

Statistics for the Sciences

ANOVA for Single Factor Designs

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January 18, 2025

Outline

- Design of Experiments
- one-way ANOVA
 - ▶ Model fit and model assumptions check
 - ▶ Effect size
 - ▶ Power and size
- Random effects one-way ANOVA
- Lab 1: Analysis of `medley.csv`
- Lab 2: Analysis of `medley_rand.csv`

Design of Experiments

- Two ways of data collection
 - ▶ observational study
 - ▶ experimental design
- Experiment: people apply some treatment and then observe its effects on the subjects (subjects in an experiments generally are called experimental units)
 - ▶ An experiment requires random assignment of subjects to treatments.
 - ▶ If done correctly, experiments provide most compelling evidence that a treatment causes an observed outcome
 - ▶ For example, in a randomized clinical study patients in the experimental groups receive the drug while patients in the control groups receive a placebo or sugar pill. The patients do not know if they are receiving the experimental treatment or placebo.

Design of Experiments

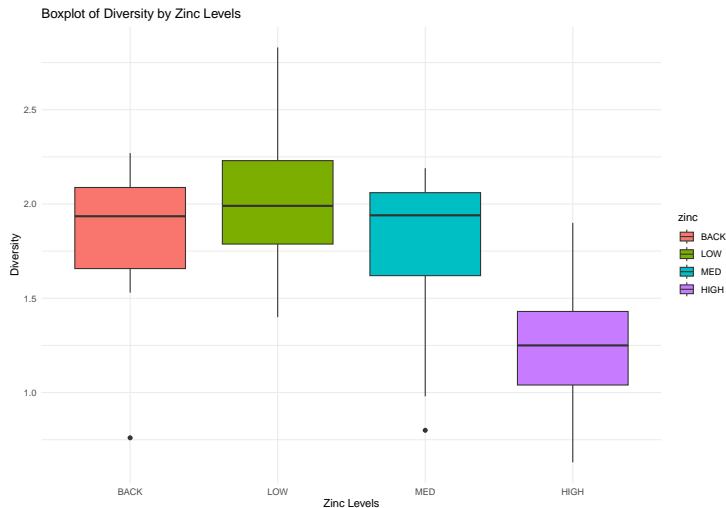
- Some terminologies:
 - ▶ An experimental unit is the object on which a measurement (or measurements) is taken.
 - ▶ The response is the variable being measured by the experimenter.
 - ▶ A factor is an independent variable whose values are controlled and varied by the experimenter.
 - ▶ A level is the intensity setting of a factor.
 - ▶ A treatment is a specific combination of factor levels.

Design of Experiments

- Example (`medley.csv`). Medley & Clements (1998) sampled a number of stations (between four and seven) on six streams known to be polluted by heavy metals in the Rocky Mountain region of Colorado, USA. They recorded zinc concentration, and species richness and species diversity of the diatom community and proportion of diatom cells that were the early-successional species, *Achanthes minutissima*.
- The first analysis compares mean diatom species diversity (response variable) across the four zinc-level groups (categorical predictor variable), zinc level treated as a fixed factor.
 - ▶ The response variable is diversity.
 - ▶ The experimenter chooses 4 levels of a single factor - zinc
 - ▶ Each level of the factor is a treatment

Design of Experiments

- Box plots to check the differences



Design of Experiments

- Calculate the group means

```
##      zinc mean_diversity
## 1 BACK      1.797500
## 2 LOW       2.032500
## 3 MED       1.717778
## 4 HIGH      1.277778
```

- Calculate the overall mean

```
## [1] 1.694118
```

Design of Experiments

- Completely randomized design:
 - ▶ one factor only; one-way classification.
 - ▶ The treatments are assigned completely randomly to the experiment units.
- The data will be like

Trt_1	Trt_2	\cdots	Trt_k
y_{11}	y_{21}	\cdots	y_{k1}
y_{12}	y_{22}	\cdots	y_{k2}
\vdots	\vdots	\vdots	\vdots
y_{1,n_1}	y_{2,n_2}	\cdots	y_{k,n_k}
$\bar{y}_{1\cdot}$	$\bar{y}_{2\cdot}$	\cdots	$\bar{y}_{k\cdot}$

1-way ANOVA

- Statistical model: By regarding each n_i observations as a random sample from an infinite population, the probability model (**means model**) is:

$$Y_{ij} = \mu_i + \varepsilon_{ij}, j = 1, 2, \dots, n_i, i = 1, 2, \dots, k,$$

where $\varepsilon_{ij} \sim N(0, \sigma^2)$ are independent normal random errors with common variance σ^2 .

- Are the k population means the same, or is at least one mean different from the others?
- We are testing $H_0 : \mu_1 = \mu_2 = \dots = \mu_k$ versus $H_a : \text{At least two means are different from each other or "Not all means are equal"}$

1-way ANOVA

- **Effects model**

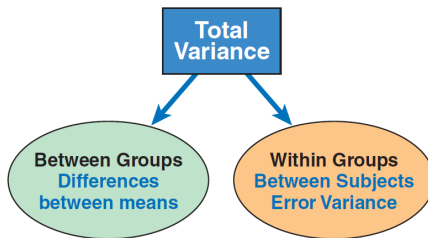
$$Y_{ij} = \mu + \tau_i + \varepsilon_{ij}, j = 1, 2, \dots, n_i, i = 1, 2, \dots, k,$$

- $\varepsilon_{ij} \sim N(0, \sigma^2)$ are independent normal random errors
- side condition: $\sum_i \tau_i = 0$
- Model fit and factor effects
 - ▶ Overall mean: $\hat{\mu} = \bar{y}_{..}$
 - ▶ Treatment: $\hat{\tau}_i = \bar{y}_{i.} - \bar{y}_{..}$, $i = 1, \dots, k$

1-way ANOVA

- The idea behind the Analysis of Variance(ANOVA) is to divide the total variation among the responses into the variance between groups, and that within groups, and see if those distributions match the null hypothesis that all groups come from the same distribution.

$$SS_{total} = SS_T + SS_E$$



Partitioning the total variance in a one-way ANOVA.

1-way ANOVA

Trt_1	Trt_2	\cdots	Trt_k	
y_{11}	y_{21}	\cdots	y_{k1}	
y_{12}	y_{22}	\cdots	y_{k2}	
\vdots	\vdots	\vdots	\vdots	
y_{1,n_1}	y_{2,n_2}	\cdots	y_{k,n_k}	
$\bar{y}_{1\cdot}$	$\bar{y}_{2\cdot}$	\cdots	$\bar{y}_{k\cdot}$	$\bar{y}_{\cdot\cdot}$

That is,

$$\sum \sum (y_{ij} - \bar{y}_{\cdot\cdot})^2 = \sum \sum (\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})^2 + \sum \sum (y_{ij} - \bar{y}_{i\cdot})^2$$

- Degrees of freedom are additive, just like the sums of squares. Partition of the dfs:

$$N - 1 = (k - 1) + (N - k),$$

where $N = \sum n_i$

1-way ANOVA

- A large value of SS_T reflects large differences in treatment means; A small value of SS_T likely indicates no differences in treatment means
- While sums of squares cannot be directly compared to test the hypothesis of equal means, mean squares can be compared.
 - ▶ A mean square is a sum of squares divided by its degrees of freedom.

$$MS_T = SS_T / (k - 1), \quad MSE = SSE / (N - k).$$

- If the treatment means are equal, the treatment and error mean squares will be (theoretically) equal.
- If treatment means differ, the treatment mean square will be larger than the error mean square.

1-way ANOVA

- Remember that σ^2 is the common variance for all a populations. The quantity $MSE = SSE/(N - k)$ is a pooled estimate of σ^2 , a weighted average of all a sample variances, whether or not H_0 is true.
- If H_0 is true, then the variation in the sample means, measured by $MS_T = SS_T/(k - 1)$, also provides an unbiased estimate of σ^2 .
- However, if H_0 is false and the population means are different, then MS_T —which measures the variance in the sample means — is unusually large. The test statistic

$$F = MS_T/MSE \sim F(df_1 = k - 1, df_2 = N - k) \text{ under } H_0$$

tends to be larger than usual.

- ▶ The F-test is a right-tailed statistical test.

1-way ANOVA

- 1-way ANOVA Table

Table 1: 1-way ANOVA Table

Source	df	SS	MS	F
Treatments	$k - 1$	SS_T	$SS_T / (k - 1)$	MS_T / MSE
Error	$N - k$	SSE	$SSE / (N - k)$	
Total	$N - 1$	SS_{total}		

- $E(MSE) = \sigma^2$, $E(MS_T) = \sigma^2 + \frac{\sum_{i=1}^k n_i \tau_i^2}{k-1}$.
- A completely randomized design is called **balanced** if the numbers of observations in all treatment are equal.
- Analysis of variance procedures are fairly **robust against departures from normality** when the design is balanced and when the data are fairly mound-shaped.

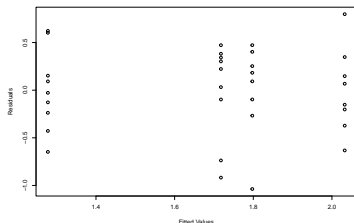
1-way ANOVA

- The Medley example

```
## Analysis of Variance Table
##
## Response: diversity
##           Df Sum Sq Mean Sq F value  Pr(>F)
## zinc        3  2.5666  0.85554   3.9387 0.01756 *
## Residuals   30  6.5164  0.21721
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```


1-way ANOVA

- Model adequacy check. We need to check the fitted values and residuals to see if the three model assumptions are satisfied.



- It can be seen that the residuals are randomly distributed and thus no pattern can be found. Therefore, the independence assumption seems reasonable. However, we can see that the spread of the residuals has a decreasing trend. But the departure from constant variance is not serious. We need statistical test, such as **Levene test**, to further **check constant variance** assumption.

1-way ANOVA

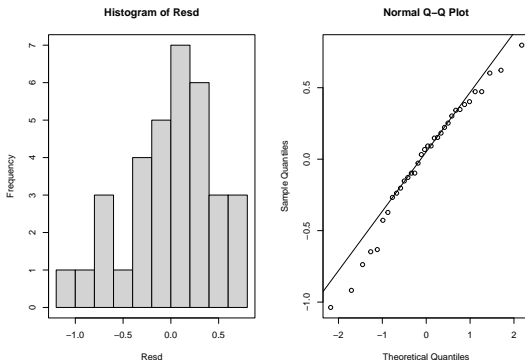
- **Levene test** is equivalent to a 1-way ANOVA with the dependent variable being the absolute value of the difference between each observation y_{ij} and the mean/median of the group to which the observation y_{ij} belongs.
 - ▶ The null of a Levene test is All groups have a same spread
 - ▶ In the Levene test, the response variable is residuals from the above ANOVA fit and the factor is the treatment in the above ANOVA.
 - ▶ By the Levene test below, constant variance assumption is valid.

```
## Levene's Test for Homogeneity of Variance (center = median)
##           Df F value Pr(>F)
## group    3  0.0195 0.9962
##           30
```

1-way ANOVA

- Normality assumption can be checked using histogram and QQ plot of the residuals.

```
par(mfrow=c(1,2));  
hist(Resd); box();  
qqnorm(Resd); qqline(Resd);
```



1-way ANOVA - Effect size

- Various types of measurement of effect sizes have been suggested; Different software (functions) may use different types of effect sizes
- Cohen's d for two samples: $\frac{\mu_1 - \mu_2}{\sigma}$
- **Effect size by Cohen's f :** Cohen's f is used to measure the effect size when there are k (>2) groups which is the standard deviation of the means divided by the standard deviation

$$f = \frac{\sigma_m}{\sigma}.$$

For equal sample sizes,

$$\sigma_m = \sqrt{\frac{\sum_{i=1}^k (\bar{y}_i - \bar{y})^2}{k}},$$

where \bar{y} is the grand mean.

- **Cohen** suggested that $f \geq 0.1$, $f \geq 0.25$, and $f \geq 0.4$ represent small, medium, and large effect sizes, respectively.
- For two groups, Cohen's f is half as large as Cohen's d , or $f = \frac{1}{2} \frac{\bar{y}_1 - \bar{y}_2}{\sigma}$

1-way ANOVA - Effect size

- **Effect size by Cohen's f^2** for F-tests in ANOVA

- ▶ **Eta squared:**

$$\eta^2 = \frac{SS_t}{SS_{total}}$$

- ▶ **Partial Eta Squared** $\eta_p^2 = \frac{SS_t}{SS_t + SS_E}$ if there are two or more factors

- ▶ **Cohen's f^2 :**

$$f^2 = \frac{\eta^2}{1 - \eta^2}$$

- For F-tests,

- ▶ $f^2 = 0.01$ ($\eta^2 = 0.01$)=small,
 - ▶ $f^2 = 0.06$ ($\eta^2 = 0.06$)=medium, and
 - ▶ $f^2 = 0.16$ ($\eta^2 = 0.14$) large effect sizes

1-way ANOVA - power and size

- Recall that ANOVA table for medley

```
## Analysis of Variance Table
##
## Response: diversity
##           Df Sum Sq Mean Sq F value Pr(>F)
## zinc       3 2.5666 0.85554   3.9387 0.01756 *
## Residuals 30 6.5164 0.21721
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

- So,

$$\eta^2 = \frac{2.5666}{2.5666 + 6.5164} = 0.283.$$

1-way ANOVA - power and size

- We use SPSS for calculation of power and size:
 - ▶ Analyze → Power Analysis → Means → One-way ANOVA

Random effects One-way ANOVA

- Random effects are another approach to designing experiments and modeling data. Random effects are appropriate when the **treatments are random** samples from a population of potential treatments.
- Consider the Medley and Clements (1998) data `medley_rand.csv`. Now we are comparing diatom species diversity across streams. **Streams** are treated as a random factor, assuming these streams represent a **random sample** of all possible streams in this part of the Rocky Mountains. The design is unbalanced with between four and seven stations (level 1 units) on each stream (level 2 clusters). In addition to diatom diversity, zinc concentration was also recorded at each station (a level 1 covariate).
 - ▶ Let's consider diversity (diatom diversity) as the response variable.

Random effects One-way ANOVA

- $\text{var}(Y_{ij}) = \sigma^2 + \sigma_\tau^2$.
- The terms σ^2 and σ_τ^2 are called components of variance or **variance components**.
- The variance components can be estimated by method of moments or maximum likelihood estimation.
 - ▶ If estimate by method of moments is negative, we just let it be 0.

Random effects One-way ANOVA

- 1-way ANOVA Table for the Random Effects Model

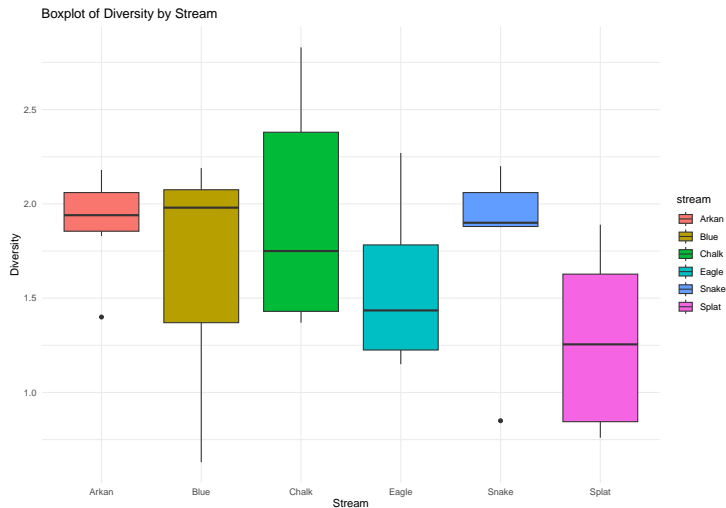
Table 2: 1-way ANOVA Table

Source	df	SS	MS	F
Treatments	$k - 1$	SS_T	$SS_T / (k - 1)$	MS_T / MSE
Error	$N - k$	SSE	$SSE / (N - k)$	
Total	$N - 1$	SS_{total}		

- $E(MSE) = \sigma^2$, $E(MS_T) = \sigma^2 + [(N - \sum_{i=1}^k n_i^2 / N) / (k - 1)]\sigma_\tau^2$, where $N = \sum_{i=1}^k n_i$.
- For balanced design, $E(MSE) = \sigma^2$, $E(MS_T) = \sigma^2 + n\sigma_\tau^2$.

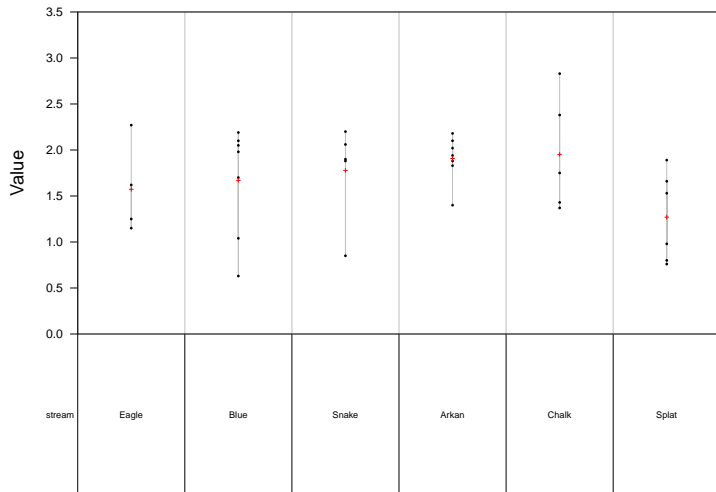
Random effects One-way ANOVA

- Visualization of the data



Random effects One-way ANOVA

- We are not interested in a population level mean for each group. Instead, we are interested in the **population of all streams**, from which these six streams may be thought of as a random (or at least representative) sample.



Random effects One-way ANOVA

- ANOVA table is identical to that of fixed-effect model

##		Df	Sum Sq	Mean Sq	F value	Pr(>F)
##	stream	5	1.828	0.3656	1.411	0.251
##	Residuals	28	7.255	0.2591		

Random effects One-way ANOVA

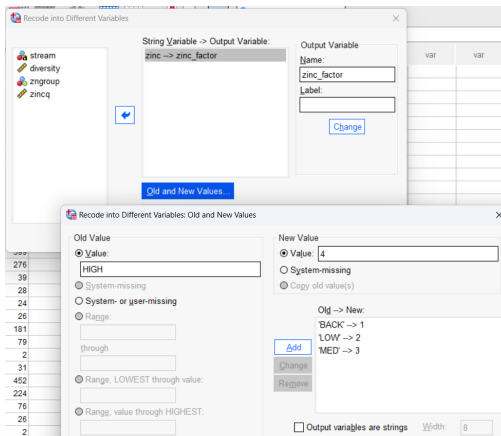
- Variance component analysis

```
##
##
## Result Variance Component Analysis:
## -----
##
##      Name    DF      SS      MS      VC      %Total    SD      CV[%]
## 1 total    31.348886      0.278044 100      0.527299 31.12526
## 2 stream    5      1.827846 0.365569 0.01893  6.80842  0.137588 8.121501
## 3 error    28      7.255178 0.259113 0.259113 93.19158 0.509032 30.04702
##
## Mean: 1.694118 (N = 34)
##
## Experimental Design: unbalanced | Method: ANOVA
```

- The variance of Y_{ij} is estimated by $0.01893 + 0.259113 = 0.278044$. Hence, only about $0.01893/0.278044 = 6.8\%$ of the total variance of the diversity is due to stream.

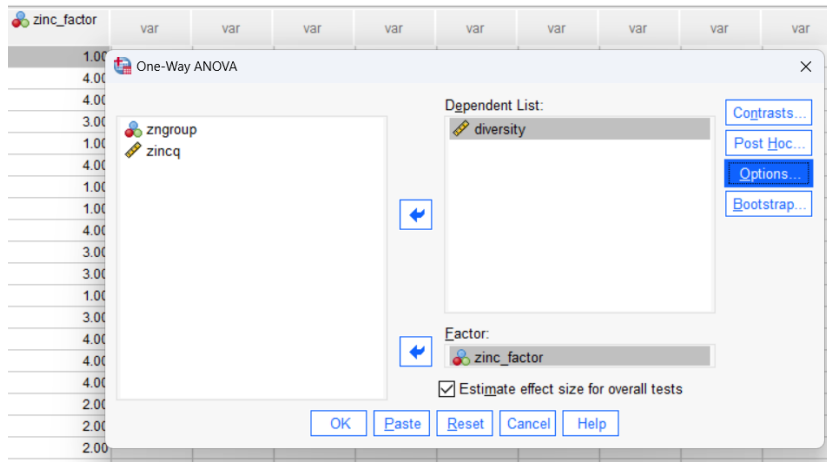
Lab 1: Analysis of medley.csv

- Consider the fixed-effect model
 - ▶ response: diversity
 - ▶ factor: zinc
- First, convert the variable zinc to a factor zinc_factor by clicking on Transform
→ Recode into Different Variables
 - ▶ zngroup was recoded from zinc



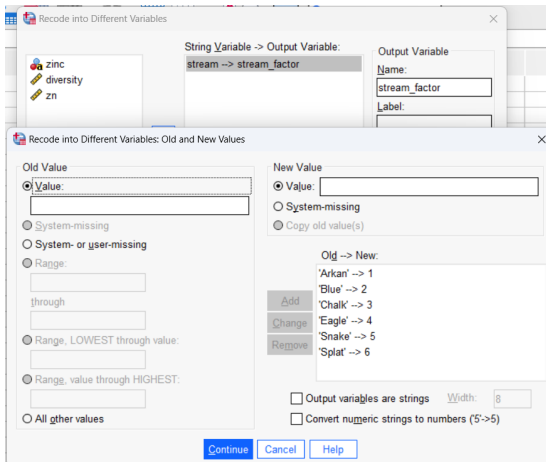
Lab 1: Analysis of medley.csv

- Analyze → Compare Means and Proportions → One-Way ANOVA



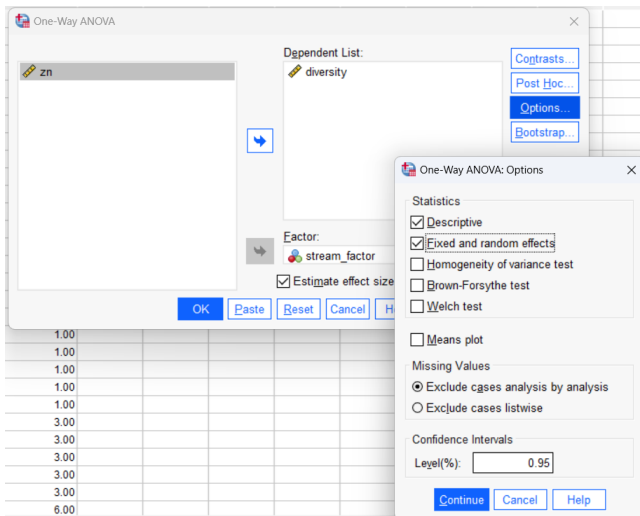
Lab 2: Analysis of medley_rand.csv

- ▶ Consider the random-effect model
 - ▶ response: diversity
 - ▶ factor: stream
- Again, like in lab 1, we convert variable `stream` to a factor

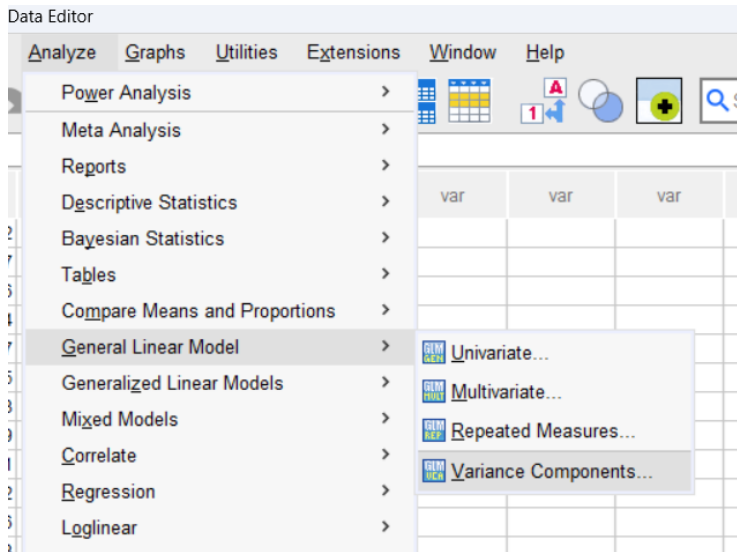


Lab 2: Analysis of medley_rand.csv

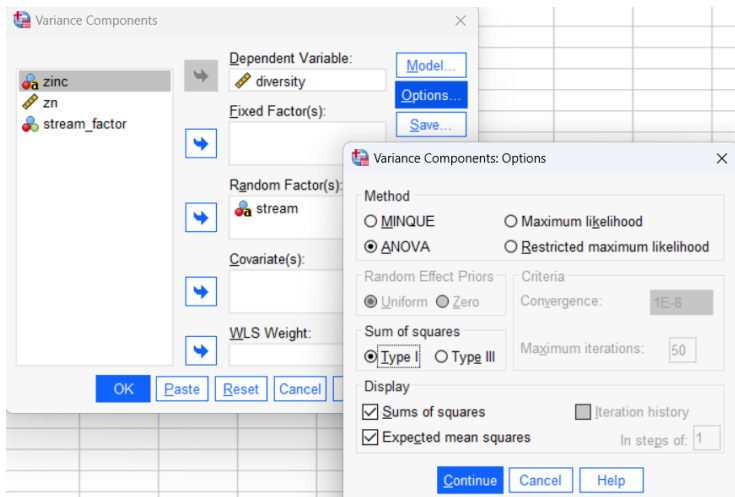
- Analyze → Compare Means and Proportions → One-Way ANOVA



Lab 2: Analysis of medley_rand.csv



Lab 2: Analysis of medley_rand.csv



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