Differential Equations & Linear Algebra
An Inquiry and Problem Based Approach

(The Second Course of a Two-Semester Sequence)

Solutions

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Chapter 0

To the Student and the Instructor

This document contains lecture notes, classroom activities, examples, and challenge problems specifically designed for Carroll College's MA334 - Differential Equations and Linear Algebra 2 class. The content herein is written and maintained by Dr. Eric Sullivan of Carroll College. Problems were either created by Dr. Sullivan, the Carroll Mathematics Department faculty, part of NSF Project Mathquest, or come from other sources and are either cited directly or cited in the LaTeX source code for the document (and are hence purposefully invisible to the student).

The notes and problems in this document are intended to serve as a text for a second semester of differential equations and linear algebra. At Carroll College the first semester materials are typically covered in MA141 - Introduction to Mathematical Modeling.

0.1 An Inquiry Based Approach

This material is written with an Inquiry-Based Learning (IBL) flavor. In that sense, this document could be used as a stand-alone set of materials for the course but these notes are not a *traditional textbook* containing all of the expected theorems, proofs, examples, and exposition. The students are encouraged to work through problems and homework, present their findings, and work together when appropriate. You will find that this document contains collections of problems with only minimal interweaving exposition. It is expected that you do every one of the problems and then use other more traditional texts as a backup when you are stuck. Let me say that again: this is not the only set of material for the course. Your brain, your peers, and the books linked in the next section are your best resources when you are stuck.

To learn more about IBL go to http://www.inquirybasedlearning.org/about/. The long and short of it is that the students in the class are the ones that are doing the work; proving theorems, writing code, working problems, leading discussions, and pushing the pace. The instructor acts as a guide who only steps in to redirect conversations or to provide necessary insight. If you are a student using this material you have the following jobs:

1. Fight! You will have to fight hard to work through this material. The fight is exactly what we're after since it is ultimately what leads to innovative thinking.

2. Screw Up! More accurately, don't be afraid to screw up. You should write code, work problems, and prove theorems then be completely unafraid to scrap what you've done and redo it from scratch. Learning this material is most definitely a non-linear path.* Embrace this!

3. Collaborate! You should collaborate with your peers with the following caveats: (a) When you are done collaborating you should go your separate ways. When you write your solution you should have no written (or digital) record of your collaboration. (b) The internet is not a collaborator. Use of the internet to help solve these problems robs you of the most important part of this class; the chance for original though.

4. Enjoy! Part of the fun of IBL is that you get to experience what it is like to think like a true mathematician / scientist. It takes hard work but ultimately this should be fun!

0.2 Online Texts and Other Resources

If you are looking for online textbooks for linear algebra and differential equations I can point you to a few. Some of the following online resources may be a good place to help you when you're stuck but they will definitely say things a bit differently. Use these resources wisely.

- The book *Differential Equations with Linear Algebra, An inquiry based approach to learning* is a nice collection of notes covering much of the material that we cover in our class. The order is a bit different but the notes are well done.
  content.byui.edu/file/664390b8-e9cc-43a4-9f3c-70362f8b9735/1/316-IBL%20(2013Spring).pdf

- The ODE Project by Thomas Juson is a nice online text that covers many (but not all) of the topics that we cover in differential equations.
  faculty.sfasu.edu/judsontw/ode/html/odeproject.html

- Elementary Differential Equations by William Trench. This book contains everything(!) but I find that it is very dense and often tough to read. It is a great resource to look up ODE techniques.
  ramanujan.math.trinity.edu/wtrench/texts/TRENCH_DIFF_EQNS_I.PDF

- A First Course in Linear Algebra by Robert Beezer. This book is very thorough but I also find it a bit quirky in some regards (personal bias). It covers everything that we do in linear algebra and more.
  linear.ups.edu/html/fcla.html

- Linear Algebra Workbook by TJ Hitchman. This is a workbook for Dr. Hitchman's class at U. Northern Iowa. Even though it is only a "workbook" it contains some

*Pun intended: our goal, after all, is really to understand that linear algebra is the glue that holds mathematics together.

nice explanations and it has embedded executable code for some problems.
theronhitchman.github.io/linear-algebra/course-materials/workbook/LinAlgWorkbook.html

0.3 To the Instructor

If you are an instructor wishing to use these materials then I only ask that you adhere to the Creative Commons license. You are welcome to use, distribute, and remix these materials for your own purposes. Thanks for considering my materials for your course!

My typical use of these materials are to let the students tackle problems in small groups during class time and to intervene when more explanation appears to be necessary or if the students appear to be missing the deeper connections behind problems. The course that I have in mind for these materials is a second semester of differential equations and linear algebra. As such, this is not a complete collection of materials for either differential equations or linear algebra in isolation. In our first course we discuss matrix operations, Gaussian elimination, the eigenvalue problem, first order linear homogeneous and non-homogeneous differential equations, and second order homogeneous differential equations. You will find that the sections in these notes covering these topics are necessarily light and are meant to only give the students a brief review of the material.

Many of the theorems in the text come without a proof. If the theorem is followed by the statement "prove the previous theorem" then I expect the students to have the skill to prove that theorem and to do so with the help of their small group. However, this course is not intended to be a proof-based mathematics course so several theorems are stated without rigorous proof. If you are looking for a proof-based linear algebra or differential equations course then I believe that these notes will not suffice. I have, however, tried to give thought provoking problems throughout so that the students can engage with the material at a level higher than just the mechanics of differential equations and linear algebra. That being said, there are also several routine exercises throughout the notes that will allow students to practice mechanical skills.

There is a toggle switch in the LaTeX code that allows you to turn on and off the solutions to problems. The line of code
`\def\ShowSoln{0}`
is a switch that, when set to 0, turns the solutions off and when set to 1 turns the solutions on. Just re-compile (`pdflatex`) the document to display the solutions. I typically do not show the solutions to the students while they're learning the material, but I allow them access during exam preparation time so they can check their understanding.

Chapter 1

First Order Differential Equations

You may recall that in an algebraic equation you are seeking to find a number, usually[*] x, so that the given equation holds true. For example, we could solve $x+2=5$ and find that $x=3$ is the only value that makes the equal sign true. As another example, we could solve $x^2-3x+2=0$ using the quadratic formula or factoring and find that $x=1$ and $x=2$ are the only solutions. Your high school algebra classes focused on the techniques necessary to solve many different types of algebraic equations and at this point you likely have the techniques down pat (right?!).

When solving differential equations we are seeking a slightly different goal. This time the unknown is a function and the equation relates the derivative(s) of the function to the function itself. For example, if we consider the simple equation $y'(t)=y(t)$ we could probably guess (using the rules of calculus) that the only functions that satisfy this equation are $y(t)=0$ and $y(t)=Ce^t$. Notice that the solution is not a number but a function. As another example consider $y''(t)=-y(t)$. In this case you can also use your intuition from calculus to guess that $y(t)$ is some combination of sines and cosines: $y(t)=C_1\sin(t)+C_2\cos(t)$. Our goal throughout this course is to build differential equations and find techniques to analyze them. As you might imagine based on the complexity of the derivative rules in calculus, the techniques to find solutions to differential equations can sometimes be quite complicated.

Let's begin by examining a few modeling-type problems where you need to write the differential equation. After we have a few differential equations we will spend some time building up the basic solution techniques.
Note: You are expected to have seen several of these techniques already. If these notes move to fast then go to the appropriate linked texts in Section 0.2.

Problem 1.1. An ant is building a tunnel. We want to create a differential equation model for the total time that it takes for the ant to build the tunnel as a function of the length of the tunnel. Which of the following would be an appropriate differential equation? Let x be the length of the tunnel and let $T(x)$ be the total time to dig a tunnel of length x.

1. $T'=kT$ (rate of change of time proportional to total time taken)

[*]See the TED Talk https://www.ted.com/talks/terry_moore_why_is_x_the_unknown to see why we use x for the unknown in Algebra.

2. $T' = kx$ (rate of change of time proportional to current length of the tunnel)
3. $T' = kx^3$ (rate of change of total time proportional to volume).
4. $T' = kS$ (rate of change of total time proportional to the surface area of the end of the tunnel)

▲

Solution: $T' = kx$ (rate is proportional to length of tunnel).

$$T' = kx \iff T(x) = \frac{k}{2}x^2$$

So the time that it takes to dig the tunnel is a quadratic function of the length.
Show a slope field with $T(0) = 0$ and various values of k.

Problem 1.2. A population of Alaskan Salmon grows according to the following rules:

- If there are no salmon then the population doesn't change (duh).
- If the population reaches the carrying capacity for the environment, M, the size of the population stops changing.
- When the population is growing and is far away from the carrying capacity the growth rate is roughly proportional to the size of the population.

Write a differential equation that models this scenario. Support your model by discussing what occurs when P is close to M and when P is close to 0.

$$\frac{dP}{dt} = \underline{\hspace{5cm}}$$

▲

Solution:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) \quad \text{or} \quad \frac{dP}{dt} = kP(M - P)$$

Note that the units of k are different in the two proposed solutions.

Problem 1.3. A spring oscillates in such a way that its acceleration is proportional to its position relative to an equilibrium point.

- If the spring is a long way from equilibrium then the acceleration is large and pointed back toward equilibrium.
- If the spring is close to equilibrium then the acceleration is small.

Let $y(t)$ be the position of the spring.

$$\frac{d^2y}{dt^2} = \underline{\hspace{5cm}}$$

Sketch a plot of the solution to this differential equation. ▲

Solution:

$$\frac{d^2y}{dt^2} = -ky$$

1.1 Differential Equation Terminology

In this chapter we will look at a few techniques for solving first-order differential equations. Before launching to the techniques let's get a little bit of terminology on the table. I am expecting that much of this terminology is familiar to you already from previous classes. We are going to cover this very quickly and we will be leaving some of the reading and remembering up to you.

Definition 1.4 (First Order Differential Equation)**.** A **first order differential equation** is a differential equation of the form

$$y'(t) = f(y,t).$$

Notice that a first order differential equation contains only the first derivative of the unknown function (hence the name). The function f can be just about anything and it depends on both $y(t)$ and maybe t explicitly.

Definition 1.5 (Autonomous Differential Equation)**.** An **autonomous differential equation** is a differential equation of the form

$$y'(t) = f(y)$$

where there is no explicit dependence of the independent variable t on the right-hand side of the equation. A differential equation that has explicit dependence on t is called **non-autonomous**.

To clarify this point consider the following examples:

$$\text{first order autonomous D.E:} \quad y'(t) = -0.2y(t) + 4 \tag{1.1}$$

$$\text{first order non-autonomous D.E:} \quad y'(t) = -0.2y(t) + 4t \tag{1.2}$$

Definition 1.6 (Linear First Order Differential Equation)**.** A **linear** first order differential equation has the form

$$y'(t) + P(t)y(t) = Q(t) \qquad \text{or} \qquad y'(t) = -P(t)y(t) + Q(t).$$

The reason for the name "linear" is that the right-hand side of this equation is literally a linear function of y. Hence the differential equation can be written as $y' = f(y)$ where $f(y) = -Py + Q$.

Definition 1.7 (Homogenous Differential Equations)**.** A differential equation is called **homogeneous** if, loosely speaking, no terms appear that do not involve the unknown function. Another way to say this is that every term in the differential equation will either contain the function $y(t)$ or its derivatives. A differential equation that is not homogeneous is called **non-homogeneous**.

Problem 1.8. It is often a good exercise to make examples associated with new definitions. For each of the above definitions create an example of a differential equation that *is* described by the definition and a differential equation that *is not* described by the definition. ▲

Problem 1.9. Come up with an example for each of the following descriptions of differential equations.

(a) A linear first order homogeneous differential equation.

(b) A non-linear first order homogeneous differential equation.

(c) A linear first order non-autonomous differential equation.

(d) A linear first order non-autonomous differential equation that is homogeneous.

(e) A linear first order non-autonomous differential equation that is non-homogeneous.

▲

Solution:

(a) $y' = -0.2y$

(b) $y' = -0.2y^2$

(c) $y' = -0.2y + t$

(d) $y' = -0.2yt$

(e) $y' = -0.2y + 3t$

Example 1.10. Here are a few examples of homogeneous and non-homogeneous differential equations.

- The differential equation $y' = -0.2y$ is homogeneous.
- The differential equation $y' = -0.2y + 3$ is non-homogeneous.
- The differential equation $y'' + 3y' - 5y = 0$ is homogeneous.
- The differential equation $y'' + 3y' - 5y = 2$ is non-homogeneous.

- The differential equation $v' = g - cv^2$ is non-homogeneous (and nonlinear).

1.2 Solution Technique: Integration

In the sections that follow we will review (or introduce) some of the primary solution techniques for first order differential equations. As has been mentioned before, it is likely that you have seen these techniques before but it is worth your time to blow the dust off of your memories and to review what you once knew.

Problem 1.11. For each of the following differential equations use the rules of Calculus find the function $y(t)$ that solve the differential equation.

$$\frac{dy}{dt} = 2t + 5 \quad \text{with} \quad y(0) = 3$$
$$\frac{dy}{dt} = \sin(t) \quad \text{with} \quad y(0) = 1$$
$$\frac{dy}{dt} = te^{-t^2} \quad \text{with} \quad y(0) = 0$$

▲

Solution:

$$y(t) = t^2 + 5t + 3$$
$$y(t) = -\cos(t) + 2$$
$$y(t) = -\frac{1}{2}e^{-t^2} + \frac{1}{2}$$

Technique 1.12 (Solution via Integration)**.** To solve

$$\frac{dy}{dt} = f(t)$$

you can first think of "multiplying by dt" to get $dy = f(t)dt$. Then integrate both sides with respect to t. Therefore, $y(t) = \int f(t)dt + C = F(t) + C$ where $F(t)$ is the antiderivative of $f(t)$ such that $F'(t) = f(t)$. Given $y(t_0) = y_0$ we can get C by

$$y_0 = F(t_0) + C \quad \implies \quad C = y_0 - F(t_0)$$

Problem 1.13. Create and solve a first order differential equation (along with an appropriate initial condition) that can be solved using the technique of integration. ▲

Problem 1.14. Solve the differential equation

$$y'(t) = e^{-2t} \quad \text{with} \quad y(0) = 1$$

▲

Solution:

$$y(t) = \int e^{-2t} dt = -\frac{1}{2}e^{-2t} + C$$
$$y(0) = 1 \implies 1 = -\frac{1}{2} + C \implies C = \frac{3}{2} \implies y(t) = -\frac{1}{2}e^{-2t} + \frac{3}{2}$$

Problem 1.15. A stone is dropped from rest at an initial height h above the surface of the earth. We want to show that the speed with which it strikes the ground is $v = \sqrt{2gh}$. Start by writing an appropriate differential equation and then use the differential equation to very this result. You do not need to include air resistance in your model. ▲

Solution:

$$\frac{dv}{dt} = -g \implies v(t) = -gt + 0 \implies \frac{dy}{dt} = -gt \implies y(t) = -\frac{gt^2}{2} + h \implies t_{ground} = \sqrt{\frac{2h}{g}}$$

Therefore,

$$v_{ground} = -g\sqrt{\frac{2h}{g}} = -\sqrt{2gh}$$

1.3 Solution Technique: Separation of Variables

Problem 1.16. Consider the differential equation

$$\frac{dy}{dt} = y$$

with the initial condition $y(0) = 1$.

(a) Putting the differential equation into words:
the derivative of some unknown function is equal to the function itself.
what is the function?

(b) Allow me to abuse some notation:
If you multiply both sides by dt and divide both sides by y we end up with

$$\frac{dy}{y} = dt.$$

Integrate both sides and solve for y.

(c) Compare your answers to parts (b) and (c).

▲

Solution: The solution is clearly $y(t) = e^t$.

Problem 1.17. In part (b) of the previous problem I said that I was “abusing notation”. What does that mean? What notation is being abused? ▲

Technique 1.18 (Separation of Variables)**.** To solve a differential equation of the form

$$\frac{dy}{dx} = f(y) \cdot g(x)$$

Separate and integrate by treating the "dy/dt" as a fraction[a]

$$\int \frac{dy}{f(y)} = \int g(x)dx$$

Notice that the right-hand side of the differential equation factors perfectly hence separating the variables into the functions f and g.

[a]Technically speaking the "dy/dt" is not a fraction it is a shorthand notation for a limit. More technically there is some sneaky chain rule happening behind the scenes here . . . can you find it.

Problem 1.19. With your partner, write a differential equation that can be solved via separation of variables. Once you have your equation trade with a different group and solve their equation. ▲

Problem 1.20. A drug is eliminated from the body via natural metabolism. Assume that there is an initial amount of A_0 drug in the body. Which of the following is the best differential equation model for the drug removal? Once you have the model solve it with the appropriate technique.

1. $A' = -kt$
2. $A' = -kA$
3. $A' = -kA(1 - A/N)$
4. $A' = -kAt$

▲

Solution: $A' = -kA$ so $A(t) = Ce^{-kt}$ with separation of variables.
Show a slope field for this and discuss stability and equilibrium. Also discuss why the other won't work.

Problem 1.21. In a local pine forest the Pine Beetle is killing the trees at a rate proportional to the number of available trees in the forest. A conservation group is attempting to curb the problem by planting 5 live trees per week. Write a differential equation describing this scenario, classify the differential equation, and determine if it can be solved with separation of variables. ▲

Solution: $T' = -kT + 5$. First order, linear, autonomous, non-homogeneous. Separation is fine since is it autonomous.

$$\int \frac{dT}{T - 5/k} = \int -kdt = -kt + C \implies \ln(T - 5/k) = -kt + C \implies T = Ce^{-kt} + \frac{5}{k}$$

Problem 1.22. True or False: Every first order autonomous differential equation is separable. Be able to defend your answer. ▲

Solution: True, but the integration may be horrible!

Problem 1.23. In the movie Interstellar, "Plan B" was for the astronauts to start a colony on a new planet. There was 1 female in the group so she would presumably carry the children. Genetic diversity was no problem because of the donor eggs. The supplies on the colony would be limited by local resources as well as what they brought with them (which minimal). Which of the following models should the astronauts use to plan their future reproduction, and what do the parameters mean? Explain your choice for the best one.

- $P' = kP$
- $P' = kt$
- $P' = -kP\ln(P/N)$
- $P' = kP(1-P/N)$

▲

Solution: A logistic model is the best choice. Either of the last two choice could work.

$$\int \frac{dP}{P\ln(P/N)} = \int -kdt \implies \int \frac{dP}{P\ln(P/N)} = -kt + C$$

$$\implies (\text{with } u = P/N) \int \frac{1}{u\ln(u)} = -kt + C \implies (\text{with } v = 1/u) \int \frac{1}{v} dv = -kt + C$$

$$\implies \ln(v) = -kt + c \implies \ln(\ln(P/N)) = -kt + C \implies \ln(P/N) = Ce^{-kt} \implies P(t) = Ne^{Ce^{-kt}}$$

For the more standard logistic model:

$$\int \frac{dP}{P(1-P/N)} = -kt + C \implies \int \frac{A}{P} + \frac{B}{1-P/N} dP = -kt + C \implies \int \frac{1}{P} + \frac{(1/N)}{1-P/N} dP = -kt + C$$

$$\implies \ln(P) - \ln(1-P/N) = -kt + C \implies \ln\left(\frac{P}{1-P/N}\right) = -kt + C$$

$$\implies \frac{P}{1-P/N} = Ce^{-kt} \implies P(t) = \frac{Ce^{-kt}}{1-Ce^{-kt}/N}$$

1.4 Solution Technique: Undetermined Coefficients

Some folks call this technique the "four step method", but in reality this is just one of many techniques for solving non-homogeneous differential equations. Instead of thinking of this in "four steps" you should really just consider this a bit of mathematical detective work.

Technique 1.24 (Solving Non-Homogeneous Differential Equations)**.** The method of *undetermined coefficients* is roughly outlined as:

1. find a solution for the homogeneous differential equation,
2. conjecture a particular solution for the non-homogeneous differential equation,
3. use the initial condition to find a relationship between some of the coefficients, and
4. use the particular solution in the differential equation to find the remaining coefficients.

Problem 1.25. Solve the following first order linear non-homogeneous differential equation by following the steps outlined.

$$\frac{dy}{dt} = -0.2y + 3 \quad \text{with} \quad y(0) = 5$$

1. First solve the homogeneous part of the equation using separation of variables: $y' = -0.2y$.

   $$y_{hom}(t) = \underline{\hspace{2in}}$$

   Solution: $y_h(t) = C_0 e^{-0.2t}$

2. Next conjecture that a *particular* solution has the same functional form as the non-homogeneity. In this case the non-homogeneity is a constant function so we guess that the particular function is a generic constant function

   $$y_{particular}(t) = C.$$

3. The full analytic solution to the differential equation is the sum of the homogeneous and particular solutions: $y(t) = y_{hom}(t) + y_{part}(t)$. Note that this is only the case for linear differential equations.

   $$y(t) = \underline{\hspace{4in}}$$

   Solution: $y(t) = C_0 e^{-0.2t} + C_1$

4. Substitute the particular solution into the differential equation and see what equation comes out

   $$\underline{\hspace{2in}} = \underline{\hspace{2in}}$$

   Solution: $0 = -0.2C_1 + 3 \implies C_1 = 3/0.2 = 15$

5. Substitute the initial condition into the analytic solution and see what equation comes out

   $$\underline{\hspace{2in}} = \underline{\hspace{2in}}$$

   Solution: $5 = C_0 + 15 \implies C_0 = -10$

6. Determine the final solution **Solution:** $y(t) = -10e^{-0.2t} + 15$

▲

Technique 1.26 (Undetermined Coefficients)**.** To solve a non-homogenous linear differential equation:

1. Solve the associated homogeneous differential equation.
2. Conjecture a *particular solution* that has the same functional form as the non-homogeneity.
3. Build the full analytic solution as a linear combination of the homogeneous and particular solutions: $y(t) = y_{hom}(t) + y_{part}(t)$.
4. Substitute the particular solution into the differential equation.
5. Subsitute the intitial condition(s) into the analytic solution.
6. Use the equations that you found in steps 4 and 5 to find the constants.

Problem 1.27. For each of the following linear non-homogeneous differential equations write the homogeneous solution and the particular solution.

(a) $y' = 3y + 4$ $y_{hom}(t) =$ ____________ and $y_{part}(t) =$ ____________

(b) $y' = 3y + 4t$ $y_{hom}(t) =$ ____________ and $y_{part}(t) =$ ____________

(c) $y' = 3y + 4\sin(t)$ $y_{hom}(t) =$ ____________ and $y_{part}(t) =$ ____________

(d) $y' = 3y + 4e^{-t}$ $y_{hom}(t) =$ ____________ and $y_{part}(t) =$ ____________

▲

Solution:

(a) $y_h(t) = C_0 e^{3t}$ and $y_p(t) = C_1$

(b) $y_h(t) = C_0 e^{3t}$ and $y_p(t) = C_1 t + C_2$

(c) $y_h(t) = C_0 e^{3t}$ and $y_p(t) = C_1 \sin(t) + C_2 \cos(t)$

(d) $y_h(t) = C_0 e^{3t}$ and $y_p(t) = C_1 e^{-t}$

Problem 1.28. Solve all of the differential equations in the previous problem using either separation of variables (if possible) or undetermined coefficients. For each one use $y(0) = 2$.

▲

Solution: Part (a)

$$y' = 3(y + \frac{4}{3}) \implies \cdots \implies y(t) = Ce^{3t} - \frac{4}{3}$$

$$y_{hom} = C_0 e^{3t} \quad y_{part} = C_1 \implies 0 = 3C_1 + 4 \implies C_1 = -\frac{4}{3}$$

Using the initial conditions we see that $2 = C_0 - \frac{4}{3} \implies C_0 = \frac{10}{3}$ so

$$y(t) = \frac{10}{3}e^{3t} - \frac{4}{3}.$$

Part (b)

$$y_{hom} = C_0 e^{3t} \quad y_{part} = C_1 t + C_2 \implies C_1 = 3(C_1 t + C_2) + 4t$$

$$\implies 3C_1 + 4 = 0 \quad \text{and} \quad C_1 = 3C_2 \implies C_1 = -\frac{4}{3}, \quad C_2 = -\frac{4}{9}$$

$$\implies y(t) = C_0 e^{3t} - \frac{4}{3}t - \frac{4}{9} \implies 2 = C_0 - \frac{4}{9} \implies C_0 = \frac{22}{9} \implies y(t) = \frac{22}{9}e^{3t} - \frac{4}{3}t - \frac{4}{9}$$

1.5 Solution Technique: Integrating Factor

It is likely that the previous two solution techniques are the ones that you recall best from previous courses. The trouble, however, is that separation of variables and undetermined coefficients can only solve certain types of differential equations. You'll find that the study of differential equations is laced with many different techniques that only work in very particular scenarios. What follows is another powerful technique that allows us to take care of many first order non-autonomous differential equation that are not separable.

Problem 1.29. At this point we have reviewed several techniques for solving first order differential equations. Remember that a linear differential equation can be written in the form

$$\frac{dy}{dt} + P(t)y = Q(t)$$

where both P and Q are functions of t. In this problem we will write first order linear differential equations that can (or cannot) be solved with different techniques.

(a) Write a first order linear differential equation that can be solved by simple integration (not separation of variables). **Solution:** Take $P(t) = 0$ and the DE can be solved directly by integration.

(b) Write a first order linear differential equation that can be solved with separation of variables. **Solution:** There are several ways to do this: (a) take $Q = 0$ or (b) take P and Q to be constants, or (c) take P to be some scalar multiple of Q, or ...
For example: (a) $y' + 3y = 0$ or (b) $y' + 3y = 5$ or (c) $y' + 3(t+1)y = 5(t+1)$.

(c) Write a first order linear differential equation that can be solved with undetermined coefficients but NOT with separation of variables. **Solution:** There are several ways to do this. The simplest is to take P to be constant and Q to be some non-constant function of t. For example: $y' + 3y = e^t$

(d) Write a first order linear differential equation that can NOT be solved with integration, separation, or undetermined coefficients. **Solution:** The simplest way is to take P and Q to be functions that are not scalar multiples of each other. For example: $y' + 3ty = e^t$.

▲

In the last part of the previous problem you likely found a differential equation where all of your known techniques fail. In this section we add a rather handy technique to your toolbox. This technique requires that you remember the product rule from calculus:

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \cdot g'(x) + g(x) \cdot f'(x).$$

Let's get to know this technique through a structured problem.

Problem 1.30. Consider the differential equation

$$\frac{dy}{dt} + 2ty = 5t \quad \text{with} \quad y(0) = 1.$$

Follow these steps to solve the differential equation. You may notice that with some clever factoring we could solve using separation of variables ...let's not do that here ...trust me.

(a) This is a linear differential equation in form $y' + P(t)y = Q(t)$. What are P and Q?

$$P(t) = \underline{\hspace{3cm}} \quad \text{and} \quad Q(t) = \underline{\hspace{3cm}}$$

Solution: $P(t) = 2t$ and $Q(t) = 5t$

(b) Create the function $\rho(t)$ defined as

$$\rho(t) = e^{\int P(t)dt} = \exp\left(\int P(t)dt\right) = \underline{\hspace{3cm}}$$

Solution: $\rho(t) = \exp\left(\int 2tdt\right) = e^{t^2}$

(c) Multiply both sides of the original differential equation by the function $\rho(t)$. **Solution:**

$$e^{t^2}y' + 2te^{t^2}y = 5te^{t^2}$$

(d) Write the derivative of the product $(\rho(t)y(t))$ using the product rule.

$$\frac{d}{dt}(\rho(t)y(t)) = \underline{\hspace{3cm}}$$

Solution: $\frac{d}{dt}(\rho y) = e^{t^2}y' + 2te^{t^2}y$

(e) We would like the write the differential equation so that the left-hand side is the derivative of some product. Fill in the blanks

$$\frac{d}{dt}(\underline{\hspace{4cm}}) = \underline{\hspace{3cm}}$$

Solution:

$$\frac{d}{dt}\left[e^{t^2}y\right] = 5te^{t^2}$$

(f) Finally, since the left-hand side is written as a derivative we can integrate both sides and solve for $y(t)$. Do this now.

$$y(t) = \underline{\hspace{5cm}}$$

Solution:

$$e^{t^2}y = 5\int te^{t^2}dt \implies e^{t^2}y = \frac{5}{2}\int e^u du = \frac{5}{2}e^{t^2} + C$$
$$\implies y(t) = \frac{5}{2} + Ce^{-t^2} \quad \text{and} \quad y(0) = 1 \implies y(t) = \frac{5}{2} - \frac{3}{2}e^{-t^2}$$

▲

Technique 1.31 (Solutions Via Integrating Factors)**.** To solve the linear differential equation

$$\frac{dy}{dt} + P(t)y = Q(t)$$

we can use the following recipe.

1. Let $\rho(t) = e^{\int P(t)dt}$. This is called an *integrating factor*.
2. Multiply both sides by this integrating factor to get

$$\rho(t)\frac{dy}{dt} + P(t)\rho(t) = Q(t)\rho(t)$$

which can be rewritten as

$$e^{\int P(t)dt}\frac{dy}{dt} + P(t)e^{\int P(t)dt}y = Q(t)e^{\int P(t)dt}$$

3. The left-hand side is the result of the product rule:

$$\frac{d}{dt}\left[y \cdot e^{\int P(t)dt}\right] = Q(t)e^{\int P(t)dt}$$

4. Integrate both sides and solve for y

$$y(t) = e^{-\int P(t)dt} \int \left(Q(t)e^{\int P(t)dt}\right) dt$$

Problem 1.32. Solve the differential equation $y' - 2y = 3$ with integrating factors. ▲

Solution: Let $\rho(t) = e^{\int -2dt} = e^{-2t}$. Hence if we multiply both sides by ρ we get

$$e^{-2t}y' + (-2)e^{-2t}y = 3e^{-2t}.$$

Observing that the left-hand side is just the result of the product rule we get

$$\frac{d}{dt}\left[e^{-2t}y\right] = 3e^{-2t}.$$

$$\implies e^{-2t}y = \int 3e^{-2t}dt \implies e^{-2t}y = -\frac{3}{2}e^{-2t} + C \implies y = -\frac{3}{2} + Ce^{2t}$$

Problem 1.33. The following differential equation can theoretically be solved with integrating factors but the integration may end up being horrible. Work this problem as far as you can.

$$y' - 2ty = 3$$

▲

Solution:

$$\rho(t) = \exp\left(\int -2tdt\right) = \exp\left(-t^2\right) = e^{-t^2} \implies \cdots \implies \frac{d}{dt}\left[e^{-t^2}y\right] = 3e^{-t^2}$$

$$\implies e^{-t^2}y = 3\int e^{-t^2}dt \implies y(t) = 3e^{t^2}\int e^{-t^2}dt$$

The final integration is impossible with the regular techniques of integration like substitution and integration by parts.

Problem 1.34. Solve each of the following first order differential equations with an appropriate solution technique. For each differential equation use $y(0) = 3$. Your goal should be to choose the technique that makes the solution *easiest* to come by.

$$\begin{aligned} &\text{Problem (a): } y' = -0.2t \\ &\text{Problem (b): } y' = -0.2y \\ &\text{Problem (c): } y' = -0.2y + 3 \\ &\text{Problem (d): } y' = -0.2y + 3t \\ &\text{Problem (e): } y' = -0.2y^2 \\ &\text{Problem (f): } y' = -0.2y \cdot t \\ &\text{Problem (g): } y' = -0.2y \cdot t + t \end{aligned}$$

▲

Solution:

$$\begin{aligned} &\text{Problem (a): } y(t) = -0.1t^2 + 3 \text{ (solved by integration)} \\ &\text{Problem (b): } y(t) = 3e^{-0.2t} \text{ (solved by separation)} \\ &\text{Problem (c): } y(t) = -12e^{-0.2t} + 15 \text{ (solve by separation)} \\ &\text{Problem (d): } y(t) = 78e^{-0.2t} + 15t - 75 \text{ (solve by undetermined coefficients)} \\ &\text{Problem (e): } y(t) = \frac{5}{t + 5/3} \text{ (solved by separation)} \\ &\text{Problem (f): } y(t) = 3e^{-0.1t^2} \text{ (solved by separation)} \\ &\text{Problem (g): } y(t) = -2e^{-0.1t^2} + 5 \text{ (solved by integrating factors)} \end{aligned}$$

The last one may be the most complicated so I'll give two full solutions. First I'll show how to get this solution via integrating factors.

$$y' + (0.2t)y = t \implies \rho(t) = \exp\left(\int 0.2t dt\right) = \exp\left(0.1t^2\right)$$

$$\therefore \frac{d}{dt}\left[e^{0.1t^2} y\right] = te^{0.1t^2} \implies e^{0.1t^2} y = \int te^{0.1t^2} dt \implies e^{0.1t^2} t = \frac{e^{0.1t^2}}{0.2} + C \implies y = 5 - 2e^{-0.1t^2}$$

Another way to get the last one would be to factor first and then to separate (this may be easier!)

$$y' = t(-0.2y + 1) \implies \int \frac{dy}{-0.2y + 1} = \int t dt \implies -5\ln(-0.2y + 1) = \frac{t^2}{2} + C + C$$

$$\implies \ln\left(\frac{1}{(-0.2y+1)^5}\right) = \frac{t^2}{2} + C \implies \frac{1}{(-0.2y+1)^5} = Ce^{0.5t^2} \implies (-0.2y+1)^5 = Ce^{-0.5t^2}$$

$$\implies -0.2y + 1 = Ce^{-0.1t^2} \implies y = Ce^{-0.1t^2} + 5 \implies y = -2e^{-0.1t^2} + 5$$

Example 1.35. Solve the differential equation $y' + 0.2ty = t$ with $y(0) = 3$ using the integrating factors technique.
Solution: From the differential equation we see that the integrating factor is

$$\rho(t) = \exp\left(\int 0.2t dt\right) = \exp\left(0.1t^2\right).$$

Multiplying both sides of the differential equation by $\rho(t)$ gives

$$e^{0.1t^2}\frac{dy}{dt} + 0.2te^{0.1t^2}y = e^{0.1t^2}t,$$

and we can immediately recognize that the left-hand side is the result of the product rule. Hence we can rewrite the differential equation as

$$\frac{d}{dt}\left[e^{0.1t^2}y\right] = te^{0.1t^2}.$$

Integrating both sides with respect to t and solving for y gives

$$e^{0.1t^2}y = \int te^{0.1t^2}dt \implies e^{0.1t^2}t = \frac{e^{0.1t^2}}{0.2} + C \implies y = 5 + Ce^{-0.1t^2}.$$

Finally we can use the initial condition to observe that $C = -2$ and the solution to the differential equation is

$$\boxed{y(t) = 5 - 2e^{-0.1t^2}.}$$

The observant reader should note that this problem is actually easier to solve with separation of variables by observing that we can initially rewrite as $y' = (-0.2y + 1)t$.

Example 1.36. Solve the differential equation $y' + 2ty = e^{-t^2}$ with $y(0) = 3$ using the integrating factors technique.
Solution: The integrating factor is $\rho(t) = e^{t^2}$ so just as in the previous example we can multiply both sides of the differential equation by this expression to get

$$e^{t^2}\frac{dy}{dt} + 2te^{t^2}y = e^{t^2}e^{-t^2}.$$

The right-hand side clearly simplifies to 1 and the left-hand side can be re-written as the result of the product rule

$$\frac{d}{dt}\left[e^{t^2}y\right] = 1.$$

Integrating both sides with respect to t and multiplying by e^{-t^2} gives

$$y(t) = te^{-t^2} + Ce^{-t^2}.$$

Using the initial condition we see that $C = 3$ and the solution is

$$\boxed{y(t) = te^{-t^2} + 3e^{-t^2}.}$$

Problem 1.37. Write a differential equation that cannot be solved with integrating factors. ▲

Solution: One example might be $y' + ty = 3$. The resulting integration requires that we find an anitderivative for e^{t^2} but one does not exist.

1.6 Mixing Problems

A classic differential equations problem is to consider the mixing of a contaminant in a tank. It is important to remember in these problems that you are strictly considering the conservation of mass. Hence, if $C(t)$ is the amount of contaminant in the tank then

$$\frac{dC}{dt} = \text{rate that } C \text{ flows in} - \text{rate the } C \text{ flows out.}$$

To determine the two rates it is often easiest to consider the units.

Problem 1.38. A 120-gallon tank initially contains 90 pounds of salt dissolved in a full tank. Brine containing 2 pounds per gallon of salt flows into the tank at a rate of 4 gallons per minute and the well stirred mixture flows out of the tank at the same rate. Write a differential equation for the amount of salt in the tank and solve your differential equation.

Remember that if $S(t)$ is the amount of salt in the tank then

$$\frac{dS}{dt} = \text{rate that salt flows in} - \text{rate that salt flows out}$$

and the units of each term all need to be the same

$$\frac{\text{pounds}}{\text{minute}} = \frac{\text{pounds}}{\text{minute}} - \frac{\text{pounds}}{\text{minute}}.$$

▲

Solution:

$$\frac{dS}{dt} = 4(2) - \frac{4S}{120} \quad \text{with} \quad S(0) = 90.$$
$$\frac{dS}{dt} = -\frac{4}{120}\left(S - 8\left(\frac{120}{4}\right)\right)$$
$$S(t) = -150e^{-4t/120} + 240$$

Problem 1.39. A 120-gallon tank initially contains 90 pounds of salt dissolved in 100 gallons of water. Brine containing 2 pounds per gallon of salt flows into the tank at a rate of 4 gallons per minute and the well stirred mixture flows out of the tank at a rate of 3 gallons per minute (so the tank is filling up). Write a differential equation for the amount of salt in the tank and solve your differential equation. ▲

Solution:

$$\frac{dS}{dt} = 8 - \left(\frac{3}{100+t}\right)S \quad \text{with} \quad S(0) = 90.$$

$$\frac{dS}{dt} + \left(\frac{3}{100+t}\right)S = 8$$

$$\rho(t) = \exp\left(\int \frac{3}{100+t}dt\right) = \exp\left(4\ln(100+t)\right) = (100+t)^4$$

Using integrating factors we get

$$\frac{d}{dt}\left[(100+t)^3 S\right] = 8(100+t)^3 \implies (100+t)^3 S = 8\int (100+t)^3 dt = 8\frac{(100+t)^4}{4} + C = 2(100+t)^4 + C$$

$$\implies S(t) = 2(100+t) + \frac{C}{(100+t)^3}$$

$$S(0) = 90 \implies 90 = 200 + \frac{C}{100^3} \implies C = (-110)(100^3) \implies C = -1.1\times 10^8$$

Problem 1.40. Canyon Ferry reservoir has a volume of approximately $V = 2.33\times 10^9$ m^3 and assume that the inflow from the Missouri river in the spring is $R = 113$ m^3/sec. Assume further that the dam leading to Hauser reservoir is open so the outflow rate is the same as the inflow rate in Canyon Ferry. A large gas tank at a marina upstream is leaking into the river so the contaminated water coming in has a concentration of $c = 0.25\text{kg/m}^3$. Write a differential equation for the amount of gas, $G(t)$, in Canyon Ferry lake at time t. Assume for simplicity that the gas is well mixed in the lake. Once you have your model solve it with an appropriate technique.
Hint #1: The rate of change of the amount of gas equals the rate in minus the rate out

$$\frac{dG}{dt} = \text{rate that the gas comes in} - \text{rate that the gas goes out}$$

Hint #2: Do not substitute the values given until you have solved the model. ▲

Solution: The correct model is:

$$\frac{dG}{dt} = R\cdot c - \frac{RG}{V}$$

where R is the flow rate, c is the contamination concentration, V is the volume of the lake, and G is the amount of gas in the water. Therefore, the correct model is

$$\frac{dG}{dt} = 113\cdot 0.25 - \frac{113\cdot G}{2.33\times 10^9}$$

The solution can be found either with separation, integrating factors, or with undetermined coefficients.

$$\frac{dG}{dt} = -\frac{R}{V}(G - cV) \implies G(t) = Ce^{-Rt/V} + cV$$

Explore with slope fields.

Extension. What if the rates are not the same and $V = V(t)$? Must use integrating factors.

Problem 1.41. A fishing pond currently has 1000 rainbow trout (R) but the Department of Fish, Wildlife, and Parks (FWP) has decided to institute a restocking plan with a mix of rainbow trout and brown trout (B) to diversify the species in the pond. Assume that the fishermen from the pond remove 50 fish per week and FWP restocks with 30 mature rainbows and 25 mature browns every week. Model the population of rainbow trout with a differential equation. Solve the differential equation using appropriate techniques and plot both populations together. ▲

Solution: We have $R(0) = 1000$ but we also observe that the number of fish in the pond is actually increasing by 5 fish per week. Therefore, the total population can be modeled as $P(t) = 1000 + 5t$.

rate of rainbows in: 30

$$\text{rate of rainbows out: } \frac{50(\text{fish/wk}) \cdot R(\text{rainbows})}{P(t)\text{total fish}} = \frac{50R}{1000+5t}$$

Therefore,

$$\frac{dR}{dt} = 30 - \frac{50R}{1000+5t} \quad \text{or} \quad \frac{dR}{dt} + \frac{50}{1000+5t}R = 30$$

This is a job for integrating factors. Let $\rho(t) = \exp\left(\int \frac{50}{1000+5t}\right) = \exp\left(10\ln(1000+5t)\right)$ and therefor

$$\rho(t) = (1000+5t)^{10}$$

Multiplying and recognizing the product rule gives

$$\frac{d}{dt}\left[(1000+5t)^{10}R\right] = 30(1000+5t)^{10} \implies (1000+5t)^{10}R = \frac{30}{55}(1000+5t)^{11} + C \implies$$

$$R(t) = \frac{30}{55}(1000+5t) + \frac{C}{(1000+5t)^{10}}$$

Using the initial condition we get

$$1000 = \frac{30}{55}(1000) + \frac{C}{1000^{10}} \implies C = \frac{25}{55}(1000)^{11} = \frac{25}{55} \times 10^{33}$$

$$R(t) = \frac{30}{55}(1000+5t) + \frac{25}{55}\frac{10^{33}}{(1000+5t)^{10}} \quad \text{and} \quad B(t) = P(t) - R(t) = (1000+5t) - R(t)$$

1.7 Existence and Uniqueness of Solutions (INCOMPLETE)

Here are a few fundamental questions about differential equations:

- When does the solution to a differential equation exist?
- If you find a solution is it the only one?
- On what domain does the solution make sense?

If we don't know that a solution exists (or worse yet, if we know that it doesn't exist) then there is no need to go searching for it. Furthermore, we often solve differential equations with numerical methods but if the solution doesn't exist then our numerical method is only giving us computational garbage. In this section we present two fundamental theorems discussing these questions for first order differential equations.

Theorem 1.42 (Existence Theorem)**.** Suppose that $f(t,y)$ is a continuous function in a rectangle of the form

$$\{(t,y) : a < t < b, c < y < d\}$$

in the ty-plane. If (t_0, y_0) is a point in the rectangle then there exists a number $\varepsilon > 0$ ad a function $y(t)$ defined for $t_0 - \varepsilon < t < t_0 + \varepsilon$ that solves the initial value problem

$$\frac{dy}{dt} = f(t,y) \quad y(t_0) = y_0.$$

Problem 1.43. What does this theorem mean? ▲

Theorem 1.44 (Uniqueness Theorem)**.** Suppose that $f(t,y)$ and $\partial f/\partial y$ are continuous function in a rectangle of the form

$$\{(t,y) : a < t < b, c < y < d\}$$

in the ty-plane. If (t_0, y_0) is a point in the rectangle and if $y_1(t)$ and $y_2(t)$ are two functions that solve the initial value problem

$$\frac{dy}{dt} = f(t,y) \quad y(t_0) = y_0$$

for all t in the interval $t_0 - \varepsilon < t < t_0 + \varepsilon$ (for $\varepsilon > 0$) then

$$y_1(t) = y_2(t)$$

for $t_0 - \varepsilon < t < t_0 + \varepsilon$. That is, the two solution must be identical

Problem 1.45. What does this theorem mean? ▲

Problem 1.46. A bucket of water has a hole in the bottom, and so the water is slowly leaking out. The height of the water in the bucket is thus a decreasing function of time $h(t)$ which changes according to the differential equation

$$\frac{dh}{dt} = -k\sqrt{h}$$

where k is a positive constant that depends on the size of the hole in and the bucket. If we start out a bucket with 25cm of water in it, then according to this model, will the bucket ever empty?

1. Yes
2. No
3. Can't tell with the given information

▲

Solution: Can't tell since the solution isn't guaranteed to exist when $h = 0$

Problem 1.47. Based upon observations, Kate developed the differential equation

$$\frac{dT}{dt} = -0.09(T - 72)$$

to predict the temperature in her vanilla chai tea. In the equation, T represents the temperature of the chai in $^\circ F$ and t is time. Kate has a cup of chai whose initial temperature is $110^\circ F$ and her friend has a cup of chai whose initial temperature is $120^\circ F$. According to Kate's model, will there be a point in time when the two cups of chai have exactly the same temperature?

1. Yes
2. No
3. Can't tell with the given information

▲

Solution: No. If so it would violate uniqueness

Chapter 2

Qualitative Methods, Numerical Methods, and Bifurcations

2.1 Equilibrium Points and Stability

A first order autonomous differential equation may have an equilibrium point where the change simply stops. In this section we will build (or remind you about) the tools to find and analyze equilibrium points for autonomous first order differential equations.

Problem 2.1. The equilibrium point of a first order autonomous differential equation $y'(t) = f(y)$ is defined as the value of y where $y' = 0$.

(a) Why is this called an equilibrium point?

(b) Find the equilibrium for the first order autonomous differential equation

$$y' = -0.2y^2 + 3$$

(c) Make a plot with y' on the vertical axis on y on the horizontal axis for the previous differential equation. Use this plot to determine whether the equilibrium point is stable or unstable.

▲

Solution: The equilibrium is $y_{eq} = \sqrt{3/0.2} = \sqrt{15}$. Stable

Problem 2.2. The air resistance on a sky diver is proportional to the square of the velocity. Newton's 2nd law can be used to get a differential equation for the velocity of the sky diver.

1. Write the differential equation for the velocity (take *down* to be positive). You might want to start with a free body diagram and consider the balance of forces. ...What Would Newton Do? (WWND)

2. Find the equilibrium (AKA: terminal velocity) in terms of the mass and the proportionality constant for air resistance. How many equilibria are there? Discuss their stability.

▲

Solution:

$$mv' = mg - kv^2$$

$$v_{eq} = \sqrt{\frac{mg}{k}}$$

So for every k there is a terminal velocity, but as k gets larger the terminal velocity goes down.

Problem 2.3. Create a first order autonomous differential equation that has 2 unstable equilibria and 1 stable equilibrium.
Hint#1: Use the plot of y' vs y to help.
Hint#2: Write the right-hand side of your DE to in factored form. ▲

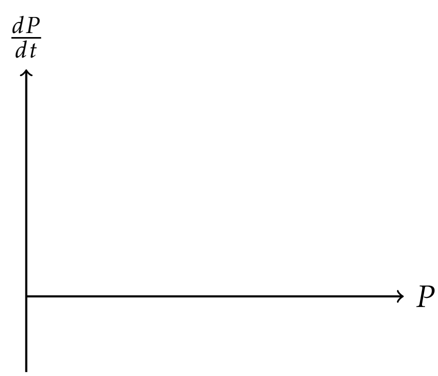
Solution: $y' = (y-1)(y-2)(y-3)$ will be unstable at $y = 1$ and $y = 3$ but stable at $y = 2$.

Problem 2.4. A trout pond has a carrying capacity of 200 fish. Suppose that the trout population can be modeled according to the logistic equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{200}\right)$$

where k is the intrinsic growth rate of the population. For the sake of simplicity in this model let's assume that $k = 0.5$ (for now).

(a) The coordinate axes below have dP/dt on the vertical axis and P on the horizontal axis. Use the differential equation to sketch this plot.



(b) Mark the intercepts on the horizontal axis in the plot above. What do they represent in the context of this problem?

(c) What does it mean about the rate of change of the population if the plot lies above the horizontal axis? What about below?

(d) Use your answer in part (c) to classify the two equilibrium points as either stable or unstable.

▲

Solution:

(a) This is a downward facing parabola with roots at 0 and 200.

(b) The intercepts are where $P' = 0$ so the rate of change is zero. These are the equilibrium points.

(c) If the curve is above the horizontal axis then the rate of change is positive. If the curve is below the horizontal axis then the rate of change is negative.

(d) 0 is unstable and 200 is stable.

Problem 2.5. Use what you learned in the previous problems to find and classify the equilibria for the first order non-linear autonomous differential equation

$$y'(t) = (y-1)(y-2)(y-3)^2.$$

▲

Solution: 1 is stable, 2 is unstable and 3 is semi-stable

Technique 2.6 (Phase Line Analysis)**.** It is often very helpful to draw a *phase diagram* (sometimes called a phase line) to analyze the equilibrium points of an autonomous differential equation. There are four possible cases shown graphically in Figure 2.1. In each of the following fill in with the word(s) "stable", "unstable", "semi-stable approaching from below", or "semi-stable approaching from above".

- In Case #1 there is a/an ________________ equilibrium at $y = 2$.
- In Case #2 there is a/an ________________ equilibrium at $y = 2$.
- In Case #3 there is a/an ________________ equilibrium at $y = 2$.
- In Case #4 there is a/an ________________ equilibrium at $y = 2$.

Solution:

- Case #1: unstable
- Case #2: semi-stable approaching from below
- Case #3: stable
- Case #2: semi-stable approaching from above

Problem 2.7. For each of the phase plots in Figure 2.2 sketch a plot on the y vs t plane of the solutions to the underlying differential equation. A few helpful markers are given to you in the first plot. ▲

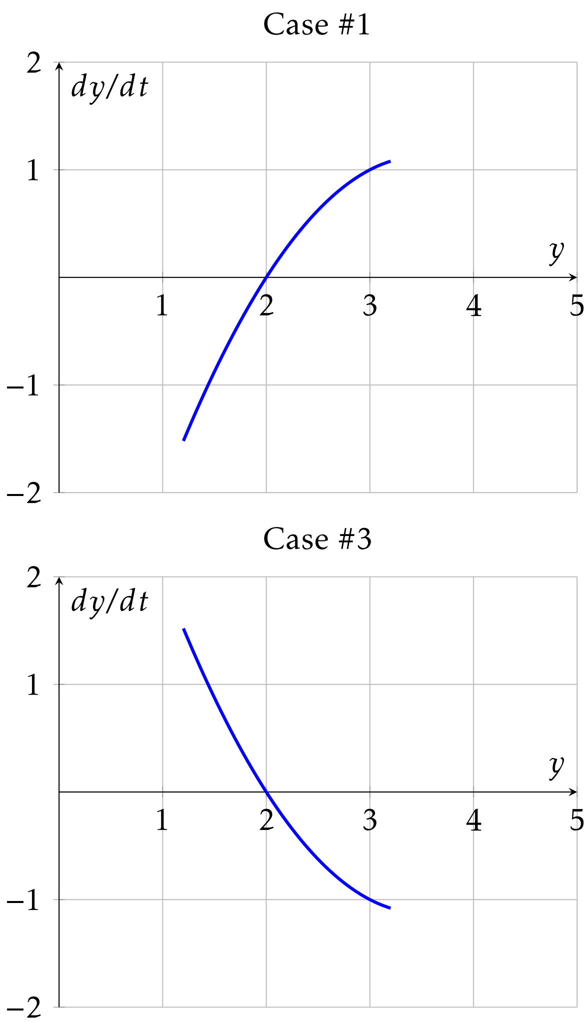


Figure 2.1. Four cases for phase line analysis. In each plot we see a small portion of the $\frac{dy}{dt}$ vs y plot for an autonomous first order differential equation: $\frac{dy}{dt} = f(y)$.

2.2 Numerical Methods

In this section we will build two numerical solvers that allow you to use a computer to approximate solutions to the differential equation

$$y' = f(t, y) \tag{2.1}$$

for a given initial condition. In the problems that follow you will be creating several MATLAB files that you will use throughout the course, so please save them in a meaningful place and share them with your group mates.

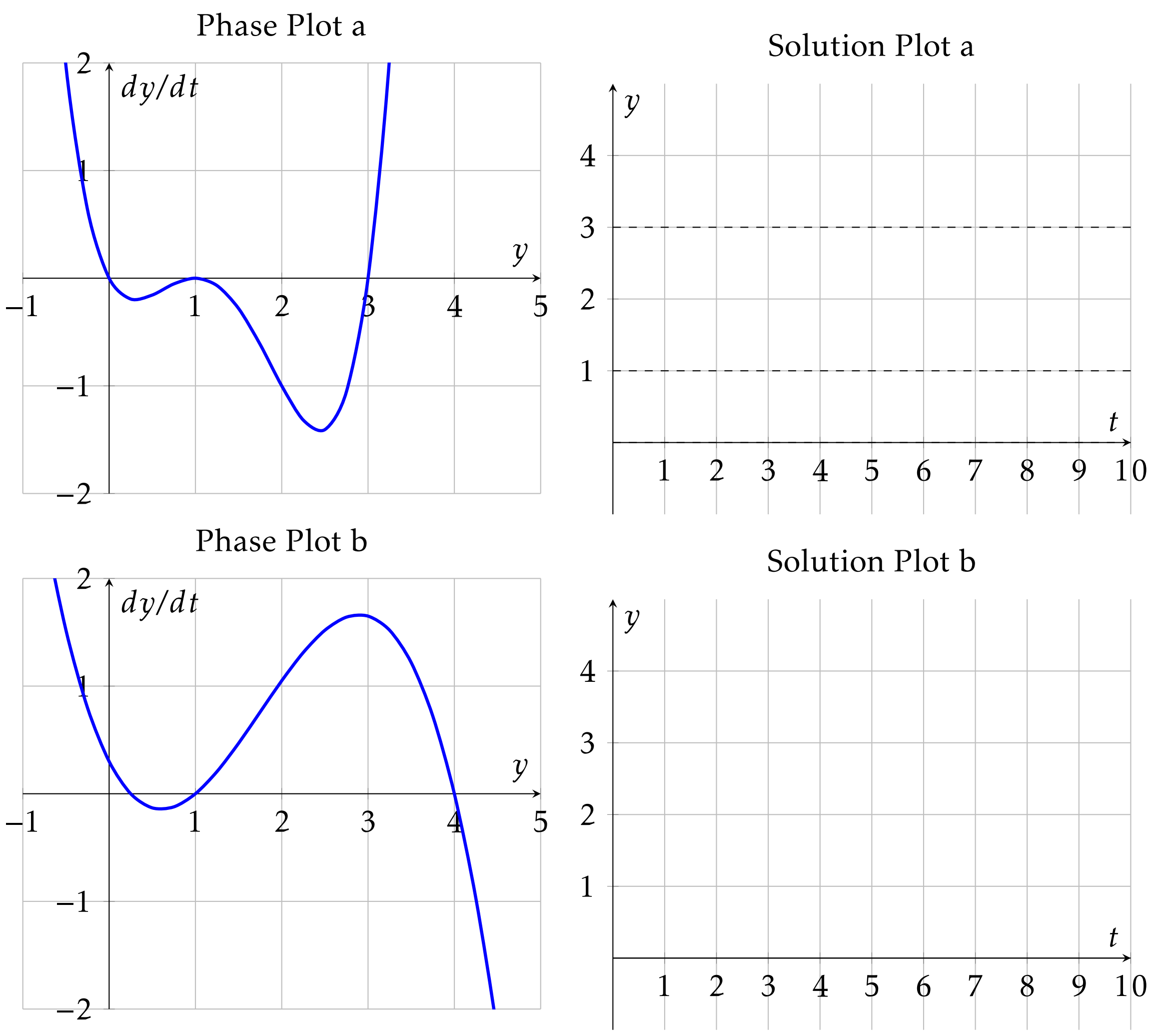


Figure 2.2. Phase plots and solution plots. On the left are the phase plots and on the right are coordinate axes to sketch the solution plots.

2.2.1 Euler's Method

Euler's method is the simplest numerical method for solving the differential equation $y' = f(t,y)$, and hence it is the best place to start! You should be familiar with Euler's method from pre-requisite courses, but this time we are not going to be using Excel (since we usually need something FAR more powerful!).

Euler's method is simple: approximate the derivative in the most naive possible way:

$$\frac{dy}{dt} \approx \frac{y_{new} - y_{old}}{\Delta t}$$

and then rewrite the differential equation (2.1) as

$$\frac{y_{new} - y_{old}}{\Delta t} \approx f(t_{old}, y_{old}).$$

After some rearrangement and relabeling we get the difference equation

$$y_{n+1} = y_n + \Delta t f(t_n, y_n).$$

Technique 2.8 (Euler's Method)**.** To approximate $y' = f(t,y)$ first choose Δt and then implmenet the difference equation

$$y_{n+1} + y_n + \Delta t f(t_n, y_n)$$

using appropriate computer software.

Remember that the only reasonble choice for Δt is to make it *very small*. The trade off to choosing Δt small is that it will take more computer memory to approximate the problem.

Problem 2.9. In Excel the process of building an Euler solver is relatively simple. In MATLAB it takes a bit more work the very first time, but trust me, the work will pay off in the long run!

(a) Open MATLAB and create a new function.

(b) Change the first line of the function so that it reads
`function [t,y] = MyEuler(f,tmin,tmax,numpoints,IC)`

(c) After the commented lines (which we'll come back to) you should complete the following code. Explain what each line of code does using comments. Some of the lines of code are likely new to you so I suggest you either use the help command or you do some basic experimentation.

```
function [t,y] = MyEuler(f,tmin,tmax,numpoints,IC)
t=linspace(tmin,tmax,numpoints);  % what does this line do?
dt=t(2)-t(1); % what does this line do?
y=zeros(size(t)); % what does this line do?
y(1) = IC; % what does this line do?
for n=1:length(t)-1 % what does this line do?
    y(n+1) = ... some code for Euler's method ...
end
```

(d) Now. Save this code in the working directory for this lab. You also need to get MATLAB to look in that working directory. The simplest way to do that is to press F5 while you're in the MyEuler function (and then ignore all of the errors that occur).

The code you just created works for ALL of the times that you ever need Euler's method. We now just need to create a short script which calls this code and uses it.

(e) Finally, let's try out your MyEuler code (and at this point you should see why you really shouldn't be using Excel for numerical differential equation solvers). I'm assuming that you have MATLAB working in the correct directory (you'll get errors otherwise!!).

We want to get an approximate solution to the differential equation

$$y' = -y \cdot \left(1 - \frac{y}{5}\right) + 0.1t \quad \text{where} \quad y(0) = 2$$

Open a new script in MATLAB and complete the following code to get your plot.

```
clear; clc; clf;
f = @(t,y) -y*(1-y/5)+0.1*t;
[t,y] = MyEuler(f, ..., ..., ..., ...)
plot(t,y)
```

(f) Run your code for several different initial conditions and overlay the plots to explore the dynamics of the differential equation. Be sure to properly label your plot. It is probably best to do this step within a loop in MATLAB. To get a legend to appear for each new plot you can use the following:

```
clear; clc; clf;
f = @(t,y) -y*(1-y/5)+0.1*t;
LegendItems = { }; % initialize the storage for the legend entries
counter=1; % set up a dummy counter
for IC= ... : ... : ...
  [t,y] = MyEuler(f, ..., ..., ..., ...)
  plot(t,y), hold on
  LegendItems{counter} = ['IC = ' , num2str(IC)]; % what does this line do?
  counter = counter+1; % what does this line do?
end
legend(LegendItems)
```

▲

Problem 2.10. Run an Euler solver on the differential equation

$$y' = -y\left(1 - \frac{y}{5}\right)^2 + \sin(t)$$

for several different initial conditions. Save your plot in an appropriate place with appropriate labels and title. ▲

2.2.2 Runge-Kutta Method

Euler's method is one of MANY different numerical differential equation solvers. The second one that we are going to study in this section is called the Runge-Kutta 4 solver. The idea is basically the same as with Euler's method: approximate the derivative and rewrite the differential equation as a difference equation. The difference here is that the algorithm is a bit more complex. ...Here it is:

Technique 2.11 (Runge Kutta Method)**.** First define the dummy variables k_1, k_2, k_3, and k_4 as

$$\begin{aligned}
k_1 &= f(t_n, y_n) \\
k_2 &= f(t_n + \frac{\Delta t}{2}, y_n + \frac{\Delta t}{2} k_1) \\
k_3 &= f(t_n + \frac{\Delta t}{2}, y_n + \frac{\Delta t}{2} k_2) \\
k_4 &= f(t_n + \Delta t, y_n + \Delta t k_3).
\end{aligned}$$

Then we build the difference equation as a weighted sum of the $k_j's$:

$$y_{n+1} = y_n + \frac{\Delta t}{6}(k_1 + 2k_2 + 2k_3 + k_4).$$

Problem 2.12. Explain the Runge Kutta method graphically. ▲

Problem 2.13. (a) Create a new function called `MyRungeKutta` in `MATLAB` and copy all of your code from your `MyEuler` function.

(b) Modify the loop from the Euler code so that you perform the Runge-Kutta iteration. The skeleton code should get you going:

```
...
for n=1:length(t)-1
k1 = ...
k2 = ...
k3 = ...
k4 = ...
y(n+1) = y(n) + ...
end
```

(c) Run your Runge-Kutta code the same way that you did your Euler code. Test it out on the same two problems as before, and be sure that you get the same qualitative solutions.

▲

What we're about to do next should seem a bit silly. We're going to compare our two numerical solvers to a differential equation where we HAVE the analytic solution. You may be asking yourself "that seems silly, if you have the analytic solution then why on earth would you need or want the numerical solution?" Well, we're going to do it here to prove a point.

Problem 2.14. The differential equation in question is: $y' = -5y$. Solve this equation now by hand (it should only take a second or two).

(a) Write MATLAB code that solves this differential equation with Euler's method and with the Runge-Kutta method for several initial conditions. Put the solutions on top of each other in one subplot.

(b) In a second subplot, plot the errors between Euler and the exact solution as well as Runge-Kutta and the exact solution. It would be best to use a logarithmically scaled y-axis (use `semilogy` instead of `plot`). Put the results together with several different initial conditions on the same plots.

(c) What conclusions can you make about the two numerical methods that we have just built? Is one better than the other? When do they have the largest amount of error in general (for any problem)?

▲

The ideas behind solving ordinary differnetial equations numerically are covered extensively in a numerical analysis course. If you're interested I highly recommend that you take this course.

2.3 Bifurcations (INCOMPLETE)

In this section we will explore the behavior of the equilirium solutions in a differential equation to changes in a parameter. In some instances the change in a parameter might cause the behavior of equilibrium points to change from stable to unstable (or possibly semi-stable). It is also possible to *spawn* new equilibrium points by changing the values of parameters. Let's start with an example of a logistic model with a harvesting term.

Problem 2.15. A population of Alaskan Salmon grows according to a logistic model:

$$\frac{dP}{dt} = kP(M-P)$$

where k is the growth rate and M is the carrying capacity.

1. What are the equilibrium points and discuss their stability.
2. Fishermen actively target these fish. Modify the model to implement a constant harvesting rate (fish per day).

$$\frac{dP}{dt} = kP(M-P) + \underline{\qquad\qquad}$$

3. Find the equilibrium points in terms of the model parameters.
4. When are there 2, 1, or 0 equilibrium points?
5. Draw a plot with the parameter h on the horizontal axis and the value of the equilibrium point(s) on the vertical axis. Use $M = 1$ and $k = 1$ for simplicity.

▲

Solution:

$$\frac{dP}{dt} = kP(M-P)$$

has equilibria $P = 0$ and $P = M$ (unstable and stable resp.)

For a harvesting model:

$$\frac{dP}{dt} = kP(m-P) - h$$

Therefore,

$$0 = kP(M-P) - h \implies -kP_k^2 MP - h = 0 \implies P = \frac{-kM \pm \sqrt{k^2M^2 - 4kh}}{-2k}$$

Plot this in GeoGebra to show the bifurcation.

Further analysis:

- When $(kM)^2 - 4kh < 0$ there are no equilibrium points. In other words,

$$h > \frac{kM^2}{4} \implies \text{no equilibrium points}$$

- When $(kM)^2 - 4kh = 0$ there is 1 equilibrium point. In other words,

$$h = \frac{kM^2}{4} \implies \text{1 equilib. point} P_{eq} = \frac{M}{2}$$

- When $(kM)^2 - 4kh > 0$ there are 2 equilibrium points. In other words,

$$h < \frac{kM^2}{4} \implies \text{2 equilibrium points}$$

...more about bifurcations when I get around to it ...

Chapter 3

Linear Systems and Matrices

In this chapter we'll assume that you are familiar with the basics of linear algebra. Hence, you can use the appropriate linked text from the Section 0.2 for any necessary explanation on these problems. I highly suggest you use your notes from when you first saw linear algebra. We will begin here with a few of the basic definitions and we will recap some of the basics from systems of equations, row reduction, linear combinations, and matrix operations. I highly suggest that you put your calculator down and get used to doing all of these techniques by hand. There is a time and place for technology and for the most part this chapter is not it.

3.1 Matrix Operations and Definitions

Definition 3.1 (Size of a Matrix)**.** If A is a matrix with m rows and n columns then we say that A has size (or dimensions) $m \times n$.

Definition 3.2 (Equality of Matrices)**.** Two matrices are equal if their corresponding entries are equal. Matrices can only be equal if the sizes are equal.

Definition 3.3 (Addition and Subtraction of Matrices)**.** Matrix addition and subtraction and done by regular addition and subtraction on the corresponding entries. Matrix addition and subtraction can only be performed on matrices of the same size.

Definition 3.4 (Scalar Multiplication)**.** If A is a matrix then cA is a scalar multiple of the matrix. Multiplying a matrix by a scalar multiplies every entry by the scalar.

Definition 3.5 (Transposition of a Matrix)**.** If A is a matrix then A^T is the transpose of the matrix found by interchanging the rows and columns of A. If A is $m \times n$ then A^T is $n \times m$.

Definition 3.6. If A is an $m \times n$ matrix and B is an $n \times p$ matrix then the product of A and B is $C = AB$ where:

- The size of AB is $m \times p$. The number of columns in A must be the same as the number of rows of B.
- The entry in row i and column j of $C = AB$ is

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

It is very important to note that in general $AB \neq BA$.

Problem 3.7. Consider the matrices A and B. Find the products AB and BA if they exist.

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 5 & -2 \\ -1 & 0 \\ 1 & 3 \end{pmatrix}$$

▲

Solution:

$$AB = \begin{pmatrix} 0 & -11 \\ 11 & -1 \end{pmatrix} \qquad BA = \begin{pmatrix} 1 & 10 & -17 \\ -1 & -2 & 3 \\ 7 & 2 & 0 \end{pmatrix}$$

Problem 3.8. Consider the matrices below.

$$A = \begin{pmatrix} 2 & -1 & 4 \\ 3 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 1 \\ 0 & -3 \\ 4 & -1 \end{pmatrix} \quad C = \begin{pmatrix} 0 & -1 \\ 3 & 2 \\ -3 & 1 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$$

(a) Determine which products are possible:

$$AB, \quad AC, \quad A\mathbf{x}, \quad BA, \quad CA, \quad \mathbf{x}A, \quad BC, \quad B\mathbf{x}, \quad CB, \quad C\mathbf{x}.$$

For each of the products that is possible find the size of the result.

(b) Write the product AB and the product BA. Does $AB = BA$?

▲

Solution: The matrix products that exist are: $AB : 2\times 2$, $AC : 2\times 2$, $A\mathbf{x} : 2\times 1$, $BA : 3\times 3$, $CA : 3\times 3$. None of the rest exist.
As expected, $AB \neq BA$.

Problem 3.9. Compute the product AB for

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

▲

Solution:

$$AB = \begin{pmatrix} 7 \\ 10 \end{pmatrix}$$

3.2 Gaussian Elimination: Reduced Row Eschelon Form

Solving systems of equations is one of the most essential applications of linear algebra. It is expected that you have experience solving systems with row reduction so as such we will cover it quickly in this section.

Problem 3.10. Consider the system of equations:

$$\begin{cases} -x_1 + x_2 - x_3 & = -6 \\ x_1 + x_3 & = 15 \\ 2x_1 - x_2 + x_3 & = 9 \end{cases}$$

We want to solve this system of equations using Gaussian elimination (row reduction). We will do so using the following steps.

(a) For the sake of practice let's first write this system as a matrix equation of the form $A\mathbf{x} = \mathbf{b}$. What are A, $\mathbf{x}$, and $\mathbf{b}$? **Solution:**

$$\begin{pmatrix} -1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -6 \\ 15 \\ 9 \end{pmatrix}$$

(b) Next write the system as an *augmented matrix*.

$$\left(\begin{array}{ccc|c} \underline{\hspace{2em}} & \underline{\hspace{2em}} & \underline{\hspace{2em}} & \underline{\hspace{2em}} \\ \underline{\hspace{2em}} & \underline{\hspace{2em}} & \underline{\hspace{2em}} & \underline{\hspace{2em}} \\ \underline{\hspace{2em}} & \underline{\hspace{2em}} & \underline{\hspace{2em}} & \underline{\hspace{2em}} \end{array}\right)$$

Solution:

$$\left(\begin{array}{ccc|c} -1 & 1 & -1 & -6 \\ 1 & 0 & 1 & 15 \\ 2 & -1 & 1 & 9 \end{array}\right)$$

(c) Our goal is to transform the augmented matrix $(A|\mathbf{b})$ to the matrix $(I|\mathbf{x})$ using only the following operations:

- multiply one row by a scalar quantity
- add a multiple of one row to another row
- interchange two rows

Discuss why we are allowed to use these operations. **Solution:** These *moves* are legal since each row represents a linear equation and we are simply manipulating the equations while making sure that the equal sign remains true.

(d) Starting with the top left corner of the augmented matrix, systematically row reduce the matrix to the form $(I|\mathbf{x})$. **Solution:**

$$\left(\begin{array}{ccc|c} -1 & 1 & -1 & -6 \\ 1 & 0 & 1 & 15 \\ 2 & -1 & 1 & 9 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 0 & 1 & 0 & 9 \\ 2 & -1 & 1 & 9 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 0 & 1 & 0 & 9 \\ 0 & 1 & -1 & -3 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 15 \\ 0 & 1 & 0 & 9 \\ 0 & 1 & -1 & -3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 15 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & -1 & -12 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & 12 \end{array}\right)$$

(e) Once you have the row reduced matrix interpret your result. **Solution:** Our row reduction gives us

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \\ 12 \end{pmatrix}$$

▲

Technique 3.11 (Practical Tips for Gaussian Elimination)**.** When performing Gaussian Elimination you should keep the following in mind:

- First try to get a 1 in the upper left-hand corner of the augmented matrix.
- Next, use the new first row to eliminate all of the non-zero entries in the first column. By the time you're done with this you should have a column with a 1 on top and zeros below.
- Next get a 1 in row 2 column 2.
- Use your new second row to eliminate all of the non-zero entries in the second column.
- Proceed in a similar fashion until you have reached the final row

Example 3.12. Let's row reduce an augmented matrix. Pay particular attention to the systematic way that we work toward getting the identity matrix on the left-hand side of the augmented matrix.

$$\left(\begin{array}{cc|c} 2 & -2 & 6 \\ 2 & 1 & 0 \end{array}\right) \xrightarrow{R_1 \leftarrow (1/2)R_1} \left(\begin{array}{cc|c} 1 & -1 & 3 \\ 2 & 1 & 0 \end{array}\right)$$
$$\xrightarrow{R_2 \leftarrow R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 3 & -6 \end{array}\right)$$
$$\xrightarrow{R_2 \leftarrow (1/3)R_2} \left(\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 1 & -2 \end{array}\right)$$
$$\xrightarrow{R_1 \leftarrow R_2 + R_1} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -2 \end{array}\right)$$

Notice further that at each step we indicate which row operations were done. Finally notice that there are no equal signs since the matrices that you create at each step are definitely not equal; they are called "row equivalent".

I leave it to you to make yourself familiar with examples of Gaussian Elimination from other texts (see the linked materials in Section 0.2 of these notes).

Problem 3.13. Consider the following three systems of equations and their row reduced forms. Describe their solution sets geometrically. If the system has a solution then give it. If the system has no solution then explain why. If the system has infinitely many solutions then give them all in a parameterized form.

$$\text{System \#1:} \quad \left(\begin{array}{cc|c} 1 & -1 & 3 \\ 2 & 1 & 0 \end{array}\right) \to \cdots \to \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -2 \end{array}\right)$$
$$\text{System \#2:} \quad \left(\begin{array}{cc|c} 1 & -1 & 3 \\ -1 & 1 & 0 \end{array}\right) \to \cdots \to \left(\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 0 & 3 \end{array}\right)$$
$$\text{System \#3:} \quad \left(\begin{array}{cc|c} 1 & -1 & 3 \\ -1 & 1 & -3 \end{array}\right) \to \cdots \to \left(\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 0 & 0 \end{array}\right)$$

▲

Solution: System #1 has one unique solution $x_1 = 1, x_2 = -2$. System #2 has no solution. System #3 has infinitely many solutions $x_1 = 3 + t, x_2 = t$ for $t \in \mathbb{R}$.

Problem 3.14. Create 3×3 systems of equations that have

(a) exactly 1 solution

(b) no solutions

(c) infinitely many solutions

▲

Problem 3.15. What is the value of k so that the linear system represented by the following matrix would have infinitely many solutions?

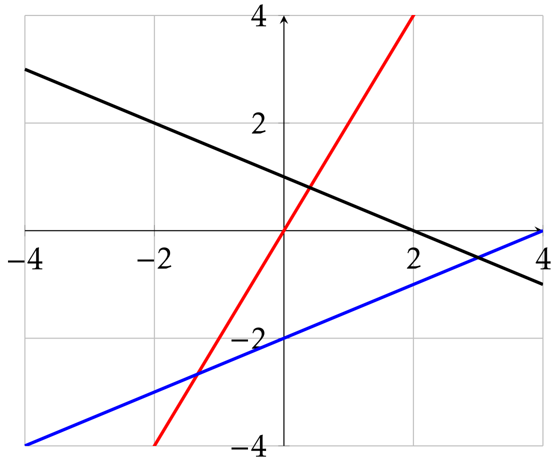
$$\left(\begin{array}{cc|c} 2 & 6 & 8 \\ 1 & k & 4 \end{array}\right)$$

Choose from the following choices:
(a) $k = 1$, (b) $k = 2$, (c) $k = 3$, (d) $k = 4$, (e) not possible, (f) there are infnitely many ways to do this ▲

Solution: $k = 3$

Problem 3.16. We have a system of three linear equations with two unknowns as plotted in the graph

How many solutions does the system have? Choose from the following:
(a) 0, (b) 1, (c) 2, (d) 3, (e) infinite

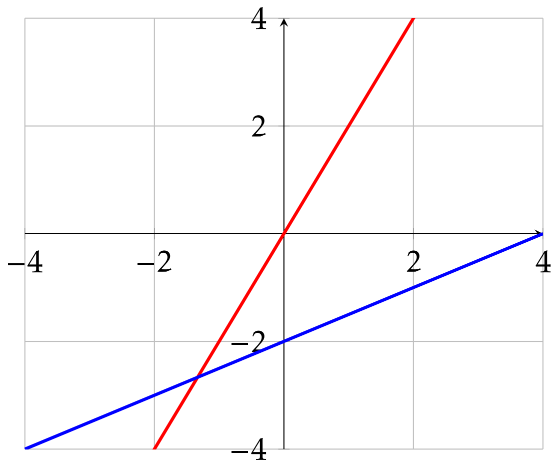


▲

Solution: No solution

Problem 3.17. We have a system of two linear equations with two unknowns as plotted in the graph

How many solutions does the system have? Choose from the following:
(a) 0, (b) 1, (c) 2, (d) 3, (e) infinite

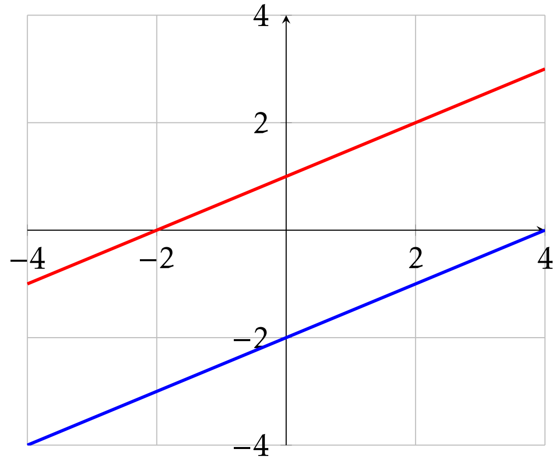


▲

Solution: One Solution

Problem 3.18. We have a system of two linear equations with two unknowns as plotted in the graph

How many solutions does the system have? Choose from the following:
(a) 0, (b) 1, (c) 2, (d) 3, (e) infinite



▲

Solution: No Solutions

Problem 3.19. The following system has infinitely many solutions. Write an equation that expresses all of them.

$$\begin{aligned} x + y &= 2 \\ -3x - 3y &= -6 \\ 2x + 2y &= 4 \end{aligned}$$

▲

Solution: $x = t$ and $y = 2 - t$ for all $t \in \mathbb{R}$.

Problem 3.20. A system of 8 linear equations and 6 variables could not have exactly ______ solution.
(a) 0, (b) 1, (c) infinite, (d) more than one of these is possible, (e) all of these are possible ▲

Solution: Infinitely many solution or no solutions

Problem 3.21. What is the solution to the system of equations represented by this augmented matrix?

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Choose from:

(a) $x = 2, y = 3, z = 4$

(b) $x = -2, y = 1, z = 1$

(c) There are infinitely many solutions

(d) There is no solution

(e) We can't tell without having the system of equations

(If there are infinitely many solutions then write expression for all of them) ▲

Solution: There are infinitely many

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} t \quad t \in \mathbb{R}$$

Problem 3.22. Solve the system of equations

$$\begin{aligned} x + 2y + z &= 0 \\ x + 3y - 2z &= 0 \end{aligned}$$

▲

Solution:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -7 \\ 3 \\ 1 \end{pmatrix} t \quad t \in \mathbb{R}$$

Problem 3.23. Let R be the reduced row echelon form of matrix A. True or False: the solutions to $R\mathbf{x} = \mathbf{0}$ are the same as the solutions to $A\mathbf{x} = \mathbf{0}$. ▲

Solution: True

Problem 3.24. Let R be the reduced row echelon form of matrix A. True or False: the solutions to $R\mathbf{x} = \mathbf{b}$ are the same as the solutions to $A\mathbf{x} = \mathbf{b}$. ▲

Solution: False

3.3 Linear Combinations

One of the most beautiful parts of linear algebra is the richness of the structure of matrices. As you already know, every system of linear equations can be written several different ways: as a system, as a matrix equation, as a vector equation, or as an augmented system.

Example 3.25. For example, we can write the system of equations

$$\begin{aligned} 2x_1 + 3x_2 &= 5 \\ 4x_1 - 6x_2 &= 6 \end{aligned}$$

equivalently in the following ways:

Algebraic System: $$\begin{aligned} 2x_1 + 3x_2 &= 5 \\ 4x_1 - 6x_2 &= 6 \end{aligned}$$

Matrix Equation: $$\begin{pmatrix} 2 & 3 \\ 4 & -6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

Vector Equation: $$x_1\begin{pmatrix} 2 \\ 4 \end{pmatrix} + x_2\begin{pmatrix} 3 \\ -6 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

Augmented System: $$\left(\begin{array}{cc|c} 2 & 3 & 5 \\ 4 & -6 & 6 \end{array}\right).$$

In this section well look in particular at the vector equation. Hiding behind a vector equation is one of the most fundamental ideas behind all of linear algebra: the linear combination. The "vector equation" above really says "some amount of $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ plus some amount of $\begin{pmatrix} 3 \\ -6 \end{pmatrix}$ gives $\begin{pmatrix} 5 \\ 6 \end{pmatrix}$ and our job is to find the amounts that make the equality true." More generally, the vector equation is saying that $\begin{pmatrix} 5 \\ 6 \end{pmatrix}$ is a linear combination of the other two vectors.

Let's first start with a simple exercise.

Problem 3.26. Write the system of equations as a vector equation.

$$\begin{aligned} x_1 + 3x_2 - 5x_3 &= 9 \\ -x_1 + x_2 &= -3 \\ 7x_1 + 2x_3 &= -\pi \end{aligned}$$

▲

Solution:

$$x_1\begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix} + x_2\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + x_3\begin{pmatrix} -5 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 9 \\ -3 \\ -\pi \end{pmatrix}$$

Definition 3.27 (Linear Combination)**.** Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_p$ be vectors in n-dimensional space and let $c_1, c_2, c_3, \ldots, c_p$ be scalar quantities. The vector $\mathbf{u}$ defined by

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_p\mathbf{v}_p = \sum_{j=1}^{p} c_j\mathbf{v}_j$$

is called a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_p$ with weights $c_1, c_2, c_3, \ldots, c_p$.

Problem 3.28. Open the GeoGebra applet: www.geogebra.org/m/WShmQvQU in a browser window.

(a) Move the vectors $\mathbf{u}$ and $\mathbf{v}$ to $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

(b) Describe all of the possible vectors $\mathbf{w} = c_1\mathbf{u} + c_2\mathbf{v}$ if $c_1 = 0$

(c) Describe all of the possible vectors $\mathbf{w} = c_1\mathbf{u} + c_2\mathbf{v}$ if $c_2 = 0$

(d) Is it possible to find c_1 and c_2 such that $\mathbf{w} = \begin{pmatrix} -6 \\ 0.5 \end{pmatrix}$. If so, what are c_1 and c_2.

▲

Solution:

(a) -

(b) This will be a line pointing in the same direction of $\mathbf{v}$

(c) This will be a line pointing in the same direction of $\mathbf{w}$

(d) For this problem we want to find c_1 and c_2 such that $\mathbf{w} = c_1\mathbf{u} + c_2\mathbf{v}$.

$$c_1\mathbf{u} + c_2\mathbf{v} = \mathbf{w}$$
$$\implies \left(\begin{array}{cc|c} 1 & 2 & -6 \\ 2 & -1 & 0.5 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 2 & -6 \\ 0 & -5 & 12.5 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 2 & -6 \\ 0 & 1 & -2.5 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -2.5 \end{array}\right)$$

so if we take $c_1 = -1$ and $c_2 = -2.5$ we get $\mathbf{w} = -\mathbf{u} - 2.5\mathbf{v}$

Problem 3.29. Write $\mathbf{u} = \begin{pmatrix} -5 \\ 3 \\ 16 \end{pmatrix}$ as a linear combination of $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} -3 \\ 2 \\ 6 \end{pmatrix}$ ▲

Solution: $\mathbf{u}$ cannot be written as a linear combination of these vectors.

Problem 3.30. Write $d = (3, -5, 10)$ as a linear combination of the vectors $a = (-1, 0, 3)$, $b = (0, 1, 5)$, and $c = (4, -2, 0)$.
Choose from:

(a) $d = -3a - 5b + c$

(b) $d = 5a - b + 2c$

(c) $d = (10/3)a + (5/2)c$

(d) d cannot be written as a linear combination of a, b, and c.

▲

Solution: b

3.4 Inverses and Determinants

Division is always a bit of a touchy subject. In the real numbers division is well defined except when the denominator is zero. The same story is true in the rational numbers: a fraction divided by a fraction is another fraction so long as the divisor is not zero. What if we wanted to stay only in the integers? Can we divide two integers and get another integer? Of course you can always divide by 1, but in most other cases division will move you into the rational numbers. Hence, division on the integers doesn't really make sense. Mathematically speaking we say that the integers are not closed under addition.

Similarly, if we try to define division on matrices we run into trouble. What does it mean to divide by a matrix? In general, that phrase is meaningless! Let's expand our view a bit.

When considering the operation of addition, we call 0 the additive identity and we call (a) the additive inverse of a since $a + (-a) = 0$. When considering multiplication, we call 1 the multiplicative identity and $1/a$ is the multiplicative inverse of a (when $a \neq 0$) since $a \cdot \frac{1}{a} = 1$. To define the matrix inverses of a square matrix A we seek the same thing: find matrix B such that $AB = I$ and $BA = I$. Where I is the identity matrix which is the multiplicative identity for matrices.

Problem 3.31. Consider the matrix $A = \begin{pmatrix} 1 & 2 \\ -4 & -6 \end{pmatrix}$. Find a matrix B such that $AB = I$ and $BA = I$.
You may recall some sort of "shortcut" from previous classes ... you are forbidden to use this shortcut! ▲

Solution: Augment with the identity and row reduce.

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -4 & -6 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 2 & 4 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1/2 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 0 & -3 & 1 \\ 0 & 1 & 2 & 1/2 \end{array}\right)$$

Technique 3.32 (Finding Matrix Inverses)**.** If A is a square matrix of size $n \times n$ what is the technique for mechanically find the A^{-1} if it exists?
Hint: Use what you did in the previous problem to give the appropriate technique.

Solution: Augment with the identity and row reduce.

Problem 3.33. Give an example of a non zero 2×2 matrix that does not have an inverse. ▲

Solution:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Problem 3.34. Which of the following matrices does not have an inverse?
(a) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, (b) $\begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix}$, (c) $\begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$, (d) $\begin{pmatrix} 0 & 4 \\ 2 & 0 \end{pmatrix}$, (e) More than one of these does not have an inverse, (f) All have inverses ▲

Solution: (b)

Problem 3.35. When we put a matrix A into row reduced echelon form, we get the following matrix. What dose this mean?

$$A \to \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

(a) Matrix A has no inverse

(b) The matrix we have found is the inverse of A

(c) Matrix A has an inverse but this isn't it

(d) This tells us nothing about whether A has an inverse.

▲

Solution: (a)

Problem 3.36. True or False: Suppose that A, B, and C are square matrices and $CA = B$ and A is invertible. This means that $C = A^{-1}B$. ▲

Solution: False

Problem 3.37. A and B are invertible matrices. If $AB = C$ then what is the inverse of C?
(a) $C^{-1} = A^{-1}B^{-1}$, (b) $C^{-1} = B^{-1}A^{-1}$, (c) $C^{-1} = AB^{-1}$, (d) $C^{-1} = BA^{-1}$,
(e) More than one of these is true,
(f) Just because A and B have inverses this doesn't mean that C has an inverse ▲

Solution: (b)

Definition 3.38 (2 × 2 Determinant)**.** Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The determinant of A is

$$\det(A) = ad - bc.$$

Problem 3.39. What is the determinant of $A = \begin{pmatrix} 4 & -1 \\ -2 & 1 \end{pmatrix}$? ▲

Solution: $\det(A) = 4 - 2 = 2$

Problem 3.40. Now consider the matrix

$$A = \begin{pmatrix} 1 & 5 & 3 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{pmatrix}.$$

- Cross out row 1 and column 1. Call the remaining 2×2 matrix A_{11}.
- Cross out row 1 and column 2. Call the remaining 2×2 matrix A_{12}.
- Cross out row 1 and column 3. Call the remaining 2×2 matrix A_{13}.

The determinant of A is

$$\det(A) = 1 \cdot \det(A_{11}) - 5 \cdot \det(A_{12}) + 3 \cdot \det(A_{13}).$$

Perform this computation. ▲

When doing determinants of square matrices you can expand along any row or column you like. The previous problem had you expand along the first row but arguably expanding along the third row would have been easier since there are several zeros. Notice, however, that there the signs on the determinant terms alternate. Hence, if you are going to expand upon a given row or column you need to keep in mind that the signs on the terms follow checkerboard pattern:

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Problem 3.41. Expand the matrix A from problem 3.40 along the third row. ▲

Problem 3.42. Find the determinant of

$$A = \begin{pmatrix} 1 & 5 & 3 \\ 2 & 0 & -5 \\ 0 & 0 & 3 \end{pmatrix}$$

▲

Solution: Expand along the second column:

$$\det(A) = -5 \det\left(\begin{pmatrix} 2 & -5 \\ 0 & 3 \end{pmatrix}\right) + 0 \det(\cdot) - 0 \det(\cdot) = (-5)(6-0) = -30.$$

Problem 3.43. What is the determinant of the $n \times n$ identity matrix? ▲

Solution: $\det(I) = 1$

Theorem 3.44 (Determinants and Invertibility)**.** The $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$.

Theorem 3.45. Determinants and the Matrix Transpose Let A be an $n \times n$ matrix. The determinant of the transpose of A is the same as the determinant of A. In other words,

$$\det(A^T) = \det(A).$$

Theorem 3.46 (Determinants of Matrix Products)**.** If A and B are $n \times n$ matrices, then

$$\det(AB) = \det(A) \cdot \det(B).$$

Problem 3.47. Suppose that the determinant of a 2×2 matrix A is equal to 3. What is the determinant of A^{-1}? Be able to defend your claim mathematically. ▲

Solution: $\det(A^{-1}) = 1/3$ since $1 = \det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1})$ so $\det(A^{-1}) = 1/\det(A)$.

Theorem 3.48 (Determinants of Inverses)**.** If A is an invertible matrix then

$$\det(A^{-1}) = \underline{\hspace{3cm}}$$

(Fill in the blank . . . you just proved this in the last problem)

Solution: $\det(A^{-1}) = 1/\det(A)$

Theorem 3.49. If λ is some scalar and A is a square matrix of size $n \times n$ then

$$\det(\lambda A) = \lambda^n \det(A).$$

Problem 3.50. What is the determinant of $5A$ where $A = \begin{pmatrix} 1 & 3 \\ -7 & 2 \end{pmatrix}$? ▲

Solution: $\det(A) = 2 + 21 = 23$ so $\det(5A) = 5^2(23) = 575$.

Problem 3.51. When using a computer to find the determinant of a square matrix you must always be careful. Not because the computation is difficult (which it sometimes is), but because of the property

$$\det(\lambda A) = \lambda^n \det(A)$$

where A is an $n \times n$ matrix.

- Why does this property tell you that you really need to be careful with computers and determinants?
- Why should you never use a computer to find a matrix inverse?

▲

Solution: If λ is the roundoff error native to a computer and A is a large matrix then the determinant computation exponentiates the round off error.

Problem 3.52. Matrix A has size $1,000,000 \times 1,000,000$ and has determinant $\det(A) = 7$. Let's say that A is stored on a computer in single precision so each value in the matrix is only accurate to 10^{-8}. That is, a number x in the matrix is actually somewhere between $(1-10^{-8})x$ and $(1+10^{-8})x$. If we actually find $\det(A)$ on this computer what is a range for the determinant computation?
Hint: the relative error is $\lambda = 1 \pm 10^{-8}$ so you should be computing $\det(\lambda A)$. ▲

Solution: $6.93 < \det(A) < 7.07$

Theorem 3.53 (Summary of Properties of Determinants)**.** Let A be a square matrix.

1. If a multiple of one row of A is added to another row to produce matrix B then $\det(A) = \det(B)$.
2. If two rows are interchanged in matrix A to produce matrix B then $\det(B) = -\det(A)$.
3. If one row of A is multiplied by k to product matrix B then $\det(B) = k\det(A)$.
4. $\det(A^T) = \det(A)$.
5. If A is an $n \times n$ matrix and $\lambda \in \mathbb{R}$ then $\det(\lambda A) = \lambda^n \det(A)$
6. $\det(A^{-1}) = 1/\det(A)$
7. $\det(AB) = \det(A)\det(B)$
8. The $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$

Problem 3.54. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\det(A) = 8$ then what is $\det(B)$ where $B = \begin{pmatrix} a & b \\ 3c & 3d \end{pmatrix}$ ▲

Solution: $\det(B) = 3 \cdot 8 = 24$

Problem 3.55. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\det(A) = 8$ then what is $\det(B)$ where $B = \begin{pmatrix} a & b \\ 2a+c & 2b+d \end{pmatrix}$
▲

Solution: 24

Problem 3.56. True or False: The determinant of A is the same as the determinant of the row reduced form of A. Explain your answer. ▲

Solution: False. Row reduction often leads swapping and scaling rows

3.5 Applications of Linear Systems

Problem 3.57. Use linear algebra to find a cubic polynomial of the form

$$f(x) = A + Bx + Cx^2 + Dx^3$$

that interpolates the data points

$$(-1,4), (1,2), (2,1), (3,16)$$

Hint: each point creates a linear equation in $A, B, C,$ and D. ▲

Solution:

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 4 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 2 & 4 & 8 & 1 \\ 1 & 3 & 9 & 27 & 16 \end{array}\right) \to \cdots \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

$f(x) = 7 - 3x - 4x^2 + 2x^3$

Problem 3.58. (a) How many data points do you need to exactly describe a unique quadratic polynomial?

(b) How many data points do you need to exactly describe a unique cubic polynomial?

(c) How many data points do you need to exactly describe a unique quartic polynomial?

(d) In general, if you have n data points that you believe form a polynomial function, what is the order of the polynomial?

▲

Solution: (a) 3, (b) 4, (c) 5, (d) $n-1$

Definition 3.59 (The Vandermonde Matrix)**.** The Vandermonde matrix is an $m \times n$ matrix of the form

$$V = \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & x_2^3 & \cdots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & x_3^3 & \cdots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & x_m^2 & x_m^3 & \cdots & x_m^{n-1} \end{pmatrix}$$

There are several interesting properties of the Vandermonde matrix but of particular interest in this section of these notes is that the Vandermonde matrix appears when doing a polynomial interpolation.

Example 3.60. Build the Vandermonde matrix associated with fitting a quadratic polynomial to the data points $(-1,3)$, $(2,5)$, and $(7,-1)$.

Solution: We will let $x_1 = -1$, $x_2 = 2$ and $x_3 = 7$ to get the equations

$$\begin{aligned} a(-1)^2 + b(-1) + c &= 3 \\ a(2)^2 + b(2) + c &= 5 \\ a(7)^2 + b(7) + c &= -1 \end{aligned}$$

since we are trying to fit the data to the quadratic function $f(x) = ax^2 + bx + c$. Rearranging the system into a matrix equation we immediately see the Vandermonde matrix appear as the coefficient matrix.

$$\begin{pmatrix} 1 & -1 & (-1)^2 \\ 1 & 2 & 2^2 \\ 1 & 7 & 7^2 \end{pmatrix} \begin{pmatrix} c \\ b \\ a \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}$$

We leave it to the reader to solve the system for a, b, and c.

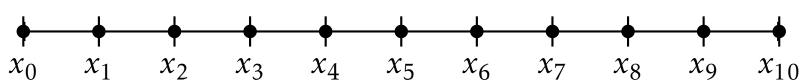
Problem 3.61. Find a polynomial that perfectly fits the following data set

x	y
-2	3
-1	5
0	-9
4	-3
5	0
8	-6

Solution: We will use a fifth order polynomial, $f(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$, since there are 6 data points and 6 unknowns. ...finish typing solution.

Problem 3.62. Imagine that we have a 1 meter long thin metal rod that has been heated to 100° on the left-hand side and cooled to 0° on the right-hand side. We want to know the temperature every 10 cm from left to right on the rod.

(a) First we break the rod into equal 10cm increments as shown.



How many unknowns are there in this picture?

(b) The temperature at each point along the rod is the average of the temperatures at the adjacent points. For example, if we let T_1 be the temperature at point x_1 then

$$T_1 = \frac{T_0 + T_2}{2}.$$

Write a system of equations for each of the unknown temperatures.

(c) Solve the system for the temperature at each unknown node.

▲

Solution:

$$\begin{aligned}
T_1 &= \frac{T_0 + T_2}{2} = \frac{100 + T_2}{2}\\
T_2 &= \frac{T_1 + T_3}{2}\\
T_3 &= \frac{T_2 + T_4}{2}\\
T_4 &= \frac{T_3 + T_5}{2}\\
T_5 &= \frac{T_4 + T_6}{2}\\
T_6 &= \frac{T_5 + T_7}{2}\\
T_7 &= \frac{T_6 + T_8}{2}\\
T_8 &= \frac{T_7 + T_9}{2}\\
T_9 &= \frac{T_8 + T_{10}}{2} = \frac{T_8}{2}
\end{aligned}$$

Solving this system we get $T_0 = 100, T_1 = 90, T_2 = 80, T_3 = 70, \cdots, T_1 0 = 0$.

Chapter 4

Vector Spaces

Words of Advice to Mathematical Scientists:

LINEAR ALGEBRA IS THE MOST IMPORANT OF ALL OF THE MATHEMATICAL SCIENCES

Why?

- Within linear algebra is the language that describes all of the how and why for all linear operations.
  - solving linear differential equations
  - the reasons why the derivative and the integral operators work so nicely
  - geometry and transformations
  - computer graphics (video games are 99% linear algebra)
  - image processing (Photoshop = fancy linear algebra package)
  - large scale stress computations (linear elasticity)
  - ...
- Linear Algebra is **far** more than just *matrices*.

4.1 What is a Vector Space

To get properly in to Linear Algebra we first need to establish some of the common notation. You are familiar with 2D and 3D Vectors but using the spatial notion of "dimension" limits how you will think about vectors in linear algebra. Instead of thinking of the spatial dimensions that we live in you should be thinking of vectors as abstract objects with mathematical properties. Your intuitive notion of "arrows" is limiting and ultimately incorrect for many purposes. This chapter starts us in the direction of abstract vector spaces, but don't be worried that we will be doing abstract mathematics. We are not abstracting a familiar notion for no good reason. It was this very abstraction that has allowed mathematics to advance over the past several centuries.

Let's briefly talk about how you learned about functions. First you were told to think of a function as a "rule" that accepted an input and gave a single output (high school algebra). Then you got used to the notion that you could add, subtract, multiply, and divide functions to get new functions (pre-calculus). When you took calculus you started to think of functions as *objects* themselves and started discussing ways to get properties of those objects. The process that you've gone through with functions is called "objectification" of a mathematical idea: you have turned functions into objects in your mind's eye. Our goal in this chapter is to "objectify" the idea of vectors. We are going to turn them into abstract objects that have mathematical properties.

Definition 4.1 (The Spaces $\mathbb{R}^n$)**.** Some common vector spaces (and common mathematical notation)

$$\begin{aligned}
\mathbb{R}^2 &= \{(x, y) : x, y \in \mathbb{R}\} \\
\mathbb{R}^3 &= \{(x, y, z) : x, y, z \in \mathbb{R}\} \\
\mathbb{R}^n &= \{(x_1, x_2, x_3, x_4) : x_j \in \mathbb{R}\} \\
&\vdots \\
\mathbb{R}^n &= \{(x_1, x_2, \dots, x_n) : x_j \in \mathbb{R}\}
\end{aligned}$$

Problem 4.2. (< 5 minutes): Make a list of all of the things that you can do to vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ and give examples of how you do them. ▲

Solution: Students likely list addition, subtraction, scalar multiplication, maybe the cross product (but this doesn't have a great meaning above 3D), projections, length, zero, etc.

The list that you made for vectors in $\mathbb{R}^n$ is really just a wish list of all of the things that you would like out of a well defined collection of mathematical objects called "vectors". We are going to abstract the idea so that we do not have to just think about arrows and lines in space. Instead, a *vector space* is a collection of abstract mathematical objects that satisfies a collection of rules (the rules that you likely already wrote down).

Definition 4.3 (Vector Space)**.** A **Vector Space** $\mathcal{V}$ is defined as a set of *mathematical objects* called vectors that follow the following 10 rules.
If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in $\mathcal{V}$ and $c_1, c_2 \in \mathbb{R}$ then

1. Closure under Addition: $\mathbf{u} + \mathbf{v} \in \mathcal{V}$
2. Closure under Scalar Multiplication: $c\mathbf{u} \in \mathcal{V}$
3. Commutativity: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
4. Associativity: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

5. Zero Vector: $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
6. Additive Inverses: $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$
7. Distributive Properties: $c_1(\mathbf{u} + \mathbf{v}) = c_1\mathbf{u} + c_1\mathbf{v}$
8. $(c_1 + c_2)\mathbf{u} = c_1\mathbf{u} + c_2\mathbf{u}$
9. $c_1(c_2\mathbf{u}) = (c_1c_2)\mathbf{u}$
10. Scalar Identity: $1\mathbf{u} = \mathbf{u}$

Problem 4.4. What other *sets of things* have the mathematical properties of vector spaces? ▲

Solution: The collection of all differentiable functions, The collection of all square matrices of the same size, the collection of all polynomials, ...

Problem 4.5. Is $\mathcal{V} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x \ge 0 \text{ and } y \ge 0 \right\}$ a vector space over the real numbers? Why or why not? ▲

Solution: no since $-1\mathbf{v} \notin \mathcal{V}$ for $\mathbf{v} \in \mathcal{V}$

Problem 4.6. Is $\mathcal{V} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : xy \ge 0 \right\}$ a vector space? Why or why not? ▲

Solution: if $\mathbf{v} = (1,1)^T$ and $\mathbf{u} = (-1,-2)^T$ then $\mathbf{v} + \mathbf{u} = (0,-1)^T \notin \mathcal{V}$

Problem 4.7. Is the collection of all polynomials of the form $p(t) = at^2$ a vector space where $a \in \mathbb{R}$? ▲

Solution: yes

Problem 4.8. Is the collection $\mathcal{V} = \{f(x) : f'(x) \text{ exists } \}$ a vector space? ▲

Solution: yes

Definition 4.9 (The Trace of a Matrix)**.** Let A be a square $n \times n$ matrix. The trace of A, denoted $\text{tr}(A)$ is the sum of the diagonal entries. For example, the trace of a 2×2 matrix is

$$\text{tr}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = a + d.$$

Problem 4.10. Let $\mathcal{V}$ be defined as

$$\mathcal{V} = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{R} \text{ and } \text{tr}(A) = 0 \right\}$$

Is $\mathcal{V}$ a vector space? ▲

Solution: yes

4.2 Linear Independence and Linear Dependence

When studying vector spaces it is useful to think about how to *build* the vector space out of the simplest possible components. In order to understand that we first need an important idea in linear algebra: *linear independence*. Roughly speaking, a collection of vectors is called linearly independent if you cannot make any of the vectors in the collection by taking linear combinations of the other vectors in the collection. Keep this in mind when you read the following formal definition.

Definition 4.11 (Linearly Independent Vectors)**.** The vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly independent if the equation

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 = \mathbf{0}$$

has only the trivial solution $c_1 = c_2 = 0$.

More generally, The vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n$ are linearly independent if the equation

$$\sum_{j=1}^{n} c_j\mathbf{u}_j = \mathbf{0}$$

has only the trivial solution $c_1 = c_2 = \cdots = c_n = 0$.
A set of vectors that is not linearly independent is called *linearly dependent*.

Problem 4.12. Write three vectors that are linearly independent in $\mathbb{R}^3$. Then write three vectors that are linearly dependent in $\mathbb{R}^3$. ▲

Problem 4.13. Consider the vector space

$$P_n = \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \,:\, a_0, a_1, \dots, a_n \in \mathbb{R}\}.$$

Let $n = 3$ and find 4 linearly independent *vectors* in this vector space. Then find 4 linearly dependent vectors in this vector space. ▲

Problem 4.14. Consider the vector space $\mathcal{V} = \{f(x) \,:\, f'(x) \text{ exists }\}$. Find three linearly independent vectors in this vector space and are also a solution to the differential equation $y' = -2y + 3t + 5$. ▲

Solution:

$$\{e^{2t}, t, 1\}$$

Problem 4.15. If $\mathbf{v}_1$ and $\mathbf{v}_2$ are vectors in $\mathbb{R}^2$ how would we show that they are linearly independent? ▲

Solution: Solve the system of equations $C_1\mathbf{v}_1 + C_2\mathbf{v}_2 = \mathbf{0}$ for C_1 and C_2. If $C_1 = C_2 = 0$ then they are linearly independent.

Theorem 4.16. Let S be a set of vectors in a vector space $\mathcal{V}$. If the zero vector, $\mathbf{0}$, is contained in S then the set S is linearly dependent.

Proof. Prove this theorem. □

Solution: Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p, \mathbf{0}\}$. Then

$$\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_p + c\mathbf{0}$$

where c is any real number. Hence the only solution to the equation $\mathbf{0} = \sum_{j=1}^{p+1} c_j \mathbf{v}_j$ is NOT the trivial solution and this means that the vectors are linearly dependent.

Theorem 4.17. Let S be a set of vectors in a vector space $\mathcal{V}$. If $\mathbf{u} \in \mathcal{V}$ and $c\mathbf{u} \in \mathcal{V}$ for some fixed real number c then the set S is linearly dependent.

Proof. Prove this theorem. □

Solution: Let $S = \{\mathbf{v}_1 = \mathbf{u}, \mathbf{v}_2 = c\mathbf{u}, \mathbf{v}_3, \mathbf{v}_4, \ldots, \mathbf{v}_p\}$. If $\mathbf{0} = \sum_{j=1}^{p} c_j \mathbf{v}_j$ then a non-trivial solution is $\mathbf{0} = -c\mathbf{u} + 1(c\mathbf{u}) + \sum_{j=3}^{p} 0\mathbf{v}_p$. Hence the vectors are not linearly independent.

Problem 4.18. Which set of vectors is linearly independent?

(a) $(2,3)$, $(8,12)$

(b) $(1,2,3)$, $(4,5,6)$, $(7,8,9)$

(c) $(-3,1,0)$, $(4,5,2)$, $(1,6,2)$

(d) None of these sets are linearly independent.

(e) Exactly two of these sets are linearly independent.

(f) All of these sets are linearly independent.

▲

Solution: None of these are linearly independent. You can tell by row reducing each of them.

$$1: \begin{pmatrix} 2 & 8 \\ 3 & 12 \end{pmatrix} \to \begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix}$$

$$2: \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$3: \begin{pmatrix} -3 & 4 & 1 \\ 1 & 5 & 6 \\ 0 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Problem 4.19. Consider the first order non-homogeneous differential equation

$$y' = -3y + 4t.$$

What are the homogeneous and particular solutions that arise from using the method of undetermined coefficients? Are these functions linearly independent? Verify that the analytic solution to the differential equations is a linear combination of these solutions. ▲

Solution: $y_{hom} = e^{-3t}$ and $y_{part} = C_1 t + C_2$.

$$y(t) = C_1 e^{-3t} + C_1 t + C_2$$

Problem 4.20. Suppose you wish to determine whether a set of vectors is linearly independent. You form a matrix with those vectors as the columns and you calculate the reduced row echelon form

$$R = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

What do you decide?

(a) The vectors are linearly independent.

(b) The vectors are not linearly independent.

▲

Solution: These vectors are not linearly independent.

Problem 4.21. To determine whether a set S of vectors is linearly independent you form a matrix which has those vectors as columns and you calculate its row reduced form. Suppose the resulting form is

$$R = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Which of the following subsets of S are linearly independent?

(a) The first, second, and third vectors

(b) The first, second, and fourth vectors

(c) The first, third, and fourth vectors

(d) The second, third, and fourth vectors

(e) All of the above

▲

Solution: The first, second, and fourth

Problem 4.22. (a) True or False: A set of 2 vectors from $\mathbb{R}^3$ must be linearly independent. **Solution:** False.

(b) True or False: A set of 3 vectors from $\mathbb{R}^3$ could be linearly independent. **Solution:** True

(c) True or False: A set of 5 vectors from $\mathbb{R}^4$ could be linearly independent. **Solution:** False

▲

Problem 4.23. Let $y_1(t) = e^{2t}$. For which of the following functions $y_2(t)$ will the set $\{y_1, y_2\}$ be linearly independent?

(a) $y_2(t) = e^{-2t}$

(b) $y_2(t) = te^{2t}$

(c) $y_2(t) = 1$

(d) $y_2(t) = e^{3t}$

(e) All of the above

(f) None of the above

▲

Solution: All

Problem 4.24. True or False: The function $h(t) = 4 + 3t$ is a linear combination of the functions $f(t) = (1+t)^2$ and $g(t) = 2 - t - 2t^2$. ▲

Solution: True. $h(t) = 2f(t) + g(t) = 2(1+2t+t^2) + (2-t-2t^2) = 4+3t$

Problem 4.25. True or False: The function $h(t) = t^2$ is a linear combination of $f(t) = (1-t)^2$ and $g(t) = (1+t)^2$. ▲

Solution: False

4.3 Span

Definition 4.26. The **span** of a collection of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is the set

$$\{c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots c_n\mathbf{u}_n : c_j \in \mathbb{R}\}$$

This is the set of all linear combinations of the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$.

Problem 4.27. Describe the span of the vectors **u** and **v** where

$$\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

1. all of $\mathbb{R}^3$
2. A plane in $\mathbb{R}^3$
3. all of $\mathbb{R}^2$
4. A line in $\mathbb{R}^2$
5. none of these

▲

Solution: All of $\mathbb{R}^2$.

Problem 4.28. Describe the span of the vectors **u** and **v** where

$$\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$$

1. all of $\mathbb{R}^3$
2. A plane in $\mathbb{R}^3$
3. all of $\mathbb{R}^2$
4. A line in $\mathbb{R}^2$
5. none of these

▲

Solution: All of $\mathbb{R}^3$

Problem 4.29. Describe the span of the vectors $\mathbf{u}_1, \mathbf{u}_2$, and $\mathbf{u}_3$ where

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{u}_3 = \begin{pmatrix} 0 \\ -4 \\ 2 \end{pmatrix}$$

1. all of $\mathbb{R}^3$
2. A plane in $\mathbb{R}^3$
3. all of $\mathbb{R}^2$

4. A line in $\mathbb{R}^2$

5. none of these

▲

Problem 4.30. What is the span of the set $S = \{e^{-2t}, 1\}$ in the space of all differentiable functions. What first order differential equation has a solution space spanned by S? ▲

Solution: $\text{span}(S) = C_1 e^{-2t} + C_2$. This is the general solution to the differential equation $y'(t) = -2y + c$.

Problem 4.31. What is the span of the set $S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$? ▲

Solution: This is the set of all 2×2 matrices with a zero on the bottom left corner.

Problem 4.32. What is the span of the set $S = \{1, x, x^2\}$? ▲

Solution: The set of all quadratic functions.

4.4 Subspaces

The structure of a vector space is filled with geometric richness and wonderful abstraction. As you have already experienced, we can use this abstraction to better understand the structure of sets that contain *mathematical things* that are not vectors in the traditional physics sense. We now examine the notion of a subspace to a vector space. The basic idea is that if we take a vector space and *zoom in* to just the right part we will find that there are subspaces embedded within most vector spaces. This is another abstract notion but, as it turns out, we have been dealing with subspaces all along. In multivariable calculus you got used to dealing with $\mathbb{R}^3$ and undoubtedly dealt with planes in $\mathbb{R}^3$ that went through the origin. Those planes were vector spaces in their own right (check the 10 rules in Definition 4.3).

To get going with the idea of a subspace we need to formalize what we mean by *zoom in*. Let's start this section with a little background terminology.

Definition 4.33 (Subset)**.** Let S be a set. A subset B of S is a collection of elements that are in S. We use the notation $B \subset S$ or $B \subseteq S$.

Example 4.34. Let $S = \{a, b, c, d\}$. Then the set $B = \{a, b\}$ is a subset of S. The set $C = \{a, b, c, e\}$ is not a subset of S since $e \notin S$.

Example 4.35. Let $S = \mathbb{R}^2$. The set $B = \{(x,y) : x \cdot y \ge 0\}$ is a subset of S since it contains things that are all in $\mathbb{R}^2$. Geometrically, B is the set of all points in the first and third quadrants of the coordinate plane whereas S is all of the coordinate plane.

Problem 4.36. Let $S = \mathbb{R}^3$. Give an example of a set S_1 that IS a subset of S and a set S_2 that IS NOT a subset of S. ▲

Solution: S_1 is any collection of things from $\mathbb{R}^3$. S_2 needs to at least contain things that are not in $\mathbb{R}^3$.

Problem 4.37. How many elements are in each of the following sets?

$$S_1 = \mathbb{R}^2 \qquad S_2 = \{\mathbb{R}^2\} \qquad S_3 = \emptyset \qquad S_4 = \{\emptyset\}$$

▲

Solution: S_1 has an uncountable infinity of elements, S_2 has exactly 1 element, S_3 has no elements, S_4 has exactly 1 element.

Throughout the following definitions you need to keep in mind the notion of a subset, but now we will be taking special subsets of vector spaces.

Definition 4.38 (Subspace)**.** If $\mathcal{V}$ is a Vector Space and S is a proper subset of $\mathcal{V}$ then S is called a **subspace** if it is a vector space in its own right.

Problem 4.39. Consider the vector space $\mathbb{R}^2$. Propose a subspace of $\mathbb{R}^2$ and be able to defend your proposition. ▲

Solution: Any line that goes through the origin

Problem 4.40. Which of the vector space criteria would need to establish to show that a set S is a subspace of a vector space $\mathcal{V}$? Look back to the vector space definition here: 4.3.
▲

Solution: We only need closure under addition and closure under scalar multiplication and the rest comes along for the ride.

Problem 4.41. Which of the following sets are subspaces of $\mathbb{R}^3$? (there are multiple answers)

1. $\{(x,0,0) : x \in \mathbb{R}\}$
2. $\{(5x+4y, 7x+2y, -8x-2y) : x,y \in \mathbb{R}\}$
3. $\{(x,y,z) : x,y,z > 0\}$
4. $\{(-6,y,z)) : y,z \in \mathbb{R}\}$
5. $\{(x,y,z) : -7x+8y-4z = -5\}$

6. $\{(x, y, z) : x + y + z = 0\}$

▲

Solution: a,b, f

Problem 4.42. The set of all 2×2 matrices with determinant equal to zero is not a vector subspace. Why?

(a) 2×2 matrices are not vectors

(b) With matrices, AB need not equal BA

(c) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 2 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & 3 \\ 2 & 2 \end{pmatrix}$ is not in the set.

(d) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not in the set.

(e) None of the above

▲

Solution: (d) since the two matrices on the left are in the set but the sum is not.

Problem 4.43. Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} 6 \\ 0 \\ -2 \end{pmatrix}$. Which of the following vectos is *not* in the subspace of $\mathbb{R}^3$ spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

(a) $(1, 0, 0)$

(b) $(4, 1, 1)$

(c) $(3, 3, 6)$

(d) All of these are in the subspace of $\mathbb{R}^3$ spanned by the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$

▲

Solution: $(1, 0, 0)^T$

Theorem 4.44. If A is an $n \times n$ matrix, then the solution set of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ is a subspace of $\mathbb{R}^n$.

Example 4.45. Consider the homogeneous system of equations

$$\begin{pmatrix} 1 & 3 & 6 \\ 1 & 0 & 0 \\ 2 & -1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We would like to know which subspace of $\mathbb{R}^3$ is spanned by the solution to this sys-

tem.
Solution: We first row reduce the augmented system

$$\left(\begin{array}{ccc|c} 1 & 3 & 6 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & -1 & -2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Hence, the solution to the system is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} t \quad \text{where} \quad t \in \mathbb{R}$$

Therefore the subspace spanned by the solution to the homogeneous system is

$$S = \operatorname{span}\left(\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}\right)$$

which is a one-dimensional subspace of $\mathbb{R}^3$. Geometrically, this subspace is a line through the origin in $\mathbb{R}^3$ pointing in the direction of the vector $(0, -2, 1)^T$.

Problem 4.46. What subspace of $\mathbb{R}^3$ is spanned by the solution space of the equations

$$\begin{cases} 3x_1 + 2x_2 + x_2 & = 0 \\ 2x_1 + 5x_2 - 3x_3 & = 0 \\ 5x_1 - 4x_2 + 9x_3 & = 0 \end{cases}$$

▲

Solution:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Therefore,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

so the subspace spanned by the solutions to the homogeneous system are

$$S = \operatorname{span}\left(\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right)$$

Theorem 4.47 (Proving that a set is a subspace)**.** Let S be a subset of a vector space $\mathcal{V}$. To prove that S is a subspace of $\mathcal{V}$ we only need to check that

- if $\mathbf{u} \in S$ then $c\mathbf{u} \in S$ for any scalar c.
- if $\mathbf{u}, \mathbf{v} \in S$ then $\mathbf{u} + \mathbf{v} \in S$.

More simply, you can check both conditions simultaneously:
If $\mathbf{u}, \mathbf{v} \in S$ and $c, d \in \mathbb{R}$ then show that $c\mathbf{u} + d\mathbf{v} \in S$.

Problem 4.48. Discuss why the technique listed above is sufficient to prove that S is a vector space in its own right (go back to the definition of a vector space). ▲

Theorem 4.49. The set containing the zero vector, $S = \{\mathbf{0}\}$, is a subspace of every vector space.

Proof. (Prove the previous theorem) □

Solution:

Proof. We will use Theorem 4.47 to prove this theorem. Let $\mathcal{V}$ be a vector space and let S be a subset of $\mathcal{V}$ that contains only the zero vector: $S = \{\mathbf{0}\}$. If $\mathbf{u}, \mathbf{v} \in S$ and $c_1, c_2 \in \mathbb{R}$ then $c_1\mathbf{u} + c_2\mathbf{v} = c_1\mathbf{0} + c_2\mathbf{0} = \mathbf{0} + \mathbf{0} = \mathbf{0} \in S$. Therefore S is a subspace of $\mathcal{V}$. □

Theorem 4.50. The span of a set of vectors is a subspace.

Proof. (You should prove this theorem) □

Solution: True.

Proof. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a set of vectors from a vector space $\mathcal{V}$. Let $\mathbf{u}, \mathbf{w} \in \text{span}(S)$ and $c_1, c_2 \in \mathbb{R}$. Since $\mathbf{u}, \mathbf{w} \in \text{span}(S)$ we know that there exists real constants $c_1, \ldots, c_k$ and $d_1, \ldots, d_k$ such that

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k \quad \text{and} \quad \mathbf{w} = d_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k.$$

Hence, if we consider the linear combination $C\mathbf{u} + D\mathbf{w}$ we get

$$\begin{aligned} C\mathbf{u} + D\mathbf{w} &= C(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k) + D(d_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k) \\ &= (Cc_1 + Dd_1)\mathbf{v}_1 + (Cc_2 + Dd_2)\mathbf{v}_2 + \cdots + (Cc_k + Dd_k)\mathbf{v}_k \in \text{span}(S). \end{aligned}$$

By Theorem 4.47 we see that span(S) is a subspace of $\mathcal{V}$. □

Example 4.51. Consider the vector space $\mathbb{R}^2$ and consider the subset of $\mathbb{R}^2$

$$S = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} : x \in \mathbb{R} \right\}.$$

Prove that S is a subspace of $\mathbb{R}^2$.

Proof. Let $\mathbf{u}, \mathbf{v} \in S$. Therefore there exists real numbers x and z such that $\mathbf{u} = \begin{pmatrix} x \\ 0 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} z \\ 0 \end{pmatrix}$. We will check both closure under addition and closure under scalar multiplication.

$$\text{Closure under addition:} \quad \mathbf{u}+\mathbf{v} = \begin{pmatrix} x \\ 0 \end{pmatrix} + \begin{pmatrix} z \\ 0 \end{pmatrix} = \begin{pmatrix} x+z \\ 0 \end{pmatrix} \in S \quad \checkmark$$

$$\text{Closure under scalar multiplication:} \quad c\mathbf{u} = \begin{pmatrix} cx \\ 0 \end{pmatrix} \in S \quad \checkmark$$

□

Example 4.52. Prove that the following subset of $\mathbb{R}^4$ is not a subspace of $\mathbb{R}^4$.

$$S = \left\{ \begin{pmatrix} 3 \\ y \\ z \\ w \end{pmatrix} : y, z, w \in \mathbb{R} \right\}$$

Proof. If $\mathbf{u} \in S$ then $\mathbf{u} = (3, y, z, w)^T$ but we see that $c\mathbf{u} \notin S$ for any c that is not 1. Hence, the set S is not closed under scalar multiplication and therefore cannot be a subspace of $\mathbb{R}^4$. □

Problem 4.53. Which of the following sets are subspaces of $\mathbb{R}^3$ and which are not? Be sure to explain your reasoning. (Hint: three of them are subspace of $\mathbb{R}^3$ and three of them are not.)

$$S_1 = \left\{ \begin{pmatrix} 8x \\ -2x \\ -9x \end{pmatrix} : x \in \mathbb{R} \right\}$$

$$S_2 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x + y + z = 0 \right\}$$

$$S_3 = \left\{ \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} : x \in \mathbb{R} \right\}$$

$$S_4 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x < y < z \right\}$$

$$S_5 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x + y + z = -8 \right\}$$

$$S_6 = \left\{ \begin{pmatrix} 7 \\ y \\ z \end{pmatrix} : y, z \in \mathbb{R} \right\}$$

▲

Solution: S_1, S_2 and S_3 are subspaces of $\mathbb{R}^3$. S_4 is not since it is not closed under scalar multiplication, S_5 is not since it does not contain the zero vector (hence not closed under scalar multiplication), and S_6 is not since it is also not closed under scalar multiplication.

Problem 4.54. Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} k \\ 8 \\ 11 \end{pmatrix}$. For how many values of k will the vector $\mathbf{w}$ be in the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

(a) No values of k. The vector $\mathbf{w}$ will never be in this subspace.

(b) Exactly one value of k will work

(c) Any value of k will work

▲

Solution: (a), the only independent vectors are $\mathbf{v}_1$ and $\mathbf{v}_2$. The vector $\mathbf{v}_3$ is $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$.

Problem 4.55. Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} 6 \\ 0 \\ -2 \end{pmatrix}$. Geometrically, what is the subspace spanned by the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?
(a) a point, (b) a line, (c) a plane, (d) a volume, (e) All of $\mathbb{R}^3$. ▲

Solution: A plane

4.5 Basis

We now come to a pivotal definition in linear algebra. Previously we hinted that we could *build* vector spaces out of simpler components. If we were to ask something of these building blocks what would we ask?

Problem 4.56. Let $\mathcal{V}$ be a vector space. We would like to find a set $\mathcal{B}$, called a **basis**, such that

- The span of $\mathcal{B}$ gives you all of $\mathcal{V}$, and
- The set $\mathcal{B}$ contains as few vectors as possible.

In order to get both of the bullets listed what property would $\mathcal{B}$ have to have? Why? ▲

Solution: It would have to be linearly independent. Otherwise there would be redundancies and the set would not be "as small as possible".

Definition 4.57 (A Basis for a Vector Space)**.** A set $\mathcal{B}$ is called a **basis** for a vector space $\mathcal{V}$ if

- span($\mathcal{B}$) = ____________
- $\mathcal{B}$ is ____________

(Fill in the blanks)

Solution: span($\mathcal{B}$) = $\mathcal{V}$ and $\mathcal{B}$ is linearly independent. Another way to say this is that a basis is a linearly independent spanning set for a vector space.

Problem 4.58. How large should the basis be for the vector space $\mathbb{R}^2$? Be able to support your answer. ▲

Solution: 2. Any larger and the set would be necessarily linearly dependent and any smaller and it wouldn't span $\mathbb{R}^2$.

Problem 4.59. How large should the basis be for the vector space $\mathbb{R}^3$? Be able to support your answer. ▲

Solution: 3 Any larger and the set would be necessarily linearly dependent and any smaller and it wouldn't span $\mathbb{R}^3$.

Problem 4.60. Find a basis for the following vector spaces:

- $\mathbb{R}^3$
- $V = \{(x, y, 0) \in \mathbb{R}^3 : x, y \in \mathbb{R}\}$
- $\mathcal{P} = \{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n : a_1, a_2, \ldots, a_n \in \mathbb{R}\}$
- $M_{2\times 2} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$
- The solution space for the differential equation $y' = -3y + 4t$

▲

Solution:

- any collection of 3 linearly independent vectors in $\mathbb{R}^3$
- $\mathcal{B} = \{(1, 0, 0), (0, 1, 0)\}$
- $\mathcal{B} = \{1, x, x^2, \ldots, x^n\}$
- $\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$
- $\mathcal{B} = \{e^{-3t}, t, 1\}$

Problem 4.61. The set $\mathcal{B} = \{e^{-0.5t}, \sin(3t), \cos(3t)\}$ is the basis for the solution space for which first order linear non-homogeneous differential equation? ▲

Solution: $y' = -0.5y + A\sin(3t) + B\cos(3t)$

Problem 4.62. The set $\mathcal{B}$ is the basis for what subspace of $\mathbb{R}^3$?

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

▲

Solution: The subspace is the $x - y$ plane in $\mathbb{R}^3$.

Problem 4.63. Which of the following sets of vectors is a basis for $\mathbb{R}^3$?

(a) $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

(b) $\{(1, 0, 1), (1, 1, 0), (1, 1, 1)\}$

(c) $\{(2, 0, 0), (0, 5, 0), (0, 0, 8)\}$

(d) All are bases for $\mathbb{R}^3$.

▲

Solution: All are bases for $\mathbb{R}^3$

Problem 4.64. With your partner create four different sets of vectors. Each vector must be in $\mathbb{R}^3$. Two of the sets must span $\mathbb{R}^3$ and two of the sets must not. ▲

Definition 4.65 (Dimension of a Vector Space)**.** The **dimension** of a vector space is the number of ____________.
(Fill in the blank)

Solution: basis vectors

Theorem 4.66. Let $\mathcal{B}$ be a basis for a vector space $\mathcal{V}$. If any set S contains more vectors than $\mathcal{B}$ then the vectors in S must be ____________.
(Fill in the blank and then prove it)

Solution: linearly dependent.

Proof. (prove the previous theorem) □

Solution: Let a basis for $\mathcal{V}$ contain n vectors. Then any vector in the vector space can be built as a linear combination of these n vectors. So, if there is an *extra* vector in the set S we can build that extra vector out of the other n vectors. That is to say, there exists scalars $c_1, \ldots, c_n$ such that $\mathbf{v}_{n+1} = \sum_{j=1}^n c_j \mathbf{v}_j$ and hence

$$\mathbf{0} = \left(\sum_{j=1}^n c_j \mathbf{v}_j \right) - \mathbf{v}_{n+1}$$

and since this isn't the trivial linear combination the vectors must be linearly dependent.

Problem 4.67. True or False: Any two bases for a vector space consist of the same number of vectors. (If this is true then I suppose we would call it a theorem and we should then prove it) ▲

Solution: True.

Proof. Assume otherwise. That is, let's assume that the vector space $\mathcal{V}$ has two bases $\mathcal{B}_1$ and $\mathcal{B}_2$ where $\mathcal{B}_1 = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ and $\mathcal{B}_2 = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ and $k \neq p$. Since both are bases for $\mathcal{V}$ we know that each set is a linearly independent spanning set of $\mathcal{V}$. Since $k \neq p$ we can assume, without loss of generality, that $k > p$ and hence by Theorem 4.66 we know that $\mathcal{B}_1$ must be a linearly dependent set. Hence we have arrived at a contradiction so our assumption that there are two bases with a different number of vectors must have been false. Therefore, any two bases for a vector space must consist of the same number of vectors.

□

Theorem 4.68 (Independence, Span, and Basis)**.** Let $\mathcal{V}$ be an n-dimensional vector space and let S be a subset of $\mathcal{V}$. Then

- If S is linearly independent and consists of n vectors, then ____________.
- If S spans $\mathcal{V}$ and consists of n vectors, then ____________.
- If S is linearly independent, then S (is contained in / is equal to / contains) a basis for $\mathcal{V}$ (choose one).
- If S spans $\mathcal{V}$, then S (contains / is / is contained in) a basis for $\mathcal{V}$ (choose one)

Solution:

- $\text{span}(S) = \mathcal{V}$ and S must be a basis for $\mathcal{V}$.
- S must be linearly independent
- is contained in
- contains

Problem 4.69. Which of teh following describes a basis for a subspace $\mathcal{V}$?

(a) A basis is a linearly independent spanning set for $\mathcal{V}$.

(b) A basis is a minimal spanning set for $\mathcal{V}$.

(c) A basis is a largest possible set of linearly independent vectors in $\mathcal{V}$.

(d) All of the above

(e) Some of the above

(f) None of the above

▲

Solution: All of the above

Problem 4.70. Consider the vector space of quadratic polynomials

$$\mathbb{P}_2 = \{a_0 + a_1 x + a_2 x^2 \,:\, a_0, a_1, a_2 \in \mathbb{R}\}.$$

A basis for $\mathbb{P}_2$ is $\mathcal{B} = \{1, x, x^2\}$ (...discuss why this is a basis ...).
Answer the following questions about subsets of $\mathbb{P}_2$:

(a) What is the dimension of $\mathbb{P}_2$?

(b) Is $\mathcal{W} = \text{span}(\{1, x\})$ a subspace of $\mathbb{P}_2$?

(c) Is the set $S = \{1+x, x^2+12, x-1, 2+x-x^2\}$ linearly independent or linearly dependent?

(d) Is the set $S = \{1+x, x^2, x-x^2\}$ a basis for $\mathbb{P}_2$? If so, prove it. If not then explain why not

▲

Solution:

- the dimension is 3
- This is a subspace since the two vectors are linearly independent and clearly span some space. The space is the collection of all linear polynomials.
- Linear dependent since there are 4 vectors and the basis only contains 3 vectors.
- yes. There are three of them and they are linearly indepenedent.

Example 4.71. (modified from [5]) Suppose that an astronaut has 4 boosters on his jet pack that point in the directions $\mathbf{v}_1 = (1,1,2)^T$, $\mathbf{v}_2 = (0,1,3)^T$, $\mathbf{v}_3 = (2,1,1)^T$, and $\mathbf{v}_4 = (-2,1,0)$. Show that the span of these vectors is all of $\mathbb{R}^3$ then select a basis for $\mathbb{R}^3$ from this set.
Solution: To show that the set $S\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4\}$ spans all of $\mathbb{R}^3$ we consider the matrix A below and row reduce to indentify the linearly indpendent vectors.

$$A = \begin{pmatrix} 1 & 0 & 2 & -2 \\ 1 & 1 & 1 & 1 \\ 2 & 3 & 1 & 0 \end{pmatrix} \to \cdots \to \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We see that there are three linearly independent vectors in the set S so we know that $\text{span}(S) = \mathbb{R}^3$. Furthermore, we see that the vectors $\mathbf{v}_1,\mathbf{v}_2$ and $\mathbf{v}_4$ are linearly indepdent and hence a basis for $\mathbb{R}^3$ is

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Example 4.72. If the fourth booster breaks in the previous example what kind of object is the span of the remaining three boosters?
Solution: This will only leave two linearly independent vectors so the astronaut will be able to travel along a plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$.

4.6 Row, Column, and Null Spaces in $\mathbb{R}^n$

Now let's restrict our attention to spaces associated with the rows and columns of matrices.

Definition 4.73 (Row Space)**.** Let A be an $m \times n$ matrix. The subspace of $\mathbb{R}^n$ spanned by the m rows of A is called the **row space** of A

Definition 4.74 (Row Rank of a Matrix)**.** The dimension of the row space is called the (row) **rank** of the matrix A

Definition 4.75 (Column Space)**.** Let A be an $m \times n$ matrix. The subspace of $\mathbb{R}^m$ spanned by the n column vectors is called the **column space** of A.

Definition 4.76 (Column Rank of a Matrix)**.** The dimension of the column space a matrix A is called the (column) **rank** of A.

Definition 4.77 (Null Space (Kernel))**.** Let A be an $m \times n$ matrix. The **null space** os a matrix A is the solution space of the homogeneous system $A\mathbf{x} = \mathbf{0}$

$$Null(A) = \{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}$$

The null space of a matrix A is often denoted $Null(A)$ of $\mathcal{N}(A)$. We will use both interchangeably throughout the remainder of these notes.

Definition 4.78 (Nullity of a Matrix)**.** The dimension of the null space of a matrix A is called the **nullity** of A.

Theorem 4.79. In the previous definitions we explicitly stated that each of the spaces are indeed subspaces. More specifically, if A is an $m \times n$ matrix then

(a) the span of the rows of A is a subspace of $\mathbb{R}^n$

(b) the span of the columns of A is a subspace of $\mathbb{R}^m$

(c) the set $\mathcal{N}(A) = \{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}$ is a subspace of $\mathbb{R}^n$.

Proof. (you should prove all three of these statements) □

Solution:

Proof. (a) The rows of the matrix A contain n entries so they are elements of $\mathbb{R}^n$. The span of a collection of vectors is always a subspace by Theorem 4.50 so the span of the rows of A is a subspace of $\mathbb{R}^n$.

(b) The columns of the matrix A contain m entries so they are elements of $\mathbb{R}^m$. The span of a collection of vectors is always a subspace by Theorem 4.50 so the span of the rows of A is a subspace of $\mathbb{R}^m$.

(c) First observe that any $\mathbf{x} \in \mathcal{N}(A)$ are vectors in $\mathbb{R}^n$ since the matrix multiplication $A\mathbf{x} = \mathbf{0}$ must make sense. Let $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{N}(A)$ and let $c_1, c_2 \in \mathbb{R}$. Observe that if we multiply the linear combination $c_1\mathbf{x}_1 + c_2\mathbf{x}_2$ by the matrix A we get

$$A(c_1\mathbf{x}_1 + c_2\mathbf{x}_2) = c_1 A\mathbf{x}_1 + c_2 A\mathbf{x}_2 = c_1\mathbf{0} + c_2\mathbf{0} = \mathbf{0}$$

so we conclude that the linear combination is in the null space of A. Hence, by Theorem 4.47 we have shown that the null space is indeed a subspace of $\mathbb{R}^n$.

□

Problem 4.80. Find a basis for the null, column, and row spaces for the matrix below. The row reduced form of the matrix is given for convenience.

$$A = \begin{pmatrix} 5 & -2 & 3 \\ -1 & 0 & -1 \\ 0 & -2 & -2 \\ -5 & 7 & 2 \end{pmatrix} \to \cdots \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

▲

Solution:

$$Nul(A) = span\left\{\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}\right\}$$

$$Col(A) = span\left\{\begin{pmatrix} 5 \\ -1 \\ 0 \\ -5 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ -2 \\ 7 \end{pmatrix}\right\}$$

$$Row(A) = span\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$$

Problem 4.81. True or False:

(a) If A is a 2×3 matrix then the zero vector $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is in the column space of A.

(b) If A is a 2×3 matrix then the zero vector $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is in the row space of A.

(c) If A is a 2×3 matrix then the zero vector $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is in the null space of A.

▲

Solution:

(a) True

(b) False

(c) False

Example 4.82. Find the row space, column space, and null space of the matrix

$$A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}.$$

Also find the rank and the nullity. **Solution:** First we observe that A row reduces to

$$A \to \cdots \to \begin{pmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Column Space: We observe that columns 1 and 3 are linearly independent so the column space of A is

$$\mathrm{Col}(A) = \mathrm{span}\left(\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}\right)$$

which is a subspace of $\mathbb{R}^3$.
(Notice that we are NOT claiming that the column space is the span of the first and third columns from the row reduced matrix. Why?)

Row Space: From the row reduced form we see that columns 1 and 2 are linearly independent. Since the row reduced form of A is “row equivalent” we see that the row space of A is

$$\mathrm{Row}(A) = \mathrm{span}\left((1,-2,0,-1,3),\,(0,0,1,2,-2)\right)$$

which is a subspace of $\mathbb{R}^5$.

Null Space: For the null space we are considering the equation $A\mathbf{x} = \mathbf{0}$. From the row reduced form of the matrix we see that the solution to the homogeneous system is

$$\begin{aligned} x_1 &= 2x_2 + x_4 - 3x_5 \\ x_3 &= -2x_4 + 2x_5 \end{aligned}$$

where $x_2, x_4, x_5 \in \mathbb{R}$. Therefore we can write the solution to the homogeneous system as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} x_4 + \begin{pmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} x_5$$

Thus the null space of A is

$$\mathcal{N}(A) = \operatorname{span}\left(\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right)$$

which is a subspace of $\mathbb{R}^5$.

Rank: The rank is the dimension of the column space. We see that the basis of the column space has 2 vectors so $\operatorname{rank}(A) = 2$.

Nullity: The nullity is the dimension of the null space. We see that the basis for the null space has three vectors so $\operatorname{nullity}(A) = 3$. Observe that

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = 5 = \text{number of columns in } A.$$

Example 4.83. Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Find $\mathcal{N}(A)$ and determine the nullity of A.

Solution: This matrix is already row reduced so we can read the solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$ as $x_2 = 0$, $x_3 = 0$ and $x_1 \in \mathbb{R}$. Therefore,

$$\mathcal{N}(A) = \operatorname{span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right)$$

Since the basis for the null space has only one vector we see that $\operatorname{nullity}(A) = 1$. One might also observe that $\operatorname{rank}(A) = 2$ and that $\operatorname{rank}(A) + \operatorname{nullity}(A) = 3$ which is the number of columns in A.

Example 4.84. Let A be a matrix that row reduces to $A \to \cdots \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Describe Col($A$) and determine the rank of A.
Solution: The column space is the span of columns 2 and 3 since these are the two linearly independent columns. Since the basis for the column space contains two vectors we see that rank(A) = 2.

Example 4.85. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & -1 & \pi \\ -30 & 1 & 17 \\ 19 & 3 & -e^2 \\ 0 & 0 & 2 \end{pmatrix}$$

The row, column, and null spaces are subspaces of which vector spaces?
Solution:

Row Space: The row space is a subspace of $\mathbb{R}^3$ since each row contains three entries.

Column Space: The column space is a subspace of $\mathbb{R}^5$ since each column contains 5 entries.

Null Space: The null space is a subspace of $\mathbb{R}^3$ since for $\mathbf{x} \in \text{Null}(A)$ the equation $A\mathbf{x} = \mathbf{0}$ must make sense. Since A is a 5×3 matrix $\mathbf{x}$ must be a 3×1 vector.

Problem 4.86. The *row space* of a matrix A is the set of vectors that can be created by taking all linear combinations of the rows of A. Which of the following vectors is in the row space of the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$?

(a) $\mathbf{x} = \begin{pmatrix} -2 & 4 \end{pmatrix}$

(b) $\mathbf{x} = \begin{pmatrix} 4 & 8 \end{pmatrix}$

(c) $\mathbf{x} = \begin{pmatrix} 0 & 0 \end{pmatrix}$

(d) $\mathbf{x} = \begin{pmatrix} 8 & 4 \end{pmatrix}$

(e) More than one of the above

(f) None of the above

▲

Solution: (e)

Problem 4.87. The *column space* of a matrix A is the set of vectors that can be created by taking all linear combinations of the columns of A. Is the vector $\mathbf{b} = \begin{pmatrix} -4 \\ 12 \end{pmatrix}$ in the column space of the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$?

(a) Yes, since we can find a vector $\mathbf{x}$ so that $A\mathbf{x} = \mathbf{b}$.

(b) Yes, since $-2\begin{pmatrix} 1 \\ 3 \end{pmatrix} - 1\begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} -4 \\ 12 \end{pmatrix}$.

(c) No, because there is no vector $\mathbf{x}$ so that $A\mathbf{x} = \mathbf{b}$.

(d) No, because we can't find c_1 and c_2 such that $c_1\begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} -4 \\ 12 \end{pmatrix}$.

(e) More than one of the above

(f) None of the above

▲

Solution: No (c and d)

Problem 4.88. True or False: The row space of a matrix A is the same as the column space of A^T. ▲

Solution: True

Problem 4.89. The row space of the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$ consists of

(a) All linear combinations of the columns of A^T.

(b) All multiples of the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

(c) All linear combinations of the rows of A.

(d) All of the above

(e) None of the above

▲

Solution: d

Problem 4.90. The column space of the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$ is

(a) the set of all linear combinations of the columns of A.

(b) a line in $\mathbb{R}^2$.

(c) the set of all multiples of the vector $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

(d) All of the above

(e) None of the above

▲

Solution: d

Problem 4.91. The *null space* of a matrix A is the set of all vectors x that are solutions of $Ax = 0$. Which of the following vectors is in the null space of the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$?

(a) $x = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

(b) $x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

(c) $x = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$

(d) All of the above

(e) None of the above

▲

Solution: d

Problem 4.92. Let $A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 1 \\ 2 & -1 & 1 & 1 \end{pmatrix}$. Which of the following vectors are in the nullspace of A?

(a) $\begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix}$

(b) $\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$

(c) $\begin{pmatrix} 2 \\ 3 \\ -1 \\ 0 \end{pmatrix}$

(d) $\begin{pmatrix} 3 \\ -1 \\ 3 \\ 2 \end{pmatrix}$

▲

Solution: c

Theorem 4.93 (Rank-Nullity Theorem)**.** Let A be an $m \times n$ matrix. The sum of the rank and the nullity of A is the number of columns of A.

$$\text{dimension}(\text{Null}(A)) + \text{dimension}(\text{Col}(A)) = n$$

All of this discussion now leads us to a fundamental theorem of matrices.

Theorem 4.94 (Invertible Matrix Theorem)**.** Let A be an $n \times n$ square matrix. Then the following statements are equivalent. That is, for a given matrix A,
the following statements are either **all true or all false**.

(a) A is an invertible matrix

(b) A can be row reduced to the $n \times n$ identity matrix

(c) A has n pivot positions

(d) The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution

(e) The columns of A form a linearly independent set

(f) The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b} \in \mathbb{R}^n$

(g) The columns of A span $\mathbb{R}^n$

(h) There is an $n \times n$ matrix C such that $CA = I$

(i) There is an $n \times n$ matrix D such that $AD = I$

(j) A^T is invertible

(k) The columns of A form a basis for $\mathbb{R}^n$

(l) $\text{Col}(A) = \mathbb{R}^n$

(m) $\dim(\text{Col}(A)) = n$

(n) $\text{rank}(A) = n$

(o) $\text{Null}(A) = \{\mathbf{0}\}$

(p) $\dim(\text{Null}(A)) = 0$

Problem 4.95. Consider the following true or false questions.

(a) The number of free variables in the solution to $A\mathbf{x} = \mathbf{0}$ is equal to the dimension of the null space of A. **Solution:** True

(b) If A is a 3×4 matrix then the row vectors belong to $\mathbb{R}^3$. **Solution:** False, they belong to $\mathbb{R}^4$

(c) If A is an $n \times n$ matrix and $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then the dimension of the null space of A is 0. **Solution:** True

(d) If S is a set of three vectors, each of this is in $\mathbb{R}^2$, then S spans $\mathbb{R}^2$. **Solution:** False, they could form a 1-dimensional subspaces of $\mathbb{R}^2$

(e) If A is an $n \times n$ matrix and the dimension of the null space is 0 then A is invertible. **Solution:** True

▲

Problem 4.96. If A is an $n \times n$ matrix and $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b} \in \mathbb{R}^n$ then which of the following are true?

- A is invertible
- The columns of A are linearly independent
- $A\mathbf{x} = \mathbf{0}$ has only the trial solution
- The columns of A span $\mathbb{R}^n$

▲

Solution: All of them are true.

Problem 4.97. Consider the matrix $A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 2 \\ 0 & 2 & 4 \end{pmatrix}$. It can be shown that $\det(A) = 0$. Based on this fact find all of the following statements that must be true about A.

(a) The matrix A is invertible.

(b) The columns of A are linearly dependent.

(c) The columns of A form a basis for $\mathbb{R}^3$.

(d) The rank of A is less than 3

(e) $A\mathbf{x} = \mathbf{0}$ ha a non-trivial solution.

▲

Solution: (b), (d), and (e) are true

Chapter 5

The Geometry of Vector Spaces

In this chapter we do stuff.

5.1 The Geometry of $\mathbb{R}^n$

At this point we have talked almost exclusively about linear combinations and the spaces associated with them. We have not, however, discussed the geometry of vector spaces. So far we haven't generalized the notions of angle and length of vectors to our large view of vector spaces. You may recall things like projections, angles, norms (lengths) from $\mathbb{R}^2$ or $\mathbb{R}^3$ as discussed in physics or in multivariable calculus but you need to keep in mind that this is only a limited view of the world of vector spaces. Let's jump right in by filling in some definitions and theorems that you likely already know

5.1.1 The Dot Product

Now let's summarize our results in the following definitions and theorems.

Definition 5.1 (Dot Product in $\mathbb{R}^n$)**.** Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. The **dot product** of $\mathbf{u}$ and $\mathbf{v}$ is defined algebraically as

$$\mathbf{u} \cdot \mathbf{v} = \underline{\hspace{6cm}}$$

(this definition doesn't explicitly mention the angle between the vectors)

Solution: $\mathbf{u} \cdot \mathbf{v} = \sum_{j=1}^{n} u_j v_j$

Definition 5.2 (Dot Product and Vector Angles)**.** If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ then the relationship between the dot product of the vectors and the angle between the vectors is

$$\mathbf{u} \cdot \mathbf{v} = \underline{\hspace{6cm}}$$

Solution: $\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||||\mathbf{v}|| \cos\theta$

Theorem 5.3 (Orthogonal Vectors in $\mathbb{R}^n$)**.** If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ then $\mathbf{u}$ is orthogonal (perpendicular) to $\mathbf{v}$ if and only if ____________.

Solution: $\mathbf{u} \cdot \mathbf{v} = 0$

Proof. (prove this theorem) □

Solution: $\mathbf{u} \cdot \mathbf{v} = ||\mathbf{v}||||\mathbf{u}|| \cos\theta$ and since $\theta = \pi/2$ we know that $\cos\theta = 0$. Hence $\mathbf{u} \cdot \mathbf{v} = 0$.

Definition 5.4 (Length of Vectors in $\mathbb{R}^n$)**.** Let $\mathbf{u} \in \mathbb{R}^n$. The **length (norm)** of $\mathbf{u}$ is

$$||\mathbf{u}|| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Solution: $||\mathbf{u}|| = \sum_{j=1}^{n} u_j^2$

Definition 5.5 (Distance Between Vectors in $\mathbb{R}^n$)**.** Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. The **distance between** $\mathbf{u}$ and $\mathbf{v}$ is

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \_\_\_\_\_\_\_\_\_\_\_\_$$

Solution: $\text{dist}(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||$

5.1.2 Projections

Finally we are going to discuss projections. When dealing with projections you should be thinking about how shadows are cast between vectors. To solidify this notion (even though you likely already know it) let's look at some projections in $\mathbb{R}^2$ before we ramp up the dimension. Take a look at Figure 5.1. We would like to project vector $\mathbf{u}$ onto vector $\mathbf{v}$ and by that we mean that we would like to draw a vector (depicted by the dashed vector $\mathbf{w}$ in the figure) that is perpendicular to $\mathbf{v}$ and meets the head of $\mathbf{u}$. This projection creates the vector $\hat{\mathbf{v}}$ so that $\hat{\mathbf{v}}$ points in exactly the same direction as $\mathbf{v}$ but $\hat{\mathbf{v}} \perp \mathbf{w}$. Since $\mathbf{v}$ and $\hat{\mathbf{v}}$ point in the same direction we know that $\hat{\mathbf{v}} = c\mathbf{v}$ for some scalar c. Furthermore, we know that $\mathbf{w} + \hat{\mathbf{v}} = \mathbf{u}$ so $\mathbf{w} = \mathbf{u} - \hat{\mathbf{v}}$. Therefore,

$$\begin{aligned} 0 &= \hat{\mathbf{v}} \cdot \mathbf{w} = c\mathbf{v} \cdot (\mathbf{u} - c\mathbf{v}) \\ &\implies c\mathbf{v} \cdot \mathbf{u} - c^2 \mathbf{v} \cdot \mathbf{v} = 0 \\ &\implies \mathbf{u} \cdot \mathbf{v} = c\mathbf{v} \cdot \mathbf{v} \\ &\implies c = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \end{aligned}$$

All of the prior discuss proves the following theorem but notice that we never made any mention explicitly about the vectors living in $\mathbb{R}^2$. In fact, the proof that we gave works generally in $\mathbb{R}^n$.

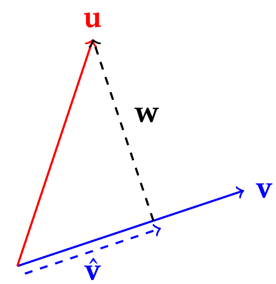


Figure 5.1. Depiction of vector projection in $\mathbb{R}^2$.

Theorem 5.6 (Orthogonal Projection)**.** Let $\mathbf{u},\mathbf{v} \in \mathbb{R}^n$. If we are to project $\mathbf{u}$ onto $\mathbf{v}$ as in Figure 5.1 we get

$$\text{proj}_{\mathbf{v}}(\mathbf{u}) = \hat{\mathbf{v}} = \left(\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \text{projection of } \mathbf{u} \text{ onto } \mathbf{v}$$
$$\mathbf{w} = \mathbf{u} - \hat{\mathbf{v}} = \mathbf{u} - \left(\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \text{projection error}$$

The vector $\mathbf{w}$ is often called the *error* in the projection.

Problem 5.7. What is the dot product of $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix}$?

(a) $\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$

(b) 5

(c) 0

(d) The dot product cannot be computed for these vectors.

▲

Solution: $(0)(4) + (1)(2) + (-1)(-3) = 2 + 3 = 5$

Problem 5.8. If $\mathbf{b} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and $y = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, then the orthogonal projection of $\mathbf{b}$ onto $\mathbf{y}$ is

(a) $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(b) $\begin{pmatrix} 3/2 \\ -1/2 \end{pmatrix}$

(c) $\begin{pmatrix} 10 \\ 5 \end{pmatrix}$

(d) $\begin{pmatrix} 1/10 \\ 3/10 \end{pmatrix}$

▲

Solution: $\text{proj}_y b = \left(\frac{b \cdot y}{y \dot{y}}\right) y = \left(\frac{5}{5}\right)\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Problem 5.9. If $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal basis for a vector space $\mathcal{V}$ then how do you write $\mathbf{x}$ as a linear combination of the basis vectors?
(Why is it advantageous to have an orthogonal basis?) Hint: Since $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$ how can you use orthogonality to solve for c_j? ▲

Solution: $c_j = \frac{\mathbf{x} \cdot \mathbf{v}_j}{\mathbf{v}_j \cdot \mathbf{v}_j}$

Problem 5.10. Implement your idea on the subspace spanned by the basis

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

where $\mathbf{x}$ is in the subspace of $\mathbb{R}^3$ spanned by $\mathcal{B}$

$$\mathbf{x} = \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}$$

▲

Solution: Since $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ so

$$c_1 = \frac{\mathbf{x} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} = \frac{-17}{26}$$

$$c_2 = \frac{\mathbf{x} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} = \frac{7}{10}$$

Now summarize the process that you built in the previous problem into the following theorem.

Theorem 5.11 (Building Vectors from an Orthogonal Basis)**.** If $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal basis for a vector space $\mathcal{V}$ then for $\mathbf{x} \in \mathcal{V}$

$$\mathbf{x} = \underline{\quad}\mathbf{v}_1 + \underline{\quad}\mathbf{v}_2 + \underline{\quad}\mathbf{v}_3 + \cdots + \underline{\quad}\mathbf{v}_n$$

(Fill in the blanks)

Solution: $c_j = \frac{\mathbf{x}\cdot\mathbf{v}_j}{\mathbf{v}_j\cdot\mathbf{v}_j}$

Theorem 5.12. If the nonzero vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_2, \ldots, \mathbf{u}_k$ are mutually orthogonal then they are linearly independent.

Proof. (prove this theorem) □

Solution: Consider $\mathbf{0} = \sum_{j=1}^{n} c_j \mathbf{u}_j$. From the previous theorem we know that $c_j = (\mathbf{0}\cdot\mathbf{u}_j)/(\mathbf{u}_j\cdot\mathbf{u}_j) = 0$. Therefore the only solution is the trivial solution and the vectors must be linearly independent.

Problem 5.13. Determine if the following set of vectors is linearly independent. Do this two different ways.

$$\left\{ \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -4 \\ 7 \end{pmatrix} \right\}$$

▲

Solution: They are mutually orthogonal so they are linearly independent.

Problem 5.14. If we have two linearly independent vectors that are NOT orthogonal, how do we find a set of two orthogonal vectors that span the same space?

For example, can we find two orthogonal vectors that span the same space as

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

▲

5.1.3 The Gram-Schmidt Process: Making Orthogonal Sets

The previous theorems and problems give us good reason to think that having an orthogonal (or orthonormal) basis for a vector space is advantageous both computationally and geometrically. In fact, we have been used to an orthonormal basis all of our matheamtical lives since that is what the regular Cartesian coordinate system is built from. The question now is this:

Given a basis $\mathcal{B}$ for a vector space $\mathcal{V}$ how can we transform that basis into a different basis for the same space but also gain orthogonality? We will build your intuition to the process via a scaffolded problem.

Problem 5.15. Build a basis for $\mathbb{R}^2$ so that it contains two orthogonal unit vectors with one of the vectors parallel to $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. ▲

Problem 5.16. Consider the vector space $\mathbb{R}^3$ with the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ given by

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

We are going to build a basis $\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ such that $\text{span}(\mathcal{U}) = \mathbb{R}^3$ but the vectors are also mutually orthogonal and all have unit length. (One should note here that the normalization step is optional but since unit vectors are so nice to work with we are leaving it here.)

(a) Define $\mathbf{u}_1$ as a unit vector that points in the same direction as $\mathbf{v}_1$.

$$\mathbf{u}_1 = \frac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1 = \begin{pmatrix} \underline{\quad} \\ \underline{\quad} \\ \underline{\quad} \end{pmatrix}$$

Solution: $\mathbf{u}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$

(b) Now we project $\mathbf{v}_2$ onto $\mathbf{u}_1$ and find the error in the projection. This would be the vector $\mathbf{w}$ in Figure 5.1. Once we have the error we should normalize it to get $\mathbf{u}_2$.

$$\mathbf{w}_2 = \mathbf{v}_2 - \text{proj}_{\mathbf{u}}(\mathbf{v}_2) = \mathbf{v}_2 - (\mathbf{v}_2 \cdot \mathbf{u}_1)\mathbf{u}_1 \quad \text{and therefore} \quad \mathbf{u}_2 = \frac{1}{\|\mathbf{w}_2\|}\mathbf{w}_2.$$

Draw a picture of what we just did.

$$\mathbf{u}_2 = \frac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1 = \begin{pmatrix} \underline{\quad} \\ \underline{\quad} \\ \underline{\quad} \end{pmatrix}$$

Solution: $\mathbf{w}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \left(\frac{2}{\sqrt{3}}\right)\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2/3 \\ 2/3 \\ 2/3 \end{pmatrix} = \begin{pmatrix} -2/3 \\ 1/3 \\ 1/3 \end{pmatrix}$. Therefore $\mathbf{u}_2 = \frac{1}{\|\mathbf{w}_2\|}\mathbf{w}_2$ so

$$\mathbf{u}_2 = \frac{\sqrt{3}}{\sqrt{2}}\begin{pmatrix} -2/3 \\ 1/3 \\ 1/3 \end{pmatrix} = \frac{1}{\sqrt{6}}\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}$$

(c) For $\mathbf{u}_3$ we project $\mathbf{v}_3$ onto both $\mathbf{u}_1$ and $\mathbf{u}_2$ and then normalize.

$$\mathbf{w}_3 = \mathbf{v}_3 - (\mathbf{v}_3 \cdot \mathbf{u}_1)\mathbf{u}_1 - (\mathbf{v}_3 \cdot \mathbf{u}_2)\mathbf{u}_2 \quad \text{and therefore} \quad \mathbf{u}_3 = \frac{1}{\|\mathbf{w}_3\|}\mathbf{w}_3$$

$$\mathbf{u}_3 = \frac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1 = \begin{pmatrix} \_\_\_ \\ \_\_\_ \\ \_\_\_ \end{pmatrix}$$

Solution:

$$\mathbf{w}_3 = \mathbf{v}_3 - \left(\frac{1}{\sqrt{3}}\right)\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} - \left(\frac{1}{\sqrt{6}}\right)\begin{pmatrix} -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix} - \begin{pmatrix} -2/6 \\ 1/6 \\ 1/6 \end{pmatrix} = \begin{pmatrix} 0 \\ -1/2 \\ 1/2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

Therefore, we can build $\mathbf{u}_3$ by normalizing

$$\mathbf{u}_3 = \frac{\sqrt{2}}{2}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

(d) Verify that indeed $\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis for $\mathbb{R}^3$. **Solution:** The orthonormal basis is:

$$\mathcal{U} = \left\{ \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}, \begin{pmatrix} -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}, \begin{pmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \right\}$$

and we leave it to the reader to verify that indeed the vectors are mutually orthogonal.

▲

Problem 5.17. Use the Gram-Schmidt process outlined in the previous problem to produce an orthogonal basis $\mathcal{U}$ for the subspace spanned by

$$\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 8 \\ 5 \\ -6 \end{pmatrix}.$$

▲

Problem 5.18. Let's build an orthogonal basis $\mathcal{U}$ for $\mathbb{R}^3$. To get started let $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. (notice that we are not normalizing this time)

(a) Create a vector $\mathbf{u}_2$ in $\mathbb{R}^3$ so that $\mathbf{u}_1 \perp \mathbf{u}_2$. **Solution:** A simple choice is $\mathbf{u}_2 = (0,1,0)^T$.

(b) Pick a vector $\mathbf{v}_3 \in \mathbb{R}^3$ such that $\mathbf{v}_3$ is linearly independent of $\mathbf{u}_1$ and $\mathbf{u}_2$. Then use one step of the Gram-Schmidt process to create $\mathbf{u}_3$ out of $\mathbf{v}_3$. **Solution:** One choice is $\mathbf{v}_3 = (2,0,1)^T$. Therefore

$$\mathbf{u}_3 = \mathbf{v}_3 - \left(\frac{\mathbf{v}_3 \cdot \mathbf{u}_1}{\|\mathbf{u}_1\|^2}\right)\mathbf{u}_1 - \left(\frac{\mathbf{v}_3 \cdot \mathbf{u}_2}{\|\mathbf{u}_2\|^2}\right)\mathbf{u}_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} - \frac{3}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - 0\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \end{pmatrix}$$

(c) Verify that the vectors in your proposed basis are indeed mutually orthogonal. If so we can use one of the previous theorems (which one) to say that the vectors are linearly independent and must therefore span $\mathbb{R}^3$. **Solution:** It is trivial to verify that the three vectors are indeed mutually orthogonal.

▲

5.2 Inner Product Spaces

Now time for some more abstraction! In this section we take the notions of geometry and abstract them to generalized vector spaces. You may have noticed that the dot product is the basic computation necessary to understand angle in $\mathbb{R}^n$ so we first have to provide a generalized version of the dot product.

Definition 5.19 (The Inner Product)**.** An **inner product** is the abstraction of a dot product to a general vector space. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in a vector space $\mathcal{V}$ and c is some real number then

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$
2. $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$
3. $\langle c\mathbf{u}, \mathbf{v} \rangle = c \langle \mathbf{u}, \mathbf{v} \rangle$
4. $\langle \mathbf{u}, \mathbf{u} \rangle \ge 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$

Problem 5.20. Verify that the dot product is indeed an inner product on the vector space $\mathbb{R}^n$. ▲

Solution:

1. The dot product is definitely symmetric: $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$. Proof:
$$\mathbf{u} \cdot \mathbf{v} = \sum_{j=1}^n u_j v_j = \sum_{j=1}^n v_j u_j = \mathbf{v} \cdot \mathbf{u}$$
2. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} + \mathbf{w})$ since
$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \sum_{j=1}^n u_j (v_j + w_j) = \sum_{j=1}^n u_j v_j + u_j w_j = \sum_{j=1}^n u_j v_j + \sum_{j=1}^n u_j w_j = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$$
3. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$ since
$$(c\mathbf{u}) \cdot \mathbf{v} = \sum_{j=1}^n (c u_j) v_j = c \sum_{j=1}^n u_j v_j = c(\mathbf{u} \cdot \mathbf{v})$$

4. If $\mathbf{u} = \mathbf{0}$ then $\mathbf{u} \cdot \mathbf{u} = \sum_{j=1}^{n} u_j = \sum_{j=1}^{n} 0 = 0$. Furthermore, $\mathbf{u} \cdot \mathbf{u} = \sum_{j=1}^{n} u_j^2$ which is the sum on non-negative real numbers which must clearly also be non-negative.

Problem 5.21. Consider the vector space of quadratic polynomials on the interval $x \in [0,1]$.

$$\mathbb{P}_2 = \{a_0 + a_1 x + a_2 x^2 : a_0, a_1, a_2 \in \mathbb{R} \text{ and } x \in [0,1]\}$$

An inner product on this vector space is

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx.$$

(a) Verify that this is indeed a proper inner product on $\mathbb{P}_2$

(b) Find the inner product of $f(x) = x^2 + 1$ and $g(x) = 2x + x^2$ in $\mathbb{P}_2$ under this inner product.

(c) Set up the necessary integrals to find the lengths of f and g in $\mathbb{P}_2$ under this inner product.

(d) Set up the necessary integrals to find the angle between f and g in $\mathbb{P}_2$ under this inner product.

(e) Is this the only inner product on $\mathbb{P}_2$?

▲

Solution:

(b) $\langle f, g\rangle = \int_0^1 (x^2+1)(2x+x^2)dx = \int_0^1 2x^3 + x^4 + 2x + x^2 dx = \frac{1}{2} + \frac{1}{5} + 1 + \frac{1}{3} = \frac{61}{30}$

(c) $||f|| = \langle f, f\rangle^{1/2} = \sqrt{\int_0^1 (x^2+1)^2 dx}$, and $||g|| = \langle g, g\rangle^{1/2} = \sqrt{\int_0^1 (2x+x^2)^2 dx}$.

(d) $\theta = \frac{\langle f,g\rangle}{||f||||g||}$

(e) since these are polynomials we could take any finite bounds of integration and we get a valid inner product.

Problem 5.22. Consider the vector space of 2×2 real matrices

$$\mathcal{V} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

along with the inner product

$$\langle A, B\rangle = tr(AB^T)$$

Note: If M is a matrix, the *trace* of the matrix, $tr(M)$, is the sum of the entries on the main diagonal.

Find an orthogonal basis for $\mathcal{V}$. That's right ... I'm asking you to find angles between matrices!! AWESOME!! ▲

Solution:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Problem 5.23 (Legendre Polynomials)**.** Consider the vector space

$$\mathcal{V} = \{a_0 + a_1 x + a_2 x^2 : a_0, a_1, a_2 \in \mathbb{R} \text{ and } x \in [-1, 1]\}$$

along with the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) dx$$

Consider the basis $\mathcal{B} = \{1, x, \frac{1}{2}(3x^2 - 1)\}$.

(a) Is this basis an orthogonal basis?

(b) Set up the necessary calculus to write $h(x) = 3x^2 + 2$ as a linear combination of vectors in $\mathcal{B}$.

▲

Solution: This is an orthogonal basis. We need to consider that

$$c_1(1) + c_2(x) + c_3(\frac{1}{2}(3x^2 - 1)) = h(x)$$

Since this is an orthogonal basis we can get each c_j by doing inner products:

$$\begin{aligned} c_1 &= \frac{\langle h(x), 1 \rangle}{\langle 1, 1 \rangle} = 3 \\ c_2 &= \frac{\langle h(x), x \rangle}{\langle x, x \rangle} = 0 \\ c_3 &= \frac{\left\langle h(x), \frac{1}{2}(3x^2 - 1) \right\rangle}{\left\langle \frac{1}{2}(3x^2 - 1), \frac{1}{2}(3x^2 - 1) \right\rangle} = 2 \end{aligned}$$

OK. I admit it. Inner product spaces are pretty darn abstract and at first glance they seem to have no purpose. To finish this section we will consider a non-abstract application of the inner product space that has changed the modern world in uncountably many ways. This application, which will arise later in these notes (in the PDE's chapter), is one of the most stunningly beautiful applications out there for everyone to see: The Fourier Series.

Consider the vector space spanned by an infinite basis of sine functions

$$\mathcal{B} = \{\sin(kx) : k \in \mathbb{N}\}$$

equipped with the inner product

$$\langle f, g\rangle = \frac{1}{\pi}\int_0^{2\pi} f(x)g(x)dx.$$

This particular basis is infinite dimensional since the natural numbers, $\mathbb{N}$, are (countably) infinite, and if we consider the span we can build any periodic function as a linear combination of sine functions of different frequencies. More specifically, since every periodic function can be written as a linear combination of the basis functions we have the infinite sum

$$f(x) = \sum_{k=1}^{\infty} C_k \sin(kx) \tag{5.1}$$

for all period functions f. The most important part of the basis $\mathcal{B}$ is that it is an othonormal basis under the inner product: an orthogonal basis made entirely of unit vectors.

The Fourier Series plays an incredibly important role in the theory and practice of signal analysis. The applications that we'll look at in the next few problems explores this idea.

Problem 5.24. Open MATLAB (or any other symbolic calculus package) and verify that the basis

$$\mathcal{B} = \{\sin(kx) : k \in \mathbb{N}\}$$

is indeed an orthogonal basis. That is, compute

$$\langle \sin(kx), \sin(jx)\rangle = \frac{1}{\pi}\int_0^{\pi} \sin(kx)\sin(jx)\,dx$$

for various values of j and k and verify that

- if $j = k$ then the inner produce is identically 1.
- if $j \neq k$ then the inner product is zero.

▲

Problem 5.25. If $f(x)$ is some periodic function then we can write it as a linear combination of the basis vectors in $\mathcal{B}$:

$$f(x) = \sum_{k=1}^{\infty} C_k \sin(kx).$$

Knowing that the sine functions in the sum form an orthonormal basis for the space of periodic functions propose a way to find each C_k. Hint: Consider Theorem 5.11. ▲

Problem 5.26. Consider the function

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < \pi \\ -1 & \text{if } \pi < x < 2\pi \end{cases}.$$

We want to build a Fourier Series for this function (called the *square wave*). Open MATLAB and complete the following partial code to plot an approximation to the Fourier series of the square wave.

```
clear; clc;
syms x
f(x) = 0*x; % this gives a placeholder for the function
N = 20; % the top end of the finite Fourier series
for k=1:N
  C(k) = ... some code to find C(k) ...
  f(x) = f(x) + C(k) * sin(k*x);
end
ezplot(f(x) , [0,2*pi])
```

Once you have the plot working, append the code

```
x = 0:0.01:150;
MySound = double(f(x));
soundsc(MySound)
```

...and turn the volume up. ▲

Problem 5.27. Now find the Fourier series the function

$$f(x) = -\frac{1}{\pi}x + 1$$

for $x \in [0, 2\pi]$ and extended periodically outside the domain. ▲

5.3 Practice Problems for Vector Spaces

Problem 5.28. Let $\mathcal{V}$ be the set of all ordered triples (x, y, z) such that $x + y + z = 3$. Show that $\mathcal{V}$ is not a subspace of $\mathbb{R}^3$. ▲

Solution: $\mathbf{0} \notin \mathcal{V}$

Problem 5.29. Show that every subspace $\mathcal{W}$ of a vector space $\mathcal{V}$ contains the zero vector. ▲

Solution: Subspaces are closed under linear combinations so taking the zero combination puts you back in the subspace. Hence $\mathbf{0} \in \mathcal{W}$ for all $\mathcal{W}$.

Problem 5.30. Assume that the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. Show that the set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly independent given that

$$\mathbf{u}_1 = \mathbf{v}_2 + \mathbf{v}_3, \quad \mathbf{u}_2 = \mathbf{v}_1 + \mathbf{v}_3, \quad \mathbf{u}_3 = \mathbf{v}_1 + \mathbf{v}_2.$$

▲

Solution: Consider the linear combination

$$\mathbf{0} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = c_1(\mathbf{v}_2 + \mathbf{v}_3) + c_2(\mathbf{v}_1 + \mathbf{v}_3) + c_3(\mathbf{v}_1 + \mathbf{v}_2) = (c_2 + c_3)\mathbf{v}_1 + (c_1 + c_3)\mathbf{v}_2 + (c_1 + c_2)\mathbf{v}_3$$

and hence $c_1 = c_2 = c_3 = 0$.

Problem 5.31. Suppose that S is a set of n vectors that span the n-dimensional vector space $\mathcal{V}$. Prove that S is a basis for $\mathcal{V}$. ▲

Solution: If $\mathcal{V}$ is n-dimensional then any set of n vectors that spans $\mathcal{V}$ must also be linearly independent. Hence, S is a basis for $\mathcal{V}$.

Problem 5.32. Explain why the $n \times n$ matrix A is invertible if and only if its rank is n. ▲

Solution: If $\text{rank}(A) = n$ then A has n linearly independent columns and hence A is invertible. If A is invertible then A must have n linearly independent columns meaning that the rank must be n.

Problem 5.33. Let $\mathcal{F}$ be the space of all real-valued functions on $\mathbb{R}$. Determine if the set of all functions f such that $f(-x) = -f(x)$ for all x is a subspace of $\mathcal{F}$. ▲

Solution: Let f, g be odd functions and let $c_1, c_2 \in \mathbb{R}$. Define $h(x) = c_1 f(x) + c_2 g(x)$ and consider that $h(-x) = c_1 f(-x) = c_2 g(-x) = -c_1 f(x) - c_2 g(x) = -h(x)$ so the set of all odd functions is a subspace of $\mathcal{F}$.

Problem 5.34. Let $M_{3\times 3}$ be the set of all 3×3 matrices. Determine if the following subsets of $M_{3\times 3}$ are subspaces

(a) The set of all diagonal 3×3 matrices. **Solution:** Yes

(b) The set of all symmetric 3×3 matrices. **Solution:** Yes

(c) The set of all singular (non-invertible) 3×3 matrices **Solution:** No. For Example if we define A and B as

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

then A and B are singular but $A + B$ is not.

▲

5.4 Linear Transformations

Definition 5.35 (Linear Transformation)**.** A **linear transformation** T from a vector space $\mathcal{V}$ into a vector space $\mathcal{W}$ is a rule that assignes to each vector $\mathbf{v} \in \mathcal{V}$ a unique vector $\mathbf{w} \in \mathcal{W}$, such that

(a) $T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2)$ for all $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$

(b) $T(c\mathbf{v}) = cT(\mathbf{v})$ for all $\mathbf{v} \in \mathcal{V}$ and all scalars c.

More simply, a linear transformation has the property that

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1 T(\mathbf{v}_1) + c_2 T(\mathbf{v}_2)$$

for all $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$ and for all scalars c_1 and c_2.

Problem 5.36. Verify that if A is an $n \times n$ matrix then the function T defined as $T(\mathbf{x}) = A\mathbf{x}$ is indeed a linear transformation. ▲

Solution: Yep. Matrix mulitplication is a linear operation.

Problem 5.37. In calculus we know of two very important linear transformations. Let $\mathcal{V}$ be the vector space of all real-valued functions f on the interval $[a,b]$ that are differentiable and continuous on $[a,b]$. Let $\mathcal{W}$ be the vector space $C[a,b]$ of all continuous functions on $[a,b]$.

- The transformation $D : \mathcal{V} \to \mathcal{W}$ is defined as $D(f) = f'$. That is, D is the transformation that takes a derivative of a function.
- The transformation $\mathcal{I} : \mathcal{W} \to \mathcal{V}$ is defined as $\mathcal{I}(f) = \int_a^x f(\tau)d\tau$. That is, $\mathcal{I}$ is the transformation that gives the antiderivative of a function.

Verify that both of these well-known transformations are indeed linear transformations. ▲

Solution: Pulling scalars and sums around in integrals and derivatives is appropriate. This is known from calc 1 but the reason is that these are linear transformations.

Definition 5.38 (Kernel of a Linear Transformation)**.** Let T be a linear transformation from the vector space $\mathcal{V}$ to the vector space $\mathcal{W}$. The **kernel** of T is defined as

$$\text{Ker}(T) = \{\mathbf{x} \in \mathcal{V} : T(\mathbf{x}) = \mathbf{0} \in \mathcal{W}\}.$$

Observe that the kernel is another name for the null space.

Problem 5.39. Let D be the linear transformation defined as $D(f) = f'$ as in problem 5.37. What is the kernel of D? ▲

Solution: The set of all constant functions.

Problem 5.40. Define $T(y)$ on $\mathcal{V} = \{y(t) : y'(t) \text{ and } y''(t) \text{ exist }\}$ and define $T(y) = \frac{d^2y}{dt^2}$. What is the kernel of T? ▲

Solution: The set of all linear functions.

Problem 5.41. Define $T : \mathcal{P}_2 \to \mathbb{R}^2$ by

$$T(p) = \begin{pmatrix} p(0) \\ p(1) \end{pmatrix}.$$

For instance, if $p(t) = 3 + 5t + 7t^2$ then $T(p) = \begin{pmatrix} 3 \\ 15 \end{pmatrix}$.

(a) Verify that T is indeed a linear transformation.

(b) Find a polynomial $p(x) \in \mathcal{P}_2$ that is in the kernel of T.

▲

Solution: (adapted from 4.2 problem 31 of [4]).

(a) $T(c_1p + c_2q) = \begin{pmatrix} c_1p(0) + c_2q(0) \\ c_1p(1) + c_2q(1) \end{pmatrix} = c_1 \begin{pmatrix} p(0) \\ q(0) \end{pmatrix} + c_2 \begin{pmatrix} p(1) \\ q(1) \end{pmatrix} = c_1T(p) + c_2T(q)$

(b) $p(x) = cx - cx^2$ would work just fine.

Problem 5.42. Let $M_{2\times 2}$ be the vector space of all 2×2 matrices and define $T(A) = A + A^T$ for $A \in M_{2\times 2}$. Let B be any matrix in $M_{2\times 2}$ such that $B^T = B$. Find a matrix A such that $T(A) = B$. Then describe the kernel of T. ▲

Solution: (adapted from 4.2 problem 33 of [4])
$A = (1/2)B$
The kernel of T consists of matrices of the form $\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$ where $a \in \mathbb{R}$.

Example 5.43. Determine if the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4x_1 - 2x_2 \\ 3|x_2| \end{pmatrix}$$

is or is not a linear transformation.
Solution: By the definition of a linear transformation we need to see check that $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}+\mathbf{v})$ and that $T(c\mathbf{u}) = cT(\mathbf{u})$ for arbitrary vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$.

Let $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and observe that

$$T(\mathbf{u}+\mathbf{v}) = T\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = T\left(\begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix}\right) = \begin{pmatrix} 4(u_1 + v_1) - 2(u_2 + v_1) \\ 3|u_2 + v_1| \end{pmatrix}.$$

We can clearly separate the first component but due to the absolute value in the second component we cannot separate the result of the previous equation to form $T(\mathbf{u}) + T(\mathbf{v})$. Therefore, T is not a linear transformation.

Theorem 5.44. If T is a linear transformation then $T(\mathbf{0}) = \mathbf{0}$.

Proof. If T is a linear transformation then $T(c\mathbf{u}) = cT(\mathbf{u})$ for any vector $\mathbf{u}$ and any scalar $c \in \mathbb{R}$. If we take $c = 0$ then $T(0\mathbf{u}) = cT(\mathbf{u})$ which implies that $T(\mathbf{0}) = \mathbf{0}$. □

Example 5.45. Determine if the transformation $T(x_1,x_2,x_3) = (1,x_2,x_3)$ is a linear transformation.
Solution: Observe that $T(0,0,0) = (1,0,0)$ so by the previous theorem we see that T is not a linear transformation.

Example 5.46. Determine if the transformation $T(x_1,x_2,x_3) = (x_1,0,x_3)$ is a linear transformation.
Solution: Since $T(\mathbf{0}) = \mathbf{0}$ it is possible that T is a linear transformation but we cannot use this to prove that T is linear. We need to check that $T(c_1\mathbf{u}+c_2\mathbf{v}) = c_1T(\mathbf{u})+c_2T(\mathbf{v})$. Indeed, let $\mathbf{u} = (u_1,u_2,u_3)$ and let $\mathbf{v} = (v_1,v_2,v_3)$ and let $c_1,c_2 \in \mathbb{R}$. Therefore,

$$T(c_1\mathbf{u}+c_2\mathbf{v}) = T((c_1u_1,c_1u_2,c_1u_3)+(c_2v_1,c_2v_2,c_2v_3)) = T((c_1u_1+c_2v_1,c_1u_2+c_2v_2,c_1u_3+c_2v_3))$$

Applying the transformation gives

$$T(c_1\mathbf{u}+c_2\mathbf{v}) = (c_1u_1+c_2v_1,0,c_1u_3+c_2v_3) = \cdots = c_1T(\mathbf{u})+c_2T(\mathbf{v})$$

which means that T is indeed a linear transformation.

In this class we have studied two particular types of questions: solving first order non-homogeneous differential equations and solving systems of equations. Let's consider the processes for these two problems side by side so that we can truly see them as the exact same problem in the language of linear transformations.

Non-homogeneous 1^{st} order ODE	**Non-homogeneous linear system**
1. Solve $y'+Py = Q(t)$	1. Solve $A\mathbf{x} = \mathbf{b}$
2. Let $T(y) = y'+Py$. We want to find y so that $T(y) = Q$.	2. Let $T(\mathbf{x}) = A\mathbf{x}$. We want to find $\mathbf{x}$ so that $T(\mathbf{x}) = \mathbf{b}$.
3. Find $y_h \in \text{Ker}(T)$	3. Find $\mathbf{x}_h \in \text{Null}(A)$
4. Find a particular y_p so that $T(y_p) = Q$.	4. Find a particular $\mathbf{x}_p$ so that $T(\mathbf{x}) = \mathbf{b}$.
5. The solution to $T(y) = Q$ is $y = y_h + y_p$	5. The solution to $T(\mathbf{x}) = \mathbf{b}$ is $\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p$

Example 5.47. In this example we will solve two problems related to linear transformations. Let $T_1(y) = y'+0.5y$ and $Q(t) = 3$. Let $T_2(\mathbf{x}) = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}\mathbf{x}$ and let $\mathbf{b} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$. Solve $T_1(y) = Q$ and $T_2(\mathbf{x}) = \mathbf{b}$.

$T_1(y) = Q$

1. The homogeneous solution is $y_h \in \text{span}\{e^{-0.5t}\}$.
2. The non-homogeneity if a constant function so $y_p \in \text{span}\{1\}$
3. The solution to $T_1(y) = Q$ is $y = C_0 e^{-0.5t} + C_1$ where $C_0, C_1 \in \mathbb{R}$.

$T_2(\mathbf{x}) = \mathbf{b}$

1. After row reducing the homogeneous solution is $\mathbf{x}_h \in \text{span}\left\{\begin{pmatrix} -3 \\ 1 \end{pmatrix}\right\}$
2. After row reducing with $\mathbf{b}$ on the right we see that the particular solution is $\mathbf{x}_p = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$
3. The solution to $T_2(\mathbf{x}) = \mathbf{b}$ is $\mathbf{x} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \end{pmatrix} t$ where $t \in \mathbb{R}$.

Example 5.48. Consider the homogeneous linear differential equation $y' + 0.5y = 0$. We can see this as a question about the kernel of a linear transformation. Indeed, if we let $T(y) = y' - 0.5y$ be a transformation from the space of differentiable functions to the space of continuous functions (on appropriate domains) then the differential equation can simply be stated as: find y in the kernel of the transformation $T(y) = y' + 0.5y$.

The kernel of this linear transformation is spanned by $y(t) = e^{-0.5t}$ since $T(y) = 0$. Therefore the solution to the differential equation is $y(t) = Ce^{-0.5t}$.

Problem 5.49. Consider the homogeneous linear differential equation $y'' + y' - y = 0$. Rewrite this differential equation as a question about the kernel of an appropriate linear transformation. ▲

Solution: $T(y) = y'' + y' - y$

Problem 5.50. For non-homogeneous linear differential equations we can re-frame them in the language of linear transformations in the following way.

- Find a function in the kernel of the transformation
- Find a particular solution that satisfies the non-homogeneous equation
- The general solution is a linear combination of the kernel solution and the particular solution.

Use this idea to solve $T(y) = \sin(t)$ where $T(y) = y' + y$ ▲

Solution: The kernel of the linear transformation is spanned by $y(t) = e^{-t}$. Therefore the solution is $C_1 e^{-t} + C_1 \sin(t) + C_2 \cos(t)$.

Problem 5.51. Let T be a linear transformation that maps vectors in $\mathbb{R}^2$ to vectors in $\mathbb{R}^2$. Symbolically we write $T : \mathbb{R}^2 \to \mathbb{R}^2$. Assume that

$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}3\\-1\end{pmatrix} \quad \text{and} \quad T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-7\\-3\end{pmatrix}.$$

Use the following hints to determine the action of T on an arbitrary vector $\begin{pmatrix}x_1\\x_2\end{pmatrix} \in \mathbb{R}^2$.

- Expand $\begin{pmatrix}x_1\\x_2\end{pmatrix}$ as a linear combination of the basis vectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$.
- Recall that if T is a linear transformation then $T(c\mathbf{u}) = cT(\mathbf{u})$ and $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$. Use this fact to write $T\begin{pmatrix}x_1\\x_2\end{pmatrix}$
- Simplify your answer to give the definition of T.

▲

Solution:

$$\begin{aligned}
T\begin{pmatrix}x_1\\x_2\end{pmatrix} &= T\left(x_1\begin{pmatrix}1\\0\end{pmatrix} + x_2\begin{pmatrix}0\\1\end{pmatrix}\right)\\
&= x_1T\begin{pmatrix}1\\0\end{pmatrix} + x_2T\begin{pmatrix}0\\1\end{pmatrix}\\
&= x_1\begin{pmatrix}3\\-1\end{pmatrix} + x_2\begin{pmatrix}-7\\-3\end{pmatrix}\\
&= \begin{pmatrix}3x_1 - 7x_2\\-x_1 - 3x_2\end{pmatrix}
\end{aligned}$$

Theorem 5.52. Let $T : \mathcal{V} \to \mathcal{W}$ be a linear transformation from a vector space $\mathcal{V}$ to a vector space $\mathcal{W}$. The action of T on any vector $\mathbf{v} \in \mathcal{V}$ is completely determined by the actions of T on the basis vectors for $\mathcal{V}$.

More clearly:

Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a basis for the vector space $\mathcal{V}$. Let T be a linear transformation from $\mathcal{V}$ to vector space $\mathcal{W}$ and assume that

$$T(\mathbf{v}_1) = \mathbf{w}_1, \quad T(\mathbf{v}_2) = \mathbf{w}_2, \quad \ldots \quad T(\mathbf{v}_k) = \mathbf{w}_k$$

where $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k \in \mathcal{W}$. If $\mathbf{v}$ is written as a linear combination of basis vectors from $\mathcal{B}$

$$\mathbf{v} = \sum_{j=1}^{k} c_j\mathbf{v}_j,$$

then

$$T(\mathbf{v}) = \sum_{j=1}^{k} c_j \mathbf{w}_j$$

Proof. The proof follows from the definition of a linear transformation.

$$\begin{aligned} T(\mathbf{v}) &= T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k) \\ &= T(c_1\mathbf{v}_1) + T(c_2\mathbf{v}_2) + \cdots + T(c_k\mathbf{v}_k) \\ &= c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_kT(\mathbf{v}_k) \\ &= c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_k\mathbf{w}_k \end{aligned}$$

□

The consequence of Theorem 5.52 is that all we really need to know is the action of a linear transformation on the basis vectors and we know the entire definition of the transformation.*

Problem 5.53. Let T be a linear transformation mapping quadratic polynomials to 2×2 matrices: $T : \mathcal{P}_2 \to M_{2\times 2}$. Recall that the set $\mathcal{B} = \{1, x, x^2\}$ is a basis for $\mathcal{P}_2$. If

$$T(1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$
$$T(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$T(x^2) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

then what is the action of T on the generic quadratic polynomial $T(ax^2 + bx + c)$? ▲

Solution:

$$\begin{aligned} T(ax^2 + bx + c) &= aT(x^2) + bT(x) + cT(1) \\ &= a\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} c & b \\ b & 2a \end{pmatrix} \end{aligned}$$

*Note here that we are implicitly assuming that the vector spaces $\mathcal{V}$ and $\mathcal{W}$ are finite dimensional. If they were infinite dimensional the theorem will still hold under suitable convergence conditions.

Chapter 6

The Eigenvalue Eigenvector Problem

In this chapter we look at the important eigenvalue-eigenvector question. In this question we wish to find vectors $\mathbf{x}$ and values λ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some given square matrix A. This has profound impact on how we understand differential equations but it also have profound impact on how we understand bases, matrix multiplication, and many other important aspects of linear algebra. Furthermore, we can extend the idea to linear operators and view certain linear differential equations as eigenvalue questions. Wow ...just wow ...you're going to love this chapter!

6.1 Introduction To Eigenvalues

Definition 6.1 (The Eigenvalues and Eigenvectors of a Matrix)**.** Let A be a square $n \times n$ matrix. In the equation $A\mathbf{x} = \lambda\mathbf{x}$, the vector $\mathbf{x} \in \mathbb{R}^n$ is called an eigenvector of the matrix A and λ is the associated eigenvalue.

Problem 6.2. In the applet http://www.geogebra.org/m/334841 you will find a way to graphically manipulate vectors to approximate eigenvectors in $\mathbb{R}^2$. Use the applet to approximate the eigenvectors and eigenvalues of $A = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}$. ▲

Solution: Eigenvectors are $(2,1)^T$ and $(-1,1)^T$ with eigenvalues 1 and -2 respectively.

Problem 6.3. Which of the following is an eigenvector of the matrix $\begin{pmatrix} 2 & -1 \\ -4 & -1 \end{pmatrix}$?

(a) $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$

(b) $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$

(c) $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$

(d) $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$

(e) None of the above

(f) More than one of the above

▲

Solution: The vector $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ is the only eigenvector and it has eigenvalue $\lambda = -2$.

Problem 6.4. Suppose the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has an eigenvalue 1 with associated eigenvector $x = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$. What is $A^{50}x$?

(a) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

(b) $\begin{pmatrix} a^{50} & b^{50} \\ c^{50} & d^{50} \end{pmatrix}$

(c) $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

(d) $\begin{pmatrix} 2^{50} \\ 3^{50} \end{pmatrix}$

(e) Way too hard to compute.

▲

Solution: choice c

Problem 6.5. Vector $\mathbf{x}$ is an eigenvector of matrix A. If $\mathbf{x} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $Ax = \begin{pmatrix} 4 \\ 12 \end{pmatrix}$, then what is the associated eigenvalue?

(a) 1

(b) 3

(c) 4

(d) Not enough information is given.

▲

Solution: eigenvalue = 4 (choice c)

Technique 6.6 (Finding Eigenvalues and Eigenvectors)**.** To find the eigenvalues of an $n \times n$ matrix A:

- Rearrange the equation $A\mathbf{x} = \lambda\mathbf{x}$ so that the right-hand side is the zero vector:

$$\underline{\hspace{3cm}} = \mathbf{0}$$

- Eigenvectors are never the zero vector so what does the previous equation imply about the matrix $A - \lambda I$?
- Form the characteristic polynomial: $p(\lambda) = \det(A - \lambda I)$ and solve for λ using algebra
- Find the associated eigenspace:

$$E_\lambda = \{\mathbf{x} : (A - \lambda I)\mathbf{x} = \mathbf{0}\}$$

Solution:

- $(A - \lambda I)\mathbf{x} = \mathbf{0}$
- The matrix $A - \lambda I$ must be singular so the determinant must be zero.

Problem 6.7. Find the eigen-pairs for the matrix

$$A = \begin{pmatrix} 5 & 7 \\ -2 & -4 \end{pmatrix}$$

▲

Solution:

$$\lambda_1 = -2, \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \lambda_2 = 3, \quad \mathbf{v}_2 = \begin{pmatrix} 7 \\ -2 \end{pmatrix}$$

Problem 6.8. The matrix $A = \begin{pmatrix} 3 & 2 \\ 4 & 10 \end{pmatrix}$ has eigenvalue $\lambda_1 = 2$. Find the associated eigenspace E_{λ_1}. In other words, find the space spanned by the associated eigenvector(s). ▲

Solution: Since $\lambda_1 = 2$ is an eigenvalue of A we need to solve the homogeneous system $(A - 2I)\mathbf{v} = \mathbf{0}$ for $\mathbf{v}$. Observe that $A - 2I = \begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}$ and we can row reduce to $A - 2I \to \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$. Therefore

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

so

$$E_{\lambda_1} = \text{span}\left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}.$$

Problem 6.9. For any integer n, what will this product be? $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}^n \begin{pmatrix} 1 \\ 5 \end{pmatrix}$

(a) $-1(3)^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 3(-2)^n \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(b) $3(-1)^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-2)3^n \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(c) $3(3)^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-2)(-1)^n \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(d) $3(3)^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-1)(-2)^n \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(e) None of the above

(f) More than one of the above

▲

Problem 6.10. $\begin{pmatrix} 4/3 \\ 1 \end{pmatrix}$ is an eigenvector of $\begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}$. What is the associated eigenvalue? (Think! Don't solve for all the eigenvalues and eigenvectors.)

(a) 4/3

(b) 5

(c) -2

▲

Solution: The eigenvalue must be 5 since the product of the matrix and the vector is 5 times the original vector.

Problem 6.11. If a vector x is in the eigenspace of A corresponding to λ, and $\lambda \neq 0$, then x is

(a) in the nullspace of the matrix A.

(b) in the nullspace of the matrix $A - \lambda I$.

(c) not the zero vector.

(d) More than one of the above correctly completes the sentence.

▲

Solution: 2: in the null space of the matrix $A - \lambda I$.

Problem 6.12. Explain why an $n \times n$ matrix can have at most n distinct eigenvalues. ▲

Solution: The characteristic polynomial is n^{th} order and by the fundamental theorem of algebra has at most n distinct roots.

Example 6.13. Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 3 & 2 \\ 4 & 10 \end{pmatrix}$.

Solution: First we'll find the roots of the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$:

$$\begin{aligned} p(\lambda) &= \det\begin{pmatrix} 3-\lambda & 2 \\ 4 & 10-\lambda \end{pmatrix} = (3-\lambda)(10-\lambda) - 8 = 30 - 13\lambda + \lambda^2 - 8 \\ &= \lambda^2 - 13\lambda + 22 = (\lambda - 2)(\lambda - 11). \end{aligned}$$

Therefore we have the eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 11$.

Next we find the associated eigenvectors.

- Eigenvector for $\lambda_1 = 2$:
  Since $\lambda_1 = 2$ is an eigenvalue of A we need to solve the homogeneous system $(A - 2I)\mathbf{v} = \mathbf{0}$ for $\mathbf{v}$. Observe that $A - 2I = \begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}$ and we can row reduce to $(A - 2I) \to \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$. Therefore $\mathbf{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$
- Eigenvector for $\lambda_2 = 11$:
  Since $\lambda_2 = 11$ is an eigenvalue of A we need to solve the homogeneous system $(A - 11I)\mathbf{v} = \mathbf{0}$ for $\mathbf{v}$. Observe that $A - 11I = \begin{pmatrix} -8 & 2 \\ 4 & -1 \end{pmatrix}$ and we can row reduce to $(A - 2I) \to \begin{pmatrix} 1 & -1/4 \\ 0 & 0 \end{pmatrix}$. Therefore $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

Example 6.14. The matrix $A = \begin{pmatrix} 3 & -1 & 1 \\ -2 & 5 & 1 \\ 2 & -3 & 4 \end{pmatrix}$ has an eigenvector $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$. What is the associated eigenvalue?

Solution: By the definition of the eigenvalue-eigenvector pair we know that $A\mathbf{v} = \lambda\mathbf{v}$ so observe that

$$A\mathbf{v} = \begin{pmatrix} 3 & -1 & 1 \\ -2 & 5 & 1 \\ 2 & -3 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 10 \end{pmatrix} = 5\mathbf{v}$$

Therefore we see that the eigenvalue associated with $\mathbf{v}$ is $\lambda = 5$.

Finally, we can extend the idea of the eigenvalue-eigenvector problem to other linear operators. The mathematical field interested in the eigenvalues of linear operators is called *spectral theory* and the name is chosen because of the deep ties between an operator's eigenvalue structure and light spectra.

There are a few eigenfunctions that we already know so let's just hint at the idea by looking at a few differential equations.

Problem 6.15. Consider the linear operator $T(y) = y'$. If we consider the eigenvalue problem $T(y) = \lambda y$ that corresponds to the differential equation $y' = \lambda y$.

(a) What are the eigenfunctions of the linear operator T?

(b) What is the meaning of the eigenvalues in this case?

▲

Solution: The eigenfunctions must be an exponential function so $y(t) \in \text{span}(e^{\lambda t})$. The eigenvalues are the growth (or decay) rates of the exponential functions.

Problem 6.16. Consider the linear operator $T(y) = y''$. If we consider the eigenvalue problem $T(y) = \lambda y$ that corresponds to the differential equation $y'' = -\lambda y$.

(a) What are the eigenfunctions of the linear operator T?

(b) What is the meaning of the eigenvalues in this case?

▲

Solution: The eigenfunctions that naturally arise from the second derivative operator are the trigonometric functions. Hence, $y(t) \in \text{span}(\sin(\sqrt{\lambda} t), \cos(\sqrt{\lambda} t))$. The square roots of the eigenvalues are the frequencies of the trig functions.

6.2 Diagonlization of Matrices

Problem 6.17. Consider the matrix A and the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Form the matrices $P = \begin{pmatrix} | & | \\ \mathbf{v}_1 & \mathbf{v}_2 \\ | & | \end{pmatrix}$ and $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ then compute the products AP and PD. What do you observe?

$A = \begin{pmatrix} 5 & -6 \\ 2 & -2 \end{pmatrix}$ with eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 1$

▲

Solution:

$$AP = \begin{pmatrix} 4 & 3 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = PD$$

Problem 6.18. Repeat the previous problem with these vectors

$A = \begin{pmatrix} 5 & -3 \\ 2 & 0 \end{pmatrix}$ with eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 2$

▲

Solution:
$$AP = \begin{pmatrix} 9 & 2 \\ 6 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} = PD$$

The previous two problems should now lead you to the following theorem. If you find that you cannot fill in the blanks then go back to the two problems and look for patterns.

Theorem 6.19 (Diagonalization of Matrices)**.** If A is $n\times n$, A has n linearly independent eigenvectors, and we form the matrix $P = [\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_n]$ then $AP =$ ___ ___

Furthermore, if you solve the previous equation for A then you get

$$A = \_\_\_\ \_\_\_\ \_\_\_$$

(Fill in the blanks)

Solution: $A = PDP^{-1}$

Another way to state the previous theorem is as follows:

Theorem 6.20. Let A be an $n\times n$ square matrix. Then the following two conditions are equivalent:

- There is a basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ for $\mathbb{R}^n$ consisting of eigenvector for A.
- It is possible to find an invertible matrix P so that $A =$ ___ ___ ___, where D is a diagonal matrix whose entries are the eigenvalues of A.

Solution: $A = PDP^{-1}$

Problem 6.21. Let $A = \begin{pmatrix} 5 & 1 \\ 0 & 3 \end{pmatrix}$. Find a basis for $\mathbb{R}^2$ that consists of eigenvectors of A. ▲

Solution: The eigenvalues of A are $\lambda_1 = 5$ and $\lambda_2 = 3$. $\mathbf{v}_1 = (1,0)^T$ and $\mathbf{v}_2 = (1,-2)^T$ so the basis for $\mathbb{R}^2$ is $\mathcal{B} = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}\right\}$.

Problem 6.22. The matrix A has eigenvalues $\lambda_1 = -3$ and $\lambda_2 = 5$ with associated eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$. Find A. ▲

Solution:
$$A = PDP^{-1} = \begin{pmatrix} 2 & 0 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} -3 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ -4 & 3 \end{pmatrix}^{-1}$$

Problem 6.23. What are the eigenvalues of $D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$?

(a) 2 and 3

(b) 0 and 2

(c) 0 and 3

(d) 5 and 6

▲

Solution: The eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 3$.

Problem 6.24. Why might we be interested in diagonalizing a matrix?

(a) Because it is easy to find the eigenvalues of a diagonal matrix.

(b) Because it is easy to compute powers of a diagonal matrix.

(c) Both of these reasons.

▲

Solution: Both!

Now that we have the tools of diagonalization we can prove the following simple theorem. The real power in this theorem comes in determining if a matrix is invertible given its eigen-structure.

Theorem 6.25. Let A be an $n \times n$ square matrix with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$. Then

$$\det(A) = \lambda_1 \cdot \lambda_2 \cdots \lambda_n.$$

That is, the determinant of the matrix is the same as the product of the eigenvalues.

Proof. (Prove the previous theorem) □

Solution: Write A as $A = PDP^{-1}$ and observe that $\det(A) = \det(PDP^{-1})$. Then use the fact that you can break apart products in determinants to get

$$\det(A) = \det(P)\det(D)\det(P^{-1}) = \det(D) = \lambda_1 \cdot \lambda_2 \cdots \lambda_n.$$

An immediate corollary to the previous theorem is the following.

Corollary 6.26. Let A be an $n \times n$ square matrix. If A has at least one zero eigenvalue then A is not invertible.

Proof. (Prove the previous theorem) □

Solution: If $\lambda_j = 0$ for some j then $\det(A) = 0$ and hence A is not invertible.

Problem 6.27. What does it mean if 0 is an eigenvalue of a matrix A?

(a) The determinant of A is zero.

(b) The columns of A are linearly dependent.

(c) There are an infinite number of solutions to the system $Ax = 0$.

(d) All of the above

(e) None of the above

▲

Solution: All of these.

Problem 6.28. Show that $A = \begin{pmatrix} 1 & -3 \\ -2 & 6 \end{pmatrix}$ is not invertible as many ways as possible. ▲

Solution: The columns are linearly dependent, the determinant is zero, the product of the eigenvalues is zero, the matrix doesn't row reduce to the identity, ...

Theorem 6.29. If λ is an eigenvalue of an invertible matrix A then $1/\lambda$ is an eigenvalue of A^{-1}.

Proof. (Prove the previous theorem. Hint: start with $A\mathbf{v} = \lambda\mathbf{v}$) □

Solution:

Proof. If λ is an eigenvalue of A then there is an associated eigenvector $\mathbf{v}$ and we know that $A\mathbf{v} = \lambda\mathbf{v}$. Multiplying both sides of this equation by A^{-1} gives $\mathbf{v} = \lambda A\mathbf{v}$. Dividing by λ (and observing that λ cannot be zero) gives the equation $\frac{1}{\lambda}\mathbf{v} = A\mathbf{v}$ and the result follows. □

6.3 Powers of Matrices

We will see in a few chapters that powers of matrices appear often in difference and differential equations. For that reason it is very handy to be able to quickly compute powers of matrices and our new-found technique of diagonalization is the right tool for the job! Let's begin with the primary theorem for this section.

Theorem 6.30. Let A be an $n \times n$ matrix with distinct eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ and associated eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. If we diagonalize A as $A = PDP^{-1}$ we know that

$$P = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & \cdots \\ 0 & 0 & \lambda_3 & \cdots \\ \vdots & & & \ddots \end{pmatrix}$$

Using this diagonalization we see that for some integer power n

$$A^n = PD^nP^{-1}$$

Proof. (Prove the previous theorem) □

Solution:

$$A^n = (PDP^{-1})^n = (PDP^{-1})(PDP^{-1})\cdots(PDP^{-1}) = PD(PP^{-1})D(PP^{-1})\cdots DP^{-1} = PD^nP^{-1}$$

Theorem 6.31. If D is a diagonal matrix

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & \cdots \\ 0 & 0 & \lambda_3 & \cdots \\ \vdots & & & \ddots \end{pmatrix}$$

then D^n has the form

$$D = \begin{pmatrix} \lambda_1^n & 0 & 0 & \cdots \\ 0 & \lambda_2^n & 0 & \cdots \\ 0 & 0 & \lambda_3^n & \cdots \\ \vdots & & & \ddots \end{pmatrix}$$

Proof. (Prove the previous theorem) □

Solution: the proof comes straight from matrix multiplication

Problem 6.32. If $D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, what is D^5?

(a) $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

(b) $\begin{pmatrix} 10 & 0 \\ 0 & 15 \end{pmatrix}$

(c) $\begin{pmatrix} 2^5 & 0 \\ 0 & 3^5 \end{pmatrix}$

(d) Too hard to compute by hand.

▲

Solution: 3

Problem 6.33. Consider the matrix

$$A = \begin{pmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{pmatrix}$$

with eigenspaces:

$$\lambda_1 = 2 \text{ with } E_{\lambda_1} = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \right\}$$

$$\lambda_2 = 3 \text{ with } E_{\lambda_2} = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Find A^{100} without technology. ▲

Problem 6.34. The matrix $A = \begin{pmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{pmatrix}$ has eigen-pairs

$$\lambda_1 = 0.5 \text{ with } \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \lambda_2 = 1 \text{ with } \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(a) If $\mathbf{x} = 3\mathbf{v}_1 + 7\mathbf{v}_2$ what is $A\mathbf{x}$? (you should not need to build $\mathbf{x}$ directly)

(b) Write an expression for $A^k\mathbf{x}$ in terms of $\mathbf{v}_1$ and $\mathbf{v}_2$.

(c) Evaluate the limit $\lim_{k\to\infty} A^k\mathbf{x}$.

▲

Solution:

(a) $A\mathbf{x} = A(3\mathbf{v}_1 + 7\mathbf{v}_2) = 3A\mathbf{v}_1 + 7A\mathbf{v}_2 = 3(0.5)\mathbf{v}_1 + 7(1)\mathbf{v}_2 = 1.5\mathbf{v}_1 + 7\mathbf{v}_2 = \begin{pmatrix} 8.5 \\ 5.5 \end{pmatrix}$

(b) $A^k\mathbf{x} = 3A^k\mathbf{v}_1 + 7A^k\mathbf{v}_2 = 3\lambda_1^k\mathbf{v}_1 + 7\lambda_2^k\mathbf{v}_2 = 3\left(\frac{1}{2}\right)^k \mathbf{v}_1 + 7\mathbf{v}_2$

(c) $\lim_{k\to\infty} A^k\mathbf{x} = 7\mathbf{v}_2 = \begin{pmatrix} 7 \\ 7 \end{pmatrix}$ since $(1/2)^k \to 0$ as $k \to \infty$.

Problem 6.35. Long ago, in a galaxy far, far away ... there are two cell-phone companies serving a town: the Evil Empire and the Rebel Alliance. The Evil Empire has terrible service, so each week 25% of their customers switch to the Rebel Alliance and 2% give up their cell phone service entirely. The Rebel Alliance loses only 5% of their customers to the Evil Empire every week due to the advertising. If there are currently 100 customers in the Evil Empire and 75 customers in the Rebel Alliance, what is the long-term enrollment in the two plans?

Write a system of difference equations and use the ideas of eigenvalues and eigenvectors to discuss the long-term behavior of the system. ▲

Solution:

$$\begin{pmatrix} E_{n+1} \\ R_{n+1} \end{pmatrix} = \begin{pmatrix} E_n \\ R_n \end{pmatrix} + \begin{pmatrix} -0.27 & 0.05 \\ 0.25 & -0.05 \end{pmatrix} \begin{pmatrix} E_n \\ R_n \end{pmatrix}$$
$$\implies \begin{pmatrix} E_{n+1} \\ R_{n+1} \end{pmatrix} = \begin{pmatrix} 0.73 & 0.05 \\ 0.25 & 0.95 \end{pmatrix} \begin{pmatrix} E_n \\ R_n \end{pmatrix}$$

The eigenvalues of the coefficient matrix are $\lambda_1 \approx 0.68$ and $\lambda_2 \approx 0.9968$. Since both of these are less than 1 we know that the eventual behavior is that both companies will lose their customers although the Evil Empire will lose at a faster rate.

Theorem 6.36 (Solving a System of Difference Equations)**.** Assume that you have a linear system of difference equations $\mathbf{x}_{n+1} = A\mathbf{x}_n$ with initial condition $\mathbf{x}_0$. If the matrix A has eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ and eigenvalues $\lambda_1, \dots, \lambda_k$ then the analytic solution to the system of difference equations is

$$\mathbf{x}_n = c_1\lambda_1^n\mathbf{v}_1 + c_2\lambda_2^n\mathbf{v}_2 + \cdots c_n\lambda_k^n\mathbf{v}_k = \sum_{j=1}^{k} c_j\lambda_j^n\mathbf{v}_j$$

which is equivalent to

$$\mathbf{x}_n = PD^nP^{-1}\mathbf{x}_0$$

Proof. (Prove the previous theorem) □

Solution: If $\mathbf{x}_{n+1} = A\mathbf{x}_n$ then it can easily be shown that $\mathbf{x}_n = A^n\mathbf{x}_0$. The result follows.

6.4 The Google Page Rank Algorithm

In this section you will discover how the PageRank algorithm works to give the most relevant information as the top hit on a Google search.

Search engines compile large indexes of the dynamic information on the Internet so they are easily searched. This means that when you do a Google search, you are not actually searching the Internet; instead, you are searching the indexes at Google.

When you type a query into Google the following two steps take place:

1. Query Module: The query module at Google converts your natural language into a language that the search system can understand and consults the various indexes at Google in order to answer the query. This is done to find the list of relevant pages.

2. Ranking Module: The ranking module takes the set of relevant pages and ranks them. The outcome of the ranking is an ordered list of web pages such that the pages near the top of the list are most likely to be what you desire from your search. This ranking is the same as assigning a *popularity score* to each web site and then listing the relevant sites by this score.

This section focuses on the Linear Algebra behind the Ranking Module developed by the founders of Google: Sergey Brin and Larry Page. Their algorithm is called the *PageRank algorithm*, and you use it every single time you use Google's search engine.

In simple terms: *A webpage is important if it is pointed to by other important pages*.

The Internet can be viewed as a directed graph (look up this term here on Wikipedia) where the nodes are the web pages and the edges are the hyperlinks between the pages. The hyperlinks into a page are called *inlinks*, and the ones pointing out of a page are called *outlinks*. In essence, a hyperlink from my page to yours is my endorsement of your page. Thus, a page with more recommendations must be more important than a page with a few links. However, the status of the recommendation is also important.

Let us now translate this into mathematics. To help understand this we first consider the small web of six pages shown in Figure 6.1 (a graph of the router level of the internet can be found here). The links between the pages are shown by arrows. An arrow pointing into a node is an *inlink* and an arrow pointing out of a node is an *outlink*. In Figure 6.1, node 3 has three outlinks (to nodes 1, 2, and 5) and 1 inlink (from node 1).

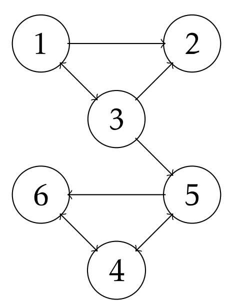


Figure 6.1. Sample graph of a web with six pages.

We will first define some notation in the PageRank algorithm:

- $|P_i|$ is the number of outlinks from page P_i
- H is the *hyperlink* matrix defined as

$$H_{ij} = \begin{cases} \frac{1}{|P_j|}, & \text{if there is a link from node } j \text{ to node } i \\ 0, & \text{otherwise} \end{cases}$$

  where the "i" and "j" are the row and column indices respectively.

- $\mathbf{x}$ is a vector that contains all of the PageRanks for the individual pages.

The PageRank algorithm works as follows:

1. Initialize the page ranks to all be equal. This means that our initial assumption is

that all pages are of equal rank. In the case of Figure 6.1 we would take $\mathbf{x}_0$ to be

$$\mathbf{x}_0 = \begin{pmatrix} 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \end{pmatrix}.$$

2. Build the hyperlink matrix.
   As an example we'll consider node 3 in Figure 6.1. There are three outlinks from node 3 (to nodes 1, 2, and 5). Hence $H_{13} = 1/3$, $H_{23} = 1/3$, and $H_{53} = 1/3$ and the partially complete hyperlink matrix is

$$H = \begin{pmatrix} - & - & 1/3 & - & - & - \\ - & - & 1/3 & - & - & - \\ - & - & 0 & - & - & - \\ - & - & 0 & - & - & - \\ - & - & 1/3 & - & - & - \\ - & - & 0 & - & - & - \end{pmatrix}$$

3. The difference equation $\mathbf{x}_{n+1} = H\mathbf{x}_n$ is used to iteratively refine the estimates of the page ranks. You can view the iterations as a person visiting a page and then following a link at random, then following a random link on the next page, and the next, and the next, etc. Hence we see that the iterations evolve exactly as expected for a difference equation.

Iteration	New Page Rank Estimation
0	$\mathbf{x}_0$
1	$\mathbf{x}_1 = H\mathbf{x}_0$
2	$\mathbf{x}_2 = H\mathbf{x}_1 = H^2\mathbf{x}_0$
3	$\mathbf{x}_3 = H\mathbf{x}_2 = H^3\mathbf{x}_0$
4	$\mathbf{x}_4 = H\mathbf{x}_3 = H^4\mathbf{x}_0$
$\vdots$	$\vdots$
k	$\mathbf{x}_k = H^k\mathbf{x}_0$

4. When a steady state is reached we sort the resulting vector $\mathbf{x}_k$ to give the page rank. The node (web page) with the highest rank will be the top search result, the second highest rank will be the second search result, and so on.

It doesn't take much to see that this process can be very time consuming. Think about your typical web search with hundreds of thousands of hits; that makes a square matrix H that has a size of hundreds of thousands of entries by hundreds of thousands of entries! The matrix multiplications alone would take many minutes (or possibly many hours) for every search! ... but Brin and Page were pretty smart dudes!!

We now state a few theorems and definitions that will help us simplify the iterative PageRank process.

Theorem 6.37. If A is an $n \times n$ matrix with n linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n$ and associated eigenvalues $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ then for any initial vector $\mathbf{x} \in \mathbb{R}^n$ we can write $A^k \mathbf{x}$ as

$$A^k \mathbf{x} = c_1 \lambda_1^k \mathbf{v}_1 + c_2 \lambda_2^k \mathbf{v}_2 + c_3 \lambda_3^k \mathbf{v}_3 + \cdots c_n \lambda_n^k \mathbf{v}_n$$

where $c_1, c_2, c_3, \dots, c_n$ are the constants found by expressing $\mathbf{x}$ as a linear combination of the eigenvectors.
Note: We can assume that the eigenvalues are ordered such that $\lambda_1 \ge \lambda_2 \ge \lambda_3 \ge \cdots \ge \lambda_n$.

Proof. (Prove the preceding theorem) □

Definition 6.38. A **probability vector** is a vector with entries on the interval $[0, 1]$ that add up to 1.

Definition 6.39. A **stochastic matrix** is a square matrix whose columns are probability vectors.

Theorem 6.40. If A is a stochastic $n \times n$ matrix then A will have n linearly independent eigenvectors. Furthermore, the largest eigenvalue of a stochastic matrix will always be $\lambda_1 = 1$ and the smallest eigenvalue will always be nonnegative: $0 \le \lambda_n < 1$.

Some of the following tasks will ask you to *prove* a statement or a theorem. This means to clearly write all of the logical and mathematical reasons why the statement is true. Your proof should be absolutely crystal clear to anyone with a similar mathematical background . . . if you are in doubt then have a peer from a different group read your proof to you out loud.

Problem 6.41. Finish writing the hyperlink matrix H from Figure 6.1. ▲

Problem 6.42. Write MATLAB code to implement the iterative process defined previously. Make a plot that shows how the rank evolves over the iterations. ▲

Problem 6.43. What must be true about a collection of n pages such that an $n \times n$ hyperlink matrix H is a stochastic matrix. ▲

The statement of the next theorem is incomplete, but the proof is given to you. Fill in the blank in the statement of the theorem and provide a few sentences supporting your answer.

Theorem 6.44. If A is an $n \times n$ stochastic matrix and $\mathbf{x}_0$ is some initial vector for the difference equation $\mathbf{x}_{n+1} = A\mathbf{x}_n$, then the steady state vector is

$$\mathbf{x}_{equilib} = \lim_{k\to\infty} A^k \mathbf{x}_0 = \underline{\hspace{3cm}}.$$

Proof. First note that A is an $n \times n$ stochastic matrix so from Theorem 6.40 we know that there are n linearly independent eigenvectors. We can then substitute the eigenvalues from Theorem 6.40 in Theorem 6.37. Noting that if $0 < \lambda_j < 1$ we have $\lim_{k\to\infty} \lambda_j^k = 0$ the result follows immediately. □

Problem 6.45. Discuss how Theorem 6.44 greatly simplifies the PageRank iterative process described previously. In other words: there is no reason to iterate at all. Instead, just find ____________. ▲

Problem 6.46.

Now use the previous two problems to find the resulting PageRank vector from the web in Figure 6.1? Be sure to rank the pages in order of importance. Compare your answer to the one that you got in problem 2. ▲

Problem 6.47. Consider the web in Figure 6.2.

(a) Write the H matrix and find the initial state $\mathbf{x}_0$,

(b) Find steady state PageRank vector using the two different methods described: one using the iterative difference equation and the other using Theorem 6.44 and the dominant eigenvector.

(c) Rank the pages in order of importance.

▲

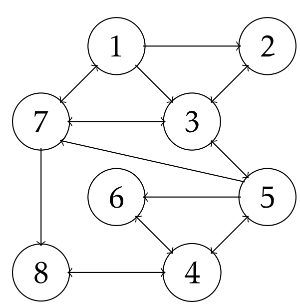


Figure 6.2. Graph of a web with eight pages.

Chapter 7

Second Order Differential Equations

In this brief chapter we will transition back to our discussion of differential equations. In Chapter 1 we discussed first order linear and nonlinear differential equations and in this chapter we will discuss second order linear differential equations. Second order differential equations arise very naturally from Newton's second law, $ma = \sum F$, since acceleration is the second derivative of position. Second order differential equations also arise naturally in circuit analysis and many other physics-based contexts. Our primary focus here will be on mechanical vibrations since that is likely a familar physics context for most students.

7.1 Intro to Second Order Differential Equations

Problem 7.1. A branch sways back and forth with position $f(t)$. Studying its motion you find that its acceleration is proportional to its position, so that when it is 8 cm to the right, it will accelerate to the left at a rate of 2 cm/s^2. Which differential equation describes the motion of the branch?

(a) $\frac{d^2f}{dt^2} = 8f$

(b) $\frac{d^2f}{dt^2} = -4f$

(c) $\frac{d^2f}{dt^2} = -2$

(d) $\frac{d^2f}{dt^2} = \frac{f}{4}$

(e) $\frac{d^2f}{dt^2} = -\frac{f}{4}$

▲

Solution: e

Problem 7.2. Which of the following is not a solution of $y'' + ay = 0$ for some value of a?

(a) $y = 4\sin 2t$

(b) $y = 8\cos 3t$

(c) $y = 2e^{2t}$

(d) all are solutions

▲

Solution: d

Problem 7.3. The motion of a mass on a spring follows the equation $mx'' = -kx$ where the displacement of the mass is given by $x(t)$. Which of the following would result in the highest frequency motion?

(a) $k = 6$, $m = 2$

(b) $k = 4$, $m = 4$

(c) $k = 2$, $m = 6$

(d) $k = 8$, $m = 6$

(e) All frequencies are equal

▲

Solution: d

Problem 7.4. Which of the following is not a solution of $\frac{d^2y}{dt^2} = -ay$ for some positive value of a?

(a) $y = 2\sin 6t$

(b) $y = 4\cos 5t$

(c) $y = 3\sin 2t + 8\cos 2t$

(d) $y = 2\sin 3t + 2\cos 5t$

▲

Solution: d

Problem 7.5. What function solves the equation $y'' + 10y = 0$?

(a) $y = 10\sin 10t$

(b) $y = 60\cos\sqrt{10}t$

(c) $y = \sqrt{10}e^{-10t}$

(d) $y = 20e^{\sqrt{10t}}$

(e) More than one of the above

▲

Solution: b

Technique 7.6 (Solving Second Order Linear ODEs)**.** To solve $ay'' + by' + cy = 0$:

- Assume that $y(t) = e^{rt}$
- Write the associated characteristic polynomial and use it to find r
- Write the solution as a linear combination of the linearly independent eigenfunctions. Use the initial conditions to find the constants in the linear combination.
- Remember to use Euler's Formula if r happens to be complex:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

It is worth it here to take a side step and discuss Euler's formula in more detail. To understand the roots of Euler's formula we first need to recall the definition of a Taylor series.

Definition 7.7 (Taylor Series)**.** Let $f(x)$ be an infinitely differentiable function at a real number $x = a$. The **Taylor Series** of $f(x)$ at $x = a$ is

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \cdots$$

The number "a" is called the center of the Taylor series and if $a = 0$ then the series is sometimes called a MacLaurin series.

The Taylor series is a useful tool for approximating functions with simple polynomials. In fact, every time you use the sine, cosine, and logarithm buttons on your calculator you are actually just evaluating their Taylor series approximations; your calculator has no idea what the sine function really is.

Example 7.8. Find the Taylor series expansions for the functions e^x, $\sin(x)$, and $\cos(x)$ centered at $a = 0$.
Solution: We leave it to the reader to take all of the requisite derivatives to verify the following.

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\ \sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ \cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \end{aligned}$$

Now we have all of the tools necessary to verify Euler's formula.

Problem 7.9. We would like to verify Euler's formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

(a) In the Taylor series for e^x replace the x with $i\theta$. **Solution:**

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!}\cdots$$

(b) Recall that since $i = \sqrt{-1}$ we have

$$i^1 = -1, \quad i^3 = -i, \quad i^4 = 1$$

and successive powers of i repeat this pattern: $i, -1, -i, 1, i, -1, -i, 1, \ldots$. Simplify each of the powers in your answer to part (a). **Solution:**

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - i\frac{\theta^7}{7!}\cdots$$

(c) Rearrange your answer in part (c) to gather the real terms together and the imaginary terms together. **Solution:**

$$e^{i\theta} = \left(1 + -\frac{\theta^2}{2} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right)$$

(d) How does your answer to part (c) verify that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$? **Solution:** From the Taylor series expansions of $\sin(x)$ and $\cos(x)$ we can see that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

▲

In Technique 7.6 the last step cryptically says to "remember to use Euler's formula is r happens to be complex." Now let's clarify that. Let's say that we have a second order linear differential equation $ay'' + by' + cy = 0$ and the roots of the characteristic polynomial are $r_1 = 2 + 3i$ and $r_2 = 2 - 3i$. This means that the general solution to the differential equation is

$$y(t) = C_1 e^{(2+3i)t} + C_2 e^{(2-3i)t}.$$

Using the algebraic rules of exponents we can observe that $e^{(2+3i)t} = e^{2t}e^{3it}$ and $e^{(2-3i)t} = e^{2t}e^{-3it}$. Therefore

$$y(t) = C_1 e^{2t}e^{3it} + C_2 e^{2t}e^{-3it}$$

and factoring e^{2t} gives

$$y(t) = e^{2t}\left(C_1 e^{3it} + C_2 e^{-3it}\right).$$

Using Euler's formula we can now expand the complex exponentials to get

$$y(t) = e^{2t}\left(C_1\cos(3t) + C_1 i\sin(3t) + C_2\cos(-3t) + C_2 i\sin(-3t)\right).$$

We can next recall some helpful trigonometric identities:

$$\cos(-\theta) = \cos(\theta) \quad \text{and} \quad \sin(-\theta) = -\sin(\theta)$$

(coming from the fact that cosine is an even function and sine is an odd function). Therefore,

$$y(t) = e^{2t}\left(C_1\cos(3t) + C_1 i\sin(3t) + C_2\cos(3t) - C_2 i\sin(3t)\right)$$

and gathering like terms gives

$$y(t) = e^{2t}\left((C_1 + C_2)\cos(3t) + (C_1 i - C_2 i)\sin(3t)\right).$$

Since i is a constant we can just relabel the coefficients as new constants and arrive at a much more convenient form of the solution:

$$y(t) = e^{2t}\left(C_3\cos(3t) + C_4\sin(3t)\right).$$

All of this discussion leads us to the following theorem.

Theorem 7.10. For $a, b, c \in \mathbb{R}$ if $y(t)$ is a twice differentiable function such that $ay'' + by' + cy = 0$ and if the characteristic polynomial $ar^2 + br + c = 0$ has complex roots

$$r_1 = \alpha + i\omega \text{ and } r_2 = \alpha - i\omega$$

then the general solution to the second-order linear differential equation is

$$y(t) = e^{\alpha t}\left(C_1\cos(\omega t) + C_2\sin(\omega t)\right)$$

where the constants C_1 and C_2 are expected to be real and are determined by the initial conditions.

To conclude this introductory section on second order differential equations it is now your turn to find the solutions to the following four problems. You will need to use Euler's formula for some of them.

Problem 7.11. Solve the differential equation $y'' - 4y' + 3y = 0$ with $y(0) = 7$ and $y'(0) = 11$ ▲

Solution: $y(t) = 2e^{3t} + 5e^{t}$

Problem 7.12. Solve the differential equation $y'' + 25y = 0$ with $y(0) = 2$ and $y'(0) = 15$ ▲

Solution: $y(t) = 2\cos(5t) + 3\sin(5t)$

Problem 7.13. Solve the differential equation $y'' - 6y' + 9y = 0$ with $y(0) = 2$ and $y'(0) = 1$ ▲

Solution: $y(t) = 2e^{3t} - 5te^{3t}$

Problem 7.14. Solve the differential equation $y'' - 4y' + 5y = 0$ with $y(0) = 2$ and $y'(0) = 3$ ▲

Solution: $y(t) = 2e^{2t}\left(\cos(t) - \sin(t)\right)$

7.2 Mechanical Vibrations

One of the principle applications of second order differential equations is to model mass-spring oscillators. First we are going change our vantage point so that only one of the bodies is oscillating. We achieve this by finding the *reduced mass* $m = \frac{m_1 m_2}{m_1+m_2}$ and then fixing our point of reference so that one of the bodies is fixed. Next, Newton's second law states that we should sum the forces on the oscillating spring.

Problem 7.15. On a mass-spring oscillator what are the primary forces controlling the motion. Use Newton's secondd law to summarize them.

$$ma = \sum F = \underline{\hspace{5cm}}$$

▲

Solution:

$$\implies F_{damping} + F_{restoring} + F_{external} = ma$$

Problem 7.16. In the previous problem you likely have two primary forces. A restoring force due to the spring and a damping force due to friction. Propose functional forms for each of these forces.

$$F_{restoring} = \underline{\hspace{5cm}}$$
$$F_{damping} = \underline{\hspace{5cm}}$$

▲

Solution: $F_{damping} \propto$ velocity
$F_{restoring} \propto$ position

Problem 7.17. The differential equation modeling a mass spring oscillator is:

$$my'' + \underline{\hspace{1.2cm}} + \underline{\hspace{1.2cm}} = F_{external}$$

where $F_{external}$ is any external driving force (like wind, periodic bumps to the spring-mass apparatus, magnetic fields, etc).
(Fill in the blanks) ▲

Solution: $my'' + by' + ky = F_{ext}$

Problem 7.18. If we guess that $y(t) = e^{rt}$ in the previous equation (with $F_{ext} = 0$) then the resulting characteristic polynomial is $p(r) = \underline{\hspace{3cm}}$. Solve for r and classify the types of roots that might occur in terms of the damping coefficient and the spring constant. The three classifications are "under damped", "over damped", and "critically damped". Which situation is which? ▲

Solution:

$$mr^2 + br + k = 0 \implies r = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

- Under Damped: $b^2 - 4mk < 0$
- Over Damped: $b^2 - 4mk > 0$
- Critically Damped: $b^2 - 4mk = 0$

Theorem 7.19. For the homogeneous mass spring oscillator equation

$$my'' + by' + ky = 0$$

with $m, k, b > 0$ there are four primary solution types.

Un-Damped ($b = 0$):

$$y(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t)$$

where $\omega = \sqrt{\frac{k}{m}}$ is called the natural frequency of the oscillator.

Under Damped (two complex roots):

$$y(t) = e^{\alpha t} \left(C_1 \cos(\omega t) + C_2 \sin(\omega t) \right)$$

where $r = \alpha \pm i\omega$

Over Damped (two real roots):

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

Critically Damped (one repeated real root):

$$y(t) = C_1 e^{rt} + C_2 t e^{rt}$$

Proof. (Prove the previous theorem) □

Solution: do the algebra ...

Problem 7.20. Give a linear algebra based reason for the algebraic form of the solution to the Critically Damped oscillator. ▲

Solution: The two solutions (homogeneous and particular) and not linearly independent so we multiply by t to make the independent.

Problem 7.21. Classify each of the second order linear differential equations as either under damped, over damped, or critically damped. After you classify each differential equation write the general form of the solution.

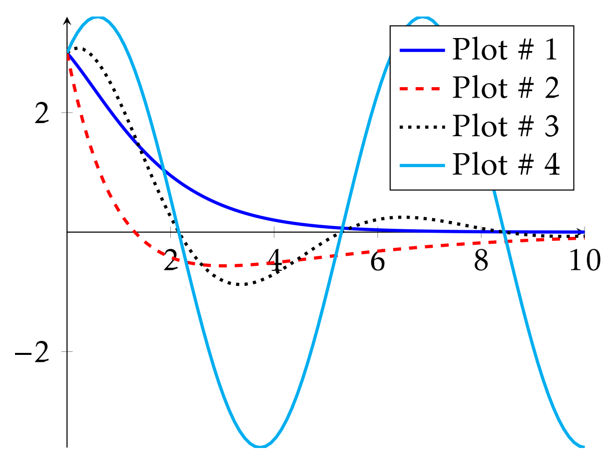
$$y'' + y' + y = 0$$
$$y'' + 2y' + y = 0$$
$$4y'' + 5y' + y = 0$$

▲

Solution:

- For $y'' + y' + y = 0$ the characteristic equation is $r^2 + r + 1 = 0$ which has roots $r = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ so this is under damped. The general form of the solution is
$$y(t) = e^{-(1/2)t}\left(C_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + C_2 \sin\left(\frac{\sqrt{3}}{2}t\right)\right).$$
- For $y'' + 2y' + y = 0$ the characteristic equation is $r^2 + 2r + 1 = 0$ which has 1 repeated root: $r = -1$ so this is a critically damped oscillator. The general form of the solution is
$$y(t) = C_1 e^{-t} + C_2 t e^{-t}.$$
- For $4y'' + 5y' + y = 0$ the characteristic equation is $4r^2 + 5r + 1 = 0$ which has roots $r_1 = -1/4$ and $r_2 = -1$ so this is an overdamped oscillator. The general form of the solution is
$$y(t) = C_1 e^{-(1/4)t} + C_2 e^{-t}.$$

Problem 7.22. Below you will find four plots of solutions to second order oscillators. Which of them would you classify as un-damped, which as under damped, which as over damped, and which as critically damped?



▲

Solution: Plot 1: critically damped, Plot 2: over damped, Plot 3: under damped

Problem 7.23. A steel ball weighing 128 pounds is suspended from a spring. This stretches the spring $\frac{128}{37}$ feet. The ball is started in motion from the equilibrium position with a downward velocity of 5 feet per second (assume that the equilibrium position is $y = 0$). The air resistance (in pounds) of the moving ball numerically equals 4 times its velocity (in feet per second). Suppose that after t second the ball is y feet below its rest position. Find y in terms of t.
Note: The positive direction for y is down and we can take as the gravitational acceleration 32 feet per second per second. ▲

Solution: If the ball pulls the spring by $\frac{128}{37}$ feet and is then at rest then we know from Newton's second law that

$$mg = k \cdot \frac{128}{37}$$

since the only acceleration acting on the ball is gravity. The weight of the steel ball is 128 pounds which is equal to the gravitational force of the ball. Hence, $mg = 128$ pounds. Therefore

$$128 = k \cdot \frac{128}{37} \implies k = 37.$$

From the other information in the problem we know that the differential equation modeling the motion is

$$4y'' + 4y' + 37y = 0 \quad \text{with} \quad y(0) = 0 \text{ and } y'(0) = 5.$$

The resulting characteristic polynomial is $4r^2 + 4r + 37 = 0$ and hence the roots are

$$r = \frac{-4 \pm \sqrt{16 - 4(4)(37)}}{8} = \frac{-4 \pm 24i}{8} = -\frac{1}{2} \pm 3i$$

so the general solution to the differential equation is

$$y(t) = e^{-(1/2)t}\left(C_1 \cos(3t) + C_2 \sin(3t)\right)$$

Using the initial condition that $y(0) = 0$ we see that $C_1 = 0$ so the solution becomes

$$y(t) = Ce^{-(1/2)t} \sin(3t)$$

The initial velocity is $y'(0) = 5$ so we next find the first derivative

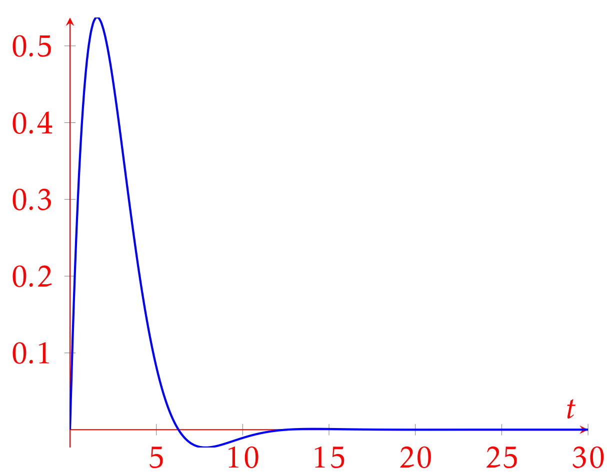
$$y'(t) = -\frac{C}{2}e^{-(1/2)t}\sin(3t) + 3Ce^{-(1/2)t}\cos(3t)$$

Using the initial velocity we get

$$5 = 3C \implies C = \frac{5}{3}.$$

Therefore the solution is

$$y(t) = \frac{5}{3}e^{-(1/2)t}\sin(3t).$$



Problem 7.24.

Consider a floating cylindrical buoy with radius r, height h, and uniform density $\rho \leq 0.5$ grams/cm^3. The buoy is initially suspended at rest with its bottom at the top surface of the water and is released at $t = 0$. Thereafter it is acted on by two forces:

1. a downward gravitational force equal to its weight: $mg = \left(\pi r^2 h\rho\right)g$, and
2. an upward force due to buoyancy equal to the weight of the displaced water: $\left(\pi r^2 x\right)g$

where x is the depth of the bottom of the buoy beneath the surface at time t. (recall that the density of water is 1 gram/cm^3)

(a) Write a differential equation modeling the depth of the buoy.

(b) What type of motion does the buoy undergo?

(c) What is the equilibrium of the oscillation?

(d) What is the period of the oscillation?

▲

Solution:

$$mx'' = \text{weight} - \text{buoyancy} \implies \rho\pi r^2 hx'' = \pi r^2 h\rho g - \pi r^2 gx$$

$$\implies x'' = g - \frac{g}{\rho h}x \implies x'' = \frac{g}{\rho h}(\rho h - x)$$

Substitute $y = \rho h - x$ so we see that $y'' = -x''$ and we get

$$y'' + \frac{g}{\rho h}y = 0 \implies k^2 + \frac{g}{\rho h} = 0 \implies k = \pm\sqrt{\frac{g}{\rho h}}i$$

$$\implies y(t) = C_1\cos\left(\sqrt{\frac{g}{\rho h}}t\right) + C_2\sin\left(\sqrt{\frac{g}{\rho h}}t\right)$$

$$\implies x(t) = \rho h - \left[C_1\cos\left(\sqrt{\frac{g}{\rho h}}t\right) + C_2\sin\left(\sqrt{\frac{g}{\rho h}}t\right)\right]$$

Harmonic motion, period $= 2\pi/\sqrt{g/(\rho h)} = 2\pi\sqrt{\frac{\rho h}{g}}$. equilibrium $= \rho h$

Problem 7.25. The previous problem does not account for the water's viscosity. We can reframe the differential equation as

$$mx'' = \text{weight} - \text{viscosity} - \text{buoyancy}$$

and assume that the retarding force due to viscosity is proportional to the velocity of the buoy.

(a) Write the resulting differential equation.

(b) If b represents the viscosity then what is the period of oscillation?

▲

Solution:

$$\rho\pi r^2 h x'' = \pi r^2 h\rho g - bx' - \pi r^2 g x \implies x'' = \frac{g}{\rho h}(\rho h - x) - \frac{b}{\rho\pi r^2 h}x'$$

$$\implies y'' + \frac{b}{\rho\pi r^2 h}y' + \frac{g}{\rho h}y = 0$$

$$\implies k = \frac{-B \pm \sqrt{B^2 - 4\frac{g}{\rho h}}}{2} = \frac{\frac{b}{\rho\pi r^2 h} \pm \sqrt{\frac{b^2}{\rho^2\pi^2 r^4 h^2} - \frac{4g}{\rho h}}}{2}$$

$$\implies \text{frequency} = \frac{1}{2}\sqrt{\frac{b^2}{\rho^2\pi^2 r^4 h^2} - \frac{4g}{\rho h}}$$

7.3 Undetermined Coefficients

In the previous section we were dealing with un-driven oscillators. That is, the oscillators had no external forces driving the oscillations aside from the initial conditions. In this section we'll consider what happens when you have an external driving force. From Newton's second law we can write the governing equation as

$$my'' + by' + ky = F_{external} \tag{7.1}$$

where $F_{external}$ in (7.1) is a driving force beyond the initial conditions, the spring constant, and the damping force.

Problem 7.26. Propose two different physical instances where an external driving force would influence the motion of an oscillator. ▲

Solution: The presence of a magnetic field on an iron oscillator, a wind force blowing against the oscillator, ...

Problem 7.27. What is the equilibrium of the differential equation $4y'' + 5y' + y = 1$? ▲

Solution: If the motion stops then the y'' and y' terms are zero and the equilibrium is $y = 1$.

Problem 7.28. Work with your partner(s) to suggest a solution technique for the non-homogenous linear second order differential equation $4y'' + 5y' + y = 1$. ▲

Solution: This problem can be solved by the method of undetermined coefficients. From Problem 7.21 we know that the homogeneous solution is

$$y_h(t) = C_1 e^{-(1/4)t} + C_2 e^{-t}$$

and since the non-homogeneity is constant we conjecture that

$$y_p(t) = C_3.$$

Therefore the general solution is

$$y(t) = C_1 e^{-(1/4)t} + C_2 e^{-t} + C_3.$$

Technique 7.29. To solve $my'' + by' + ky = f(t)$:

1. Solve the homogeneous problem (taking $f(t) = 0$)
2. Find a particular solution to the non-homogeneous problem (same functional form as $f(t)$)
3. IF the homogeneous and particular solutions are linearly independent then $y(t)$ is a linear combination of the homogeneous solutions and the particular solution

Now let's put your undetermined coefficients skills to the test by solving the following three un-damped driven oscillators.

Problem 7.30. Solve $y'' + 4y = 2e^{3t}$ with $y(0) = 0$ and $y'(0) = 1$ ▲

Solution: The natural frequency on the un-damped homogeneous oscillator is $\omega = \sqrt{4} = 2$ so the homogeneous solution is $y_h(t) = C_1\cos(2t) + C_2\sin(2t)$. The non-homogeneity is exponential so the particular solution is $y_p(t) = C_3 e^{3t}$ making the general solution

$$y(t) = C_1\cos(2t) + C_2\sin(2t) + C_3 e^{3t}.$$

Using the initial position we see that $0 = C_1 + C_3$. To utilize the initial velocity we first observe that

$$y'(t) = -2C_2\sin(2t) + 2C_2\cos(2t) + 3C_3 e^{3t}.$$

From the initial velocity we see that $1 = 2C_2 + 3C_3$.

To get the third equation necessary to find C_1, C_2, and C_3 we substitute the particular solution into the differential equation to get

$$9C_3 e^{3t} + 4C_3 e^{3t} = 2e^{3t} \implies C_3 = \frac{2}{13}.$$

Back substituting we find that $C_2 = 1 - 3C_3 = 1 - \frac{6}{13} = \frac{7}{13}$ and $C_1 = -\frac{2}{13}$. Therefore the solution is

$$y(t) = -\frac{2}{13}\cos(2t) + \frac{7}{13}\sin(2t) + \frac{2}{13}e^{3t}.$$

Problem 7.31. Solve $y'' + 4y = 5\sin(3t)$ with $y(0) = 0$ and $y'(0) = 1$ ▲

Solution: The natural frequency on the un-damped homogeneous oscillator is $\omega = \sqrt{4} = 2$ so the homogeneous solution is $y_h(t) = C_1\cos(2t) + C_2\sin(2t)$. The non-homogeneity is trigonometric with a different frequency as the natural frequency so $y_p(t) = C_3\cos(3t) + C_4\sin(3t)$ making the general solution

$$y(t) = C_1\cos(2t) + C_2\sin(2t) + C_3\cos(3t) + C_4\sin(3t).$$

Using the initial position we see that $0 = C_1 + C_3$. To utilize the initial velocity we first observe that

$$y'(t) = -2C_1\sin(2t) + 2C_2\cos(2t) - 3C_3\sin(3t) + 3C_4\cos(3t).$$

From the initial velocity we see that $1 = 2C_2 + 3C_4$.

The get the third and fourth equations necessary to find C_1, C_2, C_3 and C_4 we substitute the particular solution in to the differential equation to get

$$-9C_3\cos(3t) - 9C_4\sin(3t) + 4C_3\cos(3t) + 4C_4\sin(3t) = 5\sin(3t).$$

Matching the coefficients of the cosine terms gives $-5C_3 = 0$ and matching coefficients of the sine terms gives $-5C_4 = 5$. Hence, $C_3 = 0$ and $C_4 = -1$.

Back substituting gives $C_1 = 0$ and $C_2 = 2$. Hence the solution to the driven oscillator is

$$y(t) = 2\sin(2t) - \sin(3t).$$

Problem 7.32. Solve $y'' + 4y = 5\sin(2t)$ with $y(0) = 0$ and $y'(0) = 1$ ▲

Solution: We start again with a natural frequency of $\omega = 2$ but this time observe that the driving frequency (on the right-hand side of the differential equation) exactly matches the natural frequency of the homogeneous oscillator. Therefore the homogeneous and particular solutions will not be linearly independent and the general solution is

$$y(t) = C_1\cos(2t) + C_2\sin(2t) + C_3 t\cos(2t) + C_4 t\sin(2t).$$

From the initial position we know that $C_1 = 0$. In order to utilize the initial velocity we need $y'(t)$:

$$\begin{aligned} y'(t) &= 2C_2\cos(2t) + C_3(-2t\sin(2t) + \cos(2t)) + C_4(2t\cos(2t) + \sin(2t)) \\ &= 2C_2\cos(2t) - 2tC_3\sin(2t) + C_3\cos(2t) + 2tC_4\cos(2t) + C_4\sin(2t) \\ &= (2C_2 + C_3 + 2tC_4)\cos(2t) + (-2tC_3 + C4)\sin(2t). \end{aligned}$$

From the initial velocity we have $1 = 2C_2 + C_3$.

To find the next two equations we substitute the particular solution into the differential equation to get

$$\begin{aligned} &-4C_2\sin(2t) - 2C_3(2t\cos(2t) + \sin(2t)) - 2C_3\sin(2t) + 2C_4(-2t\sin(2t) + \cos(2t)) + 2C_4\cos(2t) \\ &+ 4C_3 t\cos(2t) + 4C_4 t\sin(2t) = 5\sin(2t). \end{aligned}$$

Matching coefficients gives

$$\begin{aligned} -4C_2 - 2C_3 - 2C_3 &= 5 \quad \text{(matching the } \sin(2t) \text{ coeffiecients)} \\ 2C_4 + 2C_4 &= 0 \quad \text{(matching the } \cos(2t) \text{ coeffiecients)} \end{aligned}$$

Putting the three equations together gives the system of equation

$$\begin{pmatrix} 2 & 1 & 0 \\ -4 & -4 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix}$$

Clearly $C_4 = 0$ so we can reduce the system to

$$\begin{pmatrix} 2 & 1 \\ -4 & -4 \end{pmatrix} \begin{pmatrix} C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \implies \left(\begin{array}{cc|c} 2 & 1 & 1 \\ -4 & -4 & 5 \end{array} \right) \to \left(\begin{array}{cc|c} 1 & 1/2 & 1/2 \\ 0 & -2 & 7 \end{array} \right) \to \left(\begin{array}{cc|c} 1 & 0 & 9/4 \\ 0 & 1 & -7/2 \end{array} \right)$$

Finally we arrive at the solution

$$y(t) = \frac{9}{4}\sin(2t) - \frac{7}{2}t\cos(2t).$$

7.4 Resonance and Beats

A particular type of forcing term occurs when there is a forcing term that matches the natural frequency of the ocscillor. In an un-damped oscillator this looks like:

$$y'' + \omega^2 y = A\cos(\omega t).$$

Notice that the natural frequency of the homogeneous differential equation is the same as the forcing term.

Problem 7.33. What is the homogeneous solution to the above equation?

$$y_{hom} = \rule{4cm}{0.4pt}$$

and since the particular solution has the same frequency they are not linearly independent of those of the homogeneous solutions. Propose a particular solution:

$$y_{particular} = \rule{4cm}{0.4pt}.$$

The term that you proposed is the root cause of *resonance*. ▲

Problem 7.34. In the differential equation $y'' + \omega^2 y = A\cos(\omega t)$ the amplitudes of the waves will grow in time. What function do the amplitudes follow? ▲

Solution: The amplitudes grow linearly due to the linear function multiplying the trigonometric terms.

Problem 7.35. Solve the differential equation $y'' + 144y = 4\cos(12t)$ with $y(0) = 0$ and $y'(0) = 0$ ▲

Solution: $y(t) = \frac{1}{6}t\sin(12t)$

Example 7.36. Resonance was responsible for the collapse of the Broughton suspension bridge near Manchester, England in 1831. The collapse occurred when a column of soldiers marched in cadence over the bridge, setting up a periodic force of rather large amplitude. The frequency of the force was approximately equal to the natural frequency of the bridge. Thus, the bridge collapsed when large oscillations occurred. For this reason soldiers are ordered to break cadence whenever they cross a bridge.

The Millennium Bridge, the first new bridge to span the Thames River in London in over 100 years, is a modern example of how resonance can effect a bridge. This pedestrian bridge, which opened to the public in June 2000, was quickly closed after the bridge experienced high amplitude horizontal oscillations during periods of high traffic. Studies by designers found that the bridge experienced high amplitude horizontal oscillations in response to horizontal forcing at a rate of one cycle per second. Typically, people walk at a rate of two steps per second, so the time between two successive steps of the left foot is about one second. Thus, if people were to walk in cadence, they would could set up strong horizontal forcing that would place a destructive load on the bridge. The engineers did not envision this to be a problem since tourists do not generally march in time. However, a video of tourists crossing the bridge revealed the opposite. When the bridge began oscillating, people tended to walk in time in order to keep their balance.
https://www.youtube.com/watch?v=gQK21572oSU

Solution: (Modified from [3])

Now we consider what happens when the natural frequency of the homogeneous oscillator is close but not exactly equal to that of the driving term. First watch the video in the example below.

Example 7.37. For an example of the *beats* phenomenon see www.youtube.com/watch?v=pRpN9uLiouI. This phenomenon occurs when the natural frequency and the forcing frequency differ by only a small amount.

$$y'' + \omega_0^2 y = A\cos(\omega t)$$

where ω_0 and ω are very close but not the same. You will be playing with this in the lab.

Solution: (Modified from [3])

Problem 7.38. Go to the GeoGebra applet: https://www.geogebra.org/m/T9yws7CB and explore what happens to the sum of the two functions $f(x) = \sin(Ax)$ and $g(x) = \sin(Bx)$ when A and B are arbitrarily close to each other. ▲

Problem 7.39. Use what you know about the previous problem to sketch the graphical solution to the differential equation $y'' + 4y = \sin(1.9t)$ with $y(0) = y'(0) = 0$. ▲

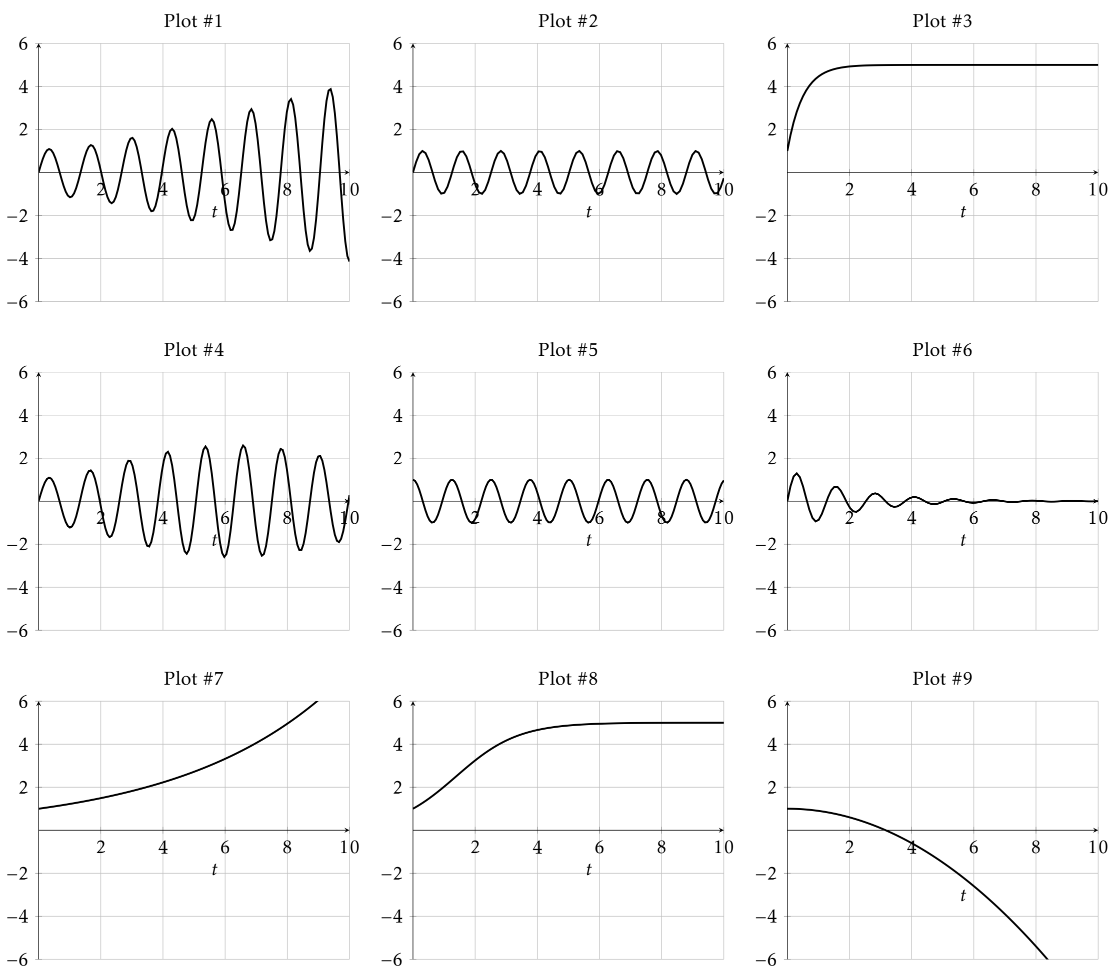
Problem 7.40. Now solve the differential equation $y''+4y = \sin(1.9t)$ with $y(0) = y'(0) = 0$. ▲

To end this chapter we consider one more problem related to all of the differential equations that you should know at this point.

Problem 7.41. Match the differential equations below to the solution plots further below. There should be no need to actually solve the differential equations, but I won't stop you if that is what you really want to do. (there are 9 plots and 8 differential equations. One plot does not have a match . . . just to keep you on your toes)

Differential Equation	Initial Conditions	Matches to Plot #
$y' = 0.2y(5-y)$	$y(0) = 1$	
$y'' + y' + 25y = 0$	$y(0) = 0$ and $y'(0) = 5$	
$y'' + 25y = 4\sin(5.5t)$	$y(0) = 0$ and $y'(0) = 5$	
$y' = -2y + 10$	$y(0) = 1$	
$y'' + 25y = 4\sin(5t)$	$y(0) = 0$ and $y'(0) = 5$	
$y'' + 25y = 0$	$y(0) = 0$ and $y'(0) = 5$	
$y' - 0.2y = 0$	$y(0) = 1$	
$y' + 0.2t = 0$	$y(0) = 1$	

Solution: 8,6,4,3,1,2,7,9



▲

Chapter 8

Systems of Differential Equations

8.1 Matrices and Linear Systems

Problem 8.1. Consider the second order differential equation

$$x'' + bx' + cx = 0$$

By substituting $y = x'$ we can get a system of differential equations. What is the system, write it as a matrix equation, and what is the meaning of y in this system? ▲

Solution:

$$\begin{aligned} x' &= y \\ y' &= -by - cx \end{aligned}$$

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 0 & 1 \\ -c & -b \end{pmatrix} \mathbf{x}$$

Problem 8.2. If

$$\begin{cases} x' &= y \\ y' &= -5x - 2y \end{cases}$$

Then what was the mass spring system associated with the system? Draw a picture of the solutions to x and y. ▲

Solution:

$$x'' + 2x' + 5x = 0$$

Observe that $b^2 - 4mk = 4 - 4(5) < 0$ so the system is underdamped and will oscilalte. Show in PPlane.

Problem 8.3. In the previous problem you likely arrived at the second order differential equation $x'' + 2x' + 5x = 0$. The discriminant on this differential equation is $b^2 - 4mk = 4 - 4(1)(5) = -16$ so we know that the system is underdamped.

(a) What are the roots of the characteristic polynomial.

(b) Using the system in the previous problem write the associated matrix equation and find the eigenvalues of the matrix. What do you notice?

(c) Make a conjecture about the roots of the characteristic polynomial and the eigenvalues of the associated matrix.

▲

Problem 8.4. Consider the second order differential equation modeling a spring-mass system:

$$x'' + 2x' + 2x = 0.$$

(a) Find the discriminant, the roots of the characteristic polynomial, classify the system, and discuss the expected behavior. **Solution:** The system is underdamped since $b^2 - 4mk < 0$. Hence, the system will exhibit long-term damped oscillations. The roots of the characteristic polynomial are

$$r = \frac{-2 \pm \sqrt{4 - 4(1)(2)}}{2} = -1 \pm i$$

(b) Write the differential equation as a first order system and find the associated matrix equation. **Solution:**

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

(c) Find the eigenstructure of A and discuss what this means about the behavior of the system.

▲

Theorem 8.5. If we transform the second order differential equation $x'' + bx' + cx = 0$ into the first order matrix equation $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -c & -b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ by making the substitution $y = x'$ then the roots of the characteristic polynomial $p(r) = r^2 + br + c$ are the same as the eigenvalues of the matrix in the first-order system.

Proof. Let $A = \begin{pmatrix} 0 & 1 \\ -c & -b \end{pmatrix}$ and observe that $\det(A - \lambda I) = \lambda^2 + \lambda b + c$. The result follows. (The reader should fill in all of the details of this proof.) □

Problem 8.6. Consider a two-tank system where brine is transferred between them according to the following rules.

- 20 L/min of fresh water enters Tank #1
- Tank #1 holds $x(t)$ kg of salt and holds a total of 100 liters of total mixture.
- Tank #2 holds $y(t)$ kg of salt and holds a total of 200 liters of total mixture.
- Mixture runs from Tank #1 to Tank #2 at a rate of 30 L/min.
- Mixture runs from Tank #2 to Tank #1 at a rate of 10 L/min.
- Tank #2 drains mixture at a rate of 20 L/min

Write a system of differential equations modeling the transfer of brine in this two-tank system. What interesting questions can you ask about this system? Finally, write the system as a matrix equation. You may want to investigate the system using software like `pplane`. ▲

Solution:

$$\frac{dx_1}{dt} = \text{rate in from } x_2 - \text{rate out to } x_2 + \text{rate in from external}$$
$$\frac{dx_2}{dt} = \text{rate in from } x_1 - \text{rate out to } x_1 + \text{rate out to external}$$

Using concentrations in each we get

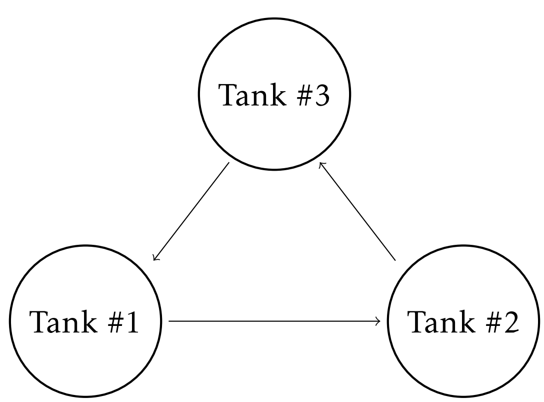
$$\frac{dx_1}{dt} = -\frac{30}{100}x_1 + \frac{10}{200}x_2 + 0 = -\frac{3}{10}x_1 - \frac{1}{20}x_2$$
$$\frac{dx_2}{dt} = \frac{30}{100}x_1 - \frac{10}{200}x_2 - \frac{20}{200}x_2 = \frac{3}{10}x_1 - \frac{3}{20}x_2$$

Show using PPlane as well as get sketches of the time solutions.

Problem 8.7. Three 100 gallon fermentation tanks are connected as shown below, and the mixtures in each tank are kept uniform by stirring. Denote $x_j(t)$ as the amount of alcohol in tank T_j at time t. Suppose that the mixture circulates between the tanks at the rate of 10 gal/min. What system of differential equations governs this closed system? Write the system as a matrix equation. ▲

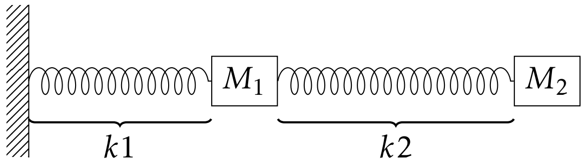
Solution:

$$\frac{d\mathbf{x}}{dt} = \frac{1}{10}\begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}\mathbf{x}$$



Problem 8.8.

A coupled spring mass system is shown below. Let $x_1(t)$ denote the displacement of mass 1 from its equilibrium and let $x_2(t)$ denote the displacement of mass 2 from its equilibrium. Assuming no damping forces complete the system of differential equations below.



$$\begin{aligned} m_1 x_1'' &= -\_\_\_\_\_\_ x_1 + k_2 \_\_\_\_\_\_ \\ m_2 x_2'' &= -\_\_\_\_\_\_ (x_2 - x_1) \end{aligned}$$

Once you have the system, write it as a matrix equation. ▲

Solution:

$$\begin{aligned} m_1 x_1'' &= -k_1 x_1 + k_2(x_2 - x_1) = -(k_1 + k_2)x_1 + k_2 x_2 \\ m_2 x_2'' &= -k_2 (x_2 - x_1) = k_2 x_1 - k_2 x_2 \end{aligned}$$

$$\frac{d^2\mathbf{x}}{dt^2} = \begin{pmatrix} -(k_1 + k_2) & k_2 \\ k_2 & -k_2 \end{pmatrix} \mathbf{x}$$

Problem 8.9. The previous problem ended in a 2×2 second order system. Make an appropriate substitution and arrive at a 4×4 first order system. ▲

Solution: Let $y_1 = x_1'$ and $y_2 = x_2'$. Therefore,

$$\begin{aligned} m_1 y_1' &= -(k_1 + k_2)x_1 + k_2 x_2 \\ x_1' &= y_1 \\ m_2 y_2' &= k_2 x_1 - k_2 x_2 \\ x_2' &= y_2 \end{aligned}$$

$$\begin{pmatrix} y_1' \\ x_1' \\ y_2' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & -(k_1 + k_2) & 0 & k_2 \\ 1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & -k_2 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ x_1 \\ y_2 \\ x_2 \end{pmatrix}$$

8.2 The Eigenvalue Method for Linear Systems

Now let's interweave the idea of eigenvalues in with systems of differential equations. As we already know, the eigenvalues and eigenvectors of a matrix A tell us the underlying structure of the matrix. In some sense they are the DNA of the matrix. In the cases that we investigate here the eigen-structure of A will tell us about the solution curves of the linear first order differential equation.

Theorem 8.10. Let λ be an eigenvalue of the matrix A for the first order linear system

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}.$$

If $\mathbf{v}$ is an eigenvector associated with eigenvalue λ then

$$\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$$

is a nontrivial solution of the system.

Proof. Let A be a real square matrix and let $\mathbf{v}$ and λ be an eigen-pair for A. To check that $\mathbf{x} = \mathbf{v}e^{\lambda t}$ is a solution to the differential equation $\mathbf{x}' = A\mathbf{x}$ we substitute $\mathbf{x}$ in on both sides and check.

On the left-hand side of the differential equation we get

$$\frac{d\mathbf{x}}{dt} = \frac{d}{dt}\left(\mathbf{v}e^{\lambda t}\right) = \mathbf{v}\frac{d}{dt}\left(e^{\lambda t}\right) = \mathbf{v}\left(\lambda e^{\lambda t}\right) = \lambda e^{\lambda t}\mathbf{v}.$$

On the right-hand side of the differential equation we get

$$A\mathbf{x} = A\left(\mathbf{v}e^{\lambda t}\right) = e^{\lambda t}A\mathbf{v} = e^{\lambda t}(\lambda\mathbf{v}) = \lambda e^{\lambda t}\mathbf{v} \quad \checkmark.$$

$\square$

Theorem 8.11. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are unique solutions to the first order linear system of equations $\mathbf{x}' = A\mathbf{x}$ then a linear combination

$$\mathbf{v} = \sum_{j=1}^{k} c_j \mathbf{v}_j$$

is also a solution of $\mathbf{x}' = A\mathbf{x}$.

Proof. Since the differential equation is linear we know that a linear combination of solutions is also a solution. $\square$

Theorem 8.12. If the matrix A has eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ and associated eigenvectors $\lambda_1, \lambda_2, \dots, \lambda_n$ then the general solution to the differential equation $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}(t) = \rule{4cm}{0.4pt}$$

Proof. (Fill in the blank above and prove the theorem) □

Solution:

$$\mathbf{x}(t) = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t} + \cdots + c_k \mathbf{v}_k e^{\lambda_k t}$$

This follows immediately from the previous two theorems.

Problem 8.13. Solve the following linear system of differential equations.

$$\begin{aligned} x_1' &= 4x_1 - x_2 \\ x_2' &= 2x_1 + x_2 \end{aligned}$$

with initial conditions $x_1(0) = 1$ and $x_2(0) = 3$. Complete the problem by giving a plots x_1 vs t, x_2 vs t, and x_2 vs x_1. Use technology to find the eigen-structure of the resulting matrix and to create the plots. ▲

Solution: First write the system as a matrix equation $\mathbf{x}' = A\mathbf{x}$:

$$\mathbf{x}' = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} \mathbf{x}.$$

Now observe that the eigen-pairs of A are

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ with } \lambda_1 = 2 \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ with } \lambda_2 = 3.$$

Hence, the solution to the differential equation is

$$\mathbf{x}(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

To find the values of the constants we need to solve the system

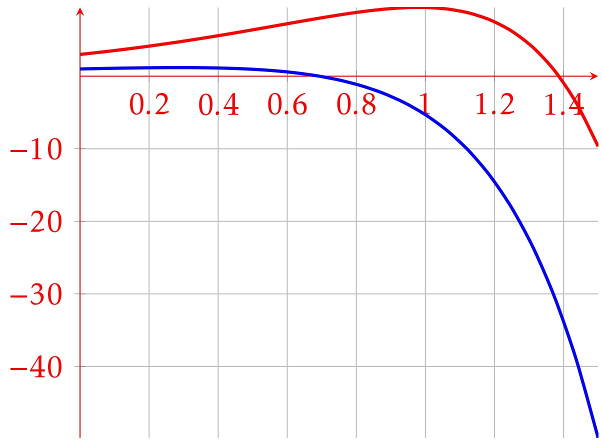
$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Augmenting and row reducing gives

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 1 & 3 \end{array} \right) \to \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -1 & 1 \end{array} \right) \to \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array} \right).$$

so $c_1 = 2$ and $c_2 = -1$ and the full solution is

$$\mathbf{x}(t) = 2e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} - e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$



Problem 8.14. Since we know that both $x_1 = x_2 = e^{3t}$ and $x_1 = e^{-t}, x_2 = -e^{-t}$ are solutions to the system

$$x_1' = x_1 + 2x_2$$
$$x_2' = 2x_1 + x_2$$

Which of the following are also solutions?

(a)

$$x_1 = 3e^{3t} - e^{-t}$$
$$x_2 = 3e^{3t} + e^{-t}$$

(b)

$$x_1 = -e^{3t} - e^{-t}$$
$$x_2 = -e^{3t} + e^{-t}$$

(c)

$$x_1 = 2e^{3t} + 4e^{-t}$$
$$x_2 = -4e^{-t} + 2e^{3t}$$

(d)

$$x_1 = 0$$
$$x_2 = 0$$

(e) None of the above

(f) All of the above.

▲

Problem 8.15. Consider the system of differential equations,

$$y'(t) = \begin{pmatrix} 14 & 0 & -4 \\ 2 & 13 & -8 \\ -3 & 0 & 25 \end{pmatrix} y(t)$$

.

Which of the following functions solve this system?

(a) $y(t) = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} e^{-4t}$

(b) $y(t) = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} e^{6t}$

(c) $y(t) = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} e^{13t}$

(d) None of the above

(e) All of the above.

▲

Problem 8.16. Two forces are fighting one another. Let x and y be the number of soldiers in each force and let a and b be the offensive fighting capacities of x and y respectively. Assume that forces are lost only to combat, and no reinforcements are brought in.

(a) Write a system of differential equations that models this scenario. Write the system as a matrix equation.

(b) Solve the system using the eigenvalue method using sensible initial conditions and values for a and b.

(c) Determine values of a and b for which army x wins and for which army y wins.

▲

Problem 8.17. In Problem 8.16 we built a model that might be really good for hand-to-hand combat. Let's tweak this model.

(a) Modify the model from Problem 8.16 to allow each army to get a constant number of recruits each day (assume time is measured in days).

(b) Propose a solution technique for this model and implement it.

(c) Propose a way to find a steady state solution to your model (if it exists) and implement your idea.

▲

In a linear system where there is a constant non-homogeneity we need to modify our solution technique. Consider the system of differential equations

$$\mathbf{x}'(t) = A\mathbf{x} + \mathbf{b} \tag{8.1}$$

where $\mathbf{x}$ is a vector of functions, A is a real matrix, and $\mathbf{b}$ is a vector of constants. Problem 8.17 should result in a model of this type and in that problem you proposed a solution technique and a method for finding the steady-state solution. Now let's summarize these techniques.

Technique 8.18. Let $\mathbf{x}$ be an $n \times n$ vector of single variable functions, let A be an $n \times n$ real matrix, and let $\mathbf{b}$ be an $n \times n$ real vector. Consider the system of n differential equations

$$\mathbf{x}' = A\mathbf{x} + \mathbf{b}$$

with initial conditions $\mathbf{x}(0) = \mathbf{x}_0$. Observe that this is a non-homogeneous differential equation with a constant non-homogeneity. The general solution to the differential equation is

$$\mathbf{x}(t) = \underline{\hspace{6cm}}$$

where . . . (complete the technique)

Solution:

$$\mathbf{x}(t) = \sum_{j=1}^{n} c_j \left(\mathbf{v}_j e^{\lambda_j t} \right) + \mathbf{w}$$

where $\mathbf{w}$ is the steady state vector resulting from solving the equation $A\mathbf{w} + \mathbf{b} = \mathbf{0}$ and the constants arise from the equation $\mathbf{x}_0 - \mathbf{w} = \sum_{j=1}^{n} c_j \mathbf{v}_j$

Problem 8.19. Consider the linear system of differential equations given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

with initial conditions $x(0) = 1$ and $y(0) = 0$. Solve the system of equations, generate a plot of the solution, and find the steady state (if it exists). ▲

Problem 8.20. Solve the second order differential equation $y'' + 4y' + 3y = 2$ by converting to a first order system. Find the equilibrium if it exists. ▲

Solution: Partial solution:
Let $x = y'$ and we can rewrite as

$$\begin{aligned} x' &= -4y + 3x + 2 \\ y' &= x \end{aligned}$$

which can be rewritten as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

The equilibrium can be found by solving

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

8.3 The Matrix Exponential

Recall that if we are solving the first order linear homogeneous differential equation $x' = rx$ we know (from separation of variables) that the solution is $x(t) = x_0 e^{rt}$. This is one of the simplest ordinary differential equation that there is! In this chapter we have encountered linear systems of differential equations that have a very similar form:

$$\mathbf{x}' = A\mathbf{x} \tag{8.2}$$

where $\mathbf{x}$ this time is a vector of functions and A is a matrix of values. More clearly, if A is an $n \times n$ real matrix then a linear system of differential equations can be written as

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}.$$

It stands to reason that the solution to (8.2) should be the same, or at least have the same form, as the solution to the simple differential equation $x' = rx$. Hence we conjecture that the solution to (8.2) is

$$\mathbf{x} = e^{At}\mathbf{x}_0 \tag{8.3}$$

where $\mathbf{x}_0$ is the vector of initial conditions. ...but wait! What is e^{At}? That's right, we have a matrix in the exponent of an exponential function!

To understand the matrix exponential we first start by recalling the Taylor series of the exponential function

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots.$$

If instead we examine the function $f(x) = e^{ax}$ where $a \in \mathbb{R}$ then it is easy to see that

$$e^{ax} = 1 + ax + a^2\frac{x^2}{2} + a^3\frac{x^3}{3!} + a^4\frac{x^4}{4!} + \cdots.$$

If we use the Taylor series as the definition of the exponential function then a natural definition for the matrix exponential is as follows.

Definition 8.21 (Matrix Exponential)**.** Let A be a square matrix and t be a real variable. The matrix exponential e^{At} is defined as

$$e^{At} = I + At + A^2\frac{t^2}{2} + A^3\frac{t^3}{3!} + A^4\frac{t^4}{4!} + \cdots. \tag{8.4}$$

Recall that if A has a complete collection of eigenvalues and eigenvectors then we can find matrices P and D such that $A = PDP^{-1}$. Therefore the definition of the matrix exponential can be rewritten as

$$\begin{aligned} e^{At} &= I + PDP^{-1}t + PD^2P^{-1}\frac{t^2}{2} + PD^3P^{-1}\frac{t^3}{3!} + PD^4P^{-1}\frac{t^4}{4!} + \cdots \\ &= P\left(I + Dt + D^2\frac{t^2}{2} + D^3\frac{t^3}{3!} + D^4\frac{t^4}{4!} + \cdots\right)P^{-1} \end{aligned} \tag{8.5}$$

Using either (8.4) or (8.5) we now have a way to find the analytic solution to a linear homogeneous system of differential equations of the form $\mathbf{x}' = A\mathbf{x}$.

Theorem 8.22. If $\mathbf{x}$ is a vector of functions and A is a real square matrix then the solution to the differential equation $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0.$$

The computation of the matrix exponential in Theorem 8.22 is potentially really obnoxious by hand, but thankfully there are built-in tools for performing the computation in the most scientific computing software. For example, in MATLAB you can use the command `expm(A)` to find the matrix exponential of the matrix A.

Example 8.23. Use the matrix exponential to solve the system of differential equations

$$\mathbf{x}'(t) = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}\mathbf{x}(t) \quad \text{with} \quad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

Solution: The following MATLAB code finds the matrix exponential

```
clear; clc;
syms t
A = [2 , 0 ; 1 , 2];
x0 = [2;-3];
```

```
x = expm(A*t) * x0   % this is the solution to the system of ODEs
```

which results in the solution

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} e^{2t} & 0 \\ te^{2t} & e^{2t} \end{pmatrix} \begin{pmatrix} 2 \\ -3 \end{pmatrix}.$$

pulling this out of matrix form we have

$$x_1(t) = 2e^{2t} \quad \text{and} \quad x_2(t) = 2te^{2t} - 3e^{2t}.$$

Finally we can plot the solution with the following code.

```
tmax = 1;
ezplot(x(1),[0,tmax])
hold on
ezplot(x(2),[0,tmax])
axis([0,1,-10,10])
```

Hint: You can use the command `latex(simplify(x))` to take your symbolic answer for $\mathbf{x}(t)$ from MATLAB and get it into LaTeX.

The following two problems give you a chance to practice using the matrix exponential to solve applied systems of differential equations problems. Have fun!!

Problem 8.24. Tank #1 is connected to Tank #2 by two separate pipes. Pure water is flowing into Tank #1 at a rate of 2 gallons per minute. Tank #1 is initially filled with 50 gallons of water with 5 pounds of salt dissolved in it. Tank #2 contains 40 gallons of water with 1 pound of salt dissolved in it. Solution flows from Tank #1 to Tank #2 at a rate of 3 gallons per minute and from Tank #2 to Tank #1 and at a rate of 1 gallon per minute. Thoroughly mixed solution is also being drained from Tank #2 at a rate of 2 gallons per minute. Let $x(t)$ be the amount of salt in Tank #1 and let $y(t)$ be the amount of salt in Tank #2 at time t.

(a) Write the linear system of differential equations that models this scenario. Be sure to include your initial conditions (it may help to draw a picture first).

(b) Solve the system of differential equations using the matrix exponential. Clearly write your answer in matrix and vector form.

(c) Solve the system again using the eigenvalue method. Clearly write your answer showing how the eigenvalues and eigenvectors play a role in the solution.

(d) Make a plot showing how the two concentrations evolve over 100 minutes.

▲

Problem 8.25. You are given a mechanical system with three springs A, B, and C and two objects F and G each of mass M. Springs A and C have spring constant k and spring B has spring constant $2k$. Spring A is attached to a wall on the left and spring C is attached to a wall on the right. Let $x_1(t)$ be the position of F from it's equilibrium ($x_1 = 0$ indicates that F is at equilibrium), and let $x_2(t)$ be the position of G from it's equilibrium. See Figure 8.1.

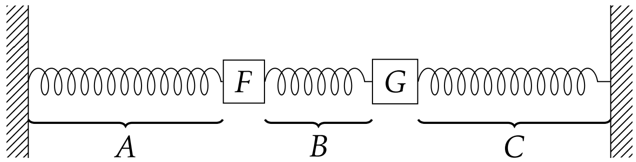


Figure 8.1. Double spring mass system

Initial conditions: Assume that we move mass F exactly 1 unit to the left and let it go. This means that $x_1(0) = -1$ and $x_3(0) = x_1'(0) = 0$. If we initially hold mass G fixed then $x_2(0) = x_4(0) = 0$.

(a) First we'll write the second order linear system of equations. This system of equations is just a statement of Hooke's law for springs: $F = -kx$, where k is the spring constant. To help you get started you can use the skeleton below:

$$\begin{aligned} Mx_1'' &= -kx_1 + (\text{some spring constant})(\text{distance between } x_2 \text{ and } x_1) \\ Mx_2'' &= -2k(x_2 - x_1) - (\text{some spring constant})x_2 \end{aligned}$$

(b) To turn this into a first-order system we introduce two new variables: $x_3 = x_1'$ and $x_4 = x_2'$. Write this system of differential equations ...I'll get you started ...

$$\begin{aligned} x_1' &= x_3 \\ x_2' &= x_4 \\ x_3' &= \ldots \\ x_4' &= \ldots \end{aligned}$$

Now write the system as a matrix equation of the form $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$.

(c) Solve the linear system of differential equations with the method of matrix exponentials.

(d) Next solve the linear system of differential equations with the eigenvalue method. You *should* find that you have 4 purely imaginary eigenvalues (I'll wait while you let MATLAB find these for you ...). This means that you will have oscillations in your solutions. Why?

(e) Plot your solutions.

▲

8.4 Complex Eigenvalues

In this brief section we consider the case where the eigenvalues of the coefficient matrix in a linear system are complex. If $\mathbf{x}' = A\mathbf{x}$ and A has complex eigenvalues $\lambda = \alpha \pm \beta i$ then the solution to the system is

$$\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2 = e^{\alpha t}\left(C_1 e^{\beta i t}\mathbf{v}_1 + C_2 e^{-\beta i t}\mathbf{v}_2\right).$$

If we use Euler's formula for the complex exponentials the solution becomes

$$\mathbf{x}(t) = e^{\alpha t}\left(C_1\left(\cos(\beta t) + i\sin(\beta t)\right)\mathbf{v}_1 + C_2\left(\cos(\beta t) - i\sin(\beta t)\right)\mathbf{v}_2\right).$$

Now if we gather the trigonometric functions the solution can be written as

$$\mathbf{x}(t) = e^{\alpha t}\left[(C_1\mathbf{v}_1 + C_2\mathbf{v}_2)\cos(\beta t) + (C_1 i\mathbf{v}_1 - C_2 i\mathbf{v}_2)\sin(\beta t)\right]. \tag{8.6}$$

Form the initial conditions we know that $\mathbf{x}(0) = C_1\mathbf{v}_1 + C_2\mathbf{v}_2$ and once we have calculated C_1 and C_2 it is straight forward to find the vectors multiplying the cosine and sine terms in the solution.

Example 8.26. Solve the system of differential equations

$$\mathbf{x}'(t) = \begin{pmatrix} -2 & -2 \\ 2 & -2 \end{pmatrix}\mathbf{x}(t)$$

with initial condition $\mathbf{x}(0) = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$.

Solution: We first need to find the eigenvalues and eigenvectors of the coefficient matrix, A.

$$\mathbf{v}_1 = \begin{pmatrix} -i \\ -1 \end{pmatrix} \quad \text{with} \quad \lambda_1 = -2 + 2i$$

$$\mathbf{v}_2 = \begin{pmatrix} i \\ -1 \end{pmatrix} \quad \text{with} \quad \lambda_2 = -2 - 2i$$

Solve the system of equation $\left(\mathbf{v}_1\, \mathbf{v}_2\right)\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \mathbf{x}(0)$ to get $C_1 = \frac{3}{2} + \frac{3}{2}i$ and $C_2 = \frac{3}{2} - \frac{3}{2}i$ (the actual computation is left to the reader). Hence,

$$C_1\mathbf{v}_1 + C_2\mathbf{v}_2 = \left(\frac{3}{2} + \frac{3}{2}i\right)\begin{pmatrix} -i \\ -1 \end{pmatrix} + \left(\frac{3}{2} - \frac{3}{2}i\right)\begin{pmatrix} i \\ -1 \end{pmatrix} = \cdots = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$$

and

$$C_1 i\mathbf{v}_1 - C_2 i\mathbf{v}_2 = \left(\frac{3}{2} + \frac{3}{2}i\right)i\begin{pmatrix} -i \\ -1 \end{pmatrix} - \left(\frac{3}{2} - \frac{3}{2}i\right)i\begin{pmatrix} i \\ -1 \end{pmatrix} = \cdots = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

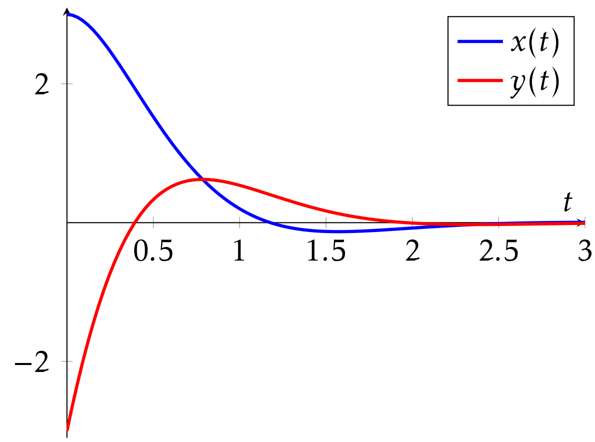


Figure 8.2. Solution curves for Example 8.26

Now we can substitute into (8.6) to get the complete solution

$$\mathbf{x}(t) = e^{-2t}\left(\cos(2t)\begin{pmatrix} 3 \\ -3 \end{pmatrix} + \sin(2t)\begin{pmatrix} -3 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} e^{-2t}(3\cos(2t) + 3\sin(2t)) \\ e^{-2t}(-3\cos(2t) + 3\sin(2t)) \end{pmatrix}.$$

The solutions to the system are shown in Figure 8.2

Chapter 9

Nonlinear Systems of Differential Equations

The world is non-linear! Well shoot. It might seem that this means that for *real* problems we can't use anything that we've done so far. Wrong! There are plenty of things that we can do with nonlinear problems. For the most part we will rely on a basic premise from Calculus: up close, a nonlinear function looks linear. You did this back in calculus when you found tangent lines and tangent planes but now we're going to do the same for matrices and nonlinear differential equations. For most of the problems in this chapter it will be helpful to have MATLAB up so you can plot phase planes and analyze equilibria graphically.

9.1 Trace-Determinant Plane

In this section we will briefly examine a technique for quickly determining the behavior of 2D linear systems. This will play a major role in our nonlinear systems since each nonlinear system will eventually be linearized as we will see in a bit.

Recall that if $\mathbf{v}_1$ and $\mathbf{v}_2$ are the eigenvectors of A with eigenvalues λ_1 and λ_2 then the solution to the first order linear system of equations $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x} = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$$

where $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. Furthermore we observe that $\mathbf{x} = \mathbf{0}$ is an equilibrium point of the system of differential equations. Use these ideas to answer the following questions.

Problem 9.1. Consider the 2×2 linear system of differential equations $\mathbf{x}' = A\mathbf{x}$. In each of the following cases what is the expected behavior of the solution? The choices are: spirals in to the origin, spirals out from the origin, decays in to the origin, diverges out from the origin, decays toward the origin but then diverges out, or circles the origin.

1. If $\lambda_1 \le \lambda_2 < 0$ then near the origin the solution ____________ **Solution:** decays in
2. If $\lambda_1 < 0 < \lambda_2$ then near the origin the solution ____________ **Solution:** decays toward the origin but then divers out

3. If $\lambda_1 \geq \lambda_2 > 0$ then near the origin the solution ____________ **Solution:** diverges out from the origin

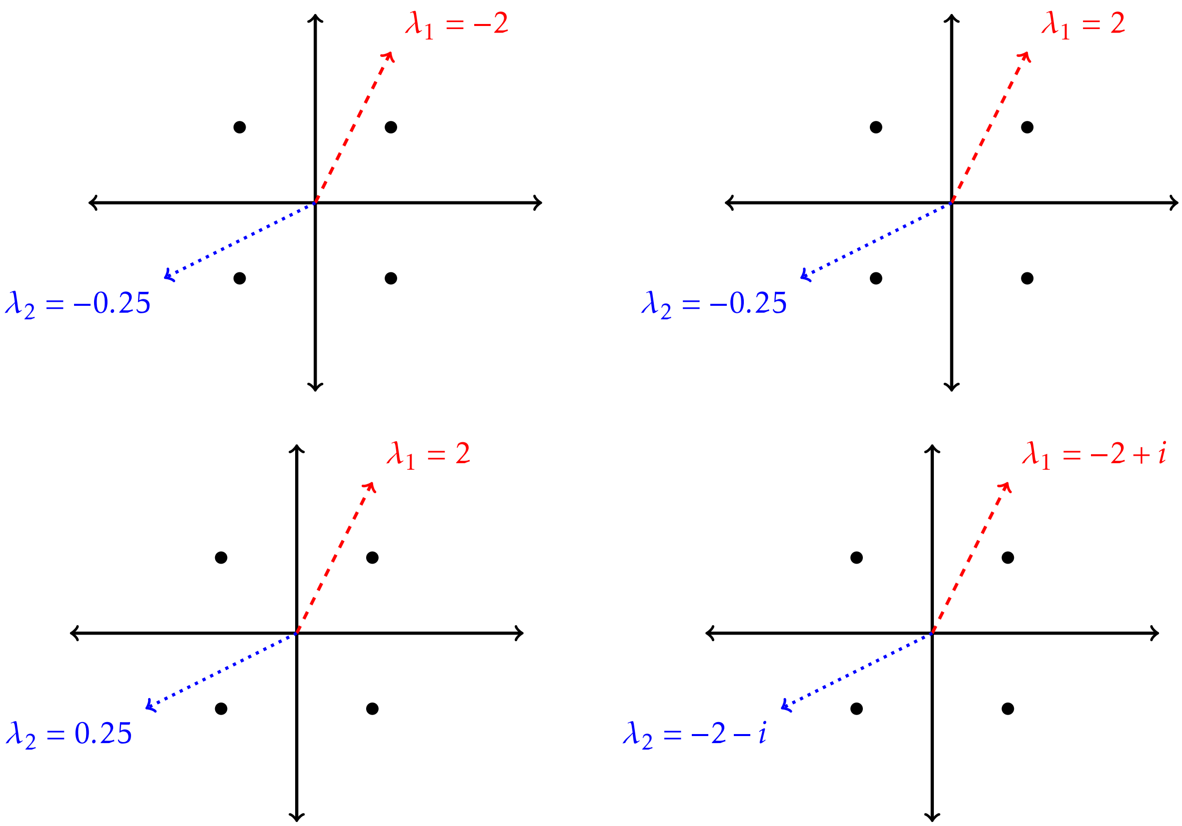
4. If $\lambda_1, \lambda_2 = \alpha \pm \beta i \, (\alpha < 0)$ then near the origin the solution ____________ **Solution:** spirals in toward the origin

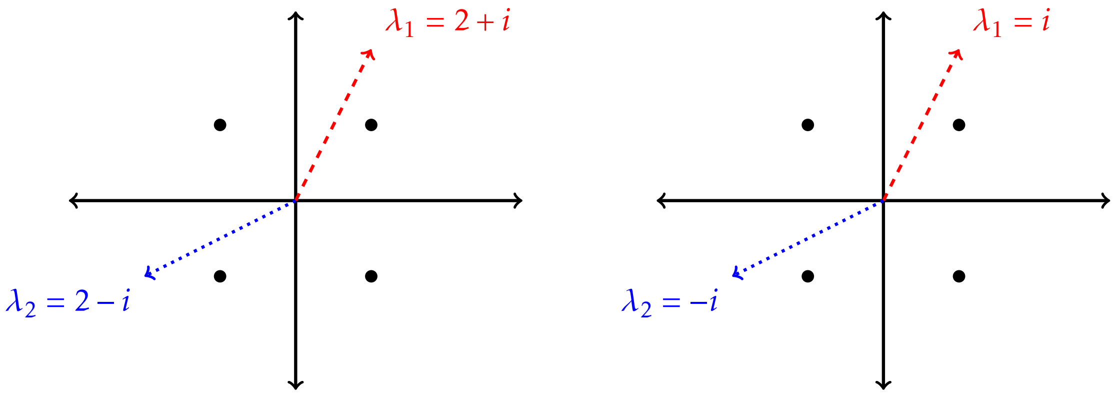
5. If $\lambda_1, \lambda_2 = \alpha \pm \beta i \, (\alpha > 0)$ then near the origin the solution ____________ **Solution:** spirals out from the origin

6. If $\lambda_1, \lambda_2 = \pm \beta i$ then near the origin the solution ____________ **Solution:** circles the origin

▲

Problem 9.2. In the following plots the two eigenvectors for a matrix A are plotted and the eigenvalues are given. Sketch the trajectory of the solution in the xy-plane starting at the given points (use your answers from the previous problem to help).





▲

Now that we know the general behavior of all 2×2 systems based on the eigenstructure let's get a faster way to make the determination. That is, if we can avoid actually finding the eigenvalues that might save some time.

Problem 9.3. Let the 2×2 linear system $\mathbf{x}' = A\mathbf{x}$ be written as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Your Tasks:

1. What is the equilibrium of this system presuming that A^{-1} exists?
2. Find the characteristic polynomial of the coefficient matrix
3. Simplify the characteristic polynomial to fill in the blanks:

$$\lambda^2 + \underline{\qquad}\lambda + \underline{\qquad} = 0$$

The blanks should be familiar features of the matrix.

▲

Solution: The solution to the system is $\mathbf{x} = \mathbf{0}$ since we solve $\mathbf{0} = A\mathbf{x}$ to find the equilibrium and if A^{-1} exists then $\mathbf{x} = \mathbf{0}$.

The characteristic polynomial is: $p(\lambda) = \lambda^2 - tr(A)\lambda + \det(A)$

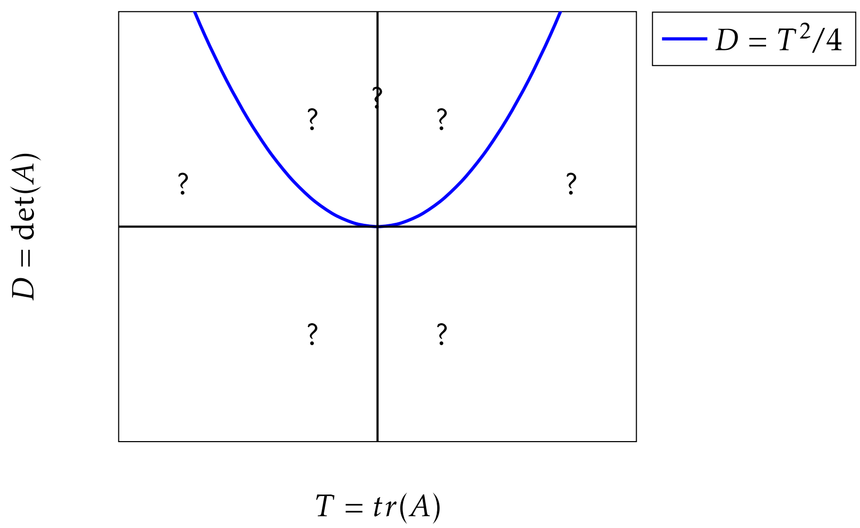
Problem 9.4. In the previous problems you found that the characteristic polynomial for a 2D linear system is $p(\lambda) = \lambda^2 - T\lambda + D$ where $T = \mathrm{tr}(A)$ and $D = \det(A)$. Solving for λ we see that

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$

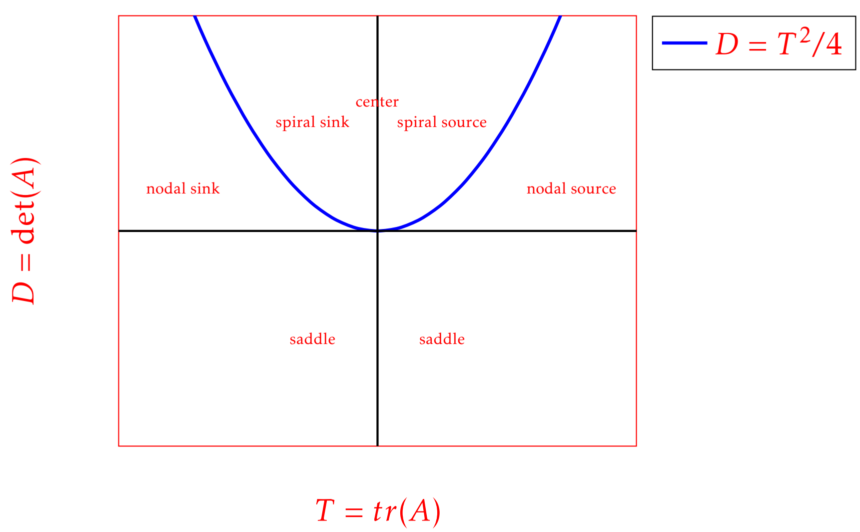
Clearly the behavior of the eigenvalues depends on the quantity $T^2 - 4D$. To visualize this we create the *trace-determinant* plane as seen in the figure immediately below. Fill in the question marks in the figure with the type of behavior you expect to see for each region of the *trace-determinant* plane?

To illustrate what we mean condsider the furthest right question mark. In that question mark the determinant, D, is greater than zero, the trace, T, is greater than zero and $D < T^2/4$. Hence $T^2 - 4D > 0$ so we expect the behavior of the system to be a nodal source since both eigenvalues will be real and positive.

Use the following words to fill in the question marks: nodal source, nodal sink, spiral source, spiral sink, center, and saddle. ▲



Solution:



Now we'll finally look at some nonlinear systems.

Problem 9.5. Suppose that x and y are the population of two distinct species that compete for the same resources. For example, two species of fish may compete for the same food in a lake or sheep and cattle competing for the same grazing land. We can model two competing species using the following system of first-order differential equations,

$$x' = 2x\left(1 - \frac{x}{2}\right) - xy$$
$$y' = 3y\left(1 - \frac{y}{3}\right) - 2xy.$$

It is reasonably easy to show that the four equilibrium solutions are $(0,0)$, $(0,3)$, $(2,0)$, and $(1,1)$. Use the `pplane` software (linked from Moodle) to analyze what happens near the equilibrium $(1,1)$. ▲

Now we'll build up the analytic tools necessary to analyze the competing species model in problem 9.5.

Definition 9.6 (Nullclines and Equilibria)**.** The **nullclines** for a linear system

$$x'(t) = f(x,y)$$
$$y'(t) = g(x,y)$$

are the curves where $f(x,y) = 0$ and $g(x,y) = 0$. When two nullclines intersect there is and equilibrium solution.

Problem 9.7. Use a graphing tool to sketch the nullclines of the system and use your graph to verify the location of the equilibrium points.

$$x' = 2x\left(1 - \frac{x}{2}\right) - xy$$
$$y' = 3y\left(1 - \frac{y}{3}\right) - 2xy.$$

▲

Solution: The nullclines are $2x(1-x/2)-xy = 0$ and $3y(1-y/3)-2xy = 0$. In the first one we either have the vertical line $x = 0$ or the line $y = 2(1-x/2)$. In the second one we either have the horizontal line $y = 0$ of the line $2x = 3(1-y/3)$. Plotting all four of these lines together we see that the intersections are $(0,0)$, $(0,3)$, $(2,0)$, and $(1,1)$.

Definition 9.8 (Jacobian Matrix)**.** Consider the system of equations

$$f(x,y) = 0$$
$$g(x,y) = 0.$$

The **Jacobian matrix** for this system is defined as

$$J(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$

where subscripts mean partial dervatives.

Problem 9.9. Find the Jacobian matrix $J(x,y)$ for the system

$$x' = 2x\left(1-\frac{x}{2}\right) - xy = 2x - x^2 - xy$$
$$y' = 3y\left(1-\frac{y}{3}\right) - 2xy = 3y - y^2 - 2xy.$$

▲

Solution:

$$J(x,y) = \begin{pmatrix} 2-2x-y & -x \\ -2y & 3-2y-2x \end{pmatrix}$$

The Jacobian describes the local linear behavior of the system near an equilibrium. That is to say that if we substitute the values (x_*, y_*) from an equilibrium point into the Jacobian then *near* the equilibrium point will behave like the linear system $\mathbf{x}' = J(x_*, y_*)\mathbf{x}$ centered at the equilibrium.

Problem 9.10. Verify the behavior of the system

$$x' = 2x\left(1-\frac{x}{2}\right) - xy$$
$$y' = 3y\left(1-\frac{y}{3}\right) - 2xy$$

near the equilibrium $(1,1)$ (you already discussed this in Problem 9.5). Then use the Jacobian matrix to discuss the behavior of the system near the other three equilibria. ▲

Solution: At $(1,1)$ we have $J(1,1) = \begin{pmatrix} -1 & -1 \\ -2 & -1 \end{pmatrix}$. Observe that $\text{tr}(J(1,1)) = -2$ and $\det(J(1,1)) = 1-2 = -1$. From the trace-determinant plane we should have a saddle near $(1,1)$.

Technique 9.11 (Equilibria and Stability of Nonlinear Systems)**.** Consider the non-linear system of differential equations

$$x'(t) = f(x,y)$$
$$y'(t) = g(x,y).$$

To find and analyze the equilibria for the system:

1. Find the equilibria by setting ______ to zero and solving for x and y. It may be necessary to use technology to solve this system of nonlinear equations.

2. Find the Jacobian matrix at each of the equilibrium points.

3. Investigate the ______ for each Jacobian matrix. Based on this investigate you can make a claim about local stability.

Example 9.12. Consider the system

$$x'(t) = x - 3y + xy^2$$
$$y'(t) = 2x - 4y - x^2y.$$

Find the nullclines, equilibria, the Jacobian, and classify the equilibrium solutions.
Solution:
The nullclines are the curves $f(x,y) = 0$ and $g(x,y) = 0$ called the x-nullcline and the y-nullcline respectively since if $f = 0$ the x-variable stops changing and if $g = 0$ the y-variable stops changing.

$$x\text{-nullcline: } 0 = x - 3y + xy^2$$
$$y\text{-nullcline: } 0 = 2x - 4y - x^2y$$

These are rather complicated curves in the xy-plane.

Using a computer algebra system the approximate equilibria are $(-1.06, -0.41)$, $(1.06, 0.41)$, and $(0,0)$ (along with a few imaginary equilibria).

The Jacobian is $J(x,y) = \begin{pmatrix} 1+y^2 & -3+2xy \\ 2-2xy & -4-x^2 \end{pmatrix}$ and at $(0,0)$ we have $J(0,0) = \begin{pmatrix} 1 & -3 \\ 2 & -4 \end{pmatrix}$. For this equilibrium point, $T(0,0) = -3$ and $D(0,0) = (-4) - (-6) = 2$. Hence $T^2/4 = 9/4 = 2.25 > D$ so according to the trace-determinant plane we much have a spiral sink at $(0,0)$.

9.2 Applied Nonlinear Systems

Let's get started with a nonlinear system. This system will be familiar in a lot of ways but we will added a small wrinkle: air resistance.

Problem 9.13. Modeling a bungee jumper is much like modeling a 1-dimensional spring mass system except for the fact that the air resistance plays a major role. According to Newton's second law as well as Hooke's law

$$mx'' = -kx + F_d \quad \text{where} \quad F_d = -ax' - b(x')^2$$

Hence, the model for the motion of the bungee jumper is

$$mx'' + ax' + b(x')^2 + ky = 0$$

Dividing by mass we get

$$x'' + \alpha x' + \beta (x')^2 + \kappa x = 0$$

Turn this into a system of differential equations with an appropriate substitution, discuss equilibria and stability, and explore it graphically. ▲

Solution: Let $x' = y$ and get the following:

$$\begin{cases} x' &= y \\ y' &= -\alpha y - \beta y^2 - \kappa x \end{cases}$$

This is a nonlinear system!!

Equlibria: Set $x' = y' = 0$

$$\begin{cases} 0 &= y \\ 0 &= -\alpha y - \beta y^2 - \kappa x \end{cases}$$

Therefore, $x = y = 0$ is the equilibrium.

Local stability: Find the Jacobian:

$$J(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\kappa & -\alpha - 2\beta y \end{pmatrix}$$

$$\implies J(0,0) = \begin{pmatrix} 0 & 1 \\ -\kappa & -\alpha \end{pmatrix}$$

and we are locally back at the same spot as the linear problem. Hence the equilibrium is locally stable.

Problem 9.14. Suppose that we have a predator-prey system consisting of a population of foxes (F) and rabbits (R)

$$R'(t) = 2R - RF$$
$$F'(t) = -5F + RF.$$

It is easy to check that have equilibrium at $R = 5$ and $F = 2$. Fully analyze the dynamics of the population assuming that it doesn't start with $(R,F) = (5,2)$. In particular, make a phase plot and determine the stability of the equilibrium. ▲

Solution: (Modified from [3])

Problem 9.15. In the previous problem, modify the rabbit population so that it follows logistic growth

$$R'(t) = 2R\left(1 - \frac{R}{10}\right) - RF.$$

Fully analyze this new system. ▲

Solution: spiral sink at $(5,1)$

Problem 9.16. Romeo and Juliet's love can be quantified as

Hysterical Hatred	-5
Disgust	-2.5
Indifference	0
Sweet Affection	2.5
Ecstatic Love	5

The characters struggle with frustrated love due to the lack of reciprocity of their feelings.

Romeo: "My feelings for Juliet decrease in proportion to her love for me."

Juliet: "My love for Romeo grows in proportion to his love for me."

Write a mathematical model for the ill-fated love of Romeo and Juliet. Discuss equilibria and stability. Explore graphically.
Assume $R(0) = 2$ and $J(0) = 0$. What do these initial conditions mean? Start your explorations with $\alpha = 0.2$ and $\beta = 0.8$. ▲

Solution:

$$\begin{cases} R' &= -\alpha J \\ J' &= \beta R \end{cases}$$

Linear, so

$$\begin{pmatrix} 0 & -\alpha \\ \beta & 0 \end{pmatrix}$$

$$\implies \lambda_{1,2} = \pm\sqrt{-\alpha\beta}$$

so there will be oscillations so long as $\alpha, \beta > 0$.
Take $\alpha = 0.2$ and $\beta = 0.8$ for an interesting exploration.

Problem 9.17. Romeo and Juliet's love can be quantified as

Hysterical Hatred	−5
Disgust	−2.5
Indifference	0
Sweet Affection	2.5
Ecstatic Love	5

The characters struggle with frustrated love due to the lack of reciprocity of their feelings.

Romeo: "My feelings for Juliet decrease in proportion to her love for me."

Juliet: "My love for Romeo grows in proportion to his love for me." But, her emotional swings lead to sleepless nights which consequently dampen her emotions.

Write a mathematical model for the ill-fated love of Romeo and Juliet. Discuss equilibria and stability. Explore graphically. ▲

Solution:

$$\begin{cases} R' &= -\alpha J \\ J' &= \beta R - \kappa J^r \end{cases}$$

If $r = 1$ then this is linear, so

$$\begin{pmatrix} 0 & -\alpha \\ \beta & \kappa \end{pmatrix}$$

$$\implies \lambda_{1,2} = \frac{-\kappa \pm \sqrt{\kappa^2 - 4\alpha\beta}}{2}$$

So there could be oscillations but since the real part is negative there will be an overall damping and the end result will be a stable equilibrium at $(R, J) = (0, 0)$.

If $r \neq 1$ then:

$$J(x, y) = \begin{pmatrix} 0 & -\alpha \\ \beta & r\kappa J^{r-1} \end{pmatrix}$$

You will still have the same local stability.

Problem 9.18. In historical battles where hand-to-hand combat was common, a mathematical model for the survival of the various forces is:

- The rate at which the RED army loses troops is proportional to the product of the sizes of the two armies
- The rate at which the BLUE army loses troops is proportional to the product of the sizes of the two armies

Write a mathematical model for the size of each army. Discuss equilibria, stability, and explore graphically. What is wrong with this model? ▲

Solution:

$$\begin{cases} R' &= -\alpha RB \\ B' &= -\beta RB \end{cases}$$

Clearly if either $R = 0$ OR if $B = 0$ then there is an equilibrium.

$$J(R, B) = \begin{pmatrix} -\alpha B & -\alpha R \\ -\beta B & \beta R \end{pmatrix}$$

If $R = 0$ then $J(0, B) = \begin{pmatrix} -\alpha B & 0 \\ -\beta B & 0 \end{pmatrix}$ and locally

$$\begin{pmatrix} R \\ B \end{pmatrix} = c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 e^{-\beta t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and any of the equilibrium points will be stable. Similar for $B = 0$.

Problem 9.19. In historical battles where hand-to-hand combat was common, a mathematical model for the survival of the various forces is:

- The rate at which the RED army loses troops is proportional to the product of the sizes of the two armies
- The rate at which the BLUE army loses troops is proportional to the product of the sizes of the two armies
- The rate at which the RED army gains recruits is proportional to the size of the RED army.

Write a mathematical model for the size of each army. Discuss equilibria, stability, and explore graphically. How do you prove stability? ▲

Solution:

$$\begin{cases} R' &= -\alpha RB + \kappa R \\ B' &= -\beta RB \end{cases}$$

Problem 9.20. The Van der Pol oscillator equation arose in the 1920's when Balthasar Van der Pol was working with oscillator circuits for radios. The equation is

$$x'' + \mu(x^2 - 1)x' + x = 0$$

where x is related to the current in an RLC-circuit. Write the Van der Pol equation as a non-linear first order system and completely investigate the behaviour of the system using $\mu = 1$. Use `pplane` plots to aid in your analysis. ▲

Solution: The first order nonlinear system is

$$x'(t) = y$$
$$y'(t) = -x - \mu(x^2 - 1)y$$

The only equilibrium point is $(x, y) = (0, 0)$ and the Jacobian is

$$J(x, y)\begin{pmatrix} 0 & 1 \\ -1 - 2\mu xy & -\mu(x^2 - 1) \end{pmatrix}$$

so at $(0, 0)$ we have

$$J(0, 0) = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}.$$

Since the trace is μ and the determinant is 1 we know that the origin is a spiral source.

Problem 9.21. A virus spreads through a dorm. Assume that there are three types of people in the dormitory population: S is the number of people susceptible to the virus, I is the number of infectious people, and R is the number of recovered people. Assume that $S + I + R = N$ is the total number of people in the dorm (and N is fixed). Build a differential equation model for the rates at which S, I, and R change assuming that

- The susceptible people get sick at a rate proportional to the interactions with infectious people.
- Infectious people recover at a fixed rate.

Once you have your model explore it graphically (using `pplane`) and analyze any equilibrium points. (Hint: you really only need 2 equations) ▲

Problem 9.22. The **Western Grasslands Model**: This is a model of the competition between "good" grass and weeds on a fixed area of rangeland where cattle are allowed to graze. The two dependent variables $g(t)$ and $w(t)$ represent, respectively, the fraction of the area colonized by the good grass and the weeds at time t. Hence, $0 \le g \le 1$ and $0 \le w \le 1\}$. The model is given by

$$\begin{aligned}\frac{dg}{dt} &= R_1 g\left(1 - g - 0.6w\frac{E+g}{0.31E+g}\right)\\ \frac{dw}{dt} &= R_2 g\left(1 - w - 1.07g\frac{0.31E+g}{E+g}\right).\end{aligned}$$

The parameters R_1 and R_2 represent the intrinsic growth rates of the grass and weeds respectively. The cattle stocking rate is introduced through the parameter E. For this problem assume that $R_1 = 0.27$, $R_2 = 0.4$, and $E = 0.3$. There are several equilibrium points that we need to analyze.

- There is an equilibrium at $(0,0)$. What does it mean physically and what type of behavior do we see near this point?
- There is an equilibrium at $(0,1)$. What does it mean physically and what type of behavior do we see near this point?
- There are two equilibria inside the domain where both weeds and grass can coexist. Find them and describe the behavior of the system near them.

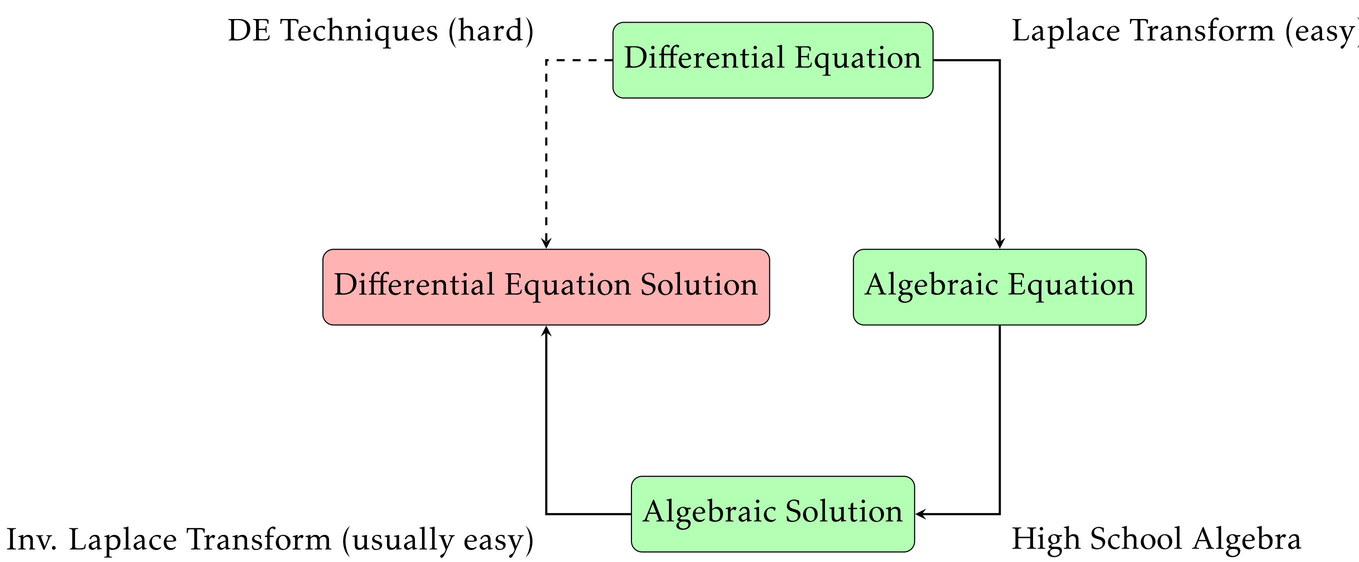
▲

Chapter 10

Laplace Transforms

10.1 Introduction to Laplace Transforms

Let's face it. Solving differential equations can be hard. Sometimes it is really hard, and sometimes it is downright impossible. Generally speaking it is much easier to solve algebraic equations like the ones you were introduced to in high school mathematics. The goal of the Laplace Transform Method for solving differential equations is to turn a linear differential equation into an algebraic equation, solve it, then turn the answer back into a solution to the differential equation. The process is depicted below. Once you get used to this technique you'll never want to use our old techniques again!



10.1.1 Where Laplace Transforms Come From

You may have run into Taylor Series in past courses. The idea is to represent a function $f(x)$ near the point $x = 0$ as an infinite series of power functions:

$$f(x) = \frac{f(0)}{0!}x^0 + \frac{f'(0)}{1!}x^1 + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots. \tag{10.1}$$

More compactly, we can write the Taylor Series as

$$f(x) = \sum_{j=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n. \tag{10.2}$$

Problem 10.1. Find the Taylor Series representations for the functions $f(x) = e^x$, $g(x) = \frac{1}{1-x}$ (for $|x| < 1$), and $h(x) = \sin(x)$ all centered at $x = 0$.

$$\begin{aligned} e^x &= \rule{6cm}{0.4pt} \\ \frac{1}{1-x} &= \rule{6cm}{0.4pt} \\ \sin(x) &= \rule{6cm}{0.4pt} \end{aligned}$$

▲

Solution:

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\ \frac{1}{1-x} &= 1 + x + x^2 + x^3 + x^4 + \cdots \\ \sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \end{aligned}$$

The Taylor Series does something else amazing! In a sense, the coefficients of the Taylor Series are the DNA of the function. That is to say: If you know the Taylor coefficients you know the function and visa versa.

Problem 10.2. Using problem 1, which function has the following sequence of Taylor coefficients?

$a(n) = \{1, 1, 1, 1, 1, \ldots\}$ corresponds to $f(x) =$ ____________

$a(n) = \{1, 1, \frac{1}{2!}, \frac{1}{3!}, \frac{1}{4!}, \ldots\}$ corresponds to $f(x) =$ ____________

$a(n) = \{0, 1, 0, -\frac{1}{3!}, 0, \frac{1}{5!}, \ldots\}$ corresponds to $f(x) =$ ____________

▲

Solution:

$$\{1,1,1,1,1,\dots\} \quad \text{corresponds to} \quad f(x) = \frac{1}{1-x}$$
$$\{1,1,\frac{1}{2!},\frac{1}{3!},\frac{1}{4!},\dots\} \quad \text{corresponds to} \quad f(x) = e^x$$
$$\{0,1,0,-\frac{1}{3!},0,\frac{1}{5!},\dots\} \quad \text{corresponds to} \quad f(x) = \sin(x)$$

Hence, if we have a sequence $a(n)$ (and the sequence has some basic properties*) then we can associate the sequence with a function $f(x)$ via the Taylor Series

$$a(n) \rightsquigarrow f(x) \quad \text{since} \quad \sum_{n=0}^{\infty} a(n)x^n = f(x).$$

Problem 10.3. You might be asking yourself "So what! Why do I need another way to represent a function?" To answer this question discuss with your partner how you think the Taylor sequence "DNA" of a function might be a useful tool in this modern age of computers (hint). ▲

Solution: There are many ways to for a computer to store and understand a transcendental function line the trigonometric functions and the exponential function. Your calculator and many computer programming languages only know these functions by storing their Taylor sequence. That requires very little computer storage and allows the computer program to calculate these functions to arbitrary precision.

The Laplace Transform

Now we're ready to create the Laplace Transform. The Laplace Transform is the continuous analog of what we just discussed: If we have a function $a(n)$ we can find a function $f(x)$ such that we replace the sum in the Taylor series with an integral:

$$\int_0^{\infty} a(n)x^n dn = f(x) \tag{10.3}$$

There are some notational conventions that we have to adjust for:

1. We don't typically use n as a continuous variable so we're going to switch it to t.
2. We typically don't use the letter "a" for functions of a continuous variable so we'll switch it to f. Then we'll make the right-hand side "F" so we can keep them straight. The integral now becomes
$$\int_0^{\infty} f(t)x^t dt = F(x)$$

*See any standard Calculus text if you don't recall the necessary conditions for a Taylor Series to converge.

3. The exponential function x^t is really inconvenient when integrating with respect to t so we'll switch it to

$$x^t = e^{\ln(x)t} \qquad \Longrightarrow \qquad \int_0^\infty f(t)e^{\ln(x)t}dt = F(x)$$

(convince yourself that this is the same thing algebraically)

4. Now the left-hand side will be a function of $\ln(x)$ and the right-hand side will be a function of x. This is rather inconvenient so let's make a change of variables: replace $\ln(x)$ with $-s$ (the negative sign gives positive values when $0 < x < 1$ and negative values otherwise). Therefore, the Laplace transform is:

$$\boxed{\int_0^\infty e^{-st} f(t)dt = F(s)} \tag{10.4}$$

The Laplace transform associates a function $f(t)$ with a function $F(s)$ just like the Taylor Series associates a sequence of numbers $a(n)$ with a function $f(x)$:

$$\boxed{\underbrace{a(n) \rightsquigarrow f(x)}_{\text{Taylor Series}} \quad \text{via} \quad \sum_{n=0}^{\infty} a(n)x^n = f(x) \qquad \text{and} \qquad \underbrace{f(t) \rightsquigarrow F(s)}_{\text{Laplace Transform}} \quad \text{via} \quad \int_0^\infty e^{-st} f(t)dt = F(s).}$$

Notationally we write the Laplace Transform of $f(t)$ as $\mathcal{L}\{f(t)\} = F(s)$.

10.1.2 Basic Laplace Transforms and Basic Properties

Now let's do a few Laplace Transforms. This exercise (as well as a few others) will be essential before we can start using Laplace transforms for differential equations.

Problem 10.4. Find the Laplace Transform of each of the following functions: (no calculator!)

(a) If $f(t) = 1$ then

$$\mathcal{L}\{f(t)\} = \underline{\hspace{3cm}}$$

Solution:

$$\int_0^\infty e^{-st} \cdot 1dt = -\frac{e^{-st}}{s}\Big|_{t=0}^{t\to\infty} = \lim_{b\to\infty} -\frac{e^{-st}}{s}\Big|_{t=0}^{t=b} = -\left(\lim_{b\to\infty} \frac{e^{-sb}}{s} - \frac{1}{s}\right) = \frac{1}{s}$$

(b) If $f(t) = e^{at}$ then

$$\mathcal{L}\{f(t)\} = \underline{\hspace{3cm}}$$

Solution:

$$\int_0^\infty e^{-st}e^{at}dt = \int_0^\infty e^{-(s-a)t}dt = \cdots = \frac{1}{s-a}$$

(c) If $f(t) = t$ then (Hint: integrate by parts)

$$\mathcal{L}\{f(t)\} = \underline{\hspace{3cm}}$$

Solution:

$$\int_0^\infty te^{-st}dt = -\frac{te^{-st}}{s}\Big|_{t=0}^{t\to\infty} - \int_0^\infty -\frac{e^{-st}}{s}dt = \frac{1}{s}\int_0^\infty e^{-st}dt = \frac{1}{s}\mathcal{L}\{1\} = \frac{1}{s^2}$$

▲

Example 10.5. Find the Laplace transform of $f(t) = t^2$.
Solution: We want $\mathcal{L}\{f(t)\}$ so we write the integral

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}t^2dt.$$

This integral requires integration by parts. Let $u = t^2$ and $dv = e^{-st}$ to get $du = 2tdt$ and $v = -\frac{1}{s}e^{-st}$ and hence

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \int_0^\infty e^{-st}t^2dt = -\frac{t^2}{s}e^{-st}\Big|_{t=0}^{t\to\infty} + \frac{2}{s}\int_0^\infty te^{-st}dt \\ &= 0 + \frac{2}{s}te^{-st}dt \\ &= \frac{2}{s}\mathcal{L}\{t\} = \frac{2}{s}\cdot\frac{1}{s^2} = \frac{2}{s^3}\end{aligned}$$

Problem 10.6. Based on what we saw in the previous problem and example you may see a convenient pattern for finding the Laplace transform of power functions. Based on this pattern let's conjecture a few more basic Laplace transforms. (Hint: look at the previous example and see what will happen every time we use integration by parts on these functions.)

$$\begin{aligned}\mathcal{L}\{1\} &= \frac{1}{s} \\ \mathcal{L}\{t\} &= \frac{1}{s^2} \\ \mathcal{L}\left\{t^2\right\} &= \frac{2}{s^3} \\ \mathcal{L}\left\{t^3\right\} &= \underline{\hspace{3cm}} \\ \mathcal{L}\left\{t^4\right\} &= \underline{\hspace{3cm}} \\ \mathcal{L}\left\{t^5\right\} &= \underline{\hspace{3cm}} \\ \mathcal{L}\{t^n\} &= \underline{\hspace{3cm}}\end{aligned}$$

▲

Solution:

$$\mathcal{L}\{1\} = \frac{1}{s}, \mathcal{L}\{t\} = \frac{1}{s^2}, \mathcal{L}\left\{t^2\right\} = \frac{2}{s^3}, \mathcal{L}\left\{t^3\right\} = \frac{3!}{s^4}, \mathcal{L}\left\{t^4\right\} = \frac{4!}{s^5}, \mathcal{L}\left\{t^5\right\} = \frac{5!}{s^6}, \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

Now we will build some of the basic properties of the Laplace transform. Many of these are intuitively obvious from the definition of the Laplace transform

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

since we know that the integral is a linear operator ...hmmm, I wonder what this means about the Laplace transform.

Theorem 10.7. If $f(t)$ and $g(t)$ are functions that have Laplace transforms then

$$\mathcal{L}\{f(t) + g(t)\} = \underline{\hspace{4cm}}.$$

(Fill in the blank with what your intuition tells you *should* happen)

Proof. (Prove the previous theorem) □

Solution:

$$\mathcal{L}\{f(t) + g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$$

since

$$\int_0^\infty e^{-st}(f(t) + g(t))dt = \int_0^\infty e^{-st} f(t)dt + \int_0^\infty e^{-st} g(t)dt$$

Theorem 10.8. If a is a scalar and $f(t)$ is a function that has a Laplace transform then

$$\mathcal{L}\{af(t)\} = \underline{\hspace{4cm}}.$$

(Fill in the blank with what your intuition tells you *should* happen)

Solution:

$$\mathcal{L}\{af(t)\} = a\mathcal{L}\{f(t)\}$$

since

$$\int_0^\infty e^{-st}(af(t))dt = a\int_0^\infty e^{-st} f(t)dt$$

Problem 10.9. The previous two theorems tell that the Laplace transform is ____________.
▲

Solution: a linear transformation

10.1.3 Some Important Theorems

We will now state a few important theorems (without proof):

Theorem 10.10 (Existence of Laplace Transforms:)**.** If $f(t)$ is a piecewise continuous function such that $|f(t)| < Me^{ct}$ for $t \geq T$ and for some non-negative constants M, c, and T, then $\mathcal{L}\{f(t)\} = F(s)$ exists for all $s > c$.

Theorem 10.10 gives us conditions for when the Laplace transform exists. It just says that the function $f(t)$ needs to *grow slower* than an exponential function.

Now that we know when the Laplace transform exists it would be handy to know if it is unique. It should be intuitively *obvious* that if we calculate $\mathcal{L}\{f(t)\}$ then we will only ever get one answer, but as mathematicians we can't just rely on our instincts for what is *obvious.* For a uniqueness theorem we state it the other way around: If two Laplace transforms are the same then they must have come from the same place. This is summarized in the following theorem.

Theorem 10.11 (Uniqueness of Laplace Transforms:)**.** If $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$ then if $F(s) = G(s)$ we MUST have $f(t) = g(t)$. In other words, the Laplace transform of a function is unique.

Finally we get to the ultimate utility of the Laplace transform. From the beginning of this chapter we stated that we want to use the Laplace transform to make solving differential equations easier. To do this we need to first convert a differential equation to an algebraic equation. The reader should see that this might be possible with the Laplace transform. At the end of the process, however, we need to do an inversion of the Laplace transform. If the inverse isn't known to exist then the whole process is going to fail and this conversation is moot. Thankfully we have the following theorem!

Theorem 10.12 (Invertibility of Laplace Transforms:)**.** Since the Laplace transform of a function is unique, the *inverse Laplace transform* exists and

$$\mathcal{L}^{-1}\{F(s)\} = f(t).$$

10.1.4 Common Laplace Transforms

Here are some common Laplace transforms. These DO NOT need to be memorized. You will be provided such a table on any exam. For a more complete table see tutorial.math.lamar.edu/pdf/Laplace_Table.pdf

Function $f(t)$	Laplace Transform $F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
$\frac{1}{\sqrt{t}}$	$\frac{\sqrt{\pi}}{\sqrt{s}}$
e^{at}	$\frac{1}{s-a}$
e^{-at}	$\frac{1}{s+a}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$
$e^{-at}\cos(bt)$	$\frac{s+a}{(s+a)^2+b^2}$
$\sin(bt)$	$\frac{b}{s^2+b^2}$
$e^{-at}\sin(bt)$	$\frac{b}{(s+a)^2+b^2}$
$t\sin(bt)$	$\frac{2bs}{(s^2+b^2)^2}$
$t\cos(bt)$	$\frac{s^2-b^2}{(s^2+b^2)^2}$
$\sin(bt)+bt\cos(bt)$	$\frac{2bs^2}{(s^2+b^2)^2}$
$\sin(bt)-bt\cos(bt)$	$\frac{2b^3}{(s^2+b^2)^2}$

Problem 10.13. Find the Laplace Transforms of the following functions. (please don't do the integration!)

(a) $f(t) = t^2 + 5$

(b) $f(t) = e^{3t+2}$

(c) $f(t) = t^3 e^{4t}$

▲

Solution:

(a) $f(t) = t^2 + 5 \implies \mathcal{L}\{f(t)\} = \frac{2}{s^3} + \frac{5}{s}$

(b) $f(t) = e^{3t+2} = e^2 \cdot e^{3t} \implies \mathcal{L}\{f(t)\} = \frac{e^2}{s-3}$

(c) $f(t) = t^3 e^{4t} \implies \mathcal{L}\{f(t)\} = \frac{6}{(s-4)^4}$

Problem 10.14. Find the Inverse Laplace Transforms of the following functions.

(a) $F(s) = \frac{3}{s^4}$

(b) $F(s) = \frac{3}{s-4}$

▲

Solution:

(a) $F(s) = \frac{3}{s^4} \implies \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2}t^3$

(b) $F(s) = \frac{3}{s-4} \implies \mathcal{L}^{-1}\{F(s)\} = 3e^{4t}$

10.2 Solving Differential Equations with Laplace Transforms

Now we get to the good stuff.

Theorem 10.15. Suppose that $f(t)$ is a continuous piecewise smooth function for $t \geq 0$ such that the Laplace transform of f exists. Under these conditions $\mathcal{L}\{f'(t)\}$ exists and

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

Proof. To prove this theorem consider the following hints:

1. Recall that $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$
2. Therefore, $\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)dt$
3. Now use integration by parts with $u = e^{-st}$ and $dv = f'(t)dt$
4. The result follows after some computation

Now prove the theorem □

Solution: With $u = e^{-st}$ and $dv = f'(t)dt$ we have $du = -se^{-st}dt$ and $v = f(t)$. Therefore

$$\begin{aligned}\mathcal{L}\{f'(t)\} &= f(t)e^{-st}\Big|_{t=0}^{t\to\infty} + s\int_0^\infty e^{-st} f(t)dt \\ &= \lim_{t\to\infty}\left(f(t)e^{-st}\right) - f(0) + s\mathcal{L}\{f(t)\} \\ &= s\mathcal{L}\{f(t)\} - f(0).\end{aligned}$$

Theorem 10.16. Suppose that $f(t)$ is a continuous piecewise smooth function for $t \geq 0$ such that the Laplace transforms of f and f' exist. Under these conditions $\mathcal{L}\{f''(t)\}$ exists and

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

Proof. Prove this theorem:
Hints:

1. Let $g(t) = f'(t)$ and find $\mathcal{L}\{g'(t)\}$
2. The result follows after some computation

□

Solution:

$$\begin{aligned}\mathcal{L}\{f''(t)\} &= \mathcal{L}\{g'(t)\} = s\mathcal{L}\{g(t)\} - g(0)\\ &= s\mathcal{L}\{f'(t)\} - f'(0)\\ &= s(s\mathcal{L}\{f(t)\} - f(0)) - f'(0)\\ &= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)\end{aligned}$$

Theorem 10.17. Suppose that $f(t)$ is a continuous piecewise smooth function for $t \geq 0$ such that the Laplace transforms of f, f', and f'' exist. Under these conditions $\mathcal{L}\{f'''(t)\}$ exists and

$$\mathcal{L}\{f'''(t)\} = s^3\mathcal{L}\{f(t)\} - s^2f(0) - sf'(0) - f''(0)$$

Problem 10.18. Looking at the previous three theorems, what is the Laplace transform of a fourth derivative? ▲

Solution: $s^4\mathcal{L}\{f(t)\} - s^3f(0) - s^2f'(0) - sf''(0) - f'''(0)$

Technique 10.19 (Solving ODEs with Laplace Transforms)**.** Follow these steps to solve a linear ordinary differential equation with Laplace transforms.

1. Take the Laplace transform of both sides.
2. Solve (algebraically) for $X(s)$.
3. Simplify the right-hand side (this typically involves partial fractions)
4. Take the inverse Laplace transform.

Problem 10.20. Solve the following differential equation with Laplace transforms.

$$y' = -4y + 3e^{2t} \quad \text{with} \quad y(0) = 1$$

▲

Solution:

$$\mathcal{L}\{y'\} = -4\mathcal{L}\{y\} + 3\mathcal{L}\left\{e^{2t}\right\} \quad \implies \quad sY - y(0) = -4Y + 3\left(\frac{1}{s-2}\right)$$

$$\implies (s+4)Y = 1 + \frac{3}{s-2} = \frac{s+1}{s-2} \implies Y(s) = \frac{s+1}{(s+4)(s-2)}$$

$$\implies Y(s) = \frac{1}{2(s-2)} + \frac{1}{2(s+4)}$$

$$\implies y(t) = \frac{1}{2}e^{2t} + \frac{1}{2}e^{-4t}$$

Problem 10.21. Solve with Laplace transforms:

$$x'' + 4x = \sin(3t) \quad \text{with} \quad x(0) = x'(0) = 0$$

▲

Solution:

$$\mathcal{L}\{x''\} + 4\mathcal{L}\{x\} = \frac{3}{s^2+9}$$

$$\implies s^2X - sx(0) - x'(0) + 4X = \frac{3}{s^2+9}$$

$$\implies (s^2+4)X = \frac{3}{s^2+9}$$

$$\implies X(s) = \frac{3}{(s^2+4)(s^2+9)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9}$$

$$\implies X(s) = \frac{3}{5}\frac{1}{s^2+4} - \frac{3}{5}\frac{1}{s^2+9}$$

$$\implies X(s) = \frac{3}{10}\frac{2}{s^2+4} - \frac{1}{5}\frac{3}{s^2+9}$$

$$\implies x(t) = \frac{3}{10}\sin(2t) - \frac{1}{5}\sin(3t)$$

Problem 10.22. Use Laplace transforms to show that the solution to the differential equation

$$x'' + 3x' + 2x = t \quad \text{with} \quad x(0) = 0 \quad \text{and} \quad x'(0) = 2$$

is

$$x(t) = 3e^{-t} - \frac{9}{4}e^{-2t} + \frac{t}{2} - \frac{3}{4}$$

▲

Solution:

$$
\begin{aligned}
& x'' + 3x' + 2x = t \\
& \implies \mathcal{L}\{x'' + 3x' + 2x\} = \mathcal{L}\{t\} \\
& \implies \mathcal{L}\{x''\} + 3\mathcal{L}\{x'\} + 2\mathcal{L}\{x\} = \frac{1}{s^2} \\
& \implies s^2\mathcal{L}\{x\} - sx(0) - x'(0) + 3s\mathcal{L}\{x\} - 3x(0) + 2\mathcal{L}\{x\} = \frac{1}{s^2} \\
& \implies s^2X + 3sX + 2X - (s+3)x(0) - x'(0) = \frac{1}{s^2} \\
& \implies (s^2 + 3s + 2)X - 2 = \frac{1}{s^2} \\
& \implies X = \frac{1}{s^2 + 3s + 2}\left(2 + \frac{1}{s^2}\right) \\
& \implies X = \frac{1}{(s+2)(s+1)}\left(\frac{2s^2+1}{s^2}\right).
\end{aligned}
$$

At this point we break apart the fraction using partial fractions. We do this in the hopes that the individual fractions will have recognizable inverse Laplace transforms.

$$
\begin{aligned}
& \frac{2s^2+1}{s^2(s+2)(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2} + \frac{D}{s+1} \\
& \implies 2s^2 + 1 = As(s+2)(s+1) + B(s+2)(s+1) + Cs^2(s+1) + Ds^2(s+2).
\end{aligned}
$$

Taking $s = 0$ we get $B = 1/2$. Taking $s = -2$ we get $9 = -4C$ so $C = -9/4$. Taking $s = -1$ we get $3 = D$. We observe that we cannot use the same technique to find the value of A. However, we observe that A will be the coefficient of an s^3 term. Expanding all of the cubic terms we see that

$$0s^3 = As^3 + Cs^3 + Ds^3 \quad \implies \quad A - \frac{9}{4} + 3 = 0 \quad \implies \quad A = \frac{9}{4} - 3 = -\frac{3}{4}.$$

Therefore we have

$$X(s) = -\frac{3}{4}\cdot\frac{1}{s} + \frac{1}{2}\cdot\frac{1}{s^2} - \frac{9}{4}\cdot\frac{1}{s+2} + 3\cdot\frac{1}{s+1}$$

and the inverse Laplace transforms are now readily apparent so the solution to the ODE is

$$x(t) = -\frac{3}{4} + \frac{t}{2} - \frac{9}{4}e^{-2t} + 3e^{-t}.$$

Problem 10.23. Use Laplace transforms to show that the solution to the differential equation

$$x'' + 6x' + 25x = 0 \quad \text{with} \quad x(0) = 2 \quad \text{and} \quad x'(0) = 3$$

is

$$x(t) = e^{-3t}\left(2\cos(4t) + \frac{9}{4}\sin(4t)\right)$$

▲

Problem 10.24. Use Laplace transforms to show that the solution to the differential equation

$$x'' + 6x' + 18x = \cos(2t) \quad \text{with} \quad x(0) = 1 \quad \text{and} \quad x'(0) = -1$$

is

$$x(t) = \frac{7}{170}\cos(2t) + \frac{3}{85}\sin(2t) + e^{-3t}\left(\frac{163}{170}\cos(3t) + \frac{307}{510}\sin(3t)\right)$$

▲

10.3 The Heaviside Function and Delayed Forcing Terms

Problem 10.25. Let $u(t)$ be defined as

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \ge 0 \end{cases}.$$

Sketch a picture of $u(t)$ to the right of the definition . . . I'll wait while you sketch. . . . Good! The function $u(t)$ is called the Heaviside function (named after a guy who's last name was Heaviside). ▲

Problem 10.26. Now let's define a shifted version, $u_a(t)$, of the Heaviside function as

$$u_a(t) = u(t-a) = \begin{cases} 0, & t < a \\ 1, & t \ge a \end{cases}.$$

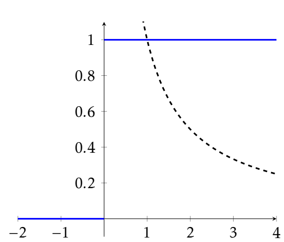
Sketch a picture of this one too . . . I'll wait. ▲

One HUGE advantage to Laplace transforms is that these functions have nice smooth and easy to handle Laplace transforms. Imagine if they showed up on the right-hand side of a differential equation before. What would you have done?!

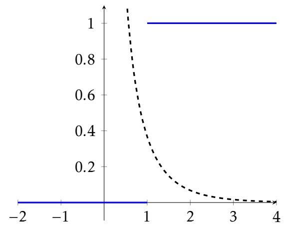
Function $f(t)$	Laplace Transform $F(s) = \mathcal{L}\{f(t)\}$
$u(t)$	$\frac{1}{s}$
$u_a(t)$	$\frac{e^{-as}}{s}$

Let's look at step functions and shifted step functions graphically.

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \ge 0 \end{cases} \rightsquigarrow \mathcal{L}\{u(t)\} = \frac{1}{s}$$



$$u_a(t) = u(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases} \rightsquigarrow \mathcal{L}\{u(t)\} = \frac{e^{-as}}{s}$$



Problem 10.27. Consider the differential equation $x' + \frac{1}{2}x = u_3(t)$ with $x(0) = 3$.

(b) Make a sketch of the solution to the differential equation.

(b) Take the Laplace transform of both sides of the differential equation and solve for $X(s)$. **Solution:**

$$\begin{aligned}
& x' + \frac{1}{2}x = u_3 \\
\implies & \mathcal{L}\{x'\} + \frac{1}{2}\mathcal{L}\{x\} = \mathcal{L}\{u_3\} \\
\implies & sX - x(0) + \frac{1}{2}X = \frac{e^{-3s}}{s} \\
\implies & \left(s + \frac{1}{2}\right)X = 3 + \frac{e^{-3s}}{s} \\
\implies & X = \frac{3}{s + 1/2} + \frac{e^{-3s}}{s(s + 1/2)}
\end{aligned}$$

(c) What do you need to be able to invert the Laplace transform? **Solution:** The first term inverts to $3e^{-0.5t}$ as expected. The second term, on the other hand, doesn't have an inverse transform that we've studied (yet).

▲

Problem 10.28. For each of the following Laplace transforms take the inverse transform and sketch the resulting function.

(a) $X(s) = \dfrac{2}{s^2 + 4} + \dfrac{e^{-4s}}{s}$ **Solution:**

$$x(t) = \sin(2t) + u_4(t)$$

(b) $X(s) = \dfrac{2}{s + 3} + 3\dfrac{e^{-s}}{s}$ **Solution:**

$$x(t) = 2e^{-3t} + 3u_1(t)$$

(c) $X(s) = \frac{e^{-s}}{s} - \frac{e^{-5s}}{s}$ **Solution:**

$$x(t) = u_1(t) - u_5(t)$$

For this one we have $x = 0$ from $t = 0$ to $t = 1$ then $x = 1$ from $t = 1$ to $t = 5$ and then back to zero.

▲

Problem 10.29. Find the laplace transform of the function

$$f(t) = -4u_3(t) - 5u_5(t) + 2u_6(t)$$

and sketch the resulting function. ▲

The reader should observe that the Laplace transform is usually a very smooth (continuous and differentiable) function. This even holds when we have discontinuous functions $f(t)$, and this fact is one of the reasons that the Laplace transform is really powerful: would you rather solve a differential equation with a discontinuous right-hand side or a smooth differentiable right-hand side?

10.4 Impulses and The Delta Function

Next we'll build up the mathematical machinery to understand shifted and impulse-type forcing terms in differential equations. We have already seen the Heaviside function, but what about a function that provides an impulse? Consider this situation: An undamped mass-spring oscillator is oscillating without the influence of any external forces until at a certain time you give the whole apparatus a bump. Before the bump you expect undamped oscillations modeled by trigonometric functions and after the bump you expect the same, but how does the bump change the behavior? The answer to this question lies in understanding the Delta function.

Problem 10.30. What do you suppose the derivative of the Heaviside function looks like? Draw a picture. ▲

Solution: A sensible derivative is zero everywhere except at the break point. At that point the derivative should be infinite since a tangent line (if it were to exist) would be vertical.

Definition 10.31 (The Dirac Delta Function)**.** The Dirac delta function $\delta_a(t)$ is define as

$$\delta_a(t) = \begin{cases} 0, & \text{if } t \neq a \\ \infty, & \text{if } t = a \end{cases}$$

where

$$\int_{-\infty}^{\infty} \delta_a(t)dt = 1.$$

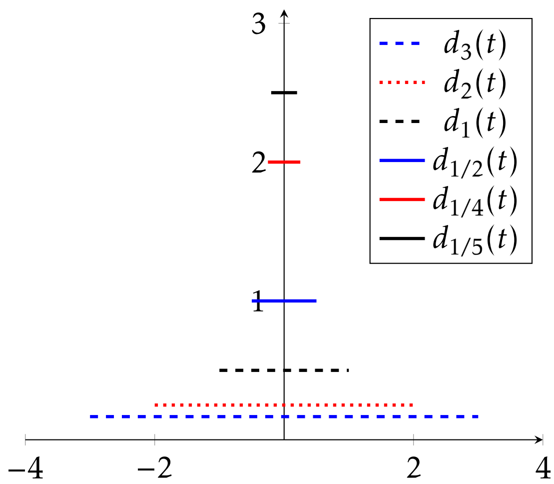


Figure 10.1. The function $d_k(t)$ for several values of k. In this limit this function approximates the Dirac delta function.

Don't think too hard about this since it should become obvious that this definition is kind of nonsense. We have an infinite spike at $t = a$ but the integral itself is actually finite ...strange. That being said, this is the proper definition of the delta function.

Now we're going to work out the Laplace transform of the delta function. This is an important step since the delta function models an impulse; an important concept in engineering and physics.

Define the function $d_k(t)$ as

$$d_k(t) = \begin{cases} \frac{1}{2k}, & \text{if } -k \leq t \leq k \\ 0, & \text{otherwise} \end{cases} .$$

In Figure 10.1 we see that the function $d_k(t)$ will *turn into* the Dirac delta function as k goes to infinity. That is,

$$\lim_{k\to\infty} d_k(t) = \begin{cases} 0, & \text{if } t \neq 0 \\ \infty, & \text{if } t = 0 \end{cases}$$

where for every k we must have

$$\int_{-\infty}^{\infty} d_k(t)dt = 1$$

since the area underneath $d_k(t)$ is a rectangle and is fixed at 1 by construction. Figure 10.1 shows plots of $d_k(t)$ for several values of k. It should be clear to the reader that d_k does indeed converge to the delta function as k approaches infinity.

This is all well and good, but what is the Laplace transform of the Dirac delta function? To answer this question we explore one more property of the delta function

$$\int_{-\infty}^{\infty} \delta_a(t) f(t)dt = f(a). \tag{10.5}$$

In words, (10.5) says that if we take the product of the delta function and a (suitably continuous) function $f(t)$ and integrate over the whole real line then we simply extract the function value of $f(t)$ located at the spike of the delta function. In this sense, the delta function just probes function values.

Problem 10.32. Provide a graphical reason why

$$\int_{-\infty}^{\infty} \delta_a(t) f(t) dt = f(a).$$

▲

From here we get a simple formula for the Laplace transform of the delta function.

$$\mathcal{L}\{\delta_a(t)\} = \int_0^{\infty} e^{-st}\delta_a(t)dt = \int_{-\infty}^{\infty} e^{-st}\delta_a(t)dt = e^{-as}.$$

Theorem 10.33 (Properties of the Dirac Delta Function)**.** Let $\delta_a(t)$ be the shifted Dirac delta function. The delta function has the following properties.

$$\delta_a(t) = \begin{cases} 0, & \text{if } t \neq a \\ \infty, & \text{if } t = a \end{cases}$$

$$\int_{-\infty}^{\infty} \delta_a(t)dt = 1$$

$$\int_{-\infty}^{\infty} \delta_a(t) f(t) dt = f(a)$$

$$\mathcal{L}\{\delta_a(t)\} = e^{-as}$$

$$\mathcal{L}\{\delta_0(t)\} = 1$$

To make the inversion of the Laplace transform of the delta function more useful we finally consider the following theorem.

Theorem 10.34. Let $f(t)$ be a function where $\mathcal{L}\{f(t)\} = F(s)$ exists.

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)f(t-a) = u_a(t)f(t-a).$$

$$\mathcal{L}\{u_a(t)f(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}$$

For example consider the expression $\frac{e^{-2s}}{s+1}$. The inverse Laplace transform of this expression is

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s+3}\right\} = u(t-2)e^{-3(t-2)} = u_2(t)e^{-3(t-2)}.$$

Problem 10.35. Explain what Theorem 10.34 says and create a few examples of this theorem in action. ▲

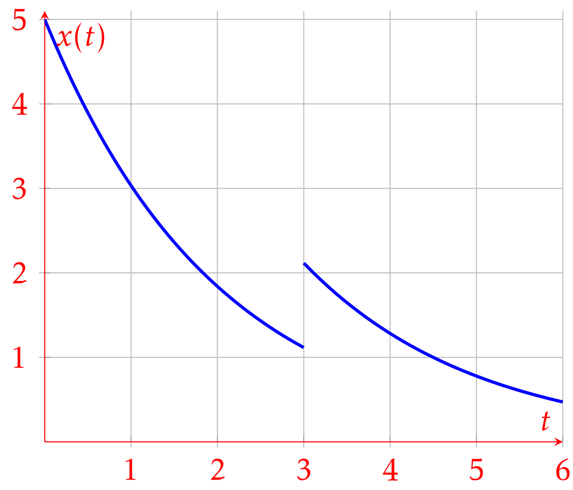
Problem 10.36. Solve the differential equation

$$x' + \frac{1}{2}x = \delta_3(t)$$

with the initial condition $x(0) = 5$. Draw a picture of your solution and explain what happened. ▲

Solution:

$$\begin{aligned}
&x' + \frac{1}{2}x = \delta_3(t) \\
&\implies sX - x(0) + \frac{1}{2}X = e^{-3s} \\
&\implies \left(s + \frac{1}{2}\right)X = 5 + e^{-3s} \\
&\implies X = \frac{5}{s+1/2} + \frac{e^{-3s}}{s+1/2} \\
&\implies x(t) = 5e^{-0.5t} + u_3(t)e^{-0.5(t-3)}
\end{aligned}$$



Problem 10.37. Let's return to the differential equation $x' + \frac{1}{2}x = u_3(t)$ with $x(0) = 5$.

(a) Return to your notes from Problem 10.27 and recall our conjecture for the plot of the solution.

(b) Take the Laplace transform of both sides and rearrange to solve for $X(s) = \mathcal{L}\{x(t)\}$.
Solution: From Problem 10.27 recall that

$$X(s) = \frac{5}{s+1/2} + \frac{e^{-3s}}{s(s+1/2)}$$

Using the idea of partial fraction we can rewrite the second fraction as

$$\frac{e^{-3s}}{s(s+1/2)} = \frac{Ae^{-3s}}{s} + \frac{Be^{-3s}}{s+1/2}$$

so if we cancel the e^{-3s} from both sides and clear the fractions we get

$$1 = A(s + 1/2) + Bs$$

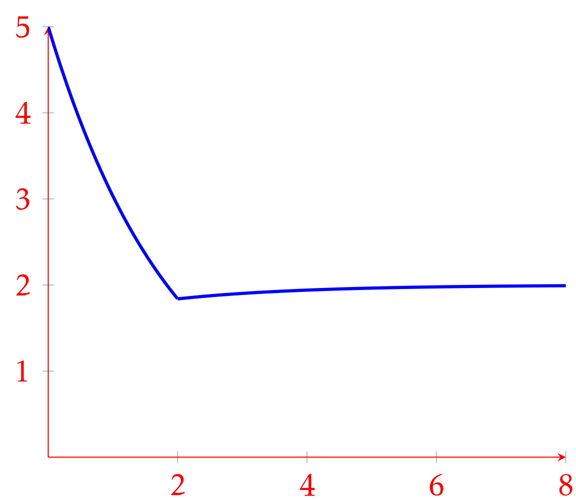
which means that $A = 2$ and $B = -2$. Therefore,

$$X(s) = \frac{5}{s + 1/2} + 2\frac{e^{-3s}}{s} - 2\frac{e^{-3s}}{s + 1/2}$$

(c) Take the inverse Laplace transform now with the help of Theorem 10.34. **Solution:**
The first term inverts to $5e^{-(1/2)t}$.
The second term inverts to $2u_3(t)$
The third term inverts to $-2u_3(t)e^{-(1/2)(t-2)}$

(d) Use MATLAB to create a plot of the solution.
Hint: the Heaviside function is the `heaviside` command in MATLAB. **Solution:**

```
t = 0:0.001:15;
x1 = 5*exp(-(1/2)*t);
x2 = 2*heaviside(t-2);
x3 = -2*heaviside(t-2).*exp(-(1/2)*(t-2));
plot(t,x1+x2+x3)
```



▲

Problem 10.38. Find the Laplace transform of

$$f(t) = \begin{cases} 0, & t < 3 \\ (t-3)^2, & t \geq 3 \end{cases}$$

▲

Problem 10.39. Find the Laplace transform of

$$f(t) = u_5(t)e^{-(t-5)}.$$

▲

Solution:

$$\mathcal{L}\left\{u_5(t)e^{-(t-5)}\right\} = \frac{e^{-5s}}{s+1}$$

Problem 10.40. Find $\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s+2}\right\}$ ▲

Solution:

$$\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s+2}\right\} = u_3(t)e^{-2(t-3)}$$

Problem 10.41. We have a system modeled as an undamped harmonic oscillator that begins at equilibrium and at rest, so $y(0) = y'(0) = 0$. The systems receives a unit impulse force at $t = 4$ so that it is modeled by the differential equation

$$y'' + 9y = \delta_4(t).$$

Find $y(t)$. ▲

Solution:

$$y(t) = u_4(t)\sin(3(t-4)).$$

Problem 10.42. Create a differential equation that can only be solved analytically using Laplace transforms. Be sure to provide sufficient initial conditions. After you've written your problem trade with another group and solve the other group's problem. ▲

Solution: Any linear differential equation that contains heaviside or delta functions will work.

10.5 Convolutions (INCOMPLETE)

. . . in this section we talk about convolutions . . . later

Chapter 11

Power Series Method: The Ultimate Guess

Solution techniques for differential equations lead most to believe that there is a certain amount of educated guesswork to get started writing a solution. To some extent this observation is correct! Think about the method of undetermined coefficients; built on an educated guess. Why don't we just make the ultimate guess:
If $y(t)$ is a solution to a differential equation and so long as y is expected to have a Taylor series representation, then why don't we just guess that $y(t)$ is a Taylor series and use some detective work to determine the coefficients. To demonstrate this consider the next problem.

11.1 Taylor Series Solutions to Diff. Equations

In the following problems we are assuming that the solution to the differential equation can be written in terms of a Taylor series centered at $t = 0$

$$y(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^n.$$

Problem 11.1. Consider the differential equation $y' = y$ with $y(0) = 1$.

(a) Solve this differential equation using any appropriate technique. **Solution:** $y(t) = e^t$

(b) Let's assume that $y(t) = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} t^n$ (a Taylor series centered at $t = 0$). Expanding the Taylor series we see that

$$y(t) = y(0) + y'(0)t + \frac{y''(0)}{2!}t^2 + \frac{y'''(0)}{3!}t^3 + \frac{y^{(iv)}(0)}{4!}t^4 + \cdots$$

From the initial condition we know that $y(0) = 1$ so we at least know that

$$y(t) = 1 + y'(0)t + \frac{y''(0)}{2!}t^2 + \frac{y'''(0)}{3!}t^3 + \frac{y^{(iv)}(0)}{4!}t^4 + \cdots$$

Using the differential equation determine $y'(0)$. **Solution:** $y'(0) = y(0) = 1$ so the Taylor series starts as $y(t) = 1 + t + \cdots$

(c) You should now have the first two terms in the Taylor series. Differentiate both sides of the differential equation and use your answer to determine $y''(0)$. **Solution:** Since $y' = y$ we see that $y'' = y'$ so $y''(0) = y'(0) = 1$ and hence the Taylor series is now

$$y(t) = 1 + t + \frac{1}{2}t^2 + \cdots$$

(d) Finally, use what you did in part (c) to determine the rest of the Taylor series. Verify your answer off of part (a). **Solution:**

$$y(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \cdots = e^t.$$

▲

The previous problem suggests a technique for building the Taylor series of a solution to a differential equation. Let's put it into action on another problem that isn't quite as easy.

Problem 11.2. Consider the differential equation

$$y' - ty = 1$$

with initial condition $y(0) = 1$.

(a) This problem would traditionally require integrating factors. Start the process of integrating factors and work the procedure until you get to an integral that cannot be evaluated. **Solution:**
For integrating factors we let $\rho(t) = e^{\int -t dt} = e^{-t^2/2}$. If we multiply both sides of the differential equation by this integrating factor and rearrange the left-hand side then

$$\frac{d}{dt}\left[e^{-t^2/2}y\right] = e^{-t^2/2}.$$

If we integrate both sides of the differential equation then

$$e^{-t^2/2}y = \int e^{-t^2/2} dt.$$

The right-hand side, however, does not have a known analytic antiderivative. Hence we have to stop here.

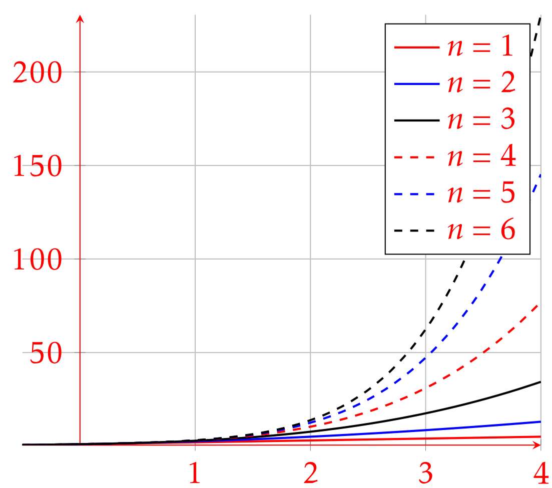
(b) Now rearrange the differential equation to solve for y' and use that rearrangement to determine $y'(0)$, $y''(0)$, $y'''(0)$, etc. **Solution:**

$$\begin{aligned}
y(0) &= 1 \\
y' &= ty + 1 \\
\implies y'(0) &= 0 \cdot y(0) + 1 = 1 \\
y''(t) &= ty'(t) + y(t) \\
\implies y''(0) &= 0 \cdot y'(0) + y(0) = 1 \\
y'''(t) &= (ty''(t) + y'(t)) + y'(t) = ty''(t) + 2y'(t) \\
\implies y'''(0) &= 0 \cdot 1 + 2 = 2 \\
y^{(iv)} &= (ty'''(t) + y''(t)) + 2y''(t) = ty'''(t) + 3y''(t) \\
\implies y^{(iv)}(0) &= 0 \cdot 2 + 3 \cdot 1 = 3 \\
y^{(v)} &= \left(ty^{(iv)} + y'''\right) + 3y''' = ty^{(iv)} + 4y''' \\
\implies y^{(v)}(0) &= 0 \cdot 3 + 4 \cdot 2 = 8 \\
y^{(vi)} &= \left(ty^{(v)} + y^{(iv)}\right) + 4y^{(iv)} = ty^{(v)} + 5y^{(iv)} \\
\implies y^{(vi)}(0) &= 0 \cdot 8 + 5 \cdot 3 = 15
\end{aligned}$$

(c) Write the Taylor series solution for the differential equation. **Solution:**

$$y(t) = 1 + t + \frac{1}{2}t^2 + \frac{2}{3!}t^3 + \frac{3}{4!}t^4 + \frac{8}{5!} + t^5 + \cdots$$

(d) Use a plotting tool to create a plot of the approximate Taylor series solution using the first several terms in the Taylor series. **Solution:**



▲

Problem 11.3. Summarize the technique of using Taylor series to approximate the solution to a differential equation. ▲

Problem 11.4. Use the Taylor series method to estimate a solution to the differential equation

$$y'' - 2ty' + 2y = 0$$

with $y(0) = 1$ and $y'(0) = -1$. ▲

Solution: Since $y(0) = 1$ and $y'(0) = -1$ the Taylor series starts as

$$y(t) = 1 - t + \cdots.$$

Since $y'' - 2ty + 2y = 0$ we can rewrite to get $y'' = 2ty' - 2y$. Hence $y''(0) = 2(0)(-1) - 2(1) = -2$. Therefore,

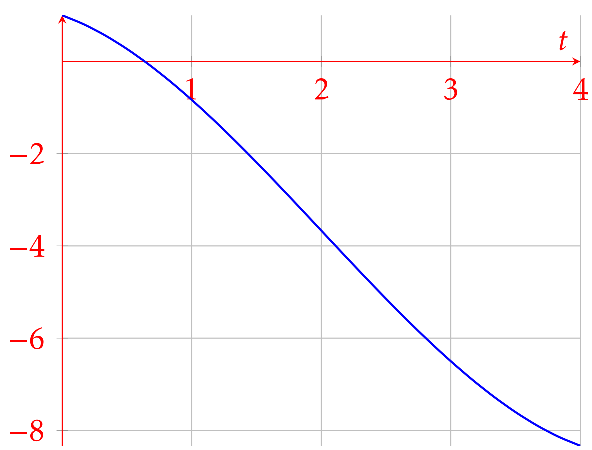
$$y(t) = 1 - t - \frac{2}{2!}t^2 + \cdots.$$

Next, $y'''(t) = 2ty'' + 2y' - 2y' = 2ty''$ so $y'''(0) = 2(0)(-2) = 0$. Therefore,

$$y(t) = 1 - t - t^2 + 0t^3 + \cdots.$$

For the next step, $y^{(iv)} = 2ty''' + 2y''$ so $y^{(iv)}(0) = 2(0)(0) + 2(-2) = 4$ so

$$y(t) = 1 - t - t^2 + 0t^3 + \frac{4}{4!}t^4 + \cdots$$



11.2 Radius of Convergence for Power Series (INCOMPLETE)

A power series is an infinite series of power functions

$$y(t) = \sum_{n=0}^{\infty} a_n t^n.$$

We observe that the Taylor series discussed previously is just a special kind of power series, but that is not the only kind that is interesting. Power series, in general, can be used to build up functions that cannot be written in terms of regular algebraic or trigonometric functions. The "infinity" in the upper bound on the sum, however, could cause some problems. It is not generally true to say that any power series will create a new function. For poor choices of the coefficients a_n the sum may diverge to infinity for all values of t. What we need is an arsenal of tools to determine whether or not a power series actually converges to something meaningful or if it diverges off to infinity. The primary tools are summarized in the following problems and theorems.

Problem 11.5. We start this investigation just looking at sums of numbers. Write computer code to determine the sums

$$\sum_{n=0}^{5} \frac{n!}{2^n} = \underline{\hspace{3cm}}$$
$$\sum_{n=0}^{50} \frac{n!}{2^n} = \underline{\hspace{3cm}}$$
$$\sum_{n=0}^{500} \frac{n!}{2^n} = \underline{\hspace{3cm}}$$
$$\sum_{n=0}^{5000} \frac{n!}{2^n} = \underline{\hspace{3cm}}$$

and now make a conjecture: Does the series

$$\sum_{n=0}^{\infty} \frac{n!}{2^n}$$

converge to a finite value or diverge to infinity? ▲

Solution: This clearly diverges

Problem 11.6. Now let's consider a more interesting series that maybe isn't so obvious. Write computer code to determine the sums

$$\sum_{n=0}^{5} \frac{n^2}{(2n-1)!} = \underline{\hspace{3cm}}$$
$$\sum_{n=0}^{50} \frac{n^2}{(2n-1)!} = \underline{\hspace{3cm}}$$
$$\sum_{n=0}^{500} \frac{n^2}{(2n-1)!} = \underline{\hspace{3cm}}$$
$$\sum_{n=0}^{5000} \frac{n^2}{(2n-1)!} = \underline{\hspace{3cm}}$$

and now make a conjecture: Does the series

$$\sum_{n=0}^{\infty} \frac{n^2}{(2n-1)!}$$

converge to a finite value or diverge to infinity? ▲

Solution: This series converges.

Problem 11.7. Each of the previous two problems were written in the form $\sum_{n=0}^{\infty} a_n$ where a_n is the sequence of numbers being summed.

1. For the sum
$$\sum_{n=0}^{\infty} \frac{n!}{2^n}$$
determine a_n and find the limit
$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$
Solution: The limit is infinite

2. For the sum
$$\sum_{n=0}^{\infty} \frac{n^2}{(2n-1)!}$$
determine a_n and find the limit
$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$
Solution: The limit is zero

▲

Problem 11.8. Consider the series

$$\sum_{n=0}^{\infty} \frac{n^2+2n+1}{3^n+2}.$$

Write computer code that finds sums for successivaly larger and larger upper bounds. Use this computer code to conjecture whether this series converges or diverges. Finally, evaluate the limit

$$\lim \left| \frac{a_{n+1}}{a_n} \right|$$

where $a_n = \frac{n^2+2n+1}{3^n+2}$. ▲

Solution: The series converges and the limit is 1/3.

The *ratio test* that follows is a test to determine if a series will converge or diverge.

Theorem 11.9 (The Ratio Test)**.** *Let a_n be a sequence of numbers and consider the sum $\sum_{n=0}^{\infty} a_n$.*

- *If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$ then the sum converges to a finite value.*
- *If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$ then the sum diverges to infinity.*
- *If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$ then the sum may either converge or diverge.*

Problem 11.10. Use the ratio test to determine whether the following series converge or diverge.

$$\sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$$
$$\sum_{n=0}^{\infty} \frac{1}{n!}$$
$$\sum_{n=1}^{\infty} \frac{n^n}{n!}$$
$$\sum_{n=1}^{\infty} \frac{7^{n+2}}{2n6^n}$$

▲

Now we switch back to our study of power series. It is not generally true that we can just build a power series and we get a meaningful result. There might only be a small region where the series converges. Consider the next problem.

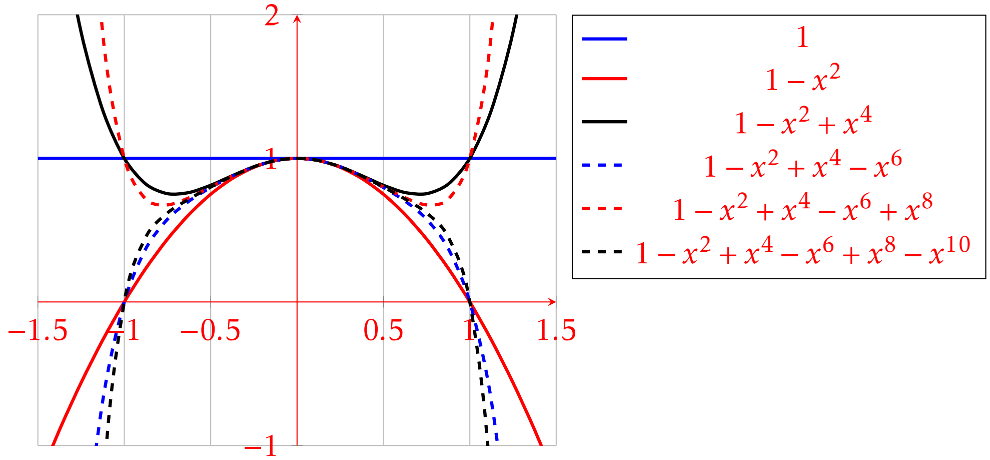
Problem 11.11. Consider the power series

$$f(x) = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

(a) Write computer code that plots a sequence of functions

$$1, \quad 1-x^2, \quad 1-x^2+x^4, \quad 1-x^2+x^4-x^6, \quad \ldots.$$

Based on your sequence of plots where does the power series converge? **Solution:**



It seems like the series is converging in $-1 < x < 1$.

(b) Let $a_n = (-1)^n x^{2n}$ and evaluate the limit

$$\lim \left| \frac{a_{n+1}}{a_n} \right|.$$

Solution: This limit evaluates to $|x|$

(c) Based on your knowledge of the ratio test, for what values of x does the series $\sum_{n=0}^{\infty} (-1)^n x^{2n}$ converge? **Solution:** This series only converges for $|x| < 1$

▲

Problem 11.12. Repeat problem 11.11 with the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$

Back up any conjectures that you make from the plots with the use of the ratio test. ▲

Solution: This is the Taylor series for e^x and the series converges for all $x \in \mathbb{R}$.

Problem 11.13. Repeat problem 11.11 with the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n n}{4^n} (x+3)^n.$$

Back up any conjectures that you make from the plots with the use of the ratio test. ▲

Solution: This series converges on $-7 < x < 1$.

Theorem 11.14 (Radius of Convergence of Power Series)**.** Given a power series $\sum a_n t^n$ and the limit

$$R(x) = \lim_{n\to\infty} \left| \frac{c_{n+1}x^{n+1}}{c_n x^n} \right|$$

exists then

- for all x such that $R(x) < 1$ the power series converges,
- for all x such that $R(x) > 1$ the power series diverges.

11.3 Power Series Solutions to Diff. Equations

Using Taylor series as we did in Section 11.1 is one technique for finding a series solution to differential equations. In this section we will look at a more general technique: using power series to build solutions. This technique streamlines the Taylor series technique and actually involves must less work for some problems. Instead of saying that $y(t)$ is a Taylor series, $y(t) = \sum_{n=0}^{\infty} \frac{y^{(n)}}{n!} t^n$ we simply start by saying that $y(t)$ is just a power series with an unknown sequences of coefficients:

$$y(t) = \sum_{n=0}^{\infty} a_n t^n.$$

Our detective work from this assumption amounts to massaging the power series and the differential equation to determine a pattern for the sequence a_n.

In order to use power series to build solutions to differential equations we need to be able to differentiate (and integrate) power series in a meaningful way. Thankfully, we are blessed with the following theorem from mathematical analysis.

Theorem 11.15. Let a_n be a sequence of real numbers such that the series $\sum_{n=0}^{\infty} a_n x^n$ converges for $x \in (-R, R)$. The number R is called the radius of convergence for the power series.

1. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ then $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$
2. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ then $f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$
3. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ then $f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n x^{n-k}$
4. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ then $\int f(x)dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} + C$

Proof. Really, the full proof of this theorem is beyond the scope of this text. Instead, convince yourself that this theorem is true for finite sums (this should be obvious) and trust the author that it holds within the radius of convergence for infinite sums. $\square$

Really, Theorem 11.15 says that so long as you are within the radius of convergence of the power series then the series can be differentiated or integrated term by term. This has an impact on the use of power series for approximating solutions to differential equations. We simply assume that we are within the radius of convergence and proceed with determining the sequence a_n that defines the power series. Let's look at an example.

Example 11.16. Consider the differential equation $y' = -0.5y$ with $y(0) = 1$ and assume that $y(t) = \sum_{n=0}^{\infty} a_n t^n$. Find the sequence a_n that defines the power series.
Solution: This is a simple differential equation and we know that solution is $y(t) = e^{-0.5t}$. Let's build the power series.
Assume that $y(t) = \sum_{n=0}^{\infty} a_n t^n$ so we see that since $y' = -0.5y$,

$$\sum_{n=1}^{\infty} n a_n t^{n-1} = -\frac{1}{2}\sum_{n=0}^{\infty} a_n t^n.$$

Expanding both sides of the summation and then matching terms we get

$$1a_1 + 2a_2t^1 + 3a_3t^2 + 4a_4t^3 + 5a_5t^4 + \cdots = -\frac{1}{2}\left(a_0 + a_1t^1 + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5 + \cdots\right)$$
$$\implies 0 = \left(-\frac{1}{2}a_0 - a_1\right) + \left(-\frac{1}{2}a_1 - 2a_2\right)t + \left(-\frac{1}{2}a_2 - 3a_3\right)t^2 + \left(-\frac{1}{2}a_3 - 4a_4\right)t^3 + \cdots$$
$$\implies a_1 = -\frac{1}{2}a_0, \quad a_2 = -\frac{1}{4}a_1 = \frac{1}{8}a_0, \quad a_3 = -\frac{1}{6}a_2 = -\frac{1}{48}a_0, \quad a_4 = -\frac{1}{8}a_3 = \frac{1}{384}a_0, \quad \ldots$$

Since $y(0) = 1$ we know that $a_0 = 1$ so

$$a_0 = 1, \quad a_1 = -\frac{1}{2}, \quad a_2 = \frac{1}{8}, \quad a_3 = -\frac{1}{48}, \quad a_4 = \frac{1}{384}, \quad \ldots$$

We conclude by writing the Taylor series approximation

$$y(t) = 1 - \frac{1}{2}t + \frac{1}{8}t^2 - \frac{1}{48}t^3 + \frac{1}{384}t^4 + \cdots.$$

We can observe that this Taylor series can be rewritten as

$$y(t) = 1 + \left(-\frac{t}{2}\right) + \frac{1}{2!}\left(-\frac{t}{2}\right)^2 + \frac{1}{3!}\left(-\frac{t}{2}\right)^3 + \frac{1}{4!}\left(-\frac{t}{2}\right)^4 + \cdots = e^{-0.5t}$$

hence recognizing the analytic solution that we knew from separation of variables.

Problem 11.17. Assume that y is represented as a power series $y(t) = \sum_{n=0}^{\infty} a_n t^n$ and find the coefficients a_n for the solution to the differential equation $y' - ty = 1$ with $y(0) = 1$. ▲

Solution: Assuming that y is a power series we get

$$\sum_{n=1}^{\infty}\left(n a_n t^{n-1}\right) - t\sum_{n=0}^{\infty}\left(a_n t^n\right) = 1.$$

Expanding the left-hand side we get

$$\begin{aligned}
&\left(a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4 + \cdots\right) - \left(a_0 t + a_1 t^2 + a_2 t^3 + a_3 t^4 + a_4 t^5 + \cdots\right) = 1 \\
&\implies a_1 + (-a_0 + 2a_2)t + (-a_1 + 3a_3)t^2 + (-a_2 + 4a_4)t^3 + (-a_3 + 5a_5)t^4 + \cdots = 1 \\
&\implies a_1 = 1, \quad 2a_2 = a_0, \quad 3a_3 = a_1, \quad 4a_4 = a_2, \quad \cdots, (k+2)a_{k+2} = a_k \\
&\implies a_{k+2} = \frac{a_k}{k+2}
\end{aligned}$$

From the initial condition we know that $a_0 = 1$ so

$$a_0 = a_1 = 1$$

$$a_2 = \frac{1}{2}, \quad a_3 = \frac{1}{3}, \quad a_4 = \frac{1}{8}, \quad a_5 = \frac{1}{15}, \quad a_6 = \frac{1}{48}, \quad \cdots$$

and the power series solution is

$$y(t) = 1 + t + \frac{1}{2}t^2 + \frac{1}{3}t^3 + \frac{1}{8}t^4 + \frac{1}{15}t^5 + \frac{1}{48}t^6 + \cdots.$$

Problem 11.18. Assume that y is represented as a power series $y(t) = \sum_{n=0}^{\infty} a_n t^n$ and find the coefficients a_n for the solution to the differential equation

$$t^2 y'' + t y' + t^2 y = 0 \quad \text{with} \quad y(0) = 1.$$

This function is called a **Bessel function of the first kind** and shows up in the study of wave phenomenon. Use a graphing utility (and enough terms in the series) to show a plot of $y(t)$ on the domain $t \in [0, 6]$. ▲

Solution: We assume that $y(t)$ can be written as a power series and expand each term on the left-hand side of teh differential equation.

$$\begin{aligned}
t^2 y'' &= t^2 \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} \\
&= t^2 \left((2)(1)a_2 + (3)(2)a_3 t + (4)(3)a_4 t^2 + \cdots + (k)(k-1)a_k t^{k-2} + \cdots\right) \\
&= (2)(1)a_2 t^2 + (3)(2)a_3 t^3 + (4)(3)a_4 t^4 + (5)(4)a_5 t^5 + \cdots + (k)(k-1)a_k t^k + \cdots \\
t y' &= t \sum_{n=1}^{\infty} n a_n t^{n-1} \\
&= t\left(1a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4 + \cdots + k a_k t^{k-1}\right) \\
&= a_1 t + 2a_2 t^2 + 3a_3 t^3 + 4a_4 t^4 + 5a_5 t^5 + \cdots + k a_k t^k + \cdots \\
t^2 y &= t^2 \sum_{n=0}^{\infty} a_n t^n \\
&= a_0 t^2 + a_1 t^3 + a_2 t^4 + a_3 t^5 + a_4 t^6 + \cdots + a_{k-2} t^k + \cdots
\end{aligned}$$

Adding these three terms we see that the k^{th} power of t has coefficient $(k)(k-1)a_k + (k)a_k + a_{k-2}$ and the fact that the right-hand side of the differential equation is zero implies that $k^2 a_k = -a_{k-2}$. Hence the coeeficients in the power series are

$$a_k = \frac{-a_{k-2}}{k^2}.$$

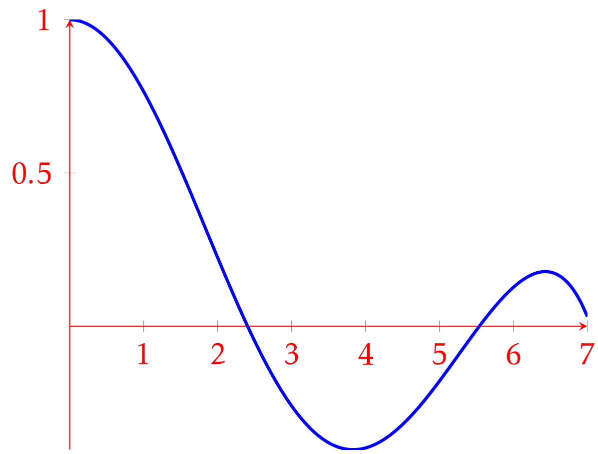
Hence the coefficients of the power series are

$$a_0 = 1, \quad a_1 = 0, \quad a_2 = -\frac{a_0}{2^2} = -\frac{1}{4}, \quad a_3 = -\frac{a_1}{3^2} = 0, \quad a_4 = -\frac{a_2}{4^2} = \frac{a_0}{2^2 4^2} = \frac{1}{2^2 4^2}$$

$$a_5 = -\frac{a_3}{5^2} = 0, \quad a_6 = -\frac{a_4}{6^2} = -\frac{a_0}{2^2 4^2 6^2} = -\frac{1}{2^2 4^2 6^2}, \quad \cdots$$

It is clear that each odd indexed coefficient will be zero ($a_1 = a_3 = a_5 = a_7 = \cdots = 0$) and for each even index

$$a_{2k} = \frac{(-1)^k}{(2k)^2(2k-2)^2(2k-4)^2 \cdots 2^2}$$



Problem 11.19. Write computer code to generate the plot of the Bessel function in the previous problem up to any amount of accuracy. ▲

Problem 11.20. Use power series to solve the equation $y'' + ty = 0$ with $y(0) = 1$ and $y'(0) = 1$. This differential equation gives rise to a function called the **Airy equation**. We saw it as the solution to one of the lab problems at the beginning of the semester. ▲

Solution:

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} = (2)(1)a_2 + (3)(2)a_3 t + (4)(3)a_4 t^2 + (5)(4)a_5 t^3 + \cdots$$

$$ty = \sum_{n=0}^{\infty} a_n t^{n+1} = a_0 t + a_1 t^2 + a_2 t^3 + a_3 t^4 + \cdots.$$

From here we see that

$$0 = 2a_2 + ((3)(2)a_3 + a_0)\,t + ((4)(3)a_4 + a_1)\,t^2 + ((5)(4)a_5 + a_2)\,t^3 + \cdots ((k)(k-1)a_k + a_{k-3})\,t^{k-2} + \cdots$$

From the initial conditions we know that $a_0 = a_1 = 1$. From the previous equation we also know that $a_2 = 0$. Generally,

$$a_k = \frac{-a_{k-3}}{k(k-1)} \quad \text{for} \quad k \geq 3.$$

Therefore,

$$a_0 = 1, \quad a_1 = 1, \quad a_2 = 0, \quad a_3 = -\frac{a_0}{(3)(2)} = -\frac{1}{(3)(2)}, \quad a_4 = -\frac{a_1}{(4)(3)} = -\frac{1}{(4)(3)}, \ldots$$

Problem 11.21. Find the first four terms in the power series expansion of the solution to the differential equation

$$\left(t^2+1\right)y'' - 4ty' + 6y = 0 \quad \text{with} \quad y(0) = y'(0) = 1.$$

▲

Solution: Assuming that $y(t) = \sum_{n=0}^{\infty} a_n t^n$ we get the following solution. From the initial conditions we know that $a_0 = a_1 = 1$. We'll start breaking the differential equation apart term by term.

$$\begin{aligned}
\left(t^2+1\right)y'' &= \left(t^2+1\right)\left((2)(1)a_2 + (3)(2)a_3 t + (4)(3)a_4 t^2 + (5)(4)a_5 t^3 + \cdots\right) \\
&= \left((2)(1)a_2 t^2 + (3)(2)a_3 t^3 + (4)(3)a_4 t^4 + (5)(4)a_5 t^5 + \cdots\right) \\
&\quad + \left((2)(1)a_2 + (3)(2)a_3 t + (4)(3)a_4 t^2 + (5)(4)a_5 t^3 + \cdots\right) \\
-4ty' &= (-4t)\left((1)a_1 + (2)a_2 t + (3)a_3 t^2 + (4)a_4 t^3 + \cdots\right) \\
&= -4a_1 t - (4)(2)a_2 t^2 - (4)(3)a_3 t^3 - (4)(4)a_4 t^4 + \cdots \\
&= -4t - (4)(2)a_2 t^2 - (4)(3)a_3 t^3 - (4)(4)a_4 t^4 + \cdots \\
-6y &= -6a_0 - 6a_1 t - 6a_2 t^2 - 6a_3 t^3 - 6a_4 t^4 - 6a_5 t^5 - \cdots \\
&= -6 - 6t - 6a_2 t^2 - 6a_3 t^3 - 6a_4 t^4 - 6a_5 t^5 - \cdots
\end{aligned}$$

Gathering like terms in the differential equation we see that

$$0 = (-6 + 2a_2) + (-6 - 4 + 6a_3)\,t + (-6a_2 - 8a_2 + 12a_4 + 2a_2)\,t^2 + \cdots$$

which implies that $a_2 = 3$, $a_3 = \frac{5}{3}$, and $a_4 = 3$. Therefore

$$y(t) \approx 1 + t + 3t^2 + \frac{5}{3}t^3 + 3t^4 + \cdots.$$

Chapter 12

Partial Differential Equations

This brief document contains class notes, explanations, examples, and problems for our brief introduction to partial differential equations (PDEs). The study of PDEs spans all fields of the mathematical sciences, and like ODEs, PDEs are the language used by scientists to model change. The change this time happens simultaneously in all three spatial dimensions as well as time.

12.1 Some Reminders from Multivariable Calculus

Problem 12.1. Let $f(x,y)$ be a differentiable multivariable function. Which of the following is the gradient of f?

(a) $\nabla f = \left\langle \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial x} \right\rangle$

(b) $\nabla f = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}$

(c) $\nabla f = \left\langle \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y} \right\rangle$

(d) $\nabla f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2}$

▲

Problem 12.2. Let $\mathcal{F}(x,y)$ be a vector function so that

$$\mathcal{F}(x,y) = \langle f_1(x,y), f_2(x,y) \rangle.$$

Which of the following is the divergences of $\mathcal{F}$?

(a) $\nabla \cdot \mathcal{F} = \dfrac{\partial f_1}{\partial x} + \dfrac{\partial f_2}{\partial y}$

(b) $\nabla \cdot \mathcal{F} = \dfrac{\partial^2 f_1}{\partial x^2} + \dfrac{\partial^2 f_2}{\partial y^2}$

(c) $\nabla \cdot \mathcal{F} = \left\langle \dfrac{\partial f_1}{\partial y} + \dfrac{\partial f_2}{\partial x}, \dfrac{\partial f_1}{\partial x} + \dfrac{\partial f_2}{\partial y} \right\rangle$

(d) $\nabla \cdot \mathcal{F} = \left\langle \dfrac{\partial f_1}{\partial x}, \dfrac{\partial f_2}{\partial y} \right\rangle$

▲

Problem 12.3. Let $f(x,y)$ be a twice differentiable function. Which of the following is the result of taking the divergence of the gradient of f?

(a) $\nabla \cdot \nabla f = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}$

(b) $\nabla \cdot \nabla f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2}$

(c) $\nabla \cdot \nabla f = \left\langle \dfrac{\partial^2 f}{\partial x^2}, \dfrac{\partial^2 f}{\partial y^2} \right\rangle$

(d) $\nabla \cdot \nabla f = \left\langle \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y} \right\rangle$

▲

Problem 12.4. The last quarter of multivariable calculus contains some beautiful properties from vector calculus. Of particular interest here is the divergence theorem:

$$\iint \mathbf{q} \cdot n dA = \iiint \nabla \cdot \mathbf{q} dV.$$

In words, this theorem says

(a) The flux of a vector field **q** through the surface of an object is equal to how the vector field **q** spreads out within the object.

(b) The amount of work done by the vector field **q** is equal to how the vector field **q** curls within the object.

(c) The amount of work gained or lost by traveling around the exterior of the object is equal to the amount that the vector field **q** spreads out within the object.

(d) The flux of a vector field **q** through the surface of an object is equal to how the vector field **q** curls within the object.

▲

Problem 12.5. The heat equation (which we will derive in a bit) is

$$\frac{\partial u}{\partial t} = k \nabla \cdot \nabla u.$$

If $u(x,y,z,t)$ is the temperature of an object then what does the heat equation say in words?
▲

12.2 Where to PDEs Come From?

This somewhat lengthy section is meant to be an introduction to many of the primary partial differential equations of interest in basic mathematical physics.

We start with a brief derivation of a *general conservation law*. The result being a partial differential equation that can be used for conservation of mass, momentum, or energy. Let u be the quantity you are trying to conserve, $\mathbf{q}$ be the flux of that quantity, and f be any source of that quantity. For example, if we are to derive a conservation of energy equation, u might be energy, $\mathbf{q}$ might be temperature flux, and f might be a temperature source (or sink).

12.2.1 Derivation of General Balance Law

Let Ω be a fixed volume and denote the boundary of this volume by $\partial\Omega$. The rate at which u is changing in time throughout Ω needs to be balanced by the rate at which u leaves the volume plus any sources of u. Mathematically, this means that

$$\frac{\partial}{\partial t}\iiint\limits_{\Omega} u\, dV = -\iint\limits_{\partial\Omega} \mathbf{q}\cdot n\, dA + \iiint\limits_{\Omega} f\, dV. \tag{12.1}$$

This is a global balance law in the sense that it holds for all volumes Ω. The troubles here are two fold: (1) there are many integrals, and (2) there are really two variables (u and q since $f = f(u,x,t)$) so the equation is not closed. In order to mitigate that fact we apply the divergence theorem to get

$$\frac{\partial}{\partial t}\iiint\limits_{\Omega} u\, dV = -\iiint\limits_{\Omega} \nabla\cdot\mathbf{q}\, dV + \iiint\limits_{\Omega} f\, dV. \tag{12.2}$$

Gathering all of the terms on the right of (12.2), interchanging the integral and the derivative on the left (since the volume is not changing in time), and rewriting gives

$$\iiint\limits_{\Omega}\left(\frac{\partial u}{\partial t} + \nabla\cdot\mathbf{q}\right) dV = \iiint\limits_{\Omega} f\, dV \tag{12.3}$$

If we presume that this equation holds for all volumes Ω then the integrands must be equal and we get the local balance law

$$\boxed{\frac{\partial u}{\partial t} + \nabla\cdot\mathbf{q} = f.} \tag{12.4}$$

In particular, the physics of energy, momentum, and mass transport are governed by (12.4). In each of these instances we need to have a suitable functional form of the flux $\mathbf{q}$. In the following subsection we will discuss one common form of $\mathbf{q}$.

12.2.2 Simplifications of the Local Balance Law

If equation (12.4) it is often assumed that the system is free of external sources. In this case we set f to zero and obtain the source-free balance law

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{q} = 0. \tag{12.5}$$

It is this form of balance law where many of the most interesting and important partial differential equations come from. In particular consider the following two cases: mass balance and energy balance.

Mass Balance

In mass balance we take u to either be the density of a substance (e.g. in the case of liquids) or the concentration of a substance in a mixture (e.g. in the case of gasses). If C is the mass concentration of a substance in a gas then the flux of that substance is given via Fick's Law as

$$\mathbf{q} = -k\nabla C. \tag{12.6}$$

Combining (12.6) with (12.5) (and assuming that k is independent of space, time, and concentration) gives

$$\frac{\partial C}{\partial t} = k\nabla \cdot \nabla C. \tag{12.7}$$

In the presenence of external sources of mass, (12.7) is

$$\frac{\partial C}{\partial t} = k\nabla \cdot \nabla C + f(x). \tag{12.8}$$

Problem 12.6. What does (12.8) equation look like in terms of spatial derivatives on the right-hand side?

$$\frac{\partial C}{\partial t} = \underline{\hspace{6cm}} \quad \text{(1 Spatial Dimension)}$$
$$\frac{\partial C}{\partial t} = \underline{\hspace{6cm}} \quad \text{(2 Spatial Dimensions)}$$
$$\frac{\partial C}{\partial t} = \underline{\hspace{6cm}} \quad \text{(3 Spatial Dimensions)}$$

▲

Energy Balance

The energy balance equation is essentially the same as the mass balance equation. If u is temperature then the flux of temperature is given by Fourier's Law for heat conduction

$$q = -k\nabla T. \tag{12.9}$$

Making the same simplifications as in the mass balance equation we arrive at

$$\frac{\partial T}{\partial t} = k\nabla \cdot \nabla T. \tag{12.10}$$

In the presence of external sources of heat, (12.10) becomes

$$\frac{\partial T}{\partial t} = k\nabla \cdot \nabla T + f(x). \tag{12.11}$$

Problem 12.7. What does (12.11) equation look like in terms of spatial derivatives on the right-hand side?

$$\frac{\partial T}{\partial t} = \underline{\hspace{5cm}} \quad \text{(1 Spatial Dimension)}$$
$$\frac{\partial T}{\partial t} = \underline{\hspace{5cm}} \quad \text{(2 Spatial Dimensions)}$$
$$\frac{\partial T}{\partial t} = \underline{\hspace{5cm}} \quad \text{(3 Spatial Dimensions)}$$

▲

12.2.3 Laplace's Equation and Poisson's Equation

Equations (12.8) and (12.11) are the same partial differential equation for two very important physical phenomenon; mass and heat transfer. In the case where time is allowed to run to infinity and no external sources of mass or energy are included these equations reach a steady state solution (no longer changing in time) and we arrive at Laplace's Equation

$$\nabla \cdot \nabla u = 0. \tag{12.12}$$

Laplace's equation is actually a statement of minimal energy as well as steady state heat or temperature. We can see this since entropy always drives systems from high energy to low energy, and if we have reached a steady state then we must have also reached a surface of minimal energy.

Problem 12.8. Equation (12.12) is sometimes denoted as $\nabla \cdot \nabla u = \nabla^2 u = \Delta u$, and in terms of the partial derivatives it is written as

$$0 = \underline{\hspace{5cm}} \quad \text{(1 Spatial Dimension)}$$
$$0 = \underline{\hspace{5cm}} \quad \text{(2 Spatial Dimensions)}$$
$$0 = \underline{\hspace{5cm}} \quad \text{(3 Spatial Dimensions)}$$

▲

If there is a time-independent external source the the right-hand side of (12.12) will be non-zero and we arrive at Poisson's equation:

$$\nabla \cdot \nabla u = -f(x). \tag{12.13}$$

Note that the negative on the right-hand side comes from the fact that $\frac{\partial u}{\partial t} = k\nabla \cdot \nabla u + f(x)$ and $\frac{\partial u}{\partial t} \to 0$. Technically we are taking absorbing the constant k into f (that is "f" is really "f/k"). Also note that in many instances the value of k is not contant and cannot therefore be pulled out of the derivative without a use of the product rule.

We will start our exploration of numerical PDEs with Laplace's and Poisson's equations. We will then layer on the temporal derivatives to explore mass and heat transport. Finally, we will explore wave phenomena as well as advection-diffusion transport models.

12.3 The 1D Heat Equation

In this section we will discuss analytic techniques for solving the heat equation in 1 spatial dimension. Most of this section is paraphrased from Richard Haberman's *Applied Partial Differential Equations* text [2].

We can think of the heat equation physically as tracking the heat diffusion in a thin rod of length L. Hence, in 1 spatial dimension equation (12.10) becomes

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \text{with } 0 < x < L \text{ and } t > 0. \tag{12.14}$$

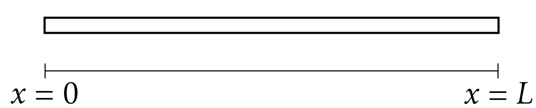


Figure 12.1. Sample geometry for the 1D heat equation (12.14).

In each of the subsequent subsections of this document we will explore different boundary conditions for equation (12.14). The boundary conditions are a way to prescribe how the heat is transfering or is otherwise being controlled at the ends of the rod.

12.3.1 1D Heat Equation with Zero Temperature Ends

For our first case, consider a 1D rod as in Figure 12.1 with $u(0,t) = 0$ and $u(L,t) = 0$. That is, let's assume that the two ends of the rod are in an ice bath held at exactly 0° (See Figure 12.2).

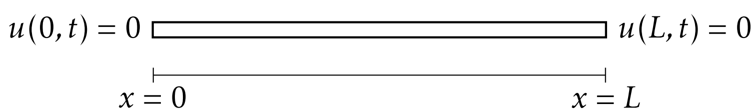


Figure 12.2. Sample geometry for the 1D heat equation (12.14).

Problem 12.9. If we were to solve the 1D heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

with boundary conditions as in Figure 12.2, what other information would we need, aside from k, in order to get a physically meaningful solution?

(a) $u(L/2, t)$: a fixed temperature at the midpoint

(b) $u(x, 0)$: an initial temperature profile along the rod

(c) $u(x, \infty)$: the steady state temperature profile

(d) $u(0, t)$: the way that the heat evolves at $x = 0$

▲

Problem 12.10. Assume that $u(x, 0) = 100$. That is, assume that the rod is initially heated to 100° from end to end. Further assume that the boundary conditions are $u(0, t) = u(L, 0) = 0$ just as in Figure 12.2. Draw several curves clearly showing how the temperature in the rod will evolve in time. ▲

Separation of Variables

With ordinary differential equation we origianlly saw separation of variables with relatively *simple* differential equations. It turns out that we can do the same thing the heat equation. We start by assuming that in order to solve $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ we assume that

$$u(x, t) = \phi(x)G(t) \tag{12.15}$$

where $\phi(x)$ is ONLY a function of space (x) and $G(t)$ is ONLY a function of time (t). This technique was invented in the 1700's, and it works becuase it reduces the PDE to two ODEs.

Problem 12.11. Let's see what happens under assumption (12.15):
Let $u(x, t) = \phi(x)G(t)$ and

- write $\dfrac{\partial u}{\partial t}$ in terms of the functions ϕ and G,
- write $\dfrac{\partial^2 u}{\partial x^2}$ in terms of the functions ϕ and G, and
- put your two answers into the heat equation $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$.
- Finally, separate the variables so that all of the expressions with G are on the left of the equation and all of the expressions with ϕ are on the right of the equation. (put the k with the G function ... trust me)

▲

The left-hand side of your result from the previous problem should only contain function of G and the right-hand side should only contain functions of ϕ. The strange thing is that the equal sign is still valid. How can this be? The left-hand side is only a function of t and the right-hand side is only a function of x.

Problem 12.12. From the previous question you should have arrived at

$$\frac{1}{kG}\frac{dG}{dt} = \frac{1}{\phi}\frac{d^2\phi}{dx^2}.$$

The left-hand side is only a function of t and the right-hand side is only a function x but the equal sign is absolutely true for all x and for all t. How can this be? ▲

Problem 12.13. From the separation of variables we arrive at two ordinary differential equations:

$$\frac{d^2\phi}{dx^2} = -\lambda\phi \tag{12.16}$$

$$\frac{dG}{dt} = -\lambda k G. \tag{12.17}$$

What types of behavior do you expect out of the solutions for equations (12.16) and (12.17).

(a) Equation (12.16): exponential decay
Equation (12.17): oscillations modeled by trig functions

(b) Equation (12.16): over damped system modeled by exponential functions
Equation (12.17): exponential decay

(c) Equation (12.16): critically damped system modeled by exponential functions
Equation (12.17): exponential decay

(d) Equation (12.16): oscillations modeled by trig functions
Equation (12.17): exponential decay

▲

Problem 12.14. Solve the time-dependent equation

$$\frac{dG}{dt} = -\lambda k G$$

where λ is (at the moment) an unknown constant. What happens if $\lambda > 0$, if $\lambda = 0$, and if $\lambda < 0$? ▲

Note: We don't expect to have solutions that grow exponentially in time so we should expect that $\lambda \geq 0$

Problem 12.15. Now we're going to solve the spatial boundary-valued problem

$$\frac{d^2\phi}{dx^2} = -\lambda\phi \quad \text{with} \quad \phi(0) = 0 \text{ and } \phi(L) = 0.$$

- Assume that $\phi(x) = e^{rx}$, find the characteristic polynomial, and find the two linearly independent solutions (these will contain λ).
- Write the solution to ϕ as a linear combination of the two linearly independent solutions.
- Apply the left-hand boundary condition $\phi(0) = 0$ to get one of the constants.
- Apply the right-hand boundary condition $\phi(L) = 0$ and find the equation that must be true in order for this boundary condition to be satisfied.
- What must λ be equal to in order for the previous equation to be satisfied?
- Write the solution to $\phi(x)$.

▲

Let me interject here for a few sentences:
Let's put the pieces together. From Problem 12.14 we know that

$$G(t) = C_1 e^{-\lambda k t} \tag{12.18}$$

and from Problem 12.15 we know that

$$\phi(x) = C_2 \sin\left(\frac{n\pi x}{L}\right). \tag{12.19}$$

Since we are assuming that $u(x,t) = \phi(x)G(t)$ we can multiply the time dependent solution $G(t)$ and the spatial solution $\phi(x)$ to get

$$u(x,t) = A_n e^{-k(n\pi/L)^2 t} \sin\left(\frac{n\pi x}{L}\right) \quad \text{for} \quad n = 1, 2, 3, \ldots. \tag{12.20}$$

Strangely enough, there is a solution for every natural number $n = 1, 2, 3, \ldots$. This is certainly the first time we've encountered this, but we know something that will help: the derivative is a linear operator so the sum of two solutions must also be a solution. Carrying this to it's logical end gives the final solution to the 1D heat equation with homogeneous boundary conditions:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} A_n e^{-k(n\pi/L)^2 t} \sin\left(\frac{n\pi x}{L}\right).} \tag{12.21}$$

This is a rather complicated solution so let's apply it to an example so we can see how it works.

Example 12.16. Solve the 1D heat equation with the following initial and boundary conditions:

$$\begin{aligned}\frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} \quad \text{with } t > 0 \text{ and } 0 < x < 1 \\ k &= 1 \quad \text{(this is called the thermal diffusivity)} \\ u(0,t) &= 0 \quad \text{for } t > 0 \\ u(L,t) &= 0 \quad \text{for } t > 0 \\ u(x,0) &= 100 \quad \text{for } 0 < x < 1.\end{aligned}$$

See Figure 12.2 for a schematic of the problem. The following problems will walk you through the solution.

Problem 12.17. We know that the general solution is:

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi/1)^2 t} \sin\left(\frac{n\pi x}{1}\right).$$

At time $t = 0$ we have

$$100 = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

since the exponential function evaluates to zero at $t = 0$. If we multiply both sides by $\sin\left(\frac{m\pi x}{L}\right)$ and integrate from 0 to 1 what do we get? (Assume that m is not necessarily the same as n). ▲

There are some really conventient trig identities that we can use next:

$$\boxed{\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0, & m \neq n \\ L/2, & m = n \end{cases}} \tag{12.22}$$

Problem 12.18. Using equation (12.22) and the result from the previous problem we can evaluate the integrals from the previous problem to get an expression for A_n. You may need to recall that $\cos(n\pi) = (-1)^n$ to find a pattern for A_n. Write down the pattern for A_n. ▲

Problem 12.19. Using your pattern from the previous problem we can expand the solution (12.21) as

$$u(x,t) = A_1 e^{-\pi^2 t}\sin(\pi x) + A_2 e^{-(2\pi)^2 t}\sin(2\pi x) + A_3 e^{-(3\pi)^2 t}\sin(3\pi x) + \ldots.$$

Write several of the terms using your patter. Then use technology to plot approximate solutions to this problem. An example plot of the time-dependent solution is shown in Figure 12.3. ▲

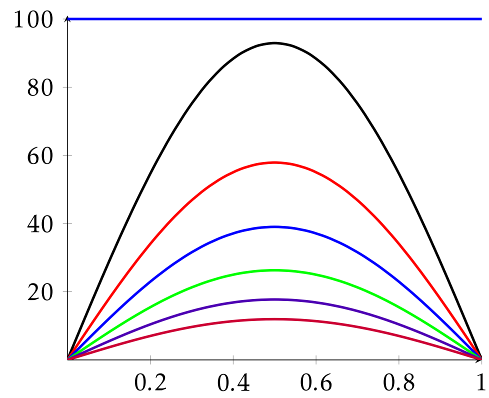


Figure 12.3. Snapshots of the solution to Example 12.16.

Summary of Separation of Variables

The following steps were used throughout the last several problems to solve a linear homogeneous PDE with linear and homogeneous boundary conditions.

1. Temporarily ignore the initial condition.
2. Separate the variables by assuming that $u(x,t) = G(t)\phi(x)$.
   (a) Rearrange the PDE to separate the functions G and ϕ.
   (b) Introduce a separation constant $-\lambda$.
   (c) Write the two separate ODEs as eigenvalue problems.
3. Determine the separation constants as eigenvalues of a boundary value problem.
4. Solve the other differential equations. Record all products of solutions of the PDE obtainable by this method.
5. Apply the principle of superposition: the general solution is a linear combination of all of the individual solutions.
6. Satisfy the initial conditions:
   (a) substitute $t = 0$ into the general solution
   (b) multiply both sides of the resulting equation by an appropriate function (it should be another basis function from the same vector space that builds the infinite sum)
   (c) integrate and take advantage of orthogonality
   (d) use the resulting integrals to find a pattern in the coefficients.
7. Use software to plot the solutions as they evolve over time.

Now it is your turn. The following two problems allow you to put these ideas to the test.

Problem 12.20. Solve the 1D heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

subject to the boundary conditions $u(0,t) = u(1,t) = 0$ with initial temperature profile $u(x,0) = 6\sin(9\pi x)$. Start by making a sketch of several time steps of the solution using what you now about the physics of the problem. ▲

Problem 12.21. Solve the 1D heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

subject to the boundary conditions $u(0,t) = u(1,t) = 0$ with initial temperature profile $u(x,0) = x(1-x)$. Start by making a sketch of several time steps of the solution using what you now about the physics of the problem. ▲

Problem 12.22. So far you have noticed that the spatial differential equation in ϕ turns out to be an eigenvalue problem

$$\frac{d^2\phi}{dx^2} = -\lambda\phi.$$

Determine the eigenvalues of the problem with the following sets of boundary conditions:

(a) $\phi(0) = 0$ and $\phi(\pi) = 0$

(b) $\phi(0) = 0$ and $\phi(1) = 0$

(c) $\frac{d\phi}{dx}(0) = 0$ and $\frac{d\phi}{dx}(1) = 0$

(d) $\phi(0) = 0$ and $\frac{d\phi}{dx}(1) = 0$

Each of these relates to a physical problem, so now go back through problems (a) - (d) and express each problem as boundary conditions for a 1D heat conducting rod. What do these boundary conditions mean physically (i.e. how are we controlling the heat at the ends of the rod)? ▲

Problem 12.23. Solve the heat equation with homogenous boundary conditions on a rod of unit length with initial condition

$$u(x,0) = \begin{cases} 0, & 0 < x < 1/3 \\ 100, & 1/3 < x < 2/3 \\ 0, & 2/3 < x < 1 \end{cases}$$

▲

12.3.2 1D Heat Equation with Insulated Ends

In this section we consider the heat equation again, but this time we consider the problem where the ends are insulated instead of being held at a fixed temperature. In Figure 12.4 we see that we are now not letting heat escape from the rod through the ends. This means that the energy within the rod from the initial condition will remain in the rod, but may spread out over time.

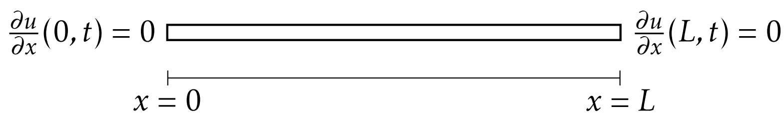


Figure 12.4. Sample geometry for the 1D heat equation (12.14) with Neumann (insulating) boundary conditions.

As you will soon see, the series solution will actually be in terms of cosines this time instead of sines. You will need the following identity:

$$\boxed{\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0, & n \neq m \\ \frac{L}{2}, & n = m \neq 0 \\ L, & n = m = 0 \end{cases}} \tag{12.23}$$

Problem 12.24. Solve the following 1D heat equation with Neumann boundary conditions.

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} \quad \text{with } t > 0 \text{ and } 0 < x < 1 \\ k &= 1 \quad \text{(this is called the thermal diffusivity)} \\ \frac{\partial u}{\partial x}(0,t) &= 0 \quad \text{for } t > 0 \\ \frac{\partial u}{\partial x}(L,t) &= 0 \quad \text{for } t > 0 \\ u(x,0) &= -\cos(2\pi x) + 1 \quad \text{for } 0 < x < 1. \end{aligned}$$

Start by sketching plots of the time evolution of the initial condition based solely on your physical intuition. Then follow the steps for solving a 1D PDE with separation of variables. ▲

12.3.3 Heat Equation on a Thin Ring

Now it is time for a new geometry. Let us formulate the appropriate initial boundary value problem for a thin wire (with lateral sides insulated) that is bent into the shape of a circle. We will let the wire have length $2L$ as shown in Figure 12.5. If the wire is thin enough then it is reasonable to assume that the temperature in the wire is constant along cross sections.

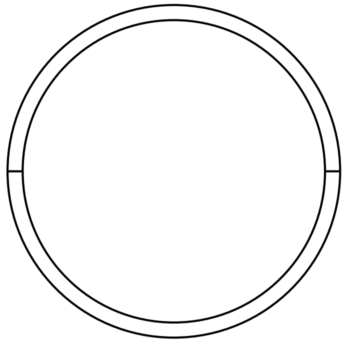


Figure 12.5. A thin circular wire of length $2L$.

The formulation for the heat equation in this case is:

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} \quad \text{with } t > 0 \text{ and } -L < x < L \\ u(-L,t) &= u(L,t) \quad \text{(since the heat must match at } x = \pm L) \\ \frac{\partial u}{\partial x}(-L,t) &= \frac{\partial u}{\partial x}(L,t) \quad \text{(since the derivative of the temp. must be continuous)} \\ u(x,0) &= f(x) \quad \text{for } -L < x < L. \end{aligned}$$

Problem 12.25. After separating the variables we again have the eigenvalue problem

$$\frac{d^2\phi}{dx^2} = -\lambda\phi.$$

Choose the proper boundary conditions on this problem?

(a) $\phi(-L) = \phi(L) = 0$ and $\frac{d\phi}{dx}(-L) = \frac{d\phi}{dx}(L) = 0$

(b) $\phi(-L) = \phi(L)$ and $\frac{d\phi}{dx}(-L) = \frac{d\phi}{dx}(L)$

(c) $\phi(-L) + \phi(L) = 0$ and $\frac{d\phi}{dx}(-L) + \frac{d\phi}{dx}(L) = 0$

(d) The moon is made of cheese

▲

Problem 12.26. Solve the boundary value problem

$$\frac{d^2\phi}{dx^2} = -\lambda\phi$$

with the boundary conditions from the previous voting question. If you get multiple solutions (hint) then remember that your final solution is actually a linear combination of the solutions. ▲

Problem 12.27. The time problem $G(t)$ has the same solutions on a ring as it does for the 1D rod. Using this information as well as your solution to the spatial boundary value problem, write the full general solution to the heat equation on a thin ring. ▲

For the heat equation on a ring you should have found that the general solution is

$$u(x,t) = A_0 + \sum_{n=1}^{\infty}\left[e^{-k(n\pi/L)^2 t}\left(A_n\cos\left(\frac{n\pi x}{L}\right) + B_n\sin\left(\frac{n\pi x}{L}\right)\right)\right], \quad \text{with} \tag{12.24}$$

$$u(x,0) = f(x) = A_0 + \sum_{n=1}^{\infty}\left[A_n\cos\left(\frac{n\pi x}{L}\right) + B_n\sin\left(\frac{n\pi x}{L}\right)\right] \tag{12.25}$$

In order to find the coefficients we need the following orthogonality identities:

$$\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) = \begin{cases} 0, & n\neq m \\ L, & n=m\neq 0 \\ 2L, & n=m=0\end{cases} \tag{12.26}$$

$$\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0, & n\neq m \\ L, & n=m\neq 0\end{cases} \tag{12.27}$$

$$\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = 0 \text{ for all } n \text{ and } m. \tag{12.28}$$

Problem 12.28. Let's take a ring with $L=1$ and $f(x)=\sin(2\pi x)$.

- Find the coefficients A_n by multiply by $\cos\left(\frac{m\pi x}{1}\right)$ and integrating from -1 to 1.
- Find the coefficients B_n by multiply by $\sin\left(\frac{m\pi x}{1}\right)$ and integrating from -1 to 1.

You are welcome to use technology to evaluate the integrals. If you can't get an exact formula for the patterns in A_n and B_n at least write down the first 8 or 10 terms of each sequence. ▲

Problem 12.29. For the ring in the last problem with $L=1$ and $f(x)=\sin(2\pi x)$, write down several terms in the solution and use technology to make a plot of the time evolution. You should start by hand-sketching a plot showing the time evolution of the heat. ▲

12.4 The Wave Equation (INCOMPLETE)

This section is incomplete. If I ever get here in MA334 I will finish this section ...until then it is going to remain a work in progress.

$$u_{tt} = \alpha^2 u_{xx}$$

...stuff about waves and such ...

12.5 1D Traveling Waves (INCOMPLETE)

Problem 12.30. Let $f(x) = e^{-x^2/0.5}$. Write computer code to animate the function $u(t,x) = f(x-at)$ for various values of a on the domain $x \in (-5,5)$. ▲

Problem 12.31. Let $f(x)$ be a function and define $u(t,x)$ be defined as $u(t,x) = f(x-at)$.

(a) Find $\dfrac{\partial u}{\partial t}$

(b) Find $\dfrac{\partial u}{\partial x}$

(c) What differential equation does $u(t,x)$ satisfy?

▲

Solution:

$$u_t = -au_x$$

Problem 12.32. Solve the differential equation $u_t + 3u_x = 0$ with initial condition $u(0,x) = \sin(x)$ on the domain $x \in [0,2\pi]$ with boundary condition $u(t,0) = 0$. ▲

Solution:

$$u(t,x) = \sin(x-3t)$$

The differential equation

$$u_t = -\alpha u_x$$

exhibits traveling wave type solutions. ... General solution is ...

$$u_t + \alpha u_x = 0 \quad \text{with} \quad u(0,x) = \eta(x) \quad \implies \quad u(t,x) = \eta(x-\alpha t)$$

since by the chain rule

$$u_t + \alpha u_x = \frac{\partial}{\partial t}(\eta(x-\alpha t)) + \alpha\frac{\partial}{\partial x}(\eta(x-\alpha t)) = -\alpha\eta(x-\alpha t) + \alpha\eta(x-\alpha t) = 0.$$

The End

Appendices

Appendix A

Partial Fractions

In this appendix we explore the algebraic notion of partial fractions. The idea is simple: How do we undo the addition of fractions?

Let's first consider some elementary arithmetic.

$$\frac{1}{3} + \frac{2}{5} = ?$$

To add the two fractions you need common denominators, in this case 15. We rewrite the fractions as

$$\frac{1}{3} + \frac{2}{5} = \frac{5}{15} + \frac{6}{15}$$

and now that we're comparing like parts we add the numerators to get

$$\frac{1}{3} + \frac{2}{5} = \frac{5}{15} + \frac{6}{15} = \frac{11}{15}.$$

What if we wanted to go the other way? That is, what if we have the fraction 11/15 and we wanted to know where it came from. If we consider the prime factorization of the denominator and conjecture that the fractions can be split up with these factors as the denominators of separate fractions then the problem becomes

$$\frac{11}{15} = \frac{11}{3 \cdot 5} = \frac{A}{3} + \frac{B}{5}$$

where A and B are just numbers that we need to find. Obviously there are many different answers to this inverse questions (since we can get infinitely many equivalent fractions). If we multiply both sides of this new equation by 15 we get

$$11 = 5A + 3B$$

and for each choice of one variable we get another. In particular, if we choose $A = 1$ then simple algebra tells us that $B = 2$ and we have successfully split the fraction 11/15 into the sum of 1/3 and 2/5.

Now let's consider the algebraic problem of taking a fraction and splitting it into a sum of fractions. The notion is still the same: conjecture that the factors of the denominator are the denominators of the separate fractions and then do some detective work to find the numerators. In the remainder of this appendix we'll give several examples of this idea. We leave it up to the reader to actually find the common denominators and do the algebra to verify that indeed the right-hand side from each example is equal to the left-hand side.

Example A.1. Use partial fractions to write $\frac{4}{x(x-3)}$ as a sum or difference of two fractions.
Solution: We start by writing the fraction as

$$\frac{4}{x(x-3)} = \frac{A}{x} + \frac{B}{x-3}.$$

This choice is made since if we were to find the common denominator of the right-hand side we would have the desired denominator on the left-hand side. Next we clear all of the fractions by multiplying the common denominator yielding

$$4 = A(x-3) + B(x).$$

At this point we know that the equal sign must be true for <u>all</u> values of x so we can choose some convenient values to tease out A and B.

- If $x = 3$ then $x - 3 = 0$ and we get $4 = 3B$ which implies that $B = \frac{4}{3}$.
- If $x = 0$ then we get $4 = -3A$ which implies that $A = -\frac{4}{3}$

Therefore

$$\frac{4}{x(x-3)} = -\frac{4}{3x} + \frac{4}{3(x-3)}.$$

Example A.2. It can be shown that

$$\frac{6x}{(x-1)(x+1)(x+2)} = \frac{1}{x-1} + \frac{3}{x+1} - \frac{4}{x+2}.$$

Partial Justification: Start by observing that the denominator of the left-hand fraction is factored so we split into three fractions with the factors as the denominators:

$$\frac{6x}{(x-1)(x+1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+2}.$$

Clearing the fractions gives

$$6x = A(x+1)(x+2) + B(x-1)(x+2) + C(x-1)(x+1).$$

Now consider convenient choices of x

- If $x = -1$ then:

$$-6 = A(0)(1) + B(-2)(1) + C(-2)(0) \quad \Longrightarrow \quad B = 3.$$

- If $x = 1$ then:

$$6 = A(2)(3) + B(0)(3) + C(0)(2) \quad \Longrightarrow \quad A = 1.$$

- If $x = -2$ then:

$$-12 = A(-1)(0) + B(-3)(0) + C(-3)(-1) \quad \Longrightarrow \quad C = -4.$$

Hence

$$\frac{6x}{(x-1)(x+1)(x+2)} = \frac{1}{x-1} + \frac{3}{x+1} - \frac{4}{x+2}.$$

Example A.3. In this problem we will see repeated linear factors.

$$\frac{x^2+1}{x(x-1)^3} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}$$

Notice that the repeated factor gets repeated for all powers. Let's clear the fractions just as before and see what we get

$$x^2 + 1 = A(x-1)^3 + Bx(x-1)^2 + Cx(x-1) + Dx.$$

If we take $x = 0$ then $A = -1$. If we take $x = 1$ then $D = 2$. However, you'll notice that these two choices do not allow us to easily find B and C so we expand the polynomial on the right-hand side, gather like terms, and match coefficients. That is

$$\begin{aligned} x^2 + 1 &= A\left(x^3 - 3x^2 + 3x - 1\right) + B\left(x^3 - 2x^2 + x\right) + C\left(x^2 - x\right) + Dx \\ \Longrightarrow \quad x^2 + 1 &= (A+B)x^3 + (-3A - 2B + C)x^2 + (3A + B - C + D)x - A \end{aligned}$$

Matching the coefficients of like terms we get

$$\begin{aligned} A + B &= 0 && \text{(cubic terms)} \\ -3A - 2B + C &= 1 && \text{(quadratic terms)} \\ 3A + B - C + D &= 0 && \text{(linear terms)} \\ -A &= 1 && \text{(constant terms)} \end{aligned}$$

Since $A = -1$ we must have $B = 1$ and therefore $C = 0$. Therefore

$$\frac{x^2+1}{x(x-1)^3} = -\frac{1}{x} + \frac{1}{x-1} + \frac{0}{(x-1)^2} + \frac{2}{(x-1)^3}$$

Example A.4. In this final example we'll show what happens with an irreducible quadratic.

$$\frac{x-3}{x(x^2+3)} = \frac{A}{x} + \frac{Bx+C}{x^2+3}.$$

Notice that the numerator associated with the irreducible quadratic is a linear function with unknown parameters. Clearing the fractions we get

$$x - 3 = A(x^2+3) + (Bx+C)(x).$$

If we take $x = 0$ then $-3 = 3A$ which implies that $A = -1$. Expanding both sides of the equation and matching like terms gives

$$x - 3 = (A+B)x^2 + Cx + 3A$$

which implies that $A + B = 0$ and $C = 1$. Therefore $B = 1$ and

$$\frac{x-3}{x(x^2+3)} = -\frac{1}{x} + \frac{x+1}{x^2+3}.$$

Technique A.5 (Partial Fractions Decomposition)**.** Below are several cases of fractions that require partial fractions along with their separated forms.

$$\begin{aligned}
\frac{px+q}{(x-a)(x-b)} &= \frac{A}{x-a} + \frac{B}{x-b} \quad \text{(for } a \neq b\text{)} \\
\frac{px+q}{(x-a)^2} &= \frac{A}{x-a} + \frac{B}{(x-a)^2} \\
\frac{px+q}{(x-a)^3} &= \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{(x-a)^3} \\
\frac{px^2+qz+r}{(x-a)(x-b)(x-c)} &= \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} \\
\frac{px^2+qz+r}{(x-a)^2(x-b)} &= \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b} \\
\frac{px^2+qx+r}{(x-a)(x^2+bx+c)} &= \frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c} \quad \text{(where } x^2+bx+c \text{ cannot be factored)}
\end{aligned}$$

Problem A.6. Now let's use partial fractions to solve a separable differential equation.

Solve the logistic population equation

$$\frac{dP}{dt} = 0.2P\left(1 - \frac{P}{10}\right) \quad \text{with} \quad P(0) = 1$$

Start by separating the variables (leaving the 0.2 on the right) and then looking up the appropriate partial fractions decomposition for splitting up the fraction that appears. After that you'll get to do a whole bunch of algebra ... have fun!! ▲

Solution:

$$\begin{aligned}
\frac{dP}{dt} &= 0.2P\left(1 - \frac{P}{10}\right) \\
\frac{dP}{P(1-P/10)} &= 0.2dt \\
\int \frac{dP}{P(1-P/10)} &= \int 0.2dt \\
\int \frac{dP}{P(1-P/10)} &= 0.2t + C
\end{aligned}$$

Now we need to split the fraction on the left.

$$\frac{1}{P(1-P/10)} = \frac{A}{P} + \frac{B}{1-P/10} \quad \Longrightarrow \quad 1 = A(1-P/10) + BP$$

If we take $P = 0$ then $A = 1$. If we take $P = 10$ then $B = 1/10$. Therefore,

$$\frac{1}{P(1-P/10)} = \frac{1}{P} + \frac{1}{10(1-P/10)}$$

and we can now integrate the left-hand side to get (with a little bit of u-substitution)

$$\int \frac{1}{P(1-P/10)} dP = \int \frac{1}{P} dP + \frac{1}{10} \int \frac{1}{1-P/10} dP = \ln(P) - \ln(1-P/10)$$

Therefore,

$$\ln(P) - \ln(1-P/10) = 0.2t + C \implies \ln\left(\frac{P}{1-P/10}\right) = 0.2t + C \implies \frac{P}{1-P/10} = Ce^{0.2t}$$

$$\implies P = Ce^{0.2t}\left(1 - \frac{P}{10}\right) \implies P + P\frac{Ce^{0.2t}}{10} = Ce^{0.2t} \implies P\left(1 + \frac{Ce^{0.2t}}{10}\right) = Ce^{0.2t}$$

$$\implies \boxed{P(t) = \frac{Ce^{0.2t}}{1 + (C/10)e^{0.2t}}}$$

Assuming now that $P(0) = 1$ we can take any of the algebraically equivalent forms of the final answer to tease out the value of C. In particular,

$$\frac{1}{1-1/10} = Ce^{0.2(0)} \implies C = \frac{1}{9/10} = \frac{10}{9} \implies P(t) = \frac{10e^{0.2t}}{9 + e^{0.2t}}$$

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