example A.1. Write the vectors $\mathtt{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathtt{w} = \begin{pmatrix} 4 & 5 & 6 & 7 \end{pmatrix}$ using MATLAB.

Solution:

```
v = [1 ; 2 ; 3]
w = [4 , 5 , 6 , 7]
w = 4:7  % this is shorthand for writing a sequence as a row vector
```

Example A.2. Consider the matrices and vectors

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 5 & 7 \\ 9 & 1 & 3 \\ 5 & 7 & 11 \end{pmatrix} \quad \text{and} \quad \mathrm{b} = \begin{pmatrix} 4 & 3 & -1 \end{pmatrix}$$

- Calculate the product AB using regular matrix multiplication

```
A = [1 , 2 , 3;
     4 , 5 , 6;
     7 , 8 , 0]
B = [3 , 5 , 7;
     9 , 1 , 3;
     5 , 7 , 11]
Product = A*B
```

- Calculate the element-by-element multiplication of A and B

```
ElementWiseProduct = A .* B
```

- Calculate the inverse of A

```
Ainv = A^(-1)
Ainv = inverse(A) % alternative
```

- Calculate the transpose of B

```
Atranspose = transpose(A)
% or as an alternative:
Atranspose = A'  % actually the conjugate transpose but if A is real then ok
```

- Solve the system of equations $A\mathrm{x} = \mathrm{b}$

```
b = [4 ; 3 ; -1]
x = A \ b
```

Example A.3. Code for a matrix of zeros

```
Z = zeros(5,5) % 5 x 5 matrix of all zeros
```

Example A.4. Code for an identity matrix

```
Ident = eye(5,5) % 5 x 5 identity matrix
```

Example A.5. Code for random matrices.

- random matrix from a uniform distribution on $[0,1]$

```
R = rand(5,5) % random 5 x 5 matrix
```

- random matrix from the standard normal distribution

```
R = randn(5,5) % random 5 x 5 matrix
```

Example A.6. A linearly spaced sequence

```
List = linspace(0,10,100)
% a list of 100 equally spaced numbers from 0 to 10
```

A.2 Looping

A loop is used when a process needs to be repeated several times.

A.2.1 For Loops

A `for loop` is code that repeats across a pre-defined sequence.

Example A.7. Write a loop that produces the squares of the first 10 integers.

```
for j = 1:10
    j^2
end
```

The output of this code will be

1
4
9
16
25
36
49
64
81
100

Example A.8. Plot the functions $f(x) = \sin(kx)$ for $k = 1, 1.5, 2, 2.5, \ldots, 5$ on the domain $x \in [0, 2\pi]$.

```
x = linspace(0,2*pi,1000);
for k = 1:0.5:5
    plot(x , sin(k*x) )
    hold on
end
```

A.2.2 The While Loop

A `while loop` is a process that only repeats while a conditional statement is true. Be careful with `while loops` since it is possible to create a loop that runs forever.

Example A.9. Build the Fibonnaci sequence up until the last term is greater than 1000.

```
F(1) = 1; % first term
F(2) = 1; % second term
n = 3;
while F(end)<1000
    F(n) = F(n-1) + F(n-2);
    n=n+1;
end
```

Example A.10. An example of a while loop that runs forever.

```
a = 1;
```

```
while a>0
    a=a+1;
end
```

Example A.11. An example of a while loop that runs forever but with a failsafe step that stops the loop after 1000 steps.

```
a = 1;
counter=1;
while a>0
    a=a+1;
    if counter >= 1000
        break
    end
    counter = counter+1;
end
```

A.3 Conditional Statements

Conditional statements are used to check if something is true or false. The output of a conditional statement is a boolean value; true (1) or false (0).

A.3.1 If Statements

Example A.12. Loop over the integers up to 100 and output only the multiples of three.

```
for j = 1:100
    if mod(j,3) == 0
        j
    end
end
```

Example A.13. Check the signs of two function values and determine if they are opposite.

```
f = @(x) x^3*(x-3);
a = 2;
b = 4;
if f(a)*f(b) < 0
```

```
    fprintf('The function values are opposite sign\n')
elseif f(a)*f(b) >0
    fprintf('The function values are the same sign\n')
else
    fprintf('The function values are both zero\n')
end
```

A.3.2 Case-Switch Statements

Example A.14. Evaluate over several cases.

```
n = 3
switch n
    case 1 % if n == 1
        fprintf('n is 1\n')
    case 2 % if n == 2
        fprintf('n is 2\n')
    case 3 % if n == 3
        fprintf('n is 3\n')
end
```

A.4 Functions

A mathematical function has a single output for every input, and in some sense a computer function is the same: one single executed process for each collection of inputs.

Example A.15. Define the function $f(x) = \sin(x^2)$ so that it can accept any type of input (symbol, number, or list of numbers).

```
f = @(x) sin(x.^2) % defines the function
f(3) % evaluates the function at x=3
x=linspace(0,pi,100);
f(x) % evaluates f at 100 points equally spaced from 0 to pi
```

Example A.16. Write a computer function that accepts two numbers as inputs and outputs the sum plus the product of the two numbers.
First write a file with the following contents.

```
function MyOutput = MyFunctionName(a,b)
    MyOutput = a + b + a*b;
end
```

Be sure that the file name is the same as the function name.
Then you can call the function by name in a script or another function.

```
SumPlusProduct = MyFunctionName(3,4)
```

which will output the number 19.

Example A.17. Write a function with three inputs that outputs the sum of the three. The third input should be optional and the default should be set to 5.

```
function AwesomeOutput = SumOfThree(a,b,c)
    if nargin < 3
        c = 5;
    end
    AwesomeOutput = a+b+c;
end
```

You can call this function with

```
SumOfThree(17,23)
```

which will output $17+25+5=47$. Notice that the third input was left off and a 5 was used in its place.

A.5 Plotting

In numerical analysis we are typically plotting numerically computed lists of numbers so as such we will give a few examples of this type of plotting here. We will not, however, give examples of symbolic plotting.

The `plot` command in MATLAB accepts a list of x values followed by a list of y values then followed by color and symbol options.
`plot(xlist , ylist , color options)`

Example A.18. Plot the function $f(x) = \sin(x^2)$ on the interval $[0, 2\pi]$ with 1000 equally spaced points. Make the plot color blue.

```
x = linspace(0,2*pi,1000);
f = @(x) sin(x.^2);
plot(x , f(x) , 'b')
```

Alternatively

```
x = linspace(0,2*pi,1000);
y = sin(x.^2);
plot(x, y, 'b')
```

Example A.19. Make a 2×2 array of 4 plots of $f(x) = \sin(kx^2)$ for $k = 1, 2, 3, 4$.

```
x = linspace(0,2*pi,1000);
for k=1:4
    subplot(2,2,k)
    plot(x , sin(k*x.^2) , 'b')
end
```

Example A.20. Plot $f(x) = \sin(kx^2)$ for $k = 1, 2, \ldots, 10$ all on the same plot.

```
x = linspace(0,2*pi,1000);
for k=1:10
    plot(x, sin(k*x.^2))
    hold on % this holds the figure window open so you can write on top of it
end
```

Example A.21. Plot the function $f(x) = e^{-x}\sin(x)$ and put a mark at the local max at $x = \pi/4$.

```
x = linspace(0,2*pi,1000); % set up the domain
f = @(x) exp(-x) .* sin(x);
plot(x,f(x),'b',pi/4,f(pi/4),'ro')
```

A.6 Animations

Example A.22. Plot $f(x) = \sin(kx^2)$ for $k = 1$ to $k = 10$ by small increments with a short pause in between each step.

```
x = linspace(0,2*pi,1000);
for k=1:0.01:10 % 1 to 10 by 0.01
    plot(x, sin(k*x.^2))
    hold on % this holds the figure window open so you can write on top of it
    drawnow % draws the plot
```

```
    % the last line gives the illusion of animation
end
```