Differential Equations & Linear Algebra
An Inquiry and Problem Based Approach

(The Second Course of a Two-Semester Sequence)

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Chapter 0

To the Student and the Instructor

This document contains lecture notes, classroom activities, examples, and challenge problems specifically designed for a second semester of differential equations and Linear Algebra. The first semester content is reviewed quickly in chapters 1-3 so with supplementation these notes could serve as a first look at both topics. The content herein is written and maintained by Dr. Eric Sullivan of Carroll College. Problems were either created by Dr. Sullivan, the Carroll Mathematics Department faculty, part of NSF Project Mathquest, or come from other sources and are either cited directly or cited in the LaTeX source code for the document (and are hence purposefully invisible to the student).

0.1 An Inquiry Based Approach

Problem 0.1 (Setting The Stage).

- Get in groups of size 3-4.
- Group members should introduce themselves.
- For each of the questions that follow I will ask you to:
    1. **Think** about a possible answer on your own
    2. **Discuss** your answers with the rest of the group
    3. **Share** a summary of each group's discussion

Questions:

Question #1: What are the goals of a university education?

Question #2: How does a person learn something new?

Question #3: What do you reasonably expect to remember from your courses in 20 years?

Question #4: What is the value of making mistakes in the learning process?

Question #5: How do we create a safe environment where risk taking is encouraged and productive failure is valued?

▲

(The previous problem is inspired by Dana Ernst's first day activity in IBL activity titled: Setting the Stage.)

> "Any creative endeavor is built in the ash heap of failure."
> –Michael Starbird

This material is written with an Inquiry-Based Learning (IBL) flavor. In that sense, this document could be used as a stand-alone set of materials for the course but these notes are not a *traditional textbook* containing all of the expected theorems, proofs, examples, and exposition. The students are encouraged to work through problems and homework, present their findings, and work together when appropriate. You will find that this document contains collections of problems with only minimal interweaving exposition. It is expected that you do every one of the problems and then use other more traditional texts as a backup when you are stuck. Let me say that again: this is not the only set of material for the course. Your brain, your peers, and the books linked in the next section are your best resources when you are stuck.

To learn more about IBL go to http://www.inquirybasedlearning.org/about/. The long and short of it is that the students in the class are the ones that are doing the work; proving theorems, writing code, working problems, leading discussions, and pushing the pace. The instructor acts as a guide who only steps in to redirect conversations or to provide necessary insight. If you are a student using this material you have the following jobs:

1. Fight! You will have to fight hard to work through this material. The fight is exactly what we're after since it is ultimately what leads to innovative thinking.

2. Screw Up! More accurately, don't be afraid to screw up. You should write code, work problems, and prove theorems then be completely unafraid to scrap what you've done and redo it from scratch. Learning this material is most definitely a non-linear path.* Embrace this!

3. Collaborate! You should collaborate with your peers with the following caveats: (a) When you are done collaborating you should go your separate ways. When you write your solution you should have no written (or digital) record of your collaboration. (b) The internet is not a collaborator. Use of the internet to help solve these problems robs you of the most important part of this class; the chance for original though.

4. Enjoy! Part of the fun of IBL is that you get to experience what it is like to think like a true mathematician / scientist. It takes hard work but ultimately this should be fun!

*Pun intended: our goal, after all, is really to understand that linear algebra is the glue that holds mathematics together.

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0.2 Online Texts and Other Resources

If you are looking for online textbooks for linear algebra and differential equations I can point you to a few. Some of the following online resources may be a good place to help you when you're stuck but they will definitely say things a bit differently. Use these resources wisely.

- The book *Differential Equations with Linear Algebra, An inquiry based approach to learning* is a nice collection of notes covering much of the material that we cover in our class. The order is a bit different but the notes are well done.
content.byui.edu/file/664390b8-e9cc-43a4-9f3c-70362f8b9735/1/316-IBL%20(2013Spring).pdf

- The ODE Project by Thomas Juson is a nice online text that covers many (but not all) of the topics that we cover in differential equations.
faculty.sfasu.edu/judsontw/ode/html/odeproject.html

- Elementary Differential Equations by William Trench. This book contains everything(!) you would ever want to look up for ordinary differential equations. It is a great resource to look up ODE techniques.
ramanujan.math.trinity.edu/wtrench/texts/TRENCH_DIFF_EQNS_I.PDF

- A First Course in Linear Algebra by Robert Beezer. This book is very thorough and covers everything that we do in linear algebra and much more.
linear.ups.edu/fcla/index.html

- Linear Algebra Workbook by TJ Hitchman. This is a workbook for Dr. Hitchman's class at U. Northern Iowa. Even though it is only a "workbook" it contains some nice explanations and it has embedded executable code for some problems.
theronhitchman.github.io/linear-algebra/course-materials/workbook/LinAlgWorkbook.html

0.3 To the Instructor

If you are an instructor wishing to use these materials then I only ask that you adhere to the Creative Commons license. You are welcome to use, distribute, and remix these materials for your own purposes. Thanks for considering my materials for your course!

My typical use of these materials are to let the students tackle problems in small groups during class time and to intervene when more explanation appears to be necessary or if the students appear to be missing the deeper connections behind problems. The course that I have in mind for these materials is a second semester of differential equations and linear algebra. As such, this is not a complete collection of materials for either differential equations or linear algebra in isolation. In our first course we discuss matrix operations, Gaussian elimination, the eigenvalue problem, first order linear homogeneous and non-homogeneous differential equations, and second order homogeneous differential equations. You will find that the sections in these notes covering these topics are necessarily light and are meant to only give the students a brief review of the material.

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Many of the theorems in the text come without a proof. If the theorem is followed by the statement "prove the previous theorem" then I expect the students to have the skill to prove that theorem and to do so with the help of their small group. However, this course is not intended to be a proof-based mathematics course so several theorems are stated without rigorous proof. If you are looking for a proof-based linear algebra or differential equations course then I believe that these notes will not suffice. I have, however, tried to give thought provoking problems throughout so that the students can engage with the material at a level higher than just the mechanics of differential equations and linear algebra. There are also several routine exercises throughout the notes that will allow students to practice mechanical skills.

There is a toggle switch in the LaTeX code that allows you to turn on and off the solutions to problems. The line of code
`\def\ShowSoln{0}`
is a switch that, when set to 0, turns the solutions off and when set to 1 turns the solutions on. Just re-compile (`pdflatex`) the document to display the solutions. I typically do not show the solutions to the students while they're learning the material, but I allow them access during exam preparation time so they can check their understanding.

Chapter 1

First Order Differential Equations

You may recall that in an algebraic equation you are seeking to find a number, usually[*] x, so that the given equation holds true. For example, we could solve $x + 2 = 5$ and find that $x = 3$ is the only value that makes the equal sign true. As another example, we could solve $x^2 - 3x + 2 = 0$ using the quadratic formula or factoring and find that $x = 1$ and $x = 2$ are the only solutions. Your high school algebra classes focused on the techniques necessary to solve many different types of algebraic equations and at this point you likely have the techniques down pat (right?!).

When solving differential equations we are seeking a slightly different goal. This time the unknown is a function and the equation relates the derivative(s) of the function to the function itself. For example, if we consider the simple equation $y'(t) = y(t)$ we could probably guess (using the rules of calculus) that the only functions that satisfy this equation are $y(t) = 0$ and $y(t) = Ce^t$. Notice that the solution is not a number but a function. As another example consider $y''(t) = -y(t)$. In this case you can also use your intuition from calculus to guess that $y(t)$ is some combination of sines and cosines: $y(t) = C_1 \sin(t) + C_2 \cos(t)$. Our goal throughout this course is to build differential equations and find techniques to analyze them. As you might imagine based on the complexity of the derivative rules in calculus, the techniques to find solutions to differential equations can sometimes be quite complicated.

[*]See the TED Talk https://www.ted.com/talks/terry_moore_why_is_x_the_unknown to see why we use x for the unknown in Algebra.

1.1 Modeling and Differential Equations

Problem 1.1. Write several examples of algebraic equations and several examples of differential equations. Explicitly state the goal in solving these equations. (You do not actually need to solve the equations) ▲

Definition 1.2 (Differential Equation)**.** A **differential equation** is an equation that relates a function to its derivative(s). The goal in solving a differential equation is to find the function that satisfies the given relationship.

Let's begin by examining a few modeling-type problems where you need to write the differential equation. After we have a few differential equations we will spend some time building up the basic solution techniques.

Problem 1.3. Write a differential equation for each of the following situations. Let $P(t)$ be a function representing the population at time t (measured in years). To help you write each differential equation think about answering the question:
How does the rate of change of the population relate to the size of the population?

(a) In a fragile population each individual has a 50% chance of surviving in any given year.

(b) The same fragile population simultaneously has an influx of 10 new members every year.

(c) The population in parts (a) and (b) has a reproduction rate of 15% each year (measured after the immigrants arrive).

▲

Problem 1.4. When dissolving a sugar cube in tea the sugar is being pulled from every face of the cube. The rate at which the sugar cube dissolves is a differential equation. Which of the following descriptions of differential equations makes the most sense physically? Assume that the temperature in the tea is roughly constant during this time.

(a) The rate of change of the volume of the sugar cube is proportional to the current volume of the sugar cube.

(b) The rate of change of the volume of the sugar cube is proportional to the current surface area of the sugar cube.

(c) The rate of change of the volume of the sugar cube is proportional to the current lengths of the edges of the sugar cube.

(d) The rate of change of the volume of the sugar cube is constant.

(e) The rate of change of the volume of the sugar cube is zero.

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Once you have a physically reasonable choice from the list above write the associated differential equation in terms of volume. ▲

Problem 1.5. Did you know that you could make spherical ice cubes ... wait, that name seems wrong ... whatever, check out THIS LINK. I have several questions.

(a) Finish this sentence:

The rate of change of the volume of the ice ball is proportional to ____________

(b) Write your answer from part (a) as a differential equation. Be sure that the left-hand and right-hand sides of your differential equation refer to the same variables.

(c) I want to know which type of ice will keep my drink cold longer: sphere-shaped or cube-shaped. Assume that both chunks of ice start with exactly the same volume. What differential equations would you need to solve to answer this question?

▲

Problem 1.6. An ant is building a tunnel. We want to create a differential equation model for the total time that it takes for the ant to build the tunnel as a function of the length of the tunnel. Which of the following would be an appropriate differential equation? Let x be the length of the tunnel and let $T(x)$ be the total time to dig a tunnel of length x.

1. $T' = kT$ (rate of change of time proportional to total time taken)
2. $T' = kx$ (rate of change of time proportional to current length of the tunnel)
3. $T' = kx^3$ (rate of change of total time proportional to volume).
4. $T' = kS$ (rate of change of total time proportional to the surface area of the end of the tunnel)

▲

Problem 1.7. A population of Alaskan Salmon grows according to the following rules:

- If there are no salmon then the population doesn't change (duh).
- If the population reaches the carrying capacity for the environment, M, the size of the population stops changing.
- When the population is growing and is far away from the carrying capacity the growth rate is roughly proportional to the size of the population.

Write a differential equation that models this scenario. Support your model by discussing what occurs when P is close to M and when P is close to 0.

$$\frac{dP}{dt} = \underline{\hspace{5cm}}$$

▲

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Problem 1.8. A spring oscillates in such a way that its acceleration is proportional to its position relative to an equilibrium point.

- If the spring is a long way from equilibrium then the acceleration is large and pointed back toward equilibrium.
- If the spring is close to equilibrium then the acceleration is small.

Let $y(t)$ be the position of the spring.

$$\frac{d^2y}{dt^2} = \underline{\hspace{5cm}}$$

Sketch a plot of the solution to this differential equation. ▲

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1.2 Differential Equation Terminology

Problem 1.9. Work with your partners to group all of the differential equations by common features. Many of the differential equations could belong to many different groups.

$$\frac{dy}{dt} = ty^2 + 5 \tag{1.1}$$
$$\dot{\theta} - 2\theta = 0 \tag{1.2}$$
$$x'(t) = \frac{1}{x} \tag{1.3}$$
$$\frac{d^2x}{dt^2} = -x + \ln(t) \tag{1.4}$$
$$\ddot{x} + 2\dot{x} + x = \sin(t) \tag{1.5}$$
$$y' + t\log(y) = 5 \tag{1.6}$$
$$x''' + 4x'' - 8x' + 9x = 0 \tag{1.7}$$
$$y'' + y = 0 \tag{1.8}$$
$$\dot{x} + x^2 = 0 \tag{1.9}$$
$$\theta'' + \sin(\theta) = 0 \tag{1.10}$$
$$xx' = 1 \tag{1.11}$$
$$\left(\frac{dy}{dt}\right)^2 + t^2 y = t + t^2 \tag{1.12}$$
$$x' = \frac{1}{2}x + 5 \tag{1.13}$$
$$\theta'\theta''\theta''' = 0 \tag{1.14}$$

▲

Problem 1.10. Now return to the differential equations in Problem 1.9 and classify them based on the following terms.

(a) linear vs nonlinear

(b) first order, second order, or third order

(c) explicitly dependent on time vs implicitly dependent on time

(d) homogeneous vs non-homogeneous

▲

Now let's get the official definitions on the table. I am expecting that much of this terminology is familiar to you already from previous classes. We are going to cover this very quickly and we will be leaving some of the reading and reviewing up to you.

Definition 1.11 (First Order Differential Equation)**.** A **first order differential equation** is a differential equation of the form

$$y'(t) = f(y,t).$$

Notice that a first order differential equation contains only the first derivative of the unknown function (hence the name). The function f can be just about anything and it depends on both $y(t)$ and maybe t explicitly.

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When we encounter new definitions in this class we will always stop and write several examples associated with that definition. It is usually most informative to give examples of some things that *are* the definition and some that *aren't* the definition. I'll get us started.

Example 1.12. Write several examples of first order differential equations and several examples of differential equations that are not first order.
Solution:
Examples of first order differential equations:

$$\frac{dy}{dt} = yt + \sin(y)$$
$$x'(t) = x$$
$$\frac{dP}{dx} = rP(1-P) + h(x)$$
$$y'(t) = y\sin(t)$$

Examples of differential equations that are not first order:

$$y''(t) = 2yt + 5y' \text{ (second order)}$$
$$\frac{d^7y}{dt^7} = \cos(y) + y'' \text{ (seventh order)}$$
$$R^{(4)}(t) = R'''(t) + R''(t) - R'(t) + 17R(t) \text{ (fourth order)}$$

Definition 1.13 (Autonomous Differential Equation)**.** An **autonomous differential equation** is a differential equation of the form

$$y'(t) = f(y)$$

where there is no explicit dependence of the independent variable t on the right-hand side of the equation. A differential equation that has explicit dependence on t is called **non-autonomous**.

Problem 1.14. Write three examples of autonomous first order differential equations and three examples of non-autonomous first order differential equations. ▲

Definition 1.15 (Linear First Order Differential Equation)**.** A **linear** first order differential equation has the form

$$y'(t) + P(t)y(t) = Q(t) \qquad \text{or} \qquad y'(t) = -P(t)y(t) + Q(t).$$

The reason for the name "linear" is that the right-hand side of this equation is literally a linear function of y. Hence the differential equation can be written as $y' = f(y)$

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where $f(y) = -Py + Q$.

Problem 1.16. Write three examples of linear first order differential equations and three examples of nonlinear first order differential equations. In the linear case identify the "P" and the "Q" functions. ▲

Definition 1.17 (Homogenous Differential Equations)**.** A differential equation is called **homogeneous** if, loosely speaking, no terms appear that do not involve the unknown function. Another way to say this is that every term in the differential equation will either contain the function $y(t)$ or its derivatives. A differential equation that is not homogeneous is called **non-homogeneous**.

Problem 1.18. Write three examples of homogeneous first order differential equations and three examples of non-homogeneous first order differential equations. ▲

Problem 1.19. Come up with an example for each of the following descriptions of differential equations.

(a) A linear first order homogeneous differential equation.

(b) A non-linear first order homogeneous differential equation.

(c) A linear first order non-autonomous differential equation.

(d) A linear first order non-autonomous differential equation that is homogeneous.

(e) A linear first order non-autonomous differential equation that is non-homogeneous.

▲

Example 1.20. Here are a few examples of homogeneous and non-homogeneous differential equations. The first four differential equations are linear and the fifth is nonlinear.

- The differential equation $y' = -0.2y$ is first order homogeneous.
- The differential equation $y' = -0.2y + 3$ is first order non-homogeneous.
- The differential equation $y'' + 3y' - 5y = 0$ is second order homogeneous.
- The differential equation $y'' + 3y' - 5y = 2$ is second order non-homogeneous.
- The differential equation $v' = g - cv^2$ is first order non-homogeneous (and nonlinear).

1.3 Solution Technique: Integration

In the sections that follow we will review (or introduce) some of the primary solution techniques for first order differential equations. As has been mentioned before, it is likely that you have seen these techniques before but it is worth your time to blow the dust off of your memories and to review what you once knew.

Problem 1.21. Consider the function $y(t) = t^2 + 5t + 7$. We know that $y'(t) = 2t + 5$ by taking the derivative with the power rule. If you encountered the differential equation

$$y' = 2t + 5$$

without any prior knowledge of $y(t)$, how would you work backwards to get $y(t)$? Would your answer be unique? ▲

Problem 1.22. For each of the following differential equations use the rules of Calculus find the function $y(t)$ that solve the differential equation.

$$\frac{dy}{dt} = 2t + 5 \quad \text{with} \quad y(0) = 3$$
$$\frac{dy}{dt} = \sin(t) \quad \text{with} \quad y(0) = 1$$
$$\frac{dy}{dt} = te^{-t^2} \quad \text{with} \quad y(0) = 0$$

▲

Technique 1.23 (Solution via Integration)**.** To solve

$$\frac{dy}{dt} = f(t)$$

you can first think of "multiplying by dt" to get $dy = f(t)dt$. Then integrate both sides with respect to t. Therefore,

$$y(t) = \int f(t)dt + C = F(t) + C$$

where $F(t)$ is the antiderivative of $f(t)$ such that $F'(t) = f(t)$. Given some additional piece of information $y(t_0) = y_0$ we can find the constant C by substituting $t = t_0$ and $y = y_0$ and solving for C. Indeed,

$$y_0 = F(t_0) + C \quad \implies \quad C = y_0 - F(t_0).$$

Problem 1.24. Create and solve a first order differential equation (along with an appropriate initial condition) that can be solved using the technique of integration. ▲

Problem 1.25. Solve the differential equation

$$y'(t) = e^{-2t} \quad \text{with} \quad y(0) = 1$$

▲

Example 1.26. Solve the differential equation $\frac{dx}{dt} = \sin(2t)$ with $x(0) = 3$.
Solution:
We first notice that the differential equation can be written as $\frac{dx}{dt} = f(t)$ where $f(t) = \sin(2t)$. This perfectly matches the form in Technique 1.23 and we see that

$$x(t) = \int f(t)dt = \int \sin(2t)dt = -\frac{1}{2}\cos(2t) + C.$$

Using the initial condition we see that $3 = -\frac{1}{2}\cos(0) + C$ and since $\cos(0) = 1$ we see that $C = 3 + \frac{1}{2} = \frac{7}{2}$. Therefore $x(t) = -\frac{1}{2}\cos(2t) + \frac{7}{2}$.

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1.4 Solution Technique: Separation of Variables

Problem 1.27. Consider the differential equation

$$\frac{dy}{dt} = y$$

with the initial condition $y(0) = 1$.

(a) Putting the differential equation into words:
the derivative of some unknown function is equal to the function itself.
what is the function?

(b) Allow me to abuse some notation:
If you multiply both sides by dt and divide both sides by y we end up with

$$\frac{dy}{y} = dt.$$

Integrate both sides and solve for y.

(c) Compare your answers to parts (a) and (b).

▲

Problem 1.28. In part (b) of the previous problem I said that I was "abusing notation". What does that mean? What notation is being abused? ▲

Problem 1.29. Use the same idea used in problem 1.27 to solve the differential equation

$$\frac{dy}{dt} = y\sin(t).$$

Once you have your answer take the derivative and verify that the function that you found is indeed a solution to the differential equation. ▲

Theorem 1.30 (Separation of Variables)**.** To solve a differential equation of the form

$$\frac{dy}{dt} = f(y) \cdot g(t)$$

Separate and integrate by treating the "dy/dt" as a fraction[a]

$$\int \frac{dy}{f(y)} = \int g(t)dt$$

Notice that the right-hand side of the differential equation factors perfectly hence separating the variables into the functions f and g.

[a] Technically speaking the "dy/dt" is not a fraction it is a shorthand notation for a limit.

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Proof. Let's examine separation of variables a bit more closely since there is some calculus funny business going on. If $\frac{dy}{dt} = f(y)g(t)$ then certainly we can rewrite as

$$\frac{dy}{dt} = f(y)g(t) \quad \Longleftrightarrow \quad \frac{1}{f(y)}\frac{dy}{dt} = g(t).$$

This is true so long as $f(y)$ is nonzero of course. Now if we integrate with respect to t on both sides we have two valid integrals with respect to time:

$$\int \frac{1}{f(y(t))}\frac{dy}{dt}dt = \int g(t)dt.$$

From calculus we recall that the quantity $\frac{dy}{dt}dt$ is the differential dy[†]. Hence we arrive at the separated form

$$\int \frac{dy}{f(y)} = \int g(t)dt$$

and this ends the proof that separation of variables is a valid technique for solving differential equations of the form $y' = f(y)g(t)$. □

Problem 1.31. With your partner, write a differential equation that can be solved via separation of variables. Once you have your equation trade with a different group and solve their equation. ▲

Problem 1.32. True or False: Every first order autonomous differential equation is separable. Be able to defend your answer. ▲

Problem 1.33. A drug is eliminated from the body via natural metabolism. Assume that there is an initial amount of A_0 drug in the body. Which of the following is the best differential equation model for the drug removal? Once you have the model solve it with the appropriate technique.

1. $A' = -kt$
2. $A' = -kA$
3. $A' = -kA(1 - A/N)$
4. $A' = -kAt$

▲

Problem 1.34 (Separation and Partial Fractions)**.** A population grows according to the differential equation

$$\frac{dP}{dt} = 2P\left(1 - \frac{P}{10}\right)$$

[†]No. We are not cancelling the "dt".

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with initial condition $P(0) = 5$. This differential equation is separable and separating the variables gives

$$\frac{dP}{P(1-P/10)} = 2dt.$$

Integrating both sides yields

$$\int \frac{dP}{P(1-P/10)} = \int 2dt.$$

The right-hand integral is really easy: $\int 2dt = 2t + C$. The left-hand integral, on the other hand, takes a bit of work. We will use the method of partial fractions (see Appendix A) to rewrite the fraction on the left-hand side as

$$\frac{1}{P(1-P/10)} = \frac{A}{P} + \frac{B}{1-P/10}.$$

Clearing the fractions gives

$$1 = A(1-P/10) + BP,$$

and choosing appropriate values of P gives

$$A = 1 \quad \text{(found by taking } P = 0)$$
$$B = \frac{1}{10} \quad \text{(found by taking } P = 10).$$

Therefore the left-hand integral becomes

$$\int \frac{dP}{P(1-P/10)} = \int \frac{1}{P} dP + \frac{1}{10} \int \frac{1}{1-P/10} dP,$$

and these are integrals that you can easily do. Finish this problem. ▲

Problem 1.35. In problem 1.4 we write the differential equation

$$\frac{dV}{dt} = kS$$

to describe the dissolution of a sugar cube. Here V represents the volume of the sugar cube and S represents the surface area of the sugar cube. The proportionality constant k describes how fast the dissolution takes place and is likely a function of many physical quantities (like what?). If we let x be the length of a side of the sugar cube recall that the volume and surface area are defined as $V = x^3$ and $S = 6x^2$. Rewrite the differential equation in terms of volume only (rewriting surface area in terms of volume) and solve the resulting differential equation with separation of variables. ▲

Problem 1.36. In a dog, an intravenous dose of 30 mg of pentobarbital sodium per kilogram of body weight will usually produce surgical anesthesia. Also in the dog, pentobarbital has a biological half-life of about 4.5 hours, due almost entirely to metabolism. You anesthetize a 14-kg dog with the above dose of pentobarbital. Two hours later the anesthesia is obviously beginning to lighten and you want to restore the original depth of anesthesia. How many milligrams of pentobarbital sodium should you inject? Write and solve a differential equation to answer this question. ▲

We'll wrap up this subsection with a few more examples.

Example 1.37. Solve the differential equation

$$\frac{dy}{dt} = 0.5y \quad \text{with} \quad y(0) = 7$$

using the method of separation of variables.
Solution: Notice that this differential equation is separable since we can separate the functions of y and the functions of t

$$\frac{dy}{y} = 0.5dt.$$

Integrating both sides of this equation we get

$$\int \frac{1}{y} dy = \int 0.5dt \quad \implies \quad \ln(y) + C_1 = 0.5t + C_2.$$

Notice that if we subtract the constant C_1 from both sides we actually just get a new arbitrary constant on the right-hand side. For this reason it is customary to only write one of the two constants when showing the work for this method

$$\ln(y) = 0.5t + C.$$

Exponentiating both sides of this equation gives

$$y(t) = e^{0.5t+C}$$

and we can now recognize that this is the same, algebraically, as

$$y(t) = e^{0.5t}e^{C}.$$

Furthermore, e^C is just another constant so we write the general solution as

$$y(t) = Ce^{0.5t}.$$

To get the value of C we substitute $t = 0$ into the equation to get $7 = Ce^0$ which implies that $C = 7$ and the solution is

$$y(t) = 7e^{0.5t}.$$

Example 1.38. Solve the differential equation

$$\frac{dy}{dt} = 3y + 12 \quad \text{with} \quad y(0) = 2$$

using separation of variables.

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Solution: We're first going to factor the right-hand side of the differential equation so that the integration that we run in to is not so hard.

$$\frac{dy}{dt} = 3(y+4).$$

Separating and integrating gives

$$\int \frac{1}{y+4} dy = \int 3 dt \implies \ln(y+4) = 3t + C.$$

Exponentiating both sides and repeating the same type of algebra as in the previous example we get

$$y + 4 = Ce^{3t}.$$

Finally, we can subtract 4 from both sides of the equation to get the general solution

$$y(t) = Ce^{3t} - 4.$$

Using the initial condition we see that

$$2 = C - 4 \implies C = 6$$

which implies that

$$y(t) = 6e^{3t} - 4.$$

Example 1.39. Solve the differential equation

$$\frac{dy}{dt} = \frac{y}{t^2} \quad \text{with} \quad y(1) = 5$$

using separation of variables.
Solution: We can separate variables, integrate, and do some algebra to get

$$\int \frac{dy}{y} = \int \frac{dt}{t^2} \implies \ln(y) = -\frac{1}{t} + C \implies y(t) = Ce^{-1/t}.$$

Using the condition $y(1) = 5$ we see that $5 = Ce^{-1}$ which implies that $C = 5e$ and the solution to the differential equation is

$$y(t) = 5e^{1-1/t}.$$

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Example 1.40. Solve the differential equation

$$\frac{dy}{dt} = \frac{2ty}{t^2+1}$$

using separation of variables.
Solution: If we separate the variables and integrate we see that

$$\int \frac{dy}{y} = \int \frac{2t}{t^2+1} dt \quad \implies \quad \ln(y) = \ln(t^2+1) + C.$$

The right-hand integral used the idea of u-substitution (you should stop now and work out the u-substitution by hand). Exponentiating both sides gives

$$y(t) = e^{\ln(t^2+1)+C} = Ce^{\ln(t^2+1)} = C(t^2+1).$$

Example 1.41 (Separation and Partial Fractions)**.** Use separation of variables to solve the differential equation

$$\frac{dx}{dt} = x(2-x).$$

Solution:
Separating the variables gives

$$\int \frac{dx}{x(2-x)} = \int dt \quad \implies \quad \int \frac{dx}{x(2-x)} = t + C.$$

Notice that the left-hand integral is not in a nice form. Let's do some algebra! If we could write the fraction $\frac{1}{x(2-x)}$ as the sum of two fractions then maybe we could integrate. If our complicated fraction $\frac{1}{x(2-x)}$ came from the sum of two fractions then the denominators must have been x and $2-x$. What we don't know are the numerators. Hence we write the following:

$$\frac{1}{x(2-x)} = \frac{A}{x} + \frac{B}{2-x}.$$

This is known as a *partial fraction decomposition* (more information about partial fractions can be found in Appendix A).

Multiplying through by the denominator on the left-hand side gives

$$1 = A(2-x) + B(x).$$

At this point we can find A and B in two different, but equivalent, ways.

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1. If the equation $1 = A(2-x)+B(x)$ is always true then it must be true in particular for $x = 1$ and for $x = 0$. Notice that if we take $x = 2$ then the equation becomes $1 = 2B$ so $B = 1/2$. Notice further that if we take $x = 0$ then we get $1 = 2A$ so $A = 1/2$. Hence
$$\frac{1}{x(2-x)} = \frac{1/2}{x} + \frac{1/2}{2-x} = \frac{1}{2}\left(\frac{1}{x} + \frac{1}{2-x}\right).$$

2. If instead we expand the equation $1 = A(2-x)+B(x)$ to $1 = (B-A)x+2A$ we can match terms on the left- and right-hand sides so that the coefficients of the x's gives $0 = (B-A)$ and the constant terms gives $1 = 2A$. Therefore we see that $A = 1/2$ and hence $B = 1/2$ just as before.

Returning now to the integration problem we have

$$\int \frac{dx}{x(2-x)} = \frac{1}{2}\int\left(\frac{1}{x} + \frac{1}{2-x}\right)dx = \frac{1}{2}\left(\ln(x) - \ln(2-x)\right) = \frac{1}{2}\ln\left(\frac{x}{2-x}\right)$$

where the last step took advantage of a property of logarithms. Putting the integrations together now we see that

$$\frac{1}{2}\ln\left(\frac{x}{2-x}\right) = t + C \quad\implies\quad \ln\left(\frac{x}{2-x}\right) = 2t + C \quad\implies\quad \frac{x}{2-x} = Ce^{2t}$$

$$\implies\quad x = Ce^{2t}(2-x) \quad\implies\quad x = 2Ce^{2t} - xCe^{2t} \quad\implies\quad x + xCe^{2t} = 2Ce^{2t}$$

$$\implies x\left(1 + Ce^{2t}\right) = 2Ce^{2t} \quad\implies\quad \boxed{x = \frac{2Ce^{2t}}{1+Ce^{2t}}.}$$

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1.5 Solution Technique: Undetermined Coefficients

Problem 1.42. For each of the following differential equations determine if we can use the method of integration or separation of variables to solve.

$$\begin{aligned}
y' &= 0.5t \\
y' &= 0.5y \\
y' &= 0.5yt \\
y' &= 0.5y + 1 \\
y' &= 0.5y + t
\end{aligned}$$

▲

We now turn our attention to solving non-homogeneous differential equations. These equations can sometimes be solved by separation of variables (like the fourth differential equation in the previous problem), but not always (e.g. the last one in the previous problem). The technique used to solve linear non-homogeneous differential equations is called the *method of undetermined coefficients* and is outlined in the next problem.

Problem 1.43. Solve the following first order linear non-homogeneous differential equation by following the steps outlined.

$$\frac{dy}{dt} = -0.2y + 3 \quad \text{with} \quad y(0) = 5$$

1. First solve the homogeneous part of the equation using separation of variables: $y' = -0.2y$.

$$y_{hom}(t) = \underline{\hspace{3cm}}$$

2. Next conjecture that a *particular* solution has the same functional form as the non-homogeneity. In this case the non-homogeneity is a constant function so we guess that the particular function is a generic constant function

$$y_{particular}(t) = C.$$

3. The full analytic solution to the differential equation is the sum of the homogeneous and particular solutions: $y(t) = y_{hom}(t) + y_{part}(t)$. Note that this is only the case for linear differential equations.

$$y(t) = \underline{\hspace{6cm}}$$

4. Substitute the particular solution into the differential equation and see what equation comes out

$$\underline{\hspace{3cm}} = \underline{\hspace{3cm}}$$

5. Substitute the initial condition into the analytic solution and see what equation comes out

$$\underline{\hspace{3cm}} = \underline{\hspace{3cm}}$$

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6. Determine the final solution

▲

Some folks call the technique outlined in the previous problem the "four step method", but I can never keep the "four steps" straight. In reality this is just one of many techniques for solving non-homogeneous differential equations, and this technique is just mathematical detective work.

Technique 1.44 (Solving Non-Homogeneous Differential Equations)**.** The method of *undetermined coefficients* is roughly outlined as:

1. find a solution for the homogeneous differential equation,
2. conjecture a particular solution for the non-homogeneous differential equation,
3. use the initial condition to find a relationship between some of the coefficients, and
4. use the particular solution in the differential equation to find the remaining coefficients.

More specifically we solve linear non-homogeneous differential equations can be summarized as follows.

Technique 1.45 (Undetermined Coefficients)**.** To solve a non-homogenous linear differential equation:

1. Solve the associated homogeneous differential equation.
2. Conjecture a *particular solution* that has the same functional form as the non-homogeneity.
3. Build the full analytic solution as a linear combination of the homogeneous and particular solutions: $y(t) = y_{hom}(t) + y_{part}(t)$.
4. Substitute the particular solution into the differential equation.
5. Subsitute the intitial condition(s) into the analytic solution.
6. Use the equations that you found in steps 4 and 5 to find the constants.

Problem 1.46. For each of the following linear non-homogeneous differential equations write the homogeneous solution and the particular solution.

(a) $y' = 3y + 4$ $\quad y_{hom}(t) =$ ____________ and $y_{part}(t) =$ ____________

(b) $y' = 3y + 4t$ $\quad y_{hom}(t) =$ ____________ and $y_{part}(t) =$ ____________

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(c) $y' = 3y + 4\sin(t)$ $\quad y_{hom}(t) =$ ____________ and $y_{part}(t) =$ ____________

(d) $y' = 3y + 4e^{-t}$ $\quad y_{hom}(t) =$ ____________ and $y_{part}(t) =$ ____________

▲

Problem 1.47. Solve all of the differential equations in the previous problem using either separation of variables (if possible) or undetermined coefficients. For each one use $y(0) = 2$. ▲

Example 1.48. Solve the differential equation $x'(t) = x(t) - 1$ with $x(0) = 5$.
Solution:
Notice that this is a linear non-homogeneous differential equation so we can use the method of undetermined coefficients.

1. The homogeneous difference equation is $x' = x$ so we know that the general homogeneous solution is $x_{hom} = C_0 \cdot e^t$.
2. The non-homogeneity is constant so we guess that the particular solution will be constant: $x_{part} = C_1$.
3. The analytic solution takes the form $x_n = C_0 \cdot e^t + C_1$.
4. Putting the initial condition into the analytic solution gives $5 = C_0 + C_1$.
5. Putting the particular solution into the original differential equation gives $C_1 - C_1 = C_1 - 1$ which implies that $C_1 = 1$. Together with the information from the previous step we also now know that $C_0 = 4$.
6. The analytic solution to the differential equation is $a_n = 4 \cdot e^t + 1$.

Example 1.49. Verify that the function $y(t) = (y_0 - 12)e^{-0.25t} + 12$ is a solutions to the differential equation $\frac{dy}{dt} = -\frac{1}{4}y + 3$ for every initial condition $y(0) = y_0$.
Solution:
The differential equation states that if we substitute the derivative of y into the left-hand side of the differential equation and y into the right-hand side then we should get a true statement. Indeed, observe that $y'(t) = -\frac{1}{4}(y_0 - 12)e^{-0.25t}$ and therefore

$$
\begin{aligned}
y'(t) &\stackrel{?}{=} -\frac{1}{4}y(t) + 12 \\
\implies -\frac{1}{4}(y_0 - 12)e^{-0.25t} &\stackrel{?}{=} -\frac{1}{4}\Big((y_0 - 12)e^{-0.25t} + 12\Big) + 3 \\
&= \left(-\frac{1}{4}(y_0 - 12)e^{-0.25t} - 3\right) + 3 \\
&= -\frac{1}{4}(y_0 - 12)e^{-0.25t} \quad \checkmark
\end{aligned}
$$

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Furthermore, for every initial condition $y(0)$ we have

$$y(0) = (y_0 - 12)e^0 + 12 = y_0 - 12 + 12 = y_0 \quad \checkmark.$$

Therefore the given solution satisfies the differential equation for every possible initial condition $y(0) = y_0$.

Example 1.50. Solve the differential equation $y' = 2y + 7t$ with $y(0) = 3$.
Solution:
We will use the method of undetermined coefficients.

1. The homogeneous differential equation is $y' = 2y$ and the associated solution is $y_{hom}(t) = Ce^{2t}$.
2. The non-homogeneity is linear so we assume that the particular solution is a linear function. The most general form of a linear function is: $y_{part}(t) = C_1 t + C_2$.
3. The full general solution is
$$y(t) = C_0 e^{2t} + C_1 t + C_2.$$
4. Putting the initial condition into the analytic solution gives
$$3 = C_0 + C_1 \cdot 0 + C_2.$$
5. Putting the particular solution into the differential equation gives
$$C_1 = 2(C_1 t + C_2) + 7t.$$
We can rewrite this as
$$0t + C_1 = 2C_1 t + 2C_2 + 7t,$$
and now we match coefficients to get
$$0 = 2C_1 + 7$$
and
$$C_1 = 2C_2.$$
From this we see that $C_1 = -7/2$ and $C_2 = -7/4$. Furthermore, we can use the result from the previous step to get $C_0 = 3 - C_2 = 3 + 7/4 = 19/4$.
6. The solution is
$$y(t) = \frac{19}{4}e^{2t} - \frac{7}{2}t - \frac{7}{4}.$$

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Example 1.51. Solve the differential equation $y'(t) = \frac{1}{3}y + \cos(t)$ with $y(0) = 1$.
Solution:
We will use the method of undetermined coefficients.

1. The homogeneous equation is $y' = \frac{1}{3}y$ and the solution is $y_{hom}(t) = C_0 e^{t/3}$.
2. The non-homogeneity is a trigonometric function so we use a linear combination of both sine and cosine for the particular solution:
$$y_{part}(t) = C_1 \cos(t) + C_2 \sin(t).$$
3. The full general solution is
$$y(t) = C_0 e^{t/3} + C_1 \cos(t) + C_2 \sin(t).$$
4. Putting the initial condition into the analytic solution gives
$$1 = C_0 + C_1 + 0C_2 \quad \implies 1 = C_0 + C_1.$$
5. Putting the particular solution into the differential equation gives
$$-C_1 \sin(t) + C_2 \cos(t) = \frac{1}{3}(C_1 \cos(t) + C_2 \sin(t)) + \cos(t).$$
At this point we do some fun algebra. The "equal sign" must be true for all t so we can match the like terms and write two equations. For the coefficients of the sine functions we must have
$$-C_1 = \frac{1}{3}C_2.$$
for the coefficients of the cosine functions we must have
$$C_2 = \frac{1}{3}C_1 + 1.$$
6. We now have three equations with three unknowns. I'll write them very carefully so we can switch to matrices and solve using Gaussian Elimination easily.
$$1C_0 + 1C_1 + 0C_2 = 1$$
$$0C_0 + 1C_1 + \frac{1}{3}C_2 = 0$$
$$0C_0 + \frac{1}{3}C_1 - 1C_2 = -1$$

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Switching to matrix notation we have the augmented system and resulting row reduction

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1/3 & 0 \\ 0 & 1/3 & -1 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1/3 & 1 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & -10/9 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 13/10 \\ 0 & 1 & 0 & -3/10 \\ 0 & 0 & 1 & 9/10 \end{array}\right).$$

7. The analytic solution is

$$y(t) = \frac{13}{10}e^{t/3} - \frac{3}{10}\cos(t) + \frac{9}{10}\sin(t).$$

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1.6 Solution Technique: Integrating Factors

It is likely that the previous two solution techniques, separation of variables and undetermined coefficients (four step method), are the ones that you recall best from previous courses. The trouble, however, is that separation of variables and undetermined coefficients can only solve certain types of differential equations. You'll find that the study of differential equations is laced with many different techniques that only work in very particular scenarios. What follows is another powerful technique that allows us to take care of many first order non-autonomous differential equation that are not separable.

Problem 1.52. At this point we have reviewed several techniques for solving first order differential equations. Remember that a linear differential equation can be written in the form

$$\frac{dy}{dt} + P(t)y = Q(t)$$

where both P and Q are functions of t. In this problem we will write first order linear differential equations that can (or cannot) be solved with different techniques.

(a) Write a first order linear differential equation that can be solved by simple integration (not separation of variables).

(b) Write a first order linear differential equation that can be solved with separation of variables.

(c) Write a first order linear differential equation that can be solved with undetermined coefficients but NOT with separation of variables.

(d) Write a first order linear differential equation that can NOT be solved with integration, separation, or undetermined coefficients.

▲

In the last part of the previous problem you likely found a differential equation where all of your known techniques fail. In this section we add a rather handy technique to your toolbox. This technique requires that you remember the product rule from calculus.

Theorem 1.53 (Product Rule)**.** Let $f(x)$ and $g(x)$ be differentiable functions. Then the derivative of the product of $f(x)$ and $g(x)$ is

$$\frac{d}{dx}[f(x)\cdot g(x)] = f(x)\cdot g'(x) + g(x)\cdot f'(x).$$

Problem 1.54. In each of the linear first order differential equations listed below you can multiply both sides by a function so that the left-hand side can be rewritten as the result of the product rule. What function do you need in each case? I'll do the first one for you. (You DO NOT need to solve these differential equations!)

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(a) Consider the differential equation

$$\frac{dy}{dt} + 2y = 7.$$

Solution: If we multiply both sides by e^{2t} the left-hand side becomes

$$e^{2t}\frac{dy}{dt} + 2ye^{2t}$$

which can be viewed as the derivative of the expression $e^{2t}y$ since

$$\frac{d}{dt}\left[e^{2t}y\right] = e^{2t}\frac{dy}{dt} + y\left(e^{2t}\cdot 2\right)$$

(b) Consider the differential equation

$$\frac{dy}{dt} + 2ty = 9.$$

(c) Consider the differential equation

$$\frac{dy}{dt} + \sin(t)y = e^t$$

▲

Problem 1.55. Based on the previous problem, if we have a linear first order differential equation of the form

$$\frac{dy}{dt} + P(t)y = Q(t)$$

what function should we multiply both sides by so that the left-hand side can be seen as the result of product rule? ▲

Now let's solve a differential equation with this idea.

Problem 1.56. Consider the differential equation

$$\frac{dy}{dt} + 2ty = 5t \quad \text{with} \quad y(0) = 1.$$

(a) What do you multiply both sides by so that the left-hand side can be seen as the result of product rule? We will call this function the *integrating factor* and use the notation $\rho(t)$.

$$\rho(t) = \underline{\hspace{3cm}}$$

(b) Rewrite the left-hand side as the derivative of a product

$$\rho(t)\frac{dy}{dt} + 2ty\rho(t) = \frac{d}{dt}\left(\underline{\hspace{3cm}}\right).$$

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(c) Now use your answer from part (b) to rewrite the differential equation

$$\frac{d}{dt}(\underline{\hspace{3cm}}) = \underline{\hspace{3cm}}$$

(d) Integrate both sides with respect to t.

(e) Solve for y and apply the initial condition.

▲

Technique 1.57 (Solutions Via Integrating Factors)**.** To solve the linear differential equation

$$\frac{dy}{dt} + P(t)y = Q(t)$$

we can use the following recipe.

1. Let $\rho(t) = e^{\int P(t)dt}$. This is called an *integrating factor*.
2. Multiply both sides by this integrating factor to get

$$\rho(t)\frac{dy}{dt} + P(t)\rho(t) = Q(t)\rho(t)$$

which can be rewritten as

$$e^{\int P(t)dt}\frac{dy}{dt} + P(t)e^{\int P(t)dt}y = Q(t)e^{\int P(t)dt}$$

3. The left-hand side is the result of the product rule:

$$\frac{d}{dt}\left[y \cdot e^{\int P(t)dt}\right] = Q(t)e^{\int P(t)dt}$$

4. Integrate both sides and solve for y

$$y(t) = e^{-\int P(t)dt}\int\left(Q(t)e^{\int P(t)dt}\right)dt$$

Problem 1.58. Solve the differential equation $y' - 2y = 3$ with integrating factors. ▲

Problem 1.59. The following differential equation can theoretically be solved with integrating factors but the integration may end up being horrible. Work this problem as far as you can.

$$y' - 2ty = 3$$

▲

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Problem 1.60. Write a differential equation that cannot be solved with integrating factors. ▲

Problem 1.61. Solve each of the following first order differential equations with an appropriate solution technique. For each differential equation use $y(0) = 3$. Your goal should be to choose the technique that makes the solution *easiest* to come by.

$$\begin{aligned}
\text{Problem (a): } & y' = -0.2t \\
\text{Problem (b): } & y' = -0.2y \\
\text{Problem (c): } & y' = -0.2y + 3 \\
\text{Problem (d): } & y' = -0.2y + 3t \\
\text{Problem (e): } & y' = -0.2y^2 \\
\text{Problem (f): } & y' = -0.2y \cdot t \\
\text{Problem (g): } & y' = -0.2y \cdot t + t
\end{aligned}$$

▲

Example 1.62. Solve the differential equation $y' + 0.2ty = t$ with $y(0) = 3$ using the integrating factors technique.
Solution: From the differential equation we see that the integrating factor is

$$\rho(t) = \exp\left(\int 0.2t\,dt\right) = \exp\left(0.1t^2\right).$$

Multiplying both sides of the differential equation by $\rho(t)$ gives

$$e^{0.1t^2}\frac{dy}{dt} + 0.2te^{0.1t^2}y = e^{0.1t^2}t,$$

and we can immediately recognize that the left-hand side is the result of the product rule. Hence we can rewrite the differential equation as

$$\frac{d}{dt}\left[e^{0.1t^2}y\right] = te^{0.1t^2}.$$

Integrating both sides with respect to t and solving for y gives

$$e^{0.1t^2}y = \int te^{0.1t^2}dt \implies e^{0.1t^2}y = \frac{e^{0.1t^2}}{0.2} + C \implies y = 5 + Ce^{-0.1t^2}.$$

Finally we can use the initial condition to observe that $C = -2$ and the solution to the differential equation is

$$\boxed{y(t) = 5 - 2e^{-0.1t^2}.}$$

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The observant reader should note that this problem is actually easier to solve with separation of variables by observing that we can initially rewrite as $y' = (-0.2y + 1)t$.

Example 1.63. Solve the differential equation $y' + 2ty = e^{-t^2}$ with $y(0) = 3$ using the integrating factors technique.
Solution: The integrating factor is $\rho(t) = e^{t^2}$ so just as in the previous example we can multiply both sides of the differential equation by this expression to get

$$e^{t^2}\frac{dy}{dt} + 2te^{t^2}y = e^{t^2}e^{-t^2}.$$

The right-hand side clearly simplifies to 1 and the left-hand side can be re-written as the result of the product rule

$$\frac{d}{dt}\left[e^{t^2}y\right] = 1.$$

Integrating both sides with respect to t and multiplying by e^{-t^2} gives

$$y(t) = te^{-t^2} + Ce^{-t^2}.$$

Using the initial condition we see that $C = 3$ and the solution is

$$\boxed{y(t) = te^{-t^2} + 3e^{-t^2}.}$$

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1.7 Mixing Problems

Problem 1.64. In the Great Lakes region, rivers flowing into the lakes carry a great deal of pollution in the form of small pieces of plastic averaging 1 millimeter in diameter. In order to understand how the amount of plastic in Lake Michigan is changing, construct and solve a differential equation model for how this type pollution has built up in the lake. You'll need the following basic facts.

Some basic facts about Lake Michigan.

- The volume of the lake is 5×10^{12} cubic meters.
- Water flows into the lake at a rate of 5×10^{10} cubic meters per year. It flows out of the lake at the same rate.
- Each cubic meter flowing in the lake contains roughly 3×10^{-8} cubic meters of plastic pollution.

Let $P(t)$ denote the amount of pollution in the lake at time t where P is measured in cubic meters and t is measured in years. For the sake of simplicity assume that $P(0) = 0$. ▲

Technique 1.65 (Mixing Problems)**.** Consider the mixing of a contaminant in a tank. If $C(t)$ is the amount of contaminant in the tank then we model the amount by considering that mass is neither created nor destroyed in the process. That is to say that we write a differential equation expressing the conservation of mass:

$$\frac{dC}{dt} = \text{rate that } C \text{ flows in} - \text{rate the } C \text{ flows out.}$$

To determine the two rates it is often easiest to consider the units. The units of the left-hand side are "amount per unit time" so the units of both terms on the right-hand side need to be "amount per unit time."

Problem 1.66. A 120-gallon tank initially contains 90 pounds of salt dissolved in a full tank. Brine containing 2 pounds per gallon of salt flows into the tank at a rate of 4 gallons per minute and the well stirred mixture flows out of the tank at the same rate. Write a differential equation for the amount of salt in the tank and solve your differential equation.

Remember that if $S(t)$ is the amount of salt in the tank then

$$\frac{dS}{dt} = \text{rate that salt flows in} - \text{rate that salt flows out}$$

and the units of each term all need to be the same

$$\frac{\text{pounds}}{\text{minute}} = \frac{\text{pounds}}{\text{minute}} - \frac{\text{pounds}}{\text{minute}}.$$

▲

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Now what if the volume of the mixing solution isn't staying constant ... this is going to be awesome!

Problem 1.67. A 120-gallon tank initially contains 90 pounds of salt dissolved in 100 gallons of water. Brine containing 2 pounds per gallon of salt flows into the tank at a rate of 4 gallons per minute and the well stirred mixture flows out of the tank at a rate of 3 gallons per minute (so the tank is filling up). Write a differential equation for the amount of salt in the tank and solve your differential equation. ▲

Example 1.68. A very large fish tank initially contains 15 liters of pure water. Brine of constant, but unknown, concentration of salt is flowing in at a rate of 2 liters per minute. The solution is mixed well and drained at a rate of R liters per minute. Let c be the concentration of the brine measured in grams/liter.

Let $x(t)$ be the amount of salt, in grams, in the fish tank after t minutes have elapsed. Write a differential equation expressing the rate of change in the amount of salt in terms of salt in the solution, the unknown concentration, and the outflow rate. What solution techniques would be necessary to solve the resulting differential equation?

Solution:

We write an equation that expresses a conservation of mass:

$$\frac{dx}{dt} = \text{rate that salt flows in} - \text{rate that salt flows out.}$$

- The rate that the salt mixture flows in is 2 liters/min and the concentration is c grams/liter so

$$\text{rate that salt flows in} = 2c \text{ grams / minute.}$$

- The rate that the salt mixture flows out is R liters/minute and the concentration of salt in the tank at time t is $x(t)/V(t)$ where $V(t)$ is the volume of mixture in the tank at time t. Note that the units of x/V are grams/liter. Therefore

$$\text{rate that salt flows out} = \frac{R \cdot x(t)}{V(t)} \text{ grams / minute.}$$

Note well that the volume of mixture in the tank depends on the outflow rate R. If we assume that the rate is constant then V is a linear function of time with $V(0) = 15$ and slope given by the difference between the inflow and outflow rates

$$V(t) = 15 - (2 - R)t.$$

Notice that if $R < 2$ liters/minute then the tank will overflow and if $R > 2$ liters/minute then the tank eventually drain.

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At last, the differential equation that we need to solve is

$$\frac{dx}{dt} = 2c - \frac{Rx}{15 - (2 - R)t}.$$

Carefully checking the units reveals that we have a sensible model:

$$\underbrace{\frac{\text{grams}}{\text{liter}}}_{dx/dt} = \underbrace{\left(\frac{\text{liters}}{\text{minute}}\right)}_{2}\underbrace{\left(\frac{\text{grams}}{\text{liter}}\right)}_{c} - \underbrace{\left(\frac{\text{liters}}{\text{minute}}\right)}_{R}\underbrace{(\text{grams})}_{x}\Big/\underbrace{(\text{liters})}_{V}.$$

In the case that $R = 2$ liters/minute we have an autonomous differential equation and we could solve with either separation of variables or with the method of undetermined coefficients. In the cases were $R \neq 2$ liters/minute we have a non-autonomous differential equation so the only technique available would be integrating factors.

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1.8 Software for Differential Equations

It is a curious fact, but in the study of differential equations it is very rare that we can find analytic solutions by hand for the most interesting problems. We'll spend a large amount of time this semester building analytic solution techniques for very common classes of differential equations – the types that come up in all sorts of areas. In reality, given any off the wall differential equation it is highly unlikely that we'll be able to actually find a solution.

That being said, modern computer algebra systems have some very nice tools that can help us to get solutions to differential equations. We'll use MATLAB here since we all have some experience with it, but there is nothing special about MATLAB's tools as compared to other software packages.

1.8.1 MATLAB's dsolve command

In this short subsection we'll give the basics of how to use the `dsolve` command to solve differential equations. There are many ways to use this tool, but what we'll present here is the best for most of our purposes.

The `dsolve` command takes two inputs:

- a differential equation, and
- an initial condition.

We will have to do a little setup to get MATLAB to recognize what you're doing. Let's just jump right in with an example.

Example 1.69. Say we want to solve the differential equation

$$\frac{dy}{dt} = -\frac{t}{y}$$

with the initial condition $y(0) = 4$. You may notice that this differential equation is separable so take a few moments to carefully solve this equation by hand. You might recognize the implicit form of the solution (before you solve for $y(t)$) from your experience with geometry. Finish off the by-hand work by solving your implicit solution for $y(t)$. Be extra careful with the sign on the square root ... remember your "$\pm$" in your algebra and think carefully how to deal with it.

That was fun, but not all differential equations lend themselves to such nice by-hand analysis. Let's see what MATLAB can do on a problem such as this.

```
syms y(t) % first define a symbolic function y(t)
dy(t) = diff(y(t) , t); % then define the derivative symbolically
IC = 4; % define the initial condition as a parameter in your code
% Now we use dsolve to find the solution
soln(t) = dsolve( dy(t) == - t / y(t) , y(0) == IC)
```

```
% and finally we can plot the solution
ezplot( soln(t), [0,4])
```

Notice the use of the double equal sign in the `dsolve` command. This is to tell MATLAB that we're trying to determine if something is equal (instead of trying to assign something to a value in memory).

Problem 1.70. You should now practice your skills on the following differential equations. Find solutions and build plots for both of these differential equations.

(a) $\dfrac{dy}{dt} = \dfrac{t}{1-y}$ with $y(0) = 4$.

(b) $2y' + y = e^t$ with $y(0) = 1$.

▲

Example 1.71. In the case of second (or higher) derivatives we can still use MATLAB. What if we wanted to solve the equation

$$y'' + 2y' + y = e^t$$

with the initial conditions $y'(0) = 0$ and $y(0) = 1$. In this case there is a second derivative involved so we'll need to define it in the code.

```
syms y(t) % first define a symbolic function y(t)
dy(t) = diff(y(t) , t); % then define the derivative symbolically
d2y(t) = diff(dy(t) , t); % then define the second derivative
IC = 1; % define the initial condition as a parameter in your code
IV = 0; % define the initial velocity as a parameter in your code
% Now we use dsolve to find the solution
soln(t) = dsolve( d2y(t) + 2*dy(t) + y(t) ==  exp(t) , y(0) == IC, dy(0) == IV)
% and finally we can plot the solution
ezplot( soln(t), [0,2])
```

Finally, let's do something a bit more interesting. Often we want to explore the behavior of a differential equation with multiple initial conditions or with multiple values of a parameter.

Example 1.72. Let's return to the differential equation $y' = -t/y$ and solve with initial conditions $y(0) = 1$, $y(0) = 2$, $\ldots$, $y(0) = 6$. You could, of course, do this one line at a time, but come on(!) ...this is a problem just screaming for a loop!

```
syms y(t)
dy(t) = diff(y(t) , t)
for IC = 1:6;
```

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```
    soln(t) = dsolve( dy(t) == -t/y(t) , y(0) == IC);
    ezplot(soln(t) , [0,10]), hold on, grid on
end
```

It should be noted that MATLAB does not always provide us with an analytical solution using elementary functions. For instance, consider the differential equation $y' = y^2 + t$. MATLAB can solve this equation but what it gives us is in terms of the "Bessel" function (which is not an elementary function like a logarithm, sine, exponential, etc).
The moral of the story:
MATLAB's `dsolve` command is a powerful tool for solving differential equations, but we still need to be able to interpret the results in order for the solution to be of any use to us. In the next subsection we'll explore two qualitative techniques that can be used, with the aid of software, to explore differential equations where it may not be possible to find an analytic solution.

1.8.2 Slope Fields and Phase Plots

Analytical solutions are the *gold standard* for differential equations, but are incredibly (and perhaps surprisingly) rare! We can use qualitative analysis (or graphical analysis) to help us learn about the behavior of differential equations that may be difficult to solve. One of our primary tools for this sort of analysis will be the java-based program `dfield`. Find the program linked from the Moodle page and download it to your computer.

Problem 1.73. Use the `dfield` software to create a slope field for each of the first order differential equations that we've played with thus far:

$$y' = \frac{t}{1-y} \quad \text{and} \quad y' = y^2 + t.$$

Use the mouse to select several initial conditions and use screen shots to get pictures of your final plots. If the solver appears to "freeze", press the `stop` button on the java applet to tell it to stop trying to solve the differential equation. Discuss the behavior of solutions to both differential equations. In particular, address each of the following questions:

- Does the solution oscillate?
- Do trajectories converge along a curve?
- Do trajectories grow without bound?
- Do trajectories change directions?
- What happens in the long term with these models?

▲

When we have *autonomous equations* (i.e. an equation for which all terms in the equation contain only the dependent variable y or its derivatives (no t's)), we can also perform equilibrium analysis in addition to the techniques in the previous problem. Recall that an equilibrium solution is a solution that does not change over time; hence $y' = 0$ at the equilibrium point.

An equilibrium solution is stable if solutions near it tend toward it as $t \to \infty$, unstable if solutions near it tend away from it as $t \to \infty$, and semi-stable if it is stable on one side and unstable on the other.

Problem 1.74. Use `dfield` as well as available analytic techniques to analyze the stability of the equilibrium points for the differential equation

$$y' = 10y\left(1 - \frac{y}{5}\right)\left(\frac{y}{10} - 1\right)^2.$$

Provide a detailed analysis in your solution. Part of the analysis should be a plot with y' on the vertical axis and y on the horizontal axis (called a *phase plot*). This plot can help to reveal the stability of the equilibrium points. If you aren't sure how to interpret this plot then now is a time for a discussion with your group mate and neighboring groups. You can also look ahead in these notes to Section 2.1. ▲

1.8.3 An Exploration with MATLAB

Now let's put your MATLAB skills to the test on a modeling problem. The following problem is meant to let you explore with the help of MATLAB. Fully investigate the problem, create all necessary plots (time vs temperature, slope fields, etc), give the verdict in a clear and concise way, and provide all mathematics in your writeup.. The evidence that you give must be iron clad leaving no room for doubt.

Problem 1.75. The butler, who hates the cook, finds the chauffeur dead at 10 a.m. When found, the temperature of the murder victim was 89°F. The thermostat in the room was set to a temperature of 68°F. The French maid, who is suspected of carrying on with the gardener, notes that an hour earlier the governess reported finding the body and at that time the temperature of the body was 90°F. After finding the body and recording its temperature, the governess fainted, and thus the body lay undiscovered until the butler found it an hour later. The wine steward, who is secretly married to the nanny, suddenly appears from the wine cellar, accompanied by the pretty young librarian who is actually his daughter by a previous marriage to the ballet teacher. He is holding a bottle of 1912 Chateau La Fette Townsend. It is known that the butler hated the chauffeur because he (the chauffeur) had replaced him (the butler) in the affections of the chamber maid who is currently vacationing in Switzerland. The butler says that although it is true that he hated the chauffeur, he claimed he had been called to the bedside of MiLady's wardrobe attendant who was suffering from an attack of shingles. The fencing master confirms that he was with the butler and the wardrobe attendant (who is actually his mistress) from midnight until 3 a.m. What time was the murder committed and could the butler have done it?
Keep in mind:

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- Thermostats don't hold the room at a constant temperature. Instead the temperature is more periodic.
- A body should cool according to Newton's lay of cooling, but you don't know the exact temperature of the chauffeur when he was alive (did he have a fever? does he run cold? . . .)
- The thermometers used are not 100% accurate (maybe to within a couple degrees).
- Does anyone really know what time it is exactly? Consider all of the times to be estimates to within a few minutes.

▲

1.9 Additional Exercises

Problem 1.76 (The Oil Slick Problem)**.** An oil slick spreads at sea. From time to time, but irregularly, a helicopter is dispatched to photograph the oil slick. On each trip, the helicopter arrives over the slick, the pilot takes a picture, waits 10 minutes, takes another, and heads home. On each of seven trips the size (in area) of the slick is measured from both photographs. The data is given below.

Size at Initial Observation (sq. mi.)	Size 10 Minutes Later (sq. mi.)
1.047	1.139
2.005	2.087
3.348	3.413
5.719	5.765
7.273	7.304
8.410	8.426
9.117	9.127

Work with your partner to create a differential equation model for the size of the oil slick. Note well that we do NOT know the time between helicopter arrivals.
Hint: What can you learn about the differential equation from a plot that shows the initial observation on the horizontal axis and the rate of change on the vertical axis? ▲

Problem 1.77. A fishing pond currently has 1000 rainbow trout (R) but the Department of Fish, Wildlife, and Parks (FWP) has decided to institute a restocking plan with a mix of rainbow trout and brown trout (B) to diversify the species in the pond. Assume that the fishermen from the pond remove 50 fish per week and FWP restocks with 30 mature rainbows and 25 mature browns every week. Model the population of rainbow trout with a differential equation. Solve the differential equation using appropriate techniques and plot both populations together. ▲

Problem 1.78. In this problem you will be given a differential equation with a proposed solution. For each, (i) test the proposed solution to make sure that it is indeed a solution to the differential equation, (ii) use the differential equation and given information to find the value(s) of the constants.

(a) Consider the differential equation $3y - ty' = 18$ and the proposed solution $y(t) = Ct^3 + k$. Find a value of k that allows the function to be a solution.

(b) Consider the differential equation $y'' - 6y' + 8y = 0$ and the proposed solution $y(t) = e^{kt}$. In this problem there are two values of k that should make the differential equation true.

▲

Problem 1.79. Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings. Suppose that the temperature of a cup of coffee obeys Newton's law of cooling. If the

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coffee has a temperature of 185 degrees Fahrenheit when freshly poured, and 2 minutes later has cooled to 166 degrees in a room at 72 degrees, determine when the coffee reaches a temperature of 141 degrees. ▲

Problem 1.80. The amount of water in a pond varies over time. Water flows in steadily so that 200 gallons flow in each day while 2% of the water evaporates each day. As a result we can model this system with the differential equation $y' = -0.02y + 200$ and this equation has a solution $y(t) = -5000e^{-0.02t} + 10000$. What are the units on y, t, y', -0.02, 200, -5000, $-0.02t$, and 10000? ▲

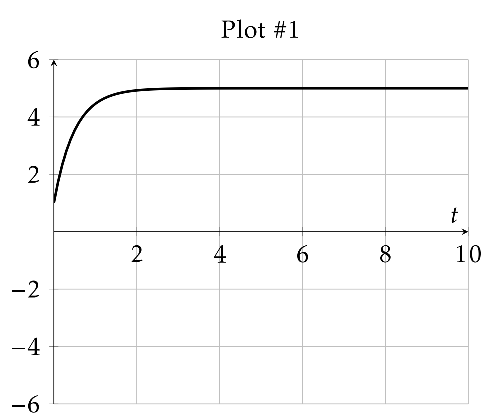
Problem 1.81. Match each of the differential equations to its solution.

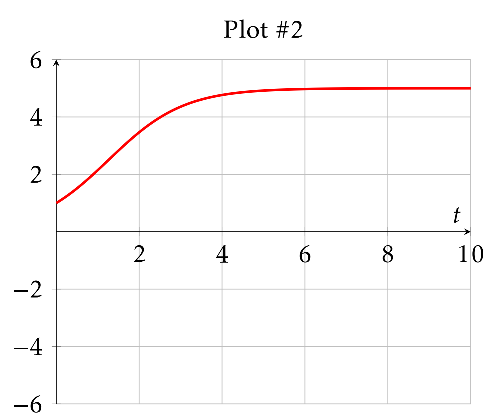
Diff. Eq. #1:	$y'' + y = 0$	Soln #1:	$y = \sin(t)$
Diff. Eq. #2:	$ty' - y = t^2$	Soln #2:	$y = \sqrt{t}$
Diff. Eq. #3:	$2t^2y'' + 3ty' = y$	Soln #3:	$y = e^{-4t}$
Diff. Eq. #4:	$y'' + 8y' + 16y = 0$	Soln #4:	$y = 3t + t^2$

▲

Problem 1.82. Match the differential equations below to the solution plots further below. There should be no need to actually solve the differential equations, but I won't stop you if that is what you really want to do.

Differential Equation	Initial Conditions	Matches to Plot #
$y' - 0.2y = 0$	$y(0) = 1$	
$y' = 0.2y(5 - y)$	$y(0) = 1$	
$y' = -2y + 10$	$y(0) = 1$	
$y' + 0.2t = 0$	$y(0) = 1$	





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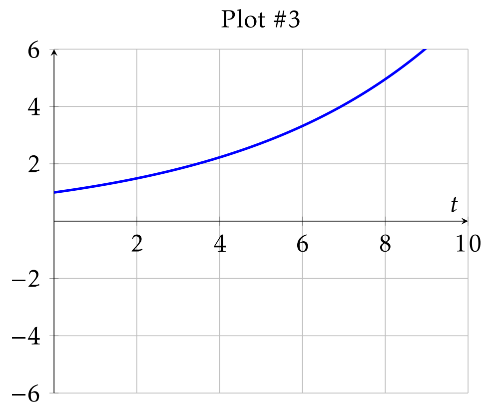

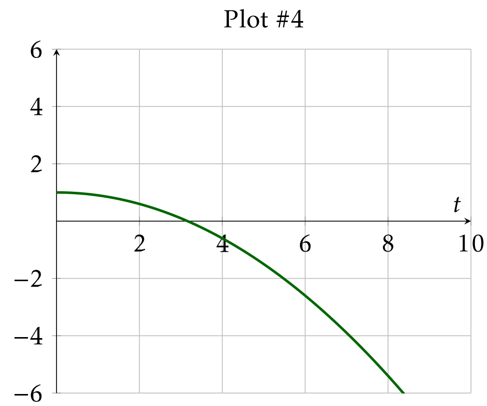

▲

Problem 1.83. For each of the following scenarios write a differential equation and give an appropriate initial condition. Do not solve these differential equations.

(a) A tank contains 10kg of salt and 2000L of water. Pure water enters the tank at a rate of 4L/min, the solution is thoroughly mixed, and solution is drained at 3 L/min. Let $S(t)$ be the amount of salt in the tank at time t. Write a differential equation describing the amount of salt in the tank.

$$\frac{dS}{dt} = \underline{\hspace{4cm}} \quad \text{with} \quad S(0) = \underline{\hspace{1.5cm}}$$

(b) Oil is pumped continuously from a well at a rate proportional to the amount of oil left in the well. Initially there were 2 million barrels of oil in the well. Let $y(t)$ be the number is barrels left in the well (measured in millions of barrels).

$$y' = \underline{\hspace{4cm}} \quad \text{with} \quad y(0) = \underline{\hspace{1.5cm}}$$

(c) Newton's Law of Cooling states that the rate of change of the temperature of a cooling body (like tea in a cup) is proportional to the difference between the current temperature and the ambient room temperature. Assume that the tea starts at 160°F and cools, eventually, to 66°F. Write the differential equation associated with Newton's Law of Cooling.

$$\frac{dT}{dt} = \underline{\hspace{4cm}} \quad \text{with} \quad T(0) = \underline{\hspace{1.5cm}}$$

(d) A pair of turtles, let's call them Adam and Eve, start a turtle colony in a small Montana pond (let's just ignore the lack of genetic diversity here). The population of turtles in the pond follows these simple rules: (1) If there are no turtles then the population clearly doesn't change, (2) the pond can support roughly 100 turtles, and (3) when the population is growing and is far away from the carrying capacity the growth rate is roughly proportional to the size of the population. Let $P(t)$ be the size of the turtle population at time t.

$$\frac{dP}{dt} = \underline{\hspace{4cm}} \quad \text{with} \quad S(0) = \underline{\hspace{1.5cm}}$$

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(e) A 120-gallon tank initially contains 90 pounds of salt dissolved in 100 gallons of water. Brine containing 2 pounds per gallon of salt flows into the tank at a rate of 4 gallons per minute and the well stirred mixture flows out of the tank at a rate of 3 gallons per minute (so the tank is filling up). Write a differential equation for the amount of salt in the tank.

$$\frac{dS}{dt} = \underline{\hspace{4cm}} \quad \text{with} \quad S(0) = \underline{\hspace{1.5cm}}$$

▲

Problem 1.84. In Problem 1.5 we built models for the melting of ice cubes. Solve both of the differential equations resulting from those models and answer the question: which type of ice cube will keep my drink colder longer. ▲

Problem 1.85. A stone is dropped from rest at an initial height h above the surface of the earth. We want to show that the speed with which it strikes the ground is $v = \sqrt{2gh}$. Start by writing an appropriate differential equation and then use the differential equation to very this result. You do not need to include air resistance in your model. ▲

Problem 1.86. In a local pine forest the Pine Beetle is killing the trees at a rate proportional to the number of available trees in the forest. A conservation group is attempting to curb the problem by planting 5 live trees per week. Write a differential equation describing this scenario, classify the differential equation, and determine if it can be solved with separation of variables. ▲

Problem 1.87. In the movie Interstellar, "Plan B" was for the astronauts to start a colony on a new planet. There was 1 female in the group so she would presumably carry the children. Genetic diversity was no problem because of the donor eggs. The supplies on the colony would be limited by local resources as well as what they brought with them (which minimal). Which of the following models should the astronauts use to plan their future reproduction, and what do the parameters mean? Explain your choice for the best one.

- $P' = kP$
- $P' = kt$
- $P' = -kP\ln(P/N)$
- $P' = kP(1-P/N)$

▲

Problem 1.88. Canyon Ferry reservoir has a volume of approximately $V = 2.33 \times 10^9$ m^3 and assume that the inflow from the Missouri river in the spring is $R = 113$ m^3/sec. Assume further that the dam leading to Hauser reservoir is open so the outflow rate is the same as the inflow rate in Canyon Ferry. A large gas tank at a marina upstream is leaking into the river so the contaminated water coming in has a concentration of $c = 0.25$kg/m^3.

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Write a differential equation for the amount of gas, $G(t)$, in Canyon Ferry lake at time t. Assume for simplicity that the gas is well mixed in the lake. Once you have your model solve it with an appropriate technique.
Hint #1: The rate of change of the amount of gas equals the rate in minus the rate out

$$\frac{dG}{dt} = \text{rate that the gas comes in} - \text{rate that the gas goes out}$$

Hint #2: Do not substitute the values given until you have solved the model. ▲

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Chapter 2

Qualitative and Numerical Methods

Let's think about the by-hand calculus techniques for differentiation and integration for a minute. If you were to ask someone for a function then so long as the person gave you the algebraic form of the function you could always differentiate it. Always! By hand derivatives, in some sense, are easy. We have differentiation techniques to take care of every algebraic function. Integration, on the other hand, is a different beast. If you are handed some random algebraic function then you would find that it is very rare that an antiderivative exists. Of course we have the ideas of u substitution, integration by parts, trig substitution, partial fractions, etc. to take care of the most common cases, but most algebraic functions simply don't have antiderivatives. What do you do in the cases where an antiderivative doesn't exist? Well, if you're trying to use the fundamental theorem of calculus to get a definite integral you can always revert to Riemann sums and use a computer. You could also use the trapezoidal rule or some other approximation technique, but that is about it; approximation is the only recourse in most instances.

The same types of troubles arise with differential equations. We saw in the first chapter that if we have a first order differential equation then we have a few techniques at our disposal to get an analytic solution, but we most certainly can't solve every differential equation. Take for example the differential equation $y' = \ln(\sin(\ln(xy)))\sin(x^2)$ ... good luck getting an analytic solution. The analytic solution to a differential equation might be the gold standard goal of differential equations, but it may come as a surprise that analytic solutions are actually rare and special unicorns. What we build in this chapter is a collection of techniques designed to find qualitative or approximate solutions to differential equations without actually trying to get analytic solutions. We will rely heavily on graphical intuition and at times we will rely on computers to do the heavy lifting. By the end of this chapter you'll see that we often times don't even need the analytic solution since our qualitative analysis can tell us a wealth of information about the differential equation.

2.1 Equilibrium Points and Stability

Definition 2.1 (Equilibrium Point(s))**.** A point $y = p$ is said to be an **equilibrium point** of a differential equation if $y' = 0$ for all time when $y = p$. That is to say that when $y = p$ the rate of change of y is zero and as such cannot change with time.

2.1.1 Classifying Equilibrium Points

Definition 2.2 (Stability)**.** An equilibrium point $y = p$ of a differential equation can be classified as either **stable**, **unstable**, or **semi-stable**.

- At a **stable equilibrium**, small deviations from $y = p$ will result in the system converging back to $y = p$ over time.
- At a **unstable equilibrium**, small deviations from $y = p$ will result in the system diverging away from $y = p$ over time.
- At a **semi-stable equilibrium**, small deviation in one direction from $y = p$ will result in the system converging back to $y = p$ over time, and small deviations in the other direction will result in the system diverging away from $y = p$ over time. We can think of a semi-stable equilibrium as being stable from one direction and unstable from another.

A first order autonomous differential equation may have an equilibrium point where the change simply stops. In this section we will build the tools necessary to find and analyze equilibrium points for autonomous first order differential equations.

Problem 2.3. The equilibrium point of a first order autonomous differential equation $y'(t) = f(y)$ is defined as the value of y where $y' = 0$.

(a) Why is this called an equilibrium point?

(b) Find the equilibrium for the first order autonomous differential equation

$$y' = -0.5y^2 + 8$$

(c) Make a plot with y' on the vertical axis on y on the horizontal axis for the previous differential equation. Use this plot to determine whether the equilibrium point is stable or unstable.

(d) Use the `dfield` software to make a slope field for this differential equation. Tie your findings from parts (b) and (c) to what you see in the slope field.

(e) Now consider the slightly different differential equation $y' = 0.5y^2 + 8$. Make a plot with y' on the vertical axis and y on the horizontal axis. Use this plot to determine whether the equilibrium is stable or unstable. Then use `dfield` to verify your classifications of the equilibria.

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▲

Problem 2.4. The air resistance on a sky diver is proportional to the square of the velocity. Newton's 2nd law can be used to get a differential equation for the velocity of the sky diver.

(a) Write the differential equation for the velocity (take *down* to be positive). You might want to start with a free body diagram and consider the balance of forces. ...What Would Newton Do? (WWND)

(b) Find the equilibrium (AKA: terminal velocity) in terms of the mass and the proportionality constant for air resistance. How many equilibria are there? Discuss their stability.

▲

Problem 2.5. Create a first order autonomous differential equation that has 2 unstable equilibria and 1 stable equilibrium.
Hint#1: Use the plot of y' vs y to help.
Hint#2: Write the right-hand side of your DE to in factored form. ▲

Problem 2.6. A trout pond has a carrying capacity of 200 fish. Suppose that the trout population can be modeled according to the logistic equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{200}\right)$$

where k is the intrinsic growth rate of the population. For the sake of simplicity in this model let's assume that $k = 0.5$ (for now).

(a) Make a plot with dP/dt on the vertical axis and P on the horizontal axis. Use the differential equation to sketch this plot.

(b) Mark the intercepts on the horizontal axis in the plot above. What do they represent in the context of this problem?

(c) What does it mean about the rate of change of the population if the plot lies above the horizontal axis? What about below?

(d) Use your answer in part (c) to classify the two equilibrium points as either stable or unstable.

▲

Problem 2.7. Use what you learned in the previous problems to find and classify the equilibria for the first order non-linear autonomous differential equation

$$y'(t) = (y-1)(y-2)(y-3)^2.$$

▲

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2.1.2 Phase Line and Slope Field Analysis

Technique 2.8 (Phase Line Analysis)**.** It is often very helpful to draw a *phase diagram* (sometimes called a phase line) to analyze the equilibrium points of an autonomous differential equation. There are four possible cases shown graphically in Figure 2.1. In each of the following fill in with the word(s) "stable", "unstable", "semi-stable approaching from below", or "semi-stable approaching from above".

- In Case #1 there is a/an ________________ equilibrium at $y = 2$.
- In Case #2 there is a/an ________________ equilibrium at $y = 2$.
- In Case #3 there is a/an ________________ equilibrium at $y = 2$.
- In Case #4 there is a/an ________________ equilibrium at $y = 2$.

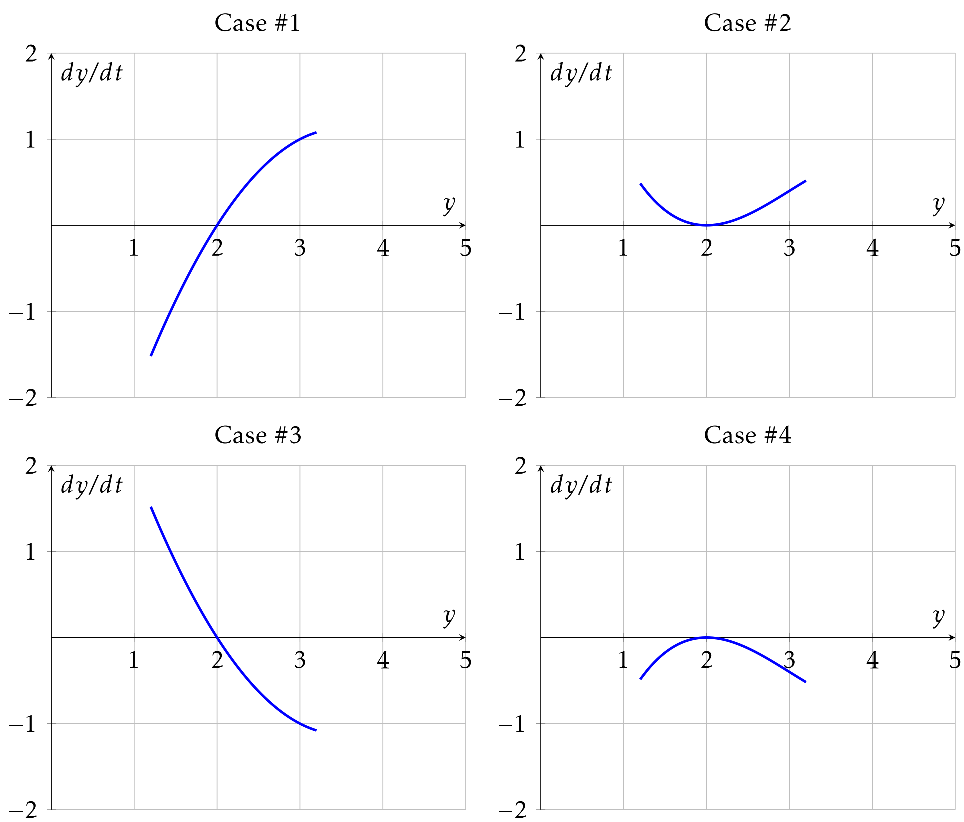

Figure 2.1. Four cases for phase line analysis. In each plot we see a small portion of the $\frac{dy}{dt}$ vs y plot for an autonomous first order differential equation: $\frac{dy}{dt} = f(y)$.

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Problem 2.9. For each of the phase plots in Figure 2.2 sketch a plot on the y vs t plane of the solutions to the underlying differential equation. A few helpful markers are given to you in the first plot. ▲

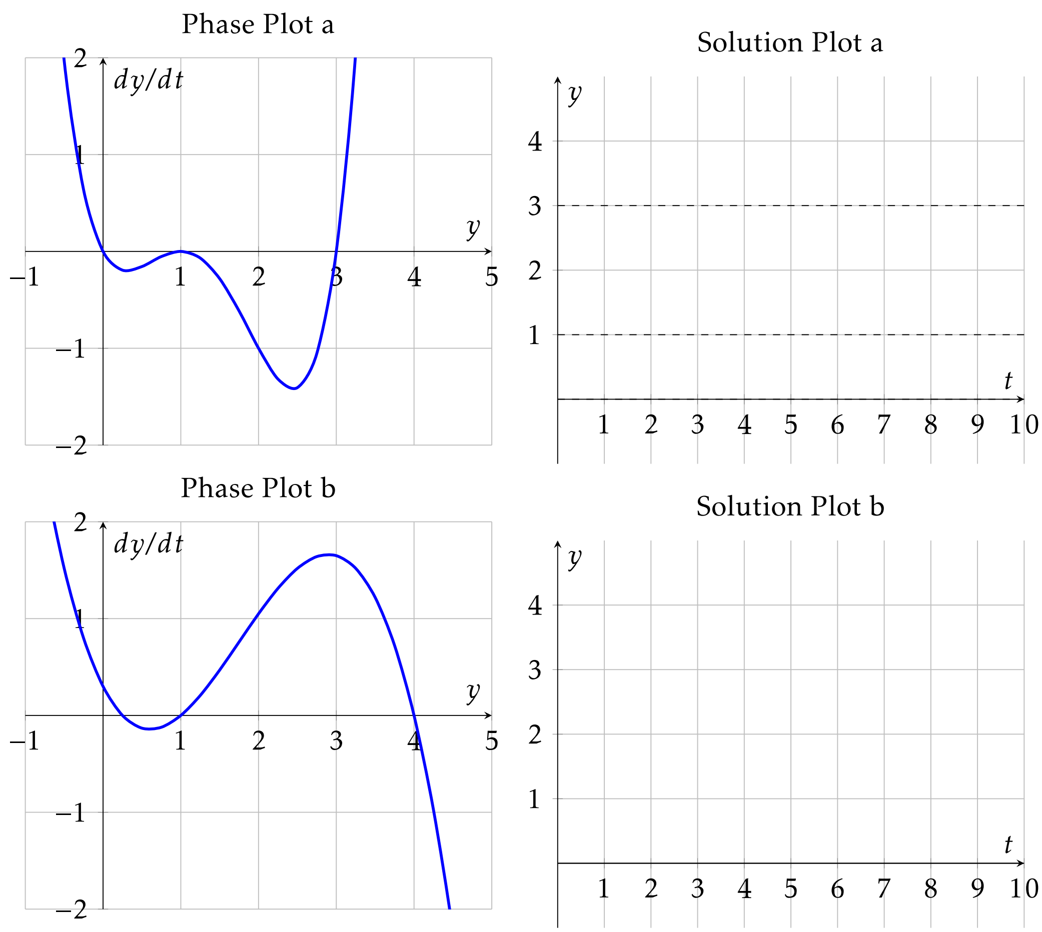

Figure 2.2. Phase plots and solution plots. On the left are the phase plots and on the right are coordinate axes to sketch the solution plots.

Example 2.10. Consider the differential equation $y' = ye^{-y^2}(y-3)$. Find and classify the equilibrium points.
Solution:
To find the equilibrium points we set y' equal to zero and solve the resulting algebraic equation. Hence we need to solve

$$0 = ye^{-y^2}(y-3)$$

which yields $y_{eq} = 0$ and $y_{eq} = 3$ (noting that the exponential is never zero). Now we can observe a phase diagram by plotting y' vs y and use this to determine stability.

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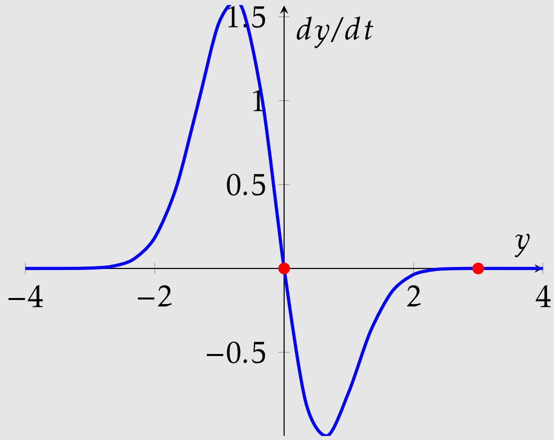
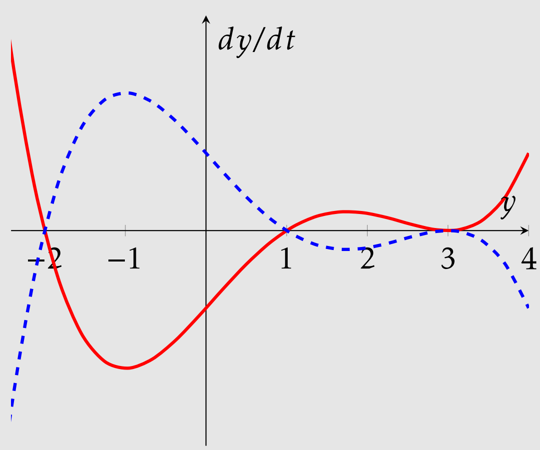


In the plot above it is clear that to the left of $y = 0$ we have $dy/dt > 0$ so the solution is increasing (moving to the right). To the right of $y = 0$ we have $dy/dt < 0$ so the solution is decreasing (moving to the left). Therefore we conclude that $y = 0$ is a stable equilibrium point. At $y = 3$, on the other hand, it is hard to tell what is happening to the right. If we put any y value larger than 3 into the right-hand side of the differential equation we will get a positive number so even though we can't see it in the graph we know that to the right of $y = 3$ the solution is increasing. To the left the solution is clearly decreasing. Hence we conclude that $y = 3$ is an unstable equilibrium point.

Example 2.11. Devise a differential equation that has 1 stable equilibrium, 1 unstable equilibrium, and 1 semi-stable equilibrium.

Solution:

We want a differential equation that, when graphed on the y' vs y plane, looks something like one of the plot below. We're going to just arbitrarily choose the equilibrium points to be $y = -2, 1$, and 3.



In both cases we can see that the point $y = 3$ is semi-stable. In on the red solid curve we see that $y = -2$ is stable and $y = 1$ is unstable. On the blue dashed curve we see that

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$y = -2$ is unstable and $y = 1$ is stable. If we assume that these curves are polynomial functions of y and we know the roots and end behavior then we can write the factored form of the functions that generate them. Indeed, the red curve is generated by the function $f(y) = (y+2)(y-1)(y-3)^2$ and the blue dashed curve is generated by the function $g(y) = -f(y)$. Therefore the differential equations are

$$y' = (y+2)(y-1)(y-3)^2 \text{ (red)}$$

and

$$y' = -(y+2)(y-1)(y-3)^2 \text{ (blue)}$$

Problem 2.12. Draw a sketch of all of the solutions to both of the differential equations generated in the previous example. For this problem the vertical axis should be y and the horizontal axis should be t. ▲

Another tool that we have for creating qualitative solutions for differential equations is the **slope field**. For every point in the $y-t$ plane we can determine the slope of the solution to the differential equation by simply putting the t and y into the right-hand side of the differential equation. At the point we can then draw a small line segment with that slope. If we repeat this process on the differential equation $\frac{dy}{dt} = -\frac{1}{2}(y-4)$ we get the image in Figure 2.3.

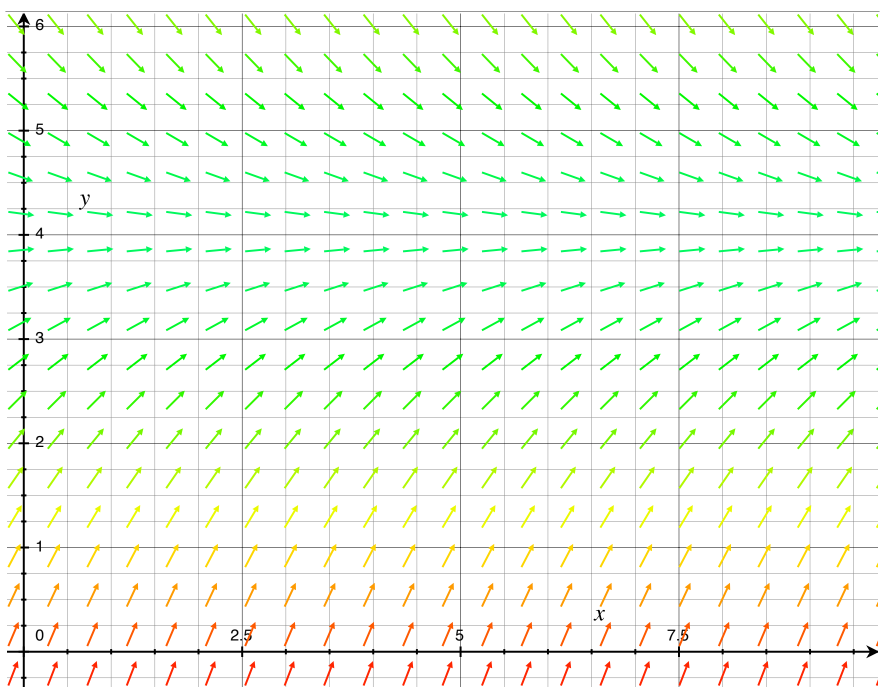

Figure 2.3. A slope field for the differential equation $\frac{dy}{dt} = -\frac{1}{2}(y-4)$.

Problem 2.13. In Figure 2.3 you see a slope field for the differential equation $\frac{dy}{dt} = -\frac{1}{2}(y-4)$.

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(a) If you pick an initial condition and draw a curve that is tangent to the slope field lines as you draw, then you will end up with a rough sketch of a solution to the differential equation. Draw solutions curves for this differential equation using $y(0) = 0, 1, 2, 3, 3.5, 4, 4.5, 5,$ and 6.

(b) Based on the slope field in Figure 2.3, your solution plots from part (a), and the phase plot in Figure 2.4, why are we classifying the equilibrium at $y = 4$ as "stable"?

▲

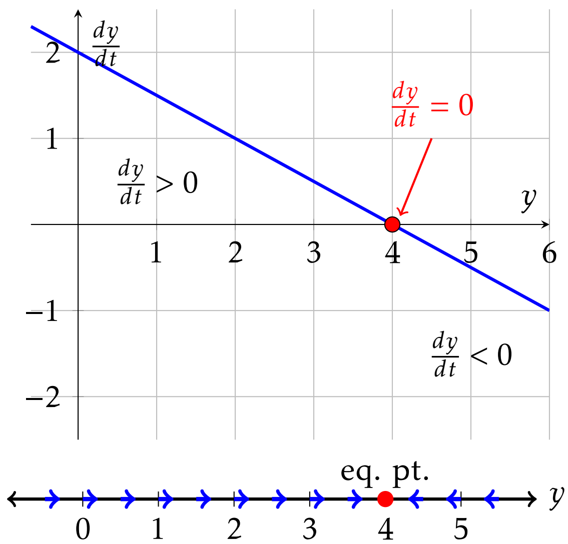

Figure 2.4. A plot of $\frac{dy}{dt}$ vs y (top) and a phase line diagram (bottom) for $\frac{dy}{dt} = -\frac{1}{2}(y-4)$

Problem 2.14. We have introduced three big ideas in the past couple of pages. Let's synthesize some of our thinking. Don't be afraid to look back to the previous examples for inspiration.

(a) To find the equilibrium for a differential equation, set the *change* to ____________. Explain your answer.

(b) When you make a graph of dy/dt vs y (called a phase plot), the equilibrium (or equilibria) of the differential equation can be found where the curve ________________________. Explain your answer.

(c) On a phase line plot you can determine the direction of the arrows by looking at the ____________ of the derivative at that value of y. Explain your answer.

(d) To get the arrows on a slope field, pick a point in the y–t plane and ________________________.

(e) To get a solution curve on a slope field, pick an initial condition and ________________________.

▲

Problem 2.15 (Crowd Sourced Qualitative Analysis)**.** Supplies Needed:

- toothpicks
- yarn
- a large table (or the floor)
- tape
- scissors
- sticky notes

Making slope fields by hand is rather cumbersome, but only ever relying on technology can detract from what you are actually doing in a slope field. In this problem you will work with a team of four to build slope fields, phase plots, and solution plots using toothpicks and yarn.

(a) Start by setting up two coordinate planes. The first one will be y vs t. The second one will be dy/dt vs y. In each case we're only interested in the first quadrant. Use your yarn to mark the axes and then tape them in place. (Make these coordinate planes a few feet by a few feet)

(b) We'll start with the differential equation
$$\frac{dy}{dt} = \frac{1}{2}y(3-y).$$
With your team do the following tasks on this DE. Divide the tasks among your group members.

  (i) Use the toothpicks to mark the slopes in the slope field (on the y vs t plot) for every integer unit of t (from 0 to 4) and every integer value of y (from 0 to 5). (that should be 30 toothpicks!)

  (ii) Use the yarn to show solution curves with initial conditions $y(0) = 0$, $y(0) = 0.1$, $y(0) = 1$, $y(0) = 2$, $y(0) = 3$, $y(0) = 4$, and $y(0) = 5$. Feel free to cut the yarn.

  (iii) In the dy/dt vs y plot use the yarn to create the phase plot.

  (iv) Indicate the location and stability of the equilibrium points with the sticky notes on both plots.

  (v) Take a picture of both of your plots when they're complete. Your instructor will tell you where to upload the photos.

(c) Create a differential equation that has 1 stable equilibrium, 1 unstable equilibrium, and 1 semi-stable equilibrium. Then work with your group to repeat part (b) of this problem with your new differential equation.

Keep in mind that a positive slope points up and to the right, while a negative slope points down and to the right. Also, a slope with an absolute value of 1 has an angle of 45°, a slope with an absolute value less than 1 is more shallow, and a slope with an absolute value greater than 1 is more steep.

▲

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2.2 Numerical Methods

In this section we will build two numerical solvers that allow you to use a computer to approximate solutions to the differential equation

$$y' = f(t, y) \tag{2.1}$$

for a given initial condition. In the problems that follow you will be creating several MATLAB files that you will use throughout the course, so please save them in a meaningful place and share them with your group mates. Numerical solutions are used when all else fails. If you have an analytic solution to your differential equation then there is no need to make a numerical solution since every numerical solution is only a means of approximating solutions – why approximate if you have the exact answer? That being said, analytic solutions to differential equations are rare so we'll have to approximate more times than not.

There are MANY approximation techniques for differential equations. Some are tailored to specific physics or engineering problems, some are tailored to specific types of initial or boundary conditions, and others are designed to work on a wide variety of problems. The two techniques that we build here are both general purpose and relatively easy to program in any programming language. The fact that they are general purpose, however, means that you can get better performance out of other methods on problems of specific types. To learn more I encourage you to take a course on numerical analysis.

2.2.1 Euler's Method

Euler's method is the simplest numerical method for solving the first-order differential equation $y' = f(t, y)$, and hence it is the best place to start! You should be familiar with Euler's method from pre-requisite courses, but this time we are not going to be using Excel (since we usually need something FAR more powerful!).

Euler's method is simple: approximate the derivative in the most naive possible way. That is, use the definition of the derivative from calculus,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

and just drop the limit. Therefore we can approximate $y'(t)$ as

$$\frac{dy}{dt} \approx \frac{y_{new} - y_{old}}{\Delta t}.$$

Then rewrite the differential equation $y' = f(t, y)$ as

$$\frac{y_{new} - y_{old}}{\Delta t} \approx f(t_{old}, y_{old}).$$

After some rearrangement and relabeling we get the difference equation

$$y_{n+1} = y_n + \Delta t f(t_n, y_n).$$

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Technique 2.16 (Euler's Method)**.** To approximate $y' = f(t, y)$ first choose Δt and then implement the difference equation

$$y_{n+1} = y_n + \Delta t f(t_n, y_n)$$

using appropriate computer software.

Remember that the only reasonable choice for Δt is to make it *very small*. The trade off to choosing Δt small is that it will take more computer memory to approximate the problem.

A way to think about Euler's method is that at a given point, the slope is approximated by the value of the right-hand side of the differential equation and then we step forward Δt units in time following that slope. Figure 2.5 shows a depiction of the idea. Notice in the figure that in regions of high curvature Euler's method will overshoot the exact solution to the differential equation. However, taking $h \to 0$ theoretically gives the exact solution at the tradeoff of needing infinite computational resources.

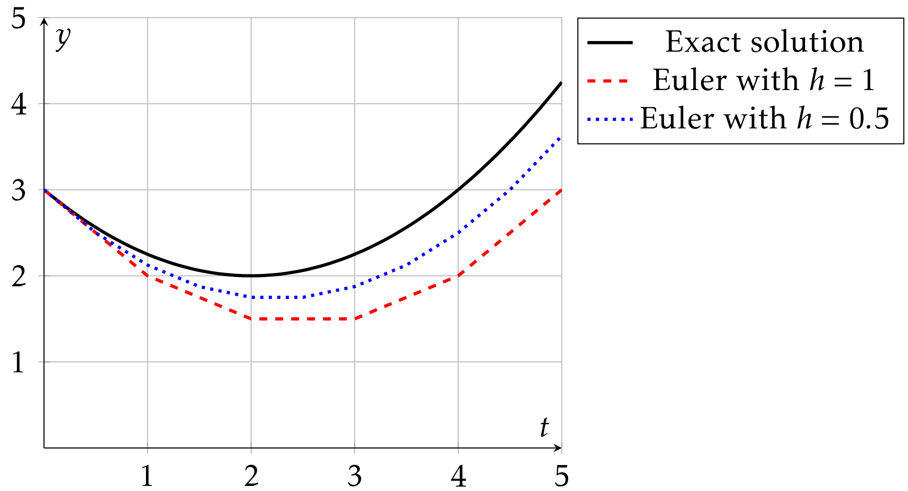

Figure 2.5. A depiction of Euler's method with step size $h = 1$ (red) and $h = 0.5$ (blue).

In Figure 2.6 we see a graphical depiction of how Euler's method works on the differential equation $y' = y$ with $\Delta t = 1$ and $y(0) = 1$. The exact solution at $t = 1$ is $y(1) = e^1 \approx 2.718$ and is shown in red in the figure.

Problem 2.17. In Excel the process of building an Euler solver is relatively simple. In MATLAB it takes a bit more work the very first time, but trust me, the work will pay off in the long run!

(a) Open MATLAB and create a new function.

(b) Change the first line of the function so that it reads
```
function [t,y] = MyEuler(f,tmin,tmax,numpoints,IC)
```

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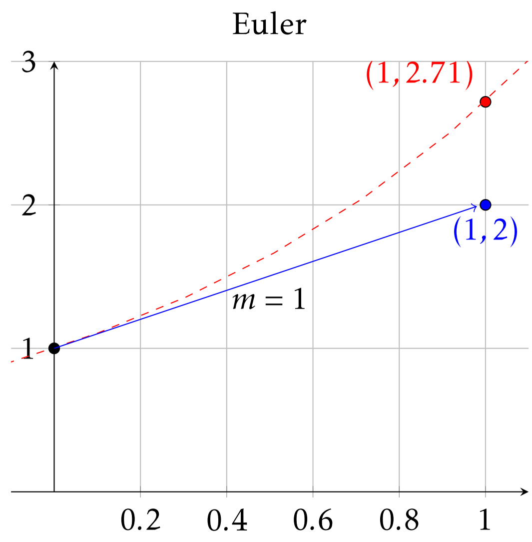

Figure 2.6. Graphical depiction of Euler's method. Here we use the simple differential equation $y' = y$ with $y(0) = 0.5$ and $\Delta t = 1$. The exact solution is shown in red.

(c) Explain what each line of code does using comments. Some of the lines of code are likely new to you so I suggest you either use the help command or you do some basic experimentation.

```
function [t,y] = MyEuler(f,tmin,tmax,numpoints,IC)
t=linspace(tmin,tmax,numpoints);  % what does this line do?
dt=t(2)-t(1); % what does this line do?
y=zeros(size(t)); % what does this line do?
y(1) = IC; % what does this line do?
for n=1:length(t)-1 % what does this line do?
    y(n+1) = ... some code for Euler's method ...
end
```

(d) Save this code in the working directory for this project. You also need to get MATLAB to look in that working directory. The simplest way to do that is to press F5 while you're in the `MyEuler` function (and then ignore all of the errors that occur).

The code you just created works for ALL of the times that you ever need Euler's method. We now just need to create a short script which calls this code and uses it.

(e) Finally, let's try out your MyEuler code (and at this point you should see why you really shouldn't be using Excel for numerical differential equation solvers). I'm assuming that you have MATLAB working in the correct directory (you'll get errors otherwise!!).

We want to get an approximate solution to the differential equation

$$y' = -y \cdot \left(1 - \frac{y}{5}\right) + 0.1t \quad \text{where} \quad y(0) = 2$$

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Open a new script in MATLAB and complete the following code to get your plot.

```
clear; clc; clf;
f = @(t,y) -y*(1-y/5)+0.1*t;
[t,y] = MyEuler(f, ..., ..., ..., ...)
plot(t,y)
grid on
xlabel(...)
ylabel(...)
title(...)
```

(f) Run your code for several different initial conditions and overlay the plots to explore the dynamics of the differential equation. Be sure to properly label your plot. It is probably best to do this step within a loop in MATLAB. To get a legend to appear for each new plot you can use the following:

```
clear; clc; clf;
f = @(t,y) -y*(1-y/5)+0.1*t;
LegendItems = { }; % initialize the storage for the legend entries
counter=1; % set up a dummy counter
for IC= ... : ... : ...
  [t,y] = MyEuler(f, ..., ..., ..., ...)
  plot(t,y), hold on
  LegendItems{counter} = ['IC = ' , num2str(IC)]; % what does this line do?
  counter = counter+1; % what does this line do?
end
legend(LegendItems)
```

▲

Problem 2.18. Run an Euler solver on the differential equation

$$y' = -y\left(1 - \frac{y}{5}\right)^2 + \sin(t)$$

for several different initial conditions. Save your plot in an appropriate place with appropriate labels and title. ▲

2.2.2 Runge-Kutta Method

Euler's method is one of MANY different numerical differential equation solvers. The second one that we are going to study in this section is called the Runge-Kutta 4 solver. The idea is basically the same as with Euler's method: approximate the derivative and rewrite the differential equation as a difference equation. The difference here is that the algorithm is a bit more complex. ...Here it is:

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Technique 2.19 (Runge Kutta Method)**.** First define the dummy variables k_1, k_2, k_3, and k_4 as

$$\begin{aligned} k_1 &= f(t_n, y_n) \\ k_2 &= f(t_n + \frac{\Delta t}{2}, y_n + \frac{\Delta t}{2} k_1) \\ k_3 &= f(t_n + \frac{\Delta t}{2}, y_n + \frac{\Delta t}{2} k_2) \\ k_4 &= f(t_n + \Delta t, y_n + \Delta t k_3). \end{aligned}$$

Then we build the difference equation as a weighted sum of the $k_j's$:

$$y_{n+1} = y_n + \frac{\Delta t}{6} (k_1 + 2k_2 + 2k_3 + k_4).$$

Before we write code to implement the RK4 method we will examine it graphically just as we did with Euler's method in Figure 2.6. For simplicity we will examine the differential equation $y' = y$ with initial condition $y(0) = 1$ and $\Delta t = 1$. In Figure 2.7 the red dashed line is the exact solution $y(t) = e^t$. In this example, $k_1 = 1$, $k_2 = 1.5$, $k_3 = 1.75$, and $k_4 = 2.75$. Hence the final slope propagating forward with $\Delta t = 1$ is

$$\frac{1}{6}(1 + 2(1.5) + 2(1.75) + 2.75) = 1.708.$$

Propagating this forward from the point $(0, 1)$ gives the new point $(1, 2.708)$. Knowing that $e \approx 2.718$ we see a very high level of accuracy even with a really large time step!

Runge-Kutta 4 Method Explanation
1. k_1 is the slope evaluated at time t_n Project this slope half a step forward from time t_n to the point y_1
2. k_2 is the slope evaluated at y_1 Project the slope k_2 half a step forward from time t_n to the point y_2
3. k_3 is the slope evaluated at the point y_2 Project the slope k_3 a full step foward from time t_n to the point y_3
4. k_4 is the slope evaluated at the point y_3
5. Project forward with slope $\frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ from time t_n

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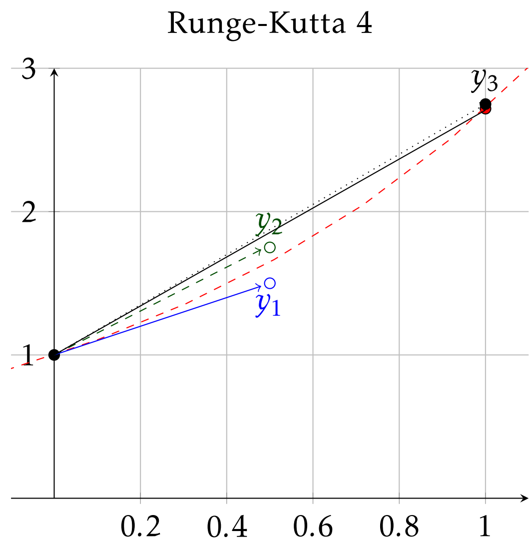

Figure 2.7. Graphical depiction of the RK4 method. The red dashed curve gives the exact solution to the differential equation $y' = y$ with initial condition $y(0) = 1$. The solid black line gives the final projection.

Problem 2.20. (a) Create a new function called `MyRungeKutta` in `MATLAB` and copy all of your code from your `MyEuler` function.

(b) Modify the loop from the Euler code so that you perform the Runge-Kutta iteration. The skeleton code should get you going:

```
...
for n=1:length(t)-1
k1 = ...
k2 = ...
k3 = ...
k4 = ...
y(n+1) = y(n) + ...
end
```

(c) Run your Runge-Kutta code the same way that you did your Euler code. Test it out on the same two problems as before, and be sure that you get the same qualitative solutions.

▲

What we're about to do next should seem a bit silly. We're going to compare our two numerical solvers to a differential equation where we HAVE the analytic solution. You may be asking yourself "that seems silly, if you have the analytic solution then why on earth would you need or want the numerical solution?" Well, we're going to do it here to prove a point.

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Problem 2.21. The differential equation in question is: $y' = -5y$. Solve this equation now by hand (it should only take a second or two).

(a) Write `MATLAB` code that solves this differential equation with Euler's method and with the Runge-Kutta method for several initial conditions. Put the solutions on top of each other in one subplot.

(b) In a second subplot, plot the errors between Euler and the exact solution as well as Runge-Kutta and the exact solution. It would be best to use a logarithmically scaled y-axis (use `semilogy` instead of `plot`). Put the results together with several different initial conditions on the same plots.

(c) What conclusions can you make about the two numerical methods that we have just built? Is one better than the other? When do they have the largest amount of error in general (for any problem)?

▲

The ideas behind solving ordinary differnetial equations numerically are covered extensively in a numerical analysis course. If you're interested I highly recommend that you take this course.

2.2.3 Numerical Explorations

In the following two problems you need to use your numerical codes to solve the differential equations and answer the associated questions. Be sure to support all of your work with sufficient mathematics, appropriate plots, and thorough exposition.

Problem 2.22 (The Combustion Problem)**.** Let T be the temperature of a combustible material (e.g. oily rags, dry hay, etc.). The conservation of energy equation states that

$$\rho c_p \frac{dT}{dt} = A_1 e^{-B/(T-T_0)} - h(T - T_a)$$

where

- T is temperature in Kelvin,
- T_0 is a reference temperature above which the fuel starts oxidizing,
- T_a is the ambient temperature of the surrounding air,
- ρ is the density of the fuel,
- c_p is the specific heat of the fuel source,
- h is a measure of the power per volume per degree Kelvin,
- A_1 is a measure of the power per unit volume, and

- B is a rate constant measured in degrees Kelvin.

If we divide both sides of the differential equation by ρc_p we arrive at the first order non-homogeneous differential equation

$$\frac{dT}{dt} = Ae^{-B/(T-T_0)} - C(T - T_a).$$

Assume that A, B, and C are all positive coefficients.
Your Tasks:

(a) Why must $T > T_0$ in order for the equation to make sense physically?

(b) Let's suppose that $T_a = 300°K$ and that T_0 is also at the ambient temperature. Let $A = 20$, $B = 600$, and $C = 0.01$. Analyze the differential equation graphically plotting the phase portrait, identifying equilibrium points, and discussing stability of each point.

(c) Discuss the physical interpretation of each equilibrium point.

(d) Use a numerical solver to get a graphical solution to the differential equation under the conditions listed in part (b).

(e) Suppose we don't know what A, B, and C are, but we do know from experiments that the three fixed points are $T_1 = T_a = 300°K$, $T_2 = 670°K$, and $T_3 = 1200°K$. What can you say about the coefficients A, B, and C?

▲

Problem 2.23 (The Spherical Flame)**.** In low gravity a candle flame does not take on the same shape as we see here on earth. While on earth there is a buoyancy effect that causes the shape of a flame to elongate. In low gravity the flame takes on a spherical shape (google it ... the pictures are awesome!).

Oxygen, which fuels the flame, enters the sphere at a rate proportional to the surface area. Combustion consumes the oxygen at a rate proportional to the volume. Therefore the differential equation modeling the radius of the spherical flame is

$$\frac{dr}{dt} = \alpha r^2 - \beta r^3$$

where α and β are the proportionality constants controlling the rate at which oxygen enters and is consumed respectively.

(a) What are the units of α and β, what are the equilibrium points of the differential equation, and discuss the stability of the equilibrium points. Clearly show your work and support your discussion of stability with an appropriate plot.

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(b) Having two parameters in the model is a bit annoying so the following is a great mathematical tool called *non-dimensionalization* wherein we choose a change of variables that allows us to effectively eliminate one parameter. In this case, let R be a new variable defined as the radius of the flame, r, divided by the non-zero equilibrium of the differential equation. That is,

$$R = \frac{r}{r_{eq}} = \frac{r}{(\alpha/\beta)} = \frac{r\beta}{\alpha}.$$

This re-scaling is done so that $R = 1$ assumes the role of the equilibrium point (and hence $0 < R < 1$ for all flames). What are the units of R?

Furthermore, show all of the necessary calculations to prove that R satisfies the differential equation

$$\frac{dR}{dt} = \frac{\alpha^2}{\beta}\left(R^2 - R^3\right).$$

Finally, since the ratio α^2/β only appears once we can rename it an hence only have one parameter in the model. Therefore, the differential equation of interest is

$$\frac{dR}{dt} = \gamma\left(R^2 - R^3\right).$$

(c) Use an appropriate numerical differential equation solver to produce plots showing several solutions with different values of γ and different initial conditions. Notice that γ has units of 1/time so values such as $\gamma = 1$, $\gamma = 10^{-1}$, $\gamma = 10^{-2}$, etc make sense. The initial conditions are the fraction of the steady state radius so values such as $R(0) = 10^{-1}, 10^{-2}, 10^{-3}$, etc make sense. Be sure to clearly label your plots and to use sufficient total time to see the interesting dynamics.

(d) Clearly explain how what you're seeing in the plots relates to what is happening physically.

▲

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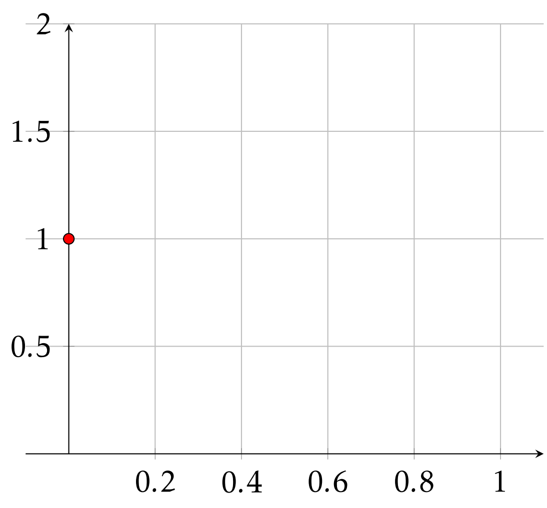
2.3 Additional Exercises

Problem 2.24. Explain how Euler's method works in clear mathematical language. ▲

Problem 2.25. It is tempting once you have written computer code for something like Euler's method to forget what is actually going on under the hood and just let it act as a *black box* computational tool. In this problem you will simply take several steps of an Euler solution by hand and graphically so you can mentally unpack what Euler's method is doing. Now would be a good time to go back and read the section on Euler's Method (Section 2.2.1).

Consider the differential equation $\frac{dy}{dt} = -\frac{ty}{2} + t^2$ with initial condition $y(0) = 1$. We would like to build an Euler approximation on the interval $t \in [0, 1]$ with a time step of $\Delta t = 0.2$. Use Euler's method to give approximations starting at $t = 0$ and ending at $t = 1$. Plot your approximate solution on the given coordinate plane. After you've done this by hand use your code to get a better approximation using Euler's method with a smaller Δt. Compare your answers.

t	Approximation of $y(t)$
0	1
1	



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Problem 2.26. Write a differential equation that has four equilibrium points: 2 unstable, 1 stable, and 1 semi-stable. Support your answer with a plot of y' vs y.

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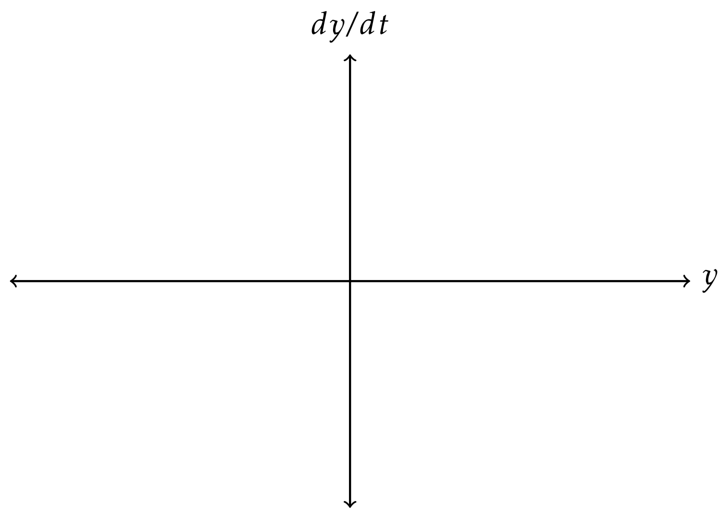
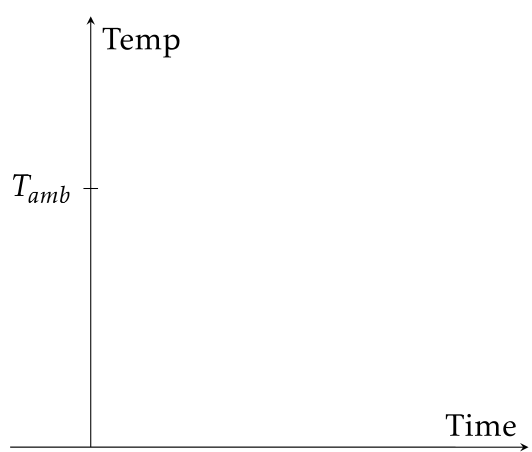


$$\frac{dy}{dt} = \underline{\qquad\qquad\qquad\qquad}$$

▲

Problem 2.27. Consider the differential equation $y' = -7(y+3)(y-4)$. What are the equilibrium values of this equation and are they stable, unstable, or semistable? What happens to the equilibrium points if the -7 were changed to 7 in the differential equation? ▲

Problem 2.28. Newton's Law of Cooling states that the rate of change of the temperature of a cooling body (like a coffee in a cup) is proportional to the difference between the current temperature and the ambient room temperature. Write the differential equation associated with Newton's Law of Cooling and sketch several solution plots. Some of your plots should be drawn assuming that the initial temperature is greater than the ambient temperature and some should be drawn assuming that the initial temperature is less than the ambient temperature.



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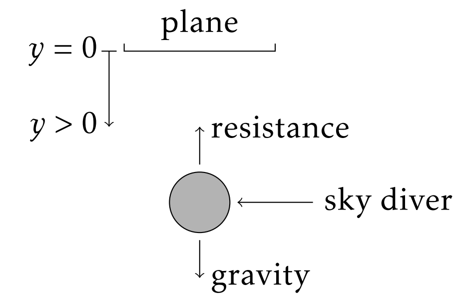
Problem 2.29. A skydiver falls out of a plane and free falls toward the ground. Newton's second law states that the product of the mass and the acceleration must be equal to the sum of the forces acting on the sky diver. The two primary forces acting on the falling sky diver are gravity and air resistance. A sensor measures the distance from the plane where it is dropped (positive distance is *down* from the sensor). See the free body diagram below.



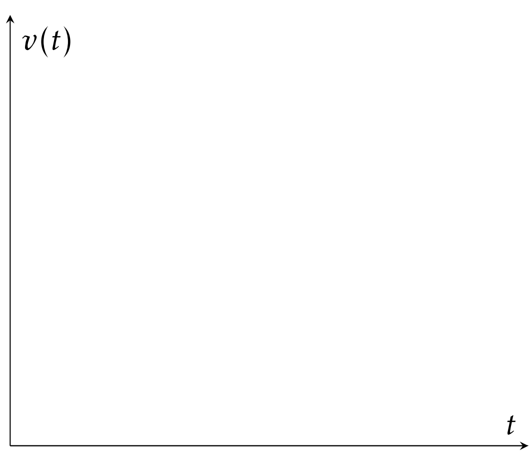
(a) It can be shown experimentally that the air resistance is proportional to the square of the velocity. What differential equation models the velocity of the falling sky diver?
(Hints: (1) Recall that if $y(t)$ is the position of the falling sky diver then $v(t) = y'(t)$ is the velocity and $a(t) = v'(t) = y''(t)$ is the acceleration.
(2) be sure to get the signs correct!)

$$m \cdot \underline{\hspace{2cm}} = \underline{\hspace{4cm}} + \underline{\hspace{4cm}}$$

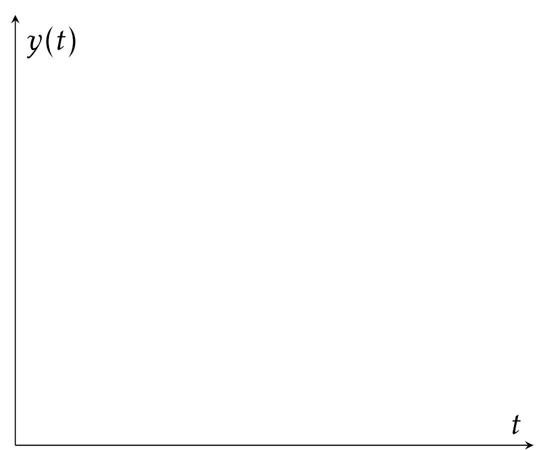
(b) What is the equilibrium solution to the differential equation that you proposed in part (a). Discuss the meaning of this equilibrium in the context of the problem. Is it stable or unstable?

(c) Sketch a plot of the velocity and the position functions vs time. (You do not need to solve the differential equation.)

Velocity vs. Time



Position vs. Time



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Chapter 3

Linear Systems and Matrices

In this chapter we'll assume that you are familiar with the basics of linear algebra. Hence, you can use the appropriate linked text from Section 0.2 for any necessary explanation on these problems. I highly suggest you use your notes from when you first saw linear algebra. We will begin here with a few of the basic definitions and we will recap some of the basics from systems of equations, row reduction, linear combinations, and matrix operations. I highly suggest that you **PUT YOUR CALCULATOR DOWN** and get used to doing all of these techniques by hand. There is a time and place for technology and for the most part this chapter is not it.

3.1 Matrix Operations and Definitions

Problem 3.1. (a) Give an example of two matrices that are equal and then give several examples of two matrices that are not equal.

(b) Give an example of two matrices where addition (or subtraction) does not make sense.

(c) Write down a non-zero matrix A and describe what it means to scalar multiply the matrix by 5.

(d) Write down a 3×2 matrix. If you swap the rows and columns of the matrix, what matrix do you get and what size is it?

(e) Give two matrices, A and B, that can be multiplied and find their product.

(f) Give two matrices, A and B, that cannot be multiplied.

▲

Definition 3.2 (Size of a Matrix)**.** If A is a matrix with m rows and n columns then we say that A has size (or dimensions) $m \times n$.

Definition 3.3 (Equality of Matrices)**.** Two matrices are equal if their corresponding entries are equal. Matrices can only be equal if the sizes are equal.

Definition 3.4 (Addition and Subtraction of Matrices)**.** Matrix addition and subtraction and done by regular addition and subtraction on the corresponding entries. Matrix addition and subtraction can only be performed on matrices of the same size.

Definition 3.5 (Scalar Multiplication)**.** If A is a matrix then cA is a scalar multiple of the matrix. Multiplying a matrix by a scalar multiplies every entry by the scalar.

Definition 3.6 (Transposition of a Matrix)**.** If A is a matrix then A^T is the transpose of the matrix found by interchanging the rows and columns of A. If A is $m \times n$ then A^T is $n \times m$.

Definition 3.7. If A is an $m \times n$ matrix and B is an $n \times p$ matrix then the product of A and B is $C = AB$ where:

- The size of AB is $m \times p$. The number of columns in A must be the same as the number of rows of B.
- The entry in row i and column j of $C = AB$ is

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

It is very important to note that in general $AB \neq BA$.

Problem 3.8. Consider the matrices A and B. Find the products AB and BA if they exist.

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 5 & -2 \\ -1 & 0 \\ 1 & 3 \end{pmatrix}$$

▲

Problem 3.9. Consider the matrices below.

$$A = \begin{pmatrix} 2 & -1 & 4 \\ 3 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 1 \\ 0 & -3 \\ 4 & -1 \end{pmatrix} \quad C = \begin{pmatrix} 0 & -1 \\ 3 & 2 \\ -3 & 1 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$$

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(a) Determine which products are possible:

$$AB, \quad AC, \quad A\mathbf{x}, \quad BA, \quad CA, \quad \mathbf{x}A, \quad BC, \quad B\mathbf{x}, \quad CB, \quad C\mathbf{x}.$$

For each of the products that is possible find the size of the result.

(b) Write the product AB and the product BA. Does $AB = BA$?

▲

Problem 3.10. Compute the product Ab for

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \quad b = \begin{pmatrix} a \\ b \end{pmatrix}$$

▲

Example 3.11. Find the product of A and B where

$$A = \begin{pmatrix} 1 & 4 & 0 \\ 3 & -1 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 \\ 2 & 3 \\ -1 & 1 \end{pmatrix}$$

Solution:

$$AB = \begin{pmatrix} 1 & 4 & 0 \\ 3 & -1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & 3 \\ -1 & 1 \end{pmatrix}$$

The matrices are 2×3 and 3×2 so the resulting product will be 2×2.

- To find the entry in row 1 column 1 we find the dot product of row 1 from matrix A and column 1 from matrix B.

$$\text{row 1 column 1: } (1)(0) + (4)(2) + (0)(-1) = 8$$

- To find the entry in row 1 column 2 we find the dot product of row 1 from matrix A and column 2 from matrix B.

$$\text{row 1 column 2: } (1)(1) + (4)(3) + (0)(1) = 13$$

- To find the entry in row 2 column 1 we find the dot product of row 2 from matrix A and column 1 from matrix B.

$$\text{row 2 column 1: } (3)(0) + (-1)(2) + (2)(-1) = -4$$

- To find the entry in row 2 column 2 we find the dot product of row 2 from matrix A and column 2 from matrix B.

$$\text{row 2 column 2: } (3)(1)+(-1)(3)+(2)(1)=2$$

Therefore the product is

$$AB=\begin{pmatrix} 8 & 13 \\ -4 & 2 \end{pmatrix}.$$

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3.2 Gaussian Elimination: Reduced Row Echelon Form

Solving systems of equations is one of the most essential applications of linear algebra. It is expected that you have experience solving systems with row reduction so as such we will cover it quickly in this section.

Problem 3.12 (Nickes and Dimes Problem)**.** Solve the following problem using any technique. Be able to clearly explain your work.
Mr. Gauss has 20 coins consisting of nickels and imes. If his nickels were dimes and his dimes were nickels he would have 70 cents more. How much are his coins worth? ▲

Problem 3.13. Consider the system of equations:

$$\begin{cases} -x_1 + x_2 - x_3 &= -6 \\ \quad x_1 + x_3 &= 15 \\ 2x_1 - x_2 + x_3 &= 9 \end{cases}$$

We want to solve this system of equations using Gaussian elimination (row reduction). We will do so using the following steps.

(a) For the sake of practice let's first write this system as a matrix equation of the form $A\mathbf{x} = \mathbf{b}$. What are A, $\mathbf{x}$, and $\mathbf{b}$?

(b) Next write the system as an *augmented matrix*.

$$\left(\begin{array}{ccc|c} \_\_ & \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ & \_\_ \end{array}\right)$$

(c) Our goal is to transform the augmented matrix $(A|\mathbf{b})$ to the matrix $(I|\mathbf{x})$ using only the following operations:

- multiply one row by a scalar quantity
- add a multiple of one row to another row
- interchange two rows

Discuss why we are allowed to use these operations.

(d) Starting with the top left corner of the augmented matrix, systematically row reduce the matrix to the form $(I|\mathbf{x})$.

(e) Once you have the row reduced matrix interpret your result.

▲

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Technique 3.14 (Practical Tips for Gaussian Elimination)**.** When performing Gaussian Elimination you should keep the following in mind:

- First try to get a 1 in the upper left-hand corner of the augmented matrix.
- Next, use the new first row to eliminate all of the non-zero entries in the first column. By the time you're done with this you should have a column with a 1 on top and zeros below.
- Next get a 1 in row 2 column 2.
- Use your new second row to eliminate all of the non-zero entries in the second column.
- Proceed in a similar fashion until you have reached the final row

Example 3.15. Let's row reduce an augmented matrix. Pay particular attention to the systematic way that we work toward getting the identity matrix on the left-hand side of the augmented matrix.

$$\left(\begin{array}{cc|c} 2 & -2 & 6 \\ 2 & 1 & 0 \end{array}\right) \xrightarrow{R_1 \leftarrow (1/2)R_1} \left(\begin{array}{cc|c} 1 & -1 & 3 \\ 2 & 1 & 0 \end{array}\right)$$
$$\xrightarrow{R_2 \leftarrow R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 3 & -6 \end{array}\right)$$
$$\xrightarrow{R_2 \leftarrow (1/3)R_2} \left(\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 1 & -2 \end{array}\right)$$
$$\xrightarrow{R_1 \leftarrow R_2 + R_1} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -2 \end{array}\right)$$

Notice further that at each step we indicate which row operations were done. Finally notice that there are no equal signs since the matrices that you create at each step are definitely not equal; they are called "row equivalent".

I leave it to you to make yourself familiar with examples of Gaussian Elimination from other texts (see the linked materials in Section 0.2 of these notes).

Problem 3.16. Consider the following three systems of equations and their row reduced forms. Describe their solution sets geometrically. If the system has a solution then give it. If the system has no solution then explain why. If the system has infinitely many solutions

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then give them all in a parameterized form.

$$\text{System \#1:} \quad \left(\begin{array}{cc|c} 1 & -1 & 3 \\ 2 & 1 & 0 \end{array}\right) \to \cdots \to \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -2 \end{array}\right)$$

$$\text{System \#2:} \quad \left(\begin{array}{cc|c} 1 & -1 & 3 \\ -1 & 1 & 0 \end{array}\right) \to \cdots \to \left(\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 0 & 3 \end{array}\right)$$

$$\text{System \#3:} \quad \left(\begin{array}{cc|c} 1 & -1 & 3 \\ -1 & 1 & -3 \end{array}\right) \to \cdots \to \left(\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 0 & 0 \end{array}\right)$$

▲

Problem 3.17. Create 3×3 systems of equations that have

(a) exactly 1 solution

(b) no solutions

(c) infinitely many solutions

▲

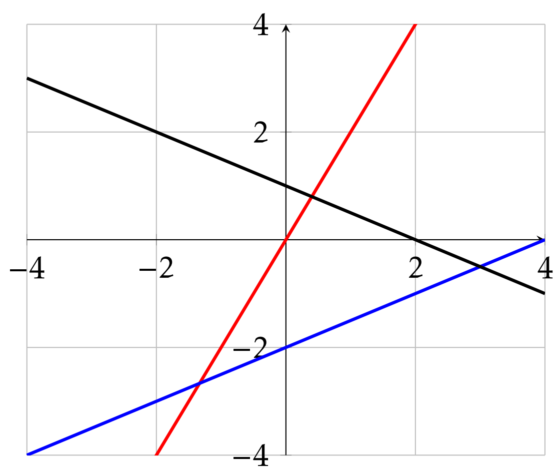
Problem 3.18. What is the value of k so that the linear system represented by the following matrix would have infinitely many solutions?

$$\left(\begin{array}{cc|c} 2 & 6 & 8 \\ 1 & k & 4 \end{array}\right)$$

Choose from the following choices:
(a) $k = 1$, (b) $k = 2$, (c) $k = 3$, (d) $k = 4$, (e) not possible, (f) there are infnitely many ways to do this ▲

Problem 3.19. We have a system of three linear equations with two unknowns as plotted in the graph

How many solutions does the system have? Choose from the following:
(a) 0, (b) 1, (c) 2, (d) 3, (e) infinite



▲

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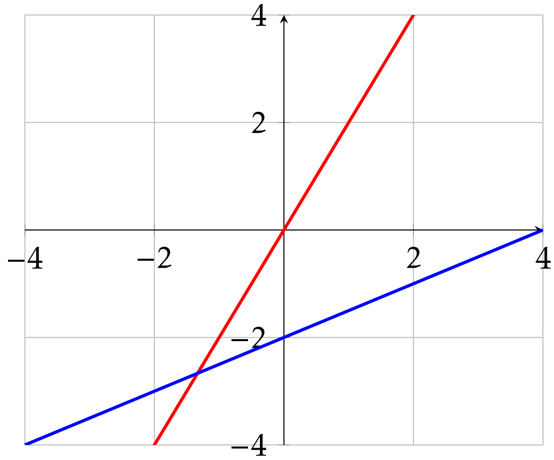
Problem 3.20. We have a system of two linear equations with two unknowns as plotted in the graph

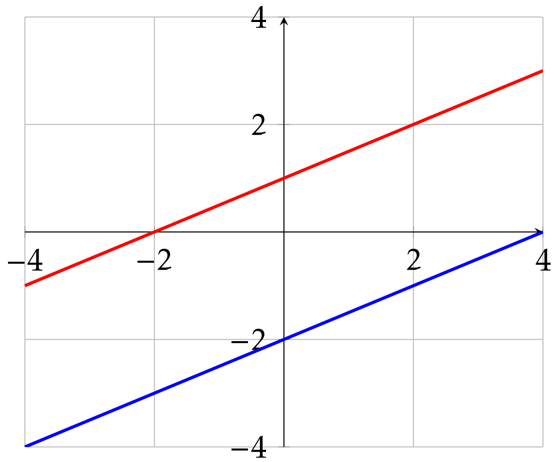
How many solutions does the system have? Choose from the following:
(a) 0, (b) 1, (c) 2, (d) 3, (e) infinite



▲

Problem 3.21. We have a system of two linear equations with two unknowns as plotted in the graph

How many solutions does the system have? Choose from the following:
(a) 0, (b) 1, (c) 2, (d) 3, (e) infinite



▲

Problem 3.22. The following system has infinitely many solutions. Write an equation that expresses all of them in a parameterized form.

$$\begin{aligned} x + y &= 2 \\ -3x - 3y &= -6 \\ 2x + 2y &= 4 \end{aligned}$$

▲

Problem 3.23. A system of 8 linear equations and 6 variables could not have exactly ______ solution(s).
(a) 0, (b) 1, (c) infinite, (d) more than one of these is possible, (e) all of these are possible ▲

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Problem 3.24. What is the solution to the system of equations represented by this augmented matrix?

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Choose from:

(a) $x = 2, y = 3, z = 4$

(b) $x = -2, y = 1, z = 1$

(c) There are infinitely many solutions

(d) There is no solution

(e) We can't tell without having the system of equations

(If there are infinitely many solutions then write expression for all of them) ▲

Problem 3.25. Solve the system of equations

$$\begin{aligned} x + 2y + z &= 0 \\ x + 3y - 2z &= 0 \end{aligned}$$

▲

Problem 3.26. Let R be the reduced row echelon form of matrix A. True or False: the solutions to $R\mathbf{x} = \mathbf{0}$ are the same as the solutions to $A\mathbf{x} = \mathbf{0}$. ▲

Problem 3.27. Let R be the reduced row echelon form of matrix A. True or False: the solutions to $R\mathbf{x} = \mathbf{b}$ are the same as the solutions to $A\mathbf{x} = \mathbf{b}$. ▲

3.3 Linear Combinations

One of the most beautiful parts of linear algebra is the richness of the structure of matrices. As you already know, every system of linear equations can be written several different ways: as a system, as a matrix equation, as a vector equation, or as an augmented system.

Example 3.28. For example, we can write the system of equations

$$\begin{aligned} 2x_1 + 3x_2 &= 5 \\ 4x_1 - 6x_2 &= 6 \end{aligned}$$

equivalently in the following ways:

$$\begin{array}{ll} \text{Algebraic System:} & \begin{aligned} 2x_1 + 3x_2 &= 5 \\ 4x_1 - 6x_2 &= 6 \end{aligned} \\ \text{Matrix Equation:} & \begin{pmatrix} 2 & 3 \\ 4 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix} \\ \text{Vector Equation:} & x_1 \begin{pmatrix} 2 \\ 4 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -6 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix} \\ \text{Augmented System:} & \left(\begin{array}{cc|c} 2 & 3 & 5 \\ 4 & -6 & 6 \end{array} \right). \end{array}$$

In this section we'll look in particular at the vector equation. Hiding behind a vector equation is one of the most fundamental ideas behind all of linear algebra: the linear combination. The "vector equation" above really says "some amount of $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ plus some amount of $\begin{pmatrix} 3 \\ -6 \end{pmatrix}$ gives $\begin{pmatrix} 5 \\ 6 \end{pmatrix}$ and our job is to find the amounts that make the equality true." More generally, the vector equation is saying that $\begin{pmatrix} 5 \\ 6 \end{pmatrix}$ is a linear combination of the other two vectors.

Let's first start with a simple exercise.

Problem 3.29. Write the system of equations as a vector equation.

$$\begin{aligned} x_1 + 3x_2 - 5x_3 &= 9 \\ -x_1 + x_2 &= -3 \\ 7x_1 + 2x_3 &= -\pi \end{aligned}$$

▲

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Definition 3.30 (Linear Combination)**.** Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_p$ be vectors in n-dimensional space and let $c_1, c_2, c_3, \dots, c_p$ be scalar quantities. The vector $\mathbf{u}$ defined by

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_p\mathbf{v}_p = \sum_{j=1}^{p} c_j\mathbf{v}_j$$

is called a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_p$ with weights $c_1, c_2, c_3, \dots, c_p$.

Problem 3.31. Open the GeoGebra applet: www.geogebra.org/m/WShmQvQU in a browser window.

(a) Move the vectors $\mathbf{u}$ and $\mathbf{v}$ to $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

(b) Describe all of the possible vectors $\mathbf{w} = c_1\mathbf{u} + c_2\mathbf{v}$ if $c_1 = 0$

(c) Describe all of the possible vectors $\mathbf{w} = c_1\mathbf{u} + c_2\mathbf{v}$ if $c_2 = 0$

(d) Is it possible to find c_1 and c_2 such that $\mathbf{w} = \begin{pmatrix} -6 \\ 0.5 \end{pmatrix}$. If so, what are c_1 and c_2.

▲

Problem 3.32. Write $\mathbf{u} = \begin{pmatrix} -5 \\ 3 \\ 16 \end{pmatrix}$ as a linear combination of $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} -3 \\ 2 \\ 6 \end{pmatrix}$ ▲

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3.4 Inverses and Determinants

Division is always a bit of a touchy subject. In the real numbers division is well defined except when the denominator is zero. The same story is true in the rational numbers: a fraction divided by a fraction is another fraction so long as the divisor is not zero. What if we wanted to stay only in the integers? Can we divide two integers and get another integer? Of course you can always divide by 1, but in most other cases division will move you into the rational numbers. Hence, division on the integers doesn't really make sense. Mathematically speaking we say that the integers are not closed under addition.

Similarly, if we try to define division on matrices we run into trouble. What does it mean to divide by a matrix? In general, that phrase is meaningless! Let's expand our view a bit.

When considering the operation of addition, we call 0 the additive identity and we call $(-a)$ the additive inverse of a since $a + (-a) = 0$. When considering multiplication, we call 1 the multiplicative identity and $1/a$ is the multiplicative inverse of a (when $a \neq 0$) since $a \cdot \frac{1}{a} = 1$. To define the matrix inverses of a square matrix A we seek the same thing: find matrix B such that $AB = I$ and $BA = I$. Where I is the identity matrix which is the multiplicative identity for matrices.

3.4.1 Inverses of Square Matrices

Problem 3.33. (a) Consider the matrix $A = \begin{pmatrix} 1 & 2 \\ -4 & -6 \end{pmatrix}$ and the vector $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Find the vector $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$ by hand using row reduction.

(b) Again consider the matrix $A = \begin{pmatrix} 1 & 2 \\ -4 & -6 \end{pmatrix}$ and the vector $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Find the vector $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$ by hand using row reduction.

(c) Consider the matrix $A = \begin{pmatrix} 1 & 2 \\ -4 & -6 \end{pmatrix}$ one more time. Find a matrix B such that $AB = I$ and $BA = I$. Use what you did in parts (a) and (b) to help set up the problem. (You may recall some sort of "shortcut" that involves switching and negating some matrix entries ... you are forbidden to use this shortcut here!)

▲

Problem 3.34. Consider the following True / False questions.

(a) True or False: If $AB = I$ and $BA = I$ then A is the inverse of B and B is the inverse of A.

(b) True or False: If $A\mathbf{x} = \mathbf{b}$ and A has an inverse then $\mathbf{x} = A^{-1}\mathbf{b}$.

(c) True or False: If $A\mathbf{x} = \mathbf{b}$ and A^{-1} exists then to find $\mathbf{x}$ we can augment A with $\mathbf{b}$ and row reduce. That is, $\left(A \mid \mathbf{b}\right) \to \cdots \to \left(I \mid \mathbf{x}\right)$.

(d) True or False: If $A\mathbf{x} = \mathbf{b}$ then $A = \mathbf{b}\mathbf{x}^{-1}$.

(e) True or False: If $AB = I$ then we can find B by augmenting A with I and row reducing? That is, $\begin{pmatrix} A & | & I \end{pmatrix} \to \cdots \to \begin{pmatrix} I & | & B \end{pmatrix}$.

▲

Technique 3.35 (Finding Matrix Inverses)**.** If A is a square matrix of size $n \times n$ then if A^{-1} exists we know that $AA^{-1} = I$. Therefore, to find A^{-1} we augment A with I and row reduce. That is, $\begin{pmatrix} A & | & I \end{pmatrix} \to \cdots \to \begin{pmatrix} I & | & B \end{pmatrix}$

If A^{-1} does not exist then we will be unable to row reduce to the identity.

Problem 3.36. Give an example of a non zero 2×2 matrix that does not have an inverse. ▲

Problem 3.37. Which of the following matrices does not have an inverse?
(a) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, (b) $\begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix}$, (c) $\begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$, (d) $\begin{pmatrix} 0 & 4 \\ 2 & 0 \end{pmatrix}$, (e) More than one of these does not have an inverse, (f) All have inverses ▲

Problem 3.38. When we put a matrix A into row reduced echelon form, we get the following matrix. What does this mean?

$$A \to \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

(a) Matrix A has no inverse

(b) The matrix we have found is the inverse of A

(c) Matrix A has an inverse but this isn't it

(d) This tells us nothing about whether A has an inverse.

▲

Problem 3.39. True or False: Suppose that A, B, and C are square matrices and $CA = B$ and A is invertible. This means that $C = A^{-1}B$. ▲

Problem 3.40. A and B are invertible matrices. If $AB = C$ then what is the inverse of C?
(a) $C^{-1} = A^{-1}B^{-1}$, (b) $C^{-1} = B^{-1}A^{-1}$, (c) $C^{-1} = AB^{-1}$, (d) $C^{-1} = BA^{-1}$,
(e) More than one of these is true,
(f) Just because A and B have inverses this doesn't mean that C has an inverse ▲

Problem 3.41. The *trick* that you might recall from a previous class for calculating 2×2 inverses is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Prove that this *trick* works by row reducing the system

$$\left(\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array} \right).$$

▲

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3.4.2 Determinants

In the demoninator of the equation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

you find the expression $ad - bd$. For a 2×2 matrix this is called the determinant. Clearly from this formula for the inverse of a 2×2 matrix if the determinant is zero then the inverse does not exist. What we'll find is that this is not unique to 2 matrices.

Definition 3.42 (2×2 Determinant)**.** Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The determinant of A is

$$\det(A) = ad - bc.$$

Problem 3.43. What is the determinant of $A = \begin{pmatrix} 4 & -1 \\ -2 & 1 \end{pmatrix}$? ▲

Problem 3.44. Find the value of k so that the matrix A is not invertible.

$$A = \begin{pmatrix} 2 & 4 \\ 3 & k \end{pmatrix}$$

▲

Problem 3.45. Given the matrix

$$B = \begin{pmatrix} 2-x & 1 \\ 4 & 2-x \end{pmatrix}$$

find all of the values of x that are solutions to the equation $\det(B) = 0$. ▲

Problem 3.46. Now consider the matrix

$$A = \begin{pmatrix} 1 & 5 & 3 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{pmatrix}.$$

- Cross out row 1 and column 1. Call the remaining 2×2 matrix A_{11}.
- Cross out row 1 and column 2. Call the remaining 2×2 matrix A_{12}.
- Cross out row 1 and column 3. Call the remaining 2×2 matrix A_{13}.

The determinant of A is

$$\det(A) = 1 \cdot \det(A_{11}) - 5 \cdot \det(A_{12}) + 3 \cdot \det(A_{13}).$$

Perform this computation. ▲

When doing determinants of square matrices you can expand along any row or column you like. The previous problem had you expand along the first row but arguably expanding along the third row would have been easier since there are several zeros. Notice, however, that there the signs on the determinant terms alternate. Hence, if you are going to expand upon a given row or column you need to keep in mind that the signs on the terms follow checkerboard pattern:

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Problem 3.47. Expand the matrix A from problem 3.46 along the third row. ▲

Problem 3.48. Find the determinant of

$$A = \begin{pmatrix} 1 & 5 & 3 \\ 2 & 0 & -5 \\ 0 & 0 & 3 \end{pmatrix}$$

▲

Example 3.49. In this example we will work through the determinant of the 3×3 matrix

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & -5 \\ 0 & 0 & 3 \end{pmatrix}$$

Solution: Let's expand along the first row:

$$\begin{aligned} \det(A) &= 1 \cdot \begin{vmatrix} 2 & -5 \\ 0 & 3 \end{vmatrix} - 0 \cdot \begin{vmatrix} 0 & -5 \\ 0 & 3 \end{vmatrix} + 3 \cdot \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} \\ &= 1 \cdot ((2)(3) - (0)(-5)) - 0 \cdot ((0)(3) - (0)(-5)) + 3 \cdot ((0)(0) - (0)(2)) \\ &= 1 \cdot 6 - 0 \cdot 0 + 3 \cdot 0 \\ &= 6. \end{aligned}$$

Also notice in this example that the entire lower triangle of the matrix is filled with zeros. When this is the case you may observe the nice pattern that the determinant is actually just the product of the entries on the main diagonal (you should prove that this is true). Hence, in this problem we know that $\det(A) = 1 \cdot 2 \cdot 3 = 6$. Be careful! If you don't have an entire triangle of zeros then this little *trick* will not work.

Problem 3.50. Find the determinant of the following matrices. Is there anything special that you can say about these matrices? Do you notice any ways to make the determinant

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computation faster on these matrices?

$$A = \begin{pmatrix} 1 & 3 \\ 6 & 2 \end{pmatrix}$$
$$B = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$$
$$C = \begin{pmatrix} 2 & 3 & 2 \\ 4 & 7 & 3 \\ 1 & 0 & 5 \end{pmatrix}$$
$$D = \begin{pmatrix} 2 & 3 & 2 \\ 0 & 0 & 3 \\ 1 & 0 & 5 \end{pmatrix}$$
$$E = \begin{pmatrix} 2 & 3 & 2 \\ 0 & 7 & 3 \\ 0 & 0 & 5 \end{pmatrix}$$

▲

The following collection of theorems give several of the primary characterizations of the determinant.

Theorem 3.51. The determinant of the identity matrix is 1.

Proof. (prove this theorem) □

Theorem 3.52 (Determinants and Invertibility)**.** The $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$.

Theorem 3.53 (Determinants and the Matrix Transpose)**.** Let A be an $n \times n$ matrix. The determinant of the transpose of A is the same as the determinant of A. In other words,

$$\det(A^T) = \det(A).$$

Proof. (prove this theorem) □

Theorem 3.54 (Determinants of Matrix Products)**.** If A and B are $n \times n$ matrices, then

$$\det(AB) = \det(A) \cdot \det(B).$$

Theorem 3.55 (Determinants of Inverses)**.** If A is an invertible matrix then

$$\det(A^{-1}) = \underline{\hspace{3cm}}$$

(Fill in the blank ... you just proved this in the last problem)

Proof. (prove your claim using the fact that $\det(AB) = \det(A)\det(B)$) □

Theorem 3.56. If λ is some scalar and A is a square matrix of size $n \times n$ then

$$\det(\lambda A) = \lambda^n \det(A).$$

Problem 3.57. Consider the matrix $A = \begin{pmatrix} 1 & 3 \\ -7 & 2 \end{pmatrix}$

(a) What is the determinant of $5A$?

(b) What is the determinant of A^T?

(c) What is the determinant of A^{-1}?

▲

Problem 3.58. When using a computer to find the determinant of a square matrix you must always be careful. Not because the computation is difficult (which it sometimes is), but because of the property

$$\det(\lambda A) = \lambda^n \det(A)$$

where A is an $n \times n$ matrix.

- Why does this property tell you that you really need to be careful with computers and determinants?
- Why should you never use a computer to find a matrix inverse?

▲

Problem 3.59. Matrix A has size $1,000,000 \times 1,000,000$ and has determinant $\det(A) = 7$. Let's say that A is stored on a computer in single precision so each value in the matrix is only accurate to 10^{-8}. That is, a number x in the matrix is actually somewhere between $(1 - 10^{-8})x$ and $(1 + 10^{-8})x$. If we actually find $\det(A)$ on this computer what is a range for the determinant computation?
Hint: the relative error is $\lambda = 1 \pm 10^{-8}$ so you should be computing $\det(\lambda A)$. ▲

There is much that we could say about the determinant, and we have proved several of the key results that you need to remember. For a summary the following theorem gives you one place to look for the key results about determinants.

Theorem 3.60 (Summary of Properties of Determinants)**.** Let A be a square matrix.

1. The determinant of the identity matrix is 1.
2. The $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$
3. $\det(A^T) = \det(A)$.
4. $\det(AB) = \det(A)\det(B)$
5. $\det(A^{-1}) = 1/\det(A)$
6. If A is an $n \times n$ matrix and $\lambda \in \mathbb{R}$ then $\det(\lambda A) = \lambda^n \det(A)$
7. If a multiple of one row of A is added to another row to produce matrix B then $\det(A) = \det(B)$.
8. If two rows are interchanged in matrix A to produce matrix B then $\det(B) = -\det(A)$.
9. If one row of A is multiplied by k to product matrix B then $\det(B) = k\det(A)$.
10. The absolute value of the determinant of a matrix A is the volume of the parallelepiped formed by the column vectors of A.

Problem 3.61. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\det(A) = 8$ then what is $\det(B)$ where $B = \begin{pmatrix} a & b \\ 3c & 3d \end{pmatrix}$ ▲

Problem 3.62. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\det(A) = 8$ then what is $\det(B)$ where $B = \begin{pmatrix} a & b \\ 2a+c & 2b+d \end{pmatrix}$
▲

Problem 3.63. True or False: The determinant of A is the same as the determinant of the row reduced form of A. Explain your answer. ▲

3.4.3 The Geometry of Determinants

The last statement of Theorem 3.60 give a bit of a deeper insight to the geometry of determinants and why invertible matrices must have a non-zero determinant. Think about a 2×2 matrix with non-zero determinant. Under matrix multiplication by this matrix, a shape with non-zero area will be transformed to another shape with non-zero area. Hence, if we were to reverse the mapping then we have all of the vectors accounted for and can reverse the transformation. If, on the other hand, a matrix has a determinant of zero then a shape with non-zero area will be transformed into a shape with zero area. Naturally, in this case, many vectors will be mapped on top of each other and any hope of reversing the transformation is lost – hence the fact that a matrix with a zero determinant is not invertible.

For example, if we consider the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, the determinant is $\det(A) = 4 - 1 = 3 \neq 0$ and we know that A^{-1} exists. To see the action on the vectors $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ see Figure 3.1. Notice that before the multiplication by A the parallelogram (square) formed by $\mathbf{u}_1$ and $\mathbf{u}_2$ is 1 (since the determinant of the identity is 1). After the multiplication by A the original square is stretched into the parallelogram on the right of Figure 3.1 and with some geometry we can see that the area of the parallelogram is 3. If we were to imagine reversing the transformation, morphing the red parallelogram back into the blue square, we can see visually how each vector in the plane gets stretched and rotated – hence giving a good meaning to *reversing* the transformation and giving us a visual sense that the inverse of A exists.

Now consider the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 0.5 \end{pmatrix}$. The determinant of this matrix is clearly zero, and geometrically in Figure 3.2 we see that the square gets *squished* into a single line segment with zero area. In fact, both of the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are mapped to exactly the same line segment and figuring out how to reverse these actions for every vector in the plane is impossible – hence giving us a sense that if the determinant is zero then the inverse of the matrix A must not exist.

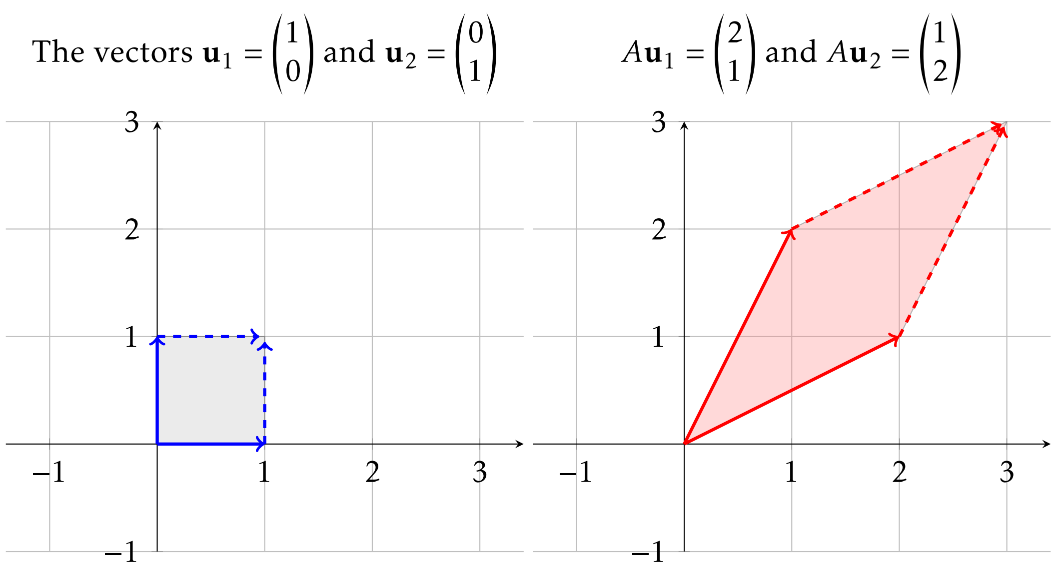

Figure 3.1. A 2D mapping from region with non-zero area to a region non-zero area.

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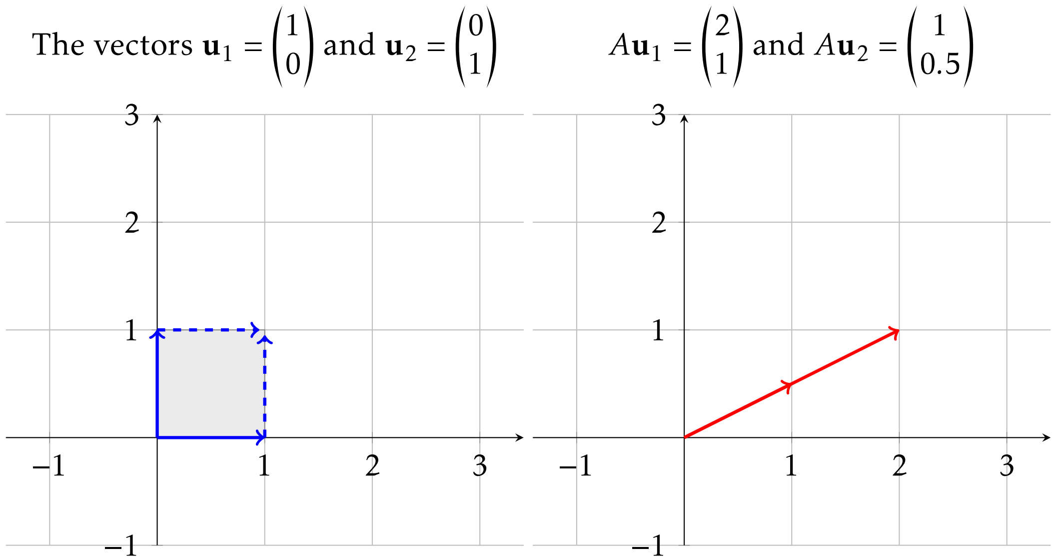


Figure 3.2. A 2D mapping from a region non-zero area to a region with zero area.

3.5 Additional Exercises

Problem 3.64. Use linear algebra to find a cubic polynomial of the form

$$f(x) = A + Bx + Cx^2 + Dx^3$$

that interpolates the data points

$$(-1,4), (1,2), (2,1), (3,16)$$

Hint: each point creates a linear equation in A, B, C, and D. ▲

Problem 3.65. (a) How many data points do you need to exactly describe a unique quadratic polynomial?

(b) How many data points do you need to exactly describe a unique cubic polynomial?

(c) How many data points do you need to exactly describe a unique quartic polynomial?

(d) In general, if you have n data points that you believe form a polynomial function, what is the order of the polynomial?

▲

Definition 3.66 (The Vandermonde Matrix)**.** The Vandermonde matrix is an $m \times n$

matrix of the form

$$V = \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & x_2^3 & \cdots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & x_3^3 & \cdots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & x_m^2 & x_m^3 & \cdots & x_m^{n-1} \end{pmatrix}$$

There are several interesting properties of the Vandermonde matrix but of particular interest in this section of these notes is that the Vandermonde matrix appears when doing a polynomial interpolation.

Example 3.67. Build the Vandermonde matrix associated with fitting a quadratic polynomial to the data points $(-1,3)$, $(2,5)$, and $(7,-1)$.
Solution: We will let $x_1 = -1$, $x_2 = 2$ and $x_3 = 7$ to get the equations

$$\begin{aligned} a(-1)^2 + b(-1) + c &= 3 \\ a(2)^2 + b(2) + c &= 5 \\ a(7)^2 + b(7) + c &= -1 \end{aligned}$$

since we are trying to fit the data to the quadratic function $f(x) = ax^2 + bx + c$. Rearranging the system into a matrix equation we immediately see the Vandermonde matrix appear as the coefficient matrix.

$$\begin{pmatrix} 1 & -1 & (-1)^2 \\ 1 & 2 & 2^2 \\ 1 & 7 & 7^2 \end{pmatrix} \begin{pmatrix} c \\ b \\ a \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}$$

We leave it to the reader to solve the system for a, b, and c.

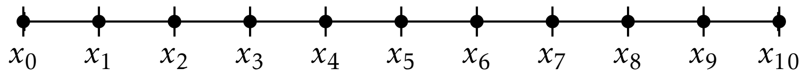
Problem 3.68. Find a polynomial that perfectly fits the following data set

x	y
-2	3
-1	5
0	-9
4	-3
5	0
8	-6

▲

Problem 3.69. Imagine that we have a 1 meter long thin metal rod that has been heated to 100° on the left-hand side and cooled to 0° on the right-hand side. We want to know the temperature every 10 cm from left to right on the rod.

(a) First we break the rod into equal 10cm increments as shown.



How many unknowns are there in this picture?

(b) The temperature at each point along the rod is the average of the temperatures at the adjacent points. For example, if we let T_1 be the temperature at point x_1 then

$$T_1 = \frac{T_0 + T_2}{2}.$$

Write a system of equations for each of the unknown temperatures.

(c) Solve the system for the temperature at each unknown node.

▲

Problem 3.70. Now imagine that we are finding the temperature on a flat (2D) metal plate that measured 1m by 1m (broken into 10cm by 10cm squares). Extend the idea from Problem 3.69 to find the temperature at each of the interior points if the left and bottom sides are heated to 100° and the top and right sides are cooled to 0°. You will likely need write computer code to solve this problem and the best possible output would be a surface (or contour) plot showing the temperature. ▲

Problem 3.71. Write $d = (3, -5, 10)$ as a linear combination of the vectors $a = (-1, 0, 3)$, $b = (0, 1, 5)$, and $c = (4, -2, 0)$.
Choose from:

(a) $d = -3a - 5b + c$

(b) $d = 5a - b + 2c$

(c) $d = (10/3)a + (5/2)c$

(d) d cannot be written as a linear combination of a, b, and c.

▲

Problem 3.72. Consider the system of 2 equations and two unknowns below:

$$\begin{cases} x + y &= 3 \\ -3x + ky &= 2 \end{cases}$$

(a) Determine the value of k for which the system has no solutions.

(b) If k were to have the value that you indicated from part (a), what would you see if you plotted both linear functions on the same coordinate plane?

▲

Problem 3.73. Consider the matrix A given below. It can be shown that $\det(A) = 0$. For each of the following true or false questions please circle the correct choice AND provide a short explanation.

$$A = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 7 & -1 & 0 & 0 \\ 7 & 1 & 0 & 0 \\ -4 & -8 & 8 & 1 \end{pmatrix} \quad \text{and we know that} \quad \det(A) = 0$$

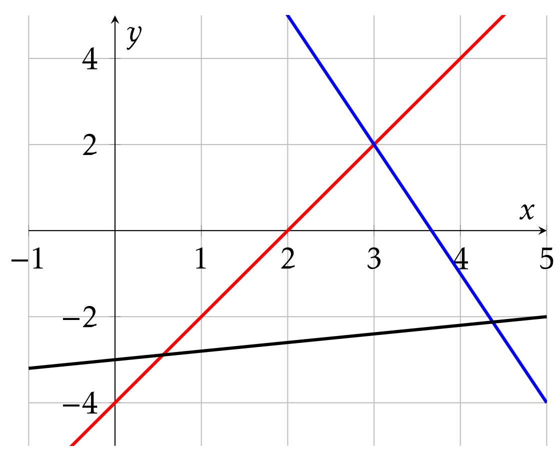
(a) TRUE or FALSE: The matrix A has an inverse.

(b) TRUE or FALSE: Let B be a 4×4 matrix and let the matrix C be defined as $C = AB$. Assume further that $\det(B) = 7$. Based on all of this, the matrix C does not have an inverse.

▲

Problem 3.74. Bonnie and Clyde, two mathematics students from the 1930's, are solving the same system of three equations in two unknowns. Bonnie is a more visual thinker so she graphs the system. Her graph is given below on the left. Clyde prefers to follow the mathematical algorithms (sometimes without much thought) and arrives at the row-reduced augmented system shown below on the right.

Bonnie's Graph



Clyde's reduced augmented system

$$\left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right)$$

(a) What is Bonnie's conclusion about the system of three equations in two unknowns? Explain in a sentence or two.

(b) What is Clyde's conclusion about the system of three equations in two unknowns? Explain in a sentence or two.

(c) It should be clear (hint hint) that Bonnie and Clyde have different solutions to the system of equations. Give a possible row-reduced matrix for Bonnie's graphical solution.

▲

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Problem 3.75. For each of the following row reduced systems of equations indicate the number of solutions AND

- if there is only 1 solution then given it.
- if there are infinitely many solutions then write an expression that gives all of them.
- if there are no solutions then explain why not.

(a)
$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

(b)
$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

(c)
$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

▲

Problem 3.76. Find a 2×2 matrix B such that

$$\begin{pmatrix} 1 & 5 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

▲

Problem 3.77. If A and B are 4×4 matrices with $\det(A) = 2$ and $\det(B) = 5$ then evaluate the following.

$$\begin{aligned} \det(AB) &= \_\_\_\_\_\_\_\_\_\_\_\_ \\ \det(3A) &= \_\_\_\_\_\_\_\_\_\_\_\_ \\ \det(A^T) &= \_\_\_\_\_\_\_\_\_\_\_\_ \\ \det(B^{-1}) &= \_\_\_\_\_\_\_\_\_\_\_\_ \\ \det(B^4) &= \_\_\_\_\_\_\_\_\_\_\_\_ \end{aligned}$$

▲

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Chapter 4

Vector Spaces

LINEAR ALGEBRA IS THE MOST IMPORANT OF ALL OF THE MATHEMATICAL SCIENCES

Why?

- Within linear algebra is the language that describes all of the how and why for all linear operations.
  - solving linear differential equations
  - the reasons why the derivative and the integral operators work so nicely
  - geometry and transformations
  - computer graphics (video games are 99% linear algebra)
  - image processing (Photoshop = fancy linear algebra package)
  - large scale stress computations (linear elasticity)
  - ...
- Linear Algebra is **far** more than just *matrices*.

4.1 What is a Vector Space

To get properly in to Linear Algebra we first need to establish some of the common notation. You are familiar with 2D and 3D Vectors but using the spatial notion of "dimension" limits how you will think about vectors in linear algebra. Instead of thinking of the spatial dimensions that we live in you should be thinking of vectors as abstract objects with mathematical properties. Your intuitive notion of "arrows" is limiting and ultimately incorrect for many purposes. This chapter starts us in the direction of abstract vector spaces, but don't be worried that we will be doing abstract mathematics. We are not abstracting a familiar notion for no good reason. It was this very abstraction that has allowed mathematics to advance over the past several centuries.

Let's briefly talk about how you learned about functions. First you were told to think of a function as a "rule" that accepted an input and gave a single output (high school algebra). Then you got used to the notion that you could add, subtract, multiply, and divide functions to get new functions (pre-calculus). When you took calculus you started to think of functions as *objects* themselves and started discussing ways to get properties of those objects. The process that you've gone through with functions is called "objectification" of a mathematical idea: you have turned functions into objects in your mind's eye. Our goal in this chapter is to "objectify" the idea of vectors. We are going to turn them into abstract objects that have mathematical properties.

Definition 4.1 (The Spaces $\mathbb{R}^n$)**.** Some common vector spaces (and common mathematical notation)

$$\begin{aligned}
\mathbb{R}^2 &= \{(x,y) : x,y \in \mathbb{R}\} \\
\mathbb{R}^3 &= \{(x,y,z) : x,y,z \in \mathbb{R}\} \\
\mathbb{R}^4 &= \{(x_1,x_2,x_3,x_4) : x_j \in \mathbb{R} \quad \text{for} \quad j \in \{1,2,3,4\}\} \\
&\vdots \\
\mathbb{R}^n &= \{(x_1,x_2,\ldots,x_n) : x_j \in \mathbb{R} \quad \text{for} \quad j \in \{1,2,\ldots,n\}\}
\end{aligned}$$

Problem 4.2. (< 5 minutes): Make a list of all of the things that you can do to vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ and give examples of how you do them. ▲

The list that you made for vectors in $\mathbb{R}^n$ is really just a wish list of all of the things that you would like out of a well defined collection of mathematical objects called "vectors". We are going to abstract the idea so that we do not have to just think about arrows and lines in space. Instead, a *vector space* is a collection of abstract mathematical objects that satisfies a collection of rules (the rules that you likely already wrote down).

Definition 4.3 (Vector Space)**.** A **Vector Space** $\mathcal{V}$ is defined as a set of *mathematical objects* called vectors that follow the following 10 rules.
If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in $\mathcal{V}$ and $c_1, c_2 \in \mathbb{R}$ then

1. Closure under Addition: $\mathbf{u}+\mathbf{v} \in \mathcal{V}$
2. Closure under Scalar Multiplication: $c\mathbf{u} \in \mathcal{V}$
3. Commutativity: $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$
4. Associativity: $\mathbf{u}+(\mathbf{v}+\mathbf{w}) = (\mathbf{u}+\mathbf{v})+\mathbf{w}$
5. Zero Vector: $\mathbf{u}+\mathbf{0} = \mathbf{0}+\mathbf{u} = \mathbf{u}$
6. Additive Inverses: $\mathbf{u}+(-\mathbf{u}) = (-\mathbf{u})+\mathbf{u} = \mathbf{0}$
7. Distributive Properties: $c_1(\mathbf{u}+\mathbf{v}) = c_1\mathbf{u}+c_1\mathbf{v}$
8. $(c_1+c_2)\mathbf{u} = c_1\mathbf{u}+c_2\mathbf{u}$

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9. $c_1(c_2\mathbf{u}) = (c_1c_2)\mathbf{u}$

10. Scalar Identity: $1\mathbf{u} = \mathbf{u}$

Problem 4.4. What other *sets of things* have the mathematical properties of vector spaces? ▲

Problem 4.5. Is $\mathcal{V} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x \ge 0 \text{ and } y \ge 0 \right\}$ a vector space over the real numbers? Why or why not? ▲

Problem 4.6. Is $\mathcal{V} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x = y = z = 0 \right\}$ a vector space over the real numbers? Why or why not? ▲

Problem 4.7. Is $\mathcal{V} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : xy \ge 0 \right\}$ a vector space? Why or why not? ▲

Problem 4.8. Is the collection of all polynomials of the form $p(t) = at^2$ a vector space where $a \in \mathbb{R}$? ▲

Problem 4.9. Is $\mathcal{V} = \{f(x) : f(x) \text{ is continuous on the interval } [a,b] \text{ and } f(a) = f(b)\}$ a vector space? ▲

Problem 4.10. Is the collection $\mathcal{V} = \{f(x) : f'(x) \text{ exists }\}$ a vector space? ▲

Problem 4.11. Is $\mathcal{V} = \left\{A \in \mathbb{R}^{2\times 2} : \det(A) \neq 0\right\}$ a vector space? ▲

Definition 4.12 (The Trace of a Matrix)**.** Let A be a square $n \times n$ matrix. The trace of A, denoted $\text{tr}(A)$ is the sum of the diagonal entries. For example, the trace of a 2×2 matrix is

$$\text{tr}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = a + d.$$

Problem 4.13. Let $\mathcal{V}$ be defined as

$$\mathcal{V} = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{R} \text{ and } \text{tr}(A) = 0 \right\}$$

Is $\mathcal{V}$ a vector space? ▲

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4.2 Linear Independence and Linear Dependence

When studying vector spaces it is useful to think about how to *build* the vector space out of the simplest possible components. In order to understand that we first need an important idea in linear algebra: *linear independence*. Roughly speaking, a collection of vectors is called linearly independent if you cannot make any of the vectors in the collection by taking linear combinations of the other vectors in the collection. Keep this in mind when you read the following formal definition.

Definition 4.14 (Linearly Independent Vectors)**.** The vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly independent if the equation

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 = \mathbf{0}$$

has only the trivial solution $c_1 = c_2 = 0$.

More generally, The vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n$ are linearly independent if the equation

$$\sum_{j=1}^{n} c_j \mathbf{u}_j = \mathbf{0}$$

has only the trivial solution $c_1 = c_2 = \cdots = c_n = 0$.
A set of vectors that is not linearly independent is called *linearly dependent*.

Problem 4.15. Write three vectors that are linearly independent in $\mathbb{R}^3$. Then write three vectors that are linearly dependent in $\mathbb{R}^3$. ▲

Problem 4.16. Consider the vector space

$$\mathcal{P}_n = \{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \, : \, a_0, a_1, \ldots, a_n \in \mathbb{R}\}.$$

Let $n = 3$ and find 4 linearly independent *vectors* in this vector space. Then find 4 linearly dependent vectors in this vector space. ▲

Problem 4.17. Consider the vector space $\mathcal{V} = \{f(x) \, : \, f'(x) \text{ exists }\}$. Find three linearly independent vectors in this vector space that are also a solution to the differential equation $y' = -2y + 3t + 5$. ▲

Problem 4.18. If $\mathbf{v}_1$ and $\mathbf{v}_2$ are vectors in $\mathbb{R}^2$ how would we show that they are linearly independent? ▲

Theorem 4.19. Let S be a set of vectors in a vector space $\mathcal{V}$. If the zero vector, $\mathbf{0}$, is contained in S then the set S is linearly dependent.

Proof. Prove this theorem. □

Theorem 4.20. Let S be a set of vectors in a vector space $\mathcal{V}$. If $\mathbf{u} \in S$ and $c\mathbf{u} \in S$ for some fixed real number c then the set S is linearly dependent.

Proof. Prove this theorem. □

Problem 4.21. Consider the first order non-homogeneous differential equation

$$y' = -3y + 4t.$$

What are the homogeneous and particular solutions that arise from using the method of undetermined coefficients? Are these functions linearly independent? Verify that the analytic solution to the differential equations is a linear combination of these solutions. ▲

Problem 4.22. Suppose you wish to determine whether a set of vectors is linearly independent. You form a matrix with those vectors as the columns and you calculate the reduced row echelon form

$$R = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

What do you decide?

(a) The vectors are linearly independent.

(b) The vectors are not linearly independent.

▲

Problem 4.23. To determine whether a set S of vectors is linearly independent you form a matrix which has those vectors as columns and you calculate its row reduced form. Suppose the resulting form is

$$R = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Which of the following subsets of S are linearly independent?

(a) The first, second, and third vectors

(b) The first, second, and fourth vectors

(c) The first, third, and fourth vectors

(d) The second, third, and fourth vectors

(e) All of the above

▲

Problem 4.24. (a) True or False: A set of 2 vectors from $\mathbb{R}^3$ must be linearly independent.

(b) True or False: A set of 3 vectors from $\mathbb{R}^3$ could be linearly independent.

(c) True or False: A set of 5 vectors from $\mathbb{R}^4$ could be linearly independent.

▲

Problem 4.25. Let $y_1(t) = e^{2t}$. For which of the following functions $y_2(t)$ will the set $\{y_1, y_2\}$ be linearly independent?

(a) $y_2(t) = e^{-2t}$

(b) $y_2(t) = te^{2t}$

(c) $y_2(t) = 1$

(d) $y_2(t) = e^{3t}$

(e) All of the above

(f) None of the above

▲

Problem 4.26. True or False: The function $h(t) = 4 + 3t$ is a linear combination of the functions $f(t) = (1+t)^2$ and $g(t) = 2 - t - 2t^2$. ▲

Problem 4.27. True or False: The function $h(t) = t^2$ is a linear combination of $f(t) = (1-t)^2$ and $g(t) = (1+t)^2$. ▲

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4.3 Span

The following sequence of problems is modified from [7].

Problem 4.28 (The Magic Carpet Ride 1)**.** You are a young traveler leaving home for the first time. Your parents want to help you on your journey, so just before your departure they give you two gifts. Specifically, they give you two forms of transportation: a hover board and a magic carpet. Your parents inform you that both the hover board and the magic carpet have restrictions in how they operate:

- If you traveled "forward" on the hover board for one hour it would move along a diagonal path that would result in a displacement of 3 miles East and 1 mile North of the starting location. Mathematically, the hover board's motion is restricted to the vector $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$
- If you traveled "forward" on the magic carpet for one hour it would move along a diagonal path that would result in a displacement of 1 mile East and 2 miles North of the starting location. Mathematically, the magic carpet's motion is restricted to the vector $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Your Uncle Euler suggests that your first adventure should be to go visit the wise man, Old Man Gauss. Uncle Euler tells you that Old Man Gauss lives in a cabin that is 107 miles East and 64 miles North of your home. Can you use the hover board and the magic carpet to get to Old Man Gauss' cabin? Be able to defend your answer. ▲

Problem 4.29 (Magic Carpet Ride 2)**.** Old Man Gauss wants to move to a cabin in a different location. You are not sure whether Gauss is just trying to test your wits at finding him or if he actually wants to hide somewhere that you can't visit him.

Are there some locations that he can hide and you cannot reach him with using the hover board and the magic carpet? Describe the places that you can reach using a combination of the hover board and the magic carpet and those you cannot. Be able to support your answers. ▲

Problem 4.30 (Magic Carpet Ride 3)**.** Suppose now that you get a third mode of transporation: a jet pack!. In this new scenario assume that your three modes of transportation work as follows:

- The hover board's motion is restricted to the vector $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.
- The magic carpet's motion is restricted to the vector $\mathbf{v}_2 = \begin{pmatrix} 4 \\ 1 \\ 6 \end{pmatrix}$.

- The jet pack's motion is restricted to the vector $\mathbf{v}_3 = \begin{pmatrix} 6 \\ 3 \\ 8 \end{pmatrix}$.

You are allowed to use each mode of transportation **EXACTLY ONCE** (in the forward or backward direction) for a fixed amount of time (c_1 on $\mathbf{v}_1$, c_2 on $\mathbf{v}_2$, and c_3 on $\mathbf{v}_3$). Find the amounts of time on each mode of transportation (c_1, c_2, and c_3 respectively) needed to go on a journey that starts and ends at home $(0,0,0)$ OR explain why it is not possible to do so. ▲

Problem 4.31 (Magic Carpet Ride 4)**.** Modify the jet pack's restriction so that it is not possible to ride each mode of transportation exactly once and end up back at home. ▲

Now let's formalize a few of the ideas that we just ran into.

Definition 4.32. The **span** of a collection of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is the set

$$\{c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots c_n\mathbf{u}_n \,:\, c_j \in \mathbb{R}\}$$

This is the set of all linear combinations of the vectors $\mathbf{u}_1, \dots, \mathbf{u}_n$.

Problem 4.33. Explain the definition of the span of a set of vectors in the context of the magic carpet ride problems. ▲

Problem 4.34. Explain the definition of the linear independence of a set of vectors in the context of the magic carpet ride problems. ▲

Problem 4.35. Span is the collection of all linear combinations of a collection of vectors. Let's build some MATLAB code that may help us visualize that. Let's consider the vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

and we'll use the following code to generate 1000 different random linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$. Fire up MATLAB and write the following code.

```
clear; clc; clf;
v1 = [1;2];
v2 = [-1;3];
plot(v1(1),v1(2),'r*'), hold on
plot(v2(1),v2(2),'k*')
for j=1:1000
    c = 20*rand(2,1)-10; % random weights between -10 and 10
    w = c(1)*v1 + c(2)*v2; % random linear combination of v1 and v2
    plot(w(1),w(2),'bo')
end
```

Based on the resulting picture, what is span$\{\mathbf{v}_1, \mathbf{v}_2\}$? ▲

Problem 4.36. In Problem 4.35 change $\mathbf{v}_2$ to

$$\mathbf{v}_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

and determine $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$. ▲

Problem 4.37. In this problem we'll take Problem 4.35 and ramp it up to three dimensions. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^3$ such that

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix}$$

Modify your code from Problem 4.35 to match the following.

```
clear; clc; clf;
v1 = [1;2;4];
v2 = [-1;3;5];
plot3(v1(1),v1(2),v1(3),'r*'), hold on
plot3(v2(1),v2(2),v2(3),'k*')
for j=1:1000
    c = 20*rand(2,1)-10; % random weights between -10 and 10
    w = c(1)*v1 + c(2)*v2; % random linear combination of v1 and v2
    plot3(w(1),w(2),w(3),'bo')
end
```

Based on the resulting picture, what is $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$? ▲

Problem 4.38. Describe the span of the vectors $\mathbf{u}$ and $\mathbf{v}$ where

$$\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

1. all of $\mathbb{R}^3$
2. A plane in $\mathbb{R}^3$
3. all of $\mathbb{R}^2$
4. A line in $\mathbb{R}^2$
5. none of these

▲

Problem 4.39. Describe the span of the vectors $\mathbf{u}$ and $\mathbf{v}$ where

$$\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$$

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1. all of $\mathbb{R}^3$
2. A plane in $\mathbb{R}^3$
3. all of $\mathbb{R}^2$
4. A line in $\mathbb{R}^2$
5. none of these

▲

Problem 4.40. Describe the span of the vectors $\mathbf{u}_1, \mathbf{u}_2$, and $\mathbf{u}_3$ where

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{u}_3 = \begin{pmatrix} 0 \\ -4 \\ 2 \end{pmatrix}$$

1. all of $\mathbb{R}^3$
2. A plane in $\mathbb{R}^3$
3. all of $\mathbb{R}^2$
4. A line in $\mathbb{R}^2$
5. none of these

▲

Problem 4.41. What is the span of the set $S = \{e^{-2t}, 1\}$ in the space of all differentiable functions. What first order differential equation has a solution space spanned by S? ▲

Problem 4.42. What is the span of the set $S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$? ▲

Problem 4.43. What is the span of the set $S = \{1, x, x^2\}$? ▲

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4.4 Subspaces

The structure of a vector space is filled with geometric richness and wonderful abstraction. As you have already experienced, we can use this abstraction to better understand the structure of sets that contain *mathematical things* that are not vectors in the traditional physics sense. We now examine the notion of a subspace to a vector space. The basic idea is that if we take a vector space and *zoom in* to just the right part we will find that there are subspaces embedded within most vector spaces. This is another abstract notion but, as it turns out, we have been dealing with subspaces all along. In multivariable calculus you got used to dealing with $\mathbb{R}^3$ and undoubtedly dealt with planes in $\mathbb{R}^3$ that went through the origin. Those planes were vector spaces in their own right (check the 10 rules in Definition 4.3).

To get going with the idea of a subspace we need to formalize what we mean by *zoom in*. Let's start this section with a little background terminology.

Definition 4.44 (Subset)**.** Let S be a set. A subset B of S is a collection of elements that are in S. We use the notation $B \subset S$ or $B \subseteq S$.

Example 4.45. Let $S = \{a, b, c, d\}$. Then the set $B = \{a, b\}$ is a subset of S. The set $C = \{a, b, c, e\}$ is not a subset of S since $e \not\in S$.

Example 4.46. Let $S = \mathbb{R}^2$. The set $B = \{(x, y) : x \cdot y \ge 0\}$ is a subset of S since it contains things that are all in $\mathbb{R}^2$. Geometrically, B is the set of all points in the first and third quadrants of the coordinate plane whereas S is all of the coordinate plane.

Problem 4.47. Let $S = \mathbb{R}^3$. Give an example of a set S_1 that IS a subset of S and a set S_2 that IS NOT a subset of S. ▲

Problem 4.48. How many elements are in each of the following sets?

$$S_1 = \mathbb{R}^2 \qquad S_2 = \{\mathbb{R}^2\} \qquad S_3 = \emptyset \qquad S_4 = \{\emptyset\}$$

▲

Throughout the following definitions you need to keep in mind the notion of a subset, but now we will be taking special subsets of vector spaces.

Definition 4.49 (Subspace)**.** If $\mathcal{V}$ is a Vector Space and S is a subset of $\mathcal{V}$ then S is called a **subspace** if it is a vector space in its own right.

Problem 4.50. Consider the vector space $\mathbb{R}^2$. Propose a subspace of $\mathbb{R}^2$ and be able to defend your proposition. ▲

Problem 4.51. Which of the vector space criteria would need to establish to show that a set S is a subspace of a vector space $\mathcal{V}$? Look back to the vector space definition here: 4.3. ▲

Problem 4.52. Which of the following sets are subspaces of $\mathbb{R}^3$? (there are multiple answers)

1. $\{(x,0,0) : x \in \mathbb{R}\}$
2. $\{(5x+4y, 7x+2y, -8x-2y) : x, y \in \mathbb{R}\}$
3. $\{(x,y,z) : x,y,z > 0\}$
4. $\{(-6,y,z)) : y,z \in \mathbb{R}\}$
5. $\{(x,y,z) : -7x+8y-4z = -5\}$
6. $\{(x,y,z) : x+y+z = 0\}$

▲

Problem 4.53. The set of all 2×2 matrices with determinant equal to zero is not a vector subspace. Why?

(a) 2×2 matrices are not vectors

(b) With matrices, AB need not equal BA

(c) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 2 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & 3 \\ 2 & 2 \end{pmatrix}$ is not in the set.

(d) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not in the set.

(e) None of the above

▲

Problem 4.54. Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} 6 \\ 0 \\ -2 \end{pmatrix}$. Which of the following vectors is *not* in the subspace of $\mathbb{R}^3$ spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

(a) $(1,0,0)$

(b) $(4,1,1)$

(c) $(3,3,6)$

(d) All of these are in the subspace of $\mathbb{R}^3$ spanned by the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$

▲

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Theorem 4.55. If A is an $n \times n$ matrix, then the solution set of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ is a subspace of $\mathbb{R}^n$.

Example 4.56. Consider the homogeneous system of equations

$$\begin{pmatrix} 1 & 3 & 6 \\ 1 & 0 & 0 \\ 2 & -1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We would like to know which subspace of $\mathbb{R}^3$ is spanned by the solution to this system.

Solution: We first row reduce the augmented system

$$\left(\begin{array}{ccc|c} 1 & 3 & 6 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & -1 & -2 & 0 \end{array} \right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

Hence, the solution to the system is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} t \quad \text{where} \quad t \in \mathbb{R}$$

Therefore the subspace spanned by the solution to the homogeneous system is

$$S = \operatorname{span}\left(\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \right)$$

which is a one-dimensional subspace of $\mathbb{R}^3$. Geometrically, this subspace is a line through the origin in $\mathbb{R}^3$ pointing in the direction of the vector $(0, -2, 1)^T$.

Problem 4.57. What subspace of $\mathbb{R}^3$ is spanned by the solution space of the equations

$$\begin{cases} 3x_1 + 2x_2 + x_3 &= 0 \\ 2x_1 + 5x_2 - 3x_3 &= 0 \\ 5x_1 - 4x_2 + 9x_3 &= 0 \end{cases}$$

▲

Theorem 4.58 (Proving that a set is a subspace)**.** Let S be a subset of a vector space $\mathcal{V}$. To prove that S is a subspace of $\mathcal{V}$ we only need to check that

- if $\mathbf{u} \in S$ then $c\mathbf{u} \in S$ for any scalar c.

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- if $\mathbf{u}, \mathbf{v} \in S$ then $\mathbf{u} + \mathbf{v} \in S$.

More simply, you can check both conditions simultaneously:
If $\mathbf{u}, \mathbf{v} \in S$ and $c, d \in \mathbb{R}$ then show that $c\mathbf{u} + d\mathbf{v} \in S$.

Problem 4.59. Discuss why the technique listed above is sufficient to prove that S is a vector space in its own right (go back to the definition of a vector space). ▲

Theorem 4.60. The set containing the zero vector, $S = \{\mathbf{0}\}$, is a subspace of every vector space.

Proof. (Prove this theorem) □

Theorem 4.61. The span of a set of vectors is a subspace.

Proof. (prove this theorem) □

Example 4.62. Consider the vector space $\mathbb{R}^2$ and consider the subset of $\mathbb{R}^2$

$$S = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} : x \in \mathbb{R} \right\}.$$

Prove that S is a subspace of $\mathbb{R}^2$.

Proof. Let $\mathbf{u}, \mathbf{v} \in S$. Therefore there exists real numbers x and z such that $\mathbf{u} = \begin{pmatrix} x \\ 0 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} z \\ 0 \end{pmatrix}$. We will check both closure under addition and closure under scalar multiplication.

$$\text{Closure under addition:} \quad \mathbf{u} + \mathbf{v} = \begin{pmatrix} x \\ 0 \end{pmatrix} + \begin{pmatrix} z \\ 0 \end{pmatrix} = \begin{pmatrix} x + z \\ 0 \end{pmatrix} \in S \quad \checkmark$$

$$\text{Closure under scalar multiplication:} \quad c\mathbf{u} = \begin{pmatrix} cx \\ 0 \end{pmatrix} \in S \quad \checkmark$$

□

Example 4.63. Prove that the following subset of $\mathbb{R}^4$ is not a subspace of $\mathbb{R}^4$.

$$S = \left\{ \begin{pmatrix} 3 \\ y \\ z \\ w \end{pmatrix} : y, z, w \in \mathbb{R} \right\}$$

Proof. If $\mathbf{u} \in S$ then $\mathbf{u} = (3, y, z, w)^T$ but we see that $c\mathbf{u} \notin S$ for any c that is not 1. Hence, the set S is not closed under scalar multiplication and therefore cannot be a subspace of $\mathbb{R}^4$. □

Problem 4.64. Which of the following sets are subspaces of $\mathbb{R}^3$ and which are not? Be sure to explain your reasoning. (Hint: three of them are subspace of $\mathbb{R}^3$ and three of them are not.)

$$S_1 = \left\{ \begin{pmatrix} 8x \\ -2x \\ -9x \end{pmatrix} : x \in \mathbb{R} \right\}$$

$$S_2 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x + y + z = 0 \right\}$$

$$S_3 = \left\{ \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} : x \in \mathbb{R} \right\}$$

$$S_4 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x < y < z \right\}$$

$$S_5 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x + y + z = -8 \right\}$$

$$S_6 = \left\{ \begin{pmatrix} 7 \\ y \\ z \end{pmatrix} : y, z \in \mathbb{R} \right\}$$

▲

Problem 4.65. Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} k \\ 8 \\ 11 \end{pmatrix}$. For how many values of k will the vector $\mathbf{w}$ be in the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

(a) No values of k. The vector $\mathbf{w}$ will never be in this subspace.

(b) Exactly one value of k will work

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(c) Any value of k will work

(d) Some values of k will work.

▲

Example 4.66. Let $\mathcal{V}$ be the set of all functions defined on a single real number. Let $S = \{f(x) \in \mathcal{V} : f'(x) \text{ exists}\}$. It is clear that S is a subset of $\mathcal{V}$. Prove that S is a subspace of the vector space $\mathcal{V}$.

Proof:

Let $f(x)$ and $g(x)$ be functions in S. That is to say that both $f'(x)$ and $g'(x)$ must exist. Observe that $h(x) = f(x)+g(x)$ also has a derivative since $h'(x) = f'(x)+g'(x)$ due to the linearity of the derivative operator. Similarly, if $c \in \mathbb{R}$ then if we define $k(x) = cf(x)$ then $k'(x) = cf'(x)$ again due to the linearity of the derivative operator. Therefore the set S is a subspace of the space of all functions.

That is to say: If you add or scale a differentiable function then you will get a differentiable function in return. This should be familiar from calculus.

Example 4.67. Let the set S be defined as

$$S = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in \mathbb{R} \right\}.$$

Is S a subspace of the vector space that contains all two-by-two matrices?

Solution:

No. Observe that the zero vector in the space of 2×2 matrices is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Observe that this zero matrix is not in the set S and therefore S cannot be a subspace of $\mathbb{R}^{2\times 2}$.

Example 4.68. Consider the following true or false questions.

(a) True or False: $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$.

(b) True or False: The xy plane is a subspace of $\mathbb{R}^3$.

(c) True or False: The xy plane in $\mathbb{R}^3$ is equal to $\mathbb{R}^2$.

Solution:

(a) False. The vectors in the space $\mathbb{R}^2$ *look like* $\begin{pmatrix} x \\ y \end{pmatrix}$ and the vectors in the space $\mathbb{R}^3$ *look like* $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$. There is no way we can reach into the space $\mathbb{R}^3$ and pull out some-

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thing that is missing a third component. Note that even if the third component is zero there is a still a third component and that doesn't mean that your vector is in $\mathbb{R}^2$.

(b) True. If we take a vector in the xy plane inside $\mathbb{R}^3$ then that vector must look like $\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$. If we add or scalar multiply vectors that look like this then we end up with another vector that looks like this.

(c) False. The things in the xy plane inside $\mathbb{R}^3$ are all vectors with three components. The things in $\mathbb{R}^2$ are all things with two components. There clearly are not the same thing.

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4.5 Basis

We now come to a pivotal definition in linear algebra. Previously we hinted that we could *build* vector spaces out of simpler components. If we were to ask something of these building blocks what would we ask?

Problem 4.69. Let $\mathcal{V}$ be a vector space. We would like to find a set $\mathcal{B}$, called a **basis**, such that

- The span of $\mathcal{B}$ gives you all of $\mathcal{V}$, and
- The set $\mathcal{B}$ contains as few vectors as possible.

In order to get both of the bullets listed what property would $\mathcal{B}$ have to have? Why? ▲

Definition 4.70 (A Basis for a Vector Space)**.** A set $\mathcal{B}$ is called a **basis** for a vector space $\mathcal{V}$ if

- span$(\mathcal{B})$ = ____________
- $\mathcal{B}$ is ____________

(Fill in the blanks)

Problem 4.71. How large should the basis be for the vector space $\mathbb{R}^2$? Be able to support your answer. ▲

Problem 4.72. How large should the basis be for the vector space $\mathbb{R}^3$? Be able to support your answer. ▲

Problem 4.73. Find a basis for the following vector spaces:

- $\mathbb{R}^3$
- $V = \{(x,y,0) \in \mathbb{R}^3 : x,y \in \mathbb{R}\}$
- $\mathcal{P} = \{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n : a_1, a_2, \ldots, a_n \in \mathbb{R}\}$
- $M_{2\times 2} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{R} \right\}$
- The solution space for the differential equation $y' = -3y + 4t$

▲

Problem 4.74. The set $\mathcal{B} = \{e^{-0.5t}, \sin(3t), \cos(3t)\}$ is the basis for the solution space for which first order linear non-homogeneous differential equation? ▲

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Problem 4.75. The set $\mathcal{B}$ is the basis for what subspace of $\mathbb{R}^3$?

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

▲

Problem 4.76. Which of the following sets of vectors is a basis for $\mathbb{R}^3$?

(a) $\{(1,0,0),(0,1,0),(0,0,1)\}$

(b) $\{(1,0,1),(1,1,0),(1,1,1)\}$

(c) $\{(2,0,0),(0,5,0),(0,0,8)\}$

(d) All are bases for $\mathbb{R}^3$.

▲

Problem 4.77. With your partner create four different sets of vectors. Each vector must be in $\mathbb{R}^3$. Two of the sets must span $\mathbb{R}^3$ and two of the sets must not. ▲

Definition 4.78 (Dimension of a Vector Space)**.** The **dimension** of a vector space is the number of ____________.
(Fill in the blank)

Theorem 4.79. *Let $\mathcal{B}$ be a basis for a vector space $\mathcal{V}$. If any set S contains more vectors than $\mathcal{B}$ then the vectors in S must be ____________.*
(Fill in the blank and then prove it)

Proof. (prove this theorem) □

Problem 4.80. True or False: Any two bases for a vector space consist of the same number of vectors. (If this is true then I suppose we would call it a theorem and we should then prove it) ▲

Theorem 4.81 (Independence, Span, and Basis)**.** *Let $\mathcal{V}$ be an n-dimensional vector space and let S be a subset of $\mathcal{V}$. Then*

- *If S is linearly independent and consists of n vectors, then ____________.*
- *If S spans $\mathcal{V}$ and consists of n vectors, then ____________.*
- *If S is linearly independent, then S (is contained in / is equal to / contains) a basis for $\mathcal{V}$ (choose one).*

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- If S spans $\mathcal{V}$, then S (contains / is / is contained in) a basis for $\mathcal{V}$ (choose one)

Problem 4.82. Which of the following describes a basis for a subspace $\mathcal{V}$?

(a) A basis is a linearly independent spanning set for $\mathcal{V}$.

(b) A basis is a minimal spanning set for $\mathcal{V}$.

(c) A basis is a largest possible set of linearly independent vectors in $\mathcal{V}$.

(d) All of the above

(e) Some of the above

(f) None of the above

▲

Problem 4.83. Consider the vector space of quadratic polynomials

$$\mathcal{P}_2 = \{a_0 + a_1 x + a_2 x^2 \,:\, a_0, a_1, a_2 \in \mathbb{R}\}.$$

A basis for $\mathcal{P}_2$ is $\mathcal{B} = \{1, x, x^2\}$ (...discuss why this is a basis ...).
Answer the following questions about subsets of $\mathcal{P}_2$:

(a) What is the dimension of $\mathcal{P}_2$?

(b) Is $\mathcal{W} = \text{span}(\{1, x\})$ a subspace of $\mathcal{P}_2$?

(c) Is the set $S = \{1 + x\,,\, x^2 + 12\,,\, x - 1\,,\, 2 + x - x^2\}$ linearly independent or linearly dependent?

(d) Is the set $S = \{1 + x\,,\, x^2\,,\, x - x^2\}$ a basis for $\mathcal{P}_2$? If so, prove it. If not then explain why not

▲

Example 4.84. Suppose that an astronaut has 4 boosters on his jet pack that point in the directions $\mathbf{v}_1 = (1,1,2)^T$, $\mathbf{v}_2 = (0,1,3)^T$, $\mathbf{v}_3 = (2,1,1)^T$, and $\mathbf{v}_4 = (-2,1,0)^T$. Show that the span of these vectors is all of $\mathbb{R}^3$ then select a basis for $\mathbb{R}^3$ from this set.
Solution: To show that the set $S\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ spans all of $\mathbb{R}^3$ we consider the matrix A below and row reduce to indentify the linearly indpendent vectors.

$$A = \begin{pmatrix} 1 & 0 & 2 & -2 \\ 1 & 1 & 1 & 1 \\ 2 & 3 & 1 & 0 \end{pmatrix} \to \cdots \to \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We see that there are three linearly independent vectors in the set S so we know that $\text{span}(S) = \mathbb{R}^3$. Furthermore, we see that the vectors $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_4$ are linearly

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indepdent and hence a basis for $\mathbb{R}^3$ is

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Example 4.85. If the fourth booster breaks in the previous example what kind of object is the span of the remaining three boosters?
Solution: This will only leave two linearly independent vectors so the astronaut will be able to travel along a plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$.

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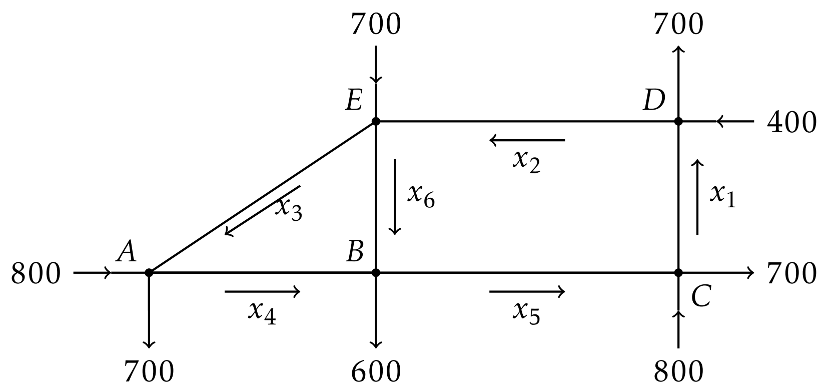
4.6 Row, Column, and Null Spaces in Euclidean Space

Now let's restrict our attention to spaces associated with the rows and columns of matrices and let's motivate a few definitions about the vector spaces associated with the rows and columns of matrices via an applied problem.

Problem 4.86. Consider the traffic flow problem. Remember in a traffic flow network you need to conserve cars. That is, cars cannot be spontaneously created or destroyed and the cars that flow into the network need to flow out so a model for each node is

$$\text{flow in} = \text{flow out}.$$



(a) We can write a system of 5 equations with 6 unknowns to model the flow of the traffic through the network. Each equation should correspond to a node in the traffic graph and each unknown corresponds to the flow of traffic along a road. To save a bit of time, the system is given below:

$$\begin{array}{rl} & \text{Flow In} = \text{Flow Out} \\ \text{Node A:} & 800 + x_3 = 700 + x_4 \\ \text{Node B:} & x_4 + x_6 = 600 + x_5 \\ \text{Node C:} & x_5 + 800 = 700 + x_1 \\ \text{Node D:} & x_1 + 400 = 700 + x_2 \\ \text{Node E:} & x_2 + 700 = x_3 + x_6 \end{array}$$

(b) Fill in the remainder of the augmented system below:

$$\left(\begin{array}{cccccc|c} 0 & 0 & 1 & -1 & 0 & 0 & -100 \\ 0 & 0 & 0 & 1 & -1 & 1 & 600 \\ -1 & 0 & 0 & 0 & 1 & 0 & -100 \\ \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} \end{array}\right)$$

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(c) Solve the system of equations using software. You should notice that the system of equations has infinitely many solutions. Write the row reduced form of the system here:

$$\left(\begin{array}{cccccc|c} \_\_ & \_\_ & \_\_ & \_\_ & \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ & \_\_ & \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ & \_\_ & \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ & \_\_ & \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ & \_\_ & \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ & \_\_ & \_\_ & \_\_ & \_\_ \end{array}\right)$$

Now write the parametric form of the solution.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \end{pmatrix} + \begin{pmatrix} \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \end{pmatrix} t + \begin{pmatrix} \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \end{pmatrix} s \qquad \text{where } s, t \in \mathbb{R}$$

(d) In the previous problem you should have found that x_5 and x_6 are *free variables* (meaning that you can choose them freely). In reality, though, you can't just choose any value for each one since we can't run the roads backward. If you choose $x_5 = 200$ and $x_6 = 0$ we get one feasible solution for the traffic flow.

$x_1 =$ ______, $x_2 =$ ______, $x_3 =$ ______, $x_4 =$ ______, $x_5 =$ ______, $x_6 =$ ______

(e) Since x_6 is another free variable you can make a choice for the amount to flow through x_6 to obtain another feasible solution. Let's still make $x_5 = 200$. How much can you push through road x_6 without causing a traffic jam?

(f) Now let's consider the case of a closed system where no cars flow in or out. What is the solution to this homogenous system of equations?

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \end{pmatrix} t + \begin{pmatrix} \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \end{pmatrix} s \qquad \text{where } s, t \in \mathbb{R}$$

▲

OK. That was fun. Now let's get some terminology and associate it with the traffic problem.

Definition 4.87 (Column Space)**.** Let A be an $m \times n$ matrix. The subspace of $\mathbb{R}^m$ spanned by the n column vectors is called the **column space** of A.

Problem 4.88. What is the column space of the traffic problem depicted in Problem 4.86? Which roads are these columns associated with? The values of the traffic flow on these roads are dependent on the values of the other *free* roads.

$$\text{Column Space of the Traffic Matrix} = \text{span}\left\{ \begin{pmatrix} \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \end{pmatrix}, \begin{pmatrix} \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \end{pmatrix}, \begin{pmatrix} \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \end{pmatrix}, \begin{pmatrix} \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \end{pmatrix} \right\}$$

▲

Definition 4.89 (Column Rank of a Matrix)**.** The dimension of the column space a matrix A is called the (column) **rank** of A.

Problem 4.90. What is the column rank of the traffic problem matrix from Problem 4.86? ▲

Definition 4.91 (Row Space)**.** Let A be an $m \times n$ matrix. The subspace of $\mathbb{R}^n$ spanned by the m rows of A is called the **row space** of A

Problem 4.92. What is the row space of the traffic problem matrix from Problem 4.86?

$$\text{Row Space of the Traffic Matrix} = \text{span}\left\{ \begin{pmatrix} \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \end{pmatrix}, \begin{pmatrix} \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \end{pmatrix}, \begin{pmatrix} \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \end{pmatrix}, \begin{pmatrix} \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \end{pmatrix} \right\}$$

▲

Definition 4.93 (Row Rank of a Matrix)**.** The dimension of the row space is called the (row) **rank** of the matrix A

Problem 4.94. What is the row rank of the traffic problem matrix from Problem 4.86? ▲

Definition 4.95 (Null Space (Kernel))**.** Let A be an $m \times n$ matrix. The **null space** is a matrix A is the solution space of the homogeneous system $A\mathbf{x} = \mathbf{0}$

$$Null(A) = \{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}$$

The null space of a matrix A is often denoted $Null(A)$ or $\mathcal{N}(A)$. We will use both interchangeably throughout the remainder of these notes.

Problem 4.96. What is the null space of the traffic problem depicted in Problem 4.86 and what does it have to do with the context of traffic flow? Pay particular attention here to the situation where only the roads indicated in the null space are turned on. What do you see in the traffic pattern?

$$\text{Null Space of the Traffic Matrix} = \text{span}\left\{ \begin{pmatrix} \_\_\_ \\ \_\_\_ \\ \_\_\_ \\ \_\_\_ \\ \_\_\_ \\ \_\_\_ \end{pmatrix}, \begin{pmatrix} \_\_\_ \\ \_\_\_ \\ \_\_\_ \\ \_\_\_ \\ \_\_\_ \\ \_\_\_ \end{pmatrix} \right\}$$

▲

Definition 4.97 (Nullity of a Matrix)**.** The dimension of the null space of a matrix A is called the **nullity** of A.

Problem 4.98. What is the nullity of the traffic problem matrix from Problem 4.86? ▲

Problem 4.99. Finally we'll make a few observations:

(a) Will the row and column ranks always be the same? Why / Why not?

(b) If you add the dimension of the column space to the dimension of the null space, what do you get and how does that relate to the original system?

▲

Theorem 4.100. The row rank is always (equal to / greater than / less than / unrelated to) the column rank of a matrix.
Circle the correct response.

Theorem 4.101 (Rank-Nullity Theorem)**.** Let A be an $m \times n$ matrix. The sum of the rank and the nullity of A is the number of columns of A.

$$\text{dimension}(\text{Null}(A)) + \text{dimension}(\text{Col}(A)) = \_\_\_\_\_\_\_\_\_\_$$

Theorem 4.102. In the previous definitions we explicitly stated that each of the spaces are indeed subspaces. More specifically, if A is an $m \times n$ matrix then

(a) the span of the rows of A is a subspace of $\mathbb{R}^n$

(b) the span of the columns of A is a subspace of $\mathbb{R}^m$

(c) the set $\mathcal{N}(A) = \{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}$ is a subspace of $\mathbb{R}^n$.

Proof. (you should prove all three of these statements) □

Problem 4.103. Find a basis for the null, column, and row spaces for the matrix below. The row reduced form of the matrix is given for convenience.

$$A = \begin{pmatrix} 5 & -2 & 3 \\ -1 & 0 & -1 \\ 0 & -2 & -2 \\ -5 & 7 & 2 \end{pmatrix} \to \cdots \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

▲

Problem 4.104. True or False:

(a) If A is a 2×3 matrix then the zero vector $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is in the column space of A.

(b) If A is a 2×3 matrix then the zero vector $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is in the row space of A.

(c) If A is a 2×3 matrix then the zero vector $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is in the null space of A.

▲

Example 4.105. Find the row space, column space, and null space of the matrix

$$A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}.$$

Also find the rank and the nullity. **Solution:** First we observe that A row reduces to

$$A \to \cdots \to \begin{pmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Column Space: We observe that columns 1 and 3 are linearly independent so the

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column space of A is

$$\text{Col}(A) = \text{span}\left(\begin{pmatrix}-3\\1\\2\end{pmatrix}, \begin{pmatrix}-1\\2\\5\end{pmatrix}\right)$$

which is a subspace of $\mathbb{R}^3$.
(Notice that we are NOT claiming that the column space is the span of the first and third columns from the row reduced matrix. Why?)

Row Space: From the row reduced form we see that columns 1 and 2 are linearly independent. Since the row reduced form of A is "row equivalent" we see that the row space of A is

$$\text{Row}(A) = \text{span}\left((1,-2,0,-1,3), (0,0,1,2,-2)\right)$$

which is a subspace of $\mathbb{R}^5$.

Null Space: For the null space we are considering the equation $A\mathbf{x} = \mathbf{0}$. From the row reduced form of the matrix we see that the solution to the homogeneous system is

$$\begin{aligned} x_1 &= 2x_2 + x_4 - 3x_5 \\ x_3 &= -2x_4 + 2x_5 \end{aligned}$$

where $x_2, x_4, x_5 \in \mathbb{R}$. Therefore we can write the solution to the homogeneous system as

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix} = \begin{pmatrix}2\\1\\0\\0\\0\end{pmatrix} x_2 + \begin{pmatrix}1\\0\\-2\\1\\0\end{pmatrix} x_4 + \begin{pmatrix}-3\\0\\2\\0\\1\end{pmatrix} x_5$$

Thus the null space of A is

$$\mathcal{N}(A) = \text{span}\left(\begin{pmatrix}2\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\-2\\1\\0\end{pmatrix}, \begin{pmatrix}-3\\0\\2\\0\\1\end{pmatrix}\right)$$

which is a subspace of $\mathbb{R}^5$.

Rank: The rank is the dimension of the column space. We see that the basis of the column space has 2 vectors so $\text{rank}(A) = 2$.

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Nullity: The nullity is the dimension of the null space. We see that the basis for the null space has three vectors so nullity$(A) = 3$. Observe that

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = 5 = \text{number of columns in } A.$$

Example 4.106. Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Find $\mathcal{N}(A)$ and determine the nullity of A.

Solution: This matrix is already row reduced so we can read the solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$ as $x_2 = 0$, $x_3 = 0$ and $x_1 \in \mathbb{R}$. Therefore,

$$\mathcal{N}(A) = \operatorname{span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right)$$

Since the basis for the null space has only one vector we see that nullity$(A) = 1$. One might also observe that rank$(A) = 2$ and that rank(A) + nullity$(A) = 3$ which is the number of columns in A.

Example 4.107. Let A be a matrix that row reduces to $A \to \cdots \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Describe Col$(A)$ and determine the rank of A.

Solution: The column space is the span of columns 2 and 3 since these are the two linearly independent columns. Since the basis for the column space contains two vectors we see that rank$(A) = 2$.

Example 4.108. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & -1 & \pi \\ -30 & 1 & 17 \\ 19 & 3 & -e^2 \\ 0 & 0 & 2 \end{pmatrix}$$

The row, column, and null spaces are subspaces of which vector spaces?

Solution:

Row Space: The row space is a subspace of $\mathbb{R}^3$ since each row contains three entries.

Column Space: The column space is a subspace of $\mathbb{R}^5$ since each column contains 5 entries.

Null Space: The null space is a subspace of $\mathbb{R}^3$ since for $\mathbf{x} \in \text{Null}(A)$ the equation $A\mathbf{x} = \mathbf{0}$ must make sense. Since A is a 5×3 matrix $\mathbf{x}$ must be a 3×1 vector.

Problem 4.109. The *row space* of a matrix A is the set of vectors that can be created by taking all linear combinations of the rows of A. Which of the following vectors is in the row space of the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$?

(a) $\mathbf{x} = \begin{pmatrix} -2 & 4 \end{pmatrix}$

(b) $\mathbf{x} = \begin{pmatrix} 4 & 8 \end{pmatrix}$

(c) $\mathbf{x} = \begin{pmatrix} 0 & 0 \end{pmatrix}$

(d) $\mathbf{x} = \begin{pmatrix} 8 & 4 \end{pmatrix}$

(e) More than one of the above

(f) None of the above

▲

Problem 4.110. The *column space* of a matrix A is the set of vectors that can be created by taking all linear combinations of the columns of A. Is the vector $\mathbf{b} = \begin{pmatrix} -4 \\ 12 \end{pmatrix}$ in the column space of the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$?

(a) Yes, since we can find a vector $\mathbf{x}$ so that $A\mathbf{x} = \mathbf{b}$.

(b) Yes, since $-2\begin{pmatrix} 1 \\ 3 \end{pmatrix} - 1\begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} -4 \\ 12 \end{pmatrix}$.

(c) No, because there is no vector $\mathbf{x}$ so that $A\mathbf{x} = \mathbf{b}$.

(d) No, because we can't find c_1 and c_2 such that $c_1\begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} -4 \\ 12 \end{pmatrix}$.

(e) More than one of the above

(f) None of the above

▲

Problem 4.111. **True or False:** The row space of a matrix A is the same as the column space of A^T. ▲

Problem 4.112. The row space of the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$ consists of

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(a) All linear combinations of the columns of A^T.

(b) All multiples of the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

(c) All linear combinations of the rows of A.

(d) All of the above

(e) None of the above

▲

Problem 4.113. The column space of the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$ is

(a) the set of all linear combinations of the columns of A.

(b) a line in $\mathbb{R}^2$.

(c) the set of all multiples of the vector $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

(d) All of the above

(e) None of the above

▲

Problem 4.114. The *null space* of a matrix A is the set of all vectors x that are solutions of $Ax = 0$. Which of the following vectors is in the null space of the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$?

(a) $x = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

(b) $x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

(c) $x = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$

(d) All of the above

(e) None of the above

▲

Problem 4.115. Let $A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 1 \\ 2 & -1 & 1 & 1 \end{pmatrix}$. Which of the following vectors are in the nullspace of A?

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(a) $\begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix}$

(b) $\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$

(c) $\begin{pmatrix} 2 \\ 3 \\ -1 \\ 0 \end{pmatrix}$

(d) $\begin{pmatrix} 3 \\ -1 \\ 3 \\ 2 \end{pmatrix}$

▲

4.7 The Invertible Matrix Theorem

Problem 4.116. Let A be an $n \times n$ square matrix and assume that A is invertible. Mark each of the following statements about the matrix A as either true or false. If a statement is true then prove it. If a statement is false then provide a counterexample.

(a) $\det(A) \neq 0$

(b) A can be row reduced to the $n \times n$ identity matrix

(c) A has n pivot positions

(d) The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution

(e) The columns of A form a linearly independent set

(f) The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b} \in \mathbb{R}^n$

(g) The columns of A span $\mathbb{R}^n$

(h) There is an $n \times n$ matrix C such that $CA = I$

(i) There is an $n \times n$ matrix D such that $AD = I$

(j) A^T is invertible

(k) The columns of A form a basis for $\mathbb{R}^n$

(l) $\text{Col}(A) = \mathbb{R}^n$

(m) $\dim(\text{Col}(A)) = n$

(n) $\text{rank}(A) = n$

(o) $\text{Null}(A) = \{\mathbf{0}\}$

(p) $\dim(\text{Null}(A)) = 0$

▲

Problem 4.117. If we assume that A is not invertible in the previous problem then which statements would be true and which would be false? Be able to defend your answers. ▲

As you undoubtedly found in the previous problem, all of the statements are true under the assumption that A was invertible. We can actually morph this into a more powerful theorem about matrices.

Theorem 4.118 (Invertible Matrix Theorem)**.** Let A be an $n \times n$ square matrix. Then the following statements are equivalent. That is, for a given matrix A,
the following statements are either **all true or all false**.

(a) A is an invertible matrix

(b) A can be row reduced to the $n \times n$ identity matrix

(c) A has n pivot positions

(d) The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution

(e) The columns of A form a linearly independent set

(f) The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b} \in \mathbb{R}^n$

(g) The columns of A span $\mathbb{R}^n$

(h) There is an $n \times n$ matrix C such that $CA = I$

(i) There is an $n \times n$ matrix D such that $AD = I$

(j) A^T is invertible

(k) The columns of A form a basis for $\mathbb{R}^n$

(l) $\text{Col}(A) = \mathbb{R}^n$

(m) $\dim(\text{Col}(A)) = n$

(n) $\text{rank}(A) = n$

(o) $\text{Null}(A) = \{\mathbf{0}\}$

(p) $\dim(\text{Null}(A)) = 0$

(q) $\det(A) \neq 0$

Problem 4.119. Consider the following true or false questions.

(a) The number of free variables in the solution to $A\mathbf{x} = \mathbf{0}$ is equal to the dimension of the null space of A.

(b) If A is a 3×4 matrix then the row vectors belong to $\mathbb{R}^3$.

(c) If A is an $n \times n$ matrix and $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then the dimension of the null space of A is 0.

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(d) If S is a set of three vectors, each of these is in $\mathbb{R}^2$, then S spans $\mathbb{R}^2$.

(e) If A is an $n \times n$ matrix and the dimension of the null space is 0 then A is invertible.

▲

Problem 4.120. If A is an $n \times n$ matrix and $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b} \in \mathbb{R}^n$ then which of the following are true?

- A is invertible
- The columns of A are linearly independent
- $A\mathbf{x} = \mathbf{0}$ has only the trial solution
- The columns of A span $\mathbb{R}^n$

▲

Problem 4.121. Consider the matrix $A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 2 \\ 0 & 2 & 4 \end{pmatrix}$. It can be shown that $\det(A) = 0$. Based on this fact find all of the following statements that must be true about A.

(a) The matrix A is invertible.

(b) The columns of A are linearly dependent.

(c) The columns of A form a basis for $\mathbb{R}^3$.

(d) The rank of A is less than 3

(e) $A\mathbf{x} = \mathbf{0}$ ha a non-trivial solution.

▲

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4.8 Additional Exercises

Problem 4.122. For each True/False question be able to give a proof or counterexample to support your answer.

(a) True or False: If an $n \times n$ matrix has a determinant of zero then its columns are linearly independent.

(b) True or False: If A is a 2×3 matrix then the zero vector $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is in the column space of A.

(c) True or False: If A is a 2×3 matrix then the zero vector $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is in the row space of A.

(d) True or False: A set of 5 vectors in $\mathbb{R}^4$ spans all of $\mathbb{R}^3$.

(e) True or False: If A is an $n \times n$ matrix and A is invertible then the sum of the dimensions of the row and null spaces of A is equal to n.

(f) True or False: The span of a set of vectors is a subspace of some vector space.

(g) True or False: the matrix $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & -2 \\ 0 & 0 & 4 \end{pmatrix}$ is invertible.

(h) True or False: If A^{-1} does not exist then the null space contains only the zero vector.

(i) If A is an $n \times n$ square matrix and A^T is invertible then $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

▲

Problem 4.123. Which set of vectors is linearly independent?

(a) $(2,3)$, $(8,12)$

(b) $(1,2,3)$, $(4,5,6)$, $(7,8,9)$

(c) $(-3,1,0)$, $(4,5,2)$, $(1,6,2)$

(d) None of these sets are linearly independent.

(e) Exactly two of these sets are linearly independent.

(f) All of these sets are linearly independent.

▲

Problem 4.124. Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} 6 \\ 0 \\ -2 \end{pmatrix}$. Geometrically, what is the subspace spanned by the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?
(a) a point, (b) a line, (c) a plane, (d) a volume, (e) All of $\mathbb{R}^3$. ▲

Problem 4.125. Which of the follow sets are subspaces of $\mathbb{R}^3$?

$$\begin{aligned}
S_1 &= \{(x,y,z) : x+y+z = 7\} \\
S_2 &= \{(x,y,z) : x+y+z = 0\} \\
S_3 &= \{(x,y,z) : -3x-4y = 0, \text{ and } -9x+7z = 0\} \\
S_4 &= \{(7x+8y, -5x+2y, -6x-2y) : x,y \in \mathbb{R}\} \\
S_5 &= \{(3,y,z) : y,z \in \mathbb{R}\} \\
S_6 &= \{(x,y,z) : x,y,z > 0\}
\end{aligned}$$

▲

Problem 4.126. We say that a vector $\mathbf{v}$ is in the span of a set of vectors if $\mathbf{v}$ can e written as a linear combination of the vectors in the set. Let

$$\mathbf{v}_1 = \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} -4 \\ -6 \\ 4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 4 \\ 12 \\ h \end{pmatrix}.$$

For what value of h is $\mathbf{v}_3$ in the plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$? ▲

Problem 4.127. Find a linearly independent spanning set for the subspace of $\mathbb{R}^3$ defined by the equation $8x_1 + 7x_2 - 6x_3 = 0$. You should also notice that this is the equation of a plane. Give a vector that is perpendicular to the plane. ▲

Problem 4.128. Find a basis of the subspace of $\mathbb{R}^5$ spanned by the vectors

$$\begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 7 \\ 0 \\ 7 \\ -6 \\ 14 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \\ -5 \\ 7 \\ -8 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 4 \\ 0 \\ 6 \end{pmatrix}$$

▲

Problem 4.129. Consider the matrix $A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 2 \\ 0 & 2 & 4 \end{pmatrix}$. It can be shown that $\det(A) = 0$. Based on this fact determine whether the following statements are true or false.

(a) True or False: The matrix A is invertible.

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(b) True or False: The columns of A are linearly dependent.

(c) True or False: The columns of A form a basis for $\mathbb{R}^3$.

(d) True or False: The rank of A is less than 3.

(e) True or False: The equation $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution.

▲

Problem 4.130. Let A be an $n \times n$ matrix and assume that $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b} \in \mathbb{R}^n$.

(a) True or False: A is invertible.

(b) True or False: The columns of A are linearly dependent.

(c) True or False: $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(d) True or False: The columns of A span $\mathbb{R}^n$

▲

Problem 4.131. Mark True or False. Do not assume that the matrix A is square unless stated so in the question.

(a) True or False: The number of free variables in the solution to $A\mathbf{x} = \mathbf{0}$ is equal to the dimension of the column space of A.

(b) True or False: If A is a 3×4 matrix then the row vectors belong to $\mathbb{R}^3$.

(c) True or False: If A is $n \times n$ and $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then the dimension of the null space of A is greater than 0.

(d) True or False: If S is a set of three vectors, each of which is in $\mathbb{R}^2$, then S spans $\mathbb{R}^2$.

(e) True or False: If A is an $n \times n$ matrix and the dimension of the null space is 0 then A is invertible.

▲

Problem 4.132. Find the null space, row space, and column space for the matrix

$$A = \begin{pmatrix} 4 & -1 & 3 \\ 12 & -6 & 9 \\ -4 & 2 & -3 \end{pmatrix}$$

▲

Problem 4.133. Find a basis for each of the following vector spaces.

(a) the set of 2×2 diagonal matrices

(b) the set of upper triangular 3×3 matrices

(c) the set of all quadratic polynomials with no linear term

(d) the set of all solutions to the differential equation $y' = -2y$

▲

Problem 4.134. Let $\mathcal{V}$ be the set of all ordered triples (x, y, z) such that $x + y + z = 3$. Show that $\mathcal{V}$ is not a subspace of $\mathbb{R}^3$. ▲

Problem 4.135. Show that every subspace $\mathcal{W}$ of a vector space $\mathcal{V}$ contains the zero vector. ▲

Problem 4.136. Assume that the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. Show that the set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly independent given that

$$\mathbf{u}_1 = \mathbf{v}_2 + \mathbf{v}_3, \quad \mathbf{u}_2 = \mathbf{v}_1 + \mathbf{v}_3, \quad \mathbf{u}_3 = \mathbf{v}_1 + \mathbf{v}_2.$$

▲

Problem 4.137. Suppose that S is a set of n vectors that span the n-dimensional vector space $\mathcal{V}$. Prove that S is a basis for $\mathcal{V}$. ▲

Problem 4.138. Explain why the $n \times n$ matrix A is invertible if and only if its rank is n. ▲

Problem 4.139. Let $\mathcal{F}$ be the space of all real-valued functions on $\mathbb{R}$. Determine if the set of all functions f such that $f(-x) = -f(x)$ for all x is a subspace of $\mathcal{F}$. ▲

Problem 4.140. Let $M_{3\times 3}$ be the set of all 3×3 matrices. Determine if the following subsets of $M_{3\times 3}$ are subspaces

(a) The set of all diagonal 3×3 matrices.

(b) The set of all symmetric 3×3 matrices.

(c) The set of all singular (non-invertible) 3×3 matrices

▲

Problem 4.141. Let $M_{3\times 3}$ be the set of all real 3×3 matrices. Prove that the set S is a subspace of $M_{3\times 3}$. Clearly show all of your work.

$$S = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$$

▲

Problem 4.142. Let $\mathcal{V}$ be the vector space of all real 2×2 matrices and let S be the subspace of all 2×2 upper triangular matrices:

$$S = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$$

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(a) Find a basis of S

(b) Based on your answer to part (a), what is the dimension of the subspace S?

(c) Explain why your basis is a basis.

▲

Problem 4.143. Which of the following describes the span of the set $\mathcal{V}$?

$$\mathcal{V} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$$

Explain your reasoning. ▲

Problem 4.144. Consider the set $\mathcal{V} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$.

(a) Which of the following best describes the span of the set $\mathcal{V}$?

(i) A point

(ii) two points

(iii) A line

(iv) two lines

(v) A plane

(vi) A planes

(vii) A 3 dimensional space

(b) Explain your reasoning from part (a).

(c) Which of the following are in the span of $\mathcal{V}$?

(i) $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

(ii) $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

(iii) $\begin{pmatrix} 0 \\ -2 \\ -4 \end{pmatrix}$

(iv) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

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(v) $3.1\begin{pmatrix}1\\2\\0\end{pmatrix} - \frac{4}{5}\begin{pmatrix}0\\1\\2\end{pmatrix}$

(vi) any vector in $\mathbb{R}^3$.

(d) Explain in general how you can deterine if a given vector is in the span of some other set of vectors.

▲

Problem 4.145. Suppose that $\mathbf{v}_1$ and $\mathbf{v}_2$ are in $\mathbb{R}^n$, c_1 and c_2 are real numbers, and the only solution to the vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$ is the trivial solutions $c_1 = c_2 = 0$.

(a) The set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is (circle one): linearly independent / linearly dependent.

(b) Can $\mathbf{v}_1$ be a scalar multiple of $\mathbf{v}_2$? Explain.

(c) What is the dimension of the span of $\{\mathbf{v}_1, \mathbf{v}_2\}$?

▲

Problem 4.146. Consider the set of vectors

$$\mathcal{W} = \left\{ \begin{pmatrix}1\\-2\\3\end{pmatrix}, \begin{pmatrix}4\\-1\\0\end{pmatrix}, \begin{pmatrix}3\\1\\-3\end{pmatrix}, \begin{pmatrix}6\\-5\\7\end{pmatrix} \right\}.$$

To detetermine whether the set is linearly independent or dependent, a student did the following correct row reduction:

$$\left(\begin{array}{cccc|c} 1 & 4 & 3 & 6 & 0 \\ -2 & -1 & 1 & -5 & 0 \\ 3 & 0 & -3 & 7 & 0 \end{array}\right) \to \cdots \to \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

(a) Is the set of vectors $\mathcal{W}$ linearly independent or linearly dependent?

(b) Explain what it is about the row reduction matrix that tells you whether or not the set $\mathcal{W}$ is linearly independent or dependent.

▲

Problem 4.147. The result of $\begin{pmatrix}1 & 0 & 3\\2 & 1 & 0\end{pmatrix}\begin{pmatrix}-1\\2\\0\end{pmatrix}$ is (circle all that apply)

- a vector in $\mathbb{R}^2$
- a vector in $\mathbb{R}^3$
- a matrix with 2 rows and 3 columns

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- a linear combination of vectors in $\mathbb{R}^2$
- a linear combination of vectors in $\mathbb{R}^3$

▲

Problem 4.148. The three equation below correspond to three planes in $\mathbb{R}^3$.

$$\begin{aligned} x+y-z&=1\\ x-y+z&=1\\ 3x+y-z&=3 \end{aligned}$$

(a) Determine if $(1,1,1)$ is a solution to the system of linear equations and explain how you know.

(b) The following is the correct row reduction of the augmented matrix corresponding to the given system of equations:

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1\\ 1 & -1 & 1 & 1\\ 3 & 1 & -1 & 3 \end{array}\right) \to \cdots \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0 \end{array}\right).$$

Describe all solutions to this system of equations.

(c) Which best describes the intersection of these planes? (circle one and explain)

- no intersection
- a point
- a line
- a plane
- other: ________________________

▲

Problem 4.149. Suppose B is a square matrix and the only solution to $B\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$. Is this enough information to determine whether or not B is invertible? Why or why not? ▲

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Chapter 5

Geometry in Vector Spaces

Vector spaces give a rich environment to explore the relationships between objects and gives natural abstractions to other types of *vectors* (e.g. functions, matrices, polynimials, etc). Most of the study of vector spaces involves abstraction and making precise ideas that spanned many decades, or centuries, in many different fields of science and mathematics. In this sense, the study of vector spaces leans heavily on past mathematical results and seeks to give them order and structure. You have seen that the study of vector spaces really acts as a unifying language to talk about a wide variety of mathematical objects.

In this chapter we layer the language of abstract vector spaces and the language of geometry. In particular, we will build and abstract the familiar ideas of angle, othogonality, lengths of vectors, and distance in general vector spaces. We will make precise the ideas of the *size of a function* or the *distance between two matrices* or *polynomials*. These notions are likely second nature in our familiar vector spaces $\mathbb{R}^2$ and $\mathbb{R}^3$, but what about in spaces of matrices? spaces of functions? spaces of polynomials? The notions of angle and distance can be abstracted in a natural and beautiful way so that our intuitive ideas still hold, but we also get something mathematically meaningful in the more abstract spaces. We'll start with an idea that is probably familiar to you: the dot product*. From the dot product we will build a similar idea is more general vector spaces so that the ideas of angle and perpendicularity come along for the ride. Here we go!

5.1 The Geometry of Euclidean Space

At this point we have talked almost exclusively about linear combinations and the spaces associated with them. We have not, however, discussed the geometry of vector spaces. So far we haven't generalized the notions of angle and length of vectors to our large view of vector spaces. You may recall things like projections, angles, norms (lengths) from $\mathbb{R}^2$ or $\mathbb{R}^3$ as discussed in physics or in multivariable calculus but you need to keep in mind that this is only a limited view of the world of vector spaces. Let's jump right in by filling in some definitions and theorems that you likely already know

*If you haven't had multivariable calculus or calculus-based physics then maybe the idea of a dot product will be new to you. Don't worry. We'll introduce everything from scratch here.

5.1.1 The Dot Product

The following are two different familiar definitions of the dot product. The first gives a purely algebraic formula for the dot product and the second gives a more geometric definition. They are indeed equivalent definitions.

Definition 5.1 (Algebraic Dot Product in $\mathbb{R}^n$)**.** Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. The **dot product** of $\mathbf{u}$ and $\mathbf{v}$ is defined algebraically as

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = \sum_{j=1}^{n} u_j v_j.$$

Notice that this definition doesn't explicitly mention the angle between the vectors.

Definition 5.2 (Geometric Dot Product in $\mathbb{R}^n$)**.** If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ then the relationship between the dot product of the vectors and the angle between the vectors is

$$\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}|| ||\mathbf{v}|| \cos\theta.$$

Notice that if you want the angle between two vectors then

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}|| ||\mathbf{v}||} \right).$$

One should note that the angle formula only holds if both $||\mathbf{u}|| \neq 0$ and $||\mathbf{v}|| \neq 0$.

Now we build the mathematical notation for orthogonality and length of vectors in $\mathbb{R}^n$. Both of these geometric concepts are build upon the dot product.

Definition 5.3 (Orthogonal Vectors)**.** Two vectors are said to be orthogonal (perpendicular) if their dot product is zero.

Problem 5.4. Use the geometric definition of the dot product to prove that statements

- "the dot product of two vectors is zero", and
- "the vectors are perpendicular"

are indeed the same based on your familiar understanding of what "perpendicular" means. ▲

Definition 5.5 (Length of Vectors in $\mathbb{R}^n$)**.** Let $\mathbf{u} \in \mathbb{R}^n$. The **length (norm)** of $\mathbf{u}$ is

$$||\mathbf{u}|| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \underline{\hspace{6cm}}$$

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(fill in the blank using the algebraic definition of the dot product)

Problem 5.6. If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ then what familiar theorem do you see in the definition of the length of a vector? Another way to put this is: the definition of the length of a vector in $\mathbb{R}^n$ is a generalization of what familiar theorem? ▲

Definition 5.7 (Distance Between Vectors in $\mathbb{R}^n$)**.** Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. The **distance between u and v** is

$$\text{dist}(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}|| = \underline{\hspace{3cm}}$$

(fill in the blank)

Definition 5.8 (Unit Vectors in $\mathbb{R}^n$)**.** Let $\mathbf{v} \in \mathbb{R}^n$. We say that $\mathbf{v}$ is a unit vector if $||\mathbf{v}|| = 1$. If you have a vector $\mathbf{v}$ that is not a unit vector (has length other than 1) then you can scale it to become a unit vector by dividing by its length

$$(\text{unit vector in the direction of } \mathbf{v}) = \mathbf{u} = \frac{\mathbf{v}}{||\mathbf{v}||}.$$

OK. Most of the geometric players are on the table: angle, distance, length, and perpendicularity. Here we give a straight forward example showing how to use these ideas in $\mathbb{R}^3$.

Problem 5.9. Let $\mathbf{u}$ and $\mathbf{v}$ be defined as. Find $\mathbf{u} \cdot \mathbf{v}$, $||\mathbf{u}||$, $||\mathbf{v}||$, and the angle between them.

$$\mathbf{u} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\mathbf{v} = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix}$$

▲

Example 5.10. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ be defined as

$$\mathbf{u} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} 0 \\ 9 \\ -3 \end{pmatrix}$$

What are $\mathbf{u} \cdot \mathbf{v}$, $||\mathbf{u}||$, $||\mathbf{v}||$, and $\text{dist}(\mathbf{u}, \mathbf{v})$? Are the two vectors orthogonal? What is the angle between them?

Solution:

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= (2)(0)+(-1)(9)+(4)(-3) = -9-12 = -21\\ ||\mathbf{u}|| &= \sqrt{2^2+(-1)^1+4^2} = \sqrt{4+1+16} = \sqrt{21}\\ ||\mathbf{v}|| &= \sqrt{0^2+9^2+(-3)^2} = \sqrt{81+9} = \sqrt{90}\\ \text{dist}(\mathbf{u},\mathbf{v}) &= ||\mathbf{u}-\mathbf{v}|| = ||\begin{pmatrix}2\\-10\\7\end{pmatrix}|| = \sqrt{4+100+49} = \sqrt{153}.\end{aligned}$$

The vectors are not orthogonal since $\mathbf{u}\cdot\mathbf{v}\neq 0$. The angle between the vectors is

$$\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||||\mathbf{v}||}\right) = \cos^{-1}\left(\frac{-21}{\sqrt{21}\sqrt{90}}\right)$$

5.1.2 Projections

Finally we are going to discuss projections. When dealing with projections you should be thinking about how shadows are cast between vectors. To solidify this notion (even though you likely already know it) let's look at some projections in $\mathbb{R}^2$ before we ramp up the dimension. Take a look at Figure 5.1. We would like to project vector $\mathbf{u}$ onto vector $\mathbf{v}$ and by that we mean that we would like to draw a vector (depicted by the dashed vector $\mathbf{w}$ in the figure) that is perpendicular to $\mathbf{v}$ and meets the head of $\mathbf{u}$. This projection creates the vector $\hat{\mathbf{v}}$ so that $\hat{\mathbf{v}}$ points in exactly the same direction as $\mathbf{v}$ but $\hat{\mathbf{v}}\perp\mathbf{w}$. Since $\mathbf{v}$ and $\hat{\mathbf{v}}$ point in the same direction we know that $\hat{\mathbf{v}} = c\mathbf{v}$ for some scalar c. Furthermore, we know that $\mathbf{w}+\hat{\mathbf{v}} = \mathbf{u}$ so $\mathbf{w} = \mathbf{u}-\hat{\mathbf{v}}$. Therefore,

$$\begin{aligned}&0 = \hat{\mathbf{v}}\cdot\mathbf{w} = c\mathbf{v}\cdot(\mathbf{u}-c\mathbf{v})\\ &\implies c\mathbf{v}\cdot\mathbf{u} - c^2\mathbf{v}\cdot\mathbf{v} = 0\\ &\implies \mathbf{u}\cdot\mathbf{v} = c\mathbf{v}\cdot\mathbf{v}\\ &\implies c = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\end{aligned}$$

All of the prior discuss proves the following theorem but notice that we never made any mention explicitly about the vectors living in $\mathbb{R}^2$. In fact, the proof that we gave works generally in $\mathbb{R}^n$.

Theorem 5.11 (Orthogonal Projection)**.** Let $\mathbf{u},\mathbf{v}\in\mathbb{R}^n$. If we are to project $\mathbf{u}$ onto $\mathbf{v}$ as

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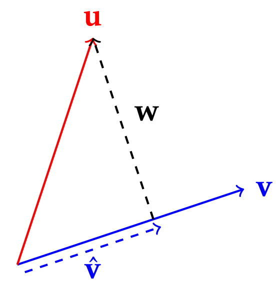

Figure 5.1. Depiction of vector projection in $\mathbb{R}^2$.

in Figure 5.1 we get

$$\text{proj}_{\mathbf{v}}(\mathbf{u}) = \hat{\mathbf{v}} = \left(\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \text{projection of } \mathbf{u} \text{ onto } \mathbf{v}$$
$$\mathbf{w} = \mathbf{u} - \hat{\mathbf{v}} = \mathbf{u} - \left(\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \text{projection error}$$

The vector $\mathbf{w}$ is often called the *error* in the projection.

Problem 5.12. If $\mathbf{b} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and $y = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, then what is the orthogonal projection of $\mathbf{b}$ onto $\mathbf{y}$? Find your solution analytically and draw of graph depicting your answer. ▲

We know what a basis is and what know what orthogonal vectors are. If you think carefully about it, all of your mathematical life you have been dealing with coordinate systems that have orthogonal bases. Every time you graph in $\mathbb{R}^2$ or $\mathbb{R}^3$ you've used the idea of othogonality without thinking much about it. Let's see why this is so incredibly useful.

Problem 5.13. If $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is an orthogonal basis for a vector space $\mathcal{V}$ then how do you write $\mathbf{x}$ as a linear combination of the basis vectors?
(Why is it advantageous to have an orthogonal basis?) Hint: Since $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$ how can you use orthogonality to solve for c_j? ▲

Problem 5.14. Implement your idea on the subspace spanned by the basis

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

where $\mathbf{x}$ is in the subspace of $\mathbb{R}^3$ spanned by $\mathcal{B}$. Specifically, let $\mathbf{x}$ be defined as

$$\mathbf{x} = \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}$$

▲

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Now summarize the process that you built in the previous problem into the following theorem.

Theorem 5.15 (Building Vectors from an Orthogonal Basis)**.** If $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is an orthogonal basis for a vector space $\mathcal{V}$ then for any vector $\mathbf{x} \in \mathcal{V}$ we can write

$$\mathbf{x} = C_1\mathbf{v}_1 + C_2\mathbf{v}_2 + \cdots + C_n\mathbf{v}_n$$

where

$$C_k = \underline{\hspace{3cm}}.$$

Proof. (prove this theorem by leveraging the fact that we have an orthogonal basis) □

Theorem 5.16. If the nonzero vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_k$ are mutually orthogonal then they are linearly independent.

Proof. (prove this theorem) □

Problem 5.17. Determine if the following set of vectors is linearly independent. Do this two different ways.

$$\left\{ \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -4 \\ 7 \end{pmatrix} \right\}$$

▲

Problem 5.18. If we have two linearly independent vectors that are NOT orthogonal, how do we find a set of two orthogonal vectors that span the same space?

For example, can we find two orthogonal vectors that span the same space as

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

▲

5.1.3 The Gram-Schmidt Process: Making Orthogonal Sets

The previous theorems and problems give us good reason to think that having an orthogonal (or orthonormal) basis for a vector space is advantageous both computationally and geometrically. In fact, we have been used to an orthonormal basis all of our matheamtical lives since that is what the regular Cartesian coordinate system is built from. The question now is this:
Given a basis $\mathcal{B}$ for a vector space $\mathcal{V}$ how can we transform that basis into a different basis for the same space but also gain orthogonality? We will build your intuition to the process via a scaffolded problem.

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Problem 5.19. Build a basis for $\mathbb{R}^2$ so that it contains two orthogonal unit vectors with one of the vectors parallel to $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. ▲

Problem 5.20. Consider the vector space $\mathbb{R}^3$ with the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ given by

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

We are going to build a basis $\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ such that $\text{span}(\mathcal{U}) = \mathbb{R}^3$ but the vectors are also mutually orthogonal and all have unit length. (One should note here that the normalization step is optional but since unit vectors are so nice to work with we are leaving it here.)

(a) Define $\mathbf{u}_1$ as a unit vector that points in the same direction as $\mathbf{v}_1$.

$$\mathbf{u}_1 = \frac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1 = \begin{pmatrix} \underline{\quad} \\ \underline{\quad} \\ \underline{\quad} \end{pmatrix}$$

(b) Now we project $\mathbf{v}_2$ onto $\mathbf{u}_1$ and find the error in the projection. This would be the vector $\mathbf{w}$ in Figure 5.1. Once we have the error we should normalize it to get $\mathbf{u}_2$.

$$\mathbf{w}_2 = \mathbf{v}_2 - \text{proj}_{\mathbf{u}}(\mathbf{v}_2) = \mathbf{v}_2 - (\mathbf{v}_2 \cdot \mathbf{u}_1)\mathbf{u}_1 \quad \text{and therefore} \quad \mathbf{u}_2 = \frac{1}{\|\mathbf{w}_2\|}\mathbf{w}_2.$$

Draw a picture of what we just did.

$$\mathbf{u}_2 = \frac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1 = \begin{pmatrix} \underline{\quad} \\ \underline{\quad} \\ \underline{\quad} \end{pmatrix}$$

(c) For $\mathbf{u}_3$ we project $\mathbf{v}_3$ onto both $\mathbf{u}_1$ and $\mathbf{u}_2$ and then normalize.

$$\mathbf{w}_3 = \mathbf{v}_3 - (\mathbf{v}_3 \cdot \mathbf{u}_1)\mathbf{u}_1 - (\mathbf{v}_3 \cdot \mathbf{u}_2)\mathbf{u}_2 \quad \text{and therefore} \quad \mathbf{u}_3 = \frac{1}{\|\mathbf{w}_3\|}\mathbf{w}_3$$

$$\mathbf{u}_3 = \frac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1 = \begin{pmatrix} \underline{\quad} \\ \underline{\quad} \\ \underline{\quad} \end{pmatrix}$$

(d) Verify that indeed $\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis for $\mathbb{R}^3$.

▲

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Problem 5.21. Use the Gram-Schmidt process outlined in the previous problem to produce an orthogonal basis $\mathcal{U}$ for the subspace spanned by

$$\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 8 \\ 5 \\ -6 \end{pmatrix}.$$

▲

Problem 5.22. Let's build an orthogonal basis $\mathcal{U}$ for $\mathbb{R}^3$. To get started let $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. (notice that we are not normalizing this time)

(a) Create a vector $\mathbf{u}_2$ in $\mathbb{R}^3$ so that $\mathbf{u}_1 \perp \mathbf{u}_2$.

(b) Pick a vector $\mathbf{v}_3 \in \mathbb{R}^3$ such that $\mathbf{v}_3$ is linearly independent of $\mathbf{u}_1$ and $\mathbf{u}_2$. Then use one step of the Gram-Schmidt process to create $\mathbf{u}_3$ out of $\mathbf{v}_3$.

(c) Verify that the vectors in your proposed basis are indeed mutually orthogonal. If so we can use one of the previous theorems (which one) to say that the vectors are linearly independent and must therefore span $\mathbb{R}^3$.

▲

5.2 Inner Product Spaces

Now time for some more abstraction! In this section we take the notions of geometry and abstract them to generalized vector spaces. You may have noticed that the dot product is the basic computation necessary to understand angle in $\mathbb{R}^n$ so we first have to provide a generalized version of the dot product.

Definition 5.23 (The Inner Product)**.** An **inner product** is the abstraction of a dot product to a general vector space. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in a vector space $\mathcal{V}$ and c is some real number then

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$
2. $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$
3. $\langle c\mathbf{u}, \mathbf{v} \rangle = c \langle \mathbf{u}, \mathbf{v} \rangle$
4. $\langle \mathbf{u}, \mathbf{u} \rangle \ge 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$

Problem 5.24. Verify that the dot product is indeed an inner product on the vector space $\mathbb{R}^n$. ▲

Problem 5.25. An inner product on $\mathcal{P}_2$ (the space of all quadratic polynomials) is

$$\langle p, q \rangle = p_0 q_0 + p_1 q_1 + p_2 q_2$$

where $p(x) = p_0 + p_1 x + p_2 x^2$ and $q(x) = q_0 + q_1 x + q_2 x^2$. Verify that this indeed is a proper inner product on $\mathcal{P}_2$. If it is then find the inner product of $p(x) = x^2 + 1$ and $q(x) = 2x + x^2$ as well as the angle between $p(x)$ and $q(x)$. ▲

Problem 5.26. Consider the vector space of quadratic polynomials on the interval $x \in [0, 1]$.

$$\mathcal{P}_2 = \{a_0 + a_1 x + a_2 x^2 \, : \, a_0, a_1, a_2 \in \mathbb{R} \text{ and } x \in [0, 1]\}$$

An inner product on this vector space is

$$\langle f, g \rangle = \int_0^1 f(x) g(x) dx.$$

(a) Verify that this is indeed a proper inner product on $\mathcal{P}_2$

(b) Find the inner product of $f(x) = x^2 + 1$ and $g(x) = 2x + x^2$ in $\mathcal{P}_2$ under this inner product.

(c) Set up the necessary integrals to find the lengths of f and g in $\mathcal{P}_2$ under this inner product.

(d) Set up the necessary integrals to find the angle between f and g in $\mathcal{P}_2$ under this inner product.

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(e) Is this the only inner product on $\mathcal{P}_2$?

▲

Problem 5.27. Consider the vector space of 2×2 real matrices

$$\mathcal{V} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

along with the inner product $\langle A, B \rangle = \text{trace}(AB^T)$. Note: If M is a matrix, the *trace* of the matrix, $tr(M)$, is the sum of the entries on the main diagonal.

To simplify your computations a bit we'll expand the definition of the inner product.

$$\langle A, B \rangle = \text{trace}(AB^T) = \text{trace}\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{pmatrix} \right) = \text{trace}\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}^T \right)$$

$$= \text{trace}\left(\begin{pmatrix} a_{11}b_{11} + a_{12}b_{12} & a_{11}b_{21} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{12} & a_{21}b_{21} + a_{22}b_{22} \end{pmatrix} \right)$$

$$= (a_{11}b_{11} + a_{12}b_{12}) + (a_{21}b_{21} + a_{22}b_{22})$$

(a) Why is this a natural choice for the inner product between 2×2 matrices?

(b) Find an orthogonal basis for $\mathcal{V}$. That's right ... I'm asking you to find angles between matrices!! AWESOME!!

▲

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5.3 Fourier Series

In this section we will consider one of the most beautiful and useful applications of inner product spaces. This application has, quite literally, changed the modern world in uncountably many ways. This application, which will arise later in these notes (in the PDE's chapter), is one of the most stunningly beautiful applications out there for everyone to see: The Fourier Series.

Fourier series were first developed by Joseph Fourier (ca 1800) as part of the development of the theory of heat transport. Since then the field of Fourier Analysis has taken over physics, engineering, signal processing, computer science, ...and the list goes on and on and on. Every signal that we transmit is touched in some way by the ideas developed by Fourier in the 1800's. It is no exaggeration that every person in our technological world is impacted daily by Fourier Analysis.

Let's get into it. Consider the vector space spanned by an infinite basis of sine functions

$$\mathcal{B} = \{\sin(kx) : k \in \mathbb{N}\}$$

equipped with the inner product

$$\langle f, g \rangle = \frac{1}{\pi} \int_0^{2\pi} f(x)g(x)dx. \tag{5.1}$$

This particular basis is infinite dimensional since the natural numbers, $\mathbb{N}$, are (countably) infinite, and if we consider the span we can build any periodic function $f(x)$ where $f(0) = 0$ as a linear combination of sine functions of different frequencies. More specifically, since every periodic function can be written as a linear combination of the basis functions we have the infinite sum

$$f(x) = \sum_{k=1}^{\infty} C_k \sin(kx) \tag{5.2}$$

for all period functions f. The most important part of the basis $\mathcal{B}$ is that it is an othonormal basis under the inner product: an orthogonal basis made entirely of unit vectors.

Theorem 5.28. If $j, k \in \mathbb{N}$ then

$$\langle \sin(kx), \sin(jx) \rangle = \frac{1}{\pi} \int_0^{2\pi} \sin(kx)\sin(jx)dx = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases} \tag{5.3}$$

Problem 5.29. What does Theorem 5.28 say about the functions $\sin(kx)$ and $\sin(jx)$ under the inner product

$$\langle f(x), g(x) \rangle = \frac{1}{\pi} \int_0^{2\pi} f(x)g(x)dx?$$

▲

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Proof. Let's prove Theorem 5.28. Notice that in Theorem 5.28 if we let $k = j$ then the inner product shows that

$$\|\sin(kx)\| = \sqrt{\langle \sin(kx), \sin(kx)\rangle} = \sqrt{\frac{1}{\pi}\int_0^{2\pi} \sin^2(kx)dx} = 1$$

which implies that each of the sine functions is a unit vector in this inner product space. Furthermore, if $j \neq k$ then $\langle \sin(kx), \sin(jx)\rangle = 0$, which shows that $\sin(kx)$ and $\sin(jx)$ are orthogonal in this space.

Let's integrate directly to verify this. First, if recall the trig identity

$$2\sin(\theta)\sin(\phi) = \cos(\theta-\phi) - \cos(\theta+\phi)$$

we can rewrite (5.3) as

$$\langle \sin(kx), \sin(jx)\rangle = \frac{1}{\pi}\left[\frac{1}{2}\int_0^{2\pi} \cos\left((k-j)x\right) - \cos\left((k+j)x\right)dx\right].$$

There are clearly two cases: when $k = j$ and when $k \neq j$. In the case that $k = j$ we observe that $k - j = 0$ so the integral becomes

$$\frac{1}{2\pi}\int_0^{2\pi} 1 - \cos(2kx)dx$$

which we can integrate to

$$\frac{1}{2\pi}\int_0^{2\pi} 1 - \cos(2kx)dx = \frac{1}{2\pi}\left[x - \frac{1}{2k}\sin(2kx)\right]_0^{2\pi} = \frac{1}{2\pi}\left[2\pi - \frac{\sin(4k\pi)}{2k} + \frac{\sin(0)}{2k}\right] = 1$$

since the sine of integer multiples of π is zero.

In the case that $k \neq j$ we have

$$\langle \sin(kx), \sin(jx)\rangle = \frac{1}{2\pi}\left[\int_0^{2\pi} \cos\left((k-j)x\right)dx - \int_0^{2\pi} \cos\left((k+j)x\right)dx\right]$$

which can be integrated to

$$\frac{1}{2\pi}\left[\frac{\sin\left((k-j)x\right)}{k-j} - \frac{\sin\left((k+j)x\right)}{k+j}\right]_0^{2\pi} = \frac{1}{2\pi}\left[\frac{\sin\left(2\pi(k-j)\right)}{k-j} - \frac{\sin\left(2\pi(k+j)\right)}{k+j}\right] = 0$$

where we again have used the fact that the sine of an integer multiple of π is zero along with the fact that $\sin(0) = 0$.

This concludes the proof and we see that the set $\mathcal{B} = \{\sin(kx) : k \in \mathbb{N}\}$ is indeed an orthonormal basis of functions under inner product (5.1). □

If you aren't happy with the analytical proof you can at least *convince* yourself that indeed Theorem 5.28 is true.

Problem 5.30. Open MATLAB (or any other symbolic calculus package) and verify that the basis

$$\mathcal{B} = \{\sin(kx) : k \in \mathbb{N}\}$$

is indeed an orthogonal basis. That is, compute

$$\langle \sin(kx), \sin(jx) \rangle = \frac{1}{\pi} \int_0^{2\pi} \sin(kx) \sin(jx)\, dx$$

for various values of j and k and verify that

- if $j = k$ then the inner produce is identically 1.
- if $j \neq k$ then the inner product is zero.

▲

It is worth it to note that we could have defined the inner product for this space as an integral from $-\pi$ to π instead of 0 to 2π and all of these results would still hold. After all, the functions are periodic with period 2π so any domain with length 2π would work just fine.

Theorem 5.31. Any periodic function $f(x)$ with period 2π and $f(0) = 0$ can be written as a linear combination of sine functions $\sin(kx)$. That is,

$$f(x) = C_1 \sin(1x) + C_2 \sin(2x) + C_3 \sin(3x) + \cdots = \sum_{k=1}^{\infty} C_k \sin(kx).$$

Proof. Indeed, we have just expanded the periodic function as a linear combination of the basis functions. □

Problem 5.32. Go to http://mathlets.org/mathlets/fourier-coefficients/, choose "Target A", "Sine", and "All terms". Then use the sliders on the right to closely match the square wave with a Fourier sine series. ▲

Problem 5.33. If $f(x)$ is some periodic function then we can write it as a linear combination of the basis vectors in $\mathcal{B}$:

$$f(x) = \sum_{k=1}^{\infty} C_k \sin(kx).$$

Knowing that the sine functions in the sum form an orthonormal basis for the space of periodic functions propose a way to find each C_k. Hint: Consider Theorem 5.15. ▲

Theorem 5.34 (Fourier Sine Series Coefficients)**.** If $f(x)$ is a 2π periodic function with $f(0) = 0$ then we can expand $f(x)$ as the series

$$f(x) = \sum_{k=1}^{\infty} C_k \sin(kx)$$

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where
$$C_k = \langle f(x), \sin(kx) \rangle = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(kx) dx.$$

Proof. This theorem is a special case of Theorem 5.15. □

Now for some fun!

Problem 5.35. Let's build a sound signal with some noise and then use the mathematics that we just discussed to work some magic.

(a) We'll start by building a clean signal.

```
clear; clc;
dt = 0.001; % set up a time step
t = 0:dt:2*pi % set up time from 0 to 2pi

% Now we'll buid a signal (like a dial tone)
% out of several sine waves with different amplitudes and
% different frequencies (make up your own)
signal = 2*sin(240*t) + 5*sin(380*t) + 2.5*sin(700*t);
soundsc(signal) % play the sound (turn up the volume a bit)
```

(b) Now let's add some noise to the signal

```
clear; clc;
dt = 0.001; % set up a time step
t = 0:dt:2*pi % set up time from 0 to 2pi

NoiseLevel = 10;
signal = 2*sin(240*t) + 5*sin(380*t) + 2.5*sin(700*t) + ...
            NoiseLevel*rand(size(t)) - NoiseLevel/2;
soundsc(signal) % play the sound (turn up the volume a bit)
```

(c) Now we'll pull the dominant frequencies out of the signal. We do this by taking advatnage of the orthogonal basis for the vector space of periodic functions. Read the code below carefully and be sure you know exactly what it happening.
Note: the `trapz` function does the trapezoidal rule to approximate the integral (yeah MATLAB!!).

```
frequences = 1:1000; % list the frequencies we will try
for n=frequencies % try every frequency
    c(n) = (1/pi)*trapz(signal .* sin(n*t))*dt; % integrate
end
```

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(d) Let's see which frequencies it found.

```
DominantFrequencies = frequencies( c > 1 )
```

(e) Let's make a plot of the signal and the Fourier weights.

```
subplot(1,3,1)
plot(t,signal) % plot noisy signal
xlabel('time'), ylabel('intensity'), title('Noisy Signal')
axis([0,0.1,-NoiseLevel , NoiseLevel])

subplot(1,3,2)
plot(frequences, c, 'r')
xlabel('frequency'), ylabel('intensity'),
title('Fourier Transform of Signal')
```

(f) And finally for the coolest part!! Let's trim out the noise!! We know what the dominant frequencies are, so we can build a signal directly from them. This is EXACTLY how digital signal processing works!!

```
CleanSignal = zeros(size(t)); % set up space for the clean signal
for n=DominantFrequencies
    CleanSignal = CleanSignal + c(n) * sin(n*t);
end
pause(2) % let matlab finish playing the old sound
soundsc(CleanSignal)

subplot(1,3,3)
plot(t,CleanSignal)
axis([0,0.1,-NoiseLevel,NoiseLevel])
```

(g) Now go have some fun! Some explorations to consider:

- How much noise can you add and still reasonably recover the original signal?
- How complex can you make the signal and still recover it?
- Can you (audibly) hide a single note but recover it perfectly?
- ...

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Problem 5.36. Consider the function

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < \pi \\ -1 & \text{if } \pi < x < 2\pi \end{cases}.$$

We'll define $f(x)$ to have the value 0 at $0, \pm\pi, \pm 2\pi, \cdots$ and extend the function periodically with period 2π forever. This way $f(x)$ is defined on all real numbers. We want to build a Fourier Series for this function (called the *square wave*)

$$f(x) = \sum_{k=1}^{\infty} C_k \sin(kx).$$

(a) From the previous problem you should have found that taking the inner product of $f(x)$ and $\sin(kx)$ for every value of k will result in the value of C_k. Use this idea to find C_1.

$$C_1 = \langle f(x), \sin(1x) \rangle = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(1x) dx = \underline{\qquad}$$

(Hint: The integral can be broken into two relatively easy integrals if you think carefully about $f(x)$.)

(b) Now find a general formula for C_k by examining the inner product

$$C_k = \langle f(x), \sin(kx) \rangle = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(kx) dx = \underline{\qquad}$$

(c) Write the first several terms of the Fourier sine series for the square wave.

(d) Use MATLAB to build a plot showing successive approximations of the square wave.

▲

Problem 5.37. Repeat the previous problem to find the Fourier series for the function

$$f(x) = -\frac{1}{\pi} x + 1$$

for $x \in [0, 2\pi]$ and extended periodically outside the domain. ▲

We will return to the idea of Fourier series in Chapter 11 where we'll use Fourier series to solve problems in heat conduction. We have also presented a bit of a limited view in this section in that we have only allowed for Fourier *sine* series. We can generalize the idea to a general Fourier series,

$$f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos(kx) + \sum_{k=0}^{\infty} b_k \sin(kx),$$

where using both sine and cosine functions allows for more flexibility in the types of functions we can model. The choice of inner product doesn't change and the cosine terms have all of the wonderful properties as we had before with the sine functions.

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5.4 Linear Transformations

Problem 5.38. Loosely speaking, a **linear transformation** in $\mathbb{R}^n$ transforms one set of basis vectors to another. Go to https://shad.io/MatVis/ and use the slider bar to experiment with how the standard basis in $\mathbb{R}^2$ transitions to other bases. Specifically experiment with:

(a) Can you create a linear transformation that only scales the coordinate system?

(b) Can you create a linear transformation that only rotates the coordinate system?

(c) What appears to happen to the determinant of the 1×1 square under your transformation? What does this tell you about how the transformation changes areas in $\mathbb{R}^2$?

▲

Definition 5.39 (Linear Transformation)**.** A **linear transformation** T from a vector space $\mathcal{V}$ into a vector space $\mathcal{W}$ is a rule that assigns to each vector $\mathbf{v} \in \mathcal{V}$ a unique vector $\mathbf{w} \in \mathcal{W}$, such that

(a) $T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2)$ for all $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$

(b) $T(c\mathbf{v}) = cT(\mathbf{v})$ for all $\mathbf{v} \in \mathcal{V}$ and all scalars c.

More simply, a linear transformation has the property that

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1 T(\mathbf{v}_1) + c_2 T(\mathbf{v}_2)$$

for all $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$ and for all scalars c_1 and c_2.

5.4.1 Matrix Transformation

Problem 5.40. Verify that if A is an $n \times n$ matrix then the function T defined as $T(\mathbf{x}) = A\mathbf{x}$ is indeed a linear transformation. ▲

Since matrix multiplication is a linear transformation let's look at some of the common matrix transformations. Matrix transformations commonly used in computer graphics are *dilation*, *shear*, and *rotation*. Notice that the common geometric transformation of *translation* is not a linear transformation since if we translate then the origin does not stay fixed.

Problem 5.41. Consider the square defined by the set of points $S = \{(0,0), (1,0), (1,1), (0,1)\}$. Let $T(\mathbf{x})$ be a linear transformation defined as $T(\mathbf{x}) = A\mathbf{x}$ where A is a matrix and $\mathbf{x} \in S$. Applying the transformation T to the points in S gives a new geometric shape. Describe the geometric action of each of the following transformations by applying them to the points in S.

(a) $T(\mathbf{x}) = A\mathbf{x}$ where
$$A = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

(b) $T(\mathbf{x}) = A\mathbf{x}$ where
$$A = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$$

(c) $T(\mathbf{x}) = A\mathbf{x}$ where
$$A = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

(d) $T(\mathbf{x}) = A\mathbf{x}$ where
$$A = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix}$$

(e) $T(\mathbf{x}) = A\mathbf{x}$ where
$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(f) $T(\mathbf{x}) = A\mathbf{x}$ where
$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

▲

Definition 5.42 (Shear Matrices)**.** The 2D linear transformation that geometrically shears parallel to the x axis is defined as

$$T(\mathbf{x}) = \begin{pmatrix} 1 & \underline{\quad} \\ \underline{\quad} & 1 \end{pmatrix}\mathbf{x}$$

The 2D linear transformation that geometrically shears parallel to the y axis is defined as

$$T(\mathbf{x}) = \begin{pmatrix} 1 & \underline{\quad} \\ \underline{\quad} & 1 \end{pmatrix}\mathbf{x}$$

(fill in the blanks)

Problem 5.43. In Figure 5.2 we applied two different shear transformations to the set $S = \{(1,0),(0,1),(-1,0),(0,-1)\}$. Which shear transformations were applied? ▲

Definition 5.44 (Dilation Matrices)**.** The 2D linear transformation that geometrically

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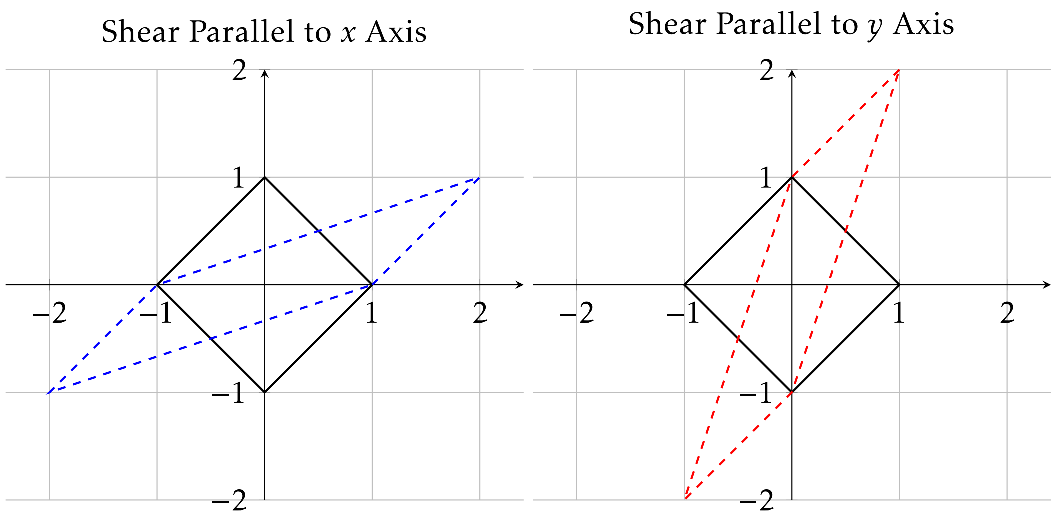

Figure 5.2. Examples of shear transformations. The blue transformation on the left is a shear parallel to the x axis. The red transformation on the right is a shear parallel to the y axis.

dilates a figures by a factor of r is defined as

$$T(\mathbf{x}) = \begin{pmatrix} \_\_\_ & 0 \\ 0 & \_\_\_ \end{pmatrix}\mathbf{x}.$$

(fill in the blanks)

Problem 5.45. In Figure 5.3 we applied two different dilation transformations to the set $S = \{(0,0),(1,1),(1,2),(0,1)\}$. Which dilation transformations were applied? ▲

Definition 5.46 (Rotation Matrices)**.** The 2D linear transformation that geometrically rotates a figure by an angle θ is defined as

$$T(\mathbf{x}) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Problem 5.47. What matrices yield to the following linear transformation?

(a) A rotation by 90° counterclockwise.

(b) A rotation by 90° clockwise.

(c) A rotation by 45° counterclockwise.

(d) A rotation by 30° clockwise.

▲

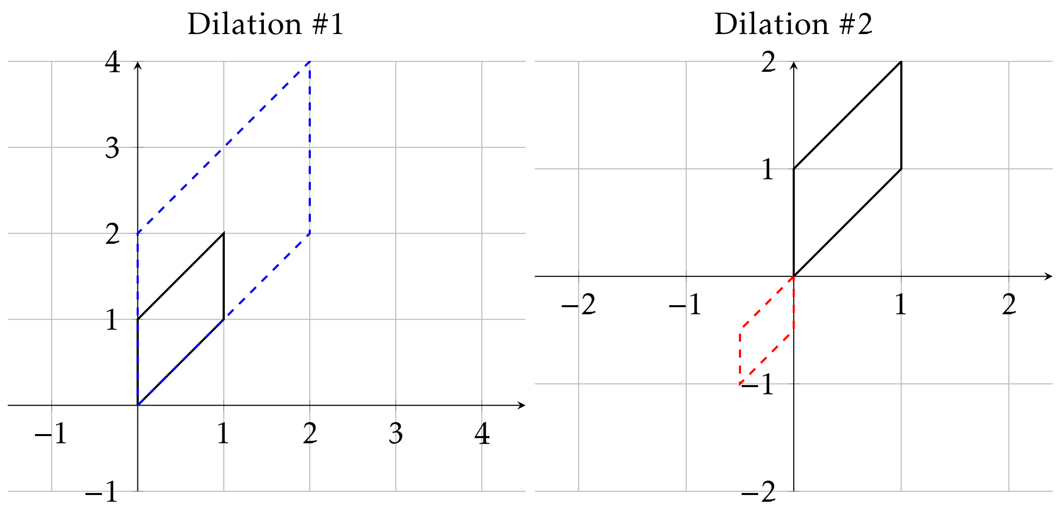

Figure 5.3. Examples of dilation transformations.

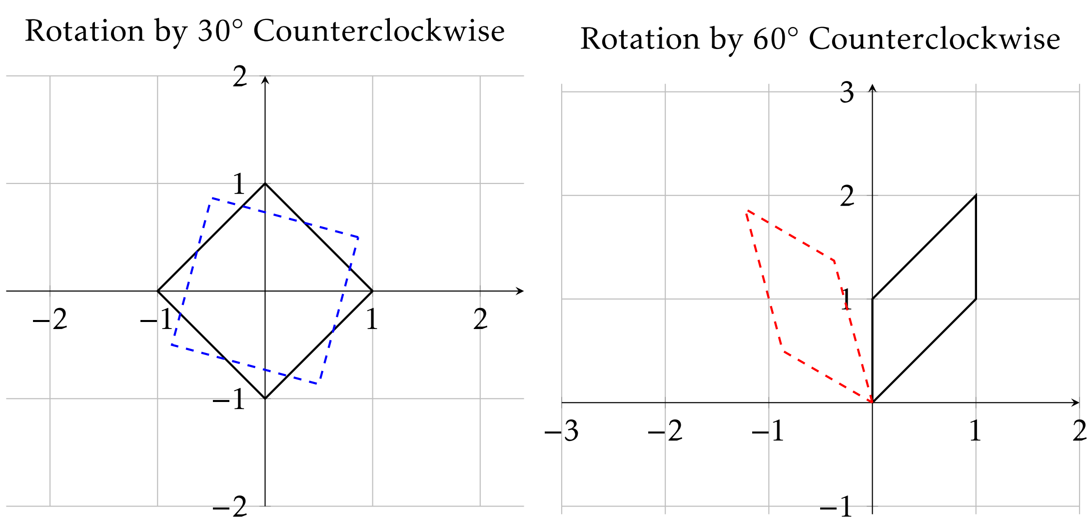

Figure 5.4. Examples of rotation transformations. The blue transformation on the left is a rotation by 30° counterclockwise. The red transformation on the right is a rotation by 60° counterclockwise.

Now that we know the back geometric linear transformations let's see what happens when we compose several of them.

Problem 5.48. Consider the set of points $S = \{(0,0),(1,0),(1,1),(0,1)\}$.
(You may want to write code to complete this problem)

(a) Shear the shape generated by S by a factor of 2 parallel to the x axis and then dilate the resulting shape by a factor of 3. Draw the resulting geometric shape and find the matrix that does these transformations as one linear transformation.

(b) Dilate the shape generated by S by a factor of 3 and then shear the resulting shape by a factor of 2 parallel to the x axis. Draw the resulting geometric shape and find the matrix that does these transformations as one linear transformation.

(c) Rotate the shape generated by S by 60° counterclockwise, then dilate the resulting shape by a factor of -1.5, and finally shear the shape by a factor of 3 parallel to the y axis. Draw the resulting geometric shape and find the matrix that does these transformations as one linear transformation.

▲

5.4.2 Linear Transformations in Abstract Vector Spaces

Problem 5.49. In calculus we know of two very important linear transformations. Let $\mathcal{V}$ be the vector space of all real-valued functions f on the interval $[a,b]$ that are differentiable and continuous on $[a,b]$. Let $\mathcal{W}$ be the vector space $C[a,b]$ of all continuous functions on $[a,b]$.

- The transformation $D : \mathcal{V} \to \mathcal{W}$ is defined as $D(f) = f'$. That is, D is the transformation that takes a derivative of a function.
- The transformation $\mathcal{I} : \mathcal{W} \to \mathcal{V}$ is defined as $\mathcal{I}(f) = \int_a^x f(\tau)d\tau$. That is, $\mathcal{I}$ is the transformation that gives the antiderivative of a function.

Verify that both of these well-known transformations are indeed linear transformations. ▲

Definition 5.50 (Kernel of a Linear Transformation)**.** Let T be a linear transformation from the vector space $\mathcal{V}$ to the vector space $\mathcal{W}$. The **kernel** of T is defined as

$$\text{Ker}(T) = \{\mathbf{x} \in \mathcal{V} : T(\mathbf{x}) = \mathbf{0} \in \mathcal{W}\}.$$

Observe that the kernel is another name for the null space.

Problem 5.51. Let D be the linear transformation defined as $D(f) = f'$ as in problem 5.49. What is the kernel of D? ▲

Problem 5.52. Define $T(y)$ on $\mathcal{V} = \{y(t) : y'(t) \text{ and } y''(t) \text{ exist }\}$ and define $T(y) = \frac{d^2y}{dt^2}$. What is the kernel of T? ▲

Problem 5.53. Define $T : \mathcal{P}_2 \to \mathbb{R}^2$ by

$$T(p) = \begin{pmatrix} p(0) \\ p(1) \end{pmatrix}.$$

For instance, if $p(t) = 3 + 5t + 7t^2$ then $T(p) = \begin{pmatrix} 3 \\ 15 \end{pmatrix}$.

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(a) Verify that T is indeed a linear transformation.

(b) Find a polynomial $p(x) \in \mathcal{P}_2$ that is in the kernel of T.

▲

Problem 5.54. Let $M_{2\times 2}$ be the vector space of all 2×2 matrices and define $T(A) = A + A^T$ for $A \in M_{2\times 2}$. Let B be any matrix in $M_{2\times 2}$ such that $B^T = B$. Find a matrix A such that $T(A) = B$. Then describe the kernel of T. ▲

Example 5.55. Determine if the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4x_1 - 2x_2 \\ 3|x_2| \end{pmatrix}$$

is or is not a linear transformation.
Solution: By the definition of a linear transformation we need to see check that $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}+\mathbf{v})$ and that $T(c\mathbf{u}) = cT(\mathbf{u})$ for arbitrary vectors $\mathbf{u},\mathbf{v} \in \mathbb{R}^2$.

Let $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and observe that

$$T(\mathbf{u}+\mathbf{v}) = T\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = T\left(\begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix}\right) = \begin{pmatrix} 4(u_1+v_1) - 2(u_2+v_1) \\ 3|u_2 + v_1| \end{pmatrix}.$$

We can clearly separate the first component but due to the absolute value in the second component we cannot separate the result of the previous equation to form $T(\mathbf{u}) + T(\mathbf{v})$. Therefore, T is not a linear transformation.

Theorem 5.56. If T is a linear transformation then $T(\mathbf{0}) = \mathbf{0}$.

Proof. If T is a linear transformation then $T(c\mathbf{u}) = cT(\mathbf{u})$ for any vector $\mathbf{u}$ and any scalar $c \in \mathbb{R}$. If we take $c = 0$ then $T(0\mathbf{u}) = cT(\mathbf{u})$ which implies that $T(\mathbf{0}) = \mathbf{0}$. □

Example 5.57. Determine if the transformation $T(x_1,x_2,x_3) = (1,x_2,x_3)$ is a linear transformation.
Solution: Observe that $T(0,0,0) = (1,0,0)$ so by the previous theorem we see that T is not a linear transformation.

Example 5.58. Determine if the transformation $T(x_1,x_2,x_3) = (x_1,0,x_3)$ is a linear transformation.
Solution: Since $T(\mathbf{0}) = \mathbf{0}$ it is possible that T is a linear transformation but we cannot use this to prove that T <u>is</u> linear. We need to check that $T(c_1\mathbf{u} + c_2\mathbf{v}) = c_1T(\mathbf{u}) + c_2T(\mathbf{v})$.

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Indeed, let $\mathbf{u} = (u_1, u_2, u_3)$ and let $\mathbf{v} = (v_1, v_2, v_3)$ and let $c_1, c_2 \in \mathbb{R}$. Therefore,

$T(c_1\mathbf{u}+c_2\mathbf{v}) = T((c_1u_1, c_1u_2, c_1u_3)+(c_2v_1, c_2v_2, c_2v_3)) = T((c_1u_1+c_2v_1, c_1u_2+c_2v_2, c_1u_3+c_2v_3))$

Applying the transformation gives

$$T(c_1\mathbf{u}+c_2\mathbf{v}) = (c_1u_1+c_2v_1, 0, c_1u_3+c_2v_3) = \cdots = c_1T(\mathbf{u})+c_2T(\mathbf{v})$$

which means that T is indeed a linear transformation.

In this class we have studied two particular types of questions: solving first order non-homogeneous differential equations and solving systems of equations. Let's consider the processes for these two problems side by side so that we can truly see them as the exact same problem in the language of linear transformations.

Non-homogeneous 1^{st} order ODE	**Non-homogeneous linear system**
1. Solve $y'+Py = Q(t)$	1. Solve $A\mathbf{x} = \mathbf{b}$
2. Let $T(y) = y'+Py$. We want to find y so that $T(y) = Q$.	2. Let $T(\mathbf{x}) = A\mathbf{x}$. We want to find $\mathbf{x}$ so that $T(\mathbf{x}) = \mathbf{b}$.
3. Find $y_h \in \mathrm{Ker}(T)$	3. Find $\mathbf{x}_h \in \mathrm{Null}(A)$
4. Find a particular y_p so that $T(y_p) = Q$.	4. Find a particular $\mathbf{x}_p$ so that $T(\mathbf{x}) = \mathbf{b}$.
5. The solution to $T(y) = Q$ is $y = y_h + y_p$	5. The solution to $T(\mathbf{x}) = \mathbf{b}$ is $\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p$

Example 5.59. In this example we will solve two problems related to linear transformations. Let $T_1(y) = y'+0.5y$ and $Q(t) = 3$. Let $T_2(\mathbf{x}) = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}\mathbf{x}$ and let $\mathbf{b} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$. Solve $T_1(y) = Q$ and $T_2(\mathbf{x}) = \mathbf{b}$.

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$T_1(y) = Q$

1. The homogeneous solution is $y_h \in \text{span}\{e^{-0.5t}\}$.
2. The non-homogeneity if a constant function so $y_p \in \text{span}\{1\}$
3. The solution to $T_1(y) = Q$ is $y = C_0 e^{-0.5t} + C_1$ where $C_0, C_1 \in \mathbb{R}$.

$T_2(\mathbf{x}) = \mathbf{b}$

1. After row reducing the homogeneous solution is $\mathbf{x}_h \in \text{span}\left\{\begin{pmatrix} -3 \\ 1 \end{pmatrix}\right\}$
2. After row reducing with $\mathbf{b}$ on the right we see that the particular solution is $\mathbf{x}_p = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$
3. The solution to $T_2(\mathbf{x}) = \mathbf{b}$ is $\mathbf{x} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \end{pmatrix} t$ where $t \in \mathbb{R}$.

Example 5.60. Consider the homogeneous linear differential equation $y' + 0.5y = 0$. We can see this as a question about the kernel of a linear transformation. Indeed, if we let $T(y) = y' - 0.5y$ be a transformation from the space of differentiable functions to the space of continuous functions (on appropriate domains) then the differential equation can simply be stated as: find y in the kernel of the transformation $T(y) = y' + 0.5y$.

The kernel of this linear transformation is spanned by $y(t) = e^{-0.5t}$ since $T(y) = 0$. Therefore the solution to the differential equation is $y(t) = Ce^{-0.5t}$.

Problem 5.61. Consider the homogeneous linear differential equation $y'' + y' - y = 0$. Rewrite this differential equation as a question about the kernel of an appropriate linear transformation. ▲

Problem 5.62. For non-homogeneous linear differential equations we can re-frame them in the language of linear transformations in the following way.

- Find a function in the kernel of the transformation
- Find a particular solution that satisfies the non-homogeneous equation
- The general solution is a linear combination of the kernel solution and the particular solution.

Use this idea to solve $T(y) = \sin(t)$ where $T(y) = y' + y$ ▲

Problem 5.63. Let T be a linear transformation that maps vectors in $\mathbb{R}^2$ to vectors in $\mathbb{R}^2$. Symbolically we write $T : \mathbb{R}^2 \to \mathbb{R}^2$. Assume that

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \quad \text{and} \quad T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -7 \\ -3 \end{pmatrix}.$$

Use the following hints to determine the action of T on an arbitrary vector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$.

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- Expand $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ as a linear combination of the basis vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
- Recall that if T is a linear transformation then $T(c\mathbf{u}) = cT(\mathbf{u})$ and $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$. Use this fact to write $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$
- Simplify your answer to give the definition of T.

▲

Theorem 5.64. Let $T : \mathcal{V} \to \mathcal{W}$ be a linear transformation from a vector space $\mathcal{V}$ to a vector space $\mathcal{W}$. The action of T on any vector $\mathbf{v} \in \mathcal{V}$ is completely determined by the actions of T on the basis vectors for $\mathcal{V}$.

More clearly:
Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a basis for the vector space $\mathcal{V}$. Let T be a linear transformation from $\mathcal{V}$ to vector space $\mathcal{W}$ and assume that

$$T(\mathbf{v}_1) = \mathbf{w}_1, \quad T(\mathbf{v}_2) = \mathbf{w}_2, \quad \ldots \quad T(\mathbf{v}_k) = \mathbf{w}_k$$

where $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k \in \mathcal{W}$. If $\mathbf{v}$ is written as a linear combination of basis vectors from $\mathcal{B}$

$$\mathbf{v} = \sum_{j=1}^{k} c_j \mathbf{v}_j,$$

then

$$T(\mathbf{v}) = \sum_{j=1}^{k} c_j \mathbf{w}_j$$

Proof. The proof follows from the definition of a linear transformation.

$$\begin{aligned} T(\mathbf{v}) &= T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k) \\ &= T(c_1\mathbf{v}_1) + T(c_2\mathbf{v}_2) + \cdots + T(c_k\mathbf{v}_k) \\ &= c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_kT(\mathbf{v}_k) \\ &= c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_k\mathbf{w}_k \end{aligned}$$

□

The consequence of Theorem 5.64 is that all we really need to know is the action of a linear transformation on the basis vectors and we know the entire definition of the transformation.[†]

[†]Note here that we are implicitly assuming that the vector spaces $\mathcal{V}$ and $\mathcal{W}$ are finite dimensional. If they were infinite dimensional the theorem will still hold under suitable convergence conditions.

Problem 5.65. Let T be a linear transformation mapping quadratic polynomials to 2×2 matrices: $T : \mathcal{P}_2 \to M_{2\times 2}$. Recall that the set $\mathcal{B} = \{1, x, x^2\}$ is a basis for $\mathcal{P}_2$. If

$$T(1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$
$$T(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$T(x^2) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

then what is the action of T on the generic quadratic polynomial $T(ax^2 + bx + c)$? ▲

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5.5 Additional Exercises

Problem 5.66 (Legendre Polynomials)**.** Consider the vector space

$$\mathcal{V} = \{a_0 + a_1 x + a_2 x^2 \, : \, a_0, a_1, a_2 \in \mathbb{R} \text{ and } x \in [-1,1]\}$$

along with the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) dx$$

Consider the basis $\mathcal{B} = \{1, x, \frac{1}{2}(3x^2 - 1)\}$.

(a) Is this basis an orthogonal basis?

(b) Set up the necessary calculus to write $h(x) = 3x^2 + 2$ as a linear combination of vectors in $\mathcal{B}$.

▲

Problem 5.67. The set S below contains three linearly independent mutually orthogonal vectors and as such we know that $\text{span}(S) = \mathbb{R}^3$

$$S = \left\{ \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -4 \\ 7 \end{pmatrix} \right\}$$

Write the vector $\mathbf{u} = \begin{pmatrix} 3 \\ 0 \\ 5 \end{pmatrix}$ as a linear combination of the vectors in S. ▲

Problem 5.68. Let $\mathcal{V} = C[-1,1]$ be the vector space of continuous functions on the closed interval $[-1,1]$. An inner product on this vector space is $\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) dx$.

Let $f(x) = x^2 + 1$ and $g(x) = x$. Set up (<u>but do not evaluate</u>) the integrals to find the angle between these two functions in $C[-1,1]$. ▲

Problem 5.69. If A and B are arbitrary $m \times n$ matrices, then the mapping $\langle A, B \rangle = \text{trace}(A^T B)$ defines an inner product in $\mathbb{R}^{m \times n}$. Use this inner product to find $\langle A, B \rangle$, the norms, $||A||$ and $||B||$, and the angle between A and B for

$$A = \begin{pmatrix} -2 & 3 \\ -1 & 3 \\ 1 & -2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -2 & 1 \\ 1 & 2 \\ 1 & 1 \end{pmatrix}.$$

▲

Problem 5.70. Find a non-trivial vector that is perpendicular to both $\mathbf{v}$ and $\mathbf{u}$ where

$$\mathbf{v} = \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{u} = \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix}$$

▲

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Problem 5.71. Determine which of the following transformations are linear transformations.

$$\begin{aligned} T_1(x,y,z) &= (x,y,-z) \\ T_2(x,y) &= (2x-3y, x+4, 5y) \\ T_3(x,y,z) &= (1,y,-z) \\ T_4(x,y) &= (4x-2y, 3|y|) \\ T_5(x,y,z) &= (x,0,z) \end{aligned}$$

▲

Problem 5.72. Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that sends the vector $\mathbf{u} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ to $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and maps $\mathbf{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ to $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$. Use properties of linear transformations to calculate the following.

(a) $T(-3\mathbf{u})$

(b) $T(8\mathbf{v})$

(c) $T(-3\mathbf{u}+8\mathbf{v})$

▲

Problem 5.73. Let $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Let $\mathbf{v} = \begin{pmatrix} -3 \\ 9 \end{pmatrix}$ and let $\mathbf{u} = \begin{pmatrix} -5 \\ -8 \end{pmatrix}$. If $T:\mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation that sends e_1 to $\mathbf{v}$ and e_2 to $\mathbf{u}$. Where does T send $\begin{pmatrix} -4 \\ 6 \end{pmatrix}$? ▲

Problem 5.74. Let $T:\mathcal{P}_2 \to \mathcal{P}_2$ be a linear transformation mapping quadratic functions to quadratic functions. Assume that

$$T(1) = 4x^2+4, \quad T(x) = 4x+3, \quad \text{and} \quad T(x^2) = 4x^2+x+3.$$

Find the image of an arbitrary quadratic polynomial ax^2+bx+c under T. That is, find $T(ax^2+bx+c)$. ▲

Problem 5.75. Assume that $T:\mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $T(\mathbf{v}) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\mathbf{v}$.

(a) Which of the following best describes what T does to the plane?

- T stretches vectors vertically by a factor of 2, then rotates them 90° clockwise.
- T rotates vectors 90° clockwise, then stretches them vertically by a factor of 2.
- T stretches vectors horizontally by a factor of 2, then rotates them 90° clockwise.
- T rotates vectors 90° clockwise, then stretches them horizontally by a factor of 2.

(b) Explain your choices from part (a)

▲

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Chapter 6

The Eigenvalue Eigenvector Problem

In this chapter we look at the important eigenvalue-eigenvector question. In this question we wish to find vectors $\mathbf{x}$ and values λ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some given square matrix A. The eigenvalue-eigenvector problem has suprisingly had a profound impact on how we understand differential equations, but it also has profound impacts on how we understand bases, matrix multiplication, and many other important aspects of linear algebra. Furthermore, we can extend the idea to linear operators and view certain linear differential equations as eigenvalue questions. Wow ... just wow ... you're going to love this chapter!

6.1 Introduction To Eigenvalues

Definition 6.1 (The Eigenvalues and Eigenvectors of a Matrix)**.** Let A be a square $n \times n$ matrix. In the equation $A\mathbf{x} = \lambda\mathbf{x}$, the vector $\mathbf{x} \in \mathbb{R}^n$ is called an eigenvector of the matrix A and λ is the associated eigenvalue.

Problem 6.2. In the applet http://www.geogebra.org/m/334841 you will find a way to graphically manipulate vectors to approximate eigenvectors in $\mathbb{R}^2$. Use the applet to approximate the eigenvectors and eigenvalues of $A = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}$. ▲

Problem 6.3. Go to the applet https://shad.io/MatVis/ and turn the Eigenvectors on. What happens to the eigenvectors under a linear transformation in $\mathbb{R}^2$?? ▲

Problem 6.4. Which of the following is an eigenvector of the matrix $\begin{pmatrix} 2 & -1 \\ -4 & -1 \end{pmatrix}$?

(a) $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$

(b) $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$

(c) $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$

(d) $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$

(e) None of the above

(f) More than one of the above

▲

Problem 6.5. Suppose the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has an eigenvalue 1 with associated eigenvector $x = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$. What is $A^{50}x$?

(a) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

(b) $\begin{pmatrix} a^{50} & b^{50} \\ c^{50} & d^{50} \end{pmatrix}$

(c) $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

(d) $\begin{pmatrix} 2^{50} \\ 3^{50} \end{pmatrix}$

(e) Way too hard to compute.

▲

Problem 6.6. Vector $\mathbf{x}$ is an eigenvector of matrix A. If $\mathbf{x} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $Ax = \begin{pmatrix} 4 \\ 12 \end{pmatrix}$, then what is the associated eigenvalue?

(a) 1

(b) 3

(c) 4

(d) Not enough information is given.

▲

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Technique 6.7 (Finding Eigenvalues and Eigenvectors)**.** To find the eigenvalues of an $n \times n$ matrix A:

- Rearrange the equation $A\mathbf{x} = \lambda\mathbf{x}$ so that the right-hand side is the zero vector:

$$\underline{\hspace{3cm}} = \mathbf{0}$$

- Eigenvectors are <u>never</u> the zero vector so what does the previous equation imply about the matrix $A - \lambda I$?
- Form the characteristic polynomial: $p(\lambda) = \det(A - \lambda I)$ and solve for λ using algebra
- Find the associated eigenspace:

$$E_\lambda = \{\mathbf{x} : (A - \lambda I)\mathbf{x} = \mathbf{0}\}$$

Problem 6.8. Find the eigen-pairs for the matrix

$$A = \begin{pmatrix} 5 & 7 \\ -2 & -4 \end{pmatrix}$$

▲

Problem 6.9. The matrix $A = \begin{pmatrix} 3 & 2 \\ 4 & 10 \end{pmatrix}$ has eigenvalue $\lambda_1 = 2$. Find the associated eigenspace E_{λ_1}. In other words, find the space spanned by the associated eigenvector(s). ▲

Problem 6.10. For any integer n, what will this product be? $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}^n \begin{pmatrix} 1 \\ 5 \end{pmatrix}$

(a) $-1(3)^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 3(-2)^n \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(b) $3(-1)^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-2)3^n \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(c) $3(3)^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-2)(-1)^n \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(d) $3(3)^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-1)(-2)^n \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(e) None of the above

(f) More than one of the above

▲

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Problem 6.11. $\begin{pmatrix} 4/3 \\ 1 \end{pmatrix}$ is an eigenvector of $\begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}$. What is the associated eigenvalue? (Think! Don't solve for all the eigenvalues and eigenvectors.)

(a) 4/3

(b) 5

(c) -2

▲

Problem 6.12. If a vector x is in the eigenspace of A corresponding to λ, and $\lambda \neq 0$, then x is

(a) in the nullspace of the matrix A.

(b) in the nullspace of the matrix $A - \lambda I$.

(c) not the zero vector.

(d) More than one of the above correctly completes the sentence.

▲

Problem 6.13. Explain why an $n \times n$ matrix can have at most n distinct eigenvalues. ▲

Example 6.14. Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 3 & 2 \\ 4 & 10 \end{pmatrix}$.

Solution: First we'll find the roots of the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$:

$$\begin{aligned} p(\lambda) &= \det\begin{pmatrix} 3-\lambda & 2 \\ 4 & 10-\lambda \end{pmatrix} = (3-\lambda)(10-\lambda) - 8 = 30 - 13\lambda + \lambda^2 - 8 \\ &= \lambda^2 - 13\lambda + 22 = (\lambda - 2)(\lambda - 11). \end{aligned}$$

Therefore we have the eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 11$.

Next we find the associated eigenvectors.

- Eigenvector for $\lambda_1 = 2$:
  Since $\lambda_1 = 2$ is an eigenvalue of A we need to solve the homogeneous system $(A - 2I)\mathbf{v} = \mathbf{0}$ for $\mathbf{v}$. Observe that $A - 2I = \begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}$ and we can row reduce to $(A - 2I) \to \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$. Therefore $\mathbf{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

- Eigenvector for $\lambda_2 = 11$:
  Since $\lambda_2 = 11$ is an eigenvalue of A we need to solve the homogeneous system

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$(A-11I)\mathbf{v} = \mathbf{0}$ for $\mathbf{v}$. Observe that $A - 11I = \begin{pmatrix} -8 & 2 \\ 4 & -1 \end{pmatrix}$ and we can row reduce to $(A-2I) \to \begin{pmatrix} 1 & -1/4 \\ 0 & 0 \end{pmatrix}$. Therefore $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

Example 6.15. The matrix $A = \begin{pmatrix} 3 & -1 & 1 \\ -2 & 5 & 1 \\ 2 & -3 & 4 \end{pmatrix}$ has an eigenvector $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$. What is the associated eigenvalue?

Solution: By the definition of the eigenvalue-eigenvector pair we know that $A\mathbf{v} = \lambda \mathbf{v}$ so observe that

$$A\mathbf{v} = \begin{pmatrix} 3 & -1 & 1 \\ -2 & 5 & 1 \\ 2 & -3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 10 \end{pmatrix} = 5\mathbf{v}$$

Therefore we see that the eigenvalue associated with $\mathbf{v}$ is $\lambda = 5$.

Finally, we can extend the idea of the eigenvalue-eigenvector problem to other linear operators. The mathematical field interested in the eigenvalues of linear operators is called *spectral theory* and the name is chosen because of the deep ties between an operator's eigenvalue structure and light spectra.

There are a few eigenfunctions that we already know so let's just hint at the idea by looking at a few differential equations.

Problem 6.16. Consider the linear operator $T(y) = y'$. If we consider the eigenvalue problem $T(y) = \lambda y$ that corresponds to the differential equation $y' = \lambda y$.

(a) What are the eigenfunctions of the linear operator T?

(b) What is the meaning of the eigenvalues in this case?

▲

Problem 6.17. Consider the linear operator $T(y) = y''$. If we consider the eigenvalue problem $T(y) = \lambda y$ that corresponds to the differential equation $y'' = -\lambda y$.

(a) What are the eigenfunctions of the linear operator T?

(b) What is the meaning of the eigenvalues in this case?

▲

Problem 6.18. Recall that the linear transformation on $\mathbb{R}^2$ defined as

$$T(\mathbf{x}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{x}$$

is a rotation by 90° counterclockwise about the origin. Find the eigenvalues and eigenvectors for the transformation matrix. The answers shouldn't surprise you. why? ▲

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Problem 6.19. Find the eigenvalues for the general rotation matrix

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Why are the eigenvalues both 1? What are the eigenvectors? ▲

6.2 Diagonlization of Matrices

Problem 6.20. Consider the matrix A and the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Form the matrices $P = \begin{pmatrix} | & | \\ \mathbf{v}_1 & \mathbf{v}_2 \\ | & | \end{pmatrix}$ and $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ then compute the products AP and PD. What do you observe?

$$A = \begin{pmatrix} 5 & -6 \\ 2 & -2 \end{pmatrix} \text{ with eigenvectors } \mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ and } \mathbf{v}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \text{ and eigenvalues } \lambda_1 = 2 \text{ and } \lambda_2 = 1$$

▲

Problem 6.21. Repeat the previous problem with these vectors

$$A = \begin{pmatrix} 5 & -3 \\ 2 & 0 \end{pmatrix} \text{ with eigenvectors } \mathbf{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \text{ and } \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ and eigenvalues } \lambda_1 = 3 \text{ and } \lambda_2 = 2$$

▲

The previous two problems should now lead you to the following theorem. If you find that you cannot fill in the blanks then go back to the two problems and look for patterns.

Theorem 6.22 (Diagonalization of Matrices)**.** If A is $n \times n$, A has n linearly independent eigenvectors, and we form the matrix $P = [\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_n]$ then $AP =$ ___ ___

Furthermore, if you solve the previous equation for A then you get

$$A = \_\_\_\ \_\_\_\ \_\_\_$$

(Fill in the blanks)

Another way to state the previous theorem is as follows:

Theorem 6.23. Let A be an $n \times n$ square matrix. Then the following two conditions are equivalent:

- There is a basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ for $\mathbb{R}^n$ consisting of eigenvector for A.
- It is possible to find an invertible matrix P so that $A =$ ___ ___ ___, where D is a diagonal matrix whose entries are the eigenvalues of A.

Problem 6.24. Let $A = \begin{pmatrix} 5 & 1 \\ 0 & 3 \end{pmatrix}$. Find a basis for $\mathbb{R}^2$ that consists of eigenvectors of A. ▲

Problem 6.25. The matrix A has eigenvalues $\lambda_1 = -3$ and $\lambda_2 = 5$ with associated eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$. Find A. ▲

Problem 6.26. What are the eigenvalues of $D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$?

(a) 2 and 3

(b) 0 and 2

(c) 0 and 3

(d) 5 and 6

▲

Problem 6.27. Why might we be interested in diagonalizing a matrix?

(a) Because it is easy to find the eigenvalues of a diagonal matrix.

(b) Because it is easy to compute powers of a diagonal matrix.

(c) Both of these reasons.

▲

Now that we have the tools of diagonalization we can prove the following simple theorem. The real power in this theorem comes in determining if a matrix is invertible given its eigen-structure.

Theorem 6.28. Let A be an $n \times n$ square matrix with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$. Then

$$\det(A) = \lambda_1 \cdot \lambda_2 \cdots \lambda_n.$$

That is, the determinant of the matrix is the same as the product of the eigenvalues.

Proof. (Prove the previous theorem) □

An immediate corollary to the previous theorem is the following.

Corollary 6.29. Let A be an $n \times n$ square matrix. If A has at least one zero eigenvalue then A is not invertible.

Proof. (Prove the previous theorem) □

Problem 6.30. What does it mean if 0 is an eigenvalue of a matrix A?

(a) The determinant of A is zero.

(b) The columns of A are linearly dependent.

(c) There are an infinite number of solutions to the system $Ax = 0$.

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(d) All of the above

(e) None of the above

▲

Problem 6.31. Show that $A = \begin{pmatrix} 1 & -3 \\ -2 & 6 \end{pmatrix}$ is not invertible as many ways as possible. ▲

Theorem 6.32. If λ is an eigenvalue of an invertible matrix A then $1/\lambda$ is an eigenvalue of A^{-1}.

Proof. (Prove the previous theorem. Hint: start with $A\mathbf{v} = \lambda\mathbf{v}$) □

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6.3 Powers of Matrices

We will see in a few chapters that powers of matrices appear often in difference and differential equations. For that reason it is very handy to be able to quickly compute powers of matrices and our new-found technique of diagonalization is the right tool for the job! Let's begin with the primary theorem for this section.

Theorem 6.33. Let A be an $n \times n$ matrix with distinct eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ and associated eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. If we diagonalize A as $A = PDP^{-1}$ we know that

$$P = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & \cdots \\ 0 & 0 & \lambda_3 & \cdots \\ \vdots & & & \ddots \end{pmatrix}$$

Using this diagonalization we see that for some integer power n

$$A^n = PD^nP^{-1}$$

Proof. (Prove the previous theorem) □

Theorem 6.34. If D is a diagonal matrix

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & \cdots \\ 0 & 0 & \lambda_3 & \cdots \\ \vdots & & & \ddots \end{pmatrix}$$

then D^n has the form

$$D = \begin{pmatrix} \lambda_1^n & 0 & 0 & \cdots \\ 0 & \lambda_2^n & 0 & \cdots \\ 0 & 0 & \lambda_3^n & \cdots \\ \vdots & & & \ddots \end{pmatrix}$$

Proof. (Prove the previous theorem) □

Problem 6.35. If $D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, what is D^5?

(a) $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

(b) $\begin{pmatrix} 10 & 0 \\ 0 & 15 \end{pmatrix}$

(c) $\begin{pmatrix} 2^5 & 0 \\ 0 & 3^5 \end{pmatrix}$

(d) Too hard to compute by hand.

▲

Problem 6.36. Consider the matrix

$$A = \begin{pmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{pmatrix}$$

with eigenspaces:

$$\lambda_1 = 2 \text{ with } E_{\lambda_1} = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \right\}$$

$$\lambda_2 = 3 \text{ with } E_{\lambda_2} = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Find A^{100} without technology. ▲

Problem 6.37. The matrix $A = \begin{pmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{pmatrix}$ has eigen-pairs

$$\lambda_1 = 0.5 \text{ with } \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \lambda_2 = 1 \text{ with } \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(a) If $\mathbf{x} = 3\mathbf{v}_1 + 7\mathbf{v}_2$ what is $A\mathbf{x}$? (you should not need to build $\mathbf{x}$ directly)

(b) Write an expression for $A^k\mathbf{x}$ in terms of $\mathbf{v}_1$ and $\mathbf{v}_2$.

(c) Evaluate the limit $\lim_{k\to\infty} A^k\mathbf{x}$.

▲

Theorem 6.38 (Solving a System of Difference Equations)**.** Assume that you have a linear system of difference equations $\mathbf{x}_{n+1} = A\mathbf{x}_n$ with initial condition $\mathbf{x}_0$. If the matrix A has eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ and eigenvalues $\lambda_1, \ldots, \lambda_k$ then the analytic solution to the system of difference equations is

$$\mathbf{x}_n = c_1\lambda_1^n\mathbf{v}_1 + c_2\lambda_2^n\mathbf{v}_2 + \cdots c_n\lambda_k^n\mathbf{v}_k = \sum_{j=1}^{k} c_j\lambda_j^n\mathbf{v}_j$$

which is equivalent to

$$\mathbf{x}_n = PD^nP^{-1}\mathbf{x}_0$$

Proof. (Prove the previous theorem) □

6.4 The Google Page Rank Algorithm

In this section you will discover how the PageRank algorithm works to give the most relevant information as the top hit on a Google search.

Search engines compile large indexes of the dynamic information on the Internet so they are easily searched. This means that when you do a Google search, you are not actually searching the Internet; instead, you are searching the indexes at Google.

When you type a query into Google the following two steps take place:

1. Query Module: The query module at Google converts your natural language into a language that the search system can understand and consults the various indexes at Google in order to answer the query. This is done to find the list of relevant pages.

2. Ranking Module: The ranking module takes the set of relevant pages and ranks them. The outcome of the ranking is an ordered list of web pages such that the pages near the top of the list are most likely to be what you desire from your search. This ranking is the same as assigning a *popularity score* to each web site and then listing the relevant sites by this score.

This section focuses on the Linear Algebra behind the Ranking Module developed by the founders of Google: Sergey Brin and Larry Page. Their algorithm is called the *PageRank algorithm*. The PageRank algorithm was the *state of the art* in internet search up until about 2010. More modern methods build upon this method but also use more advanced statistical techniques and machine learning.

In simple terms: *A webpage is important if it is pointed to by other important pages.*

The Internet can be viewed as a directed graph (look up this term here on Wikipedia) where the nodes are the web pages and the edges are the hyperlinks between the pages. The hyperlinks into a page are called *inlinks*, and the ones pointing out of a page are called *outlinks*. In essence, a hyperlink from my page to yours is my endorsement of your page. Thus, a page with more recommendations must be more important than a page with a few links. However, the status of the recommendation is also important.

Let us now translate this into mathematics. To help understand this we first consider the small web of six pages shown in Figure 6.1 (a graph of the router level of the internet can be found here). The links between the pages are shown by arrows. An arrow pointing into a node is an *inlink* and an arrow pointing out of a node is an *outlink*. In Figure 6.1, node 3 has three outlinks (to nodes 1, 2, and 5) and 1 inlink (from node 1).

We will first define some notation in the PageRank algorithm:

- $|P_i|$ is the number of outlinks from page P_i

- H is the *hyperlink* matrix defined as

$$H_{ij} = \begin{cases} \frac{1}{|P_j|}, & \text{if there is a link from node } j \text{ to node } i \\ 0, & \text{otherwise} \end{cases}$$

  where the "i" and "j" are the row and column indices respectively.

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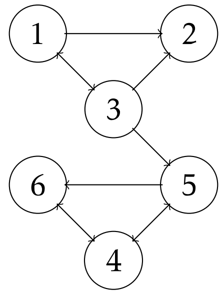

Figure 6.1. Sample graph of a web with six pages.

- $\mathbf{x}$ is a vector that contains all of the PageRanks for the individual pages.

The PageRank algorithm works as follows:

1. Initialize the page ranks to all be equal. This means that our initial assumption is that all pages are of equal rank. In the case of Figure 6.1 we would take $\mathbf{x}_0$ to be

$$\mathbf{x}_0 = \begin{pmatrix} 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \end{pmatrix}.$$

2. Build the hyperlink matrix.
   As an example we'll consider node 3 in Figure 6.1. There are three outlinks from node 3 (to nodes 1, 2, and 5). Hence $H_{13} = 1/3$, $H_{23} = 1/3$, and $H_{53} = 1/3$ and the partially complete hyperlink matrix is

$$H = \begin{pmatrix} - & - & 1/3 & - & - & - \\ - & - & 1/3 & - & - & - \\ - & - & 0 & - & - & - \\ - & - & 0 & - & - & - \\ - & - & 1/3 & - & - & - \\ - & - & 0 & - & - & - \end{pmatrix}$$

3. The difference equation $\mathbf{x}_{n+1} = H\mathbf{x}_n$ is used to iteratively refine the estimates of the page ranks. You can view the iterations as a person visiting a page and then following a link at random, then following a random link on the next page, and the next, and the next, etc. Hence we see that the iterations evolve exactly as expected for a difference equation.

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Iteration	New Page Rank Estimation
0	$\mathbf{x}_0$
1	$\mathbf{x}_1 = H\mathbf{x}_0$
2	$\mathbf{x}_2 = H\mathbf{x}_1 = H^2\mathbf{x}_0$
3	$\mathbf{x}_3 = H\mathbf{x}_2 = H^3\mathbf{x}_0$
4	$\mathbf{x}_4 = H\mathbf{x}_3 = H^4\mathbf{x}_0$
$\vdots$	$\vdots$
k	$\mathbf{x}_k = H^k\mathbf{x}_0$

4. When a steady state is reached we sort the resulting vector $\mathbf{x}_k$ to give the page rank. The node (web page) with the highest rank will be the top search result, the second highest rank will be the second search result, and so on.

It doesn't take much to see that this process can be very time consuming. Think about your typical web search with hundreds of thousands of hits; that makes a square matrix H that has a size of hundreds of thousands of entries by hundreds of thousands of entries! The matrix multiplications alone would take many minutes (or possibly many hours) for every search! . . . but Brin and Page were pretty smart dudes!!

We now state a few theorems and definitions that will help us simplify the iterative PageRank process.

Theorem 6.39. If A is an $n \times n$ matrix with n linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n$ and associated eigenvalues $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ then for any initial vector $\mathbf{x} \in \mathbb{R}^n$ we can write $A^k\mathbf{x}$ as

$$A^k\mathbf{x} = c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2 + c_3\lambda_3^k\mathbf{v}_3 + \cdots c_n\lambda_n^k\mathbf{v}_n$$

where $c_1, c_2, c_3, \dots, c_n$ are the constants found by expressing $\mathbf{x}$ as a linear combination of the eigenvectors.
Note: We can assume that the eigenvalues are ordered such that $\lambda_1 \ge \lambda_2 \ge \lambda_3 \ge \cdots \ge \lambda_n$.

Proof. (Prove the preceding theorem) □

Definition 6.40. A **probability vector** is a vector with entries on the interval $[0, 1]$ that add up to 1.

Definition 6.41. A **stochastic matrix** is a square matrix whose columns are probability vectors.

Theorem 6.42. If A is a stochastic $n \times n$ matrix then A will have n linearly independent eigenvectors. Furthermore, the largest eigenvalue of a stochastic matrix will always be $\lambda_1 = 1$ and the smallest eigenvalue will always be nonnegative: $0 \le \lambda_n < 1$.

Some of the following tasks will ask you to *prove* a statement or a theorem. This means to clearly write all of the logical and mathematical reasons why the statement is true. Your proof should be absolutely crystal clear to anyone with a similar mathematical background ... if you are in doubt then have a peer from a different group read your proof to you out loud.

Problem 6.43. Finish writing the hyperlink matrix H from Figure 6.1. ▲

Problem 6.44. Write MATLAB code to implement the iterative process defined previously. Make a plot that shows how the rank evolves over the iterations. ▲

Problem 6.45. What must be true about a collection of n pages such that an $n \times n$ hyperlink matrix H is a stochastic matrix. ▲

The statement of the next theorem is incomplete, but the proof is given to you. Fill in the blank in the statement of the theorem and provide a few sentences supporting your answer.

Theorem 6.46. If A is an $n \times n$ stochastic matrix and $\mathbf{x}_0$ is some initial vector for the difference equation $\mathbf{x}_{n+1} = A\mathbf{x}_n$, then the steady state vector is

$$\mathbf{x}_{equilib} = \lim_{k\to\infty} A^k \mathbf{x}_0 = \underline{\hspace{3cm}}.$$

Proof. First note that A is an $n \times n$ stochastic matrix so from Theorem 6.42 we know that there are n linearly independent eigenvectors. We can then substitute the eigenvalues from Theorem 6.42 in Theorem 6.39. Noting that if $0 < \lambda_j < 1$ we have $\lim_{k\to\infty} \lambda_j^k = 0$ the result follows immediately. □

Problem 6.47. Discuss how Theorem 6.46 greatly simplifies the PageRank iterative process described previously. In other words: there is no reason to iterate at all. Instead, just find ____________. ▲

Problem 6.48.

Now use the previous two problems to find the resulting PageRank vector from the web in Figure 6.1? Be sure to rank the pages in order of importance. Compare your answer to the one that you got in problem 2. ▲

Problem 6.49. Consider the web in Figure 6.2.

(a) Write the H matrix and find the initial state $\mathbf{x}_0$,

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(b) Find steady state PageRank vector using the two different methods described: one using the iterative difference equation and the other using Theorem 6.46 and the dominant eigenvector.

(c) Rank the pages in order of importance.

▲

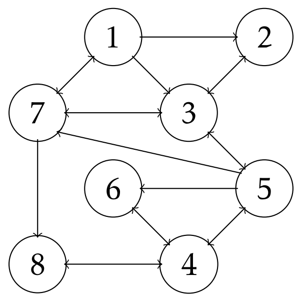

Figure 6.2. Graph of a web with eight pages.

Problem 6.50. One thing that we didn't consider in this version of the Google Page Rank algorithm is the random behavior of humans. One, admittedly slightly naive, modification that we can make to the present algorithm is to assume that the person surfing the web will randomly jump to any other page in the web at any time. For example, if someone is on page 1 in Figure 6.2 then they could randomly jump to any page 2 - 8. They also have links to pages 2, 3, and 7. That is a total of 10 possible next steps for the web surfer. There is a 2/10 chance of heading to page 2. One of those is following the link from page 1 to page 2 and the other is a random jump to page 2 without following the link. Similarly, there is a 2/10 chance of heading to page 3, 2/10 chance of heading to page 7, and a 1/10 chance of randomly heading to any other page.

Implement this new algorithm, called the *random surfer algorithm*, on the web in Figure 6.2. Compare your ranking to the non-random surfer results from the previous problem. ▲

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6.5 Additional Exercises

Problem 6.51. Let $\mathbf{v}_1 = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ be eigenvectors of a matrix A corresponding to eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 5$ respectively. Compute $A(\mathbf{v}_1 + \mathbf{v}_2)$ and $A(3\mathbf{v}_1)$. ▲

Problem 6.52. Long ago, in a galaxy far, far away ... there are two cell-phone companies serving a town: the Evil Empire and the Rebel Alliance. The Evil Empire has terrible service, so each week 25% of their customers switch to the Rebel Alliance and 2% give up their cell phone service entirely. The Rebel Alliance loses only 5% of their customers to the Evil Empire every week due to the advertising. If there are currently 100 customers in the Evil Empire and 75 customers in the Rebel Alliance, what is the long-term enrollment in the two plans?

Write a system of difference equations and use the ideas of eigenvalues and eigenvectors to discuss the long-term behavior of the system. ▲

Problem 6.53. Find a 2×2 matrix such that $\mathbf{v}_1 = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are eigenvectors of A with eigenvalues 5 and -1 respectively. ▲

Problem 6.54. The Cayley-Hamilton Theorem states that a matrix will satisfy its own characteristic polynomial. Recall that for a matrix A the characteristic polynomial is $p(\lambda) = \det(A - \lambda I)$ and to find the eigenvalues λ we solve the equation $p(\lambda) = 0$. The Cayley-Hamilton Theorem states that $p(A) = 0$ where the "0" here means that zero matrix.

(a) Let $A = \begin{pmatrix} -4 & 4 \\ -4 & -3 \end{pmatrix}$. Use the Cayley-Hamilton Theorem to find a quadratic polynomial p such that $p(A) = 0$.

(b) For a matrix A assume that $(A - 2I)(A + 3I) = 0$. There are infinitely many matrices A for which this algebraic equation is true. What characteristics do all of the matrices have?

▲

Theorem 6.55. Let A be an $n \times n$ matrix. For every positive integer k, A^k will have the same eigenvectors as A.

Problem 6.56. (a) Prove Theorem 6.55.

(b) If $\mathbf{v}$ is an eigenvector of the matrix A with eigenvalue λ then what is the eigenvalue of $\mathbf{v}$ associated with A^k for some positive integer k?

▲

Theorem 6.57. Let A be an $n \times n$ matrix. For every real number c the matrices $(A + cI)$ and cA will have the same eigenvectors as A.

Problem 6.58. (a) Prove Theorem 6.57.

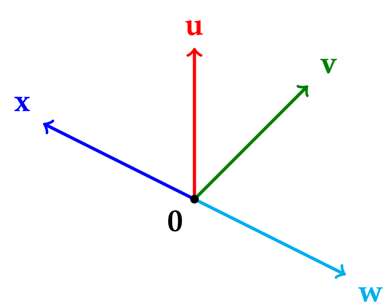
(b) If $\mathbf{v}$ is an eigenvector of the matrix A with eigenvalue λ then what are the eigenvalues of $\mathbf{v}$ associated with $(A + cI)$ and cA for some integer c?

▲

Problem 6.59. Is $\lambda = 2$ an eigenvalue of $\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$? Why or why not? ▲

Problem 6.60. Suppose that the vector $\mathbf{x}$ is a real-valued eigenvector of the matrix M and that the entries of M are also real valued.

(a) Considering the vectors plotted below, what could be the result of the product $M\mathbf{x}$? Circle all that apply.



(i) $M\mathbf{x}$ could be $\mathbf{u}$.

(ii) $M\mathbf{x}$ could be $\mathbf{v}$.

(iii) $M\mathbf{x}$ could be $\mathbf{w}$.

(iv) $M\mathbf{x}$ could be $\mathbf{0}$.

(v) $M\mathbf{x}$ could be $\mathbf{x}$.

(vi) None of the above

(b) Explain your reasoning for your choice(s) in part (a).

▲

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Chapter 7

Second Order Differential Equations

In this brief chapter we will transition back to our discussion of differential equations. In Chapter 1 we discussed first order linear and nonlinear differential equations and in this chapter we will discuss second order linear differential equations. Second order differential equations arise very naturally from Newton's second law, $ma = \sum F$, since acceleration is the second derivative of position. Second order differential equations also arise naturally in circuit analysis and many other physics-based contexts. Our primary focus here will be on mechanical vibrations since that is likely a familar physics context for most students.

7.1 Intro to Second Order Differential Equations

Problem 7.1. In Figure 7.1 you see three mass spring systems. Assume that all three masses have the same density. You give each of the masses a bump upward at time $t = 0$. Make a plot of the resulting motion for each of the mass-spring systems (time on the horizontal axis and displacement on the vertical axis). ▲

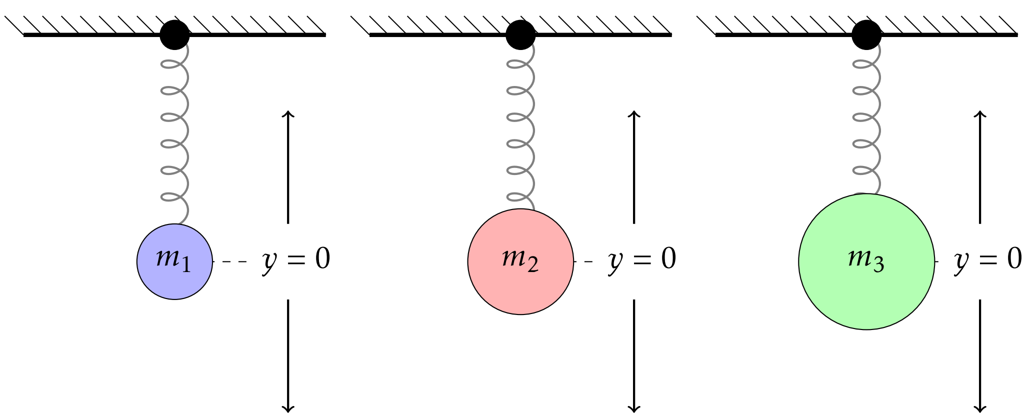

Figure 7.1. Three mass and spring oscillating systems.

Problem 7.2. On a mass-spring oscillator what are the primary forces controlling the motion. Use Newton's second law to summarize them.

$$ma = \sum F = \underline{\hspace{4cm}}$$

▲

Problem 7.3. In the previous problem you likely have two primary forces. A restoring force due to the spring and a damping force due to friction. Propose functional forms for each of these forces.

$$F_{restoring} = \underline{\hspace{4cm}}$$
$$F_{damping} = \underline{\hspace{4cm}}$$

▲

Problem 7.4. The differential equation modeling a mass spring oscillator is:

$$my'' + \underline{\hspace{1cm}} + \underline{\hspace{1cm}} = F_{external}$$

where $F_{external}$ is any external driving force (like wind, periodic bumps to the spring-mass apparatus, magnetic fields, etc).
(Fill in the blanks) ▲

Problem 7.5. Compare the values of the parameters in the model from the previous problem using the mass-spring oscillators depicted in Figure 7.1. ▲

Problem 7.6. The motion of an un-damped mass spring oscillator is driven physically by Hooke's law: the restoring force is proportional to the displacement. This leads to the differential equation $my'' = -ky$.

(a) Explain the minus sign in the context of the problem.

(b) We know that the solution to the differential equation is a trigonometric function, for example $y(t) = A\sin(\omega t)$. What is the frequency, ω, of this oscillator in terms of Hooke's law coefficient k and the mass m?

▲

Problem 7.7. The motion of a mass on a spring follows the equation $mx'' = -kx$ where the displacement of the mass is given by $x(t)$. Which of the following would result in the highest frequency motion?

(a) $k = 6$, $m = 2$

(b) $k = 4$, $m = 4$

(c) $k = 2$, $m = 6$

(d) $k = 8$, $m = 6$

(e) All frequencies are equal

▲

Problem 7.8. What function solves the equation $y'' + 10y = 0$?

(a) $y = 10\sin 10t$

(b) $y = 60\cos\sqrt{10}t$

(c) $y = \sqrt{10}e^{-10t}$

(d) $y = 20e^{\sqrt{10}t}$

(e) More than one of the above

▲

Problem 7.9. In a damped spring-mass system we know that the solution cannot simply be exponential. Why? Conjecture an algebraic form of the solution. ▲

Now that we have the basic intuition about mass-spring oscillators let's build up the mathematical machinery necessary to actually solve them.

Technique 7.10 (Solving Second Order Linear ODEs)**.** To solve $ay'' + by' + cy = 0$:

- Assume that $y(t) = e^{rt}$
- Write the associated characteristic polynomial and use it to find r
- Write the solution as a linear combination of the linearly independent eigenfunctions. Use the initial conditions to find the constants in the linear combination.
- Remember to use Euler's Formula if r happens to be complex:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

It is worth it here to take a side step and discuss Euler's formula in more detail. To understand the roots of Euler's formula we first need to recall the definition of a Taylor series.

Definition 7.11 (Taylor Series)**.** Let $f(x)$ be an infinitely differentiable function at a real number $x = a$. The **Taylor Series** of $f(x)$ at $x = a$ is

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \cdots$$

The number "a" is called the center of the Taylor series and if $a = 0$ then the series is

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sometimes called a MacLaurin series.

The Taylor series is a useful tool for approximating functions with simple polynomials. In fact, every time you use the sine, cosine, and logarithm buttons on your calculator you are actually just evaluating their Taylor series approximations; your calculator has no idea what the sine function really is.

Example 7.12. Find the Taylor series expansions for the functions e^x, $\sin(x)$, and $\cos(x)$ centered at $a = 0$.
Solution: We leave it to the reader to take all of the requisite derivatives to verify the following.

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\ \sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ \cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \end{aligned}$$

Now we have all of the tools necessary to verify Euler's formula.

Problem 7.13. We would like to verify Euler's formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

(a) In the Taylor series for e^x replace the x with $i\theta$.

(b) Recall that since $i = \sqrt{-1}$ we have

$$i^1 = -1, \quad i^3 = -i, \quad i^4 = 1$$

and successive powers of i repeat this pattern: $i, -1, -i, 1, i, -1, -i, 1, \ldots$. Simplify each of the powers in your answer to part (a).

(c) Rearrange your answer in part (c) to gather the real terms together and the imaginary terms together.

(d) How does your answer to part (c) verify that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$?

▲

In Technique 7.10 the last step cryptically says to "remember to use Euler's formula is r happens to be complex." Now let's clarify that. Let's say that we have a second order linear differential equation $ay'' + by' + cy = 0$ and the roots of the characteristic polynomial are $r_1 = 2 + 3i$ and $r_2 = 2 - 3i$. This means that the general solution to the differential equation is

$$y(t) = C_1 e^{(2+3i)t} + C_2 e^{(2-3i)t}.$$

Using the algebraic rules of exponents we can observe that $e^{(2+3i)t} = e^{2t}e^{3it}$ and $e^{(2-3i)t} = e^{2t}e^{-3it}$. Therefore

$$y(t) = C_1 e^{2t} e^{3it} + C_2 e^{2t} e^{-3it}$$

and factoring e^{2t} gives

$$y(t) = e^{2t}\left(C_1 e^{3it} + C_2 e^{-3it}\right).$$

Using Euler's formula we can now expand the complex exponentials to get

$$y(t) = e^{2t}\left(C_1\cos(3t) + C_1 i\sin(3t) + C_2\cos(-3t) + C_2 i\sin(-3t)\right).$$

We can next recall some helpful trigonometric identities:

$$\cos(-\theta) = \cos(\theta) \quad \text{and} \quad \sin(-\theta) = -\sin(\theta)$$

(coming from the fact that cosine is an even function and sine is an odd function). Therefore,

$$y(t) = e^{2t}\left(C_1\cos(3t) + C_1 i\sin(3t) + C_2\cos(3t) - C_2 i\sin(3t)\right)$$

and gathering like terms gives

$$y(t) = e^{2t}\left((C_1 + C_2)\cos(3t) + (C_1 i - C_2 i)\sin(3t)\right).$$

Since i is a constant we can just relabel the coefficients as new constants and arrive at a much more convenient form of the solution:

$$y(t) = e^{2t}\left(C_3\cos(3t) + C_4\sin(3t)\right).$$

All of this discussion leads us to the following theorem.

Theorem 7.14. For $a, b, c \in \mathbb{R}$ if $y(t)$ is a twice differentiable function such that $ay'' + by' + cy = 0$ and if the characteristic polynomial $ar^2 + br + c = 0$ has complex roots

$$r_1 = \alpha + i\omega \text{ and } r_2 = \alpha - i\omega$$

then the general solution to the second-order linear differential equation is

$$y(t) = e^{\alpha t}\left(C_1\cos(\omega t) + C_2\sin(\omega t)\right)$$

where the constants C_1 and C_2 are expected to be real and are determined by the initial conditions.

To conclude this introductory section on second order differential equations it is now your turn to find the solutions to the following four problems. You will need to use Euler's formula for some of them.

Problem 7.15. Solve the differential equation $y'' - 4y' + 3y = 0$ with $y(0) = 7$ and $y'(0) = 11$ ▲

Problem 7.16. Solve the differential equation $y'' + 25y = 0$ with $y(0) = 2$ and $y'(0) = 15$ ▲

Problem 7.17. Solve the differential equation $y'' - 6y' + 9y = 0$ with $y(0) = 2$ and $y'(0) = 1$ ▲

Problem 7.18. Solve the differential equation $y'' - 4y' + 5y = 0$ with $y(0) = 2$ and $y'(0) = 3$ ▲

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7.2 Types of Mechanical Vibrations

One of the principle applications of second order differential equations is to model mass-spring oscillators. First we are going change our vantage point so that only one of the bodies is oscillating. We achieve this by finding the *reduced mass* $m = \frac{m_1 m_2}{m_1+m_2}$ and then fixing our point of reference so that one of the bodies is fixed. Next, Newton's second law states that we should sum the forces on the oscillating spring. The second-order linear differential equations resulting from this analysis is

$$my'' + by' + ky = F_{external}$$

where my'' is the "mass times acceleration" term of Newton's second law, the by' is a damping term driven by drag, and the ky term is Hooke's law driving the resoration of toward the equilibrium.

Problem 7.19. If we guess that $y(t) = e^{rt}$ in the previous equation (with $F_{ext} = 0$) then the resulting characteristic polynomial is $p(r) =$ ____________. Solve for r and classify the types of roots that might occur in terms of the damping coefficient and the spring constant. The three classifications are "under damped", "over damped", and "critically damped". Which situation is which? ▲

Theorem 7.20. For the homogeneous mass spring oscillator equation

$$my'' + by' + ky = 0$$

with $m, k, b > 0$ there are four primary solution types.

Un-Damped ($b = 0$):

$$y(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t)$$

where $\omega = \sqrt{\frac{k}{m}}$ is called the natural frequency of the oscillator.

Under Damped (two complex roots):

$$y(t) = e^{\alpha t}\left(C_1 \cos(\omega t) + C_2 \sin(\omega t)\right)$$

where $r = \alpha \pm i\omega$

Over Damped (two real roots):

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

Critically Damped (one repeated real root):

$$y(t) = C_1 e^{rt} + C_2 t e^{rt}$$

Proof. (Prove the previous theorem) □

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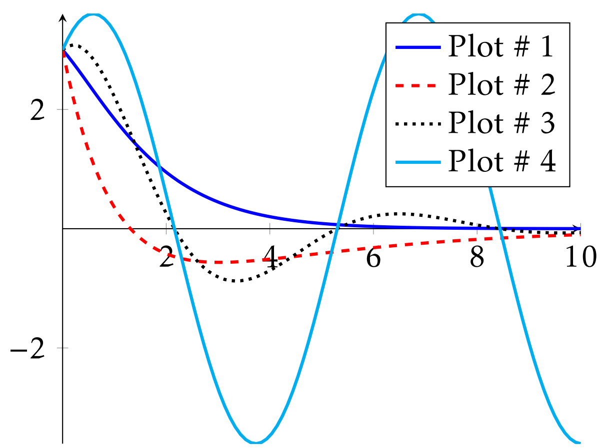
Problem 7.21. Give a linear algebra based reason for the algebraic form of the solution to the Critically Damped oscillator. ▲

Problem 7.22. Classify each of the second order linear differential equations as either under damped, over damped, or critically damped. After you classify each differential equation write the general form of the solution.

$$\begin{aligned} y'' + y' + y &= 0 \\ y'' + 2y' + y &= 0 \\ 4y'' + 5y' + y &= 0 \end{aligned}$$

▲

Problem 7.23. Below you will find four plots of solutions to second order oscillators. Which of them would you classify as un-damped, which as under damped, which as over damped, and which as critically damped?



▲

Problem 7.24. A steel ball weighing 128 pounds is suspended from a spring. This stretches the spring $\frac{128}{37}$ feet. The ball is started in motion from the equilibrium position with a downward velocity of 5 feet per second (assume that the equilibrium position is $y = 0$). The air resistance (in pounds) of the moving ball numerically equals 4 times its velocity (in feet per second). Suppose that after t second the ball is y feet below its rest position. Find y in terms of t.
Note: The positive direction for y is down and we can take as the gravitational acceleration 32 feet per second per second. ▲

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7.3 Undetermined Coefficients

In the previous section we were dealing with un-driven oscillators. That is, the oscillators had no external forces driving the oscillations aside from the initial conditions. In this section we'll consider what happens when you have an external driving force. From Newton's second law we can write the governing equation as

$$my'' + by' + ky = F_{external} \tag{7.1}$$

where $F_{external}$ in (7.1) is a driving force beyond the initial conditions, the spring constant, and the damping force.

Problem 7.25. Propose two different physical instances where an external driving force would influence the motion of an oscillator. ▲

Problem 7.26. What is the equilibrium of the differential equation $4y'' + 5y' + y = 1$? ▲

Problem 7.27. Work with your partner(s) to suggest a solution technique for the non-homogenous linear second order differential equation $4y'' + 5y' + y = 1$. ▲

Technique 7.28. To solve $my'' + by' + ky = f(t)$:

1. Solve the homogeneous problem (taking $f(t) = 0$)
2. Find a particular solution to the non-homogeneous problem (same functional form as $f(t)$)
3. IF the homogeneous and particular solutions are linearly independent then $y(t)$ is a linear combination of the homogeneous solutions and the particular solution

Now let's put your undetermined coefficients skills to the test by solving the following three un-damped driven oscillators.

Problem 7.29. Solve $y'' + 4y = 2e^{3t}$ with $y(0) = 0$ and $y'(0) = 1$ ▲

Problem 7.30. Solve $y'' + 4y = 5\sin(3t)$ with $y(0) = 0$ and $y'(0) = 1$ ▲

Problem 7.31. Solve $y'' + 4y = 5\sin(2t)$ with $y(0) = 0$ and $y'(0) = 1$ ▲

7.4 Resonance and Beats

A particular type of forcing term occurs when there is a forcing term that matches the natural frequency of the ocscillor. In an un-damped oscillator this looks like:

$$y'' + \omega^2 y = A\cos(\omega t).$$

Notice that the natural frequency of the homogeneous differential equation is the same as the forcing term.

Problem 7.32. What is the homogeneous solution to the above equation?

$$y_{hom} = \underline{\hspace{6cm}}$$

and since the particular solution has the same frequency they are not linearly independent of those of the homogeneous solutions. Propose a particular solution:

$$y_{particular} = \underline{\hspace{6cm}}.$$

The term that you proposed is the root cause of *resonance*. ▲

Problem 7.33. In the differential equation $y'' + \omega^2 y = A\cos(\omega t)$ the amplitudes of the waves will grow in time. What function do the amplitudes follow? ▲

Problem 7.34. Solve the differential equation $y'' + 144y = 4\cos(12t)$ with $y(0) = 0$ and $y'(0) = 0$ ▲

Example 7.35. Resonance was responsible for the collapse of the Broughton suspension bridge near Manchester, England in 1831. The collapse occurred when a column of soldiers marched in cadence over the bridge, setting up a periodic force of rather large amplitude. The frequency of the force was approximately equal to the natural frequency of the bridge. Thus, the bridge collapsed when large oscillations occurred. For this reason soldiers are ordered to break cadence whenever they cross a bridge.

The Millennium Bridge, the first new bridge to span the Thames River in London in over 100 years, is a modern example of how resonance can effect a bridge. This pedestrian bridge, which opened to the public in June 2000, was quickly closed after the bridge experienced high amplitude horizontal oscillations during periods of high traffic. Studies by designers found that the bridge experienced high amplitude horizontal oscillations in response to horizontal forcing at a rate of one cycle per second. Typically, people walk at a rate of two steps per second, so the time between two successive steps of the left foot is about one second. Thus, if people were to walk in cadence, they would could set up strong horizontal forcing that would place a destructive load on the bridge. The engineers did not envision this to be a problem since tourists do not generally march in time. However, a video of tourists crossing the bridge revealed the opposite. When the bridge began oscillating, people tended to walk in time in order to keep their balance.
https://www.youtube.com/watch?v=gQK21572oSU

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Now we consider what happens when the natural frequency of the homogeneous oscillator is close but not exactly equal to that of the driving term. First watch the video in the example below.

Example 7.36. For an example of the *beats* phenomenon see www.youtube.com/watch?v=pRpN9uLiouI. This phenomenon occurs when the natural frequency and the forcing frequency differ by only a small amount.

$$y'' + \omega_0^2 y = A\cos(\omega t)$$

where ω_0 and ω are very close but not the same. You will be playing with this in the lab.

Problem 7.37. Go to the GeoGebra applet: https://www.geogebra.org/m/T9yws7CB and explore what happens to the sum of the two functions $f(x) = \sin(Ax)$ and $g(x) = \sin(Bx)$ when A and B are arbitrarily close to each other. ▲

Problem 7.38. Use what you know about the previous problem to sketch the graphical solution to the differential equation $y'' + 4y = \sin(1.9t)$ with $y(0) = y'(0) = 0$. ▲

Problem 7.39. Now solve the differential equation $y''+4y = \sin(1.9t)$ with $y(0) = y'(0) = 0$. ▲

To end this chapter we consider one more problem related to all of the differential equations that you should know at this point.

Problem 7.40. Match the differential equations below to the solution plots further below. There should be no need to actually solve the differential equations, but I won't stop you if that is what you really want to do. (there are 9 plots and 8 differential equations. One plot does not have a match ...just to keep you on your toes)

Differential Equation	Initial Conditions	Matches to Plot #
$y' = 0.2y(5-y)$	$y(0) = 1$	
$y'' + y' + 25y = 0$	$y(0) = 0$ and $y'(0) = 5$	
$y'' + 25y = 4\sin(5.5t)$	$y(0) = 0$ and $y'(0) = 5$	
$y' = -2y + 10$	$y(0) = 1$	
$y'' + 25y = 4\sin(5t)$	$y(0) = 0$ and $y'(0) = 5$	
$y'' + 25y = 0$	$y(0) = 0$ and $y'(0) = 5$	
$y' - 0.2y = 0$	$y(0) = 1$	
$y' + 0.2t = 0$	$y(0) = 1$	

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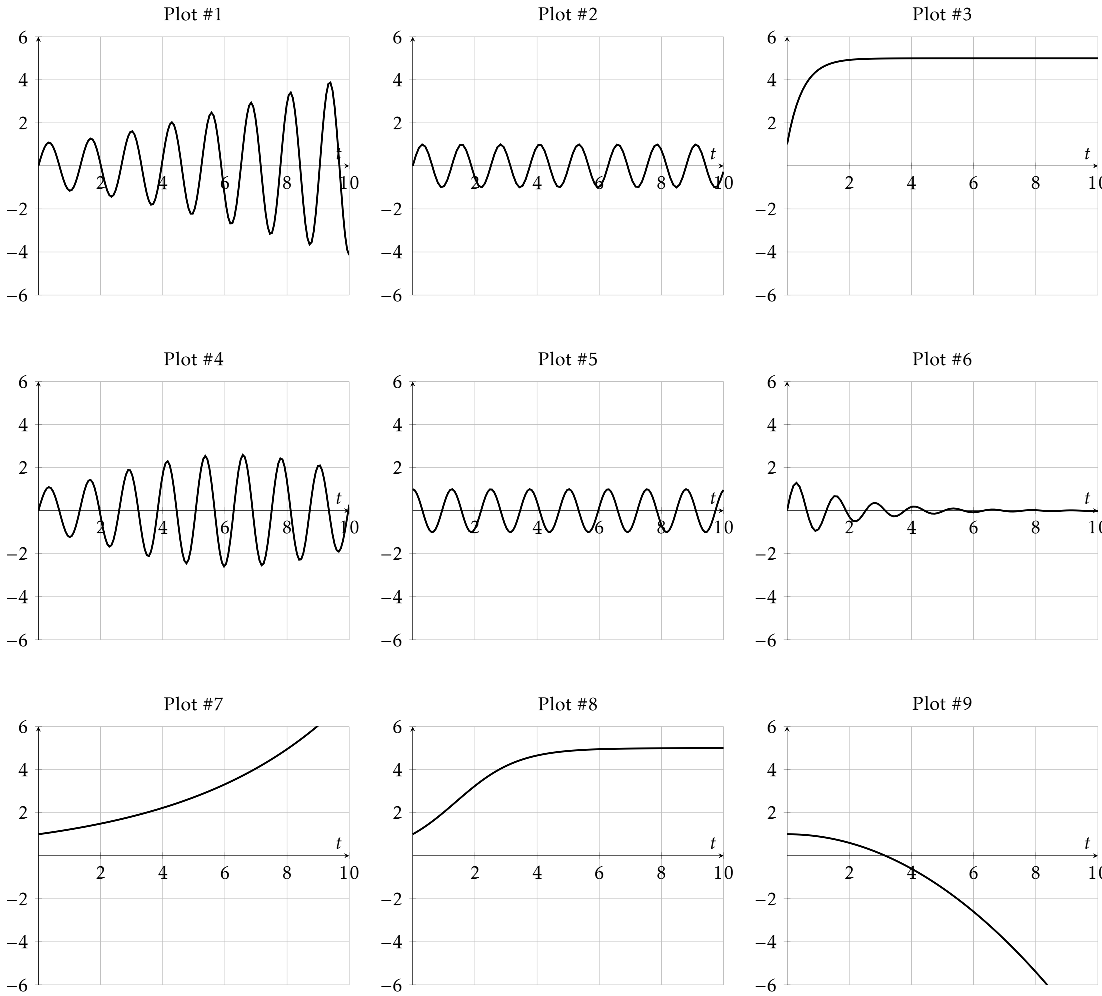
Plot #1
Plot #2
Plot #3
Plot #4
Plot #5
Plot #6
Plot #7
Plot #8
Plot #9
t
6
4
2
−2
−4
−6
2
4
6
8
10

▲

7.5 Additional Exercises

Problem 7.41.

Consider a floating cylindrical buoy with radius r, height h, and uniform density $\rho \leq 0.5$ grams/cm^3. The buoy is initially suspended at rest with its bottom at the top surface of the water and is released at $t = 0$. Thereafter it is acted on by two forces:

1. a downward gravitational force equal to its weight: $mg = \left(\pi r^2 h \rho\right) g$, and
2. an upward force due to buoyancy equal to the weight of the displaced water: $\left(\pi r^2 x\right) g$

where x is the depth of the bottom of the buoy beneath the surface at time t. (recall that the density of water is 1 gram/cm^3)

(a) Write a differential equation modeling the depth of the buoy.

(b) What type of motion does the buoy undergo?

(c) What is the equilibrium of the oscillation?

(d) What is the period of the oscillation?

▲

Problem 7.42. The previous problem does not account for the water's viscosity. We can reframe the differential equation as

$$mx'' = \text{weight} - \text{viscosity} - \text{buoyancy}$$

and assume that the retarding force due to viscosity is proportional to the velocity of the buoy.

(a) Write the resulting differential equation.

(b) If b represents the viscosity then what is the period of oscillation?

▲

Problem 7.43. A branch sways back and forth with position $f(t)$. Studying its motion you find that its acceleration is proportional to its position, so that when it is 8 cm to the right, it will accelerate to the left at a rate of 2 cm/s^2. Write a differential equation that describes the motion of the branch? ▲

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Chapter 8

Linear and Nonlinear Systems of ODEs

In this chapter we will consider qualitative and quantitative techniques for solving and understanding systems of differential equations. Systems arise naturally in many settings, but let's set the stage with a motivating example. In this example we will built two differential equations that rely on each other.

Example 8.1. A stream carries water at a rate of 50 gallons per minute first into the Upper Lake, then flows at this rate out of the Upper Lake and into the Lower Lake, then flows at this rate out of the Lower Lake. The Upper Lake is initially filled with 25,000 gallons of pure water, and the Lower Lake is initially filled with 50,000 gallons of pure water. A chemical pollutant suddenly appears into the stream flowing into the Upper Lake with a concentration of 2.5 milligrams per gallon. Let $U(t)$ be the amount of the pollutant in the upper lake, and $L(t)$ be the amount of the pollutant in the lower lake. Create a system of differential equations with initial conditions to model this scenario.
Solution:
In each of the lakes we build a differential equation that models the conservation of mass of the pollutant. The equations are all written in the form

$$\text{Rate of change of mass of pollutant} = \text{rate in} - \text{rate out}.$$

By carefully tracking the units in the problem we arrive at the differential equations

$$\begin{aligned}
\frac{dU}{dt} &= \left(2.5\frac{mg}{gal}\right)\left(50\frac{gal}{min}\right) - \left(\frac{U}{25000}\frac{mg}{gal}\right)\left(50\frac{gal}{min}\right) \\
\frac{dL}{dt} &= \left(\frac{U}{25000}\frac{mg}{gal}\right)\left(50\frac{gal}{min}\right) - \left(\frac{L}{50000}\frac{mg}{gal}\right)\left(50\frac{gal}{min}\right)
\end{aligned}$$

with initial conditions $U(0) = 0$ milligrams and $L(0) = 0$ milligrams.

Notice in this example that the equation for the upper lake only involves the variable U, but the equation for the lower lake depends on both U and L. In other words, to be able to solve for both U and L we need to look at the interplay between the two lakes.

In this case you may notice that we could just solve the U equation on its own (using separation of variables perhaps) and then substitute the answer into the L equation and solve what results (using undetermined coefficients or integrating factors perhaps). As we'll see, we won't be able to do this in every system.

In this chapter we will build, analyze, solve, and explore similar systems of differential equations. Let's start with a familiar type of ODE: a second order mass-spring oscillator.

8.1 From Second Order ODE's to Systems

Problem 8.2. Consider the differential equation

$$x'' + 3x' + 2x = 0.$$

(a) What is the characteristic polynomial for this differential equation?

(b) What are the roots of the characteristic polynomial?

(c) Based on the roots of the characteristic polynomial, how would you classify this differential equation? (undamped, underdamped, overdamped, or critically damped)

(d) What is the basis for the solution space of this differential equation?

(e) What is the general form of the solution to the differential equation?

▲

Problem 8.3. Let's consider the same differential equation

$$x'' + 3x' + 2x = 0,$$

but this time we're going to do a little clever substitution to transform the problem into a different form.

(a) Let $y = x'$. Taking the derivative of both sides of this substitution gives $y' =$ ____________.

(b) What is the physical meaning of y in the context of this problem? (recall that x is the displacement in a mass-spring oscillator)

(c) What is the physical meaning of y' in the context of this problem? (again recall that x is the displacement in a mass-spring oscillator)

(d) Rewrite the original differential equation using the substitutions from part (a): $y = x'$ and $y' = x''$.

(e) You should now have two differential equations. The primary advantage you should find here is that both of the differential equations are first order (why is this an advantage?). You should also notice that both differential equations are linear.

$$x' = \_\_\_\_\_\_\_\_\_\_\_\_$$
$$y' = \_\_\_\_\_\_\_\_\_\_\_\_$$

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(f) Now let's leverage the language of linear algebra to rewrite this system of equations.

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \_\_\_\_\_\_ & \_\_\_\_\_\_ \\ \_\_\_\_\_\_ & \_\_\_\_\_\_ \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(g) Find the characteristic polynomial for the matrix in part (f). What do you notice? What do you wonder?

(h) What are the eigenvalues of the matrix in part (f)? What do you notice?

▲

Problem 8.4. Consider the second order differential equation

$$x'' + bx' + cx = 0$$

By substituting $y = x'$ we can get a system of differential equations. What is the system, write it as a matrix equation, and what is the meaning of y in this system? ▲

Theorem 8.5. If we transform the second order differential equation $x'' + bx' + cx = 0$ into the first order matrix equation $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -c & -b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ by making the substitution $y = x'$ then the roots of the characteristic polynomial $p(r) = r^2 + br + c$ are the same as the eigenvalues of the matrix in the first-order system.

Proof. (Prove this theorem) □

Problem 8.6. If

$$\begin{cases} x' &= y \\ y' &= -5x - 2y \end{cases}$$

then what was the mass-spring differential equation associated with the system? Based on the eigenvalues of the associated matrix, is the mass-spring system undamped, underdamped, overdamped, or critically damped? Investigate the system using a tool like PPlane. Investigate the system using a tool like PPlane. ▲

Problem 8.7. In the previous problem you likely arrived at the second order differential equation $x'' + 2x' + 5x = 0$. The discriminant on this differential equation is $b^2 - 4mk = 4 - 4(1)(5) = -16$ so we know that the system is underdamped.

(a) What are the roots of the characteristic polynomial.

(b) Using the system in the previous problem write the associated matrix equation and find the eigenvalues of the matrix. What do you notice?

(c) Make a conjecture about the roots of the characteristic polynomial and the eigenvalues of the associated matrix.

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▲

Problem 8.8. Consider the second order differential equation modeling a spring-mass system:

$$x'' + 2x' + 2x = 0.$$

(a) Find the discriminant, the roots of the characteristic polynomial, classify the system, and discuss the expected behavior.

(b) Write the differential equation as a first order system and find the associated matrix equation.

(c) Find the eigenstructure of A and discuss what this means about the behavior of the system.

▲

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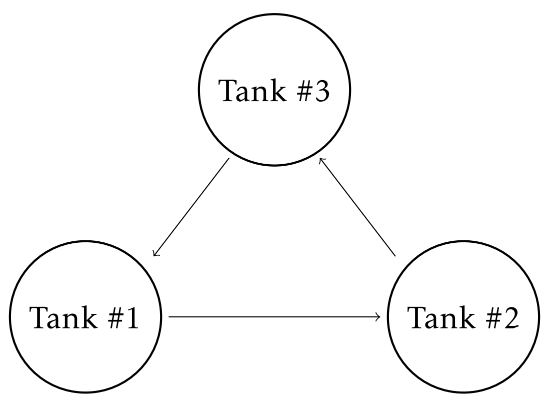
8.2 Matrices and Linear Systems

Problem 8.9. Consider a two-tank system where brine is transferred between them according to the following rules.

- 20 L/min of fresh water enters Tank #1
- Tank #1 holds $x(t)$ kg of salt and holds a total of 100 liters of total mixture.
- Tank #2 holds $y(t)$ kg of salt and holds a total of 200 liters of total mixture.
- Mixture runs from Tank #1 to Tank #2 at a rate of 30 L/min.
- Mixture runs from Tank #2 to Tank #1 at a rate of 10 L/min.
- Tank #2 drains mixture at a rate of 20 L/min

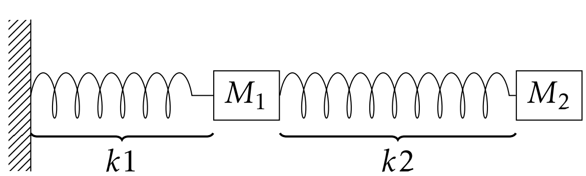
Write a system of differential equations modeling the transfer of brine in this two-tank system. What interesting questions can you ask about this system? Finally, write the system as a matrix equation. You may want to investigate the system using software like `pplane`. ▲

Problem 8.10. Three 100 gallon fermentation tanks are connected as shown below, and the mixtures in each tank are kept uniform by stirring. Denote $x_j(t)$ as the amount of alcohol in tank T_j at time t. Suppose that the mixture circulates between the tanks at the rate of 10 gal/min. What system of differential equations governs this closed system? Write the system as a matrix equation. ▲



Problem 8.11.

A coupled spring mass system is shown below. Let $x_1(t)$ denote the displacement of mass 1 from its equilibrium and let $x_2(t)$ denote the displacement of mass 2 from its equilibrium. Assuming no damping forces complete the system of differential equations below.



$$m_1 x_1'' = -\_\_\_\_\_\_ x_1 + k_2 \_\_\_\_\_\_$$
$$m_2 x_2'' = -\_\_\_\_\_\_ (x_2 - x_1)$$

Once you have the system, write it as a matrix equation. ▲

Problem 8.12. The previous problem ended in a 2×2 second order system. Make an appropriate substitution and arrive at a 4×4 first order system. ▲

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8.3 The Eigenvalue Method for Linear Systems

Now let's interweave the idea of eigenvalues in with systems of differential equations. As we already know, the eigenvalues and eigenvectors of a matrix A tell us the underlying structure of the matrix. In some sense they are the DNA of the matrix. In the cases that we investigate here the eigen-structure of A will tell us about the solution curves of the linear first order differential equation.

Problem 8.13. Let's return to the second order differential equation $x''+3x'+2x=0$. Here is what we learned in the last section

- we can rewrite this equation as the matrix equation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

- the characteristic polynomial of the coefficient matrix is $p(\lambda)=\lambda^2+3\lambda+2$ with roots $\lambda_1=-1$ and $\lambda_2=-2$.
- the eigenvalues are the same as the roots to the characteristic polynomial $p(r)=r^2+3r+2$ found by guessing that $x(t)=e^{rt}$ is a solution to the differential equation

Now let's take this a bit further:
The eigen-pairs of the coefficient matrix are

$$\lambda_1 = -1 \text{ with } \mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -2 \text{ with } \mathbf{v}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

(a) Verify that the the vector $e^{-1t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is a solution to the matrix form of the differential equation by substituting it into both sides.

(b) Verify that the the vector $e^{-2t}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ is a solution to the matrix form of the differential equation by substituting it into both sides.

(c) What do you notice from the answers to parts (a) and (b)?

(d) Since this is a linear differential equation and we have two linearly independent solution vectors, we can write the general solution to this differential equation as

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{\_\_t} \begin{pmatrix} \_\_ \\ \_\_ \end{pmatrix} + C_2 e^{\_\_t} \begin{pmatrix} \_\_ \\ \_\_ \end{pmatrix}$$

▲

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Theorem 8.14. Let λ be an eigenvalue of the matrix A for the first order linear system

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}.$$

If $\mathbf{v}$ is an eigenvector associated with eigenvalue λ then

$$\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$$

is a nontrivial solution of the system.

Proof. Let A be a real square matrix and let $\mathbf{v}$ and λ be an eigen-pair for A. To check that $\mathbf{x} = \mathbf{v}e^{\lambda t}$ is a solution to the differential equation $\mathbf{x}' = A\mathbf{x}$ we substitute $\mathbf{x}$ in on both sides and check.

On the left-hand side of the differential equation we get

$$\frac{d\mathbf{x}}{dt} = \frac{d}{dt}\left(\mathbf{v}e^{\lambda t}\right) = \mathbf{v}\frac{d}{dt}\left(e^{\lambda t}\right) = \mathbf{v}\left(\lambda e^{\lambda t}\right) = \lambda e^{\lambda t}\mathbf{v}.$$

On the right-hand side of the differential equation we get

$$A\mathbf{x} = A\left(\mathbf{v}e^{\lambda t}\right) = e^{\lambda t}A\mathbf{v} = e^{\lambda t}(\lambda\mathbf{v}) = \lambda e^{\lambda t}\mathbf{v} \quad \checkmark.$$

□

Theorem 8.15. If $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are unique solutions to the first order linear system of equations $\mathbf{x}' = A\mathbf{x}$ then a linear combination

$$\mathbf{v} = \sum_{j=1}^{k} c_j \mathbf{v}_j$$

is also a solution of $\mathbf{x}' = A\mathbf{x}$.

Proof. Since the differential equation is linear we know that a linear combination of solutions is also a solution. □

Theorem 8.16. If the matrix A has eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ and associated eigenvectors $\lambda_1, \lambda_2, \dots, \lambda_n$ then the general solution to the differential equation $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}(t) = \underline{\hspace{6cm}}$$

Proof. (Fill in the blank above and prove the theorem) □

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Problem 8.17. Solve the following linear system of differential equations.

$$x_1' = 4x_1 - x_2$$
$$x_2' = 2x_1 + x_2$$

with initial conditions $x_1(0) = 1$ and $x_2(0) = 3$. Complete the problem by giving a plots x_1 vs t, x_2 vs t, and x_2 vs x_1. Use technology to find the eigen-structure of the resulting matrix and to create the plots. ▲

Problem 8.18. Since we know that both $x_1 = x_2 = e^{3t}$ and $x_1 = e^{-t}, x_2 = -e^{-t}$ are solutions to the system

$$x_1' = x_1 + 2x_2$$
$$x_2' = 2x_1 + x_2$$

Which of the following are also solutions?

(a)

$$x_1 = 3e^{3t} - e^{-t}$$
$$x_2 = 3e^{3t} + e^{-t}$$

(b)

$$x_1 = -e^{3t} - e^{-t}$$
$$x_2 = -e^{3t} + e^{-t}$$

(c)

$$x_1 = 2e^{3t} + 4e^{-t}$$
$$x_2 = -4e^{-t} + 2e^{3t}$$

(d)

$$x_1 = 0$$
$$x_2 = 0$$

(e) None of the above

(f) All of the above.

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▲

Problem 8.19. Consider the system of differential equations,

$$y'(t) = \begin{pmatrix} 14 & 0 & -4 \\ 2 & 13 & -8 \\ -3 & 0 & 25 \end{pmatrix} y(t)$$

.

Which of the following functions solve this system?

(a) $y(t) = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} e^{-4t}$

(b) $y(t) = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} e^{6t}$

(c) $y(t) = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} e^{13t}$

(d) None of the above

(e) All of the above.

▲

Problem 8.20. Two forces are fighting one another. Let x and y be the number of soldiers in each force and let a and b be the offensive fighting capacities of x and y respectively. Assume that forces are lost only to combat, and no reinforcements are brought in.

(a) Write a system of differential equations that models this scenario. Write the system as a matrix equation.

(b) Solve the system using the eigenvalue method using sensible initial conditions and values for a and b.

(c) Determine values of a and b for which army x wins and for which army y wins.

▲

Problem 8.21. In Problem 8.20 we built a model that might be really good for hand-to-hand combat. Let's tweak this model.

(a) Modify the model from Problem 8.20 to allow each army to get a constant number of recruits each day (assume time is measured in days).

(b) Propose a solution technique for this model and implement it.

(c) Propose a way to find a steady state solution to your model (if it exists) and implement your idea.

▲

In a linear system where there is a constant non-homogeneity we need to modify our solution technique. Consider the system of differential equations

$$\mathbf{x}'(t) = A\mathbf{x} + \mathbf{b} \tag{8.1}$$

where $\mathbf{x}$ is a vector of functions, A is a real matrix, and $\mathbf{b}$ is a vector of constants. Problem 8.21 should result in a model of this type and in that problem you proposed a solution technique and a method for finding the steady-state solution. Now let's summarize these techniques.

Technique 8.22. Let $\mathbf{x}$ be an $n \times n$ vector of single variable functions, let A be an $n \times n$ real matrix, and let $\mathbf{b}$ be an $n \times n$ real vector. Consider the system of n differential equations

$$\mathbf{x}' = A\mathbf{x} + \mathbf{b}$$

with initial conditions $\mathbf{x}(0) = \mathbf{x}_0$. Observe that this is a non-homogeneous differential equation with a constant non-homogeneity. The general solution to the differential equation is

$$\mathbf{x}(t) = \underline{\hspace{6cm}}$$

where ... (complete the technique)

Problem 8.23. Consider the linear system of differential equations given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

with initial conditions $x(0) = 1$ and $y(0) = 0$. Solve the system of equations, generate a plot of the solution, and find the steady state (if it exists). ▲

Problem 8.24. Solve the second order differential equation $y'' + 4y' + 3y = 2$ by converting to a first order system. Find the equilibrium if it exists. ▲

Problem 8.25. Return to the "Upper and Lower Lake" Example 8.1 and solve the system of equations with the tools discussed in this section. ▲

Problem 8.26. Four ants are initially on the grid points $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$. All four ants follow the strange behavior where their current velocity vector is always proportional to the vector that points at the ant 90° to their right. For example, if the ant starting at $(0,0)$ is called "ant 1" and has position (x_1, y_1) then it's velocity vector is given as

$$\left(\frac{dx_1}{dt}, \frac{dy_1}{dt} \right) = k\left[(x_2, y_2) - (x_1, y_1) \right]$$

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where (x_2, y_2) is the current location of the ant that started at $(1,0)$ and k is the proportionality constant that dictates the speed of the ant. Therefore there are two differential equations for the first ant:

$$\frac{dx_1}{dt} = k(x_2 - x_1)$$
$$\frac{dy_1}{dt} = k(y_2 - y_1)$$

(a) Complete the system of differential equations by writing two equations for each additional ant.

(b) Write the system of 8 differential equations as a matrix equation.

(c) Before solving this system you should stop and sketch what the expected solution should look like. You should have four expected trajectories – one for each ant. Do you expect there to be a steady state? Do you expect there to be a point where the ants converge or do you expect the ants to diverge from each other?

(d) Write code to get an approximate solution using Euler's method.

(e) Find the eigenvectors and eigenvalues of the coefficient matrix and use them to write the full solution to the system of equations. (note that it is entirely possible that some of the eigenvectors and eigenvalues are complex ... what does this mean in this context?)

(f) Write code to build an animated solution to the system of equations.

(g) Now for the fun part. Extend your system from part (b) and your code from part (d) or part (f) to allow for n ants who's starting positions are uniformly distributed along the circumference of a unit circle. Animate your solution with several different values of n.

▲

8.4 The Matrix Exponential

Recall that if we are solving the first order linear homogeneous differential equation $x' = rx$ we know (from separation of variables) that the solution is $x(t) = x_0 e^{rt}$. This is one of the simplest ordinary differential equation that there is! In this chapter we have encountered linear systems of differential equations that have a very similar form:

$$\mathbf{x}' = A\mathbf{x} \tag{8.2}$$

where $\mathbf{x}$ this time is a vector of functions and A is a matrix of values. More clearly, if A is an $n \times n$ real matrix then a linear system of differential equations can be written as

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}.$$

It stands to reason that the solution to (8.2) should be the same, or at least have the same form, as the solution to the simple differential equation $x' = rx$. Hence we conjecture that the solution to (8.2) is

$$\mathbf{x} = e^{At}\mathbf{x}_0 \tag{8.3}$$

where $\mathbf{x}_0$ is the vector of initial conditions. ...but wait! What is e^{At}? That's right, we have a matrix in the exponent of an exponential function!

To understand the matrix exponential we first start by recalling the Taylor series of the exponential function

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots.$$

If instead we examine the function $f(x) = e^{ax}$ where $a \in \mathbb{R}$ then it is easy to see that

$$e^{ax} = 1 + ax + a^2\frac{x^2}{2} + a^3\frac{x^3}{3!} + a^4\frac{x^4}{4!} + \cdots.$$

If we use the Taylor series as the definition of the exponential function then a natural definition for the matrix exponential is as follows.

Definition 8.27 (Matrix Exponential)**.** Let A be a square matrix and t be a real variable. The matrix exponential e^{At} is defined as

$$e^{At} = I + At + A^2\frac{t^2}{2} + A^3\frac{t^3}{3!} + A^4\frac{t^4}{4!} + \cdots. \tag{8.4}$$

Recall that if A has a complete collection of eigenvalues and eigenvectors then we can find matrices P and D such that $A = PDP^{-1}$. Therefore the definition of the matrix exponential

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can be rewritten as

$$
\begin{aligned}
e^{At} &= I + PDP^{-1}t + PD^2P^{-1}\frac{t^2}{2} + PD^3P^{-1}\frac{t^3}{3!} + PD^4P^{-1}\frac{t^4}{4!} + \cdots \\
&= P\left(I + Dt + D^2\frac{t^2}{2} + D^3\frac{t^3}{3!} + D^4\frac{t^4}{4!} + \cdots\right)P^{-1} \qquad (8.5)
\end{aligned}
$$

Using either (8.4) or (8.5) we now have a way to find the analytic solution to a linear homogeneous system of differential equations of the form $\mathbf{x}' = A\mathbf{x}$.

Theorem 8.28. If $\mathbf{x}$ is a vector of functions and A is a real square matrix then the solution to the differential equation $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0.$$

The computation of the matrix exponential in Theorem 8.28 is potentially really obnoxious by hand, but thankfully there are built-in tools for performing the computation in the most scientific computing software. For example, in MATLAB you can use the command `expm(A)` to find the matrix exponential of the matrix A.

Example 8.29. Use the matrix exponential to solve the system of differential equations

$$\mathbf{x}'(t) = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}\mathbf{x}(t) \quad \text{with} \quad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

Solution: The following MATLAB code finds the matrix exponential

```
clear; clc;
syms t
A = [2 , 0 ; 1 , 2];
x0 = [2;-3];
x = expm(A*t) * x0   % this is the solution to the system of ODEs
```

which results in the solution

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} e^{2t} & 0 \\ te^{2t} & e^{2t} \end{pmatrix}\begin{pmatrix} 2 \\ -3 \end{pmatrix}.$$

pulling this out of matrix form we have

$$x_1(t) = 2e^{2t} \quad \text{and} \quad x_2(t) = 2te^{2t} - 3e^{2t}.$$

Finally we can plot the solution with the following code.

```
tmax = 1;
ezplot(x(1),[0,tmax])
```

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```
hold on
ezplot(x(2),[0,tmax])
axis([0,1,-10,10])
```

Hint: You can use the command `latex(simplify(x))` to take your symbolic answer for $\mathbf{x}(t)$ from MATLAB and get it into LaTeX.

The following two problems give you a chance to practice using the matrix exponential to solve applied systems of differential equations problems. Have fun!!

Problem 8.30. Tank #1 is connected to Tank #2 by two separate pipes. Pure water is flowing into Tank #1 at a rate of 2 gallons per minute. Tank #1 is initially filled with 50 gallons of water with 5 pounds of salt dissolved in it. Tank #2 contains 40 gallons of water with 1 pound of salt dissolved in it. Solution flows from Tank #1 to Tank #2 at a rate of 3 gallons per minute and from Tank #2 to Tank #1 at a rate of 1 gallon per minute. Thoroughly mixed solution is also being drained from Tank #2 at a rate of 2 gallons per minute. Let $x(t)$ be the amount of salt in Tank #1 and let $y(t)$ be the amount of salt in Tank #2 at time t.

(a) Write the linear system of differential equations that models this scenario. Be sure to include your initial conditions (it may help to draw a picture first).

(b) Solve the system of differential equations using the matrix exponential. Clearly write your answer in matrix and vector form.

(c) Solve the system again using the eigenvalue method. Clearly write your answer showing how the eigenvalues and eigenvectors play a role in the solution.

(d) Make a plot showing how the two concentrations evolve over 100 minutes.

▲

Problem 8.31. You are given a mechanical system with three springs A, B, and C and two objects F and G each of mass M. Springs A and C have spring constant k and spring B has spring constant $2k$. Spring A is attached to a wall on the left and spring C is attached to a wall on the right. Let $x_1(t)$ be the position of F from it's equilibrium ($x_1 = 0$ indicates that F is at equilibrium), and let $x_2(t)$ be the position of G from it's equilibrium. See Figure 8.1.

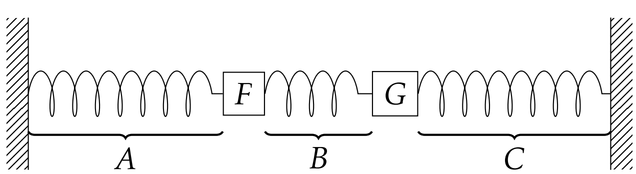

Figure 8.1. Double spring mass system

Initial conditions: Assume that we move mass F exactly 1 unit to the left and let it go. This means that $x_1(0) = -1$ and $x_3(0) = x_1'(0) = 0$. If we initially hold mass G fixed then $x_2(0) = x_4(0) = 0$.

(a) First we'll write the second order linear system of equations. This system of equations is just a statement of Hooke's law for springs: $F = -kx$, where k is the spring constant. To help you get started you can use the skeleton below:

$$\begin{aligned} Mx_1'' &= -kx_1 + (\text{some spring constant})(\text{distance between } x_2 \text{ and } x_1) \\ Mx_2'' &= -2k(x_2 - x_1) - (\text{some spring constant})x_2 \end{aligned}$$

(b) To turn this into a first-order system we introduce two new variables: $x_3 = x_1'$ and $x_4 = x_2'$. Write this system of differential equations ... I'll get you started ...

$$\begin{aligned} x_1' &= x_3 \\ x_2' &= x_4 \\ x_3' &= \ \ldots \\ x_4' &= \ \ldots \end{aligned}$$

Now write the system as a matrix equation of the form $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$.

(c) Solve the linear system of differential equations with the method of matrix exponentials.

(d) Next solve the linear system of differential equations with the eigenvalue method. You *should* find that you have 4 purely imaginary eigenvalues (I'll wait while you let MATLAB find these for you ...). This means that you will have oscillations in your solutions. Why?

(e) Plot your solutions.

▲

8.5 Complex Eigenvalues for Linear Systems of ODEs

In this brief section we consider the case where the eigenvalues of the coefficient matrix in a linear system are complex. If $\mathbf{x}' = A\mathbf{x}$ and A has complex eigenvalues $\lambda = \alpha \pm \beta i$ then the solution to the system is

$$\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2 = e^{\alpha t}\left(C_1 e^{\beta i t}\mathbf{v}_1 + C_2 e^{-\beta i t}\mathbf{v}_2\right).$$

If we use Euler's formula for the complex exponentials the solution becomes

$$\mathbf{x}(t) = e^{\alpha t}\left(C_1\left(\cos(\beta t) + i\sin(\beta t)\right)\mathbf{v}_1 + C_2\left(\cos(\beta t) - i\sin(\beta t)\right)\mathbf{v}_2\right).$$

Now if we gather the trigonometric functions the solution can be written as

$$\mathbf{x}(t) = e^{\alpha t}\left[(C_1\mathbf{v}_1 + C_2\mathbf{v}_2)\cos(\beta t) + (C_1 i\mathbf{v}_1 - C_2 i\mathbf{v}_2)\sin(\beta t)\right]. \tag{8.6}$$

Form the initial conditions we know that $\mathbf{x}(0) = C_1\mathbf{v}_1 + C_2\mathbf{v}_2$ and once we have calculated C_1 and C_2 it is straight forward to find the vectors multiplying the cosine and sine terms in the solution.

Example 8.32. Solve the system of differential equations

$$\mathbf{x}'(t) = \begin{pmatrix} -2 & -2 \\ 2 & -2 \end{pmatrix}\mathbf{x}(t)$$

with initial condition $\mathbf{x}(0) = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$.

Solution: We first need to find the eigenvalues and eigenvectors of the coefficient matrix, A.

$$\mathbf{v}_1 = \begin{pmatrix} -i \\ -1 \end{pmatrix} \quad \text{with} \quad \lambda_1 = -2 + 2i$$

$$\mathbf{v}_2 = \begin{pmatrix} i \\ -1 \end{pmatrix} \quad \text{with} \quad \lambda_2 = -2 - 2i$$

Solve the system of equation $\begin{pmatrix} \mathbf{v}_1 \, \mathbf{v}_2 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \mathbf{x}(0)$ to get $C_1 = \frac{3}{2} + \frac{3}{2}i$ and $C_2 = \frac{3}{2} - \frac{3}{2}i$ (the actual computation is left to the reader). Hence,

$$C_1\mathbf{v}_1 + C_2\mathbf{v}_2 = \left(\frac{3}{2} + \frac{3}{2}i\right)\begin{pmatrix} -i \\ -1 \end{pmatrix} + \left(\frac{3}{2} - \frac{3}{2}i\right)\begin{pmatrix} i \\ -1 \end{pmatrix} = \cdots = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$$

and

$$C_1 i\mathbf{v}_1 - C_2 i\mathbf{v}_2 = \left(\frac{3}{2} + \frac{3}{2}i\right)i\begin{pmatrix} -i \\ -1 \end{pmatrix} - \left(\frac{3}{2} - \frac{3}{2}i\right)i\begin{pmatrix} i \\ -1 \end{pmatrix} = \cdots = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

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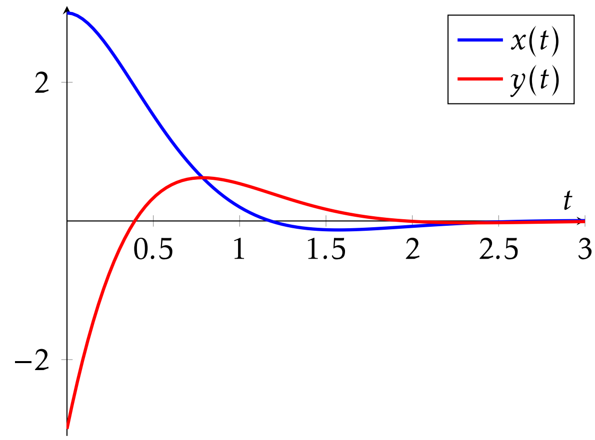

Figure 8.2. Solution curves for Example 8.32

Now we can substitute into (8.6) to get the complete solution

$$\mathbf{x}(t) = e^{-2t}\left(\cos(2t)\begin{pmatrix} 3 \\ -3 \end{pmatrix} + \sin(2t)\begin{pmatrix} -3 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} e^{-2t}\,(3\cos(2t) + 3\sin(2t)) \\ e^{-2t}\,(-3\cos(2t) + 3\sin(2t)) \end{pmatrix}.$$

The solutions to the system are shown in Figure 8.2

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8.6 Trace-Determinant Plane

In this section we will examine a technique for quickly determining the behavior of 2D linear systems. This will play a major role in our nonlinear systems since each nonlinear system will eventually be linearized as we will see in a bit.

Recall that if $\mathbf{v}_1$ and $\mathbf{v}_2$ are the eigenvectors of A with eigenvalues λ_1 and λ_2 then the solution to the first order linear system of equations $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x} = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$$

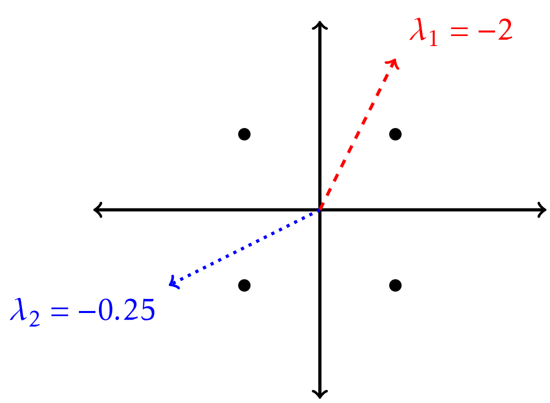
where $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. Furthermore we observe that $\mathbf{x} = \mathbf{0}$ is an equilibrium point of the system of differential equations. Use these ideas to answer the following questions.

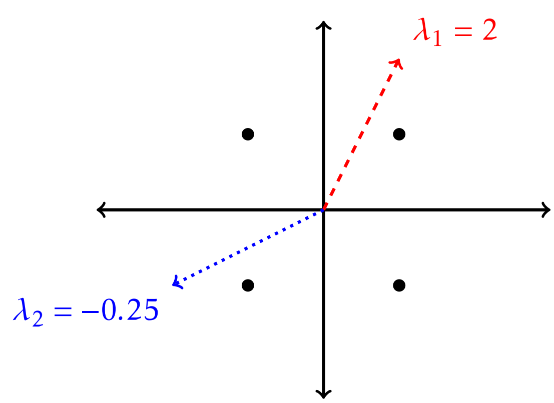
Problem 8.33. Consider the 2×2 linear system of differential equations $\mathbf{x}' = A\mathbf{x}$. In each of the following cases what is the expected behavior of the solution? The choices are: spirals in to the origin, spirals out from the origin, decays in to the origin, diverges out from the origin, decays toward the origin but then diverges out, or circles the origin.

1. If $\lambda_1 \le \lambda_2 < 0$ then near the origin the solution ____________
2. If $\lambda_1 < 0 < \lambda_2$ then near the origin the solution ____________
3. If $\lambda_1 \ge \lambda_2 > 0$ then near the origin the solution ____________
4. If $\lambda_1, \lambda_2 = \alpha \pm \beta i\, (\alpha < 0)$ then near the origin the solution ____________
5. If $\lambda_1, \lambda_2 = \alpha \pm \beta i\, (\alpha > 0)$ then near the origin the solution ____________
6. If $\lambda_1, \lambda_2 = \pm\beta i$ then near the origin the solution ____________

▲

Problem 8.34. In the following plots the two eigenvectors for a matrix A are plotted and the eigenvalues are given. Sketch the trajectory of the solution in the xy-plane starting at the given points (use your answers from the previous problem to help).





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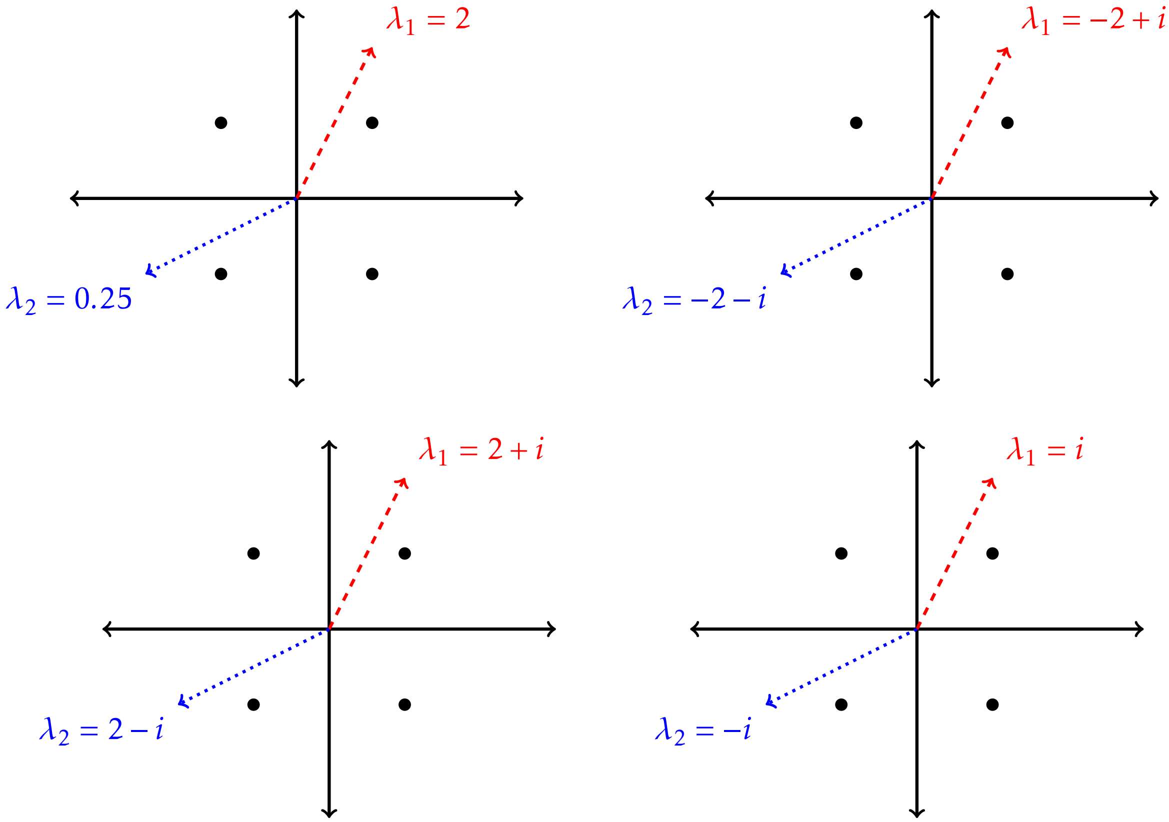

▲

Now that we know the general behavior of all 2×2 systems based on the eigenstructure let's get a faster way to make the determination. That is, if we can avoid actually finding the eigenvalues that might save some time.

Problem 8.35. Let the 2×2 linear system $\mathbf{x}' = A\mathbf{x}$ be written as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Your Tasks:

1. What is the equilibrium of this system presuming that A^{-1} exists?
2. Find the characteristic polynomial of the coefficient matrix
3. Simplify the characteristic polynomial to fill in the blanks:

$$\lambda^2 + \underline{\qquad\quad}\lambda + \underline{\qquad\quad} = 0$$

The blanks should be familiar features of the matrix.

▲

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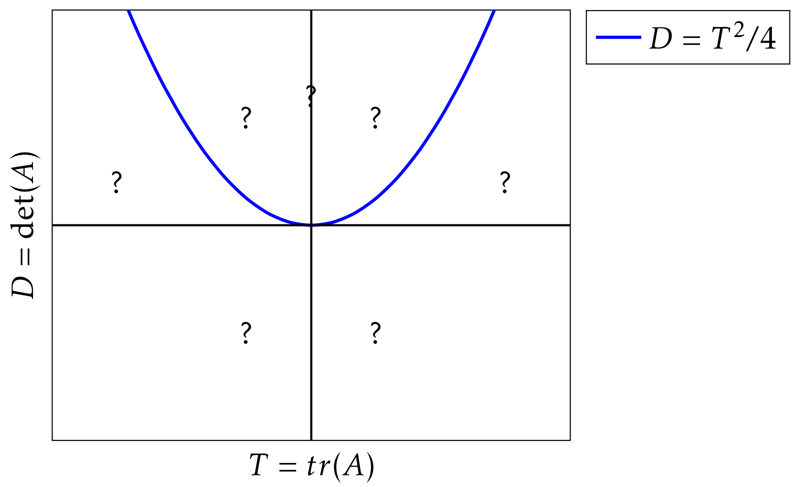
Problem 8.36. In the previous problems you found that the characteristic polynomial for a 2D linear system is $p(\lambda) = \lambda^2 - T\lambda + D$ where $T = \text{tr}(A)$ and $D = \det(A)$. Solving for λ we see that

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$

Clearly the behavior of the eigenvalues depends on the quantity $T^2 - 4D$. To visualize this we create the *trace-determinant* plane as seen in the figure immediately below. Fill in the question marks in the figure with the type of behavior you expect to see for each region of the *trace-determinant* plane?

To illustrate what we mean condsider the furthest right question mark. In that question mark the determinant, D, is greater than zero, the trace, T, is greater than zero and $D < T^2/4$. Hence $T^2 - 4D > 0$ so we expect the behavior of the system to be a nodal source since both eigenvalues will be real and positive.

Use the following words to fill in the question marks: nodal source, nodal sink, spiral source, spiral sink, center, and saddle. ▲



Problem 8.37. In a 2D linear system of differential equations $\mathbf{x}' = A\mathbf{x}$ we know that the origin is the only equilibrium if the matrix A is invertible (you should pause and prove this). For each of the following systems, describe the behaviour of the solution curves near the origin. Use the Trace-Determinant Plane to find your answers, and then justify your conclusions by actually finding the eigenvalues using the characteristic polynomial equation $\lambda^2 - T\lambda + D = 0$.

(a) $\mathbf{x}' = A\mathbf{x}$ with $\text{trace}(A) = 1$ and $\det(A) = 9$

(b) $\mathbf{x}' = A\mathbf{x}$ with $\text{trace}(A) = 9$ and $\det(A) = 1$

(c) $\mathbf{x}' = A\mathbf{x}$ with $\text{trace}(A) = -9$ and $\det(A) = 1$

(d) $\mathbf{x}' = A\mathbf{x}$ with $\text{trace}(A) = -1$ and $\det(A) = 9$

(e) $\mathbf{x}' = A\mathbf{x}$ with $\text{trace}(A) = -1$ and $\det(A) = -9$

(f) $\mathbf{x}' = A\mathbf{x}$ with $\text{trace}(A) = 1$ and $\det(A) = -9$

▲

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8.7 Building ODE Models for Nonlinear Systems

The world is non-linear! Well shoot. It might seem that this means that for *real* problems we can't use anything that we've done so far. Wrong! There are plenty of things that we can do with nonlinear problems. For the most part we will rely on a basic premise from Calculus: up close, a nonlinear function looks linear. You did this back in calculus when you found tangent lines and tangent planes but now we're going to do the same for matrices and nonlinear differential equations. For most of the problems in the remainder of this chapter it will be helpful to have MATLAB up so you can plot phase planes and analyze equilibria graphically.

Problem 8.38. Watch the video https://youtu.be/NSNWDUXN2p4 to see a simulation where a 150 person population has an outbreak and the virus is spread via close proximity contact. Notice, in particular, the homogeneous mixing.

(a) (Optional Challenge) Write computer code to produce a simulation similar to what you see in Problem

(b) There are three distinct populations in this problem: Susceptible (S), Infected (I), and Recovered (R). Write a system of differential equations for the experiment that we ran and remember to keep in mind that we were homogeneously mixing the population the entire time. Think very carefully about how a susceptible person is actually infected.

$$\frac{dS}{dt} = \underline{\hspace{5cm}} \quad \text{with} \quad S(0) = \underline{\hspace{2.5cm}}$$
$$\frac{dI}{dt} = \underline{\hspace{5cm}} \quad \text{with} \quad I(0) = \underline{\hspace{2.5cm}}$$
$$\frac{dR}{dt} = \underline{\hspace{5cm}} \quad \text{with} \quad R(0) = \underline{\hspace{2.5cm}}$$

▲

Problem 8.39. Consider the system from the previous problem.

(a) Is the system linear or nonlinear? Why?

(b) What is the expected long-term behavior of this system? Why?

▲

Problem 8.40. In the SIR model from the previous problems, one model that captures this behavior is as follows:

$$S' = -\alpha S I$$
$$I' = \alpha S I - \beta I$$
$$R' = \beta I$$

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where α is a parameter related to the likelihood that someone will get infected (subject to the homogeneous mixing) and β is the recovery rate. Let's take $\alpha = 0.1$ and $\beta = 0.4$ and create a numerical simulation for this problem. We'll use Euler's method to approximate the derivatives as follows:

$$\begin{aligned} S_{n+1} &= S_n + \Delta t\,(-\alpha S_n I_n) \\ I_{n+1} &= I_n + \Delta t\,(\alpha S_n I_n - \beta I_n) \\ R_{n+1} &= R_n + \Delta t\,(\beta I_n) \end{aligned}$$

Some partially complete MATLAB code to get you started is given below. Play with the values of α and β in your model. Be able to defend the meaning of these parameters in the context of the problem. Lastly, propose modifications to the model and test these modifications to see how the system behaves. ▲

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```
clear; clc; clf;
dt = 0.01;
tmax = 5;
alpha = 0.1;
beta = 0.4;
t = 0:dt:tmax;
S = zeros(size(t));
I = zeros(size(t));
R = zeros(size(t));
S(1) = 99;
I(1) = 1;
R(1) = 0;
for n=1:length(t)-1
  S(n+1) = S(n) + dt*( );
  I(n+1) = I(n) + dt*( );
  R(n+1) = R(n) + dt*( );
end
subplot(2,2,1)
plot(t,S,'bo',t,I,'r*',t,R,'kp'), grid on
xlabel('time'), ylabel('population')
subplot(2,2,2)
plot(S,I,'k*'), grid on
xlabel('Susceptible'), ylabel('Infected')
subplot(2,2,3)
plot(S,R,'k.'), grid on
xlabel('Susceptible'), ylabel('Recovered')
subplot(2,2,4)
plot(I,R,'kp'), grid on
xlabel('Infected'), ylabel('Recovered')
figure
plot3(S,I,R,'k.-')
box on
grid on
xlabel('Susceptible'), ylabel('Infected'), zlabel('Recovered')
```

Now let's look at another nonlinear model: the pendulum! This may seem simple, but the physics actually lends itself to a nonlinear differential equation in the case where the angle is potentially *large*. A really cool video by Three Blue One Brown that talks directly about this problem can be found here: https://youtu.be/p_di4Zn4wz4.

Problem 8.41. Consider a pendulum of length L with mass m and angle θ as shown in Figure 8.3. If we balance the forces on the pendulum and model the angle as the pendulum moves we get

$$m\theta''(t) = -\frac{mg}{L}\sin(\theta(t)) - b\theta'(t)$$

which can be rearranged to the nonlinear differential equation

$$\theta'' + b\theta' + \frac{g}{L}\sin\theta = 0$$

where b is a (linear) drag coefficient acting to slow the pendulum over time.

(a) Turn the nonlinear pendulum equation into a system of first order differential equations by making the substitution $\omega = \theta'$.

$$\theta' = \underline{\hspace{6cm}} \tag{8.7}$$
$$\omega' = \underline{\hspace{6cm}} \tag{8.8}$$

(b) If gravity is 9.8m/s^2, L is 1m, and b is 0.5kg·m/s then build a numerical approximation to the system of differential equations.

(c) Demonstrate qualitatively that there is a slight difference in the behavior of the nonlinear pendulum modeled above and the linear pendulum often used in physics where we use the *small angle approximation of sine* $\sin\theta \approx \theta$.

▲

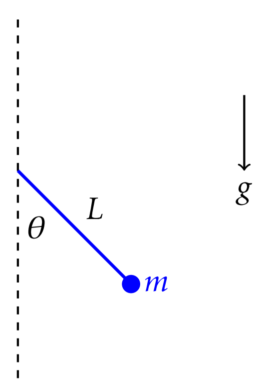

Figure 8.3. A pendulum

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```
clear; clc; clf;
dt = 0.01;
tmax = 25;
g=9.8;
L = 1;
b = 0.5;
t = 0:dt:tmax;
theta = zeros(size(t));
omega = zeros(size(t));
theta(1) = pi/4;
omega(1) = 0;
for n=1:length(t)-1
  theta(n+1) = theta(n) + dt*( );
  omega(n+1) = omega(n) + dt*( );
end
subplot(1,2,1)
plot(t,theta,'b--')
xlabel('time'), ylabel('angle')
subplot(1,2,2)
plot(theta,omega,'k.-')
xlabel('theta'), ylabel('omega')
```

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8.8 Equilibria and Linearization of Nonlinear Systems

Let's finally put some of our tools together. In this section we'll consider nonlinear systems but we will use the primary tool of calculus, linearization, and our tools from earlier in the chapter to understand what happens locally near an equilibrium point.

Problem 8.42. Suppose that x and y are the population of two distinct species that compete for the same resources. For example, two species of fish may compete for the same food in a lake or sheep and cattle competing for the same grazing land. We can model two competing species using the following system of first-order differential equations,

$$x' = 2x\left(1 - \frac{x}{2}\right) - xy$$
$$y' = 3y\left(1 - \frac{y}{3}\right) - 2xy.$$

(a) Write an Euler solver to numerically approximate the solutions to this system of differential equations. Plot your numerical solutions for several different initial conditions and verify (qualitatively) that this system of differential equations exhibits the behaviour you would expect from a competing species model.

(b) To find the equilibria for a system of first order differential equations we set the derivatives to zero. In this case we end up with the following nonlinear system of algebraic equations

$$2x\left(1 - \frac{x}{2}\right) - xy = 0$$
$$3y\left(1 - \frac{y}{3}\right) - 2xy = 0.$$

There are four solutions to this algebraic system. Do the algebra to find them.

(c) Use your Euler solver from part (a) and your equilibrium points from part (b) to classify each equilibrium as either a source, sink, spiral in, spiral out, saddle, or orbit.
Hint: Pick several initial conditions which are *near* the equilibrium and plot the solution in the xy plane.

▲

Now we'll build up the analytic tools necessary to analyze the competing species model in problem 8.42.

Definition 8.43 (Nullclines and Equilibria)**.** The **nullclines** for a linear system

$$x'(t) = f(x,y)$$
$$y'(t) = g(x,y)$$

are the curves where $f(x,y) = 0$ and $g(x,y) = 0$. When two nullclines intersect there is

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and equilibrium solution.

Problem 8.44. Use a graphing tool to sketch the nullclines of the system and use your graph to verify the location of the equilibrium points.

$$x' = 2x\left(1 - \frac{x}{2}\right) - xy$$
$$y' = 3y\left(1 - \frac{y}{3}\right) - 2xy.$$

▲

Definition 8.45 (Jacobian Matrix)**.** Consider the system of equations

$$f(x,y) = 0$$
$$g(x,y) = 0.$$

The **Jacobian matrix** for this system is defined as

$$J(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$

where subscripts mean partial dervatives.

Problem 8.46. Find the Jacobian matrix $J(x,y)$ for the system

$$x' = 2x\left(1 - \frac{x}{2}\right) - xy = 2x - x^2 - xy$$
$$y' = 3y\left(1 - \frac{y}{3}\right) - 2xy = 3y - y^2 - 2xy.$$

▲

The Jacobian describes the local linear behavior of the system near an equilibrium. That is to say that if we substitute the values (x_*, y_*) from an equilibrium point into the Jacobian then *near* the equilibrium point will behave like the linear system $\mathbf{x}' = J(x_*, y_*)\mathbf{x}$ centered at the equilibrium.

Problem 8.47. Verify the behavior of the system

$$x' = 2x\left(1 - \frac{x}{2}\right) - xy$$
$$y' = 3y\left(1 - \frac{y}{3}\right) - 2xy$$

near the equilibrium $(1,1)$ (you already discussed this in Problem 8.42). Then use the Jacobian matrix to discuss the behavior of the system near the other three equilibria. ▲

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Technique 8.48 (Equilibria and Stability of Nonlinear Systems)**.** Consider the nonlinear system of differential equations

$$x'(t) = f(x, y)$$
$$y'(t) = g(x, y).$$

To find and analyze the equilibria for the system:

1. Find the equilibria by setting ______ to zero and solving for x and y. It may be necessary to use technology to solve this system of nonlinear equations.

2. Find the Jacobian matrix at each of the equilibrium points.

3. Investigate the ______ for each Jacobian matrix. Based on this investigate you can make a claim about local stability.

Example 8.49. Consider the system

$$x'(t) = x - 3y + xy^2$$
$$y'(t) = 2x - 4y - x^2y.$$

Find the nullclines, equilibria, the Jacobian, and classify the equilibrium solutions.
Solution:
The nullclines are the curves $f(x, y) = 0$ and $g(x, y) = 0$ called the x-nullcline and the y-nullcline respectively since if $f = 0$ the x-variable stops changing and if $g = 0$ the y-variable stops changing.

$$x\text{-nullcline: } 0 = x - 3y + xy^2$$
$$y\text{-nullcline: } 0 = 2x - 4y - x^2y$$

These are rather complicated curves in the xy-plane.

Using a computer algebra system the approximate equilibria are $(-1.06, -0.41)$, $(1.06, 0.41)$, and $(0, 0)$ (along with a few imaginary equilibria).

The Jacobian is $J(x, y) = \begin{pmatrix} 1 + y^2 & -3 + 2xy \\ 2 - 2xy & -4 - x^2 \end{pmatrix}$ and at $(0, 0)$ we have $J(0, 0) = \begin{pmatrix} 1 & -3 \\ 2 & -4 \end{pmatrix}$.
For this equilibrium point, $T(0, 0) = -3$ and $D(0, 0) = (-4) - (-6) = 2$. Hence $T^2/4 = 9/4 = 2.25 > D$ so according to the trace-determinant plane we much have a spiral sink at $(0, 0)$.

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8.9 Applied Nonlinear Systems

Let's get started with a nonlinear system. This system will be familiar in a lot of ways but we will added a small wrinkle: air resistance.

Problem 8.50. Modeling a bungee jumper is much like modeling a 1-dimensional spring mass system except for the fact that the air resistance plays a major role. According to Newton's second law as well as Hooke's law

$$mx'' = -kx + F_d \quad \text{where} \quad F_d = -ax' - b\,(x')^2$$

Hence, the model for the motion of the bungee jumper is

$$mx'' + ax' + b\,(x')^2 + ky = 0$$

Dividing by mass we get

$$x'' + \alpha x' + \beta\,(x')^2 + \kappa x = 0$$

Turn this into a system of differential equations with an appropriate substitution, discuss equilibria and stability, and explore it graphically. ▲

Problem 8.51. Suppose that we have a predator-prey system consisting of a population of foxes (F) and rabbits (R)

$$\begin{aligned} R'(t) &= 2R - RF \\ F'(t) &= -5F + RF. \end{aligned}$$

It is easy to check that have equilibrium at $R = 5$ and $F = 2$. Fully analyze the dynamics of the population assuming that it doesn't start with $(R,F) = (5,2)$. In particular, make a phase plot and determine the stability of the equilibrium. ▲

Problem 8.52. In the previous problem, modify the rabbit population so that it follows logistic growth

$$R'(t) = 2R\left(1 - \frac{R}{10}\right) - RF.$$

Fully analyze this new system. ▲

Problem 8.53. Romeo and Juliet's love can be quantified as

Hysterical Hatred	-5
Disgust	-2.5
Indifference	0
Sweet Affection	2.5
Ecstatic Love	5

The characters struggle with frustrated love due to the lack of reciprocity of their feelings.

Romeo: "My feelings for Juliet decrease in proportion to her love for me."

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Juliet: "My love for Romeo grows in proportion to his love for me."

Write a mathematical model for the ill-fated love of Romeo and Juliet. Discuss equilibria and stability. Explore graphically.
Assume $R(0) = 2$ and $J(0) = 0$. What do these initial conditions mean? Start your explorations with $\alpha = 0.2$ and $\beta = 0.8$. ▲

Problem 8.54. Romeo and Juliet's love can be quantified as

Hysterical Hatred	-5
Disgust	-2.5
Indifference	0
Sweet Affection	2.5
Ecstatic Love	5

The characters struggle with frustrated love due to the lack of reciprocity of their feelings.

Romeo: "My feelings for Juliet decrease in proportion to her love for me."

Juliet: "My love for Romeo grows in proportion to his love for me." But, her emotional swings lead to sleepless nights which consequently dampen her emotions.

Write a mathematical model for the ill-fated love of Romeo and Juliet. Discuss equilibria and stability. Explore graphically. ▲

Problem 8.55. In historical battles where hand-to-hand combat was common, a mathematical model for the survival of the various forces is:

- The rate at which the RED army loses troops is proportional to the product of the sizes of the two armies
- The rate at which the BLUE army loses troops is proportional to the product of the sizes of the two armies

Write a mathematical model for the size of each army. Discuss equilibria, stability, and explore graphically. What is wrong with this model? ▲

Problem 8.56. In historical battles where hand-to-hand combat was common, a mathematical model for the survival of the various forces is:

- The rate at which the RED army loses troops is proportional to the product of the sizes of the two armies
- The rate at which the BLUE army loses troops is proportional to the product of the sizes of the two armies
- The rate at which the RED army gains recruits is proportional to the size of the RED army.

Write a mathematical model for the size of each army. Discuss equilibria, stability, and explore graphically. How do you prove stability? ▲

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8.10 Additional Exercises

Problem 8.57. The Van der Pol oscillator equation arose in the 1920's when Balthasar Van der Pol was working with oscillator circuits for radios. The equation is

$$x'' + \mu(x^2 - 1)x' + x = 0$$

where x is related to the current in an RLC-circuit. Write the Van der Pol equation as a non-linear first order system and completely investigate the behaviour of the system using $\mu = 1$. Use `pplane` plots to aid in your analysis. ▲

Problem 8.58. A virus spreads through a dorm. Assume that there are three types of people in the dormitory population: S is the number of people susceptible to the virus, I is the number of infectious people, and R is the number of recovered people. Assume that $S + I + R = N$ is the total number of people in the dorm (and N is fixed). Build a differential equation model for the rates at which S, I, and R change assuming that

- The susceptible people get sick at a rate proportional to the interactions with infectious people.
- Infectious people recover at a fixed rate.

Once you have your model explore it graphically (using `pplane`) and analyze any equilibrium points. (Hint: you really only need 2 equations) ▲

Problem 8.59. The **Western Grasslands Model**: This is a model of the competition between "good" grass and weeds on a fixed area of rangeland where cattle are allowed to graze. The two dependent variables $g(t)$ and $w(t)$ represent, respectively, the fraction of the area colonized by the good grass and the weeds at time t. Hence, $0 \le g \le 1$ and $0 \le w \le 1\}$. The model is given by

$$\begin{aligned}\frac{dg}{dt} &= R_1 g\left(1 - g - 0.6w\frac{E+g}{0.31E+g}\right)\\ \frac{dw}{dt} &= R_2 g\left(1 - w - 1.07g\frac{0.31E+g}{E+g}\right).\end{aligned}$$

The parameters R_1 and R_2 represent the intrinsic growth rates of the grass and weeds respectively. The cattle stocking rate is introduced through the parameter E. For this problem assume that $R_1 = 0.27$, $R_2 = 0.4$, and $E = 0.3$. There are several equilibrium points that we need to analyze.

- There is an equilibrium at $(0,0)$. What does it mean physically and what type of behavior do we see near this point?
- There is an equilibrium at $(0,1)$. What does it mean physically and what type of behavior do we see near this point?
- There are two equilibria inside the domain where both weeds and grass can coexist. Find them and describe the behavior of the system near them.

▲

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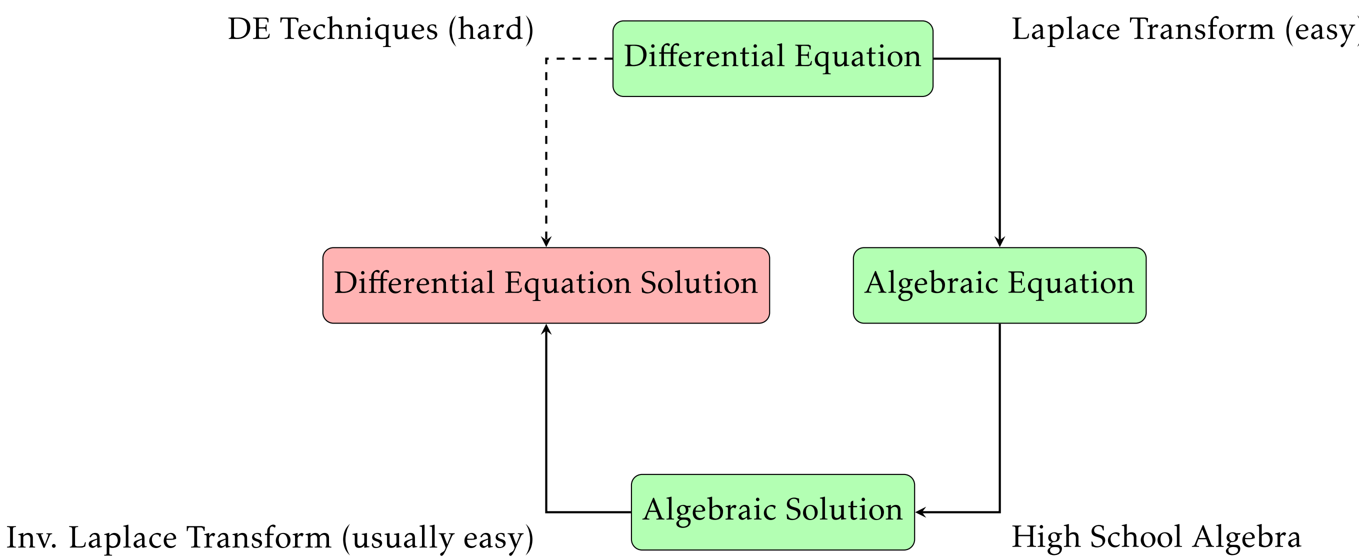
Chapter 9

Laplace Transforms

9.1 Introduction to Laplace Transforms

Let's face it. Solving differential equations can be hard. Sometimes it is really hard, and sometimes it is downright impossible. Generally speaking it is much easier to solve algebraic equations like the ones you were introduced to in high school mathematics. The goal of the Laplace Transform Method for solving differential equations is to turn a linear differential equation into an algebraic equation, solve it, then turn the answer back into a solution to the differential equation. The process is depicted below. Once you get used to this technique you'll (almost) never want to use our old techniques again!



9.2 Basic Laplace Transforms and Basic Properties

Let's start the discussion with the definition of the Laplace Transform. This linear transformation is actually defined as an integral as follows:

Definition 9.1 (The Laplace Transform)**.** Let $f(t)$ be a function of a single real variable. The **Laplace Transform** of $f(t)$ is

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt.$$

Our ultimate goal in this chapter is to use the Laplace transform to solve differential equations. In order to get there, we need to first understand how to simply *take* a Laplace transform of a function ... the application to differential equations will come later.

Now let's do a few Laplace Transforms. This exercise (as well as a few others) will be essential before we can start using Laplace transforms for differential equations.

Problem 9.2. Find the Laplace Transform of each of the following functions: (no calculator!)

(a) If $f(t) = 1$ then
$$\mathcal{L}\{f(t)\} = \underline{\hspace{3cm}}$$

(b) If $f(t) = e^{at}$ then
$$\mathcal{L}\{f(t)\} = \underline{\hspace{3cm}}$$

(c) If $f(t) = t$ then (Hint: integrate by parts)
$$\mathcal{L}\{f(t)\} = \underline{\hspace{3cm}}$$

▲

Example 9.3. Find the Laplace transform of $f(t) = t^2$.
Solution: We want $\mathcal{L}\{f(t)\}$ so we write the integral

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}t^2dt.$$

This integral requires integration by parts. Let $u = t^2$ and $dv = e^{-st}$ to get $du = 2tdt$

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and $v = -\frac{1}{s}e^{-st}$ and hence

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \int_0^\infty e^{-st}t^2 dt = -\frac{t^2}{s}e^{-st}\Big|_{t=0}^{t\to\infty} + \frac{2}{s}\int_0^\infty te^{-st}dt \\ &= 0 + \frac{2}{s}te^{-st}dt \\ &= \frac{2}{s}\mathcal{L}\{t\} = \frac{2}{s}\cdot\frac{1}{s^2} = \frac{2}{s^3}\end{aligned}$$

Problem 9.4. Based on what we saw in the previous problem and example you may see a convenient pattern for finding the Laplace transform of power functions. Based on this pattern let's conjecture a few more basic Laplace transforms. (Hint: look at the previous example and see what will happen every time we use integration by parts on these functions.)

$$\begin{aligned}\mathcal{L}\{1\} &= \frac{1}{s} \\ \mathcal{L}\{t\} &= \frac{1}{s^2} \\ \mathcal{L}\left\{t^2\right\} &= \frac{2}{s^3} \\ \mathcal{L}\left\{t^3\right\} &= \underline{\hspace{3cm}} \\ \mathcal{L}\left\{t^4\right\} &= \underline{\hspace{3cm}} \\ \mathcal{L}\left\{t^5\right\} &= \underline{\hspace{3cm}} \\ \mathcal{L}\{t^n\} &= \underline{\hspace{3cm}}\end{aligned}$$

▲

Now we will build some of the basic properties of the Laplace transform. Many of these are intuitively obvious from the definition of the Laplace transform

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt$$

since we know that the integral is a linear operator ... hmmm, I wonder what this means about the Laplace transform.

Theorem 9.5. If $f(t)$ and $g(t)$ are functions that have Laplace transforms then

$$\mathcal{L}\{f(t)+g(t)\} = \underline{\hspace{6cm}}.$$

(Fill in the blank with what your intuition tells you *should* happen)

Proof. (Prove the previous theorem) □

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Theorem 9.6. If a is a scalar and $f(t)$ is a function that has a Laplace transform then

$$\mathcal{L}\{af(t)\} = \underline{\hspace{4cm}}.$$

(Fill in the blank with what your intuition tells you *should* happen)

Problem 9.7. The previous two theorems tell that the Laplace transform is ____________.
▲

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9.3 Important Theorems for Laplace Transforms

We will now state a few important theorems (without proof):

Theorem 9.8 (Existence of Laplace Transforms:)**.** If $f(t)$ is a piecewise continuous function such that $|f(t)| < Me^{ct}$ for $t \geq T$ and for some non-negative constants $M, c,$ and T, then $\mathcal{L}\{f(t)\} = F(s)$ exists for all $s > c$.

Theorem 9.8 gives us conditions for when the Laplace transform exists. It just says that the function $f(t)$ needs to *grow slower* than an exponential function.

Now that we know when the Laplace transform exists it would be handy to know if it is unique. It should be intuitively *obvious* that if we calculate $\mathcal{L}\{f(t)\}$ then we will only ever get one answer, but as mathematicians we can't just rely on our instincts for what is *obvious*. For a uniqueness theorem we state it the other way around: If two Laplace transforms are the same then they must have come from the same place. This is summarized in the following theorem.

Theorem 9.9 (Uniqueness of Laplace Transforms:)**.** If $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$ then if $F(s) = G(s)$ we MUST have $f(t) = g(t)$. In other words, the Laplace transform of a function is unique.

Finally we get to the ultimate utility of the Laplace transform. From the beginning of this chapter we stated that we want to use the Laplace transform to make solving differential equations easier. To do this we need to first convert a differential equation to an algebraic equation. The reader should see that this might be possible with the Laplace transform. At the end of the process, however, we need to do an inversion of the Laplace transform. If the inverse isn't known to exist then the whole process is going to fail and this conversation is moot. Thankfully we have the following theorem!

Theorem 9.10 (Invertibility of Laplace Transforms:)**.** Since the Laplace transform of a function is unique, the *inverse Laplace transform* exists and

$$\mathcal{L}^{-1}\{F(s)\} = f(t).$$

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9.4 Common Laplace Transforms

Here are some common Laplace transforms. These DO NOT need to be memorized. You will be provided such a table on any exam. For a more complete table see tutorial.math.lamar.edu/pdf/Laplace_Table.pdf

Function $f(t)$	Laplace Transform $F(s) = \mathcal{L}\{f(t)\}$
1	$\dfrac{1}{s}$
t	$\dfrac{1}{s^2}$
t^n	$\dfrac{n!}{s^{n+1}}$
$\dfrac{1}{\sqrt{t}}$	$\dfrac{\sqrt{\pi}}{\sqrt{s}}$
e^{at}	$\dfrac{1}{s-a}$
e^{-at}	$\dfrac{1}{s+a}$
$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$
$\cos(bt)$	$\dfrac{s}{s^2+b^2}$
$e^{-at}\cos(bt)$	$\dfrac{s+a}{(s+a)^2+b^2}$
$\sin(bt)$	$\dfrac{b}{s^2+b^2}$
$e^{-at}\sin(bt)$	$\dfrac{b}{(s+a)^2+b^2}$
$t\sin(bt)$	$\dfrac{2bs}{(s^2+b^2)^2}$
$t\cos(bt)$	$\dfrac{s^2-b^2}{(s^2+b^2)^2}$
$\sin(bt)+bt\cos(bt)$	$\dfrac{2bs^2}{(s^2+b^2)^2}$
$\sin(bt)-bt\cos(bt)$	$\dfrac{2b^3}{(s^2+b^2)^2}$

Problem 9.11. Find the Laplace Transforms of the following functions. (please don't do the integration!)

(a) $f(t) = t^2 + 5$

(b) $f(t) = e^{3t+2}$

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(c) $f(t) = t^3 e^{4t}$

▲

Problem 9.12. Find the Inverse Laplace Transforms of the following functions.

(a) $F(s) = \frac{3}{s^4}$

(b) $F(s) = \frac{3}{s-4}$

▲

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9.5 Solving Differential Equations with Laplace Transforms

Now we get to the good stuff.

Theorem 9.13. Suppose that $f(t)$ is a continuous piecewise smooth function for $t \ge 0$ such that the Laplace transform of f exists. Under these conditions $\mathcal{L}\{f'(t)\}$ exists and
$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

Proof. To prove this theorem consider the following hints:

1. Recall that $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$
2. Therefore, $\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)dt$
3. Now use integration by parts with $u = e^{-st}$ and $dv = f'(t)dt$
4. The result follows after some computation

Now prove the theorem □

Theorem 9.14. Suppose that $f(t)$ is a continuous piecewise smooth function for $t \ge 0$ such that the Laplace transforms of f and f' exist. Under these conditions $\mathcal{L}\{f''(t)\}$ exists and
$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

Proof. Prove this theorem:
Hints:

1. Let $g(t) = f'(t)$ and find $\mathcal{L}\{g'(t)\}$
2. The result follows after some computation

□

Theorem 9.15. Suppose that $f(t)$ is a continuous piecewise smooth function for $t \ge 0$ such that the Laplace transforms of f, f', and f'' exist. Under these conditions $\mathcal{L}\{f'''(t)\}$ exists and

$$\mathcal{L}\{f'''(t)\} = s^3\mathcal{L}\{f(t)\} - s^2 f(0) - sf'(0) - f''(0)$$

Problem 9.16. Looking at the previous three theorems, what is the Laplace transform of a fourth derivative? ▲

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Technique 9.17 (Solving ODEs with Laplace Transforms)**.** Follow these steps to solve a linear ordinary differential equation with Laplace transforms.

1. Take the Laplace transform of both sides.
2. Solve (algebraically) for $X(s)$.
3. Simplify the right-hand side (this typically involves partial fractions)
4. Take the inverse Laplace transform.

Problem 9.18. Solve the following differential equation with Laplace transforms.

$$y' = -4y + 3e^{2t} \quad \text{with} \quad y(0) = 1$$

▲

Problem 9.19. Solve with Laplace transforms:

$$x'' + 4x = \sin(3t) \quad \text{with} \quad x(0) = x'(0) = 0$$

▲

Problem 9.20. Use Laplace transforms to show that the solution to the differential equation

$$x'' + 3x' + 2x = t \quad \text{with} \quad x(0) = 0 \quad \text{and} \quad x'(0) = 2$$

is

$$x(t) = 3e^{-t} - \frac{9}{4}e^{-2t} + \frac{t}{2} - \frac{3}{4}$$

▲

Problem 9.21. Use Laplace transforms to show that the solution to the differential equation

$$x'' + 6x' + 25x = 0 \quad \text{with} \quad x(0) = 2 \quad \text{and} \quad x'(0) = 3$$

is

$$x(t) = e^{-3t}\left(2\cos(4t) + \frac{9}{4}\sin(4t)\right)$$

▲

9.6 The Heaviside Function and Delayed Forcing Terms

Problem 9.22. Let $u(t)$ be defined as

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}.$$

Sketch a picture of $u(t)$ to the right of the definition … I'll wait while you sketch. … Good! The function $u(t)$ is called the Heaviside function (named after a guy who's last name was Heaviside). ▲

Problem 9.23. Now let's define a shifted version, $u_a(t)$, of the Heaviside function as

$$u_a(t) = u(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}.$$

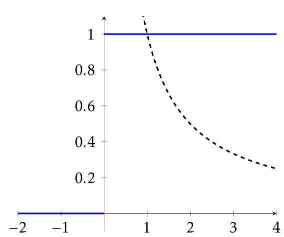
Sketch a picture of this one too … I'll wait. ▲

One HUGE advantage to Laplace transforms is that these functions have nice smooth and easy to handle Laplace transforms. Imagine if they showed up on the right-hand side of a differential equation before. What would you have done?!

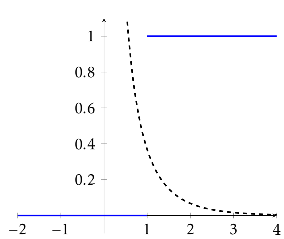
Function $f(t)$	Laplace Transform $F(s) = \mathcal{L}\{f(t)\}$
$u(t)$	$\frac{1}{s}$
$u_a(t)$	$\frac{e^{-as}}{s}$

Let's look at step functions and shifted step functions graphically.

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases} \rightsquigarrow \mathcal{L}\{u(t)\} = \frac{1}{s}$$



$$u_a(t) = u(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases} \rightsquigarrow \mathcal{L}\{u(t)\} = \frac{e^{-as}}{s}$$



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Problem 9.24. Consider the differential equation $x' + \frac{1}{2}x = u_3(t)$ with $x(0) = 3$.

(b) Make a sketch of the solution to the differential equation.

(b) Take the Laplace transform of both sides of the differential equation and solve for $X(s)$.

(c) What do you need to be able to invert the Laplace transform?

▲

Problem 9.25. For each of the following Laplace transforms take the inverse transform and sketch the resulting function.

(a) $X(s) = \dfrac{2}{s^2+4} + \dfrac{e^{-4s}}{s}$

(b) $X(s) = \dfrac{2}{s+3} + 3\dfrac{e^{-s}}{s}$

(c) $X(s) = \dfrac{e^{-s}}{s} - \dfrac{e^{-5s}}{s}$

▲

Problem 9.26. Find the Laplace transform of the function

$$f(t) = -4u_3(t) - 5u_5(t) + 2u_6(t)$$

and sketch the resulting function. ▲

The reader should observe that the Laplace transform is usually a very smooth (continuous and differentiable) function. This even holds when we have discontinuous functions $f(t)$, and this fact is one of the reasons that the Laplace transform is really powerful: would you rather solve a differential equation with a discontinuous right-hand side or a smooth differentiable right-hand side?

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9.7 Impulses and The Delta Function

Next we'll build up the mathematical machinery to understand shifted and impulse-type forcing terms in differential equations. We have already seen the Heaviside function, but what about a function that provides an impulse? Consider this situation: An undamped mass-spring oscillator is oscillating without the influence of any external forces until at a certain time you give the whole apparatus a bump. Before the bump you expect undamped oscillations modeled by trigonometric functions and after the bump you expect the same, but how does the bump change the behavior? The answer to this question lies in understanding the Delta function.

Problem 9.27. What do you suppose the derivative of the Heaviside function looks like? Draw a picture. ▲

Definition 9.28 (The Dirac Delta Function)**.** The Dirac delta function $\delta_a(t)$ is define as

$$\delta_a(t) = \begin{cases} 0, & \text{if } t \neq a \\ \infty, & \text{if } t = a \end{cases}$$

where

$$\int_{-\infty}^{\infty} \delta_a(t)dt = 1.$$

Don't think too hard about this since it should become obvious that this definition is kind of nonsense. We have an infinite spike at $t = a$ but the integral itself is actually finite . . . strange. That being said, this is the proper definition of the delta function.

Now we're going to work out the Laplace transform of the delta function. This is an important step since the delta function models an impulse; an important concept in engineering and physics.

Define the function $d_k(t)$ as

$$d_k(t) = \begin{cases} \frac{1}{2k}, & \text{if } -k \leq t \leq k \\ 0, & \text{otherwise} \end{cases} .$$

In Figure 9.1 we see that the function $d_k(t)$ will *turn into* the Dirac delta function as k goes to infinity. That is,

$$\lim_{k\to\infty} d_k(t) = \begin{cases} 0, & \text{if } t \neq 0 \\ \infty, & \text{if } t = 0 \end{cases}$$

where for every k we must have

$$\int_{-\infty}^{\infty} d_k(t)dt = 1$$

since the area underneath $d_k(t)$ is a rectangle and is fixed at 1 by construction. Figure 9.1 shows plots of $d_k(t)$ for several values of k. It should be clear to the reader that d_k does indeed converge to the delta function as k approaches infinity.

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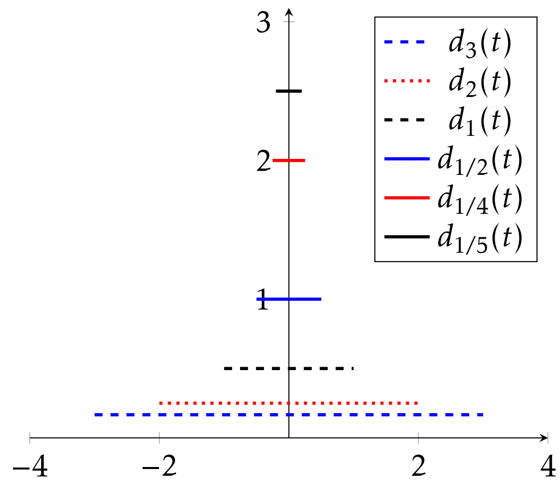

Figure 9.1. The function $d_k(t)$ for several values of k. In this limit this function approximates the Dirac delta function.

This is all well and good, but what is the Laplace transform of the Dirac delta function? To answer this question we explore one more property of the delta function

$$\int_{-\infty}^{\infty} \delta_a(t) f(t) dt = f(a). \tag{9.1}$$

In words, (9.1) says that if we take the product of the delta function and a (suitably continuous) function $f(t)$ and integrate over the whole real line then we simply extract the function value of $f(t)$ located at the spike of the delta function. In this sense, the delta function just probes function values.

Problem 9.29. Provide a graphical reason why

$$\int_{-\infty}^{\infty} \delta_a(t) f(t) dt = f(a).$$

▲

From here we get a simple formula for the Laplace transform of the delta function.

$$\mathcal{L}\{\delta_a(t)\} = \int_0^{\infty} e^{-st}\delta_a(t)dt = \int_{-\infty}^{\infty} e^{-st}\delta_a(t)dt = e^{-as}.$$

Theorem 9.30 (Properties of the Dirac Delta Function)**.** Let $\delta_a(t)$ be the shifted Dirac

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delta function. The delta function has the following properties.

$$\delta_a(t) = \begin{cases} 0, & \text{if } t \neq a \\ \infty, & \text{if } t = a \end{cases}$$
$$\int_{-\infty}^{\infty} \delta_a(t)dt = 1$$
$$\int_{-\infty}^{\infty} \delta_a(t)f(t)dt = f(a)$$
$$\mathcal{L}\{\delta_a(t)\} = e^{-as}$$
$$\mathcal{L}\{\delta_0(t)\} = 1$$

To make the inversion of the Laplace transform of the delta function more useful we finally consider the following theorem.

Theorem 9.31. Let $f(t)$ be a function where $\mathcal{L}\{f(t)\} = F(s)$ exists.

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)f(t-a) = u_a(t)f(t-a).$$

$$\mathcal{L}\{u_a(t)f(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}$$

For example consider the expression $\frac{e^{-2s}}{s+1}$. The inverse Laplace transform of this expression is

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s+3}\right\} = u(t-2)e^{-3(t-2)} = u_2(t)e^{-3(t-2)}.$$

Problem 9.32. Explain what Theorem 9.31 says and create a few examples of this theorem in action. ▲

Problem 9.33. Solve the differential equation

$$x' + \frac{1}{2}x = \delta_3(t)$$

with the initial condition $x(0) = 5$. Draw a picture of your solution and explain what happened. ▲

Problem 9.34. Let's return to the differential equation $x' + \frac{1}{2}x = u_3(t)$ with $x(0) = 5$.

(a) Return to your notes from Problem 9.24 and recall our conjecture for the plot of the solution.

(b) Take the Laplace transform of both sides and rearrange to solve for $X(s) = \mathcal{L}\{x(t)\}$.

(c) Take the inverse Laplace transform now with the help of Theorem 9.31.

(d) Use MATLAB to create a plot of the solution.
Hint: the Heaviside function is the `heaviside` command in MATLAB.

▲

Problem 9.35. Find the Laplace transform of

$$f(t) = \begin{cases} 0, & t < 3 \\ (t-3)^2, & t \geq 3 \end{cases}$$

▲

Problem 9.36. Find the Laplace transform of

$$f(t) = u_5(t)e^{-(t-5)}.$$

▲

Problem 9.37. Find $\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s+2}\right\}$ ▲

Problem 9.38. We have a system modeled as an undamped harmonic oscillator that begins at equilibrium and at rest, so $y(0) = y'(0) = 0$. The systems receives a unit impulse force at $t = 4$ so that it is modeled by the differential equation

$$y'' + 9y = \delta_4(t).$$

Find $y(t)$. ▲

Problem 9.39. Create a differential equation that can only be solved analytically using Laplace transforms. Be sure to provide sufficient initial conditions. After you've written your problem trade with another group and solve the other group's problem. ▲

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9.8 Additional Exercises

Problem 9.40. Use Laplace transforms to show that the solution to the differential equation

$$x'' + 6x' + 18x = \cos(2t) \quad \text{with} \quad x(0) = 1 \quad \text{and} \quad x'(0) = -1$$

is

$$x(t) = \frac{7}{170}\cos(2t) + \frac{3}{85}\sin(2t) + e^{-3t}\left(\frac{163}{170}\cos(3t) + \frac{307}{510}\sin(3t)\right)$$

▲

Problem 9.41. Solve the differential equation

$$x'' + 4x' + 13x = \delta_3(t) \quad \text{with} \quad x(0) = 1 \quad \text{and} \quad x'(0) = 0.$$

▲

Problem 9.42. (Modified from [5])
A harmonic oscillator with natural frequency $\omega_0 = 2$, initially at rest, is forced by the *ramp function* $f(t)$ defined as

$$f(t) = t(1 - u_1(t)).$$

Prove that the position x as a function of time t is

$$x(t) = \begin{cases} \frac{1}{4}t - \frac{1}{8}\sin(2t), & 0 < t < 1 \\ -\frac{1}{8}\sin(2t) + \frac{1}{8}\sin(2(t-1)) + \frac{1}{4}\cos(2(t-1)), & t > 1 \end{cases}$$

(Suggestion: Draw a picture of what the *ramp function* looks like before you start) ▲

Problem 9.43. (Modified from [5])
A 25-gallon tank is initially filled with water containing one pound of salt dissolved in it. It is desired to increase the salt concentration from $1/25 = 0.04$ pounds per gallon to 0.20 pounds per gallong. With $x(t)$ equal to the pounds of salt in the tank at time t, this would require that $x(t)$ be 5 pounds for the desired concentration.

For five minutes, a solution containing 0.5 pounds of alt per gallon is allowed to run into the tank at a rate of one gallon per minute, and the solution in the tank is allowed to drain out at the same rate. This seems to be taking too long, so the operators decide to dump 5 pounds of salt into the input water, and this well-stirred solution is all fed in over the next minute. At the end of the 6 minutes, the concentration is too high, so pure water is run in to lower it. Find the time at which the concentration is back to 0.2 pounds per gallon.

The function $H(t)$ defined below might be helpful:

$$H(t) = \frac{1}{2} + 4.5u_5(t) - 5u_6(t).$$

Draw a picture of this function and discuss what role it plays in this model. Once you know what $H(t)$ does, draw a picture of what you think the function $x(t)$ will look like. ▲

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Problem 9.44. An oral drug is given in periodic doses with the first dose being 2mg. The person's metabolism is such that half of the drug is left in the body after 5 hours. At every 5 hour mark another dose of the drug is given.

(a) Fill in the blanks for the differential equation where D is the dose (in mg) and t is the time (in hours).

$$\frac{dD}{dt} = -k_1\underline{\hspace{2cm}} + k_2\underline{\hspace{2cm}} + k_3\underline{\hspace{2cm}} + k_4\underline{\hspace{2cm}} + \cdots.$$

The first blank correponds to the way that the drug is removed from the system, the second blank corresponds to the dose at 5 hours, the third blank corresponds to the dose at 10 hours, etc.

(b) Use Laplace Transforms to solve your differential equation.

(c) Find the constant k_1 from the initial information.

(d) Find the constants $k_2, k_3, \ldots$ by assuming that right after each dose the amount in the patient's system jumps back to 2.

▲

Problem 9.45. In Problem 9.44 we build a differential equation for a drug dosing problem. The trouble with the model built in that problem is that it doesn't account for the time that it takes for the drug to go from the stomach to the blood stream. Instead, consider the following system of differential equations.

$$\begin{aligned}\frac{dS}{dt} &= -CS + k_2\delta(t-5) + k_3\delta(t-10) + k_4\delta(t-15) + \cdots \\ \frac{dB}{dt} &= CS - k_1 B\end{aligned}$$

where $S(t)$ is the amount of drug in the stomach and $B(t)$ is the amount of drug in the blood stream at time t.

(a) Explain each term in the model. Notice that the sum of the two models gives us exactly the same model that we had in Problem 9.44.

(b) We will now analyze this system of differential equations numerically since we don't have all of the tools to solve a system with Laplace transforms (...so much math but only finite time ...). We will use the same values of $k_1, k_2, \ldots$ for this problem. We need a constant for C (the rate at which the drug moves from the stomach to the blood stream) so at first pass let's just use $C = 0.1$.

Complete the following block of code to build an Euler solver for the system of differential equations. When the code is complete, run it to verify that what you see has the correct qualitative solution based on your understanding of basic biology. What is wrong with the new model? What is better aboud the new model? How would we improve the downsides to this model?

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▲

```
clear; clc;
k1 =  % fill in this line with the value of k1 from the prev. problem
C = 0.1;
dt = 0.01;
t = 0:dt:40;
Stomach = zeros(length(t),1);
Blood = zeros(length(t),1);
Stomach(1) = 2; % initial condition
Blood(1) = 0; % initial condition
for n=1:length(t)-1
    if mod(t(n),5) == 0 && t(n)>0
        Stomach(n+1) = Stomach(n) + dt*(-C*Stomach(n)) + 1; % what does this do
    else
        Stomach(n+1) = Stomach(n) + dt*( ... ) % finish the Euler solver
    end
    Blood(n+1) = Blood(n) + dt*( ...  ) % finish the Euler solver
end
plot(t,Stomach,'b',t,Blood,'r')
```

Chapter 10

Power Series Method: The Ultimate Guess

Solution techniques for differential equations lead most to believe that there is a certain amount of educated guesswork to get started writing a solution. To some extent this observation is correct! Think about the method of undetermined coefficients; built on an educated guess. Why don't we just make the ultimate guess:
If $y(t)$ is a solution to a differential equation and so long as y is expected to have a Taylor series representation, then why don't we just guess that $y(t)$ is a Taylor series and use some detective work to determine the coefficients. To demonstrate this consider the next problem.

10.1 Taylor Series Solutions to Diff. Equations

In the following problems we are assuming that the solution to the differential equation can be written in terms of a Taylor series centered at $t = 0$

$$y(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^n.$$

Problem 10.1. Consider the differential equation $y' = y$ with $y(0) = 1$.

(a) Solve this differential equation using any appropriate technique.

(b) Let's assume that $y(t) = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} t^n$ (a Taylor series centered at $t = 0$). Expanding the Taylor series we see that

$$y(t) = y(0) + y'(0)t + \frac{y''(0)}{2!}t^2 + \frac{y'''(0)}{3!}t^3 + \frac{y^{(iv)}(0)}{4!}t^4 + \cdots$$

From the initial condition we know that $y(0) = 1$ so we at least know that

$$y(t) = 1 + y'(0)t + \frac{y''(0)}{2!}t^2 + \frac{y'''(0)}{3!}t^3 + \frac{y^{(iv)}(0)}{4!}t^4 + \cdots$$

Using the differential equation determine $y'(0)$.

(c) You should now have the first two terms in the Taylor series. Differentiate both sides of the differential equation and use your answer to determine $y''(0)$.

(d) Finally, use what you did in part (c) to determine the rest of the Taylor series. Verify your answer off of part (a).

▲

The previous problem suggests a technique for building the Taylor series of a solution to a differential equation. Let's put it into action on another problem that isn't quite as easy.

Problem 10.2. Consider the differential equation

$$y' - ty = 1$$

with initial condition $y(0) = 1$.

(a) This problem would traditionally require integrating factors. Start the process of integrating factors and work the procedure until you get to an integral that cannot be evaluated.

(b) Now rearrange the differential equation to solve for y' and use that rearrangement to determine $y'(0)$, $y''(0)$, $y'''(0)$, etc.

(c) Write the Taylor series solution for the differential equation.

(d) Use a plotting tool to create a plot of the approximate Taylor series solution using the first several terms in the Taylor series.

▲

Problem 10.3. Summarize the technique of using Taylor series to approximate the solution to a differential equation. ▲

Problem 10.4. Use the Taylor series method to estimate a solution to the differential equation

$$y'' - 2ty' + 2y = 0$$

with $y(0) = 1$ and $y'(0) = -1$. ▲

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10.2 Radius of Convergence for Power Series

A power series is an infinite series of power functions

$$y(t) = \sum_{n=0}^{\infty} a_n t^n.$$

We observe that the Taylor series discussed previously is just a special kind of power series, but that is not the only kind that is interesting. Power series, in general, can be used to build up functions that cannot be written in terms of regular algebraic or trigonometric functions. The "infinity" in the upper bound on the sum, however, could cause some problems. It is not generally true to say that any power series will create a new function. For poor choices of the coefficients a_n the sum may diverge to infinity for all values of t. What we need is an arsenal of tools to determine whether or not a power series actually converges to something meaningful or if it diverges off to infinity. The primary tools are summarized in the following problems and theorems.

Problem 10.5. We start this investigation just looking at sums of numbers. Write computer code to determine the sums

$$\sum_{n=0}^{5} \frac{n!}{2^n} = \underline{\hspace{3cm}}$$
$$\sum_{n=0}^{50} \frac{n!}{2^n} = \underline{\hspace{3cm}}$$
$$\sum_{n=0}^{500} \frac{n!}{2^n} = \underline{\hspace{3cm}}$$
$$\sum_{n=0}^{5000} \frac{n!}{2^n} = \underline{\hspace{3cm}}$$

and now make a conjecture: Does the series

$$\sum_{n=0}^{\infty} \frac{n!}{2^n}$$

converge to a finite value or diverge to infinity? ▲

Problem 10.6. Now let's consider a more interesting series that maybe isn't so obvious.

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Write computer code to determine the sums

$$\sum_{n=0}^{5} \frac{n^2}{(2n-1)!} = \_\_\_\_\_\_\_\_\_\_$$

$$\sum_{n=0}^{50} \frac{n^2}{(2n-1)!} = \_\_\_\_\_\_\_\_\_\_$$

$$\sum_{n=0}^{500} \frac{n^2}{(2n-1)!} = \_\_\_\_\_\_\_\_\_\_$$

$$\sum_{n=0}^{5000} \frac{n^2}{(2n-1)!} = \_\_\_\_\_\_\_\_\_\_$$

and now make a conjecture: Does the series

$$\sum_{n=0}^{\infty} \frac{n^2}{(2n-1)!}$$

converge to a finite value or diverge to infinity? ▲

Problem 10.7. Each of the previous two problems were written in the form $\sum_{n=0}^{\infty} a_n$ where a_n is the sequence of numbers being summed.

1. For the sum
$$\sum_{n=0}^{\infty} \frac{n!}{2^n}$$
determine a_n and find the limit
$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

2. For the sum
$$\sum_{n=0}^{\infty} \frac{n^2}{(2n-1)!}$$
determine a_n and find the limit
$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

▲

Problem 10.8. Consider the series

$$\sum_{n=0}^{\infty} \frac{n^2+2n+1}{3^n+2}.$$

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Write computer code that finds sums for successivaly larger and larger upper bounds. Use this computer code to conjecture whether this series converges or diverges. Finally, evaluate the limit

$$\lim \left| \frac{a_{n+1}}{a_n} \right|$$

where $a_n = \frac{n^2+2n+1}{3^n+2}$. ▲

The *ratio test* that follows is a test to determine if a series will converge or diverge.

Theorem 10.9 (The Ratio Test)**.** Let a_n be a sequence of numbers and consider the sum $\sum_{n=0}^{\infty} a_n$.

- If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$ then the sum converges to a finite value.
- If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$ then the sum diverges to infinity.
- If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$ then the sum may either converge or diverge.

Problem 10.10. Use the ratio test to determine whether the following series converge or diverge.

$$\sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$$

$$\sum_{n=0}^{\infty} \frac{1}{n!}$$

$$\sum_{n=1}^{\infty} \frac{n^n}{n!}$$

$$\sum_{n=1}^{\infty} \frac{7^{n+2}}{2n6^n}$$

▲

Now we switch back to our study of power series. It is not generally true that we can just build a power series and we get a meaningful result. There might only be a small region where the series converges. Consider the next problem.

Problem 10.11. Consider the power series

$$f(x) = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

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(a) Write computer code that plots a sequence of functions

$$1, \quad 1-x^2, \quad 1-x^2+x^4, \quad 1-x^2+x^4-x^6, \quad \ldots.$$

Based on your sequence of plots where does the power series converge?

(b) Let $a_n = (-1)^n x^{2n}$ and evaluate the limit

$$\lim \left| \frac{a_{n+1}}{a_n} \right|.$$

(c) Based on your knowledge of the ratio test, for what values of x does the series $\sum_{n=0}^{\infty} (-1)^n x^{2n}$ converge?

▲

Problem 10.12. Repeat problem 10.11 with the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$

Back up any conjectures that you make from the plots with the use of the ratio test. ▲

Problem 10.13. Repeat problem 10.11 with the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n n}{4^n} (x+3)^n.$$

Back up any conjectures that you make from the plots with the use of the ratio test. ▲

Theorem 10.14 (Radius of Convergence of Power Series)**.** Given a power series $\sum a_n t^n$ and the limit

$$R(x) = \lim_{n\to\infty} \left| \frac{c_{n+1}x^{n+1}}{c_n x^n} \right|$$

exists then

- for all x such that $R(x) < 1$ the power series converges,
- for all x such that $R(x) > 1$ the power series diverges.

Problem 10.15. Devise a power series that has a domain of convergence of $|x| < 2$. ▲

Problem 10.16. What are the radii of convergence for the Taylor series of the exponential function, the sine function, and the cosine function? ▲

10.3 Power Series Solutions to Diff. Equations

Using Taylor series as we did in Section 10.1 is one technique for finding a series solution to differential equations. In this section we will look at a more general technique: using power series to build solutions. This technique streamlines the Taylor series technique and actually involves must less work for some problems. Instead of saying that $y(t)$ is a Taylor series, $y(t) = \sum_{n=0}^{\infty} \frac{y^{(n)}}{n!} t^n$ we simply start by saying that $y(t)$ is just a power series with an unknown sequences of coefficients:

$$y(t) = \sum_{n=0}^{\infty} a_n t^n.$$

Our detective work from this assumption amounts to massaging the power series and the differential equation to determine a pattern for the sequence a_n.

In order to use power series to build solutions to differential equations we need to be able to differentiate (and integrate) power series in a meaningful way. Thankfully, we are blessed with the following theorem from mathematical analysis.

Theorem 10.17. Let a_n be a sequence of real numbers such that the series $\sum_{n=0}^{\infty} a_n x^n$ converges for $x \in (-R, R)$. The number R is called the radius of convergence for the power series.

1. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ then $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$
2. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ then $f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$
3. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ then $f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n x^{n-k}$
4. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ then $\int f(x)dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} + C$

Proof. The full proof of this theorem is beyond the scope of this text. Instead, convince yourself that this theorem is true for finite sums (this should be obvious) and trust the author that it holds within the radius of convergence for infinite sums. □

Theorem 10.17 says that so long as you are within the radius of convergence of the power series then the series can be differentiated or integrated term by term. This has an impact on the use of power series for approximating solutions to differential equations. We simply assume that we are within the radius of convergence and proceed with determining the sequence a_n that defines the power series. Let's look at an example.

Example 10.18. Consider the differential equation $y' = -0.5y$ with $y(0) = 1$ and assume that $y(t) = \sum_{n=0}^{\infty} a_n t^n$. Find the sequence a_n that defines the power series.
Solution: This is a simple differential equation and we know that solution is $y(t) = e^{-0.5t}$. Let's build the power series.

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Assume that $y(t) = \sum_{n=0}^{\infty} a_n t^n$ so we see that since $y' = -0.5y$,

$$\sum_{n=1}^{\infty} n a_n t^{n-1} = -\frac{1}{2}\sum_{n=0}^{\infty} a_n t^n.$$

Expanding both sides of the summation and then matching terms we get

$$\begin{aligned}
&1a_1 + 2a_2t^1 + 3a_3t^2 + 4a_4t^3 + 5a_5t^4 + \cdots = -\frac{1}{2}\left(a_0 + a_1t^1 + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5 + \cdots\right) \\
&\implies 0 = \left(-\frac{1}{2}a_0 - a_1\right) + \left(-\frac{1}{2}a_1 - 2a_2\right)t + \left(-\frac{1}{2}a_2 - 3a_3\right)t^2 + \left(-\frac{1}{2}a_3 - 4a_4\right)t^3 + \cdots \\
&\implies a_1 = -\frac{1}{2}a_0, \quad a_2 = -\frac{1}{4}a_1 = \frac{1}{8}a_0, \quad a_3 = -\frac{1}{6}a_2 = -\frac{1}{48}a_0, \quad a_4 = -\frac{1}{8}a_3 = \frac{1}{384}a_0, \quad \ldots
\end{aligned}$$

Since $y(0) = 1$ we know that $a_0 = 1$ so

$$a_0 = 1, \quad a_1 = -\frac{1}{2}, \quad a_2 = \frac{1}{8}, \quad a_3 = -\frac{1}{48}, \quad a_4 = \frac{1}{384}, \quad \ldots$$

We conclude by writing the Taylor series approximation

$$y(t) = 1 - \frac{1}{2}t + \frac{1}{8}t^2 - \frac{1}{48}t^3 + \frac{1}{384}t^4 + \cdots.$$

We can observe that this Taylor series can be rewritten as

$$y(t) = 1 + \left(-\frac{t}{2}\right) + \frac{1}{2!}\left(-\frac{t}{2}\right)^2 + \frac{1}{3!}\left(-\frac{t}{2}\right)^3 + \frac{1}{4!}\left(-\frac{t}{2}\right)^4 + \cdots = e^{-0.5t}$$

hence recognizing the analytic solution that we knew from separation of variables.

Problem 10.19. Assume that y is represented as a power series $y(t) = \sum_{n=0}^{\infty} a_n t^n$ and find the coefficients a_n for the solution to the differential equation $y' - ty = 1$ with $y(0) = 1$. ▲

Problem 10.20. Assume that y is represented as a power series $y(t) = \sum_{n=0}^{\infty} a_n t^n$ and find the coefficients a_n for the solution to the differential equation

$$t^2y'' + ty' + t^2y = 0 \quad \text{with} \quad y(0) = 1.$$

This function is called a **Bessel function of the first kind** and shows up in the study of wave phenomenon. Use a graphing utility (and enough terms in the series) to show a plot of $y(t)$ on the domain $t \in [0, 6]$. ▲

Problem 10.21. Write computer code to generate the plot of the Bessel function in the previous problem up to any amount of accuracy. ▲

Problem 10.22. Use power series to solve the equation $y'' + ty = 0$ with $y(0) = 1$ and $y'(0) = 1$. This differential equation gives rise to a function called the **Airy equation**. We saw it as the solution to one of the lab problems at the beginning of the semester. ▲

Problem 10.23. Find the first four terms in the power series expansion of the solution to the differential equation

$$\left(t^2+1\right)y''-4ty'+6y=0 \quad \text{with} \quad y(0)=y'(0)=1.$$

▲

Problem 10.24. Determine the radius of convergence for the solutions to each of the previous power series problems. ▲

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10.4 Additional Exercises

Chapter 11

Partial Differential Equations

This brief document contains class notes, explanations, examples, and problems for our brief introduction to partial differential equations (PDEs). The study of PDEs spans all fields of the mathematical sciences, and like ODEs, PDEs are the language used by scientists to model change. The change this time happens simultaneously in all three spatial dimensions as well as time.

11.1 Some Reminders from Multivariable Calculus

Problem 11.1. Let $f(x,y)$ be a differentiable multivariable function. Which of the following is the gradient of f?

(a) $\nabla f = \left\langle \frac{\partial f}{\partial y}, \frac{\partial f}{\partial x} \right\rangle$

(b) $\nabla f = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$

(c) $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$

(d) $\nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$

▲

Problem 11.2. Let $\mathcal{F}(x,y)$ be a vector function so that

$$\mathcal{F}(x,y) = \langle f_1(x,y), f_2(x,y) \rangle.$$

Which of the following is the divergences of $\mathcal{F}$?

(a) $\nabla \cdot \mathcal{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}$

(b) $\nabla \cdot \mathcal{F} = \dfrac{\partial^2 f_1}{\partial x^2} + \dfrac{\partial^2 f_2}{\partial y^2}$

(c) $\nabla \cdot \mathcal{F} = \left\langle \dfrac{\partial f_1}{\partial y} + \dfrac{\partial f_2}{\partial x}, \dfrac{\partial f_1}{\partial x} + \dfrac{\partial f_2}{\partial y} \right\rangle$

(d) $\nabla \cdot \mathcal{F} = \left\langle \dfrac{\partial f_1}{\partial x}, \dfrac{\partial f_2}{\partial y} \right\rangle$

▲

Problem 11.3. Let $f(x,y)$ be a twice differentiable function. Which of the following is the result of taking the divergence of the gradient of f?

(a) $\nabla \cdot \nabla f = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}$

(b) $\nabla \cdot \nabla f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2}$

(c) $\nabla \cdot \nabla f = \left\langle \dfrac{\partial^2 f}{\partial x^2}, \dfrac{\partial^2 f}{\partial y^2} \right\rangle$

(d) $\nabla \cdot \nabla f = \left\langle \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y} \right\rangle$

▲

Problem 11.4. The last quarter of multivariable calculus contains some beautiful properties from vector calculus. Of particular interest here is the divergence theorem:

$$\iint \mathbf{q} \cdot n dA = \iiint \nabla \cdot \mathbf{q} dV.$$

In words, this theorem says

(a) The flux of a vector field **q** through the surface of an object is equal to how the vector field **q** spreads out within the object.

(b) The amount of work done by the vector field **q** is equal to how the vector field **q** curls within the object.

(c) The amount of work gained or lost by traveling around the exterior of the object is equal to the amount that the vector field **q** spreads out within the object.

(d) The flux of a vector field **q** through the surface of an object is equal to how the vector field **q** curls within the object.

▲

Problem 11.5. The heat equation (which we will derive in a bit) is

$$\frac{\partial u}{\partial t} = k \nabla \cdot \nabla u.$$

If $u(x,y,z,t)$ is the temperature of an object then what does the heat equation say in words?
▲

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11.2 Where to PDEs Come From?

This somewhat lengthy section is meant to be an introduction to many of the primary partial differential equations of interest in basic mathematical physics.

We start with a brief derivation of a *general conservation law*. The result being a partial differential equation that can be used for conservation of mass, momentum, or energy. Let u be the quantity you are trying to conserve, $\mathbf{q}$ be the flux of that quantity, and f be any source of that quantity. For example, if we are to derive a conservation of energy equation, u might be energy, $\mathbf{q}$ might be temperature flux, and f might be a temperature source (or sink).

11.2.1 Derivation of General Balance Law

Let Ω be a fixed volume and denote the boundary of this volume by $\partial\Omega$. The rate at which u is changing in time throughout Ω needs to be balanced by the rate at which u leaves the volume plus any sources of u. Mathematically, this means that

$$\frac{\partial}{\partial t}\iiint_{\Omega} u\,dV = -\iint_{\partial\Omega} \mathbf{q}\cdot n\,dA + \iiint_{\Omega} f\,dV. \tag{11.1}$$

This is a global balance law in the sense that it holds for all volumes Ω. The troubles here are two fold: (1) there are many integrals, and (2) there are really two variables (u and q since $f = f(u,x,t)$) so the equation is not closed. In order to mitigate that fact we apply the divergence theorem to get

$$\frac{\partial}{\partial t}\iiint_{\Omega} u\,dV = -\iiint_{\Omega} \nabla\cdot\mathbf{q}\,dV + \iiint_{\Omega} f\,dV. \tag{11.2}$$

Gathering all of the terms on the right of (11.2), interchanging the integral and the derivative on the left (since the volume is not changing in time), and rewriting gives

$$\iiint_{\Omega}\left(\frac{\partial u}{\partial t} + \nabla\cdot\mathbf{q}\right)dV = \iiint_{\Omega} f\,dV \tag{11.3}$$

If we presume that this equation holds for all volumes Ω then the integrands must be equal and we get the local balance law

$$\boxed{\frac{\partial u}{\partial t} + \nabla\cdot\mathbf{q} = f.} \tag{11.4}$$

In particular, the physics of energy, momentum, and mass transport are governed by (11.4). In each of these instances we need to have a suitable functional form of the flux $\mathbf{q}$. In the following subsection we will discuss one common form of $\mathbf{q}$.

11.2.2 Simplifications of the Local Balance Law

If equation (11.4) it is often assumed that the system is free of external sources. In this case we set f to zero and obtain the source-free balance law

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{q} = 0. \tag{11.5}$$

It is this form of balance law where many of the most interesting and important partial differential equations come from. In particular consider the following two cases: mass balance and energy balance.

Mass Balance

In mass balance we take u to either be the density of a substance (e.g. in the case of liquids) or the concentration of a substance in a mixture (e.g. in the case of gasses). If C is the mass concentration of a substance in a gas then the flux of that substance is given via Fick's Law as

$$\mathbf{q} = -k\nabla C. \tag{11.6}$$

Combining (11.6) with (11.5) (and assuming that k is independent of space, time, and concentration) gives

$$\frac{\partial C}{\partial t} = k\nabla \cdot \nabla C. \tag{11.7}$$

In the presenence of external sources of mass, (11.7) is

$$\frac{\partial C}{\partial t} = k\nabla \cdot \nabla C + f(x). \tag{11.8}$$

Problem 11.6. What does (11.8) equation look like in terms of spatial derivatives on the right-hand side?

$$\frac{\partial C}{\partial t} = \underline{\hspace{4cm}} \quad \text{(1 Spatial Dimension)}$$
$$\frac{\partial C}{\partial t} = \underline{\hspace{4cm}} \quad \text{(2 Spatial Dimensions)}$$
$$\frac{\partial C}{\partial t} = \underline{\hspace{4cm}} \quad \text{(3 Spatial Dimensions)}$$

▲

Energy Balance

The energy balance equation is essentially the same as the mass balance equation. If u is temperature then the flux of temperature is given by Fourier's Law for heat conduction

$$q = -k\nabla T. \tag{11.9}$$

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Making the same simplifications as in the mass balance equation we arrive at

$$\frac{\partial T}{\partial t} = k\nabla \cdot \nabla T. \tag{11.10}$$

In the presence of external sources of heat, (11.10) becomes

$$\frac{\partial T}{\partial t} = k\nabla \cdot \nabla T + f(x). \tag{11.11}$$

Problem 11.7. What does (11.11) equation look like in terms of spatial derivatives on the right-hand side?

$$\frac{\partial T}{\partial t} = \underline{\hspace{6cm}} \quad \text{(1 Spatial Dimension)}$$
$$\frac{\partial T}{\partial t} = \underline{\hspace{6cm}} \quad \text{(2 Spatial Dimensions)}$$
$$\frac{\partial T}{\partial t} = \underline{\hspace{6cm}} \quad \text{(3 Spatial Dimensions)}$$

▲

11.2.3 Laplace's Equation and Poisson's Equation

Equations (11.8) and (11.11) are the same partial differential equation for two very important physical phenomenon; mass and heat transfer. In the case where time is allowed to run to infinity and no external sources of mass or energy are included these equations reach a steady state solution (no longer changing in time) and we arrive at Laplace's Equation

$$\nabla \cdot \nabla u = 0. \tag{11.12}$$

Laplace's equation is actually a statement of minimal energy as well as steady state heat or temperature. We can see this since entropy always drives systems from high energy to low energy, and if we have reached a steady state then we must have also reached a surface of minimal energy.

Problem 11.8. Equation (11.12) is sometimes denoted as $\nabla \cdot \nabla u = \nabla^2 u = \Delta u$, and in terms of the partial derivatives it is written as

$$0 = \underline{\hspace{6cm}} \quad \text{(1 Spatial Dimension)}$$
$$0 = \underline{\hspace{6cm}} \quad \text{(2 Spatial Dimensions)}$$
$$0 = \underline{\hspace{6cm}} \quad \text{(3 Spatial Dimensions)}$$

▲

If there is a time-independent external source the the right-hand side of (11.12) will be non-zero and we arrive at Poisson's equation:

$$\nabla \cdot \nabla u = -f(x). \tag{11.13}$$

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Note that the negative on the right-hand side comes from the fact that $\frac{\partial u}{\partial t} = k\nabla \cdot \nabla u + f(x)$ and $\frac{\partial u}{\partial t} \to 0$. Technically we are taking absorbing the constant k into f (that is "f" is really "f/k"). Also note that in many instances the value of k is not contant and cannot therefore be pulled out of the derivative without a use of the product rule.

We will start our exploration of numerical PDEs with Laplace's and Poisson's equations. We will then layer on the temporal derivatives to explore mass and heat transport. Finally, we will explore wave phenomena as well as advection-diffusion transport models.

11.3 The 1D Heat Equation

In this section we will discuss analytic techniques for solving the heat equation in 1 spatial dimension. Most of this section is paraphrased from Richard Haberman's *Applied Partial Differential Equations* text [2].

We can think of the heat equation physically as tracking the heat diffusion in a thin rod of length L. Hence, in 1 spatial dimension equation (11.10) becomes

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \text{with } 0 < x < L \text{ and } t > 0. \tag{11.14}$$

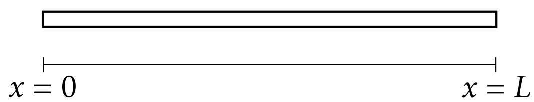

Figure 11.1. Sample geometry for the 1D heat equation (11.14).

In each of the subsequent subsections of this document we will explore different boundary conditions for equation (11.14). The boundary conditions are a way to prescribe how the heat is transfering or is otherwise being controlled at the ends of the rod.

11.3.1 1D Heat Equation with Zero Temperature Ends

For our first case, consider a 1D rod as in Figure 11.1 with $u(0,t) = 0$ and $u(L,t) = 0$. That is, let's assume that the two ends of the rod are in an ice bath held at exactly 0° (See Figure 11.2).

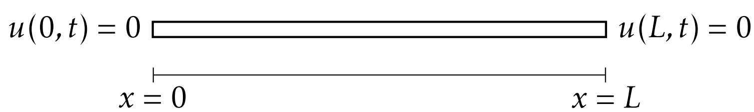

Figure 11.2. Sample geometry for the 1D heat equation (11.14).

Problem 11.9. If we were to solve the 1D heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

with boundary conditions as in Figure 11.2, what other information would we need, aside from k, in order to get a physically meaningful solution?

(a) $u(L/2,t)$: a fixed temperature at the midpoint

(b) $u(x,0)$: an initial temperature profile along the rod

(c) $u(x,\infty)$: the steady state temperature profile

(d) $u(0,t)$: the way that the heat evolves at $x=0$

▲

Problem 11.10. Assume that $u(x,0)=100$. That is, assume that the rod is initially heated to 100° from end to end. Further assume that the boundary conditions are $u(0,t) = u(L,0) = 0$ just as in Figure 11.2. Draw several curves clearly showing how the temperature in the rod will evolve in time. ▲

Separation of Variables

With ordinary differential equation we origianlly saw separation of variables with relatively *simple* differential equations. It turns out that we can do the same thing the heat equation. We start by assuming that in order to solve $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ we assume that

$$u(x,t) = \phi(x)G(t) \tag{11.15}$$

where $\phi(x)$ is ONLY a function of space (x) and $G(t)$ is ONLY a function of time (t). This technique was invented in the 1700's, and it works becuase it reduces the PDE to two ODEs.

Problem 11.11. Let's see what happens under assumption (11.15):
Let $u(x,t)=\phi(x)G(t)$ and

- write $\dfrac{\partial u}{\partial t}$ in terms of the functions ϕ and G,
- write $\dfrac{\partial^2 u}{\partial x^2}$ in terms of the functions ϕ and G, and
- put your two answers into the heat equation $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$.
- Finally, separate the variables so that all of the expressions with G are on the left of the equation and all of the expressions with ϕ are on the right of the equation. (put the k with the G function ... trust me)

▲

The left-hand side of your result from the previous problem should only contain function of G and the right-hand side should only contain functions of ϕ. The strange thing is that the equal sign is still valid. How can this be? The left-hand side is only a function of t and the right-hand side is only a function of x.

Problem 11.12. From the previous question you should have arrived at

$$\frac{1}{kG}\frac{dG}{dt} = \frac{1}{\phi}\frac{d^2\phi}{dx^2}.$$

The left-hand side is only a function of t and the right-hand side is only a function x but the equal sign is absolutely true for all x and for all t. How can this be? ▲

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Problem 11.13. From the separation of variables we arrive at two ordinary differential equations:

$$\frac{d^2\phi}{dx^2} = -\lambda\phi \tag{11.16}$$

$$\frac{dG}{dt} = -\lambda k G. \tag{11.17}$$

What types of behavior do you expect out of the solutions for equations (11.16) and (11.17).

(a) Equation (11.16): exponential decay
Equation (11.17): oscillations modeled by trig functions

(b) Equation (11.16): over damped system modeled by exponential functions
Equation (11.17): exponential decay

(c) Equation (11.16): critically damped system modeled by exponential functions
Equation (11.17): exponential decay

(d) Equation (11.16): oscillations modeled by trig functions
Equation (11.17): exponential decay

▲

Problem 11.14. Solve the time-dependent equation

$$\frac{dG}{dt} = -\lambda k G$$

where λ is (at the moment) an unknown constant. What happens if $\lambda > 0$, if $\lambda = 0$, and if $\lambda < 0$? ▲

Note: We don't expect to have solutions that grow exponentially in time so we should expect that $\lambda \ge 0$

Problem 11.15. Now we're going to solve the spatial boundary-valued problem

$$\frac{d^2\phi}{dx^2} = -\lambda\phi \quad \text{with} \quad \phi(0) = 0 \text{ and } \phi(L) = 0.$$

- Assume that $\phi(x) = e^{rx}$, find the characteristic polynomial, and find the two linearly independent solutions (these will contain λ).
- Write the solution to ϕ as a linear combination of the two linearly independent solutions.
- Apply the left-hand boundary condition $\phi(0) = 0$ to get one of the constants.
- Apply the right-hand boundary condition $\phi(L) = 0$ and find the equation that must be true in order for this boundary condition to be satisfied.

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- What must λ be equal to in order for the previous equation to be satisfied?
- Write the solution to $\phi(x)$.

▲

Let me interject here for a few sentences:
Let's put the pieces together. From Problem 11.14 we know that

$$G(t) = C_1 e^{-\lambda k t} \tag{11.18}$$

and from Problem 11.15 we know that

$$\phi(x) = C_2 \sin\left(\frac{n\pi x}{L}\right). \tag{11.19}$$

Since we are assuming that $u(x,t) = \phi(x)G(t)$ we can multiply the time dependent solution $G(t)$ and the spatial solution $\phi(x)$ to get

$$u(x,t) = A_n e^{-k(n\pi/L)^2 t} \sin\left(\frac{n\pi x}{L}\right) \quad \text{for} \quad n = 1,2,3,\ldots. \tag{11.20}$$

Strangely enough, there is a solution for every natural number $n = 1,2,3,\ldots$. This is certainly the first time we've encountered this, but we know something that will help: the derivative is a linear operator so the sum of two solutions must also be a solution. Carrying this to it's logical end gives the final solution to the 1D heat equation with homogeneous boundary conditions:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} A_n e^{-k(n\pi/L)^2 t} \sin\left(\frac{n\pi x}{L}\right).} \tag{11.21}$$

This is a rather complicated solution so let's apply it to an example so we can see how it works.

Example 11.16. Solve the 1D heat equation with the following initial and boundary conditions:

$$\begin{aligned} \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} &\quad \text{with } t > 0 \text{ and } 0 < x < 1 \\ k = 1 &\quad \text{(this is called the thermal diffusivity)} \\ u(0,t) = 0 &\quad \text{for } t > 0 \\ u(L,t) = 0 &\quad \text{for } t > 0 \\ u(x,0) = 100 &\quad \text{for } 0 < x < 1. \end{aligned}$$

See Figure 11.2 for a schematic of the problem. The following problems will walk you through the solution.

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Problem 11.17. We know that the general solution is:

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi/1)^2 t} \sin\left(\frac{n\pi x}{1}\right).$$

At time $t = 0$ we have

$$100 = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

since the exponential function evaluates to zero at $t = 0$. If we multiply both sides by $\sin\left(\frac{m\pi x}{L}\right)$ and integrate from 0 to 1 what do we get? (Assume that m is not necessarily the same as n). ▲

There are some really conventient trig identities that we can use next:

$$\boxed{\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0, & m \neq n \\ L/2, & m = n \end{cases}} \tag{11.22}$$

Problem 11.18. Using equation (11.22) and the result from the previous problem we can evaluate the integrals from the previous problem to get an expression for A_n. You may need to recall that $\cos(n\pi) = (-1)^n$ to find a pattern for A_n. Write down the pattern for A_n. ▲

Problem 11.19. Using your pattern from the previous problem we can expand the solution (11.21) as

$$u(x,t) = A_1 e^{-\pi^2 t}\sin(\pi x) + A_2 e^{-(2\pi)^2 t}\sin(2\pi x) + A_3 e^{-(3\pi)^2 t}\sin(3\pi x) + \dots.$$

Write several of the terms using your patter. Then use technology to plot approximate solutions to this problem. An example plot of the time-dependent solution is shown in Figure 11.3. ▲

Summary of Separation of Variables

The following steps were used throughout the last several problems to solve a linear homogeneous PDE with linear and homogeneous boundary conditions.

1. Temporarily ignore the initial condition.
2. Separate the variables by assuming that $u(x,t) = G(t)\phi(x)$.
   (a) Rearrange the PDE to separate the functions G and ϕ.
   (b) Introduce a separation constant $-\lambda$.
   (c) Write the two separate ODEs as eigenvalue problems.
3. Determine the separation constants as eigenvalues of a boundary value problem.

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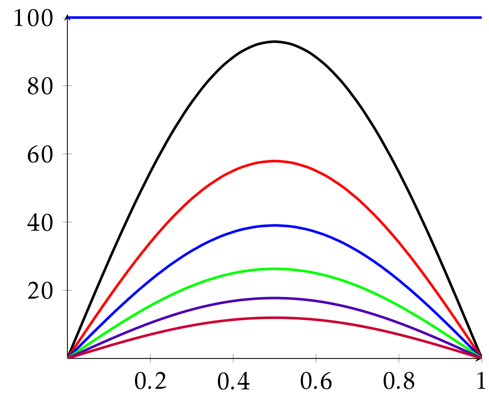

Figure 11.3. Snapshots of the solution to Example 11.16.

4. Solve the other differential equations. Record all products of solutions of the PDE obtainable by this method.
5. Apply the principle of superposition: the general solution is a linear combination of all of the individual solutions.
6. Satisfy the initial conditions:
   (a) substitute $t = 0$ into the general solution
   (b) multiply both sides of the resulting equation by an appropriate function (it should be another basis function from the same vector space that builds the infinite sum)
   (c) integrate and take advantage of orthogonality
   (d) use the resulting integrals to find a pattern in the coefficients.
7. Use software to plot the solutions as they evolve over time.

Now it is your turn. The following two problems allow you to put these ideas to the test.

Problem 11.20. Solve the 1D heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

subject to the boundary conditions $u(0,t) = u(1,t) = 0$ with initial temperature profile $u(x,0) = 6\sin(9\pi x)$. Start by making a sketch of several time steps of the solution using what you now about the physics of the problem. ▲

Problem 11.21. Solve the 1D heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

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subject to the boundary conditions $u(0,t) = u(1,t) = 0$ with initial temperature profile $u(x,0) = x(1-x)$. Start by making a sketch of several time steps of the solution using what you now about the physics of the problem. ▲

Problem 11.22. So far you have noticed that the spatial differential equation in ϕ turns out to be an eigenvalue problem
$$\frac{d^2\phi}{dx^2} = -\lambda\phi.$$
Determine the eigenvalues of the problem with the following sets of boundary conditions:

(a) $\phi(0) = 0$ and $\phi(\pi) = 0$

(b) $\phi(0) = 0$ and $\phi(1) = 0$

(c) $\frac{d\phi}{dx}(0) = 0$ and $\frac{d\phi}{dx}(1) = 0$

(d) $\phi(0) = 0$ and $\frac{d\phi}{dx}(1) = 0$

Each of these relates to a physical problem, so now go back through problems (a) - (d) and express each problem as boundary conditions for a 1D heat conducting rod. What do these boundary conditions mean physically (i.e. how are we controlling the heat at the ends of the rod)? ▲

Problem 11.23. Solve the heat equation with homogenous boundary conditions on a rod of unit length with initial condition

$$u(x,0) = \begin{cases} 0, & 0 < x < 1/3 \\ 100, & 1/3 < x < 2/3 \\ 0, & 2/3 < x < 1 \end{cases}$$

▲

11.3.2 1D Heat Equation with Insulated Ends

In this section we consider the heat equation again, but this time we consider the problem where the ends are insulated instead of being held at a fixed temperature. In Figure 11.4 we see that we are now not letting heat escape from the rod through the ends. This means that the energy within the rod from the initial condition will remain in the rod, but may spread out over time.

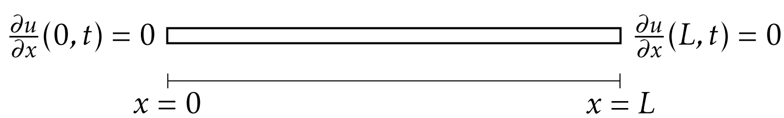

Figure 11.4. Sample geometry for the 1D heat equation (11.14) with Neumann (insulating) boundary conditions.

As you will soon see, the series solution will actually be in terms of cosines this time instead of sines. You will need the following identity:

$$\boxed{\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0, & n \neq m \\ \frac{L}{2}, & n = m \neq 0 \\ L, & n = m = 0 \end{cases}} \tag{11.23}$$

Problem 11.24. Solve the following 1D heat equation with Neumann boundary conditions.

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} \quad \text{with } t > 0 \text{ and } 0 < x < 1 \\ k &= 1 \quad \text{(this is called the thermal diffusivity)} \\ \frac{\partial u}{\partial x}(0,t) &= 0 \quad \text{for } t > 0 \\ \frac{\partial u}{\partial x}(L,t) &= 0 \quad \text{for } t > 0 \\ u(x,0) &= -\cos(2\pi x) + 1 \quad \text{for } 0 < x < 1. \end{aligned}$$

Start by sketching plots of the time evolution of the initial condition based solely on your physical intuition. Then follow the steps for solving a 1D PDE with separation of variables. ▲

11.3.3 Heat Equation on a Thin Ring

Now it is time for a new geometry. Let us formulate the appropriate initial boundary value problem for a thin wire (with lateral sides insulated) that is bent into the shape of a circle. We will let the wire have length $2L$ as shown in Figure 11.5. If the wire is thin enough then it is reasonable to assume that the temperature in the wire is constant along cross sections.

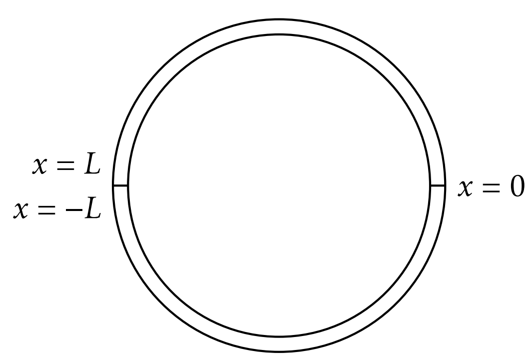

Figure 11.5. A thin circular wire of length $2L$.

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The formulation for the heat equation in this case is:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} \quad \text{with } t>0 \text{ and } -L<x<L \\
u(-L,t) &= u(L,t) \quad \text{(since the heat must match at } x=\pm L\text{)} \\
\frac{\partial u}{\partial x}(-L,t) &= \frac{\partial u}{\partial x}(L,t) \quad \text{(since the derivative of the temp. must be continuous)} \\
u(x,0) &= f(x) \quad \text{for } -L<x<L.
\end{aligned}$$

Problem 11.25. After separating the variables we again have the eigenvalue problem

$$\frac{d^2\phi}{dx^2} = -\lambda\phi.$$

Choose the proper boundary conditions on this problem?

(a) $\phi(-L) = \phi(L) = 0$ and $\frac{d\phi}{dx}(-L) = \frac{d\phi}{dx}(L) = 0$

(b) $\phi(-L) = \phi(L)$ and $\frac{d\phi}{dx}(-L) = \frac{d\phi}{dx}(L)$

(c) $\phi(-L) + \phi(L) = 0$ and $\frac{d\phi}{dx}(-L) + \frac{d\phi}{dx}(L) = 0$

(d) The moon is made of cheese

▲

Problem 11.26. Solve the boundary value problem

$$\frac{d^2\phi}{dx^2} = -\lambda\phi$$

with the boundary conditions from the previous voting question. If you get multiple solutions (hint) then remember that your final solution is actually a linear combination of the solutions. ▲

Problem 11.27. The time problem $G(t)$ has the same solutions on a ring as it does for the 1D rod. Using this information as well as your solution to the spatial boundary value problem, write the full general solution to the heat equation on a thin ring. ▲

For the heat equation on a ring you should have found that the general solution is

$$u(x,t) = A_0 + \sum_{n=1}^{\infty}\left[e^{-k(n\pi/L)^2 t}\left(A_n\cos\left(\frac{n\pi x}{L}\right) + B_n\sin\left(\frac{n\pi x}{L}\right)\right)\right], \quad \text{with} \tag{11.24}$$

$$u(x,0) = f(x) = A_0 + \sum_{n=1}^{\infty}\left[A_n\cos\left(\frac{n\pi x}{L}\right) + B_n\sin\left(\frac{n\pi x}{L}\right)\right] \tag{11.25}$$

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In order to find the coefficients we need the following orthogonality identities:

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) = \begin{cases} 0, & n \neq m \\ L, & n = m \neq 0 \\ 2L, & n = m = 0 \end{cases} \tag{11.26}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0, & n \neq m \\ L, & n = m \neq 0 \end{cases} \tag{11.27}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) dx = 0 \text{ for all } n \text{ and } m. \tag{11.28}$$

Problem 11.28. Let's take a ring with $L = 1$ and $f(x) = \sin(2\pi x)$.

- Find the coefficients A_n by multiply by $\cos\left(\frac{m\pi x}{1}\right)$ and integrating from -1 to 1.
- Find the coefficients B_n by multiply by $\sin\left(\frac{m\pi x}{1}\right)$ and integrating from -1 to 1.

You are welcome to use technology to evaluate the integrals. If you can't get an exact formula for the patterns in A_n and B_n at least write down the first 8 or 10 terms of each sequence. ▲

Problem 11.29. For the ring in the last problem with $L = 1$ and $f(x) = \sin(2\pi x)$, write down several terms in the solution and use technology to make a plot of the time evolution. You should start by hand-sketching a plot showing the time evolution of the heat. ▲

11.4 Additional Exercises

. . . none yet . . .

Appendices

Appendix A

Partial Fractions

In this appendix we explore the algebraic notion of partial fractions. The idea is simple: How do we undo the addition of fractions?

Let's first consider some elementary arithmetic.

$$\frac{1}{3} + \frac{2}{5} = ?$$

To add the two fractions you need common denominators, in this case 15. We rewrite the fractions as

$$\frac{1}{3} + \frac{2}{5} = \frac{5}{15} + \frac{6}{15}$$

and now that we're comparing like parts we add the numerators to get

$$\frac{1}{3} + \frac{2}{5} = \frac{5}{15} + \frac{6}{15} = \frac{11}{15}.$$

What if we wanted to go the other way? That is, what if we have the fraction 11/15 and we wanted to know where it came from. If we consider the prime factorization of the denominator and conjecture that the fractions can be split up with these factors as the denominators of separate fractions then the problem becomes

$$\frac{11}{15} = \frac{11}{3 \cdot 5} = \frac{A}{3} + \frac{B}{5}$$

where A and B are just numbers that we need to find. Obviously there are many different answers to this inverse questions (since we can get infinitely many equivalent fractions). If we multiply both sides of this new equation by 15 we get

$$11 = 5A + 3B$$

and for each choice of one variable we get another. In particular, if we choose $A = 1$ then simple algebra tells us that $B = 2$ and we have successfully split the fraction 11/15 into the sum of 1/3 and 2/5.

Now let's consider the algebraic problem of taking a fraction and splitting it into a sum of fractions. The notion is still the same: conjecture that the factors of the denominator are the denominators of the separate fractions and then do some detective work to find the numerators. In the remainder of this appendix we'll give several examples of this idea. We leave it up to the reader to actually find the common denominators and do the algebra to verify that indeed the right-hand side from each example is equal to the left-hand side.

Example A.1. Use partial fractions to write $\frac{4}{x(x-3)}$ as a sum or difference of two fractions.
Solution: We start by writing the fraction as

$$\frac{4}{x(x-3)} = \frac{A}{x} + \frac{B}{x-3}.$$

This choice is made since if we were to find the common denominator of the right-hand side we would have the desired denominator on the left-hand side. Next we clear all of the fractions by multiplying the common denominator yielding

$$4 = A(x-3) + B(x).$$

At this point we know that the equal sign must be true for <u>all</u> values of x so we can choose some convenient values to tease out A and B.

- If $x = 3$ then $x-3 = 0$ and we get $4 = 3B$ which implies that $B = \frac{4}{3}$.
- If $x = 0$ then we get $4 = -3A$ which implies that $A = -\frac{4}{3}$

Therefore

$$\frac{4}{x(x-3)} = -\frac{4}{3x} + \frac{4}{3(x-3)}.$$

Example A.2. It can be shown that

$$\frac{6x}{(x-1)(x+1)(x+2)} = \frac{1}{x-1} + \frac{3}{x+1} - \frac{4}{x+2}.$$

Partial Justification: Start by observing that the denominator of the left-hand fraction is factored so we split into three fractions with the factors as the denominators:

$$\frac{6x}{(x-1)(x+1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+2}.$$

Clearing the fractions gives

$$6x = A(x+1)(x+2) + B(x-1)(x+2) + C(x-1)(x+1).$$

Now consider convenient choices of x

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- If $x = -1$ then:

$$-6 = A(0)(1) + B(-2)(1) + C(-2)(0) \implies B = 3.$$

- If $x = 1$ then:

$$6 = A(2)(3) + B(0)(3) + C(0)(2) \implies A = 1.$$

- If $x = -2$ then:

$$-12 = A(-1)(0) + B(-3)(0) + C(-3)(-1) \implies C = -4.$$

Hence

$$\frac{6x}{(x-1)(x+1)(x+2)} = \frac{1}{x-1} + \frac{3}{x+1} - \frac{4}{x+2}.$$

Example A.3. In this problem we will see repeated linear factors.

$$\frac{x^2+1}{x(x-1)^3} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}$$

Notice that the repeated factor gets repeated for all powers. Let's clear the fractions just as before and see what we get

$$x^2 + 1 = A(x-1)^3 + Bx(x-1)^2 + Cx(x-1) + Dx.$$

If we take $x = 0$ then $A = -1$. If we take $x = 1$ then $D = 2$. However, you'll notice that these two choices do not allow us to easily find B and C so we expand the polynomial on the right-hand side, gather like terms, and match coefficients. That is

$$\begin{aligned} x^2+1 &= A\left(x^3 - 3x^2 + 3x - 1\right) + B\left(x^3 - 2x^2 + x\right) + C\left(x^2 - x\right) + Dx \\ \implies x^2+1 &= (A+B)x^3 + (-3A-2B+C)x^2 + (3A+B-C+D)x - A \end{aligned}$$

Matching the coefficients of like terms we get

$$\begin{aligned} A + B &= 0 && \text{(cubic terms)} \\ -3A - 2B + C &= 1 && \text{(quadratic terms)} \\ 3A + B - C + D &= 0 && \text{(linear terms)} \\ -A &= 1 && \text{(constant terms)} \end{aligned}$$

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Since $A = -1$ we must have $B = 1$ and therefore $C = 0$. Therefore

$$\frac{x^2+1}{x(x-1)^3} = -\frac{1}{x} + \frac{1}{x-1} + \frac{0}{(x-1)^2} + \frac{2}{(x-1)^3}$$

Example A.4. In this final example we'll show what happens with an irreducible quadratic.

$$\frac{x-3}{x(x^2+3)} = \frac{A}{x} + \frac{Bx+C}{x^2+3}.$$

Notice that the numerator associated with the irreducible quadratic is a linear function with unknown parameters. Clearing the fractions we get

$$x - 3 = A(x^2+3) + (Bx+C)(x).$$

If we take $x = 0$ then $-3 = 3A$ which implies that $A = -1$. Expanding both sides of the equation and matching like terms gives

$$x - 3 = (A+B)x^2 + Cx + 3A$$

which implies that $A + B = 0$ and $C = 1$. Therefore $B = 1$ and

$$\frac{x-3}{x(x^2+3)} = -\frac{1}{x} + \frac{x+1}{x^2+3}.$$

Technique A.5 (Partial Fractions Decomposition)**.** Below are several cases of fractions that require partial fractions along with their separated forms.

$$\begin{aligned}
\frac{px+q}{(x-a)(x-b)} &= \frac{A}{x-a} + \frac{B}{x-b} \quad \text{(for } a \neq b\text{)} \\
\frac{px+q}{(x-a)^2} &= \frac{A}{x-a} + \frac{B}{(x-a)^2} \\
\frac{px+q}{(x-a)^3} &= \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{(x-a)^3} \\
\frac{px^2+qz+r}{(x-a)(x-b)(x-c)} &= \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} \\
\frac{px^2+qz+r}{(x-a)^2(x-b)} &= \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b} \\
\frac{px^2+qx+r}{(x-a)(x^2+bx+c)} &= \frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c} \quad \text{(where } x^2+bx+c \text{ cannot be factored)}
\end{aligned}$$

Problem A.6. Now let's use partial fractions to solve a separable differential equation.

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Solve the logistic population equation

$$\frac{dP}{dt} = 0.2P\left(1 - \frac{P}{10}\right) \quad \text{with} \quad P(0) = 1$$

Start by separating the variables (leaving the 0.2 on the right) and then looking up the appropriate partial fractions decomposition for splitting up the fraction that appears. After that you'll get to do a whole bunch of algebra ... have fun!! ▲

Appendix B

MATLAB Basics

In this appendix we'll go through a few of the basics in MATLAB. This is by no means meant to be an all-encompassing resource for MATLAB programming. A few more thorough resources for MATLAB are listed here.

- https://www.mathworks.com/help/pdf_doc/matlab/matlab_prog.pdf
- https://www.mathworks.com/products/matlab/examples.html
- https://en.wikibooks.org/wiki/MATLAB_Programming
- http://gribblelab.org/scicomp/scicomp.pdf (this is a personal favorite)

In this appendix we'll give examples of some of the more common coding practices that the reader will run into while working through the exercises and problems in these notes.

B.1 Vectors and Matrices

Example B.1. Write the vectors $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 4 & 5 & 6 & 7 \end{pmatrix}$ using MATLAB.

Solution:

```
v = [1 ; 2 ; 3]
w = [4 , 5 , 6 , 7]
w = 4:7  % this is shorthand for writing a sequence as a row vector
```

Example B.2. Consider the matrices and vectors

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 5 & 7 \\ 9 & 1 & 3 \\ 5 & 7 & 11 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 4 & 3 & -1 \end{pmatrix}$$

- Calculate the product AB using regular matrix multiplication

```
A = [1 , 2 , 3;
     4 , 5 , 6;
     7 , 8 , 0]
B = [3 , 5 , 7;
     9 , 1 , 3;
     5 , 7 , 11]
Product = A*B
```

- Calculate the element-by-element multiplication of A and B

```
ElementWiseProduct = A .* B
```

- Calculate the inverse of A

```
Ainv = A^(-1)
Ainv = inverse(A) % alternative
```

- Calculate the transpose of B

```
Atranspose = transpose(A)
% or as an alternative:
Atranspose = A'  % actually the conjugate transpose but if A is real then ok
```

- Solve the system of equations $A\mathbf{x} = \mathbf{b}$

```
b = [4 ; 3 ; -1]
x = A \ b
```

Example B.3. Code for a matrix of zeros

```
Z = zeros(5,5) % 5 x 5 matrix of all zeros
```

Example B.4. Code for an identity matrix

```
Ident = eye(5,5) % 5 x 5 identity matrix
```

Example B.5. Code for random matrices.

- random matrix from a uniform distribution on $[0,1]$

```
R = rand(5,5) % random 5 x 5 matrix
```

- random matrix from the standard normal distribution

```
R = randn(5,5) % random 5 x 5 matrix
```

Example B.6. A linearly spaced sequence

```
List = linspace(0,10,100)
% a list of 100 equally spaced numbers from 0 to 10
```

B.2 Looping

A loop is used when a process needs to be repeated several times.

B.2.1 For Loops

A `for loop` is code that repeats across a pre-defined sequence.

Example B.7. Write a loop that produces the squares of the first 10 integers.

```
for j = 1:10
    j^2
end
```

The output of this code will be

```
1
4
9
16
25
36
49
64
81
100
```

Example B.8. Plot the functions $f(x) = \sin(kx)$ for $k = 1, 1.5, 2, 2.5, \ldots, 5$ on the domain $x \in [0, 2\pi]$.

```
x = linspace(0,2*pi,1000);
for k = 1:0.5:5
    plot(x , sin(k*x) )
    hold on
end
```

B.2.2 The While Loop

A `while loop` is a process that only repeats while a conditional statement is true. Be careful with `while loops` since it is possible to create a loop that runs forever.

Example B.9. Build the Fibonnaci sequence up until the last term is greater than 1000.

```
F(1) = 1; % first term
F(2) = 1; % second term
n = 3;
while F(end)<1000
    F(n) = F(n-1) + F(n-2);
    n=n+1;
end
```

Example B.10. An example of a while loop that runs forever.

```
a = 1;
```

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```
while a>0
    a=a+1;
end
```

Example B.11. An example of a while loop that runs forever but with a failsafe step that stops the loop after 1000 steps.

```
a = 1;
counter=1;
while a>0
    a=a+1;
    if counter >= 1000
        break
    end
    counter = counter+1;
end
```

B.3 Conditional Statements

Conditional statements are used to check if something is true or false. The output of a conditional statement is a boolean value; true (1) or false (0).

B.3.1 If Statements

Example B.12. Loop over the integers up to 100 and output only the multiples of three.

```
for j = 1:100
    if mod(j,3) == 0
        j
    end
end
```

Example B.13. Check the signs of two function values and determine if they are opposite.

```
f = @(x) x^3*(x-3);
a = 2;
b = 4;
if f(a)*f(b) < 0
```

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```
    fprintf('The function values are opposite sign\n')
elseif f(a)*f(b) >0
    fprintf('The function values are the same sign\n')
else
    fprintf('The function values are both zero\n')
end
```

B.3.2 Case-Switch Statements

Example B.14. Evaluate over several cases.

```
n = 3
switch n
    case 1 % if n == 1
        fprintf('n is 1\n')
    case 2 % if n == 2
        fprintf('n is 2\n')
    case 3 % if n == 3
        fprintf('n is 3\n')
end
```

B.4 Functions

A mathematical function has a single output for every input, and in some sense a computer function is the same: one single executed process for each collection of inputs.

Example B.15. Define the function $f(x) = \sin(x^2)$ so that it can accept any type of input (symbol, number, or list of numbers).

```
f = @(x) sin(x.^2) % defines the function
f(3) % evaluates the function at x=3
x=linspace(0,pi,100);
f(x) % evaluates f at 100 points equally spaced from 0 to pi
```

Example B.16. Write a computer function that accepts two numbers as inputs and outputs the sum plus the product of the two numbers.
First write a file with the following contents.

```
function MyOutput = MyFunctionName(a,b)
    MyOutput = a + b + a*b;
end
```

Be sure that the file name is the same as the function name.
Then you can call the function by name in a script or another function.

```
SumPlusProduct = MyFunctionName(3,4)
```

which will output the number 19.

Example B.17. Write a function with three inputs that outputs the sum of the three. The third input should be optional and the default should be set to 5.

```
function AwesomeOutput = SumOfThree(a,b,c)
    if nargin < 3
        c = 5;
    end
    AwesomeOutput = a+b+c;
end
```

You can call this function with

```
SumOfThree(17,23)
```

which will output $17+25+5 = 47$. Notice that the third input was left off and a 5 was used in its place.

B.5 Plotting

In numerical analysis we are typically plotting numerically computed lists of numbers so as such we will give a few examples of this type of plotting here. We will not, however, give examples of symbolic plotting.

The `plot` command in MATLAB accepts a list of x values followed by a list of y values then followed by color and symbol options.
`plot(xlist , ylist , color options)`

Example B.18. Plot the function $f(x) = \sin(x^2)$ on the interval $[0, 2\pi]$ with 1000 equally spaced points. Make the plot color blue.

```
x = linspace(0,2*pi,1000);
f = @(x) sin(x.^2);
plot(x , f(x) , 'b')
```

Alternatively

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```
x = linspace(0,2*pi,1000);
y = sin(x.^2);
plot(x, y, 'b')
```

Example B.19. Make a 2×2 array of 4 plots of $f(x) = \sin(kx^2)$ for $k = 1, 2, 3, 4$.

```
x = linspace(0,2*pi,1000);
for k=1:4
    subplot(2,2,k)
    plot(x , sin(k*x.^2) , 'b')
end
```

Example B.20. Plot $f(x) = \sin(kx^2)$ for $k = 1, 2, \ldots, 10$ all on the same plot.

```
x = linspace(0,2*pi,1000);
for k=1:10
    plot(x, sin(k*x.^2))
    hold on % this holds the figure window open so you can write on top of it
end
```

Example B.21. Plot the function $f(x) = e^{-x}\sin(x)$ and put a mark at the local max at $x = \pi/4$.

```
x = linspace(0,2*pi,1000); % set up the domain
f = @(x) exp(-x) .* sin(x);
plot(x,f(x),'b',pi/4,f(pi/4),'ro')
```

B.6 Animations

Example B.22. Plot $f(x) = \sin(kx^2)$ for $k = 1$ to $k = 10$ by small increments with a short pause in between each step.

```
x = linspace(0,2*pi,1000);
for k=1:0.01:10 % 1 to 10 by 0.01
    plot(x, sin(k*x.^2))
    hold on % this holds the figure window open so you can write on top of it
    drawnow % draws the plot
```

```
    % the last line gives the illusion of animation
end
```

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Appendix C

LaTeX

In this appendix we give the basics of writing with LaTeX.

C.1 Equation Environments and Cross Referencing

When working with equations is it often times convenient and necessary to cross-reference the equations that you're talking about. A simple example is:

Example C.1. Recall the Pythagorean Theorem: If a and b are the legs of a right triangle and c is the hypotenuse, then

$$a^2 + b^2 = c^2. \tag{C.1}$$

Let $a = 3$ and $b = 4$ in equation (C.1). If that is the case then ...
The LaTeX code for this is

```
Recall the Pythagorean Theorem: If $a$ and $b$ are the legs of a
right triangle and $c$ is the hypotenuse, then
\begin{flalign}
    a^2 + b^2 = c^2
    \label{eqn:pythag}
\end{flalign}

Let $a=3$ and $b=4$ in equation \eqref{eqn:pythag}.
If that is the case then \dots
```

Note in Example C.1 that the equations and the equation reference are part of the sentence. In fact, these are always part of the grammatical structure of your writing.

Other numbered environments include `align`, `flalign`, `eqnarray`, `equation` and several others. The modern convention for LaTeX is to use `align` or `flalign` for all equations. If you want to use one of these environments without numbers then use the *. In otherwords `align*` will align in the same way without numbering the equations. If you only want a number on one line then you can use `\notag` at the beginning of that line.

To align equations use the “align” enviroment, which requires the amsmath package. Align supersedes eqnarray. The ampersands control the vertical alignment:

```
\begin{align}
\frac{\partial{x}}{\partial{s}}&=zx  & \text{with} && x(0)=1\\
\frac{\partial{y}}{\partial{s}}&=x^2y & \text{with} && y(0)=t\\
\frac{\partial{z}}{\partial{s}}&=xyz & \text{with} && z(0)=t^2
\end{align}
```

$$\frac{\partial x}{\partial s} = zx \qquad \text{with} \qquad x(0) = 1 \tag{C.2}$$
$$\frac{\partial y}{\partial s} = x^2 y \qquad \text{with} \qquad y(0) = t \tag{C.3}$$
$$\frac{\partial z}{\partial s} = xyz \qquad \text{with} \qquad z(0) = t^2 \tag{C.4}$$

A few more math-related typesetting examples are included below:

- Inline math with and without numbering

```
\[ \sum_{j=1}^\infty \frac{1}{j^2} = \frac{\pi^2}{6} \]
```

$$\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6}$$

```
\begin{flalign}
    \sum_{j=1}^\infty \frac{1}{j^2} = \frac{\pi^2}{6}
    \label{eqn:sample_equation}
\end{flalign}
```

$$\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6} \tag{C.5}$$

```
\begin{subequations}
    \begin{eqnarray}
        \sin \left( \frac{\pi}{6} \right) &=& \frac{\sqrt{3}}{2}
        \label{eqn:sine} \\
        \cos \left( \frac{\pi}{6} \right) &=& \frac{1}{2}
        \label{eqn:cosine}
    \end{eqnarray}
    \label{eqn:trig}
\end{subequations}
```

$$\sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \tag{C.6a}$$
$$\cos\left(\frac{\pi}{6}\right) = \frac{1}{2} \tag{C.6b}$$

This second example allows you to cross reference equations like (C.5) using (`\ref{eqn:sample_e` or, more simply, (C.5) using `\eqref{eqn:sample_equation}`. The third set of equations allows for multiple types of references. Like:
The sine equation, (C.6a) (`\eqref{eqn:sine}`), and the cosine equation, (C.6b) (`\eqref{eqn:cosine}`), are grouped together via equation (C.6) (`\eqref{eqn:trig}`).

- Matrices

  `\[ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \]` $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

  `\[ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \]` $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

  `\[ \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \]` $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

- Including basic graphics
  Be sure that the graphics file is in the same directory as your TeX file. Your picture should be a *.eps or *.pdf file. If not then some TeX compilers will complain (plus, *.jpg usually looks horrible).

```
\begin{center}
    \includegraphics[width=0.9\columnwidth]{filename.eps}
\end{center}
```

  OR

```
\begin{center}
    \includegraphics[height=3in]{filename.eps}
\end{center}
```

  There are many options for `\includegraphics`, but these two work for many pictures. Sometimes, though, it is desired to trim an image that you've saved from elsewhere. The basic syntax for `trim` and `clip` is

```
\begin{center}
    \includegraphics[trim = 1cm 2cm 1cm 2cm, clip=true,
        width=0.9\columnwidth]{filename.eps}
\end{center}
```

  The four measurements after the trim command are the amount to trim from the left, bottom, right, and top (in that order).

- Leaving white space:
  - horizontal Spacing: `\hspace{0.5in}`
  - vertical Spacing: `\vspace{2in}`

C.2 Tables, Tabular, Figures, Shortcuts, and Other Environments

C.2.1 Tables and Tabular Environments

Tables can be rather annoying in LATEX, but it is important to get the basics down before moving on.

Example C.2. In this example we want the table to be place *here* [h*], the first column left justified, the middle column centered, and the last column right justified with vertical bars between each column.

```
\begin{center}
    \begin{tabular}[h*]{|l|c|r|}
        \hline
        Title 1 & Title 2 & Title 3 \\ \hline \hline
        Hello & Ni Hao & Bonjour \\ \hline
        good bye & zia jian & adieux \\ \hline
    \end{tabular}
\end{center}
```

Title 1	Title 2	Title 3
Hello	Ni Hao	Bonjour
good bye	zia jian	adieux

In Example C.2 we used the tabular environment. This builds the table. If you want to build a table where there is a caption and the environment *floats* to various parts of the page then you need to use the `table` command.

Example C.3. In this example we build the same table as in Example C.2 but this time we allow it to float and we want a caption. The code is:

```
\begin{table}
    \centering
        \begin{tabular}[h*]{|l|c|r|}
            \hline
            Title 1 & Title 2 & Title 3 \\ \hline \hline
```

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Title 1	Title 2	Title 3
Hello	Ni Hao	Bonjour
good bye	zia jian	adieux

Table C.1. This is the amazing table of doom

```
            Hello & Ni Hao & Bonjour \\ \hline
            good bye & zia jian & adieux \\ \hline
        \end{tabular}
    \caption{This is the amazing table of doom}
    \label{tab:MyLabel}
\end{table}
```

C.2.2 Excel To LATEX

One tool that is often overlooked is the `ExcelToLaTeX` macro for Excel. I'm leaving this one up to you. Google `ExcelToLaTeX` , download it, add it to the macros for your version of Excel, and have fun with it. This tool will allow you to convert Excel-based tables to LATEX tables.

C.2.3 Figures

The `figure` environment in LATEX is almost identical to that for `table`. For example:

Example C.4. This figure simply shows a MatLab plot of the sine and cosine functions together in all of their shared glory. The file type was eps, which is notoriously hard to handle on Windows machines and on Overleaf. Be sure to use the `epstopdf` package if you're using epsfile types.

```
\begin{figure}[ht!]
    \centering
    \includegraphics[width=0.7\columnwidth]{SampleFigure.eps}
    \caption{Figure for Example \ref{ex:C3:fig}
    \label{fig:C3:fig}
\end{figure}
```

C.2.4 New Commands: Shortcuts are AWESOME!

You can save yourself a vast amount of typing by defining new commands which meet your specific need. It is easy. The newcommand command goes in the preamble (before the `\begin{document}`). The examples that follow are a few handy ones that I've used in the past. The world is your oyster here, so make any shortcut for a LATEX command that is cumbersome to type.

- Derivatives

```
\newcommand{\dd}[2] {\frac{d #1}{d #2}}
\newcommand{\ddd}[2] {\frac{d^2 #1}{d #2^2}}

\dd{y}{x}
\ddd{y}{x}
```

The results are:

$$\frac{dy}{dx}$$
$$\frac{d^2y}{dx^2}$$

- Partial derivatives

```
\newcommand{\pd}[2] {\frac{\partial{#1}}{\partial{#2}}}
\newcommand{\pdd}[2] {\frac{\partial^2{#1}}{\partial{#2}^2}}
\newcommand{\pddm}[3] {\frac{\partial^2{#1}}{\partial{#2}\partial{#3}}}

\pd{y}{x}
\pdd{y}{x}
\pddm{y}{x}{z}
```

The results are:

$$\frac{\partial y}{\partial x}$$
$$\frac{\partial^2 y}{\partial x^2}$$
$$\frac{\partial^2 y}{\partial x \partial z}$$

- Some of the common number sets

```
\newcommand{\cc}{\mathbb{C}}
\newcommand{\rr}{\mathbb{R}}
\newcommand{\nn}{\mathbb{N}}
\newcommand{\qq}{\mathbb{Q}}
\newcommand{\zz}{\mathbb{Z}}

\cc \hspace{1cm} \nn \hspace{1cm} \qq \hspace{1cm} \rr \hspace{1cm} \zz
```

The result is:

$$\mathbb{C} \hspace{1cm} \mathbb{N} \hspace{1cm} \mathbb{Q} \hspace{1cm} \mathbb{R} \hspace{1cm} \mathbb{Z}$$

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- Grouping symbols (parentheses, brackets, etc)

```
\newcommand{\lp}{\left(}
\newcommand{\rp}{\right)}
\newcommand{\lb}{\left[}
\newcommand{\rb}{\right]}

\lp\pdd{F}{x}\rp
compared to
(\pdd{F}{x})
```

The result is:

$$\left(\frac{\partial^2 F}{\partial x^2}\right)$$

compared to

$$(\frac{\partial^2 F}{\partial x^2})$$

- Common conjunctions

```
\newcommand{\andd}[1]{\quad\text{and}\quad}
\newcommand{\orr}[1]{\quad\text{or}\quad}
\newcommand{\forr}[1]{\quad\text{for}\quad}
\newcommand{\st}[1]{\quad\text{such that}\quad}
\newcommand{\conj}[1]{\quad\text{#1}\quad}

% Implies
\newcommand{\ra}{\quad\Rightarrow\quad}

A\ra B\st C\ne D
```

The result is:

$$A \quad \Rightarrow \quad B \quad \text{such that} \quad \neq D$$

C.3 Graphics in LATEX

In this chapter we will focus on several tools that extend your knowledge of figures beyond just `includegraphics` and move you toward the domain of professional publications. The tools that we'll cover are:

1. The `tikz` package,
2. The `pgfplots` package,
3. Using `GeoGebra` to generate `tikz` code, and

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4. Using MatLab to generate tikz code.

These tools take a lot of work, but the end result is well worth it.

There is nothing worse or more distracting than a poorly done figure.

There is more to these packages than we could possibly cover in a few days. It is imperative that you use the internet to its fullest extent with these packages. You can get yourself into a pickle with some of the internet-based examples, but starting with someone else's code for these packages is über helpful sometimes!

What I'll present here are simply a few examples to get you going.

C.3.1 The Tikz and PGFPlots Packages

The Tikz package is made for doing line drawings. The simplest mode of operation with Tikz is to do point-by-point drawings on a Cartesian grid.

Example C.5. Say we want to draw a coordinate plane with a few geometric shapes. Inside the figure environment we include a tikzpicture environment around the code for the picture. Be sure to end every Tikz line with a semicolon; Figure C.1 shows the results.

```
\begin{tikzpicture}
    \draw[color=gray] (-3,-3) grid (3,3);
    \draw[thick, <->] (-3,0) -- (3,0) node[anchor=west]{$x$};
    \draw[thick, <->] (0,-3) -- (0,3) node[anchor=south]{$y$};
    \draw[very thick, blue, fill=blue!50] (0,0) --
        (2,1) -- (1,3) -- cycle;
    \draw[very thick, dashed, color=red,
        fill=red!20!blue, opacity=0.5] (-2,0) circle(1cm);
\end{tikzpicture}
```

For more examples about the Tikz package, see http://www.texample.net/tikz/examples/ . . . texample is your new best friend.

You don't have to plot in MatLab, Excel, or any other tool when writing a technical document! Say this to yourself 100 times and be sure that you're sitting down.

Example C.6. This first example shows a simple way to plot functions.

```
\begin{tikzpicture}
    \begin{axis}[axis lines=center, xlabel={x},
          title={My Awesome Plot},
          domain=-2*pi:2*pi, ymin=-1.5, ymax=2, grid]
        \addplot[blue, thick, smooth] {sin(deg(x))};
        \addlegendentry{$f(x)=\sin(x)$};
        \addplot[red, thick, smooth] {cos(deg(x))};
```

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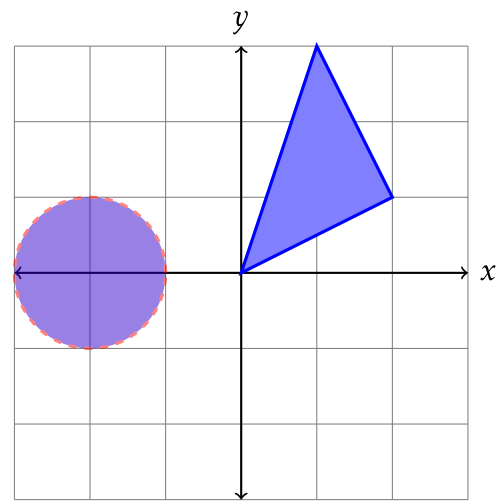

Figure C.1. A simple Tikz picture

```
        \addlegendentry{$g(x)=\cos(x)$};
        \addplot[black, dashed, thick, smooth] {0.1*exp(-x)};
        \addlegendentry{$h(x)=0.1\text{exp}(-x)$};
    \end{axis}
\end{tikzpicture}
```

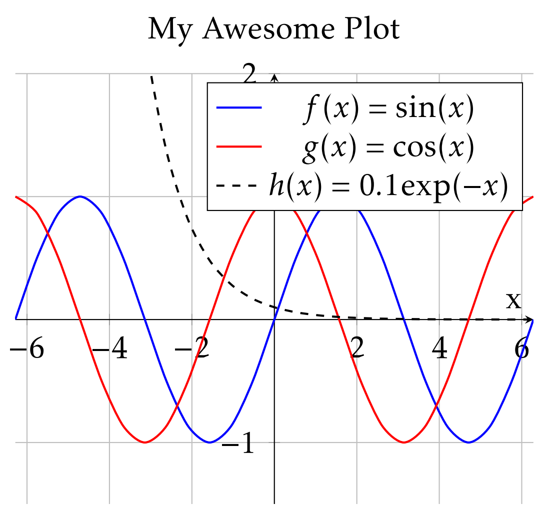

Figure C.2. A figure drawn with the `tikzpicture` and `axis` commands (leveraging the `pgfplots` package in the backgroud).

Next we'll follow with several more examples. Some of them are very advanced and some are beautifully simple.

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Example C.7. Draw a bar chart for the following table of the world's largest producers of gem-quality diamonds in 2010. The solution is shown in Figure C.3.

Country	Millions of Carats
Botswana	25.0
Russia	17.8
Angola	12.5
Canada	11.8
Congo	5.5

Souce: USGS Mineral Commodity Summaries.

```
\usetikzlibrary{patterns}
\pgfplotsset{width=12cm,height=8cm}
\begin{tikzpicture}
    \begin{axis}[
            ybar,
            bar width=10mm,
            enlargelimits=0.15,
            xlabel={\Large{Country}},
            ylabel={\Large{Millions of Carats}},
            title={\Large{World's Largest Diamond Producers 2010}},
            xtick=data,
            symbolic x coords={Botswana,Russia,Angola,Canada,Congo},
            nodes near coords,
            axis lines*=left
        ]
        \addplot [pattern=crosshatch dots,pattern color=red!80!white,
            draw=red] coordinates {(Botswana,25)
            (Russia,17.8) (Angola,12.5) (Canada,11.8) (Congo,5.5)};
    \end{axis}
\end{tikzpicture}
```

C.4 Bibliography Management

There are two primary ways to manage a bibliography file in LATEX. In both ways you need to remember that (as usual) you have full control over everything! Two rules of thumb:

1. If you are using a short bibliography or if this paper stands alone then you probably want to use an embedded bibliography.

2. If you have a collection of references that will be used for several papers then you should consider using a BibTeX database.

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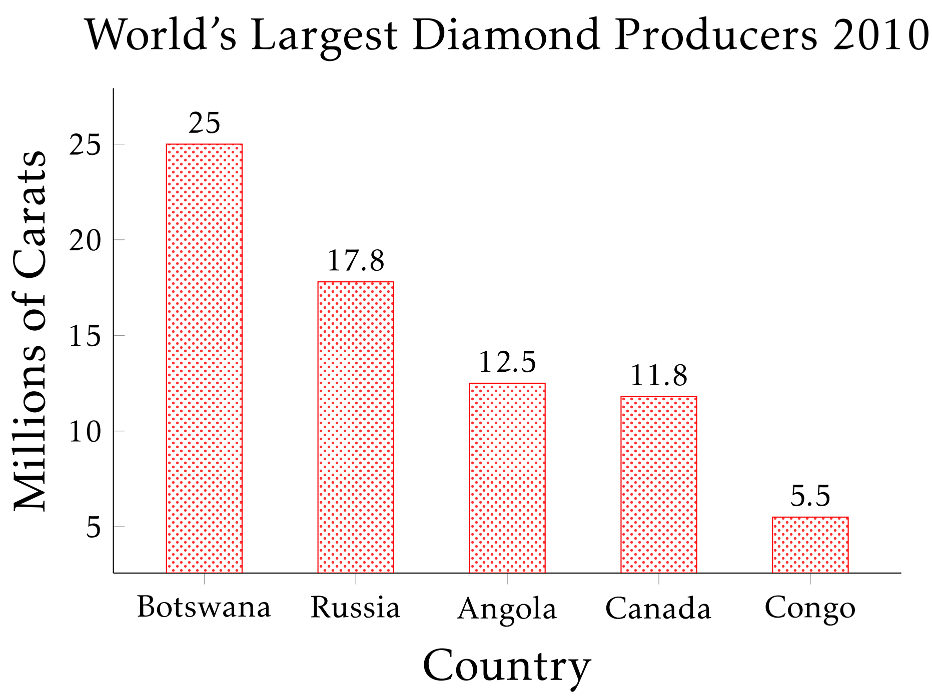

Figure C.3. Figure for Example C.7

Both types of bibliographies will save huge amounts of time and allow for very simple citation formats.

As usual, there is MUCH more to writing a good bibliography than what can possibly be listed here. A really good source is the wiki page for the latex bibliography: http://en.wikibooks.org/wiki/LaTeX/Bibliography_Management.

C.4.1 Embedded Bibliography

If you're using an embedded bib for a stand-alone paper then just before the `\end{document}` you include all of the bibliography information. A simple example (with 1 paper) is included here:

```
\begin{thebibliography}{9}

\bibitem{lamport94}
  Leslie Lamport,
  \emph{\LaTeX: a document preparation system},
  Addison Wesley, Massachusetts,
  2nd edition,
  1994.

\end{thebibliography}
```

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Use the `\cite{ }` command to cite items that are listed labeled inside the curly braces after `\bibitem`. For example, if we type `\cite{lamport94}` then we get a citation like this: [**?**].

C.4.2 Bibliography Database: BibTeX

BibTeX is a way for you to keep all of your bibliography materials in one place. The basic idea is as follows:

1. Start a file called `MyBib.bib` and follow the instructions from the link below to build your bibliography:
   http://ccm.ucdenver.edu/wiki/How_to_write_BibTeX_files

2. In your LaTeX file you can cite bib items with the `\cite{ }` command. As you cite works and compile you will build the bibliography automatically. You will need to compile MANY times to get all of the cross referencing and citations to appear.

3. Be sure that the `*.bib` file is in the same working directory as your LaTeX document (or at least give a path).

The primary utility of a bibtex file is that you can simply build it once when you're working on a large project and the citations will draw only the parts that are necessary for the current paper.

Appendix D

Miscellaneous Topics

In this chapter I have gathered several sections that were previously part of the main book, but are either no longer being taught as a regular part of the class, are just for fun, or are in need of significant revision. More than anything, this is just a storage location for sections that might be of interest to curious readers or instructors looking for extra topics. Take note that ALL of these sections are incomplete in some way.

D.1 Existence and Uniqueness of Solutions to First Order ODEs

Here are a few fundamental questions about differential equations:

- When does the solution to a differential equation exist?
- If you find a solution is it the only one?
- On what domain does the solution make sense?

If we don't know that a solution exists (or worse yet, if we know that it doesn't exist) then there is no need to go searching for it. Furthermore, we often solve differential equations with numerical methods but if the solution doesn't exist then our numerical method is only giving us computational garbage. In this section we present two fundamental theorems discussing these questions for first order differential equations.

Theorem D.1 (Existence Theorem)**.** Suppose that $f(t,y)$ is a continuous function in a rectangle of the form

$$\{(t,y) : a < t < b\,,\, c < y < d\}$$

in the ty-plane. If (t_0, y_0) is a point in the rectangle then there exists a number $\varepsilon > 0$

ad a function $y(t)$ defined for $t_0 - \varepsilon < t < t_0 + \varepsilon$ that solves the initial value problem

$$\frac{dy}{dt} = f(t,y) \quad y(t_0) = y_0.$$

Problem D.2. What does theorem D.1 mean? ▲

Theorem D.3 (Uniqueness Theorem)**.** Suppose that $f(t,y)$ and $\partial f/\partial y$ are continuous function in a rectangle of the form

$$\{(t,y) : a < t < b, c < y < d\}$$

in the ty-plane. If (t_0, y_0) is a point in the rectangle and if $y_1(t)$ and $y_2(t)$ are two functions that solve the initial value problem

$$\frac{dy}{dt} = f(t,y) \quad y(t_0) = y_0$$

for all t in the interval $t_0 - \varepsilon < t < t_0 + \varepsilon$ (for $\varepsilon > 0$) then

$$y_1(t) = y_2(t)$$

for $t_0 - \varepsilon < t < t_0 + \varepsilon$. That is, the two solution must be identical

Problem D.4. What does theorem D.3 mean? ▲

Problem D.5. A bucket of water has a hole in the bottom, and so the water is slowly leaking out. The height of the water in the bucket is thus a decreasing function of time $h(t)$ which changes according to the differential equation

$$\frac{dh}{dt} = -k\sqrt{h}$$

where k is a positive constant that depends on the size of the hole in and the bucket. If we start out a bucket with 25cm of water in it, then according to this model, will the bucket ever empty?

1. Yes
2. No
3. Can't tell with the given information

▲

Problem D.6. Based upon observations, Kate developed the differential equation

$$\frac{dT}{dt} = -0.09(T - 72)$$

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to predict the temperature in her vanilla chai tea. In the equation, T represents the temperature of the chai in $^\circ F$ and t is time. Kate has a cup of chai whose initial temperature is $110^\circ F$ and her friend has a cup of chai whose initial temperature is $120^\circ F$. According to Kate's model, will there be a point in time when the two cups of chai have exactly the same temperature?

1. Yes

2. No

3. Can't tell with the given information

▲

D.2 Bifurcations in First Order Differential Equations

In this section we will explore the behavior of the equilirium solutions in a differential equation to changes in a parameter. In some instances the change in a parameter might cause the behavior of equilibrium points to change from stable to unstable (or possibly semi-stable). It is also possible to *spawn* new equilibrium points by changing the values of parameters.

It is often the case in real models that parameters are not known with 100% certainty. Sometimes the parameters come from scientific measurements, sometimes they are estimated from data, and sometimes they are just a guess. If you are in any of these cases with a first order autonomous differential equation model then you should do a bifurcation analysis to see how the equilibrium solutions change due to the natural variability in the parameters.

Problem D.7 (Fish.net)**.** A mathematician at a fish hatchery has been using the differential equation

$$\frac{dP}{dt} = 0.5P\left(1 - \frac{P}{200}\right)$$

as a model for predicting the number of fish that a hatchery can expect to find in their pond.

(a) What are the equilibrium points of the mathematician's model. Discuss their stability. Show the equilibrium points with a plot of $\frac{dP}{dt}$ vs. P, a phase line, and a slope field.

(b) What does the differential equation predict about the future of the fish population for several different initial populations?

(c) Recently the hatchery was bought out by `fish.net` and the new owners are planning to allow the public to catch fish at the hatchery (for a fee of course). This means that the previous differential equation used to predict future fish populations needs to modified to reflect this new plan. For the sake of simplicity, assume that the managers from `fish.net` are going to allow a constant annual harvest rate k. Which of the four modified differential equations makes the most sense to you and why?

$$(i) \qquad \frac{dP}{dt} = 0.5P\left(1 - \frac{P}{200}\right) - k$$

$$(ii) \qquad \frac{dP}{dt} = 0.5P\left(1 - \frac{P}{200}\right) - kP$$

$$(iii) \qquad \frac{dP}{dt} = 0.5P\left(1 - \frac{P}{200}\right) - kt$$

$$(iv) \qquad \frac{dP}{dt} = 0.5P\left(1 - \frac{P-k}{200}\right)$$

(d) Using the modified differential equation agreed upon from the previous problem, discuss the implications that various choices of k will have on future fish populations. In particular, are there choices of k that change the nature or number of the equilibrium points? Defend your work graphically and analytically.

▲

Problem D.8. For each of the following differential equations demonstrate both graphically and analytically the way(s) in which the solutions change as the value of r changes. Identify the precise value(s) of r for which there is either a change in the number of equilibrium solution(s) or a change in the type of equilibrium solution(s).

$$\frac{dy}{dt} = (y-3)^2 + r \tag{D.1}$$

$$\frac{dy}{dt} = y^2 - ry + 1 \tag{D.2}$$

$$\frac{dy}{dt} = ry + y^3 \tag{D.3}$$

$$\frac{dy}{dt} = y^6 - 2y^4 + r \tag{D.4}$$

▲

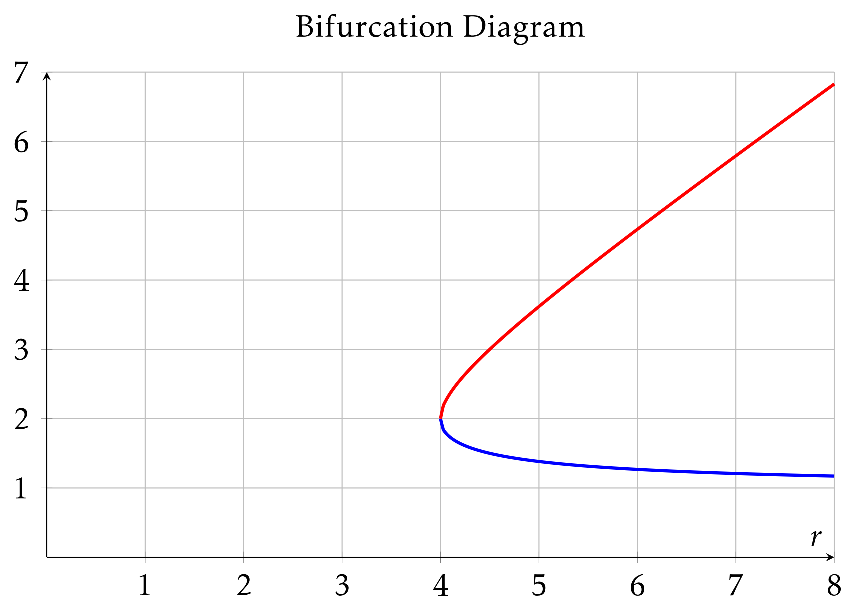
Problem D.9. Reconsider differential equation (D.1) from Problem D.8. Sketch a graph with the value of the equilibrium solution on the vertical axis and the value of r on the horizontal axis. Such a graph is referred to as a "bifurcation diagram." Connect what you see in the bifurcation diagram to your arguments from Problem D.8. ▲

Problem D.10. Reconsider differential equation (D.2) from Problem D.8. Sketch a bifurcation diagram with r on the horizontal axis and the value of the equilibrium point on the vertical axis. Connect what you see in this plot to the original discussion in Problem D.8. ▲

Problem D.11. Consider the differential equation

$$\frac{dy}{dt} = y^2 - ry + r$$

with unknown parameter r. The bifurcation diagram is shown below.



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(a) How many equilibrium solutions are there to the differential equation when $r < 4$? When $r = 4$? When $r > 4$?

(b) Make a plot of $\frac{dy}{dt}$ vs. y assuming that $r > 4$. Is the blue part of the bifurcation diagram stable, unstable or semi-stable? What about the red part?

(c) Make a plot of $\frac{dy}{dt}$ vs. y assuming that $r = 4$. Is the single equilibrium point stable, unstable, or semi-stable?

▲

D.3 Isomorphic Vector Spaces

What we'll explore in this section is the following bold claim.

> Every vector space of dimension n is *geometrically the same* as $\mathbb{R}^n$.

The consequence of this statement is that, in reality, studying any finite dimensional vector space is the same as studying the Euclidean spaces $\mathbb{R}^n$.

To make this clear let's look at a few examples. Consider the vector space $\mathcal{V} = \mathbb{R}^3$ and the vector space $\mathcal{W} = \mathcal{P}_2$ (polynomials of order 2). We claim that the set of all polynomials of order 2 is really the same vector space as $\mathbb{R}^3$.

The standard bases for these two vector spaces are

$$\mathcal{B}_{\mathcal{V}} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \quad \text{and} \quad \mathcal{B}_{\mathcal{W}} = \left\{ 1, x, x^2 \right\}.$$

Notice that there is a natural mapping between these two bases:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \leftrightarrow 1 \qquad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \leftrightarrow x \qquad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \leftrightarrow x^2.$$

Hence if we have any polynomial $p(x) \in \mathcal{P}_2$ we can use this natural mapping to find the corresponding vector in $\mathbb{R}^3$. For a specific example, let's take the polynomial $p(x) = 3x^2 + 5x - 1$. Explicitly stating $p(x)$ as a linear combination of basis vectors from the set $\{1, x, x^2\}$ we see that

$$p(x) = -1(1) + 5(x) + 3(x^2),$$

and hence $p(x)$ maps to

$$p(x) \mapsto -1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 5 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix}.$$

That is to say that the polynomial $p(x) = 3x^2 + 5x - 1$ in $\mathcal{P}_2$ is the *same vector* as $\begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix}$ in $\mathbb{R}^3$.

More generally, the polynomial $a_0 + a_1 x + a_2 x^2 \in \mathcal{P}_2$ is the *same vector* as $\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} \in \mathbb{R}^3$. In this way we can say that any vector in $\mathcal{P}_2$ has a geometric copy in $\mathbb{R}^3$.

The words *geometrically the same* are not very mathematical. Instead a mathematician would use the word *isomorphic* meaning *same form*.

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Definition D.12 (Isomorphism)**.** An **isomorphism** is an invertible mapping between two sets.

Definition D.13 (Isomorphic Vector Spaces)**.** Two vector spaces are said to be **isomorphic** if there is an isomorphic mapping between their bases.

Since the word *isomorphic* means *same form* we can loosely describe isomorphic vector spaces as having the same geometric form – whatever geometry makes sense in one vector space makes the same sense in the other.

Let's explore another example using our new terminology. Consider the vector spaces $\mathbb{R}^4$ and $M_{2\times 2}$ (the collection of 2×2 matrices). The standard bases for these vector spaces are

$$\mathcal{B}_{\mathbb{R}^4} = \left\{ \begin{pmatrix} 1\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\1\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix} \right\} \quad \text{and} \quad \mathcal{B}_{M_{2\times 2}} = \left\{ \begin{pmatrix} 1 & 0\\0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1\\0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0\\1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0\\0 & 1 \end{pmatrix} \right\}$$

and we note that both of these vector spaces are four dimensional. There is a clear natural isomorphism between the two basis sets

$$\begin{pmatrix} 1\\0\\0\\0 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0\\0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0\\1\\0\\0 \end{pmatrix} \leftrightarrow \begin{pmatrix} 0 & 1\\0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0\\0\\1\\0 \end{pmatrix} \leftrightarrow \begin{pmatrix} 0 & 0\\1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 0 & 0\\0 & 1 \end{pmatrix}.$$

Hence we see that any 2×2 matrix has an isomorphic copy in $\mathbb{R}^4$ and the invertible mapping is

$$\begin{pmatrix} a & b\\c & d \end{pmatrix} \leftrightarrow \begin{pmatrix} a\\b\\c\\d \end{pmatrix}.$$

More specifically, this means that the matrix $\begin{pmatrix} 2 & 0\\3 & -1 \end{pmatrix}$ is geometrically equivalent (isomorphic) to the vector $\begin{pmatrix} 2\\0\\3\\-1 \end{pmatrix}$.

Theorem D.14. Any vector space of dimension n is isomorphic to $\mathbb{R}^n$.

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Proof. Let $\mathcal{V}$ be an n dimensional vector space. From the definition of dimension this means that $\mathcal{V}$ has a basis that contains n vectors. Let the basis $\mathcal{B}_\mathcal{V}$ be denoted as

$$\mathcal{B}_\mathcal{V} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}.$$

The natural isomorphism between the basis for $\mathcal{V}$ and $\mathbb{R}^n$ is therefore

$$\mathbf{v}_1 \leftrightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 \leftrightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \cdots \quad \mathbf{v}_n \leftrightarrow \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

□

Problem D.15. Show that $\mathcal{P}_3$ (the space of third order polynomials) is isomorphic to $M_{2\times 2}$. ▲

Now for some fun.

Problem D.16. Let $\mathcal{P}_2$ be the vector space of second order polynomials and let T be a linear transformation $T : \mathcal{P}_2 \to \mathcal{P}_1$ that maps a quadratic polynomial to its derivative (which of course is linear). For example, $T(3x^2 + 5x - 1) = 6x + 5$.

(a) Using the natural isomorphism between $\mathcal{P}_2$ and $\mathbb{R}^3$ map the polynomial $p(x) = 3x^2 + 5x - 1$ to a vector $\mathbf{u} \in \mathbb{R}^3$.

(b) Using the natural isomorphism between $\mathcal{P}_1$ and $\mathbb{R}^2$ map the polynomial $q(x) = 6x + 5$ to a vector $\mathbf{w} \in \mathbb{R}^2$.

(c) There is a matrix A such that $A\mathbf{u} = \mathbf{w}$. Find it.

(d) Use your answer to part (c) to take the derivative of $p(x) = ax^2 + bx + c$. That is: map $p(x)$ to $\mathbb{R}^3$, apply the matrix transformation from part (c), then map your answer from $\mathbb{R}^2$ to $\mathcal{P}_1$.

(e) The linear transformation $T : \mathcal{P}_n \to \mathcal{P}_{n-1}$ that takes derivatives of n^{th} order polynomials is equivalent to what matrix transformation?

(f) ... stop and think about this for a second ... you just showed that taking derivatives of a polynomial is really just a matrix operation ... mind = blown!

▲

Problem D.17. Come up with a matrix transformation that has the same action as taking an antiderivative. ▲

Problem D.18. True or False: The function $h(t) = 4 + 3t$ is a linear combination of the functions $f(t) = (1+t)^2$ and $g(t) = 2 - t - 2t^2$. Answer this question by using the fact that the polynomial spaces are isomorphic to the Euclidean spaces. ▲

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Problem D.19. True or False: The function $h(t) = t^2$ is a linear combination of $f(t) = (1-t)^2$ and $g(t) = (1+t)^2$. Answer this question by using the fact that the polynomial spaces are isomorphic to the Euclidean spaces. ▲

Problem D.20. The dot product in $\mathbb{R}^2$ is easy remember, but now that we know that $\mathbb{R}^2$ is isomorphic to the vector space of all linear functions it is natural to ask if there is a natural inner product on $\mathcal{P}_1$ that is analogous to the dot product. Let $p(x) = p_0 + p_1 x$ and $q(x) = q_0 + q_1 x$. Map both p and q to $\mathbb{R}^2$ via the natural isomorphism and take the dot product. Does the resulting formula give a valid inner product on $\mathcal{P}_1$? You'll need to check the following (according to Definition 5.23).

1. $\langle p, q\rangle = \langle q, p\rangle$ for $p, q \in \mathcal{P}_1$?
2. $\langle p, q+r\rangle = \langle p, q\rangle + \langle p, r\rangle$ for $p, q, r \in \mathcal{P}_1$?
3. $\langle cp, q\rangle = c\langle p, q\rangle$ for $p, q \in \mathcal{P}_1$ and $c \in \mathbb{R}$?
4. $\langle p, p\rangle \ge 0$ and $\langle p, p\rangle = 0$ iff $p = 0$ for $p \in \mathcal{P}_1$.

▲

Problem D.21. Consider the vector space $M_{2\times 2}$ of all 2×2 matrices along with the inner product

$$\begin{aligned}
\langle A, B\rangle = \text{trace}(AB^T) &= \text{trace}\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix} b_{11} & b_{21} \\ b_{12} & b_{22}\end{pmatrix}\right) = \text{trace}\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix}^T\right) \\
&= \text{trace}\left(\begin{pmatrix} a_{11}b_{11} + a_{12}b_{12} & a_{11}b_{21} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{12} & a_{21}b_{21} + a_{22}b_{22}\end{pmatrix}\right) \\
&= (a_{11}b_{11} + a_{12}b_{12}) + (a_{21}b_{21} + a_{22}b_{22})
\end{aligned}$$

(a) Why is this a natural choice for the inner product between 2×2 matrices?

(b) Complete these sentences:

- Finding the magnitude of 2×2 matrices with this inner product is geometrically the same as ____________.
- Projecting a 2×2 matrix onto another 2×2 matrix is geometrically the same as ____________.
- Finding angles between 2×2 matrices is geometrically the same as ____________.

▲

Problem D.22. An valid inner product on $\mathcal{P}_1[0,1]$ (the set of all linear functions on the interval $[0,1]$) is

$$\langle p, q\rangle = \int_0^1 p(x)q(x)\,dx.$$

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Let $p(x) = p_0 + p_1 x$ and $q(x) = q_0 + q_1 x$ and observe that

$$\begin{aligned}\int_0^1 p(x)q(x) &= \int_0^1 (p_0 + p_1 x)(q_0 + q_1 x)\,dx \\ &= \int_0^1 \left(p_0 q_0 + (p_0 q_1 + p_1 q_0)x + p_1 q_1 x^2\right) dx \\ &= \left(p_0 q_0 x + \frac{(p_0 q_1 + p_1 q_0)}{2} x^2 + \frac{p_1 q_1}{3} x^3\right)\Big|_0^1 \\ &= p_0 q_0 + \frac{1}{2}(p_0 q_1 + p_1 q_0) + \frac{1}{3}(p_1 q_1).\end{aligned}$$

While this inner product naturally defines a geometry in the space of all linear polynomials it doesn't correspond to our usual Euclidean geometry in the isomorphic space $\mathbb{R}^2$. That is, if $\mathbf{u}, \mathbf{w} \in \mathbb{R}^2$ then we can map both vectors to the vector space $\mathcal{P}_1$ via the natural isomorphism and then take the inner product via the integral given above. Since $\mathbb{R}^2$ and $\mathcal{P}_1$ are isomorphic they are *geometrically equivalent*, but which geometry do we get when we use the integral inner product?

(a) What is the angle between $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ in this new geometry? Are these vectors still orthogonal?

(b) What is the length of the vector $\begin{pmatrix}1\\1\end{pmatrix}$ in this new geometry?

(c) If $\mathbf{u} \perp \mathbf{w}$ in Euclidean geometry then what do we know about them in this new geometry?

(d) If $p \perp q$ in $\mathcal{P}_1$ under the integral inner product, what does that mean about their images in $\mathbb{R}^2$ under Euclidean geometry?

▲

The previous two problems should point out an interesting fact about abstract vector spaces: in many cases the natural geometry from Euclidean space carries over perfectly to the abstract setting, and sometimes there is more to be gained in the abstract setting an your Euclidean intuition doesn't hold. Remember that abstract vector spaces are not abstractions for the sake of abstractions. Instead, the content in this chapter was a revolutionary leap forward in the understanding of mathematics in the 17th century and it has allowed us to make many modern mathematical discoveries. It is no exaggeration to say that

> much of modern mathematics (and hence modern science) is built upon the idea of an abstract vector space.

In this section we have built up the intuition for isomorphic vector spaces by using the natural standard bases. We could have also built up these ideas using any other basis and everything that we've said thus far is true. The down side to using non-standard bases, though, is that the notation and subsequent computations becomes more cumbersome. For a more in-depth discussion of these ideas see [4].

D.4 Higher Dimensions are Geometrically Weird

At this point you have likely come to the conclusion that linear algebra is beautiful but possibly very hard to visualize. This is especially true in more than three dimensions. In this brief section we'll explore a few strange facts about geometry in higher dimensions.

Problem D.23. Let's first recall some basic geometry but in each of the following problems we'll intentionally the language of vector spaces.

(a) If you are standing at the origin on a 1-dimensional Euclidean vector space and you reach out R units in each direction, how much *volume* do you capture?

(b) If you are standing at the origin of a 2-dimensional Euclidean vector space and you reach out R units in every direction, how much *volume* do you capture?

(c) If you are standing at the origin of a 3-dimensional Euclidean vector space and you reach out R units in every direction, how much *volume* do you capture?

▲

Problem D.24. Now let's make a few conjectures. Let $V_{n,ball}(R)$ be the volume an n-dimensional ball of radius R and let $V_{n,cube}(R)$ be the volume of an n-dimensional cube of side length $2R$. To make this more concrete, in 2D $V_{2,ball}(R)$ is the area of a circle of radius R and $V_{2,cube}(R)$ is the area of a square with side length $2R$. In 3D, $V_{3,ball}(R)$ is the volume of a sphere with radius R and $V_{3,cube}(R)$ is the volume of a cube with side length $2R$.

(a) Without any formal computation, approximately how much area does a circle inscribed in a square take up? In other words, what is $V_{2,ball}(R)/V_{2,cube}(R)$? Verify your conjecture with formal computation.

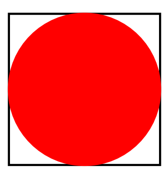

Figure D.1. A circle inscribed in a square. The ratio of the area of the square to the area of the circle is $V_{2,ball}/V_{2,cube}$.

(b) Without any formal computation, approximately how much area does a sphere inscribed in a cube take up? In other words, what is $V_{3,ball}(R)/V_{3,cube}(R)$? Verify your conjecture with formal computation.

(c) Make a conjecture for the value of the limit:

$$\lim_{n\to\infty} \frac{V_{n,ball}(R)}{V_{n,cube}(R)}.$$

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▲

In higher dimensions (4D and up) we have a hard time visualizing anything without using tricks such as color or time-like variables.

Definition D.25 (Volume of an n-Dimensional Ball)**.** For the volume of an n-dimensional ball we have the recursive formula

$$V_n(R) = \frac{2\pi R^2}{n} V_{n-2}(R) \tag{D.5}$$

where n is the dimension and R is the radius.

Problem D.26. Given that $V_1 = 2R$ and $V_2 = \pi R^2$, use the recursive formula (D.5) to verify the formula for the volume of a sphere of radius R.

$$V_3(R) = \underline{\hspace{3cm}}$$

▲

Definition D.27 (Volume of an n-Dimensional Cube)**.** For a cube of n dimensions with side length $2R$ we always have the simple formula

$$V_n(R) = (2R)^n = 2^n R^n. \tag{D.6}$$

A moment's reflection reveals that in a one dimension Euclidean vector space the volume of a cube and the volume of the sphere are the same: $V_{1,ball} = V_{1,cube} = 2R$. In two dimensions we have

$$V_{2,square} = (2R)^2 = 4R^2 \quad \text{and}$$
$$V_{2,circle} = \pi R^2,$$

which implies that the fraction of the volume of the square filled by the inscribed circle is

$$\frac{V_{2,circle}}{V_{2,square}} = \frac{\pi R^2}{4R^2} = \frac{\pi}{4} \approx 0.785$$

so the circle fills approximately 78.5% of the square. Examining Figure D.1 visually should verify that this *appears to be* correct.

In three spatial dimension we have

$$V_{3,cube} = (2R)^3 = 8R^3 \quad \text{and}$$
$$V_{3,sphere} = \frac{4}{3}\pi R^3,$$

which implies that

$$\frac{V_{3,sphere}}{V_{3,cube}} = \frac{\frac{4}{3}\pi R^3}{8R^3} = \frac{\frac{4}{3}\pi}{8} = \frac{\pi}{6} \approx 0.524$$

so we see that a sphere inscribed in a cube fills up about 52.4% of the available volume. These results are summarized in the table below.

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Dimension (n)	Volume of a Ball	Volume of a Cube	Ratio: ($V_{n,ball}/V_{n,cube}$)
1	$2R$	$2R$	1
2	πR^2	$4R^2$	$\frac{\pi}{4} \approx 0.785$
3	$\frac{4}{3}\pi R^3$	$8R^3$	$\frac{\pi}{6} \approx 0.524$
4	$\frac{2\pi R^2}{4} \cdot \left(\pi R^2\right) = \frac{\pi^2 R^4}{2}$	$16R^4$	$\frac{\pi^2}{32} \approx 0.308$
5	$\frac{2\pi R^2}{5} \cdot \frac{4}{3}\pi R^3 = \frac{8\pi^2 R^5}{15}$	$32R^5$	$\frac{\pi^2}{60} \approx 0.164$
6			
7			
8			
$\vdots$			

Problem D.28. The previous table shows the ratio $V_{n,ball}/V_{n,cube}$ for dimensions 1 through 5. Use equations (D.5) and (D.6) to complete the computation for dimensions 6 - 8 and conjecture the value of

$$\lim_{n\to\infty} \frac{V_{n,ball}(R)}{V_{n,cube}(R)}.$$

Explain the meaning of the limit geometrically (however counterintuitive it is). ▲

Problem D.29. If you want to see a cool video about higher dimensional visualization, check out this one by *3 Blue 1 Brown:* https://youtu.be/zwAD6dRSVyI. ▲

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D.5 Where Laplace Transforms Come From

You may have run into Taylor Series in past courses. The idea is to represent a function $f(x)$ near the point $x = 0$ as an infinite series of power functions:

$$f(x) = \frac{f(0)}{0!}x^0 + \frac{f'(0)}{1!}x^1 + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots. \tag{D.7}$$

More compactly, we can write the Taylor Series as

$$f(x) = \sum_{j=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n. \tag{D.8}$$

Problem D.30. Find the Taylor Series representations for the functions $f(x) = e^x$, $g(x) = \frac{1}{1-x}$ (for $|x| < 1$), and $h(x) = \sin(x)$ all centered at $x = 0$.

$$e^x = \rule{6cm}{0.4pt}$$
$$\frac{1}{1-x} = \rule{6cm}{0.4pt}$$
$$\sin(x) = \rule{6cm}{0.4pt}$$

▲

The Taylor Series does something else amazing! In a sense, the coefficients of the Taylor Series are the DNA of the function. That is to say: If you know the Taylor coefficients you know the function and visa versa.

Problem D.31. Using problem 1, which function has the following sequence of Taylor coefficients?

$$a(n) = \{1, 1, 1, 1, 1, \dots\} \quad \text{corresponds to} \quad f(x) = \rule{2cm}{0.4pt}$$
$$a(n) = \{1, 1, \frac{1}{2!}, \frac{1}{3!}, \frac{1}{4!}, \dots\} \quad \text{corresponds to} \quad f(x) = \rule{2cm}{0.4pt}$$
$$a(n) = \{0, 1, 0, -\frac{1}{3!}, 0, \frac{1}{5!}, \dots\} \quad \text{corresponds to} \quad f(x) = \rule{2cm}{0.4pt}$$

▲

Hence, if we have a sequence $a(n)$ (and the sequence has some basic properties*) then we can associate the sequence with a function $f(x)$ via the Taylor Series

$$a(n) \rightsquigarrow f(x) \quad \text{since} \quad \sum_{n=0}^{\infty} a(n)x^n = f(x).$$

Problem D.32. You might be asking yourself "So what! Why do I need another way to represent a function?" To answer this question discuss with your partner how you think the Taylor sequence "DNA" of a function might be a useful tool in this modern age of computers (hint). ▲

*See any standard Calculus text if you don't recall the necessary conditions for a Taylor Series to converge.

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The Laplace Transform

Now we're ready to create the Laplace Transform. The Laplace Transform is the continuous analog of what we just discussed: If we have a function $a(n)$ we can find a function $f(x)$ such that we replace the sum in the Taylor series with an integral:

$$\int_0^\infty a(n)x^n dn = f(x) \tag{D.9}$$

There are some notational conventions that we have to adjust for:

1. We don't typically use n as a continuous variable so we're going to switch it to t.
2. We typically don't use the letter "a" for functions of a continuous variable so we'll switch it to f. Then we'll make the right-hand side "F" so we can keep them straight. The integral now becomes
$$\int_0^\infty f(t)x^t dt = F(x)$$
3. The exponential function x^t is really inconvenient when integrating with respect to t so we'll switch it to
$$x^t = e^{\ln(x)t} \qquad \Longrightarrow \qquad \int_0^\infty f(t)e^{\ln(x)t} dt = F(x)$$
(convince yourself that this is the same thing algebraically)
4. Now the left-hand side will be a function of $\ln(x)$ and the right-hand side will be a function of x. This is rather inconvenient so let's make a change of variables: replace $\ln(x)$ with $-s$ (the negative sign gives positive values when $0 < x < 1$ and negative values otherwise). Therefore, the Laplace transform is:
$$\boxed{\int_0^\infty e^{-st} f(t)dt = F(s)} \tag{D.10}$$

The Laplace transform associates a function $f(t)$ with a function $F(s)$ just like the Taylor Series associates a sequence of numbers $a(n)$ with a function $f(x)$:

$$\boxed{\underbrace{a(n) \rightsquigarrow f(x)}_{\text{Taylor Series}} \quad \text{via} \quad \sum_{n=0}^\infty a(n)x^n = f(x) \qquad \text{and} \qquad \underbrace{f(t) \rightsquigarrow F(s)}_{\text{Laplace Transform}} \quad \text{via} \quad \int_0^\infty e^{-st} f(t)dt = F(s).}$$

Notationally we write the Laplace Transform of $f(t)$ as $\mathcal{L}\{f(t)\} = F(s)$.

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D.6 Convolutions with Laplace Transforms

. . . no content here yet, but I intend to write this up eventually . . .

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