Contents

Although we consider this a review and we will move quickly, nevertheless you are expected to know this material for all tests.

1.1 Constants and Variables

A quantity whose value remains *unchanged* is called a **constant**. For example, 2, -5, $\sqrt{7}$, π are constants.

A **variable** is a quantity to which an unlimited number of values can be assigned. Variables are denoted by the later letters of the alphabet (lowercase in general). Thus, in the equation of a straight line,

$$\frac{x}{a} + \frac{y}{b} = 1,$$

x and y may be considered as the variable coordinates of a point moving along the line.

Arbitrary constants, or **parameters**, are constants to which any one of an unlimited set of numerical values may be assigned, and they are supposed to *have these assigned values throughout the investigation.* They are usually denoted by the earlier letters of the alphabet. Thus, for every pair of values arbitrarily assigned to a and b, the equation

$$\frac{x}{a} + \frac{y}{b} = 1,$$

represents some particular straight line.

1.2 Division by Zero Excluded

$\frac{0}{0}$ is indeterminate. For the quotient of two numbers is that number which multiplied by the divisor will give the dividend. But any number whatever multiplied by zero gives zero, and the quotient is indeterminate; that is, any number whatever may be considered as this quotient, a result which is of no value.

Let a be a nonzero real number. $\frac{a}{0}$ has no meaning, for there exists no number such that if it be multiplied by zero, the product will equal a.

Therefore division by zero is not an admissible operation. **Care should be taken not to divide by zero inadvertently**. The following fallacy is an illustration.

Assume that

$$a = b, a \neq 0, b \neq 0.$$

Then evidently

$$ab = a^2.$$

Subtracting b^2,

$$ab - b^2 = a^2 - b^2.$$

Factoring,

$$b(a - b) = (a + b)(a - b).$$

Dividing by $a - b$,

$$b = a + b = b + b = 2b, \quad \text{since } a = b.$$

Dividing by b, we have

$$1 = 2.$$

The result is absurd, and is caused by the fact that we divided by $ab = 0$, which is illegal.

1.3 Interval of a Variable

Very often we confine ourselves to a portion only of the number system. For example, we may restrict our variable so that it shall take on only such real values as lie between a and b. Remember when writing or reading interval notation:

Using the left square bracket [means the start value is included in the set;

Using the left parenthesis (means the start value is not included in the set ;

Using the right square bracket] means the end value is included in the set;

Using the right parenthesis) means the end value is not included in the set.

For example, we shall employ the symbol $[a, b]$, a being less than b, to represent the numbers a, b, and all the numbers between them.

The interval $(5, 10]$ for x means $5 < x \leq 10$; $[5, 10)$ means $5 \leq x < 10$; $(5, 10)$ means $5 < x < 10$; And $[5, 10]$ means $5 \leq x \leq 10$;

A variable x is said to **vary continuously** through an interval $[a, b]$, when x takes *all real values* between a and b with both a and b are included.

That is, x starts with the value a and increases until it takes on the value b in such a manner as to assume the value of every number between a and b in the order of their magnitudes. This may be illustrated geometrically as a line segment.

The set of all real numbers in interval notation is $(-\infty, \infty)$. Infinity ∞ is not a real number, in this case it just means "continuing on ..." or there is no bound in that direction. So in our course, we never use ∞ as a closed end point in an interval.

Let a be a real number.

- (a, ∞) is equivalent to $\{x|x > a\}$;

- $[a, \infty)$ is equivalent to $\{x|x \geq a\}$;
- $(-\infty, a)$ is equivalent to $\{x|x < a\}$;
- $(-\infty, a]$ is equivalent to $\{x|x \leq a\}$.

1.4 Necessity and Sufficiency

That a statement is true or not is always under some condition(s). In the following, we use P and Q to denote a statement.

Definition. If P is true, then Q is true; we say that P is **sufficient** for Q. Mathematically, we write "$P \Rightarrow Q$" and read "P implies Q".

For example, $a > 0, b > 0 \Rightarrow ab > 0$.

Note that **sufficient conditions may not be unique**. For example, $a < 0, b < 0 \Rightarrow ab > 0$.

Definition. If P is true, then Q is true (If Q is not true then P is not true); we say that Q is **necessary** for P. Mathematically, we write "$\overline{Q} \Rightarrow \overline{P}$" (or $P \Rightarrow Q$), where $\overline{Q}$ (not Q) means Q is not true.

Note that necessary conditions may not be sufficient. For example, $ab > 0 \nRightarrow a > 0, b > 0$.

In mathematical prose for instance, several necessary conditions that, taken together, constitute a sufficient condition (i.e., individually necessary and jointly sufficient).

Definition. If $P \Rightarrow Q$ and $Q \Rightarrow P$, we say P is a **necessary and sufficient condition** of Q. We write $P \Leftrightarrow Q$ and read it as P is true **iff (if and only if)** Q is true.

For example, $a > b \Leftrightarrow a - b > 0$.
Another example, $P \Rightarrow Q \Leftrightarrow \overline{Q} \Rightarrow \overline{P}$.

1.5 Points in the Cartesian Coordinate Plane

Definition. A **set** is a collection of objects, where the objects are called the **elements** of the set. The elements can be numbers or vectors.

- In *listing* the elements of a set, it is customary of enclose the listed elements in braces, $\{\}$, and separate them by commas. For example, the set of natural numbers: $\{1, 2, 3, 4, \ldots\}$.
- We can describe a set of real numbers using set-builder notation, which would look like this: $\{x|10 \leq x < 30\}$. The vertical bar $|$ is read as such that, so altogether we would read as the set of real x-values such that x is greater than or equal to 10 and less than 30.
- Interval notation as described in the last subsection.

Definition. An **ordered pair** of real numbers is a pair of real numbers in which the order is specified.

- The ordered pair (a, b) has first component a and second component b.
- Two ordered pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$.

Definition. The sets of ordered pairs of real numbers are identified with points on a plane called the **coordinate plane** or the **Cartesian plane**.

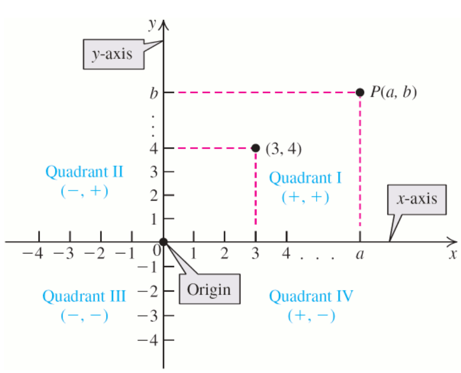


Figure 1: The Cartesian plane

We refer to the point corresponding to the ordered pair (a, b) as the **graph** of the ordered pair (a, b) in the coordinate system. The notation $P(a, b)$ designates the point P in the coordinate plane whose x-coordinate is a and whose y-coordinate is b.

1.6 Functions

1.6.1 Definition of Functions

Reading: Text Section 1.1 and 1.2.

Definition. A (real-valued) **function** (or map) f of the real numbers $\mathbb{R}$ is a well-defined rule that assigns each input $x \in \mathbb{R}$ exactly one output, called $f(x) \in \mathbb{R}$.

A map (function) is therefore a **many-to-one** relation.

Example. Write the equation of the circle centered at $(3, -8)$ with radius 4. Does the equation define a function y of x.

Vertical Line Test The **vertical line test** is a handy way to think about whether a graph defines the vertical output as a function of the horizontal input. Imagine drawing vertical lines through the graph. If any vertical line would cross the graph more than once, then the graph does not define only one vertical output for each horizontal input.

Definition (Domain and Range)**. Domain**: The set of all input values to a function f.

Range: The set of all output values of a function f. It is determined by the function and the domain of the function.

Example. $f(x) = x^2, x \in D = [-1, 1]$.

Note. If the domain D of a function f is not specified, we take

$$D = \{\text{all points for which } f \text{ is meaningful}\}$$

which is the largest possible domain of definition of the function.

Usually functions are described algebraically using some formula (such as $f(x) = x^2$, for all real numbers x) but it doesn't have to be so simple. For example,

$$f(x) = \begin{cases} x^2, & \text{if } x \text{ is an integer} \\ 0, & \text{otherwise} \end{cases}$$

is a function on $\mathbb{R}$ but it is given by a relatively complicated rule. Namely, the rule f tells you to associate to a number x the value 0 unless x is an integer, in which case you are to associate the value x^2. This type of function is called *piecewise functions*.

Definition. For a real-valued function $f : D \to R$ defined on a subset D of $\mathbb{R}$, the **graph** of f consists of all the points $(x, f(x))$ in the xy-plane.

That is, graphs of functions are typically created with the input quantity along the horizontal $x-$axis and the output quantity along the vertical $y-$axis.

Example. Draw the graph of the absolute value function $y = |x|$.

Notation of functions:

For each x in the domain of a function f, there corresponds a unique y in its range.

- The number y is denoted by $f(x)$ read as "f of x" or "f at x". We call $f(x)$ the value of f at the number x.
- We say that y is a function of x, or write $y = f(x)$ when the function is named f.
- y is sometimes referred to as the **dependent variable** and x as the **independent variable**.

Definition. Let V and W be sets and $f : V \to W$ be a map between them. The function f is called **one-one** iff (if and only if) $x_1 = x_2$ whenever $f(x_1) = f(x_2)$.

The function f is called **onto** iff for every $y \in W$ there is always an $x \in V$ such that $f(x) = y$. A function is called **one-one onto** iff it is both one-one and onto.

Remark 1.1. Think of the equation $y = f(x)$ as a problem to be solved for x. Then:

$$\text{the function } f : V \to W \text{ is } \left\{ \begin{array}{c} \text{one-one} \\ \text{onto} \\ \text{one-one onto} \end{array} \right\}$$

if and only if for every $y \in W$ the equation

$$y = f(x) \text{ has } \left\{ \begin{array}{c} \text{at most} \\ \text{at least} \\ \text{exactly} \end{array} \right\} \text{one solution } x \in V.$$

Remark 1.2. All functions in this course are onto. You can easily construct a function which is not onto.

Example. The function

$$\mathbb{R} \to \mathbb{R} : x \mapsto f(x) = x^3$$

is both one-one and onto since the equation

$$y = x^3$$

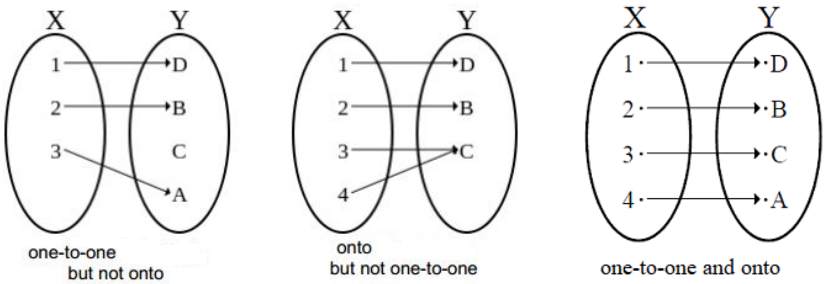


Figure 2: What is a onto function?

possesses the unique solution $y^{\frac{1}{3}} \in \mathbb{R}$ for every $y \in \mathbb{R}$.

Example. In contrast, the function

$$\mathbb{R} \to \mathbb{R} : x \to y = x^2$$

is not one-one since the equation

$$4 = x^2$$

has *two* distinct solutions, namely $x = 2$ and $x = -2$. It is also not onto since $-4 \in \mathbb{R}$, but the equation

$$-4 = x^2$$

has *no* real solutions $x \in \mathbb{R}$.

Definition. The **composition** $f \circ g$ or $f(g(\cdot))$ of two functions

$$g : U \to V, \quad f : V \to W$$

is the map

$$f \circ g : U \to W$$

defined by

$$(f \circ g)(u) = f(g(u)) \quad \text{for } u \in U.$$

For any set V the **identity function** denoted by

$$I_V : V \to V$$

is defined by

$$\mathbf{I_V}(\mathbf{v}) = \mathbf{v} \quad \text{for } v \in V.$$

It satisfies the identities

$$I_V \circ S = S$$

for $S : U \to V$ and

$$T \circ I_V = T$$

for $T : V \to W$.

Example. Let $f(x) = x^2$ and $g(x) = \ln(x)$. Find $g \circ f$.

1.6.2 Inverse Functions

Reading: Text Section 1.3.

Definition (Inverse)**.** Let $f : D \to R$ be a one-to-one and onto function. Its **inverse** is a function f^{-1} such that whenever $y = f(x)$, then $f^{-1}(y) = x$.

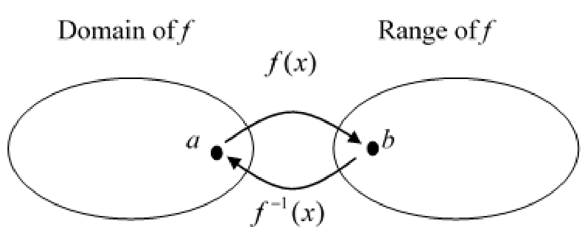


Figure 3: Inverse function

Notice that original function f and the inverse function f^{-1} undo each other. If $f(a) = b$, then $f^{-1}(b) = a$, returning us to the original input. More simply put, if you compose these functions together you get the original input as your answer. and

Remark 1:

$$f^{-1} \circ f = I_D \quad \text{and} \quad f \circ f^{-1} = I_R.$$

Remark 2: Since the outputs of the function f are the inputs to f^{-1}, the range of f is also the domain of f^{-1}. Likewise, since the inputs to f are the outputs of f^{-1}, the domain of f is the range of f^{-1}.

Horizontal Line Test: A function f has an inverse only if it is **one-to-one**. Once you have determined that a graph defines a function, an easy way to determine if it is a one-to-one function is to use the horizontal line test. Draw horizontal lines through the graph. If any horizontal line crosses the graph more than once, then the graph does not define a one-to-one function.

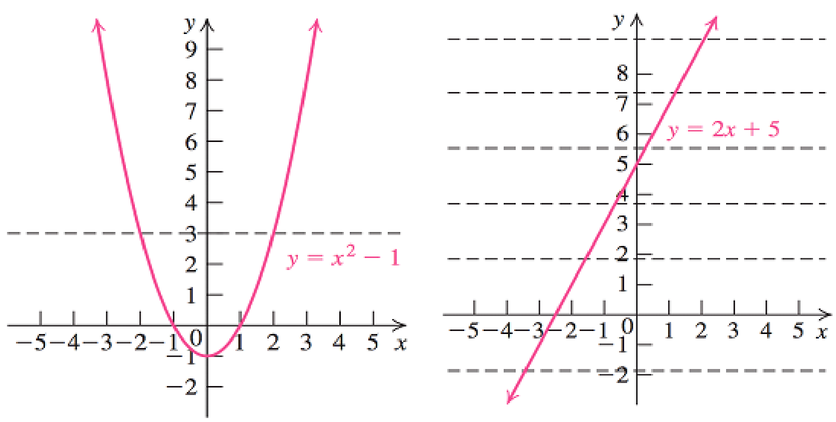


Figure 4: Horizontal Line Test

Procedure for Finding f^{-1}: It is a convention to use symbol x as the independent variable (input) and use symbol y as the dependent variable (output). So we follow these steps to find f^{-1} of f.

Step 1: Replace $f(x)$ by y in the equation for $f(x)$ if needed.

Step 2: **Interchange** x **and** y.

Step 3: Solve the equation in Step 2 for y.

Step 4: Replace y with $f^{-1}(x)$ if required.

Example Find the inverse of the function $f(x) = \sqrt{x-1}$.

Solution.

$\square$

Graphing Inverse Functions Note that the domain and range of the function do correspond with the range and domain of the inverse function on the limited domain. In fact, if we graph $f(x)$ on the restricted domain and $f^{-1}(x)$ on the same axes, we can notice symmetry: the graph of $f^{-1}(x)$ is the graph of $f(x)$ reflected over the line $y = x$.

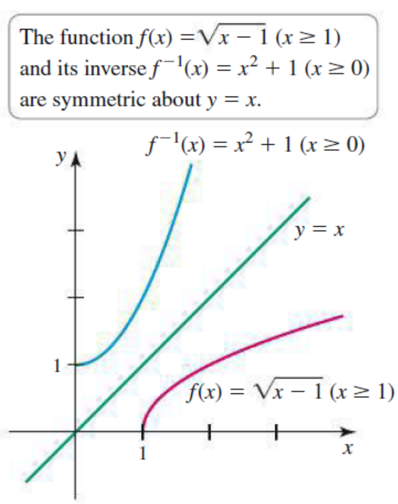


Figure 5: Graphs of f and f^{-1}

Here, symmetric about $y = x$ means if (a, b) is on the graph of f, then (b, a) is on the graph of $f^{-1}(x)$. This confirms to the definition of inverse functions.

Recall that the inverse function of the exponential function $f(x) = b^x, b > 0, b \neq 1$ is the Logarithmic Function

$$y = \log_b(x), b > 0, b \neq 1.$$

For this, let's recall the properties of **exponential functions** and **logarithmic functions**.

Exponents

Here we go over the definition of x^y when x and y are arbitrary real numbers, with $x > 0$.

For any real number x and any positive integer $n = 1, 2, 3, \ldots$ one defines

$$x^n = \overbrace{x \cdot x \cdot \cdots \cdot x}^{n \text{ times}}$$

and, if $x \neq 0$,

$$x^{-n} = \frac{1}{x^n}.$$

One defines $x^0 = 1$ for any $x \neq 0$.

To define $x^{p/q}$ for a general fraction $\frac{p}{q}$ one must assume that the number x is positive. One then defines

$$x^{p/q} = \sqrt[q]{x^p}.$$

It is shown in precalculus texts that the power functions satisfy the following properties:

$$\boxed{(xy)^a = x^a y^a, \quad x^a x^b = x^{a+b}, \quad \frac{x^a}{x^b} = x^{a-b}, \quad \left(x^a\right)^b = x^{ab}}$$

provided a and b are fractions. And these properties still hold if a and b are real numbers (not necessarily fractions.) We won't go through the proofs here.

Now instead of considering x^a as a function of x we can pick a positive number a and consider the **exponential function**

$$\boxed{f(x) = a^x.}$$

This function is defined for all real numbers x (as long as the base a is positive.).

The exponential function

$$\boxed{f(x) = e^x}$$

with base e (Euler's constant ≈ 2.718) is so prevalent in the sciences that it is often referred to as the **exponential function** or **the natural exponential function**.

Logarithmic Functions

The **logarithm function**, written $\log_a x$, is the inverse of the exponential function a^x. The function

$$f(x) = \log_a x,$$

is called the **logarithmic function with base** a.

Remark: $y = \log_a x \Leftrightarrow x = a^y$.

Theorem (Properties of Logs)**.** The following properties can be derived from the exponential function.

1. $\log_a(a^x) = x$
2. $a^{\log_a x} = x$
3. $\log_a(x^p) = p \log_a x$
4. $\log_a(A) = \dfrac{\log_b(A)}{\log_b(a)}$ (change of base)
5. $\log_a(B) + \log_a(C) = \log_a(BC)$
6. $\log_a(B) - \log_a(C) = \log_a(\frac{B}{C})$

The logarithm with base e is called the **Natural Logarithm**, and is written

$$\ln x = \log_e x.$$

Thus we have

$$\boxed{e^{\ln x} = x \qquad \ln e^x = x}$$

where the second formula holds for all real numbers x but the first one only makes sense for $x > 0$.

1.6.3 Graphical Behavior of Functions

Definition (Symmetry in Functions)**.** A function f is called an **even function** if $f(-x) = f(x)$ for all x in the domain. The graph of an even function is symmetric about the y-axis. A function f is called an **odd function** if $f(-x) = -f(x)$ for all x in the domain. The graph of an even function is symmetric about the the origin.

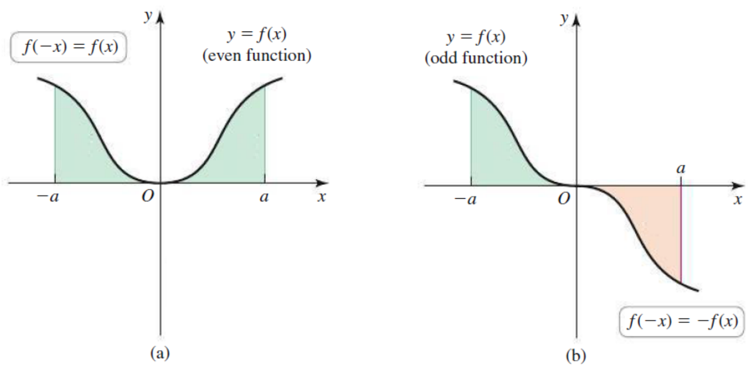


Figure 6: Even functions and odd functions

For example, $y = x^2$ is an even function and $y = x^3$ is an odd function.

Example. $f(x) = x^3 - 3x + 1$. Is the function even, odd, or neither?

Solution.

$\square$

Definition (Increasing and Decreasing Functions)**.** Let f be a function defined on an interval I.

1. f is increasing on I if for every $a < b$ in I, $f(a) \leq f(b)$.
2. f is decreasing on I if for every $a < b$ in I, $f(a) \geq f(b)$.

A function is **strictly increasing** when $a < b$ in I implies $f(a) < f(b)$, with a similar definition holding for **strictly decreasing**.

Example. Finding intervals of increasing/decreasing for the quadratic function.
Solution.

□

Definition (Periodic Functions)**.** A **periodic function** is a function for which a specific horizontal shift, P, results in the original function: $f(x + P) = f(x)$ for all values of x. When this occurs we call the smallest such horizontal shift with $P > 0$ the **period** of the function.

Remark: All trigonometric functions are periodic functions.

- Sine function: $y = \sin x$, period=2π.
- Cosine function: $y = \cos x$, period=2π.
- Tangent function: $y = \tan x$, $x \neq \frac{\pi}{2} + k\pi$ where k is an integer, period=π.
- Cotangent function: $y = \cot x$ (reciprocal of $\tan x$), $x \neq k\pi$, period=π

In mathematics, a function f defined on some set X with real or complex values is called bounded if the set of its values is bounded.

Definition (Bounded Functions)**.** Let $f(x)$ be a real function.

- The function f is **bounded above** if there is some number M such that

$$f(x) \leq M \qquad \text{for all } x.$$

- The function f is **bounded below** if there is some number m such that

$$m \leq f(x) \qquad \text{for all } x.$$

- If the function f is both bounded above and bounded below, then the function f is said to be **bounded**.

For example both $f(x) = \sin x$ and $g(x) = \cos x$ are bounded.

1.6.4 Basic Functions

Reading: Text Section 1.4.

When working with functions, it is helpful to have a base set of elements to build from.

(1) **Linear function**: $y = f(x) = ax + c$, where a and c are constants. Especially,

- Constant: $f(x) = c$.
- Identity: $f(x) = x$.

(2) **Absolute Value function**: $y = |x|$.

(3) **Power functions**: $y = x^m$, where m is a real number.

- Quadratic: $y = x^2$.

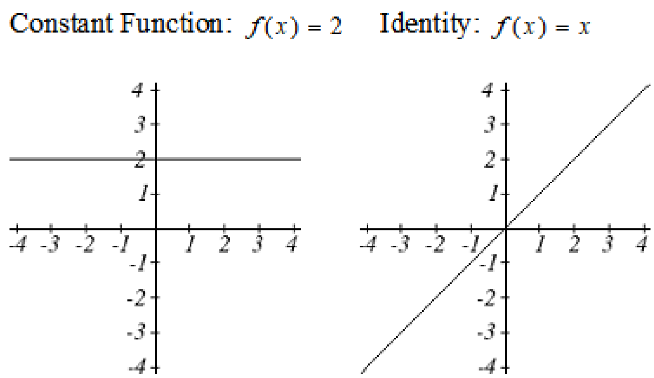


Figure 7: Linear Functions

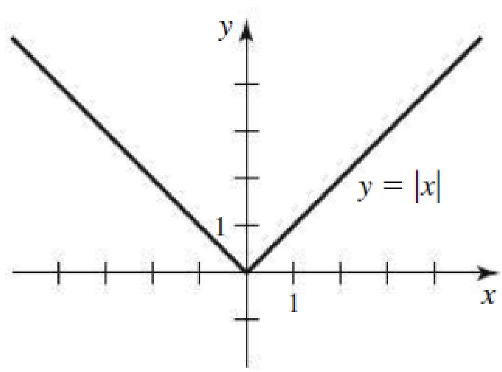


Figure 8: The Absolute Value Function

- Cubic: $y = x^3$.
- Reciprocal: $y = \dfrac{1}{x}, x \neq 0$.
- Reciprocal squared: $y = \dfrac{1}{x^2}, x \neq 0$.
- Square root: $y = \sqrt{x}, x \geq 0$.
- Cube root: $y = \sqrt[3]{x} = x^{1/3}$.

(4) **Exponential functions**: $f(x) = b^x$, where $b > 0$ and $b \neq 1$.

- Natural Exponential function: $y = \exp(x) = e^x$, where e is the Euler's number $e \approx 2.71828$.

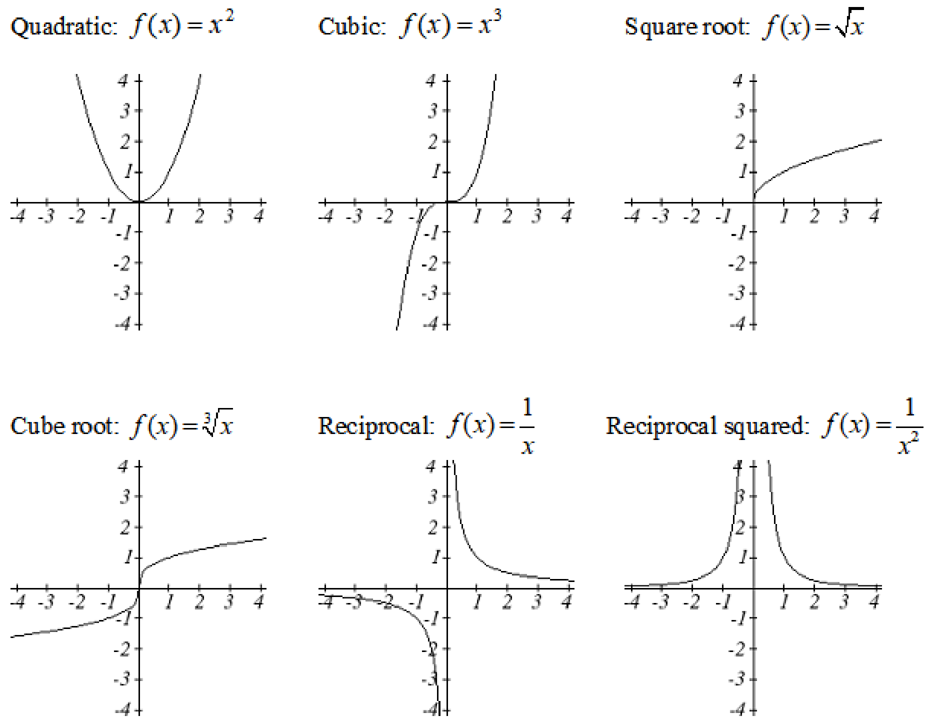


Figure 9: Power Functions

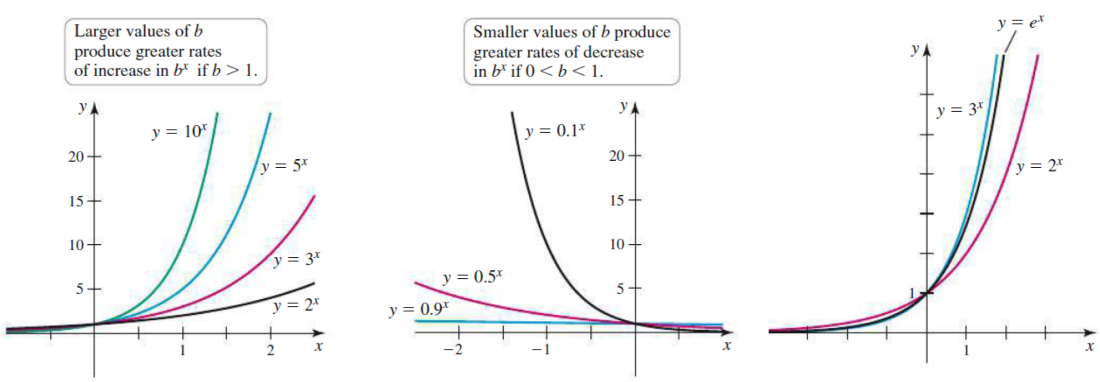


Figure 10: Exponential Functions

(5) **Logarithmic functions**: $y = \log_b x, x > 0, b > 0, b \neq 1$.

Natural Logarithmic function: $y = \ln x$.

(6) **Polynomial functions**: A **polynomial** is an expression of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, a_n \neq 0$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are constants. Such a polynomial is said to have degree

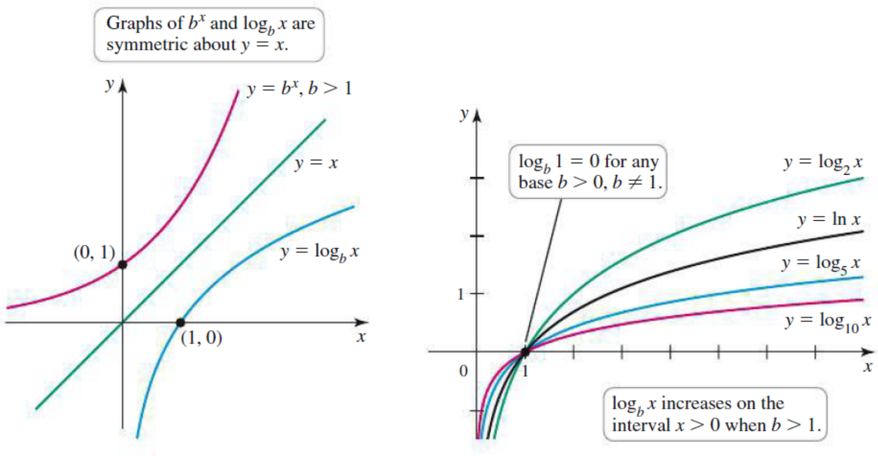


Figure 11: Log Functions

n,

$$\deg P(x) = n, \quad \text{provided } a_n \neq 0.$$

The domain of a polynomial function is the set of all real numbers. The graph of a polynomial function is a continuous curve.

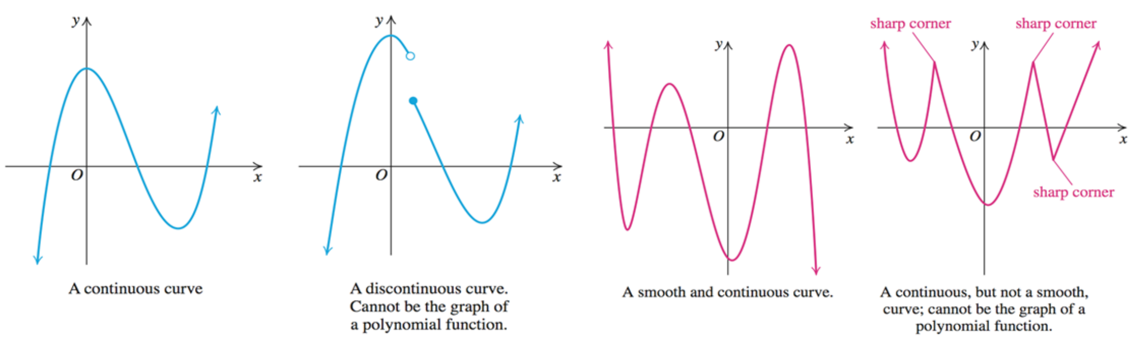


Figure 12: Polynomial Functions

(7) **Rational functions**: A **rational function** is a function that can be expressed as the ratio of two polynomials.

$$f(x) = \frac{P(x)}{Q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}, a_n \neq 0, b_m \neq 0.$$

(8) Trigonometric Functions:

- Sine function: $y = \sin x$.

- Cosine function: $y = \cos x$.
- Tangent function: $y = \tan x$, $x \neq \frac{\pi}{2} + k\pi$ where k is an integer.
- Cotangent function: $y = \cot x$ (reciprocal of $\tan x$), $x \neq k\pi$.
- Cosecant function: $y = \csc x = \dfrac{1}{\sin x}$, $x \neq k\pi$.
- Secant function: $y = \sec x = \dfrac{1}{\cos x}$, $x \neq \frac{\pi}{2} + k\pi$.

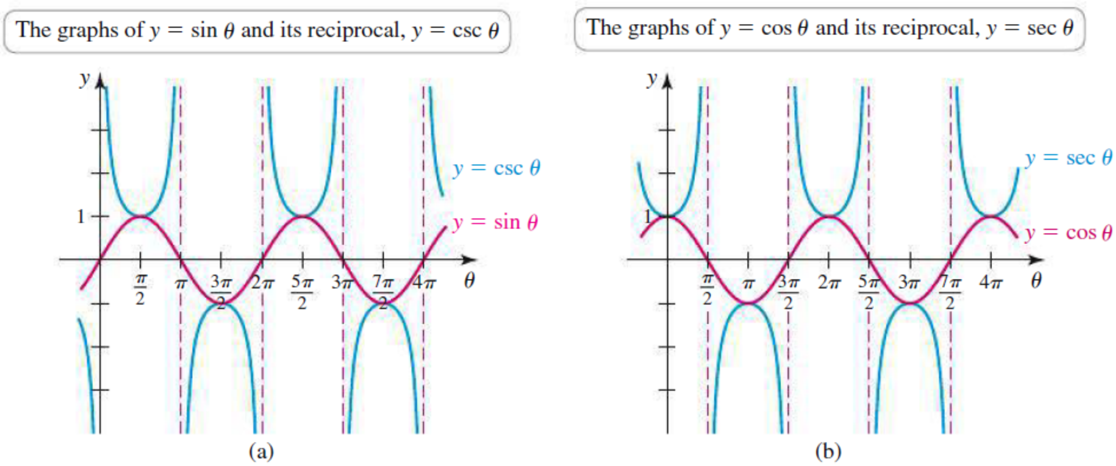


Figure 13: Sine and Cosine

(9) Inverse Trigonometric Functions:

- Inverse of Sine: $y = \arcsin x$ or $\sin^{-1} x$ which is the inverse of $y = \sin x, x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.
- Inverse of Cosine: $y = \arccos x$ or $\cos^{-1} x$ which is the inverse of $y = \cos x, x \in [0, \pi]$.
- Inverse of Tangent: $y = \arctan x$ or $\tan^{-1} x$ which is the inverse of $y = \tan x, x \in (-\frac{\pi}{2}, \frac{\pi}{2})$.
- Inverse of Cotangent: $y = \mathit{arccot}\ x$ or $\cot^{-1} x$ which is the inverse of $y = \cot x, x \in (0, \pi)$.

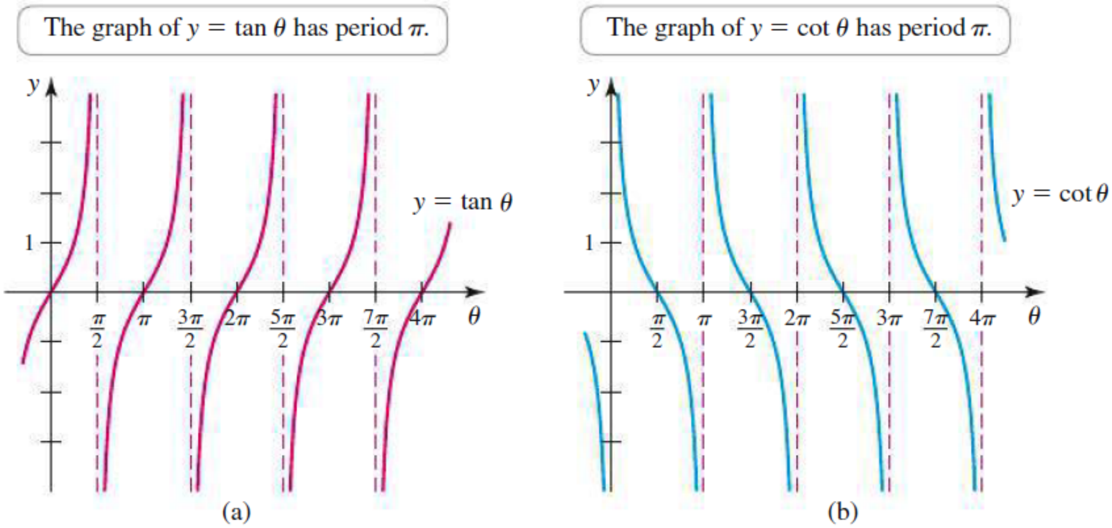


Figure 14: Tangent and Cotangent

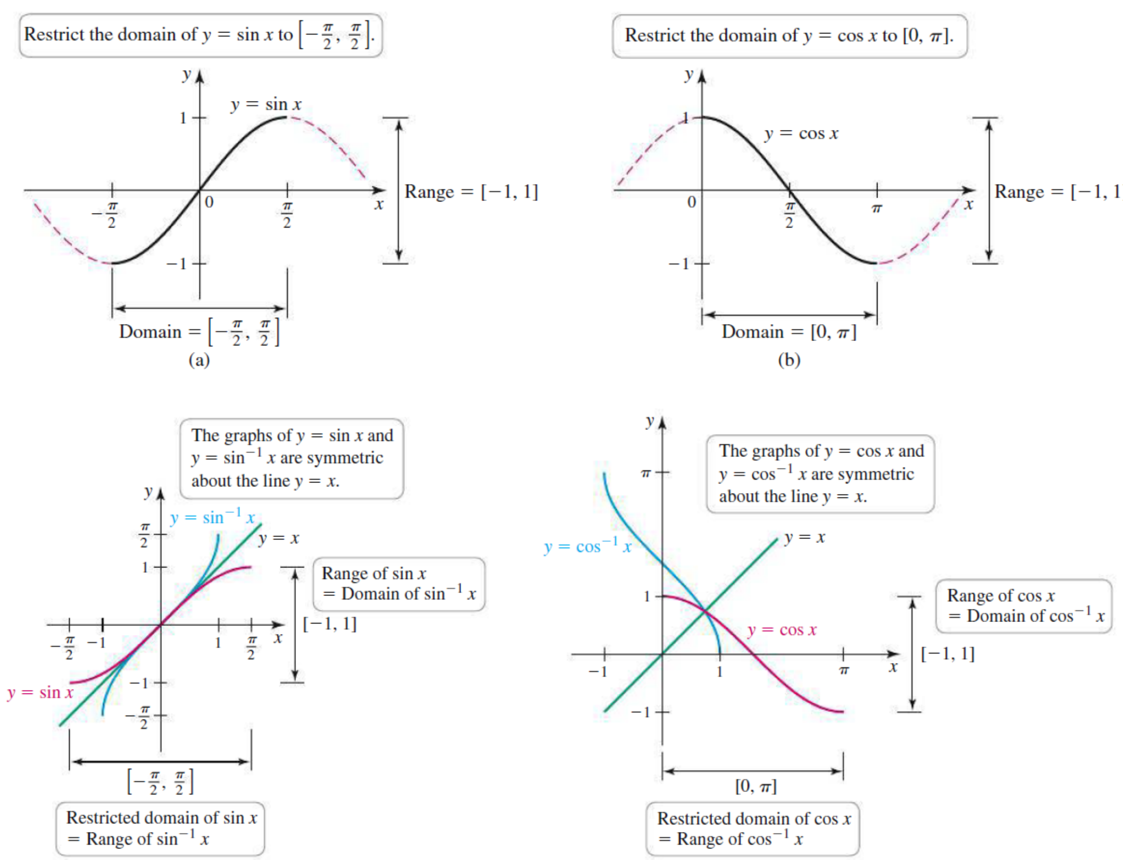


Figure 15: Graphs of Arcsine and Arccosine

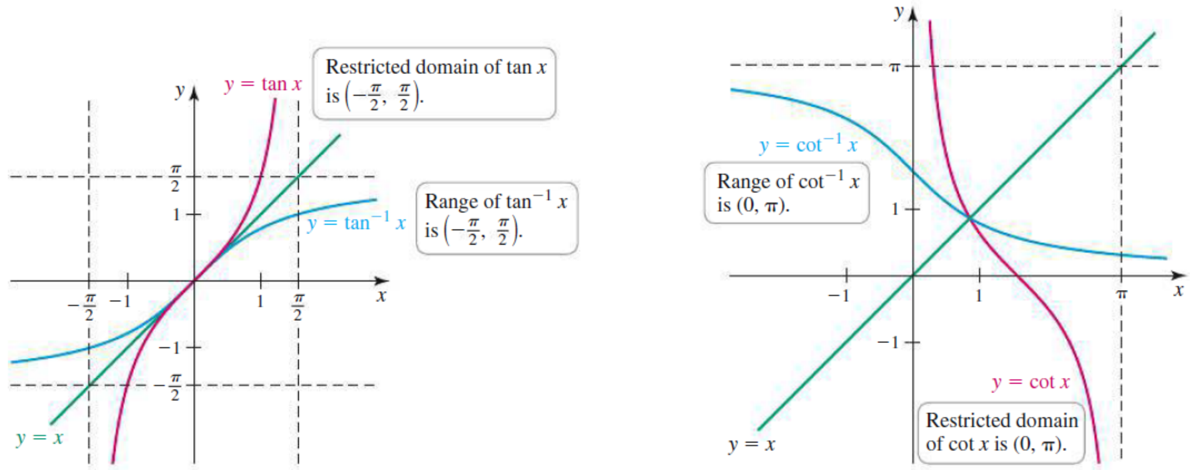


Figure 16: Graph of Arctangent and Arccot

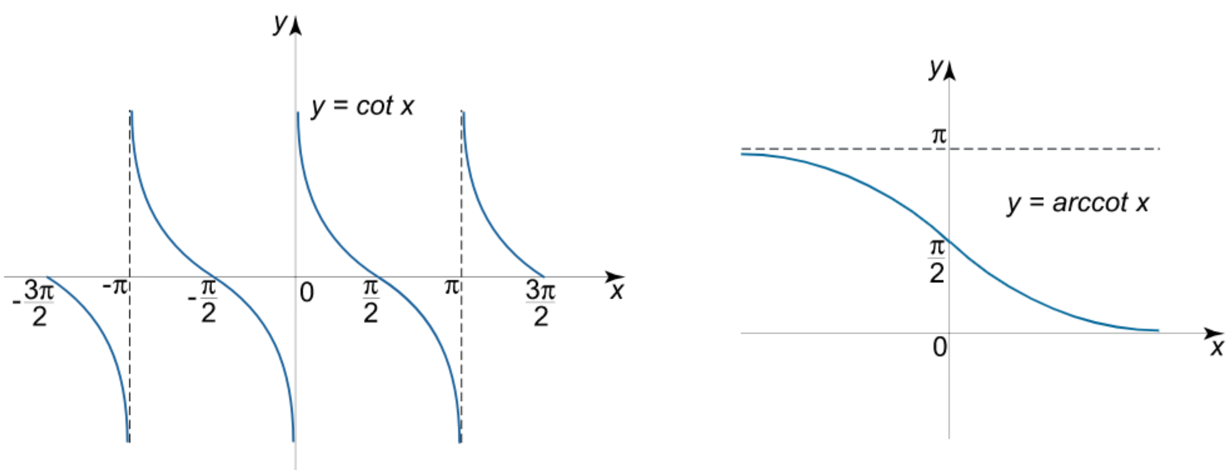


Figure 17: Graph of Arccot

Contents

Calculus means "a method of calculation or reasoning." The concept of a limit or limiting process, essential to the understanding of calculus, has been around for thousands of years. In fact, early mathematicians used a limiting process to obtain better and better approximations of areas of circles. Yet, the formal definition of a limit - as we know and understand it today - did not appear until the late 19th century.

If a variable v takes on successively a series of values that approach nearer and nearer to a constant value L in such a manner that $|v - L|$ becomes and remains less than any assigned arbitrarily small positive quantity, then v is said to approach the limit L, or to converge to the limit L. Symbolically this is written

$$\lim_{v \to L} v = L \text{ or } v \to L.$$

In this chapter, we study the limit of a function $f(x)$ when the independent variable x approaches a constant x_0 or ∞.

2.1 The Idea of Limits

Consider the function $f(x) = \dfrac{\sin x}{x}$.
(1) When x is close to the value 1, what value (if any) is $f(x)$ close to?

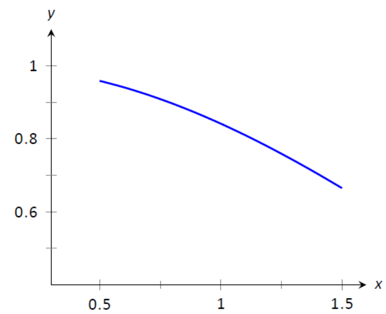


Figure 1: $\sin x/x$ near $x = 1$

We first look at the graph of this function to approximate the appropriate y values. Consider Figure 1 where $y = \frac{\sin x}{x}$ is graphed. For values of x near 1, it seems that y takes on values near 0.85. In fact, when $x = 1$, then $y = \frac{\sin 1}{1} = \sin 1 \approx 0.8415$. so it makes sense that when x is "near" 1, y will be "near" $\sin 1$.

(2) When x is close to the value 0, what value (if any) is $f(x)$ close to?

By considering Figure 2, one can see that it seems that y takes on values near 1. But what happens when $x = 0$? We have

$$y \to \frac{\sin 0}{0} = \frac{0}{0}$$

The expression $\dfrac{0}{0}$ has no value; it is **indeterminate**. Such an expression gives no information about what is going on with the function nearby. We cannot find out how y behaves near $x = 0$ for this function simply by letting $x = 0$. It can be shown that later that the limit is 1. But the function $y = \frac{\sin x}{x}$ never reaches 1.

Finding a limit entails understanding how a function behaves near a particular value of x. Before continuing, it will be useful to establish some notation. The expression "the limit of y as x approaches x_0" describes a number L (if exits), that y

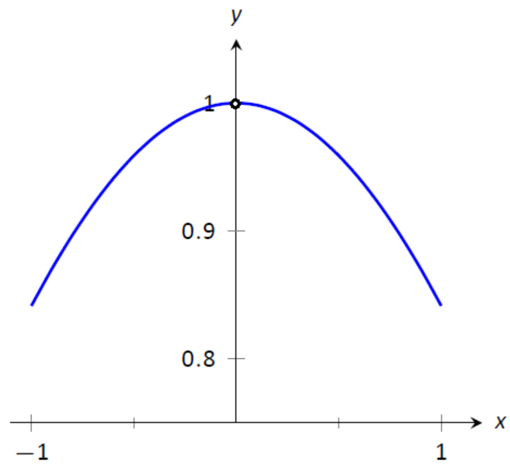


Figure 2: $\sin x/x$ near $x = 0$

approaches as x approaches x_0. We write all this as

$$\lim_{x \to x_0} f(x) = L.$$

Above, we have (to be shown rigorously later)

$$\lim_{x \to 1} \frac{\sin x}{x} = \sin 1 \quad \text{and} \quad \lim_{x \to 0} \frac{\sin x}{x} = 1.$$

For now, we will approximate both limits numerically.

x	$\sin(x)/x$
0.9	0.870363
0.99	0.844471
0.999	0.841772
1	**0.841471**
1.001	0.84117
1.01	0.838447
1.1	0.810189

x	$\sin(x)/x$
-0.1	0.9983341665
-0.01	0.9999833334
-0.001	0.9999998333
0	**not defined**
0.001	0.9999998333
0.01	0.9999833334
0.1	0.9983341665

Figure 3: Numerical approximation of $\lim_{x \to 1} \frac{\sin x}{x}$ and $\lim_{x \to 0} \frac{\sin x}{x}$

This numerical method gives confidence to say that 1 is a good approximation of $\lim_{x \to 0} \frac{\sin x}{x}$; that is,

$$\lim_{x \to 0} \frac{\sin x}{x} \approx 1.$$

Later we will be able to prove that the limit is exactly 1.

Example. Use graphical and numerical methods to approximate

$$\lim_{x \to 3} \frac{x^2 - x - 6}{6x^2 - 19x + 3}.$$

Solution.

$\square$

2.2 Definitions of Limits

2.2.1 Limit of a Function

Intuitively, $\lim_{x \to a} f(x) = L$ means that $f(x)$ **approaches** L as x approaches a.

Definition (Limit of a Function (Preliminary))**.** Suppose the function f is defined for all x near a except possibly at a. If $f(x)$ is arbitrarily close to L (as close to L as we like) for all x sufficiently close (but not equal)to a, we write

$$\lim_{x \to a} f(x) = L$$

and say the limit of $f(x)$ as x approaches a equals L.

The problem with the definition is that the words "approach" and "close to" are not exact. In what way does the variable x approach, a? How close do x and $f(x)$ have to be to a and L, respectively?

First, the preliminary definition is equivalent to

If x is within a certain *tolerance level* of a, then the corresponding value $y = f(x)$ is within a certain *tolerance level* of L;

or

($\star$) If x is within δ units of a, then the corresponding value of y is within ϵ units of L.

For this, we have to use the **absolute value function**.

We can write "x is within δ units of a" mathematically as

$$|x - a| < \delta, \qquad \text{which is equivalent to} \qquad a - \delta < x < a + \delta.$$

Thus, we can rewrite ($\star$) as

$$|x - a| < \delta \Rightarrow |y - L| < \epsilon \qquad \text{or} \qquad a - \delta < x < a + \delta \Rightarrow L - \epsilon < y < L + \epsilon.$$

The point is that δ and ϵ, being tolerances, can be any **positive** (but typically **small**) values. Finally, we have the formal definition of the limit with the notation seen in the previous section.

Definition (Limit of a Function)**.** Suppose the function f is defined for all x in some interval containing a except possibly at a. We say the **limit of** $f(x)$ **as** x **approaches** a **is** L, denoted by

$$\lim_{x \to a} f(x) = L$$

if for any given $\epsilon > 0$, there exists (at least one) $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.

Remark: Note the order in which ϵ and δ are given. In the definition, the y-tolerance ϵ is given *first* and then the limit will exist **if** we can find an x-tolerance δ that works. To show that $\lim_{x \to a} f(x) = L$, we need to find a δ which is typically depending on ϵ.

Mathematicians often enjoy writing ideas without using any words. Here is the wordless definition of the limit:

$$\lim_{x \to a} f(x) = L \iff \forall \epsilon > 0, \exists \delta > 0 \ s.t. \ 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon.$$

This course is not mainly for math major students, proof of a limit of a function using the "epsilon-delta" definition is not required in any test.

Example Use the numerical method and the definition Show that $\lim_{x \to 4} \sqrt{x} = 2$.

Solution.

$\square$

2.2.2 One-Sided Limits

The limit $\lim_{x\to a} f(x) = L$ is referred to as a *two-sided* limit because$f(x)$ approaches L as x approaches a for values of x less than a and for values of x greater than a. For some functions, it makes sense to examine *one-sided* limits called *right-sided* and *left-sided* limits. In the following, we skip the mathematical "epsilon-delta" definitions for simplicity.

Definition (**One-Sided Limits**)**.** Let $f(x)$ be a real function.

1 **Right-sided limit:** Suppose f is defined for all x near a with $x > a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x > a$, we write

$$\lim_{x\to a^+} f(x) = L$$

and say the limit of $f(x)$ as x approaches a from the right equals L.

2 **Left-sided limit:** Suppose f is defined for all x near a with $x < a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x < a$, we write

$$\lim_{x\to a^-} f(x) = L$$

and say the limit of $f(x)$ as x approaches a from the left equals L.

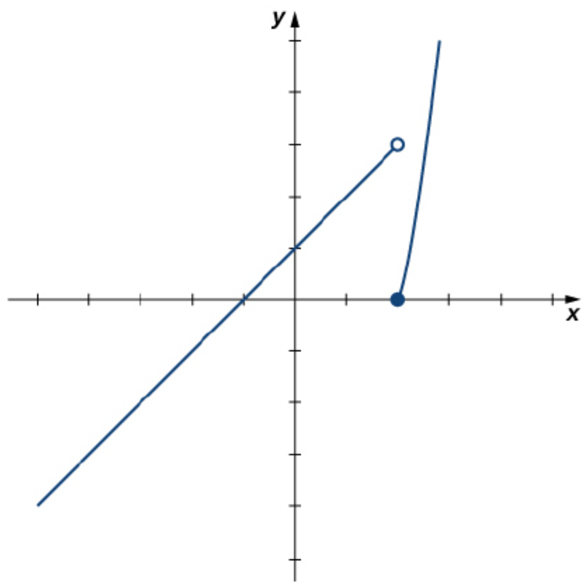
Example. For the function $f(x) = \begin{cases} x+1, \text{ if } x < 2 \\ x^2 - 4, \text{ if } x \geq 2 \end{cases}$, evaluate each of the following limits. Does $\lim_{x\to 2} f(x)$ exist?

- $\lim_{x\to 2^-} f(x)$
- $\lim_{x\to 2^+} f(x)$

Solution.

x	$f(x) = x + 1$		x	$f(x) = x^2 - 4$
1.9	2.9		2.1	0.41
1.99	2.99		2.01	0.0401
1.999	2.999		2.001	0.004001
1.9999	2.9999		2.0001	0.00040001
1.99999	2.99999		2.00001	0.0000400001

Table of Functional Values for $f(x) = \begin{cases} x + 1 \text{ if } x < 2 \\ x^2 - 4 \text{ if } x \geq 2 \end{cases}$



The graph of $f(x) = \begin{cases} x + 1 \text{ if } x < 2 \\ x^2 - 4 \text{ if } x \geq 2 \end{cases}$ has a break at $x = 2$.

$\square$

Based on the previous example, it seems that if the limit from the right and the limit from the left have a common value, then that common value is the limit of the function at that point. You might wonder whether the limits $\lim_{x \to a^-} f(x)$, $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a} f(x)$ always exist and are equal.

Theorem (Relating One-Sided and Two-Sided Limits)**.** Suppose f is defined for all x near a except possibly at a. then

$$\lim_{x\to a} f(x) = L \Leftrightarrow \lim_{x\to a^-} f(x) = L \text{ and } \lim_{x\to a^+} f(x) = L.$$

Remark If one of the one-sided limits does not exist or the two one-sided limits are not equal, then $\lim_{x\to a} f(x)$ does not exit. See the example above.

2.2.3 The Existence of a Limit

As we consider the limit in the next example, keep in mind that for the limit of a function to exist at a point, the **functional values must approach a single real-number value at that point**. If the functional values do not approach a single value when $x \to a$, then the limit $\lim_{x\to a} f(x)$ does not exist.

There are three ways in which a limit may fail to exist.

- The function $f(x)$ may approach different values on either side of a (See the theorem in the last subsection). That is, $\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x)$.
- The function may grow without upper or lower bound as x approaches a.
- The function may oscillate as x approaches a.

Example 1(Different Values Approached From Left and Right)
Explore why $\lim_{x\to 1} f(x)$ does not exist, where

$$f(x) = \begin{cases} x^2 - 2x + 3, & x \leq 1 \\ x, & x > 1 \end{cases}.$$

Solution.

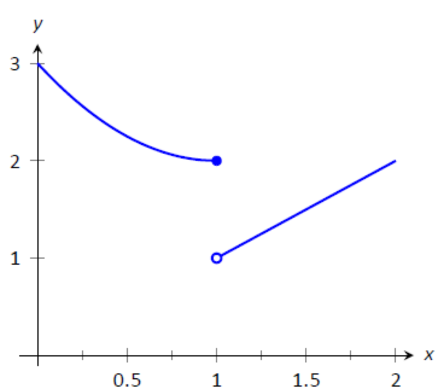


Figure 4: one-sided limits are different

Example 2 (The Function Grows Without Bound) Explore why $\lim_{x\to 1} 1/(x-1)^2$ does not exist.

Solution.

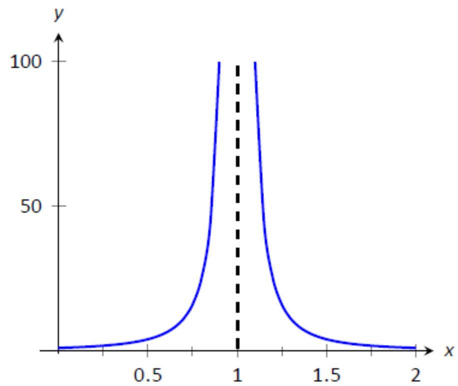
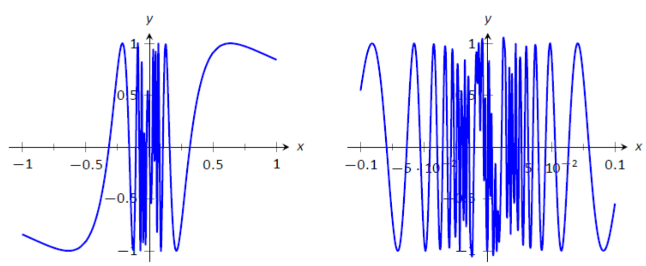


Figure 5: The Function Grows Without Bound

Example 3 (The Function Oscillates) Explore why $\lim_{x\to 0} \sin(1/x)$ does not exist.

Solution.



x	$\sin(1/x)$
0.1	−0.544021
0.01	−0.506366
0.001	0.82688
0.0001	−0.305614
$1. \times 10^{-5}$	0.0357488
$1. \times 10^{-6}$	−0.349994
$1. \times 10^{-7}$	0.420548

(a) (b) (c)

Observing that $f(x) = \sin(1/x)$ has no limit as $x \to 0$

Figure 6: The Function Oscillates

2.3 Techniques for Computing Limits

We explored the concept of the limit without a strict definition, meaning we could only make approximations: make a really good approximation either graphically or numerically. And our approximation can be verified using a ϵ-δ proof (not required for this course). This section gives a series of theorems which allow us to find limits much more quickly and intuitively.

Theorem (Basic Limit Properties)**.** Let b, c, L and K be real numbers, let n be a positive integer, and let f and g be functions defined on an open interval I containing c with the following limits:

$$\lim_{x\to c} f(x) = L \text{ and } \lim_{x\to c} g(x) = K.$$

The following limits hold.

1. Constants: $\lim_{x\to c} b = b$
2. Identity $\lim_{x\to c} x = c$
3. Sums/Differences: $\lim_{x\to c} (f(x) \pm g(x)) = L \pm K$
4. Scalar Multiples: $\lim_{x\to c} b \cdot f(x) = bL$
5. Products: $\lim_{x\to c} f(x) \cdot g(x) = LK$
6. Quotients: $\lim_{x\to c} f(x)/g(x) = L/K$, $(K \neq 0)$
7. Compositions: Adjust our previously given limit situation to:

   $$\lim_{x\to c} f(x) = L,\ \lim_{x\to L} g(x) = K \text{ and } g(L) = K.$$

   Then $\lim_{x\to c} g(f(x)) = K$.
8. Powers: $\lim_{x\to c} f(x)^n = L^n$
9. Fractional power: $\lim_{x\to c} f(x)^{n/m} = L^{n/m}$
   (provided $f(x) \geq 0$ for x near c, if m is even and n/m is reduced to lowest terms.)

Remark From 1,2,3 and 4, for any linear function $f(x) = mx + b$, $\lim_{x\to x_0} (mx + b) = mx_0 + b$.

Example 1. Evaluate the limit $\lim_{x\to 3}(\frac{1}{2}x-7)$

Solution.

$\square$

Example 2. Suppose $\lim_{x\to 2} f(x)=4$, $\lim_{x\to 2} g(x)=5$ and $\lim_{x\to 2} h(x)=8$. Use the limit laws to compute each limit.

1. $\lim_{x\to 2}\dfrac{f(x)-g(x)}{h(x)}$
2. $\lim_{x\to 2}[6f(x)g(x)+h(x)]$
3. $\lim_{x\to 2}[g(x)]^3$

Solution.

$\square$

Limits of Polynomial and Rational Functions

Consider a polynomial function $p(x) = 3x^2 - 5x + 7$, find $\lim_{x\to 2} p(x)$. Here we combine the Power, Scalar Multiple, Sum/Difference and Constant Rules. We show quite a few steps, but in general these can be omitted:

$$\begin{aligned}
\lim_{x\to 2} p(x) &= \lim_{x\to 2}(3x^2 - 5x + 7) \\
&= \lim_{x\to 2} 3x^2 - \lim_{x\to 2} 5x + \lim_{x\to 2} 7 \\
&= 3 \cdot 2^2 - 5 \cdot 2 + 7 \\
&= 9
\end{aligned}$$

This example demonstrates how the limit of a quadratic polynomial can be determined using the basic limit laws. Not only that, recognize that

$$\lim_{x\to 2} p(x) = 9 = p(2);$$

i.e., the limit at 2 was found just by plugging 2 into the function. This holds true for all polynomials, and also for rational functions (which are quotients of polynomials), as stated in the following theorem.

Theorem (Limits of Polynomial and Rational Functions)**.** Let $p(x)$ and $q(x)$ be polynomials and c a real number. Then:

1. $\lim_{x\to c} p(x) = p(c)$
2. $\lim_{x\to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}$, where $q(c) \neq 0$.

Example 3. Evaluate $\lim_{x\to 2} \frac{3x^2 - 4x}{5x^3 - 36}$.

Solution.

Example 4. Evaluate $\lim\limits_{x\to 2} \dfrac{\sqrt{2x^3+9}+3x-1}{4x+1}$.

Solution.

Using algebra to evaluate a limit

Another way to evaluate a challenging limit is to replace it with an equivalent limit that can be evaluated by direct substitution.

Example 5. Evaluate the following limits.

- $\lim\limits_{x\to 2} \dfrac{x^2-6x+8}{x^2-4}$ (Factor and cancel)
- $\lim\limits_{x\to 1} \dfrac{\sqrt{x}-1}{x-1}$ (Use conjugates)

Solution.

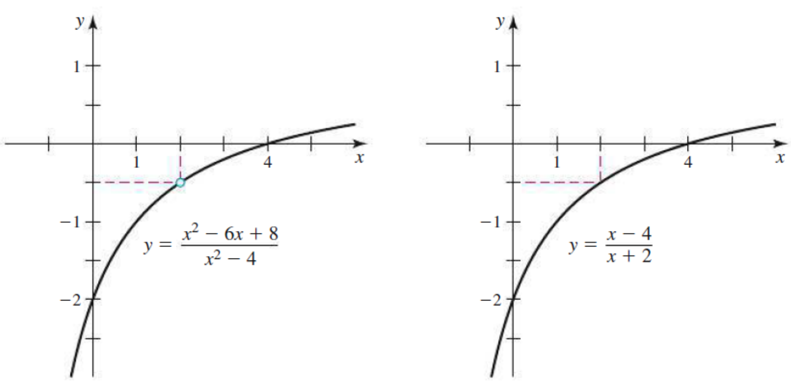


Figure 7: Using algebra to evaluate a limit

$\square$

Motivated by the above two examples, we have the following theorem.

Theorem (Limits of Functions Equal At All But One Point)**.** Let $g(x) = f(x)$ for all x in an open interval, except possibly at c, and let $\lim_{x\to c} g(x) = L$ for some real number L. Then

$$\lim_{x\to c} f(x) = L.$$

One-Sided Limits

Limit Laws 1-8, and Limit laws of Polynomial and Rational Functions also hold for left-sided and right-sided limits. In other words, these laws remain valid if we replace $\lim_{x\to c}$ with $\lim_{x\to c^-}$ or $\lim_{x\to c^+}$. But Law 9 must be modified slightly for one-sided limits, as shown in the next theorem.

Theorem (Limit Laws for One-Sided Limits)**.** Laws 1-8 hold with $\lim\limits_{x\to c}$ replaced with $\lim\limits_{x\to c^-}$ or $\lim\limits_{x\to c^+}$. Law 9 is modified as follows.
Assume $m > 0$ and $n > 0$ are integers.

9. Fractional power

1. $\lim\limits_{x\to c^+} f(x)^{n/m} = \left(\lim\limits_{x\to c^+} f(x)\right)^{n/m}$, provided $f(x) \geq 0$ for x near c with $x > c$, if m is even and n/m is reduced to lowest terms.

2. $\lim\limits_{x\to c^-} f(x)^{n/m} = \left(\lim\limits_{x\to c^-} f(x)\right)^{n/m}$, provided $f(x) \geq 0$ for x near c with $x < c$, if m is even and n/m is reduced to lowest terms.

Example 6. Let $f(x) = \begin{cases} -2x+4, & \text{if } x \leq 1 \\ \sqrt{x-1}, & \text{if } x > 1 \end{cases}$. Find the values of $\lim\limits_{x\to 1^-} f(x)$, $\lim\limits_{x\to 1^+} f(x)$, and $\lim\limits_{x\to 1} f(x)$, or state that they do not exist.

Solution.

□

Polynomial and rational functions are not the only functions to behave in a predictable way. The following theorem gives a list of functions whose behavior is particularly "nice" in terms of limits.

Theorem (Special Limits 1)**.** Let c be a real number in the domain of the given function and let n be a positive integer. The following limits hold:

1. $\lim_{x\to c} \sin x = \sin c$
2. $\lim_{x\to c} \cos x = \cos c$
3. $\lim_{x\to c} \tan x = \tan c$
4. $\lim_{x\to c} \csc x = \csc c$
5. $\lim_{x\to c} \sec x = \sec c$
6. $\lim_{x\to c} \cot x = \cot c$
7. $\lim_{x\to c} a^x = a^c \ (a > 0)$
8. $\lim_{x\to c} \ln x = \ln c$
9. $\lim_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}$

Example 7. Evaluate the following limits.

1. $\lim_{x\to \pi} \cos x$
2. $\lim_{x\to 3} (\sec^2 x - \tan^2 x)$
3. $\lim_{x\to \pi/2} \cos x \sin x$
4. $\lim_{x\to 1} e^{\ln x}$

Solution.

$\square$

Theorem (Special Limits 2)**.** More special limits are listed as follows.

1. $\displaystyle\lim_{x\to 0}\frac{\sin x}{x}=1$
2. $\displaystyle\lim_{x\to 0}\frac{\cos x-1}{x}=0$
3. $\displaystyle\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$
4. $\displaystyle\lim_{x\to 0}\frac{e^x-1}{x}=1$

The section could have been titled "Using Known Limits to Find Unknown Limits." By knowing certain limits of functions, we can find limits involving sums, products, powers, etc., of these functions. We further the development of such comparative tools with the Squeeze Theorem, a clever and intuitive way to find the value of some limits.

Before stating this theorem formally, suppose we have functions f, g and h where g always takes on values between f and h; that is, for all x in an interval,

$$f(x)\le g(x)\le h(x).$$

If f and h have the same limit at c, and g is always "squeezed" between them, then g must have the same limit as well. That is what the Squeeze Theorem states.

Theorem (Squeeze Theorem)**.** Let f, g and h be functions on an open interval I containing c such that for all x in I,

$$f(x)\le g(x)\le h(x).$$

If

$$\lim_{x\to c}f(x)=L=\lim_{x\to c}h(x),$$

then

$$\lim_{x\to c}g(x)=L.$$

Example 8. A geometric argument may be used to show that for $-\pi/2 < x < \pi/2$,

$$-|x| \leq \sin x \leq |x| \textbf{ and } 0 \leq 1 - \cos x \leq |x|.$$

Use the Squeeze Theorem to confirm the following limits.

1. $\lim_{x \to 0} \sin x = 0$.
2. $\lim_{x \to 0} \cos x = 1$.

Solution.

$\square$

Example 9. Use the Squeeze Theorem to verify that

$$\lim_{x \to 0} x^2 \sin(1/x) = 0.$$

Solution.

$\square$

2.4 Infinite Limits

Two more limit scenarios are frequently encountered in calculus and are discussed in this and the following section. An infinite limit occurs when function values increase or decrease without bound near a point. The other type of limit, known as a limit at infinity, occurs when the independent variable x increases or decreases without bound. The ideas behind infinite limits and limits at infinity are quite different. Therefore, it is important to distinguish these limits and the methods used to calculate them.

2.4.1 Infinitesimals and Infinity

The concept of Infinitesimals:

Definition 2.1. A variable v whose limit is zero is called an infinitesimal. This is written as

$$\lim_{v\to 0} \text{ or } v \to 0.$$

An infinitesimal v means that the successive absolute values of v ultimately become and remain less than any positive number however small. Such a variable is said to become "arbitrarily small."

Remark 1: Therefore a constant, no matter how small it may be, is not an infinitesimal.

Remark 2: The difference between a variable and its limit is an infinitesimal; Conversely, if the difference between a variable and a constant is an infinitesimal, then the variable approaches the constant as a limit. That is (suppose $v = f(x)$),

$$\lim_{x\to x_0} v = L \Leftrightarrow \lim_{x\to x_0} (v - L) = 0.$$

The concept of infinity:

Let v be an independent or dependent variable.

- If the variable v ultimately becomes and remains greater than any assigned positive number, however large, we say v is "unbounded and positive" (or "increases without limit"), and write

$$\lim_{v\to\infty} \text{ or } v \to \infty.$$

- If the variable v ultimately becomes and remains smaller than any assigned negative number, we say "unbounded and negative" (or "v decreases without limit"), and write

$$\lim_{v\to-\infty} \text{ or } v \to -\infty.$$

Remark: Infinity (∞) is not a number; it simply serves to characterize a particular mode of variation of a variable by virtue of which it becomes arbitrarily large.

2.4.2 Infinite Limits

As a motivating example, consider $f(x) = 1/x^2$,as shown in Figure 8. Note how, as x

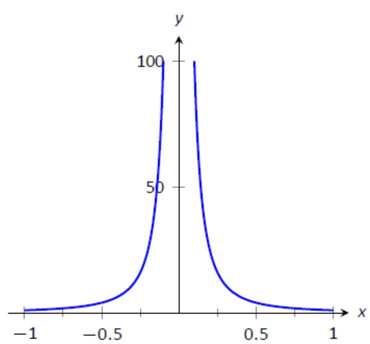


Figure 8: $f(x) = 1/x^2$

approaches 0, $f(x)$ grows very, very large – in fact, it grows without bound. It seems appropriate, and descriptive, to state that

$$\lim_{x\to 0} \frac{1}{x^2} = \infty.$$

Definition (Infinite Limits)**.** Suppose f is defined for all x near a. If $f(x)$ grows arbitrarily large for all x sufficiently close (but not equal) to a, we write

$$\lim_{x\to a} f(x) = \infty.$$

and say the limit of $f(x)$ as x approaches a is infinity.

Suppose f is negative and grows arbitrarily large in magnitude for all x sufficiently close (but not equal) to a, we write

$$\lim_{x\to a} f(x) = -\infty.$$

and say the limit of $f(x)$ as x approaches a is negative infinity.

Or some of us prefer more rigorous definitions.

Definition (Infinite Limits)**.** Let I be an open interval containing a, and let f be a function defined on I, except possibly at a.

- The **limit of** $f(x)$**, as** x **approaches** a**, is infinity**, denoted by

  $$\lim_{x\to a} f(x) = \infty,$$

  means that given any $M > 0$, there exists $\delta > 0$ such that if $|x-a| < \delta$ and $x \neq a$, then $f(x) > M$.

- The **limit of** $f(x)$**, as** x **approaches** a**, is negative infinity**, denoted by

  $$\lim_{x\to a} f(x) = -\infty,$$

  means that given any $M < 0$, there exists $\delta > 0$ such that if $|x-a| < \delta$ and $x \neq a$, then $f(x) < M$.

The definition is similar to the ϵ–δ definition of the limit of a function with finite limit.

Remark We are not asserting that a limit exists when we write $\lim_{x\to a} f(x) = \infty$ or $\lim_{x\to a} f(x) = -\infty$. We are describing the behavior of the function using the notation. That said, if for example, $\lim_{x\to a} f(x) = \infty$, we always write $\lim_{x\to a} f(x) = \infty$ rather than $\lim_{x\to a} f(x)$ DNE.

Example 1. Analyze $\lim_{x\to 1} \dfrac{x}{(x^2-1)^2}$ and $\lim_{x\to -1} \dfrac{x}{(x^2-1)^2}$ using the graph of the function.

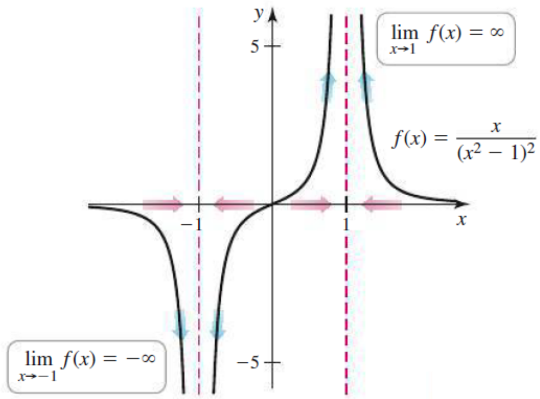


Figure 9: graph of $f(x) = x/(x^2 - 1)^2$

Solution.

□

Definition (One-Sided Infinite Limits)**.** Suppose f is defined for all x near a with $x > a$. If $f(x)$ becomes arbitrarily large for all x sufficiently close to a with $x > a$, we write

$$\lim_{x\to a^+} f(x) = \infty.$$

The one-sided infinite limits $\lim_{x\to a^+} f(x) = -\infty$, $\lim_{x\to a^-} f(x) = \infty$, and $\lim_{x\to a^-} f(x) = -\infty$ are defined analogously.

Vertical asymptotes

If the limit of $f(x)$ as x approaches a from either the left or right (or both) is ∞ or $-\infty$, then the line $x = a$ is a **vertical asymptote** of f. In all the infinite limits illustrated in Figure 10, the line $x = a$ is a vertical asymptote.

Definition (Vertical Asymptote)**.** If $\lim_{x\to a} f(x) = \pm\infty$, $\lim_{x\to a^+} f(x) = \pm\infty$, or $\lim_{x\to a^-} f(x) = \pm\infty$, the line $x = a$ is called a **vertical asymptote** of f.

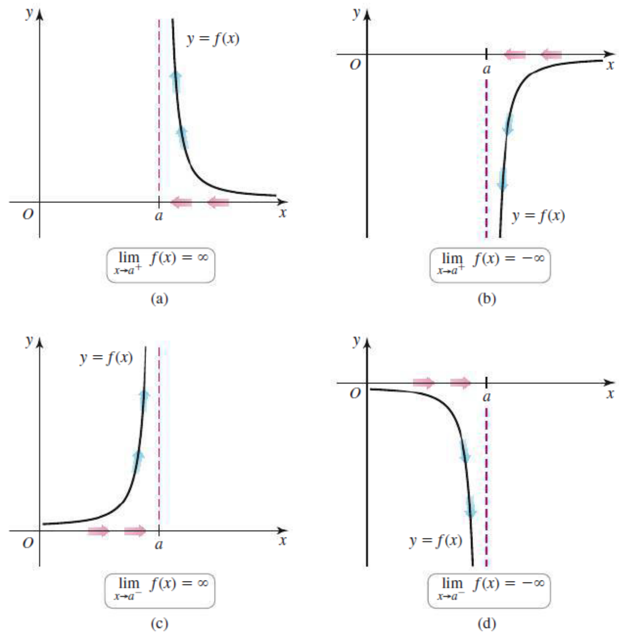


Figure 10: One-Sided Infinite Limits

Example 2. Find the vertical asymptotes of $f(x) = \dfrac{3x}{x^2-4}$.

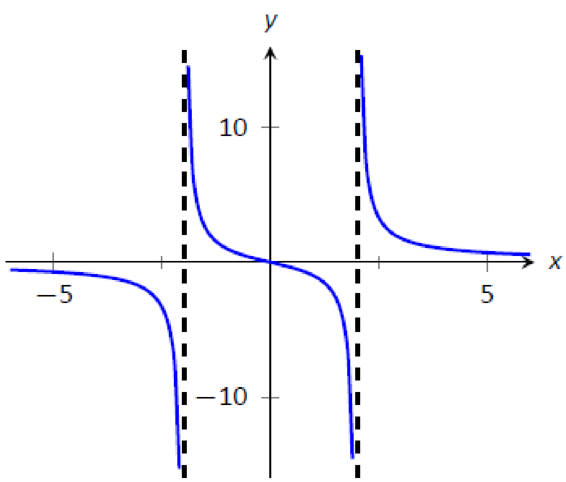


Figure 11: Graphing $f(x) = \dfrac{3x}{x^2-4}$

Finding Infinite Limits Analytically

Many infinite limits are analyzed using a simple arithmetic property: The fraction a/b grows arbitrarily large in magnitude if b approaches 0 while a remains nonzero and relatively constant. For example, consider the fraction $(5+x)/x$ for values of x approaching 0 from the right.

x	$\dfrac{5+x}{x}$
0.01	$\dfrac{5.01}{0.01} = 501$
0.001	$\dfrac{5.001}{0.001} = 5001$
0.0001	$\dfrac{5.0001}{0.0001} = 50{,}001$
$\downarrow$	$\downarrow$
0^+	∞

Figure 12: Behaviour of $\dfrac{5+x}{x}$ as $x \to 0^+$

Example 3. Find the following limits.

- $\displaystyle\lim_{x\to 3^+} \frac{2-5x}{x-3}$
- $\displaystyle\lim_{x\to 3^-} \frac{2-5x}{x-3}$

Solution.

$\square$

Example 4. Find $\lim_{x \to -4^+} \dfrac{-x^3 + 5x^2 - 6x}{-x^3 - 4x^2}$. Hint: simplify the function first.

Solution.

$\square$

Example 5. Let $f(x) = \dfrac{x^2 - 4x + 3}{x^2 - 1}$. Determine the following limits and find the vertical asymptotes of f. Verify your work with a graphing utility.

a. $\lim_{x \to 1} f(x)$

b. $\lim_{x \to -1^-} f(x)$

c. $\lim_{x \to -1^+} f(x)$

Solution.

$\square$

Example 6. Find the following limits.

- $\lim\limits_{\theta \to 0^+} \cot\theta$
- $\lim\limits_{\theta \to 0^-} \cot\theta$

Solution.

$\square$

When a rational function has a vertical asymptote at $x = a$, we can conclude that the denominator is 0 at $x = a$. However, just because the denominator is 0 at a certain point does not mean there is a vertical asymptote there. For instance, $f(x) = (x^2 - 1)/(x - 1)$ does not have a vertical asymptote at $x = 1$, as shown in Figure 13. While the denominator does get small near $x = 1$, the numerator gets small too, matching the denominator step for step. In fact, factoring the numerator, we get

$$f(x) = \frac{(x-1)(x+1)}{x-1}.$$

Canceling the common term, we get that $f(x) = x + 1$ for $x \neq 1$. So there is clearly no asymptote; rather, a hole exists in the graph at $x = 1$.

The above example may seem a little contrived. Another example demonstrating this important concept is $f(x) = \sin x/x$. We have considered this function several times in the previous sections. We found that $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$; i.e., there is no vertical asymptote. And no simple algebraic cancellation makes this fact obvious.

If the denominator is 0 at a certain point but the numerator is not, then there will usually be a vertical asymptote at that point. On the other hand, if the numerator

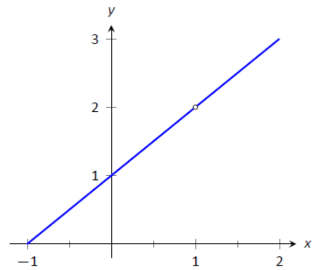


Figure 13: Graphically showing that $f(x) = \dfrac{x^2-1}{x-1}$ does not have an asymptote at $x = 1$

and denominator are both zero at that point, then there may or may not be a vertical asymptote at that point. This case where the numerator and denominator are both zero returns us to an important topic.

Indeterminate Forms

The limits

$$\lim_{x\to 0}\frac{\sin x}{x} \quad \text{and} \quad \lim_{x\to 1}\frac{x^2-1}{x-1}$$

each return the indeterminate form "0/0" when we blindly plug in $x = 0$ and $x = 1$, respectively. However, $0/0$ is not a valid arithmetical expression. It gives no indication that the respective limits are 1 and 2.

A key concept to understand is that such limits do not really return $0/0$. Rather, keep in mind that we are taking *limits*. What is really happening is that the numerator is shrinking to 0 while the denominator is also shrinking to 0. The respective rates at which they do this are very important and determine the actual value of the limit.

An indeterminate form indicates that one needs to do more work in order to compute the limit. That work may be algebraic (such as factoring and canceling) or it may require a tool such as the Squeeze Theorem. In a later section we will learn a technique called L'Hôspital's Rule that provides another way to handle indeterminate forms.

Some other common indeterminate forms are $\infty - \infty$, $\infty \cdot 0$, ∞/∞, 0^0, ∞^0 and 1^∞. Again, keep in mind that these are the "blind" results of evaluating a limit, and each, in and of itself, has no meaning. The expression $\infty - \infty$ does not really mean "subtract infinity from infinity." Rather, it means "One quantity is subtracted from the other, but both are growing without bound." What is the result? It is possible to get every value between $-\infty$ and ∞.

Note that $1/0$ and $\infty/0$ are not indeterminate forms, though they are not exactly valid mathematical expressions, either. In each, the function is growing without bound, indicating that the limit will be ∞, $-\infty$, or simply not exist if the left- and right-hand limits do not match.

2.5 Limits at Infinity

Again, consider the motivating example $f(x) = 1/x^2$,as shown in Figure 14. As x

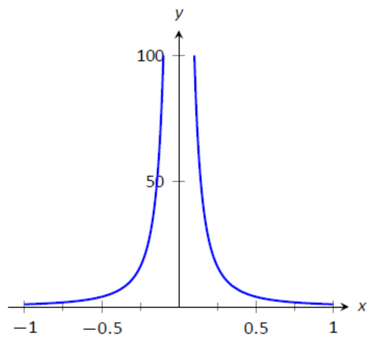


Figure 14: $f(x) = 1/x^2$

approaches 0, $f(x)$ grows without bound: $\lim\limits_{x\to 0} \dfrac{1}{x^2} = \infty$. Also note that as x gets very large, $f(x)$ gets very, very small. We could represent this concept with notation such as

$$\lim_{x\to\infty} \frac{1}{x^2} = 0.$$

Graphically, it concerns the behavior of the function to the "far right" of the graph. We make this notion more explicit in the following definition.

Definition (Limits at Infinity and Horizontal Asymptotes)**.** If $f(x)$ becomes arbitrarily close to a finite number L for all sufficiently large and positive x, then we write

$$\lim_{x\to\infty} f(x) = L.$$

In this case, the line $y = L$ is called a **horizontal asymptote** of f;

Similarly, if $f(x)$ becomes arbitrarily close to a finite number L for all sufficiently small and negative x, then we write

$$\lim_{x\to-\infty} f(x) = L.$$

Again, the line $y = L$ is a **horizontal asymptote** of f.

That is, if $\lim\limits_{x\to\infty} f(x) = L$ or $\lim\limits_{x\to-\infty} f(x) = L$, we say the line $y = L$ is a **horizontal asymptote** of f.

One may prefer a rigorous mathematical definition.

Definition (Limits at Infinity and Horizontal Asymptotes)**.** Let L be a real number.

1. Let f be a function defined on (a, ∞) for some number a. The **limit of f at infinity is** L, or $\lim\limits_{x\to\infty} f(x) = L$, means for every $\epsilon > 0$ there exists $M > a$ such that if $x > M$, then $|f(x) - L| < \epsilon$.

2. Let f be a function defined on $(-\infty, b)$ for some number b. The **limit of f at negative infinity is** L, or $\lim\limits_{x\to-\infty} f(x) = L$, means for every $\epsilon > 0$ there exists $M < b$ such that if $x < M$, then $|f(x) - L| < \epsilon$.

3. If $\lim\limits_{x\to\infty} f(x) = L$ or $\lim\limits_{x\to-\infty} f(x) = L$, we say the line $y = L$ is a **horizontal asymptote** of f.

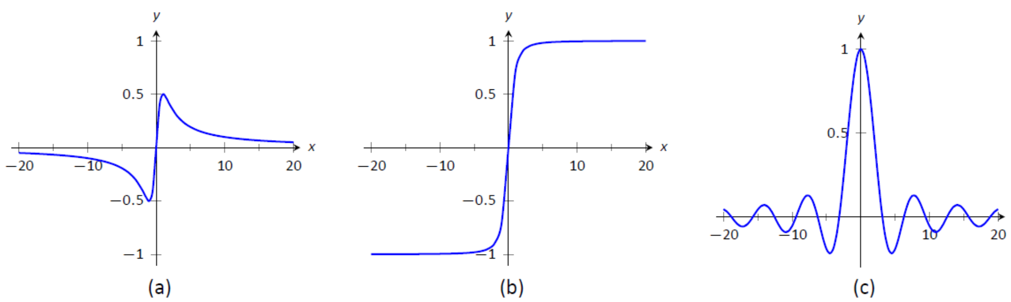


Figure 15: Considering different types of horizontal asymptotes

Horizontal asymptotes can take on a variety of forms. Figure 15(a) shows that $f(x) = x/(x^2 + 1)$ has a horizontal asymptote of $y = 0$, where 0 is approached from both above and below.

Figure 15(b) shows that $f(x) = x/\sqrt{x^2 + 1}$ has two horizontal asymptotes; one at $y = 1$ and the other at $y = -1$.

Figure 15(c) shows that $f(x) = \sin x/x$ has even more interesting behavior than at just $x = 0$; as x approaches $\pm\infty$, $f(x)$ approaches 0, but oscillates as it does this.

Example 1. Find the following limits.

- $\lim\limits_{x\to-\infty} \left(2 + \dfrac{10}{x^2}\right)$
- $\lim\limits_{x\to\infty} \left(5 + \dfrac{\sin x}{\sqrt{x}}\right)$

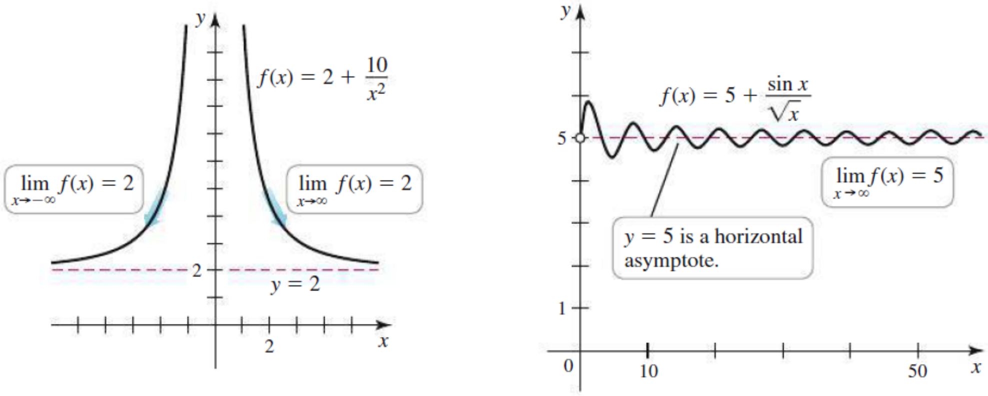


Figure 16: Graphing $2 + \dfrac{10}{x^2}$ and $5 + \dfrac{\sin x}{\sqrt{x}}$

Solution.

$\square$

Definition (Infinite Limits at Infinity)**.** If $f(x)$ grows arbitrarily large as x becomes arbitrarily large, then we write

$$\lim_{x\to\infty} f(x) = \infty.$$

The limits $\lim_{x\to\infty} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = \infty$, and $\lim_{x\to-\infty} f(x) = -\infty$ are defined similarly.

Or some of us prefer more rigorous definitions.

Definition (Infinite Limits at Infinity)**.** Let $f(x)$ be a function defined on $\mathbb{R}$.

- The **limit of** $f(x)$**, as** x **approaches** ∞**, is infinity**, denoted by

$$\lim_{x\to\infty} f(x) = \infty,$$

  means that given any $M > 0$,no matter how large, there exists an $x > 0$ such that $f(x) > M$.

The limits $\lim_{x\to\infty} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = \infty$, and $\lim_{x\to-\infty} f(x) = -\infty$ are defined similarly.

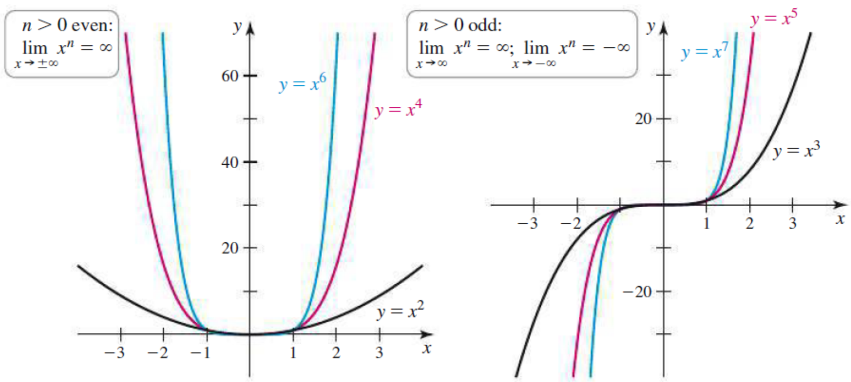


Figure 17: Long-run behavior of power functions

Infinite limits at infinity tell us about the behavior of polynomials for large magnitude values of x. First, consider power functions $f(x) = x^n$, where n is a positive integer. Figure 17 shows that when n is even, $\lim_{x\to\pm\infty} x^n = \infty$; when n is odd, $\lim_{x\to\infty} x^n = \infty$ and $\lim_{x\to-\infty} x^n = -\infty$. It follows that reciprocals of power functions $f(x) = 1/x^n = x^{-n}$, where n is a positive integer, behave as follows:

$$\lim_{x\to\pm\infty} \frac{1}{x^n} = \lim_{x\to\pm\infty} x^{-n} = 0.$$

To find the long-run behavior of any polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0, a_n \neq 0$. We re-write $p(x)$ as

$$p(x) = x^n \left(a_n + \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right), a_n \neq 0.$$

It is easy to see that the long-run behavior of $p(x)$ is depending on the leading term $a_n x^n$ only.

Theorem (Limits at Infinity of Powers and Polynomials)**.** Let n be a positive integer and let $p(x)$ be the polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0, a_n \neq 0$.

1. $\lim_{x\to\pm\infty} x^n = \infty$ when n is even.
2. $\lim_{x\to\infty} x^n = \infty$ and $\lim_{x\to-\infty} x^n = -\infty$ when n is odd.
3. $\lim_{x\to\pm\infty} \dfrac{1}{x^n} = \lim_{x\to\pm\infty} x^{-n} = 0$.
4. $\lim_{x\to\pm\infty} p(x) = \lim_{x\to\pm\infty} a_n x^n = \pm\infty$, depending on the degree of the polynomial and the sign of the leading coefficient a_n.

Now suppose we need to compute the following limit:

$$\lim_{x\to\infty} \frac{x^3 + 2x + 1}{4x^3 - 2x^2 + 9}.$$

A good way of approaching this is to divide through the numerator and denominator by x^3 (hence multiplying by 1), which is the largest power of x to appear in the function. Doing this, we get

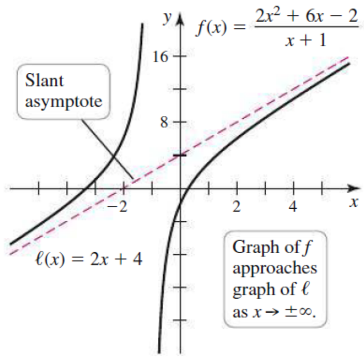
$$\begin{aligned}
\lim_{x\to\infty} \frac{x^3 + 2x + 1}{4x^3 - 2x^2 + 9} &= \lim_{x\to\infty} \frac{1/x^3}{1/x^3} \cdot \frac{x^3 + 2x + 1}{4x^3 - 2x^2 + 9} \\
&= \lim_{x\to\infty} \frac{x^3/x^3 + 2x/x^3 + 1/x^3}{4x^3/x^3 - 2x^2/x^3 + 9/x^3} \\
&= \lim_{x\to\infty} \frac{1 + 2/x^2 + 1/x^3}{4 - 2/x + 9/x^3}.
\end{aligned}$$

Then using the rules for limits (which also hold for limits at infinity), as well as the fact about limits of $1/x^n$, we see that the limit becomes

$$\frac{1 + 0 + 0}{4 - 0 + 0} = \frac{1}{4}.$$

Example (Slant asymptotes) Determine the end behavior of the function

$$f(x) = \frac{2x^2 + 6x - 2}{x + 1}.$$



➤ More generally, a line $y = \ell(x)$ (with finite and nonzero slope) is a slant asymptote of a function f if $\lim_{x\to\infty} (f(x) - \ell(x)) = 0$ or $\lim_{x\to-\infty} (f(x) - \ell(x)) = 0$.

Figure 18: Slant asymptotes

Solution.

□

Definition (slant asymptote)**.** If the graph of a function $f(x)$ approaches a line (with finite and nonzero slope) as $x \to \pm\infty$, then that line is called a **slant asymptote**, or **oblique asymptote**, of $f(x)$.

The above discussions work for any rational function. In fact, they give us the following theorem.

Theorem (End Behavior and Asymptotes of Rational Functions)**.** Let $f(x)$ be a rational function of the following form:

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0}{b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0},$$

where any of the coefficients may be 0 except for a_m and b_n.

1. If $m = n$, then $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = \frac{a_m}{b_n}$. And $\frac{a_m}{b_n}$ is a horizontal asymptote of $f(x)$.

2. If $m < n$, then $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$. And $y = 0$ is a horizontal asymptote of $f(x)$.

3. If $m > n$, then $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ are both infinite. And $f(x)$ has no horizontal asymptote.

4. **Slant asymptote** If $m = n+1$, then $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ are both infinite. $f(x)$ has no horizontal asymptote, but $f(x)$ has a slant asymptote

5. **Vertical asymptotes** Assuming that f is in reduced form ($p(x)$ and $q(x)$ share no common factors), vertical asymptotes occur at the zeros of $q(x)$.

Example Determine the end behavior of the function

$$f(x) = \frac{10x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}}.$$

Solution.

$\square$

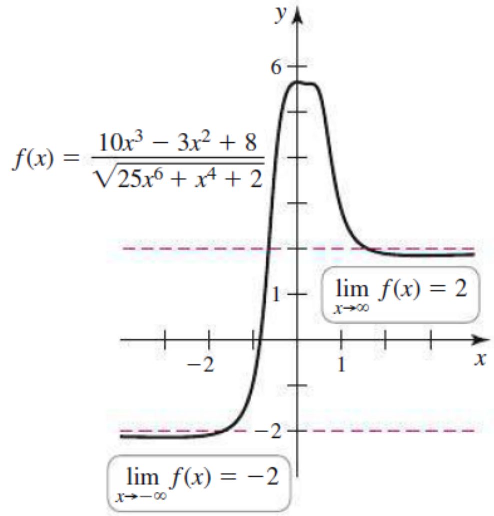


Figure 19: Graphing the algebraic function $\frac{10x^3-3x^2+8}{\sqrt{25x^6+x^4+2}}$

Theorem (End Behavior of e^x, e^{-x} and $\ln x$)**.** The end behavior for e^x and e^{-x} on $(-\infty, \infty)$ and $\ln x$ on $(0, \infty)$ is given by the following limits:

1. $\lim_{x\to\infty} e^x = \infty$
2. $\lim_{x\to\infty} e^{-x} = 0$
3. $\lim_{x\to 0^+} \ln x = -\infty$
4. $\lim_{x\to-\infty} e^x = 0$
5. $\lim_{x\to-\infty} e^{-x} = \infty$
6. $\lim_{x\to\infty} \ln x = \infty$

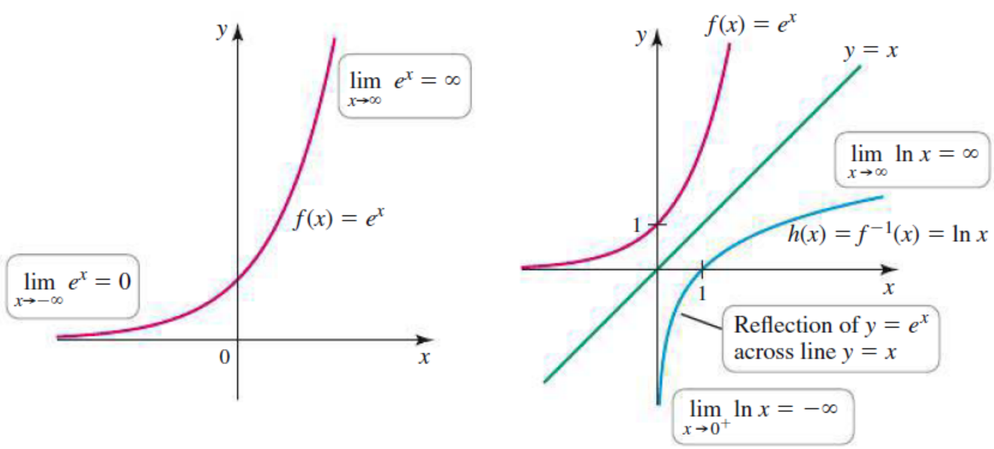


Figure 20: End behavior of e^x, e^{-x} and $\ln x$

2.6 Continuity

As we have studied limits, we have gained the intuition that limits measure "where a function is heading." That is, if $\lim_{x\to 1} f(x) = 3$, then as x is close to 1, $f(x)$ is close to 3. We have seen, though, that this is not necessarily a good indicator of what $f(1)$ actually is. This can be problematic; functions can tend to one value but attain another. This section focuses on functions that *do not* exhibit such behavior.

2.6.1 Concept of Continuity

Continuity on an Open Interval

Definition (Continuity on an Open Interval)**.** Let f be a function defined on an open interval I containing a.

1. f is **continuous at** a if $\lim_{x\to a} f(x) = f(a)$.
2. f is **continuous on** I if f is continuous at a for all values of a in I. If f is continuous on $(-\infty, \infty)$, we say f is **continuous everywhere**.

A useful way to establish whether or not a function f is continuous at a is to verify the following three things.

Continuity Check list:

1. $\lim_{x\to a} f(x)$ exists,

2. $f(a)$ is defined, and

3. $\lim_{x\to a} f(x) = f(a)$.

For example, $f(x) = \sin x/x$ is not continuous at 0 because it is not defined at 0 though $\lim_{x\to a} f(x)$ exists.

Example (Identifying discontinuities) Determine whether the following functions are continuous at a. Justify each answer using the continuity checklist.

1. $f(x) = \dfrac{3x^2 + 2x + 1}{x - 1}$; $a = 1$.

2. $g(x) = \dfrac{3x^2 + 2x + 1}{x - 1}$; $a = 2$.

3. $h(x) = \begin{cases} x \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$; $a = 0$.

Solution.

$\square$

Continuity on a Closed Interval

We just defined continuity of a function at an interior point in an open interval. We can extend the definition of continuity to closed intervals by considering two-sided limits of all **interior points** and the appropriate one-sided limits at the **endpoints**.

Definition (Continuity at Endpoints)**.** A function $f(x)$ is **continuous from the left** (or **left-continuous**) at a if $\lim\limits_{x \to a^-} f(x) = f(a)$, and $f(x)$ is **continuous from the right** (or **right-continuous**) at a if $\lim\limits_{x \to a^+} f(x) = f(a)$.

Definition (Continuity on a closed Interval)**.** Let f be defined on the closed interval $[a, b]$ for some real numbers $a < b$. f is **continuous on** $[a, b]$ if:

1. f is continuous on (a, b),

2. $\lim\limits_{x \to a^+} f(x) = f(a)$ and

3. $\lim_{x \to b^-} f(x) = f(b)$.

We can make the appropriate adjustments to talk about continuity on half–open intervals such as $[a, b)$ or $(a, b]$ if necessary.

Example (**step function**)Finding intervals of continuity

The *floor function*, $f(x) = \lfloor x \rfloor$, returns the largest integer smaller than, or equal to, the input x. (For example, $f(\pi) = \lfloor \pi \rfloor = 3$.) The graph of f in Figure 21 demonstrates why this is often called a "step function."

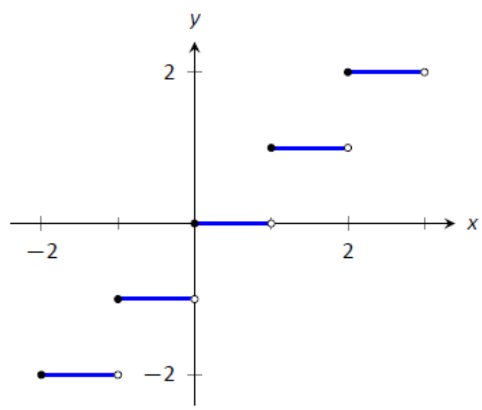


Figure 21: Graphing the floor function $\lfloor x \rfloor$

Solution.

$\square$

2.6.2 Properties of Continuity

Theorem (Continuity Rules)**.** Let f and g be **continuous** functions on an interval I, let c be a real number and let n be a positive integer. The following functions are continuous on I.

1. Sums/Differences: $f \pm g$
2. Constant Multiples: $c \cdot f$
3. Products: $f \cdot g$
4. Quotients: f/g (as long as $g \neq 0$ on I)
5. Powers: $f^{\,n}$
6. Roots: $f^{n/m}$, $m > 0$ and $n > 0$ are integers with no common factors

   (If m is even then require $f(x) \geq 0$ on I. if m is odd, then true for all values of f on I.)

Theorem (Continuity of Polynomial and Rational Functions)**. 1.** A polynomial function is continuous for all x.

2. A rational function (a function of the form $\dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials) is continuous for all x for which $q(x) \neq 0$.

Theorem (Continuity of Composite Functions)**.** Let f be continuous on I, where the range of f on I is J, and let g be continuous on J. Then $g \circ f$, i.e., $g(f(x))$, is continuous on I. That is, for any $a \in I$,

$$\lim_{x \to a} g(f(x)) = g\left(\lim_{x \to a} f(x))\right).$$

Continuity is inherently tied to the properties of limits. All the theorems above can be proved by the properties of limits.

Example. For what values of x is the function $f(x) = \dfrac{x}{x^2 - 7x + 12}$ continuous?

Solution.

$\square$

Example. Evaluate $\lim_{x \to 0} \left(\dfrac{x^4 - 2x + 2}{x^6 + 2x^4 + 1} \right)^{10}$.

Solution.

$\square$

Example. Evaluate the following limits.

a. $\lim_{x \to -1} \sqrt{2x^2 - 1}$. **b.** $\lim_{x \to 2} \cos\left(\dfrac{x^2 - 4}{x - 2} \right)$.

Solution.

$\square$

Example. For what values of x are the following functions continuous?

a. $g(x) = \sqrt{9 - x^2}$. **b.** $f(x) = (x^2 - 2x + 4)^{2/3}$.

Solution.

$\square$

Theorem (Continuity of Inverse Functions)**.** If a function $f(x)$ is continuous on an interval I and has an inverse on I, then its inverse $f^{-1}(x)$ is also continuous (on the interval consisting of the points $f(x)$, where $x \in I$).

Continuous functions are important as they behave in a predictable fashion: functions attain the value they approach. Because continuity is so important, most of the functions you have likely seen in the past are continuous on their domains.

Theorem (Continuous Functions)**.** Let n be a positive integer. The following functions are continuous on their domains.

1. $\sin x, \quad \cos x$
2. $\tan x, \quad \cot x$
3. $\sec x, \quad \csc x$
4. $\sin^{-1} x, \quad \cos^{-1} x$
5. $\tan^{-1} x, \quad \cot^{-1} x$
6. $\sec^{-1} x, \quad \csc^{-1} x$
7. $f(x) = a^x \ (a > 0)$
8. $f(x) = \ln x$
9. $f(x) = \sqrt[n]{x}$

Example. Analyze the following limits after determining the continuity of the functions involved.

a. $\lim_{x\to 0} \dfrac{\cos^2 x - 1}{\cos x - 1}$. **b.** $\lim_{x\to 1} \left(\sqrt[4]{\ln x} + \tan^{-1} x\right)$.

Solution.

$\square$

2.6.3 The Intermediate Value Theorem

A common way of thinking of a continuous function is that "its graph can be sketched without lifting your pencil." That is, its graph forms a "continuous" curve, without holes, breaks or jumps. While beyond the scope of this text, this pseudo–definition glosses over some of the finer points of continuity. Very strange functions are continuous that one would be hard pressed to actually sketch by hand.

This intuitive notion of continuity does help us understand another important concept as follows. Suppose f is defined on $[1, 2]$ and $f(1) = -10$ and $f(2) = 5$. If f is continuous on $[1, 2]$ (i.e., its graph can be sketched as a continuous curve from $(1, -10)$ to $(2, 5)$) then we know intuitively that somewhere on $[1, 2]$ f must be equal to -9, and -8, and -7, -6, $\ldots$, 0, $1/2$, etc. In short, f takes on all *intermediate* values between -10 and 5. It may take on more values; f may actually equal 6 at some time, for instance, but we are guaranteed all values between -10 and 5.

While this notion seems intuitive, it is not trivial to prove and its importance is profound. Therefore the concept is stated in the form of a theorem.

Theorem (Intermediate Value Theorem)**.** Let f be a continuous function on $[a, b]$ and, without loss of generality, let $f(a) < f(b)$. Then for every value $y = L$, where $f(a) < L < f(b)$, there is at least one value c in (a, b) such that $f(c) = L$.

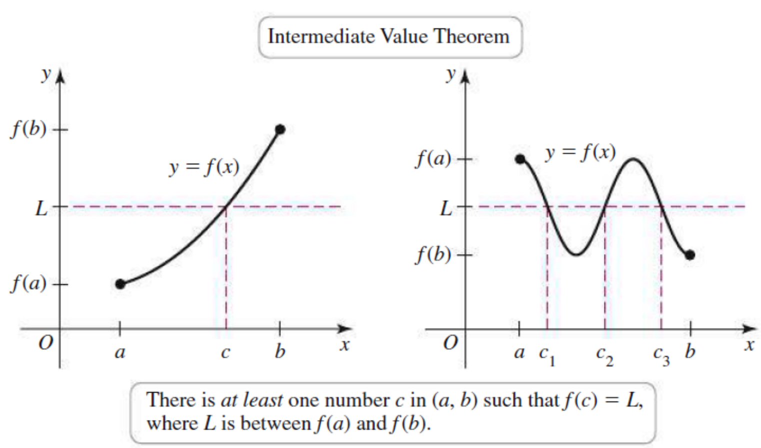


Figure 22: Intermediate Value Theorem

Example. Let $f(x) = x + e^x, x \in [-1, 0]$.

(i) Use the Intermediate Value Theorem to show that the $f(x) = 0$ has solution on the given interval.

(ii) Use a graphing utility to find all the solutions to the equation on the given interval.

Solution.

$\square$

Contents

The previous chapter introduced the most fundamental of calculus topics: the limit. This chapter introduces the second most fundamental of calculus topics: the derivative. Limits describe where a function is going; derivatives describe how fast the function is going.

3.1 Instantaneous Rates of Change: The Derivative

Average rate of change: A rate of change describes how the output quantity y changes in relation to the input quantity x. The units on a rate of change are "output units per input units". Let (x_1, y_1) and (x_2, y_2), $x_1 \neq x_2$ be points on the graph of a function. The **average rate of change** between two input values is the total change of the function values (output values) divided by the change in the input values

$$\text{Average rate of change } = \frac{y_2 - y_1}{x_2 - x_1}, x_1 \neq x_2.$$

For example, we know how to calculate an average velocity:

$$\frac{\text{change in distance}}{\text{change in time}} = \text{average velocity}.$$

A common amusement park ride lifts riders to a height then allows them to free fall a certain distance before safely stopping them. Suppose such a ride drops riders from a height of 150 feet. Students of physics may recall that the height (in feet) of the riders, t seconds after free fall (and ignoring air resistance, etc.) can be accurately modeled by $f(t) = -16t^2 + 150$.

Using this formula, it is easy to verify that, without intervention, the riders will hit the ground at $t = 2.5\sqrt{1.5} \approx 3.06$ seconds. Suppose the designers of the ride decide to begin slowing the riders' fall after 2 seconds (corresponding to a height of 86 ft.). How fast will the riders be traveling at that time?

We have been given a *position* function, but what we want to compute is a velocity at a specific point in time, i.e., we want an *instantaneous velocity*. We do not currently know how to calculate this.

We can approximate the instantaneous velocity at $t = 2$ by considering the average velocity over some time period containing $t = 2$. If we make the time interval small, we will get a good approximation. (This fact is commonly used. For instance, high speed cameras are used to track fast moving objects. Distances are measured over a fixed number of frames to generate an accurate approximation of the velocity.)

Now, consider the interval from $t = 2$ to $t = 3$ (just before the riders hit the ground). On that interval, the average velocity is

$$\frac{f(3) - f(2)}{3 - 2} = \frac{f(3) - f(2)}{1} = -80 \text{ ft/s},$$

where the minus sign indicates that the riders are moving *down*. By narrowing the interval we consider, we will likely get a better approximation of the instantaneous velocity. On $[2, 2.5]$ we have

$$\frac{f(2.5) - f(2)}{2.5 - 2} = \frac{f(2.5) - f(2)}{0.5} = -72 \text{ ft/s}.$$

We can do this for smaller and smaller intervals of time. For instance, over a time span of $1/10^{\text{th}}$ of a second, i.e., on $[2, 2.1]$, we have

$$\frac{f(2.1) - f(2)}{2.1 - 2} = \frac{f(2.1) - f(2)}{0.1} = -65.6 \text{ ft/s}.$$

Over a time span of $1/100^{\text{th}}$ of a second, on $[2, 2.01]$, the average velocity is

$$\frac{f(2.01) - f(2)}{2.01 - 2} = \frac{f(2.01) - f(2)}{0.01} = -64.16 \text{ ft/s}.$$

What we are really computing is the average velocity on the interval $[2, 2+h]$ for small values of h. That is, we are computing

$$\frac{f(2+h) - f(2)}{h}$$

where h is small.

We really want to use $h = 0$, but this, of course, returns the familiar "0/0" indeterminate form. So we employ a limit, as we did in the last chapter.

h	Average Velocity ft/s
1	-80
0.5	-72
0.1	-65.6
0.01	-64.16
0.001	-64.016

Figure 1: Approximating the instantaneous velocity with average velocities over a small time period h.

We can approximate the value of this limit numerically with small values of h as seen in Figure 1. It looks as though the velocity is approaching -64 ft/s. Computing the limit directly gives

$$\begin{aligned}
\lim_{h\to 0} \frac{f(2+h) - f(2)}{h} &= \lim_{h\to 0} \frac{-16(2+h)^2 + 150 - (-16(2)^2 + 150)}{h} \\
&= \lim_{h\to 0} \frac{-64h - 16h^2}{h} \\
&= \lim_{h\to 0} (-64 - 16h) \\
&= -64.
\end{aligned}$$

Graphically, we can view the average velocities we computed numerically as the slopes of secant lines on the graph of f going through the points $(2, f(2))$ and $(2 + h, f(2+h))$. In Figure 2, the secant line corresponding to $h = 1$ is shown in three contexts. Figure 2(a) shows a "zoomed out" version of f with its secant line. In (b), we zoom in around the points of intersection between f and the secant line. Notice

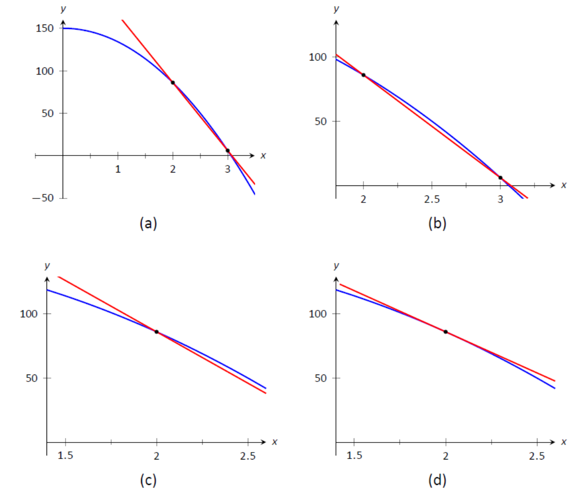


Figure 2: Parts (a), (b) and (c) show the secant line to $f(x)$ with $h = 1$, zoomed in different amounts. Part (d) shows the tangent line to f at $x = 2$.

how well this secant line approximates f between those two points – it is a common practice to approximate functions with straight lines.

As $h \to 0$, these secant lines approach the *tangent line*, a line that goes through the point $(2, f(2))$ with the special slope of -64. In parts (c) and (d) of Figure 2, we zoom in around the point $(2, 86)$. In (c) we see the secant line, which approximates f well, but not as well the tangent line shown in (d).

In this example, we see that **instantaneous rate of change** of the distance function is the **instantaneous velocity** and it is also the **slope of the tangent line** at the point.

Definition (Derivative at a Point)**.** Let f be a continuous function on an open interval I and let a be in I. The **derivative of f at** a, denoted $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

provided the limit exists. If the limit exists, we say that f **is differentiable at** a; if the limit does not exist, then f **is not differentiable at** a. If f is differentiable at every point in I, then f **is differentiable on** I.

If f is differentiable at a, then $f'(a)$ is the slope of the tangent line at $(a, f(a))$.

Definition (Tangent Line)**.** Let f be continuous on an open interval I and differentiable at a, for some a in I. The line with equation

$$y = \ell(x) = f^{'}(a)(x-a) + f(a)$$

is the **tangent line** to the graph of f at a; that is, it is the line through $(a, f(a))$ whose slope is the derivative of f at a.

Example Find an equation of the line tangent to the graph of $f(x) = x^3 + 4x$ at $(1, 5)$.

Solution.

□

Definition (The Derivative Function)**.** Let f be a differentiable function on an open interval I. That is, f is differentiable at every point of the open interval I. Then the function

$$f^{'}(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

is called **the derivative of** f.

Derivative Notation In the definition of derivative of a function, h is the change in the x-coordinate. A standard notation for change is the symbol Δ (uppercase Greek letter delta). So we can replace h with Δx to represent the change in x. Similarly,

$f(x+h) - f(x) = f(x+\Delta x) - f(x)$ is the change in y, denoted Δy. Therefore, for any given x,

$$f^{'}(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}.$$

The new notation for the derivative is $\dfrac{dy}{dx}$. It is one symbol; **it is not the fraction** dy/dx. This notation reminds us that $f^{'}(x)$ is the limit of $\frac{\Delta y}{\Delta x}$ as $\Delta x \to 0$.

In addition to the notation $f^{'}(x)$ and $\dfrac{dy}{dx}$, other common ways of writing the derivative include

$$\frac{df}{dx}, \quad \frac{d}{dx}(f(x)), \quad D_x(f(x)), \quad \text{and } y^{'}(x).$$

The following notations represents the derivative of f evaluated at a.

$$\left.\frac{df}{dx}\right|_{x=a}, \quad \left.\frac{dy}{dx}\right|_{x=a}, \quad f^{'}(a), \quad \text{and } y^{'}(a).$$

Example. Find the derivative of $f(x) = \sin x$

Solution.

$$\begin{aligned}
f^{'}(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} && \left(\begin{array}{c}\text{Use trig identity } \sin(x+h) = \\ \sin x \cos h + \cos x \sin h\end{array}\right) \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} && \text{(regroup)} \\
&= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} && \text{(split into two fractions)} \\
&= \lim_{h\to 0} \left(\frac{\sin x(\cos h - 1)}{h} + \frac{\cos x \sin h}{h}\right) && \left(\text{use } \lim_{h\to 0} \frac{\cos h - 1}{h} = 0 \text{ and } \lim_{h\to 0} \frac{\sin h}{h} = 1\right) \\
&= \sin x \cdot 0 + \cos x \cdot 1 \\
&= \cos x\ !
\end{aligned}$$

□

We have found that when $f(x) = \sin x$, $f^{'}(x) = \cos x$. This should be somewhat surprising; the result of a tedious limit process and the sine function is a nice function. Then again, perhaps this is not entirely surprising. The sine function is periodic – it repeats itself on regular intervals. Therefore its rate of change also repeats itself on the same regular intervals. We should have known the derivative would be periodic; we now know exactly which periodic function it is.

Example. Let $f(x) = \sqrt{x}$. Compute $\dfrac{df}{dx}$.

Solution.

$\square$

Example. Let $g(t) = 1/t^2$. Compute $g^{'}(t)$.

Solution.

$\square$

3.2 Continuity and Differentiability

The derivative of $f'(x)$ of a function $f(x)$ measures the instantaneous rate of change of f with respect to x. Put another way, the derivative answers "When x changes, at what rate does f change?" Graphically, $f'(a)$ is the slope of the tangent line that intersects $f(x)$ only once at $x = a$.

In this section, we explain the important relationship between continuity and differentiability.

Theorem (Differentiable Implies Continuous)**.** If f is differentiable at a, then f is continuous at a.

Proof.

$\square$

Remark: If f is not continuous at a, then f is not differentiable at a (Figure 3).

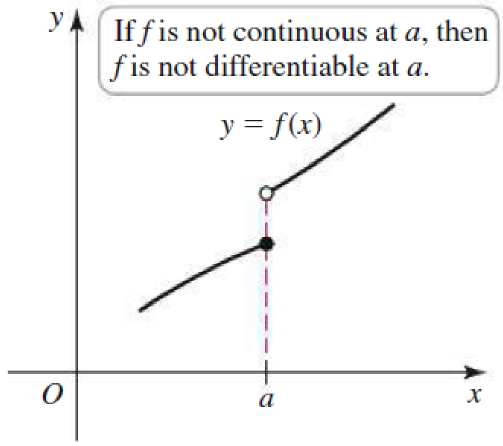


Figure 3: Case 1: $f(x)$ is not continuous at a thus not differentiable at a

By the theorem, if f is continuous at a point, f is not necessarily differentiable at that point. There are two possible cases that f is not necessarily differentiable at a point when f is continuous there:

Case 2 Suppose $f(x)$ has a sharp corner $(a, f(a)$ as shown in 4, it is easy to see that $f(x)$ is continuous at a. But because of the abrupt change in the slope of the curve at a, f is not differentiable at a: The limit that defines f' does not exist at a.

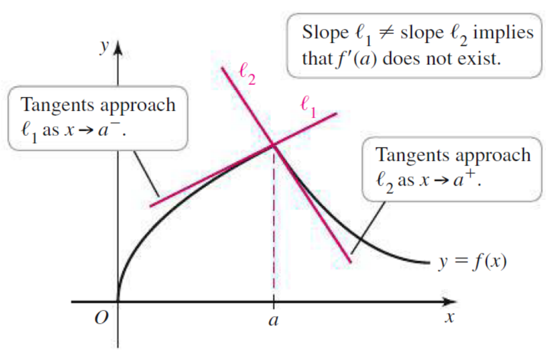


Figure 4: Case 2: $f(x)$ is not differentiable at the sharp corner $(a, f(a))$

Case 3 Another common situation occurs when the graph of a function f has a vertical tangent line at a. In this case, $f'(a)$ is undefined because the slope of a vertical line is undefined. A vertical tangent line may occur at a sharp point on the curve called a **cusp** (for example, the function $f(x) = \sqrt{|x|}$ in Figure 5(a)). In other cases, a vertical tangent line may occur without a cusp (for example, the function $f(x) = \sqrt[3]{x}$ in Figure 5 (b)).

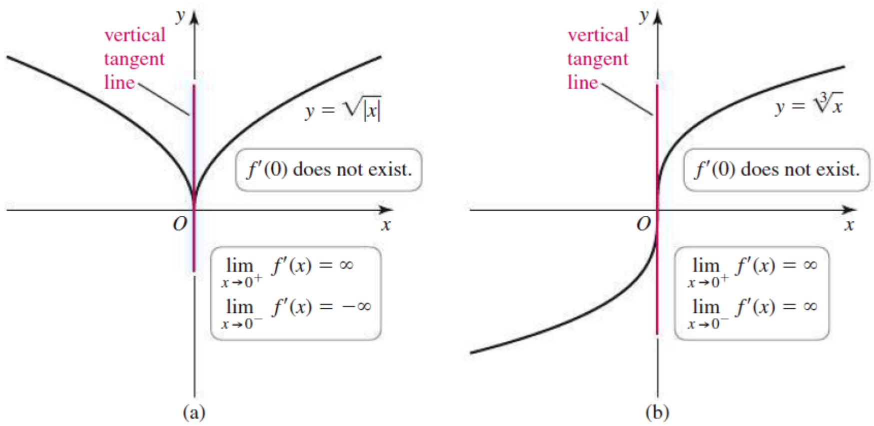


Figure 5: Case 3: $f(x)$ has a vertical tangent at a

When Is a Function Not Differentiable at a Point?

A function f is not differentiable at a if at least one of the following conditions holds:

a. f is not continuous at a(Figure 3).

b. f has a corner at a(Figure 4).

c. f has a vertical tangent at a(Figure 5).

Example[Continuous and differentiable] Consider the graph of g in Figure 6.

a. Find the values of x in the interval (-4, 4) at which g is not continuous.

b. Find the values of x in the interval (-4, 4) at which g is not differentiable.

Solution.

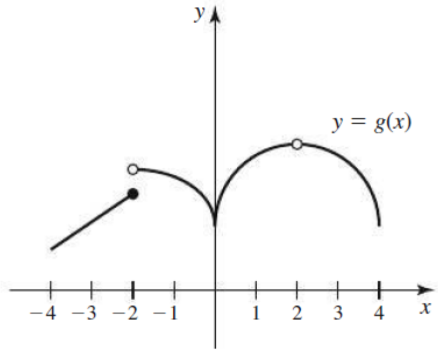


Figure 6: Continuous and differentiable

□

3.3 Rules of Differentiation

The derivative is a powerful tool but is admittedly awkward given its reliance on limits. Fortunately, one thing mathematicians are good at is *abstraction.* For instance, instead of continually finding derivatives at a point, we abstracted and found the derivative function.

Let's practice abstraction on linear functions, $y = mx + b$. What is y'? Without limits, recognize that linear function are characterized by being functions with a constant rate of change (the slope). The derivative, y', gives the instantaneous rate of change; with a linear function, this is constant, m. Thus $y' = m$.

Let's abstract once more. Let's find the derivative of the general quadratic function, $f(x) = ax^2 + bx + c$. Using the definition of the derivative, we have:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{a(x+h)^2 + b(x+h) + c - (ax^2 + bx + c)}{h} \\ &= \lim_{h\to 0} \frac{ah^2 + 2ahx + bh}{h} \\ &= \lim_{h\to 0} ah + 2ax + b \\ &= 2ax + b. \end{aligned}$$

So if $y = 6x^2 + 11x - 13$, we can immediately compute $y' = 12x + 11$.

In this section (and in some sections to follow) we will learn some of what mathematicians have already discovered about the derivatives of certain functions and how derivatives interact with arithmetic operations. We start with a theorem.

Theorem (Derivatives of Common Functions)**.** Derivatives of some common functions are listed below.

1. **Constant Rule:**
   $\dfrac{d}{dx}\big(c\big) = 0$, where c is a constant.
2. **Power Rule:**
   $\dfrac{d}{dx}\big(x^n\big) = nx^{n-1}$, where n is an integer, $n > 0$.
3. $\dfrac{d}{dx}(\sin x) = \cos x$
4. $\dfrac{d}{dx}(\cos x) = -\sin x$
5. $\dfrac{d}{dx}\left(e^x\right) = e^x$
6. $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$

This theorem starts by stating an intuitive fact: constant functions have no rate of change as they are *constant*. Therefore their derivative is 0 (they change at the rate of 0). The theorem then states some fairly amazing things. The Power Rule states that the derivatives of Power Functions (of the form $y = x^n$) are very straightforward: multiply by the power, then subtract 1 from the power. We see something incredible about the function $y = e^x$: it is its own derivative. We also see a new connection between the sine and cosine functions.

One special case of the Power Rule is when $n = 1$, i.e., when $f(x) = x$. What is $f^{'}(x)$? According to the Power Rule,

$$f^{'}(x) = \frac{d}{dx}\big(x\big) = \frac{d}{dx}\big(x^1\big) = 1 \cdot x^0 = 1.$$

In words, we are asking "At what rate does f change with respect to x?" Since f *is* x, we are asking "At what rate does x change with respect to x?" The answer is: 1. They change at the same rate. We proved that $\Big(\sin x\Big)^{'} = \cos x$ and $\Big(\cos x\Big)^{'} = -\sin x$ can be proved similarly.

That $(e^x)' = e^x$ can be shown by using the definition of derivatives and the fact that

Theorem (the number e)**.**

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

Example. Find the derivative of $f(x) = x^3$, and sketch f and f'.

Solution.

$\square$

The above theorem gives useful information, but we will need much more. For instance, using the theorem, we can easily find the derivative of $y = x^3$, but it does not tell how to compute the derivative of $y = 2x^3$, $y = x^3 + \sin x$ nor $y = x^3 \sin x$. The following theorem helps with the first two of these examples (the third is answered in the next section).

Theorem (Properties of the Derivative)**.** Let f and g be differentiable on an open interval I and let c be a real number. Then:

1. **Sum/Difference Rule:**
$$\frac{d}{dx}\Big(f(x) \pm g(x)\Big) = \frac{d}{dx}\Big(f(x)\Big) \pm \frac{d}{dx}\Big(g(x)\Big) = f'(x) \pm g'(x)$$
2. **Constant Multiple Rule:**
$$\frac{d}{dx}\Big(c \cdot f(x)\Big) = c \cdot \frac{d}{dx}\Big(f(x)\Big) = c \cdot f'(x).$$

Example. Evaluate the following derivatives.

$\frac{d}{dx}\left(-\frac{7x^{11}}{8}\right)$ **b.** $\frac{d}{dt}\left(\frac{3}{8}\sqrt{t}\right)$

Solution.

□

Example. Determine $\dfrac{d}{dw}\left(2w^3+9w^2-6w+4\right)$

Solution.

□

Example. Write an equation of the line tangent to the graph of $f(x)=2x-\frac{e^x}{2}$ at the point $(0,\frac{-1}{2})$.

Solution.

□

Example. Let $f(x)=2x^3-15x^2+24x$. For what values of x does the line tangent to the graph of f have a slope of 6?

Solution.

□

Higher Order Derivatives

The derivative of a function f is itself a function, therefore we can take its derivative. The following definition gives a name to this concept and introduces its notation.

Definition (Higher Order Derivatives)**.** Let $y = f(x)$ be a differentiable function on I. The following are defined, provided the corresponding limits exist.

1. The **second derivative of** f is:

$$f''(x) = \frac{d}{dx}\Big(f'(x)\Big) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = y''.$$

2. The **third derivative of** f is:

$$f'''(x) = \frac{d}{dx}\Big(f''(x)\Big) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = y'''.$$

3. The $\mathbf{n^{th}}$ **derivative of** f is:

$$f^{(n)}(x) = \frac{d}{dx}\left(f^{(n-1)}(x)\right) = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^ny}{dx^n} = y^{(n)}.$$

Remark In general, when finding the fourth derivative and on, we resort to the $f^{(4)}(x)$ notation, not $f''''(x)$; too many ticks is confusing.

Example. Find the third derivative of the following functions.

a. $f(x) = 3x^3 - 5x + 12$ **b.** $y = 3t + 2e^t$

Solution.

□

Interpreting Higher Order Derivatives

What do higher order derivatives *mean*? What is the practical interpretation?

It is easy to see that

> The second derivative of a function f is the rate of change of the rate of change of f.

One way to grasp this concept is to let f describe a position function. Then, as stated in the position/velocity example, f' describes the rate of position change: velocity. We now consider f'', which describes the rate of velocity change. Sports car enthusiasts talk of how fast a car can go from 0 to 60 mph; they are bragging about the *acceleration* of the car.

It can be difficult to consider the meaning of the third, and higher order, derivatives. The third derivative is "the rate of change of the rate of change of the rate of change of f." That is essentially meaningless to the uninitiated. In the context of our position/velocity/acceleration example, the third derivative is the "rate of change of acceleration," commonly referred to as "jerk."

Make no mistake: higher order derivatives have great importance even if their practical interpretations are hard (or "impossible") to understand. The mathematical topic of *series* (in Calculus II) makes extensive use of higher order derivatives.

3.4 The Product and Quotient Rules

The previous section showed that, in some ways, derivatives behave nicely. The Constant Multiple and Sum/Difference Rules established that the derivative of $f(x) = 5x^2 + \sin x$ was not complicated. We neglected computing the derivative of things like $g(x) = 5x^2 \sin x$ and $h(x) = \frac{5x^2}{\sin x}$ on purpose; their derivatives are *not* as straightforward. (If you had to guess what their respective derivatives are, you would probably guess wrong.) For these, we need the Product and Quotient Rules, respectively, which are defined in this section.

We begin with the Product Rule.

Theorem (Product Rule)**.** Let f and g be differentiable functions on an open interval I. Then fg is a differentiable function on I, and

$$\frac{d}{dx}\Big(f(x)g(x)\Big) = f^{'}(x)g(x) + f(x)g^{'}(x).$$

Important: $\frac{d}{dx}\Big(f(x)g(x)\Big) \neq f^{'}(x)g^{'}(x)$! While this answer is simpler than the Product Rule, it is wrong.

Proof. By the limit definition, we have

$$\frac{d}{dx}\Big(f(x)g(x)\Big) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}.$$

We now do something a bit unexpected; add 0 to the numerator (so that nothing is changed) in the form of $-f(x+h)g(x) + f(x+h)g(x)$, then do some regrouping as shown.

$$\begin{aligned}
\frac{d}{dx}\Big(f(x)g(x)\Big) &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \quad \text{(now add 0 to the numerator)}\\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \quad \text{(regroup)}\\
&= \lim_{h\to 0} \frac{\Big(f(x+h)g(x+h) - f(x+h)g(x)\Big) + \Big(f(x+h)g(x) - f(x)g(x)\Big)}{h}\\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x)}{h} \quad \text{(factor)}\\
&= \lim_{h\to 0} f(x+h)\frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} g(x) \quad \text{(apply limits)}\\
&= f(x)g^{'}(x) + f^{'}(x)g(x).
\end{aligned}$$

□

Example. Find and simplify the following derivatives.
a. $\dfrac{d}{dv}\Big(v^2(2\sqrt{v}+1)\Big)$ **b.** $\dfrac{d}{dx}(x^2e^x)$
Solution.

$\square$

We have learned how to compute the derivatives of sums, differences, and products of functions. We now learn how to find the derivative of a quotient of functions.

Theorem (Quotient Rule)**.** Let f and g be differentiable functions defined on an open interval I, where $g(x) \neq 0$ on I. Then f/g is differentiable on I, and

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f^{'}(x)g(x) - f(x)g^{'}(x)}{[g(x)]^2}.$$

Remark The Quotient Rule is not hard to use, although it might be a bit tricky to remember. The proof can be obtained using the Product rule by writing $\dfrac{f(x)}{g(x)} = f(x) \cdot \dfrac{1}{g(x)}$. So essentially, the Quotient Rule is equivalent to the product rule.

Example. Find and simplify the following derivatives.

a. $\dfrac{d}{dx}\left(\dfrac{x^2+3x+4}{x^2-1}\right)$ **b.** $\dfrac{d}{dx}(e^{-x})$

Solution.

$\square$

Example. Using the Quotient Rule to find $\frac{d}{dx}\big(\tan x\big) = \frac{d}{dx}\big(\frac{\sin x}{\cos x}\big)$.

Solution.

$\square$

We include this result in the following theorem about the derivatives of the trigonometric functions.

Derivatives of Trigonometric Functions:

1. $\dfrac{d}{dx}\big(\sin x\big) = \cos x$
2. $\dfrac{d}{dx}\big(\cos x\big) = -\sin x$
3. $\dfrac{d}{dx}\big(\tan x\big) = \sec^2 x$
4. $\dfrac{d}{dx}\big(\cot x\big) = -\csc^2 x$
5. $\dfrac{d}{dx}\big(\sec x\big) = \sec x \tan x$
6. $\dfrac{d}{dx}\big(\csc x\big) = -\csc x \cot x$

Extending the Power Rule to Negative Integers

Example. Find the derivatives of the following functions.

1. $f(x) = \dfrac{1}{x}$
2. $f(x) = \dfrac{1}{x^n}$, where $n > 0$ is an integer.

Solution.

$\square$

Theorem (Power Rule with Integer Exponents)**.** Let $f(x) = x^n$, where $n \neq 0$ is an integer. Then

$$f'(x) = n \cdot x^{n-1}.$$

Example Find the following derivatives.
a. $\frac{d}{dx}\left(\frac{9}{x^5}\right)$ **b.** $\frac{d}{dt}\left(\frac{3t^{16}-4}{t^6}\right)$
Solution.

$\square$

Some situations call for the use of multiple differentiation rules.

Example Combining Derivative Rules.
Find derivative of $y = \frac{4xe^x}{x^2+1}$
Solution.

$\square$

3.5 The Chain Rule

We have covered almost all of the derivative rules that deal with combinations of two (or more) functions. The operations of addition, subtraction, multiplication (including by a constant) and division led to the Sum and Difference rules, the Constant Multiple Rule, the Power Rule, the Product Rule and the Quotient Rule. To complete the list of differentiation rules, we look at the last way two (or more) functions can be combined: the process of composition (i.e. one function "inside" another).

One example of a composition of functions is $f(x) = \sin(x^2)$. We currently do not know how to compute this derivative. If forced to guess, one would likely guess $f'(x) = \cos(2x)$, where we recognize $\cos x$ as the derivative of $\sin x$ and $2x$ as the derivative of x^2. However, this is not the case; $f'(x) \neq \cos(2x)$. We'll see the correct answer, which employs the new rule this section introduces, the **Chain Rule**.

Before we define this new rule, recall the notation for composition of functions. We write $(f \circ g)(x)$ or $f(g(x))$, read as "f of g of x," to denote composing f with g. In shorthand, we simply write $f \circ g$ or $f(g)$ and read it as "f of g." Before giving the corresponding differentiation rule, we note that the rule extends to multiple compositions like $f(g(h(x)))$ or $f(g(h(j(x))))$, etc.

To motivate the rule, let's look at two derivatives we can already compute.

Example Find the derivatives of $F_1(x) = (1-x)^2$ and $F_2(x) = (1-x)^3$.

Theorem (The Chain Rule)**.** Let $u = g(x)$ be a differentiable function on an interval I, let the range of $g(x)$ be a subset of the interval J, and let $y = f(u)$ be a differentiable function on J. Then $y = f(g(x))$ is a differentiable function on I, and

$$y^{'} = f^{'}(g(x)) \cdot g^{'}(x),$$

or written as

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

A proof of the Chain Rule is given in the text book. To understand why the chain rule is true, we look at a specific example, derivative of $f(x) = \sin(x^2)$, using the definition of derivatives.

We first set up the limit that would give us the derivative at a specific value a in the domain of $\sin(x^2)$.

$$f^{'}(a) = \lim_{x \to a} \frac{\sin(x^2) - \sin(a^2)}{x - a}.$$

This expression does not seem particularly helpful; however, we can modify it by multiplying and dividing by the expression $x^2 - a^2$ to obtain

$$f^{'}(a) = \lim_{x \to a} \frac{\sin(x^2) - \sin(a^2)}{x^2 - a^2} \cdot \frac{x^2 - a^2}{x - a}.$$

From the definition of the derivative, we can see that the second factor is the derivative of x^2 at $x = a$. That is,

$$\lim_{x \to a} \frac{x^2 - a^2}{x - a} = \frac{d}{dx}(x^2)\Big|_{x=a} = 2a.$$

However, it might be a little more challenging to recognize that the first term is also a derivative. We can see this by letting $u = x^2$ and observing that as $x \to a$, $u \to a^2$.

$$\lim_{x \to a} \frac{\sin(x^2) - \sin(a^2)}{x^2 - a^2} = \lim_{u \to a^2} \frac{\sin(u) - \sin(a^2)}{u - a^2} = \frac{d}{du}(\sin u)\Big|_{u=a^2} = \cos(a^2).$$

Therefore,

$$f^{'}(a) = \lim_{x \to a} \frac{\sin(x^2) - \sin(a^2)}{x^2 - a^2} \cdot \frac{x^2 - a^2}{x - a} = \lim_{x \to a} \frac{\sin(x^2) - \sin(a^2)}{x^2 - a^2} \cdot \lim_{x \to a} \frac{x^2 - a^2}{x - a} = \cos(a^2) \cdot 2a.$$

In other words, if $f(x) = \sin(x^2)$, then $f^{'}(x) = \cos(x^2) \cdot 2x$.

Example. Use the chain rule find the derivative of the function $f(x) = \sin(x^2)$, $F_1(x) = (1-x)^2$ and $F_2(x) = (1-x)^3$.

Procedure of Applying the Chain Rule: Assume the differentiable function $y = f(g(x))$ is given.

1. Identify an outer function f and an inner function g, and let $u = g(x)$.

2. Find $f'(u)$ and evaluate it at $u = g(x)$ to obtain $f'(g(x))$.

3. Find $g'(x)$.

4. Write $y' = f'(g(x)) \cdot g'(x)$.

Example Find the derivative y' of the following functions.

a. $y = (5x+4)^3$ **b.** $y = \sin^3 x$ **c.** $y = \sin(x^3)$

Solution.

$\square$

Example Find the derivative y' of the following functions.

a. $y = (6x^3 + 3x + 1)^{10}$ **b.** $y = \sqrt{5x^2 + 1}$ **c.** $y = \left(\dfrac{5t^2}{3t^2 + 2}\right)^3$

Solution.

$\square$

The Chain Rule is used often in taking derivatives. Because of this, one can become familiar with the basic process and learn patterns that facilitate finding derivatives quickly. For instance,

$$\frac{d}{dx}\Big(\ln(\text{anything})\Big) = \frac{1}{\text{anything}} \cdot (\text{anything})' = \frac{(\text{anything})'}{\text{anything}}.$$

A concrete example of this is

$$\frac{d}{dx}\Big(\ln(3x^{15} - \cos x + e^x)\Big) = \frac{45x^{14} + \sin x + e^x}{3x^{15} - \cos x + e^x}.$$

While the derivative may look intimidating at first, look for the pattern. The denominator is the same as what was inside the natural log function; the numerator is simply its derivative.

This pattern recognition process can be applied to lots of functions. In general, instead of writing "anything", we use u as a generic function of x. We then say

$$\frac{d}{dx}\Big(\ln u\Big) = \frac{u'}{u}.$$

The following is a short list of how the Chain Rule can be quickly applied to familiar functions.

1. $\dfrac{d}{dx}\Big(u^n\Big) = n \cdot u^{n-1} \cdot u'$.
2. $\dfrac{d}{dx}\Big(e^u\Big) = e^u \cdot u'$.
3. $\dfrac{d}{dx}\Big(\sin u\Big) = \cos u \cdot u'$.
4. $\dfrac{d}{dx}\Big(\cos u\Big) = -\sin u \cdot u'$.
5. $\dfrac{d}{dx}\Big(\tan u\Big) = \sec^2 u \cdot u'$.

Of course, the Chain Rule can be applied in conjunction with any of the other rules we have already learned. We practice this next.

Example. Using the Product, Quotient and Chain Rules

Find the derivatives of the following functions.

1. $f(x) = x^5 \sin 2x^3$ 2. $f(x) = \dfrac{5x^3}{e^{-x^2}}$.

Solution.

$\square$

Theorem (The derivative of e^{kx})**.** For real numbers k,

$$\frac{d}{dx}(e^{kx}) = ke^{kx}.$$

The Chain Rule leads to a general derivative rule for powers of differentiable functions. In fact, we have already used it in several examples. Consider the function $f(x) = (g(x))^n$, where n is an integer. Letting $f(u) = u^n$ be the outer function and $u = g(x)$ be the inner function, we obtain the Chain Rule for powers of functions.

Theorem (Chain Rule for Powers)**.** If g is differentiable for all x in its domain and n is an integer, then

$$\frac{d}{dx}\Big((g(x))^n\Big) = n(g(x))^{n-1}g^{'}(x).$$

Example. Find $\frac{d}{dx}(\tan x + 10)^{21}$.

Solution.

$\square$

The Composition of Three or More Functions

We can differentiate the composition of three or more functions by applying the Chain Rule repeatedly, as shown in the following example.

Example. Find $\dfrac{d}{dx}\sin(e^{\cos x})$.

Solution.

$\square$

Example. Find $\dfrac{d}{dx}x^2\sqrt{x^2+1}$.

Solution.

$\square$

Remark: Chain Rule Notation

It is instructive to understand what the Chain Rule "looks like" using "$\frac{dy}{dx}$" notation instead of y' notation. Suppose that $y = f(u)$ is a function of u, where $u = g(x)$ is a function of x, as stated in Chain Rule Theorem. Then, through the composition $f \circ g$, we can think of y as a function of x, as $y = f(g(x))$. Thus the derivative of y with respect to x makes sense; we can talk about $\frac{dy}{dx}$. This leads to an interesting progression of notation:

$$
\begin{aligned}
y' &= f'(g(x)) \cdot g'(x) \\
\frac{dy}{dx} &= y'(u) \cdot u'(x) \quad \text{(since } y = f(u) \text{ and } u = g(x)\text{)} \\
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \quad \text{(using "fractional" notation for the derivative)}
\end{aligned}
$$

Here the "fractional" aspect of the derivative notation stands out. On the right hand side, it seems as though the "du" terms cancel out, leaving

$$\frac{dy}{dx} = \frac{dy}{dx}.$$

It is important to realize that we *are not* canceling these terms; the derivative notation of $\frac{dy}{du}$ is one symbol. It is equally important to realize that this notation was chosen precisely because of this behavior. It makes applying the Chain Rule easy with multiple variables. For instance,

$$\frac{dy}{dt} = \frac{dy}{d\bigcirc} \cdot \frac{d\bigcirc}{d\triangle} \cdot \frac{d\triangle}{dt}.$$

where $\bigcirc$ and $\triangle$ are any variables you like to use.

3.6 Implicit Differentiation

In the previous sections we learned to find the derivative, $\frac{dy}{dx}$, or y', when y is given *explicitly* as a function of x. That is, if we know $y = f(x)$ for some function f, we can find y'. For example, given $y = 3x^2 - 7$, we can easily find $y' = 6x$. (Here we explicitly state how x and y are related. Knowing x, we can directly find y.)

Sometimes the relationship between y and x is not explicit; rather, it is *implicit*. And the *implicit* relationship between x and y could be complicated: there is no way to express y explicitly in terms of x.

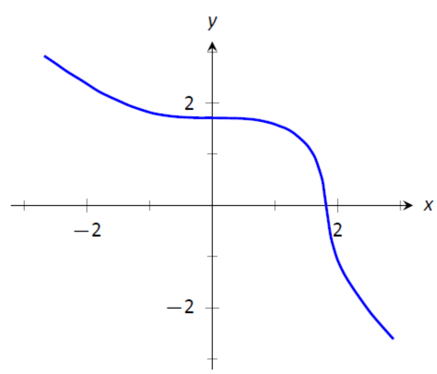


Figure 7: A graph of the implicit function $\sin(y) + y^3 = 6 - x^3$

For example, suppose we are given $\sin(y) + y^3 = 6 - x^3$. A graph of this implicit function is given in Figure 7. In this case there is absolutely no way to solve for y in terms of elementary functions. The surprising thing is, however, that we can still find y' via a process known as **implicit differentiation**. (Verify if you can get the answer $y' = \frac{-3x^2}{\cos y + 3y^2}$)

Implicit differentiation is a technique based on the Chain Rule that is used to find a derivative when the relationship between the variables is given implicitly rather than explicitly (solved for one variable in terms of the other).

We begin by reviewing the Chain Rule. Let f and g be functions of x. Then

$$\frac{d}{dx}\Big(f(g(x))\Big) = f^{'}(g(x)) \cdot g^{'}(x).$$

Suppose now that $y = g(x)$. We can rewrite the above as

$$\frac{d}{dx}\Big(f(y)\Big) = f^{'}(y) \cdot y^{'}, \quad \text{or} \quad \frac{d}{dx}\Big(f(y)\Big) = f^{'}(y) \cdot \frac{dy}{dx}. \tag{3.1}$$

These equations look strange; the key concept to learn here is that we can find $y^{'}$ even if we don't exactly know how y and x relate.

Example.

a. Calculate $\frac{dy}{dx}$ directly from the equation for the unit circle

b. Find the slope of the unit circle at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

Solution.

$\square$

Implicit functions are generally harder to deal with than explicit functions. With an explicit function, given an x value, we have an explicit formula for computing the corresponding y value. With an implicit function, one often has to find x and y values *at the same time* that satisfy the equation. It is much easier to demonstrate that a given point satisfies the equation than to actually find such a point.

Example. Find $y'(x)$ when $\sin xy = x^2 + y$.

Solution.

$\square$

Slopes of Tangent Lines: Derivatives obtained by implicit differentiation typically depend on x and y. Therefore, the slope of a curve at a particular point (a, b) requires both the x- and y-coordinates of the point. These coordinates are also needed to find an equation of the tangent line at that point.

Example. Find an equation of the line tangent to the curve $x^2 + xy - y^3 = 7$ at $(3, 2)$.

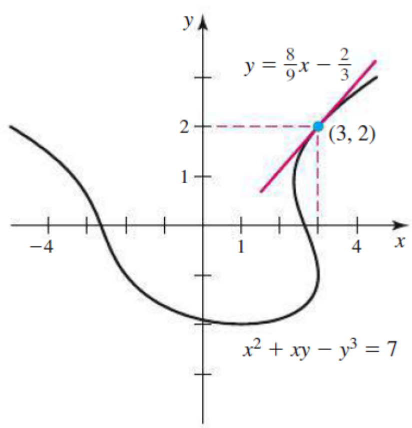


Figure 8: The graph of the implicit function $x^2 + xy - y^3 = 7$ and the tangent line at $(3, 2)$

Solution.

□

Implicit Differentiation and Higher-Order Derivatives

We can use implicit differentiation to find higher order derivatives. In theory, this is simple: first find $\frac{dy}{dx}$, then take its derivative with respect to x to find $\frac{d^2y}{dx^2}$, $\cdots$. In practice, it is not hard, but it often requires a bit of algebra.

Example. Find $\frac{d^2y}{dx^2}$ if $x^2 + y^2 = 1$.

Solution.

□

Implicit differentiation can also be used to further our understanding of "regular" differentiation.

One hole in our current understanding of derivatives is this: what is the derivative of the square root function? That is,

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{d}{dx}\left(x^{1/2}\right) = ?$$

We allude to a possible solution, as we can write the square root function as a power function with a rational (or, fractional) power. We are then tempted to apply the Power Rule and obtain

$$\frac{d}{dx}\left(x^{1/2}\right) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}.$$

The trouble with this is that the Power Rule was initially defined only for positive integer powers, $n > 0$. While we did not justify this at the time, generally the Power Rule is proved using something called the Binomial Theorem, which deals only with positive integers. The Quotient Rule allowed us to extend the Power Rule to negative integer powers. Implicit Differentiation allows us to extend the Power Rule to rational powers, as shown below.

Let $y = x^{m/n}$, where m and n are integers with no common factors (so $m = 2$ and $n = 5$ is fine, but $m = 2$ and $n = 4$ is not). We can rewrite this explicit function implicitly as $y^n = x^m$. Now apply implicit differentiation.

$$\begin{aligned}
y &= x^{m/n} \\
y^n &= x^m \\
\frac{d}{dx}\left(y^n\right) &= \frac{d}{dx}\left(x^m\right) \\
n \cdot y^{n-1} \cdot y' &= m \cdot x^{m-1} \\
y' &= \frac{m}{n}\frac{x^{m-1}}{y^{n-1}} \qquad \text{(now substitute } x^{m/n} \text{ for } y\text{)} \\
&= \frac{m}{n}\frac{x^{m-1}}{\left(x^{m/n}\right)^{n-1}} \qquad \text{(apply lots of algebra)} \\
&= \frac{m}{n}x^{(m-n)/n} \\
&= \frac{m}{n}x^{\frac{m}{n}-1}.
\end{aligned}$$

The above derivation is the key to the proof extending the Power Rule to rational powers. Using limits, we can extend this once more to include *all* powers, including irrational (even transcendental!) powers, giving the following theorem.

Theorem (Power Rule for Rational Exponents)**.** Let $f(x) = x^{p/q}$, where p and q are integers with $q \neq 0$. Then f is differentiable on its domain, except possibly at $x = 0$, and

$$f^{'}(x) = \frac{p}{q} \cdot x^{\frac{p}{q}-1}.$$

Example Calculate $\dfrac{dy}{dx}$ for the following functions.

a. $y = \dfrac{1}{\sqrt{x}}$ **b.** $y = (x^6 + 3x)^{2/3}$

Solution.

$\square$

Example Find the slope of the curve $2(x + y)^{1/3} = y$ at the point $(4, 4)$.

Solution.

$\square$

3.7 Derivatives of Logarithmic and Exponential Functions

In this section, we first discover how to differentiate the natural logarithmic function. From there, we treat general exponential and logarithmic functions.

The Derivative of $y = \ln x$.

We know that $(\ln x)' = \frac{1}{x}$. Now we show how it is obtained.

To find the derivative of $y = \ln x$, we begin with the inverse property and write

$$x = e^y, x > 0.$$

Differentiate both sides with respect to x,

$$1 = e^y \cdot \frac{dy}{dx}.$$

Solve for $\frac{dy}{dx}$ and use $x = e^y$,

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}.$$

Therefore,

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

An important extension is obtained by considering the function $\ln |x|$, which is defined for all $x \neq 0$. By the definition of the absolute value,

$$\ln |x| = \begin{cases} \ln x, & \text{if } x > 0 \\ \ln(-x), & \text{if } x < 0 \end{cases}.$$

For $x > 0$, it follows immediately that

$$\frac{d}{dx}(\ln |x|) = \frac{d}{dx}(\ln x) = \frac{1}{x}.$$

When $x < 0$, a similar calculation using the Chain Rule reveals that

$$\frac{d}{dx}(\ln |x|) = \frac{d}{dx}(\ln(-x)) = \frac{1}{-x}(-1) = \frac{1}{x}.$$

Theorem (Derivative of $\ln x$)**.**

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \text{ for } x > 0 \qquad \frac{d}{dx}(\ln |x|) = \frac{1}{x}, \text{ for } x \neq 0$$

If u is differentiable at x and $u(x) \neq 0$, then

$$\frac{d}{dx}(\ln|u(x)|) = \frac{u'(x)}{u(x)}.$$

Example Find $\dfrac{dy}{dx}$ for the following functions.

a. $y = \ln(4x)$ **b.** $y = x\ln x$ **c.** $y = \ln|\sec x|$ **d.** $y = \dfrac{\ln x^2}{x^2}$

Solution.

$\square$

The Derivative of b^x

A rule similar to $(e^x)' = e^x$ exists for computing the derivative of b^x, where $b > 0, b \neq 1$. Because $b^x = e^{\ln(b^x)} = e^{x \ln b}$, so by the chain rule

$$\frac{d}{dx}(b^x) = \frac{d}{dx}(e^{\ln(b^x)}) = e^{\ln(b^x)} \cdot \ln b = b^x \cdot \ln b.$$

Theorem (Derivative of b^x)**.** If $b > 0$ and $b \neq 1$, then for all x,

$$\frac{d}{dx}(b^x) = b^x \cdot \ln b.$$

Example Find $\frac{dy}{dx}$ for the following functions.
a. $y = 3^x$ **b.** $g(t) = 108 \cdot 2^{t/12}$

Solution.

$\square$

The General Power Rule

We derived the power rule for rational powers. Now we see a generalized power rule for both rational and irrational powers.

Theorem (The General Power Rule)**.** Let $f(x) = x^p$, where $x > 0$ and p is a non-zero real number. Then

$$f'(x) = px^{p-1}.$$

Example Find $\dfrac{dy}{dx}$ for the following functions.
a. $y = x^\pi$ **b.** $y = \pi^x$ **c.** $y = (x^2 + 4)^e$

Solution.

$\square$

Example. (General exponential functions) Let $f(x) = x^{\sin x}, x > 0$.

a. Find $f^{'}(x)$ **b.** Evaluate $f^{'}(\frac{\pi}{2})$

Solution.

$\square$

Example. (General exponential functions) Determine whether the graph of $f(x) = x^x, x > 0$ has any horizontal tangent lines.

Solution.

$\square$

Logarithmic Differentiation

Theorem (Derivative of $\log_b x$)**.** If $b > 0$ and $b \neq 1$, then

$$\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}, \text{ for } x > 0 \qquad \text{and } \frac{d}{dx}(\log_b |x|) = \frac{1}{x \ln b}, \text{ for } x \neq 0$$

Example. (Derivatives with general logarithms) Compute the derivative of the following functions.

a. $f(x) = \log_5(2x + 1)$ **b.** $g(x) = x \log_2(x)$

Solution.

$\square$

Logarithmic Differentiation Technique

Products, quotients, and powers of functions are usually differentiated using the derivative rules of the same name (perhaps combined with the Chain Rule). There are times, however, when the direct computation of a derivative is very tedious. Consider the function

$$f(x) = \frac{(x^3-1)^4\sqrt{3x-1}}{x^2+4}.$$

We would need the Quotient, Product, and Chain Rules just to compute $f'(x)$, and simplifying the result would require additional work.

A differentiation technique known as *logarithmic differentiation* becomes useful here. The basic principle is this: take the natural log of both sides of an equation $y = f(x)$, then use implicit differentiation to find y'. Recall that this technique is actually applied to the two examples of **General Exponential Functions** we just saw.

We further demonstrate this technique in this example.

Example. Let $f(x) = \dfrac{(x^3-1)^4\sqrt{3x-1}}{x^2+4}$. Calculate $f'(x)$.

Solution.

$\square$

3.8 Derivatives of Inverse Functions

Recall that a function $y = f(x)$ is said to be *one to one* if it passes the horizontal line test; that is, for two different x values x_1 and x_2, we do *not* have $f(x_1) = f(x_2)$. In some cases the domain of f must be restricted so that it is one to one. For instance, consider $f(x) = x^2$. Clearly, $f(-1) = f(1)$, so f is not one to one on its regular domain, but by restricting f to $(0, \infty)$, f is one to one.

Now recall that one to one functions have *inverses*. That is, if f is one to one, it has an inverse function, denoted by f^{-1}, such that if $f(a) = b$, then $f^{-1}(b) = a$. The domain of f^{-1} is the range of f, and vice-versa. Note that,

$$f\big(f^{-1}(x)\big) = x \quad \text{and} \quad f^{-1}\big(f(x)\big) = x.$$

This gives us a convenient way to check if two functions are inverses of each other: compose them and if the result is x (identity function), then they are inverses (on the appropriate domains.)

Let $y = f^{-1}(x)$. We want to find $y^{'}$. Since $y = f^{-1}(x)$, we know that $f(y) = x$. Using the Chain Rule and Implicit Differentiation, take the derivative of both sides of this last equality.

$$\begin{aligned} \frac{d}{dx}\Big(f(y)\Big) &= \frac{d}{dx}\Big(x\Big) \\ f^{'}(y) \cdot y^{'} &= 1 \\ y^{'} &= \frac{1}{f'(y)} \\ y^{'} &= \frac{1}{f'(f^{-1}(x))}. \end{aligned}$$

This leads us to the following theorem.

Theorem (Derivatives of Inverse Functions)**.** Let f be differentiable and one-to-one on an open interval I, where $f^{'}(x) \neq 0$ for all x in I, let J be the range of f on I, let f^{-1} be the inverse function of f, and let $f(a) = b$ for some a in I. Then f^{-1} is a differentiable function on J, and in particular,

1. $\left(f^{-1}\right)^{'}(b) = \dfrac{1}{f'(a)}$ and **2.** $\left(f^{-1}\right)^{'}(x) = \dfrac{1}{f'(f^{-1}(x))}$

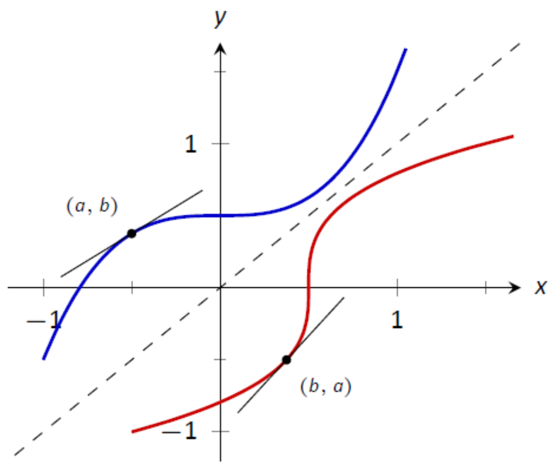


Figure 9: Corresponding tangent lines drawn to f and f^{-1}

Example. Linear functions, inverses, and derivatives

Consider the general linear function $y = f(x) = mx + b$, where $m \neq 0$ and b are constants.

a. Write the inverse of f in the form $y = f^{-1}(x)$.

b. Find the derivative of the inverse $\frac{d}{dx}(f^{-1}(x))$.

c. Consider the specific case $f(x) = 2x - 6$. Graph f and f^{-1}, and find the slope of each line.

Solution.

$\square$

Example. Derivative of an inverse function

The function $f(x) = \sqrt{x} + x^2 + 1$ is one-to-one, for $x \geq 0$, and has an inverse on that interval. Find the slope of the curve $y = f^{-1}(x)$ at the point $(3, 1)$.

Solution.

$\square$

We then apply the theorem to develop the derivatives of the six inverse trigonometric functions.

Theorem (Derivative of Inverse Sine)**.**

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, \text{ for } -1 < x < 1$$

Proof. Let $f(x) = \sin x$ and $f^{-1} = \sin^{-1} x$ or $\arcsin x$. Thus $f'(x)) = \cos x$. Applying the theorem, we have

$$\begin{aligned}(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} \\ &= \frac{1}{\cos(\arcsin x)}.\end{aligned}$$

This last expression is not immediately illuminating. Drawing a figure will help, as shown in Figure 10. Recall that the sine function can be viewed as taking in an angle and returning a ratio of sides of a right triangle, specifically, the ratio “opposite over hypotenuse.” This means that the arcsine function takes as input a ratio of sides and returns an angle. The equation $y = \arcsin x$ can be rewritten as $y = \arcsin(x/1)$; that is, consider a right triangle where the hypotenuse has length 1 and the side opposite of

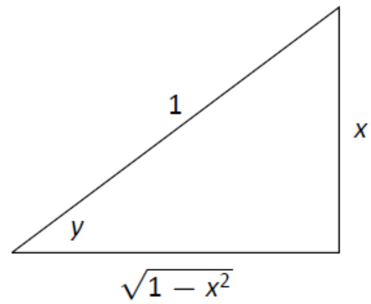


Figure 10: A right triangle defined by $y = \sin -1(\frac{x}{1})$ with the length of the third leg found using the Pythagorean Theorem.

the angle with measure y has length x. This means the final side has length $\sqrt{1-x^2}$, using the Pythagorean Theorem.

Therefore $\cos(\sin^{-1} x) = \cos y = \sqrt{1-x^2}/1 = \sqrt{1-x^2}$, resulting in

$$\frac{d}{dx}\big(\arcsin x\big) = (f^{-1})'(x) = \frac{1}{\sqrt{1-x^2}}.$$

$\square$

Remark The derivative of the inverse cosine can be derived from the identity $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$.

Example Compute the following derivatives.

a. $\frac{d}{dx} \sin^{-1}(x^2 - 1)$ **b.** $\frac{d}{dx} \cos(\sin^{-1} x)$

Solution.

$\square$

Using similar techniques and the identities $\cot^{-1} x + \tan^{-1} x = \frac{\pi}{2}$ and $\csc^{-1} x + \sec^{-1} x = \frac{\pi}{2}$, we can find the derivatives of all the inverse trigonometric functions.

Please read the textbook for the derivations. In Figure 11 we show the restrictions of the domains of the standard trigonometric functions that allow them to be invertible.

Function	Domain	Range	Inverse Function	Domain	Range
$\sin x$	$[-\pi/2, \pi/2]$	$[-1, 1]$	$\sin^{-1} x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos x$	$[0, \pi]$	$[-1, 1]$	$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan x$	$(-\pi/2, \pi/2)$	$(-\infty, \infty)$	$\tan^{-1} x$	$(-\infty, \infty)$	$(-\pi/2, \pi/2)$
$\cot x$	$(0, \pi)$	$(-\infty, \infty)$	$\cot^{-1} x$	$(-\infty, \infty)$	$(0, \pi)$
$\csc x$	$[-\pi/2, 0) \cup (0, \pi/2]$	$(-\infty, -1] \cup [1, \infty)$	$\csc^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[-\pi/2, 0) \cup (0, \pi/2]$
$\sec x$	$[0, \pi/2) \cup (\pi/2, \pi]$	$(-\infty, -1] \cup [1, \infty)$	$\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi/2) \cup (\pi/2, \pi]$

Figure 11: Domains and ranges of the trigonometric and inverse trigonometric functions.

Theorem (Derivatives of Inverse Trigonometric Functions)**.** The inverse trigonometric functions are differentiable on all open sets contained in their domains (as listed in Figure 11) and their derivatives are as follows:

1. $\dfrac{d}{dx}\left(\sin^{-1} x\right) = \dfrac{1}{\sqrt{1-x^2}},\ -1 < x < 1$
2. $\dfrac{d}{dx}\left(\sec^{-1} x\right) = \dfrac{1}{|x|\sqrt{x^2-1}},\ |x| > 1$
3. $\dfrac{d}{dx}\left(\tan^{-1} x\right) = \dfrac{1}{1+x^2},\ -\infty < x < \infty$
4. $\dfrac{d}{dx}\left(\cos^{-1} x\right) = -\dfrac{1}{\sqrt{1-x^2}},\ -1 < x < 1$
5. $\dfrac{d}{dx}\left(\csc^{-1} x\right) = -\dfrac{1}{|x|\sqrt{x^2-1}}, |x| > 1$
6. $\dfrac{d}{dx}\left(\cot^{-1} x\right) = -\dfrac{1}{1+x^2},\ -\infty < x < \infty$

Example. Derivatives of inverse trigonometric functions

a. Evaluate $f'(2\sqrt{3})$, where $f(x) = x\tan^{-1}(x/2)$.

b. Find an equation of the line tangent to the graph of $g(x) = \sec^{-1}(2x)$ at the point $(1, \pi/3)$.

Solution.

$\square$

In this chapter we have defined the derivative, given rules to facilitate its computation, and given the derivatives of a number of standard functions. We restate the most important of these in the following theorem, intended to be a reference for further work.

Glossary of Derivatives of Elementary Functions

Let $u = u(x)$ and $v = v(x)$ be differentiable functions of x, and let a, c and n be real numbers, $a > 0$, $n \neq 0$.

1. $\frac{d}{dx}\big(cu\big) = cu'$
2. $\frac{d}{dx}\big(u \pm v\big) = u' \pm v'$
3. $\frac{d}{dx}\big(u \cdot v\big) = uv' + u'v$
4. $\frac{d}{dx}\big(\frac{u}{v}\big) = \frac{u'v - uv'}{v^2}$
5. $\frac{d}{dx}\big(u(v)\big) = u'(v)v'$
6. $\frac{d}{dx}\big(c\big) = 0$
7. $\frac{d}{dx}\big(x\big) = 1$
8. $\frac{d}{dx}\big(x^n\big) = nx^{n-1}$
9. $\frac{d}{dx}\big(e^x\big) = e^x$
10. $\frac{d}{dx}\big(a^x\big) = \ln a \cdot a^x$
11. $\frac{d}{dx}\big(\ln x\big) = \frac{1}{x}$
12. $\frac{d}{dx}\big(\log_a x\big) = \frac{1}{\ln a} \cdot \frac{1}{x}$
13. $\frac{d}{dx}\big(\sin x\big) = \cos x$
14. $\frac{d}{dx}\big(\cos x\big) = -\sin x$
15. $\frac{d}{dx}\big(\sec x\big) = \sec x \tan x$
16. $\frac{d}{dx}\big(\csc x\big) = -\csc x \cot x$
17. $\frac{d}{dx}\big(\tan x\big) = \sec^2 x$
18. $\frac{d}{dx}\big(\cot x\big) = -\csc^2 x$
19. $\frac{d}{dx}\big(\sin^{-1} x\big) = \frac{1}{\sqrt{1-x^2}}$
20. $\frac{d}{dx}\big(\cos^{-1} x\big) = -\frac{1}{\sqrt{1-x^2}}$
21. $\frac{d}{dx}\big(\sec^{-1} x\big) = \frac{1}{|x|\sqrt{x^2-1}}$
22. $\frac{d}{dx}\big(\csc^{-1} x\big) = -\frac{1}{|x|\sqrt{x^2-1}}$
23. $\frac{d}{dx}\big(\tan^{-1} x\big) = \frac{1}{1+x^2}$
24. $\frac{d}{dx}\big(\cot^{-1} x\big) = -\frac{1}{1+x^2}$

Contents

Our study of limits led to continuous functions, which is a certain class of functions that behave in a particularly nice way. Limits then gave us an even nicer class of functions, functions that are differentiable. In this chapter we explore the applications of the derivatives.

4.1 Maxima and Minima

Given any quantity described by a function, we are often interested in the largest and/or smallest values that quantity attains. For instance, if a function describes the speed of an object, it seems reasonable to want to know the fastest/slowest the object traveled. If a function describes the value of a stock, we might want to know the highest/lowest values the stock attained over the past year. We call such values *extreme values.*

Definition (Absolute Maximum and Minimum)**.** Let f be defined on an interval D containing c.

1. $f(c)$ is the **minimum** (also, **absolute minimum**) of f on D if $f(c) \le f(x)$ for all x in D.

2. $f(c)$ is the **maximum** (also, **absolute maximum**) of f on D if $f(c) \ge f(x)$ for all x in D.

The maximum and minimum values are the **extreme values**, or **extrema**, of f on D.

Note: The extreme values of a function are "y" values, values the function attains, not the input values.

Consider Figure 1. The function displayed in (a) has a maximum, but no minimum, as the interval over which the function is defined is open. In (b), the function has a minimum, but no maximum; there is a discontinuity in the "natural" place for the maximum to occur. Finally, the function shown in (c) has both a maximum and a minimum; note that the function is continuous and the interval on which it is defined is closed.

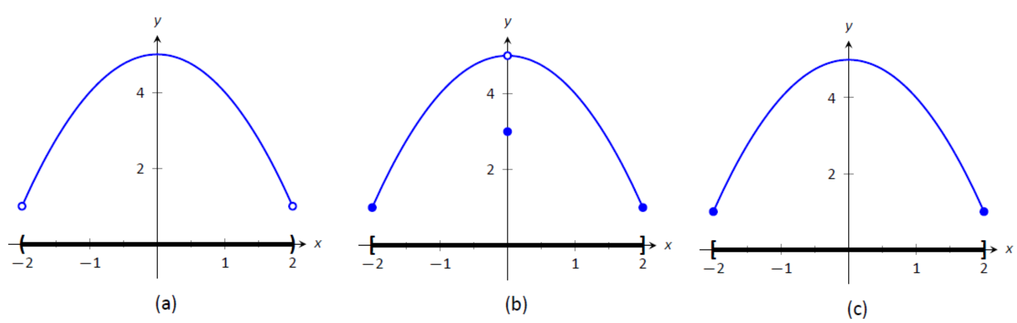


Figure 1: Graphs of functions with and without extreme values.

It turns out that two conditions ensure the existence of absolute maximum and minimum values on an interval: The function must be continuous on the interval, and the interval must be closed and bounded.

Theorem (Extreme Value Theorem)**.** Let f be a continuous function defined on a closed interval $[a, b]$. Then f has both an absolute maximum and minimum value on $[a, b]$.

This theorem states that f has extreme values, but it does not offer any advice about how/where to find these values. The process can seem to be fairly easy, as the next example illustrates. After the example, we will draw on lessons learned to form a more general and powerful method for finding extreme values.

Example. Approximating extreme values

Consider $f(x) = 2x^3 - 9x^2$ on $I = [-1, 5]$, as graphed in Figure 2. Approximate the extreme values of f.

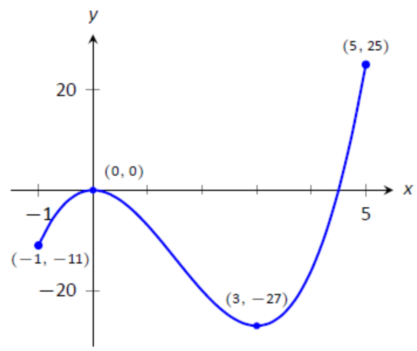


Figure 2: A graph of $f(x) = 2x^3 - 9x^2$

Solution. The graph Figure 2 is drawn in such a way to draw attention to certain points. It certainly seems that the smallest y value is -27, found when $x = 3$. It also seems that the largest y value is 25, found at the endpoint of I, $x = 5$. We use the word *seems*, for by the graph alone we cannot be sure the smallest value is not less than -27. Since the problem asks for an approximation, we approximate the extreme values to be 25 and -27. □

Notice how the minimum value came at "the bottom of a hill," and the maximum value came at an endpoint. Also note that while 0 is not an extreme value, it would be if we narrowed our interval to $[-1, 4]$. The idea that the point $(0, 0)$ is the location of an extreme value for some interval is important, leading us to a definition of a *relative maximum*. In short, a "relative max" is a y-value that's the largest y-value "nearby."

Note: The terms *local minimum* and *local maximum* are often used as synonyms for *relative minimum* and *relative maximum*.

Definition (Local Maximum and Minimum Values)**.** Let f be defined on an interval D containing c.

1. If there is an open interval I containing c such that $f(c)$ is the minimum value on I, then $f(c)$ is a **relative minimum** of f. We also say that f has a relative minimum at $(c, f(c))$.

2. If there is an open interval I containing c such that $f(c)$ is the maximum value on I, then $f(c)$ is a **relative maximum** of f. We also say that f has a relative maximum at $(c, f(c))$.

The relative maximum and minimum values comprise the **relative extrema** of f.

Note. In this book, we adopt the convention that **local maximum values and local minimum values occur only at interior points** of the interval(s) of interest. We need to check the end points to see if local minimum/maximum values are absolute minimum/maximum values.

Example. Approximating relative extrema Consider $f(x) = (3x^4 - 4x^3 - 12x^2 + 5)/5$, as shown in Figure 3. Approximate the relative extrema of f. At each of these points, evaluate f'.

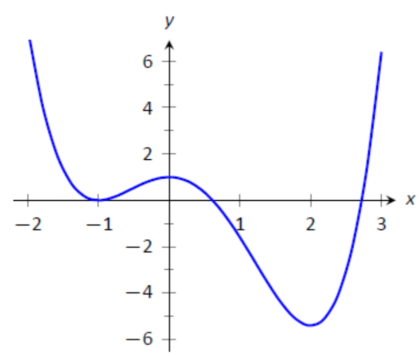


Figure 3: A graph of $f(x) = (3x^4 - 4x^3 - 12x^2 + 5)/5$

Solution. We still do not have the tools to exactly find the relative extrema, but the graph does allow us to make reasonable approximations. It seems f has relative minima at $x = -1$ and $x = 2$, with values of $f(-1) = 0$ and $f(2) = -5.4$. It also seems that f has a relative/local maximum at the point $(0, 1)$.

We approximate the relative minima to be 0 and -5.4; we approximate the relative maximum to be 1.

It is straightforward to evaluate $f'(x) = \frac{1}{5}(12x^3 - 12x^2 - 24x)$ at $x = 0, 1$ and 2. In each case, $f'(x) = 0$. $\square$

Example. Approximating relative extrema Approximate the relative extrema of $f(x) = (x-1)^{2/3} + 2$, shown in Figure 4. At each of these points, evaluate f'.

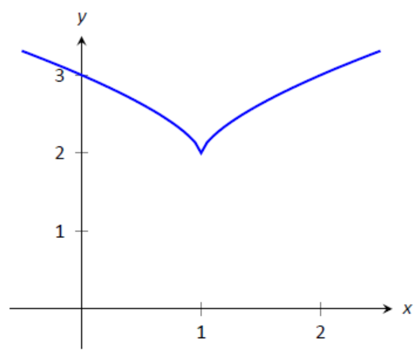


Figure 4: A graph of $f(x) = (x-1)^{2/3} + 2$

Solution. The figure implies that f does not have any relative maxima, but has a relative minimum at $(1, 2)$. In fact, the graph suggests that not only is this point a relative minimum, $y = f(1) = 2$ is *the* minimum value of the function.

We compute $f'(x) = \frac{2}{3}(x-1)^{-1/3}$. When $x = 1$, f' is undefined. $\square$

What can we learn from the previous two examples? We were able to visually approximate relative extrema, and at each such point, the derivative was either 0 or it was not defined. This observation holds for all functions, leading to a definition and a theorem.

Definition (Critical Points)**.** Let f be defined at c. The value c is a **critical number** (or **critical value**) of f if $f'(c) = 0$ or $f'(c)$ is not defined.
If c is a critical number of f, then the point $(c, f(c))$ is a **critical point** of f.

Theorem (Local Extreme Value Theorem)**.** Let a function f be defined on an open interval I containing c, and let f have a local extremum at the point $(c, f(c))$. Then c is a critical number/value of f.

Example Find the critical points of $f(x) = x^2 \ln x$.

Solution.

$\square$

By the Extreme Value Theorem, a continuous function on a closed interval will have both an absolute maximum and an absolute minimum. Common sense tells us "extrema occur either at the endpoints or somewhere in between." It is easy to check for extrema at endpoints, but there are infinitely many points to check that are "in between." Our theory tells us we need only check at the critical points that are in between the endpoints. We combine these concepts to offer a strategy for finding extrema.

Locating Absolute Maxima and Minima

Let f be a continuous function defined on a closed interval $[a, b]$. To find the maximum and minimum values of f on $[a, b]$:

1. Evaluate f at the endpoints a and b of the interval.
2. Find the critical numbers of f in (a, b), the points c such that $f'(c) = 0$ or $f'(c)$ does not exist.
3. Evaluate f at each critical number.

4. The absolute maximum of f is the largest of these values, and the absolute minimum of f is the least of these values.

We practice these ideas in the next example.

Example Find the absolute maximum and minimum values of the following functions.

a. $f(x) = x^4 - 2x^3$ on the interval $[-2, 2]$

b. $g(x) = x^{2/3}(2 - x)$ on the interval $[-1, 2]$

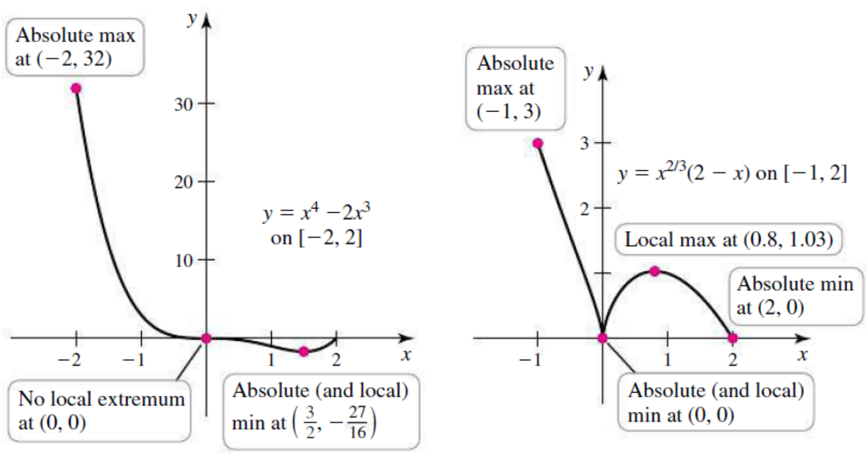


Figure 5: Examples of finding absolute extreme values

Solution.

□

4.2 The Mean Value Theorem

The Mean Value Theorem is a cornerstone in the theoretical framework of calculus. Several critical theorems rely on the Mean Value Theorem; this theorem also appears in practical applications. We begin with a preliminary result known as Rolle's Theorem.

Theorem (Rolle's Theorem)**.** Let f be continuous on $[a, b]$ and differentiable on (a, b), where $f(a) = f(b)$. There is some c in (a, b) such that $f'(c) = 0$.

Consider Figure 6 where the graph of a function f is given, where $f(a) = f(b)$. It should make intuitive sense that if f is differentiable (and hence, continuous) that there would be a value c in (a, b) where $f'(c) = 0$; that is, there would be a relative maximum or minimum of f in (a, b). Rolle's Theorem guarantees at least one; there may be more.

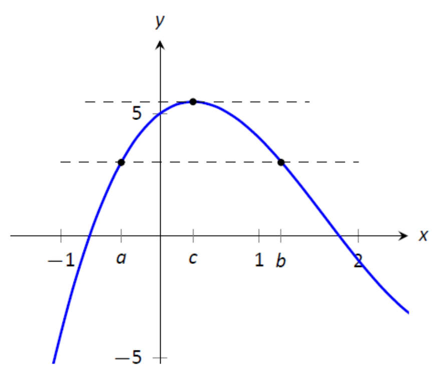


Figure 6: A graph of $f(x) = x^3 - 5x^2 + 3x + 5$, where $f(a) = f(b)$. Note the existence of c, where $a < c < b$, where $f'(c) = 0$.

Proof of Rolle's Theorem

Let f be differentiable on (a, b) where $f(a) = f(b)$. We consider two cases.

Case 1: Consider the case when f is constant on $[a, b]$; that is, $f(x) = f(a) = f(b)$ for all x in $[a, b]$. Then $f'(x) = 0$ for all x in $[a, b]$, showing there is at least one value c in (a, b) where $f'(c) = 0$.

Case 2: Now assume that f is not constant on $[a, b]$. The Extreme Value Theorem guarantees that f has a maximal and minimal value on $[a, b]$, found either at the endpoints or at a critical value in (a, b). Since $f(a) = f(b)$ and f is not constant, it is clear that the maximum and minimum cannot *both* be found at the endpoints. Assume, without loss of generality, that the maximum of f is not found at the endpoints. Therefore there is a c in (a, b) such that $f(c)$ is the maximum value of f. By the Local Extreme Value Theorem, c must be a critical number of f; since f is differentiable, we have that $f'(c) = 0$, completing the proof of the theorem. □

Why does Rolle's Theorem require continuity? A function that is not continuous on $[a, b]$ may have identical values at both endpoints and still not have a horizontal tangent line at any point on the interval (Figure 7 a). Similarly, a function that is continuous on $[a, b]$ but not differentiable at a point of (a, b) may also fail to have a horizontal tangent line (Figure 7 b).

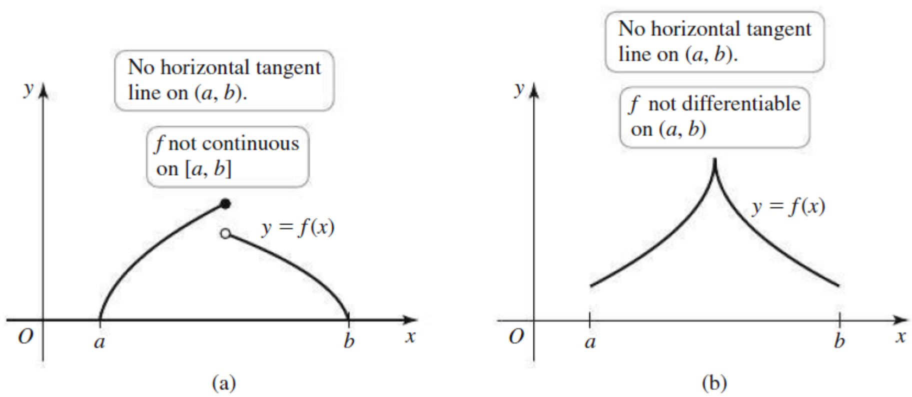


Figure 7: Functions that the Rolle's theorem cannot be applied

Example. Find an interval I on which Rolle's Theorem applies to $f(x) = x^3 - 7x^2 + 10x$. Then find all points c in I at which $f'(c) = 0$.

Solution.

□

Mean Value Theorem

The Mean Value Theorem is easily understood with the aid of a picture. Figure 8 shows a function f differentiable on (a, b) with a secant line passing through $(a, f(a))$ and $(b, f(b))$; the slope of the secant line is the average rate of change of f over $[a, b]$. The Mean Value Theorem claims that there exists a point c in (a, b) at which the slope of the tangent line at c is equal to the slope of the secant line. In other words, we can find a point on the graph of f where the tangent line is parallel to the secant line.

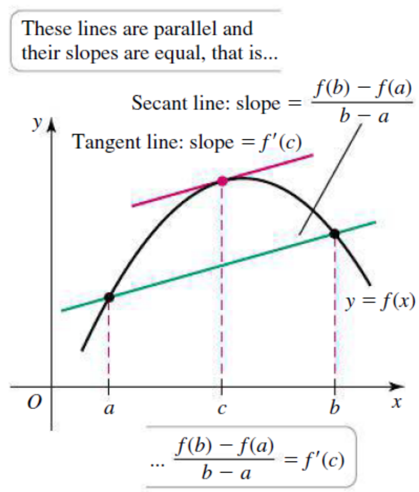


Figure 8: The Mean Value Theorem

Theorem (The Mean Value Theorem of Differentiation)**.** Let $y = f(x)$ be a continuous function on the closed interval $[a, b]$ and differentiable on the open interval (a, b). There exists a value c, $a < c < b$, such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

That is, there is a value c in (a, b) where the instantaneous rate of change of f at c is equal to the average rate of change of f on $[a, b]$.

The proof of the Mean Value Theorem below uses the Rolle's Theorem.

Proof of the Mean Value Theorem

Define the function
$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}x.$$
We know g is differentiable on (a, b) and continuous on $[a, b]$ since f is. We can show $g(a) = g(b)$ (it is actually easier to show $g(b) - g(a) = 0$, which suffices). We can then apply Rolle's theorem to guarantee the existence of c in (a, b) such that $g'(c) = 0$. But note that
$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}\ ;$$
hence
$$f'(c) = \frac{f(b) - f(a)}{b - a},$$
which is what we sought to prove. □

Example Determine whether the function $f(x) = 2x^3 - 3x + 1$ satisfies the conditions of the Mean Value Theorem on the interval $[-2, 2]$. If so, find the point(s) guaranteed to exist by the theorem.

Solution.

□

4.3 Monotonicity and Concavity

4.3.1 Increasing and Decreasing Functions

A function's monotonicity refers to whether the function is increasing or decreasing. Recall that we know the definition of Increasing and Decreasing Functions.

Definition (Increasing and Decreasing Functions)**.** Let f be a function defined on an interval I.

1. f is **increasing** on I if for every $x_1 < x_2$ in I, $f(x_1) < f(x_2)$.
2. f is **decreasing** on I if for every $x_1 < x_2$ in I, $f(x_1) > f(x_2)$.

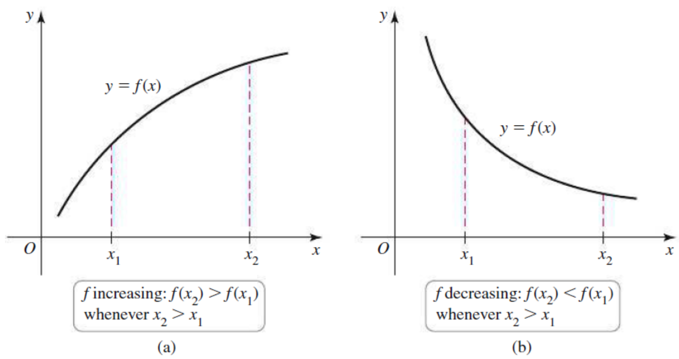


Figure 9: Increasing and Decreasing Functions

Informally, a function is increasing if as x gets larger (i.e., looking left to right) $f(x)$ gets larger; a function is decreasing if as x gets larger $f(x)$ gets smaller.

Intervals of Increase and Decrease

The graph of a function f gives us an idea of the intervals on which f is increasing and decreasing. But how do we determine those intervals precisely? This question is answered by making a connection to the derivative. Recall that the derivative of a function gives the slopes of tangent lines.

To find intervals of increase and decrease, we again consider secant lines. Let f be an increasing, differentiable function on an open interval I, such as the one shown in Figure 10, and let $a < b$ be given in I. The secant line on the graph of f from $x = a$

to $x = b$ is drawn; it has a slope of $(f(b) - f(a))/(b - a)$. But note:

$$\frac{f(b)-f(a)}{b-a} \Rightarrow \frac{\text{numerator } > 0}{\text{denominator } > 0} \Rightarrow \begin{array}{c}\text{slope of the}\\ \text{secant line}\\ > 0\end{array} \Rightarrow \begin{array}{c}\text{Average rate}\\ \text{of change of}\\ f \text{ on } [a,b] \text{ is}\\ > 0.\end{array}$$

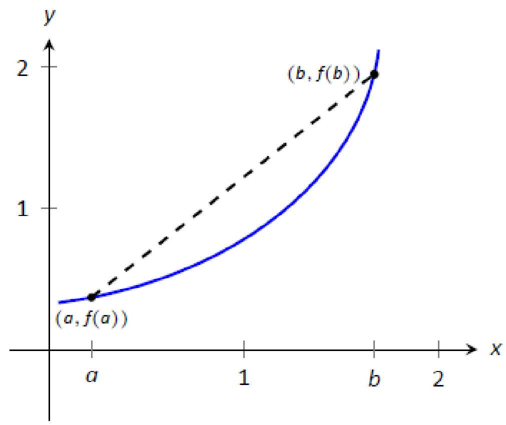


Figure 10: Examining the secant line of an increasing function

We have shown mathematically what may have already been obvious: when f is increasing, its secant lines will have a positive slope. Now recall the Mean Value Theorem guarantees that there is a number c, where $a < c < b$, such that

$$f^{'}(c) = \frac{f(b)-f(a)}{b-a} > 0.$$

By considering all such secant lines in I, we strongly imply that $f^{'}(x) > 0$ on I. A similar statement can be made for decreasing functions.

Our above logic can be summarized as "If f is increasing, then $f^{'}$ is probably positive." The Theorem below turns this around by stating "If $f^{'}$ is positive, then f is increasing." This leads us to a method for finding when functions are increasing and decreasing.

Theorem (Test For Increasing/Decreasing Functions)**.** Let f be a continuous function on $[a, b]$ and differentiable on (a, b).

1. If $f'(c) > 0$ for all c in (a, b), then f is increasing on $[a, b]$.

2. If $f'(c) < 0$ for all c in (a, b), then f is decreasing on $[a, b]$.

3. If $f'(c) = 0$ for all c in (a, b), then f is constant on $[a, b]$.

Note: Parts 1 & 2 of the Theorem also hold if $f'(c) = 0$ for a finite number of values of c in $[a, b]$.

Let f be differentiable on an interval I and let a and b be in I where $f'(a) > 0$ and $f'(b) < 0$. If f' is continuous on $[a, b]$, it follows from the Intermediate Value Theorem that there must be some value c between a and b where $f'(c) = 0$. (It turns out that this is still true even if f' is not continuous on $[a, b]$.) This leads us to the following method for finding intervals on which a function is increasing or decreasing.

Finding Intervals on Which f is Increasing or Decreasing

Let f be a differentiable function on an interval I. To find intervals on which f is increasing and decreasing:

1. Find the critical values of f. That is, find all c in I where $f'(c) = 0$ or f' is not defined.

2. Use the critical values to divide I into subintervals.

3. Pick any point p in each subinterval, and find the sign of $f'(p)$.

   (a) If $f'(p) > 0$, then f is increasing on that subinterval.

   (b) If $f'(p) < 0$, then f is decreasing on that subinterval.

We demonstrate using this process in the following example.

Example. Find the intervals on which the following functions are increasing and decreasing.

a. $f(x) = xe^{-x}$ **b.** $f(x) = 2x^3 + 3x^2 + 1$

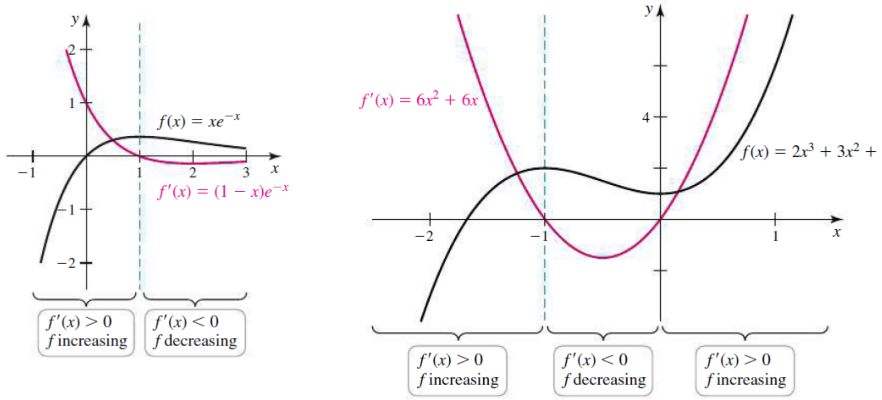


Figure 11: Graphs of $f(x) = xe^{-x}$ and $f(x) = 2x^3 + 3x^2 + 1$

Solution.

$\square$

One is justified in wondering why so much work is done when the graph seems to make the intervals very clear. We give three reasons why the above work is worthwhile.

First, the points at which f switches from increasing to decreasing are not precisely known given a graph. The graph shows us something significant happens near $x = -1$ and $x = 0.3$, but we cannot determine exactly where from the graph.

One could argue that just finding critical values is important; once we know the significant points are $x = -1$ and $x = 1/3$, the graph shows the increasing/decreasing traits just fine. That is true. However, the technique prescribed here helps reinforce the relationship between increasing/decreasing and the sign of f'. Once mastery of this concept (and several others) is obtained, one finds that either (a) just the critical points are computed and the graph shows all else that is desired, or (b) a graph is never produced, because determining increasing/decreasing using f' is straightforward and the graph is unnecessary. So our second reason why the above work is worthwhile is this: once mastery of a subject is gained, one has *options* for finding needed information. We are working to develop mastery.

Finally, our third reason: many problems we face "in the real world" are very complex. Solutions are tractable only through the use of computers to do many calculations for us. Computers do not solve problems "on their own," however; they need to be taught (i.e., *programmed*) to do the right things. It would be beneficial to give a function to a computer and have it return maximum and minimum values, intervals on which the function is increasing and decreasing, the locations of relative maxima, etc. The work that we are doing here is easily programmable. It is hard to teach a computer to "look at the graph and see if it is going up or down." It is easy to teach a computer to "determine if a number is greater than or less than 0."

In Section 4.1 we learned the definition of relative maxima and minima and found that they occur at critical points. We are now learning that functions can switch from increasing to decreasing (and vice–versa) at critical points. This new understanding of increasing and decreasing creates a great method of determining whether a critical point corresponds to a maximum, minimum, or neither. Imagine a function increasing until a critical point at $x = c$, after which it decreases. A quick sketch helps confirm that $f(c)$ must be a relative maximum. A similar statement can be made for relative minimums. We formalize this concept in a theorem.

Theorem (First Derivative Test)**.** Let f be differentiable on an interval I and let c be a critical number in I.

1. If the sign of f' switches from positive to negative at c, then $f(c)$ is a local maximum of f.

2. If the sign of f' switches from negative to positive at c, then $f(c)$ is a local minimum of f.

3. If f' is positive (or, negative) before and after c, then $f(c)$ is not a local extrema of f.

Example Consider the function $f(x) = 3x^4 - 4x^3 - 6x^2 + 12x + 1$

a. Find the intervals on which f is increasing and decreasing.

b. Identify the local extrema of f.

Solution.

$\square$

Example Find the local extrema of the function $g(x) = x^{2/3}(2 - x)$.

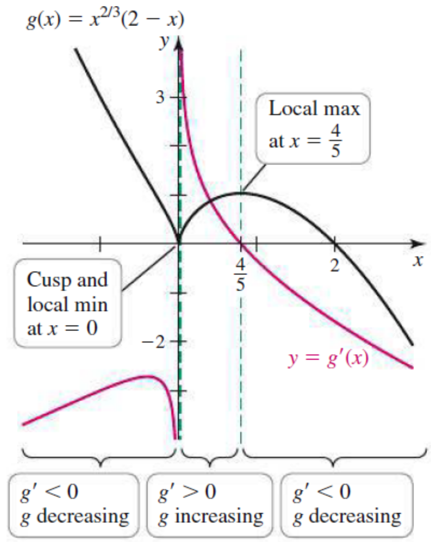


Figure 12: Graphing $g(x) = x^{2/3}(2 - x)$ and its derivative

Solution.

□

Absolute extreme Values on Any Interval

The Extreme Value Theorem (4.1) guarantees the existence of absolute extreme values only on closed intervals. What can be said about absolute extrema on intervals that are not closed? The following theorem provides a valuable test.

Theorem (One Local Extremum Implies Absolute Extremum)**.** Suppose f is continuous on an interval I that contains exactly one local extremum at c.

- If a local maximum occurs at c, then $f(c)$ is the absolute maximum of f on I.
- If a local minimum occurs at c, then $f(c)$ is the absolute minimum of f on I.

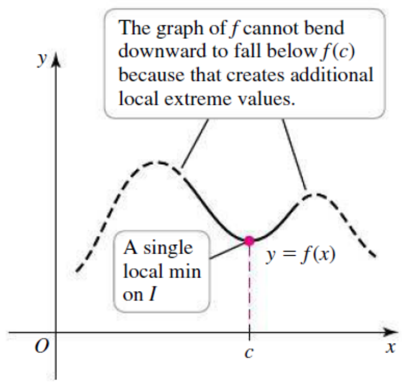


Figure 13: One Local Extremum Implies Absolute Extremum

Example Verify that $f(x) = x^x$ has an absolute extreme value on its domain.

Solution.

$\square$

4.3.2 Concavity and the Second Derivative

Our study of "nice" functions continues. The previous sub-section showed how the first derivative of a function, f', can relay important information about f. We now apply the same technique to f' itself, and learn what this tells us about f.

The key to studying f' is to consider its derivative, namely f'', which is the second derivative of f. When $f'' > 0$, f' is increasing. When $f'' < 0$, f' is decreasing. f' has relative maxima and minima where $f'' = 0$ or f'' is undefined.

This section explores how knowing information about f'' gives information about f.

Concavity

We begin with a definition, then explore its meaning.

Definition (Concave Up and Concave Down)**.** Let f be differentiable on an interval I. The graph of f is **concave up** on I if f' is increasing. The graph of f is **concave down** on I if f' is decreasing. If f' is constant then the graph of f is said to have **no concavity**.

If f is continuous at c and f changes concavity at c (from up to down, or vice versa), then f has an **inflection point** at c .

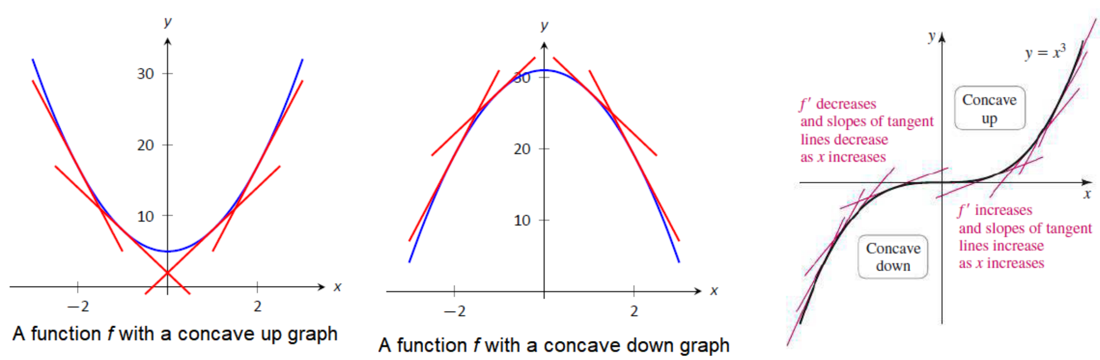


Figure 14: Concave Up and Concave Down

Note: We often state that "f is concave up" instead of "the graph of f is concave up" for simplicity.

The graph of a function f is *concave up* when f' is *increasing.* That means as one looks at a concave up graph from left to right, the slopes of the tangent lines will be increasing. If a function is decreasing and concave up, then its rate of decrease is slowing; it is "leveling off." If the function is increasing and concave up, then the *rate* of increase is increasing. The function is increasing at a faster and faster rate.

Now consider a function which is concave down. We essentially repeat the above paragraphs with slight variation.

The graph of a function f is *concave down* when f' is *decreasing.* That means as one looks at a concave down graph from left to right, the slopes of the tangent lines will be decreasing. If a function is increasing and concave down, then its rate of increase is slowing; it is "leveling off." If the function is decreasing and concave down, then the *rate* of decrease is decreasing. The function is decreasing at a faster and faster rate.

Note: A mnemonic for remembering what concave up/down means is: "Concave up is like a cup; concave down is like a frown." It is admittedly terrible, but it works.

Note: Geometrically speaking, a function is concave up if its graph lies above its tangent lines. A function is concave down if its graph lies below its tangent lines.

Our definition of concave up and concave down is given in terms of when the first derivative is increasing or decreasing. We can apply the results of the previous section and to find intervals on which a graph is concave up or down. That is, we recognize that f' is increasing when $f'' > 0$, etc.

Theorem (Test for Concavity)**.** Let f be twice differentiable on an interval I.

- If $f'' > 0$ on I, f is concave up.
- if $f'' < 0$ on I, f is concave down.

If knowing where a graph is concave up/down is important, it makes sense that the **Points of Inflection**, places where the graph changes from one to the other, is also important.

If the concavity of f changes at a point $(c, f(c))$, then f' is changing from increasing to decreasing (or, decreasing to increasing) at $x = c$. That means that the sign of f'' is changing from positive to negative (or, negative to positive) at $x = c$. This leads to the following theorem.

Theorem (Points of Inflection)**.** If $(c, f(c))$ is a point of inflection on the graph of f, then either $f''(c) = 0$ or f'' is not defined at c.

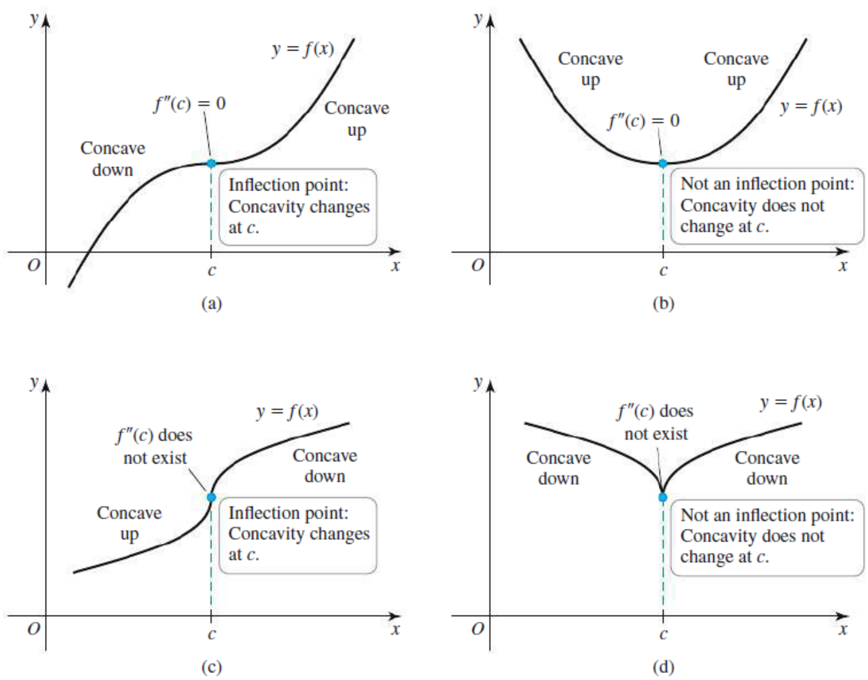


Figure 15: Concavity and inflection points

We have identified the concepts of concavity and points of inflection. It is now time to practice using these concepts; given a function, we should be able to find its points of inflection and identify intervals on which it is concave up or down. We do so in the following example.

Example Identify the intervals on which the following functions are concave up or concave down. Then locate the inflection points.

a. $f(x) = 3x^4 - 4x^3 - 6x^2 + 12x + 1$

b. $f(x) = \sin^{-1} x$ on $(-1, 1)$

Solution.

$\square$

The Second Derivative Test

The first derivative of a function gave us a test to find if a critical value corresponded to a relative maximum, minimum, or neither. The second derivative gives us another way to test if a critical point is a local maximum or minimum. The following theorem officially states something that is intuitive: if a critical value occurs in a region where a function f is concave up, then that critical value must correspond to a relative minimum of f, etc. See Figure 16 for a visualization of this.

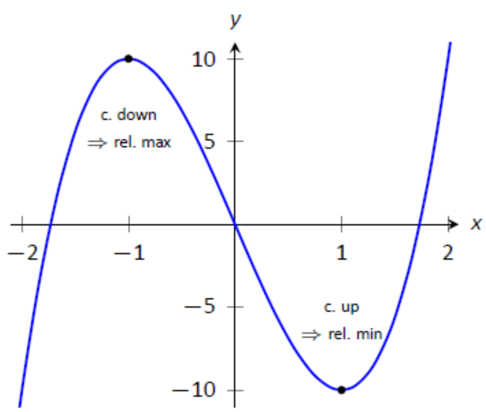


Figure 16: Demonstrating the fact that relative maxima occur when the graph is concave down and relative minima occur when the graph is concave up.

Theorem (The Second Derivative Test)**.** Let c be a *critical value* of f where $f''(c)$ is defined.

1. If $f''(c) > 0$, then f has a local minimum at $(c, f(c))$.
2. If $f''(c) < 0$, then f has a local maximum at $(c, f(c))$.
3. If $f''(c) = 0$, then the test is inconclusive; f may have a local maximum, local minimum, or neither at c.

Proof. Assume $f''(c) > 0$. Because f'' is continuous on an interval containing c, it follows that $f'' > 0$ on some open interval I containing c and that f' is increasing on I. Because $f'(c) = 0$, it follows that f' changes sign at c from negative to positive, which, by the First Derivative Test, implies that f has a local minimum at c. The proofs of the second and third statements are similar. $\square$

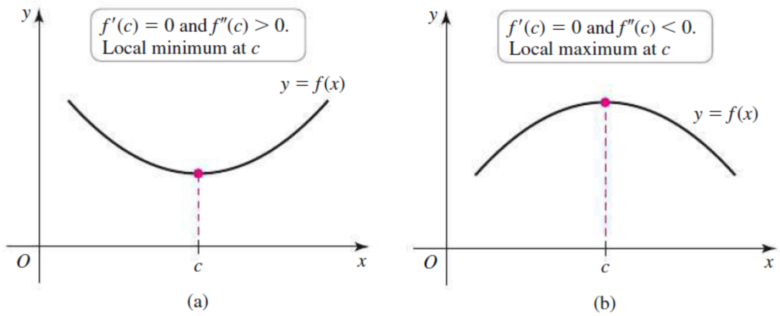


Figure 17: The Second Derivative Test for Local Extrema.

Example Use the Second Derivative Test to locate the local extrema of the following functions.

a. $f(x) = 3x^4 - 4x^3 - 6x^2 + 12x + 1$ on $[-2, 2]$ **b.** $f(x) = \sin^2 x$

Solution.

$\square$

Recap of Derivative Properties

This section has demonstrated that the first and second derivatives of a function provide valuable information about its graph. The relationships among a function's derivatives and its extreme values and concavity are summarized in Figure 18.

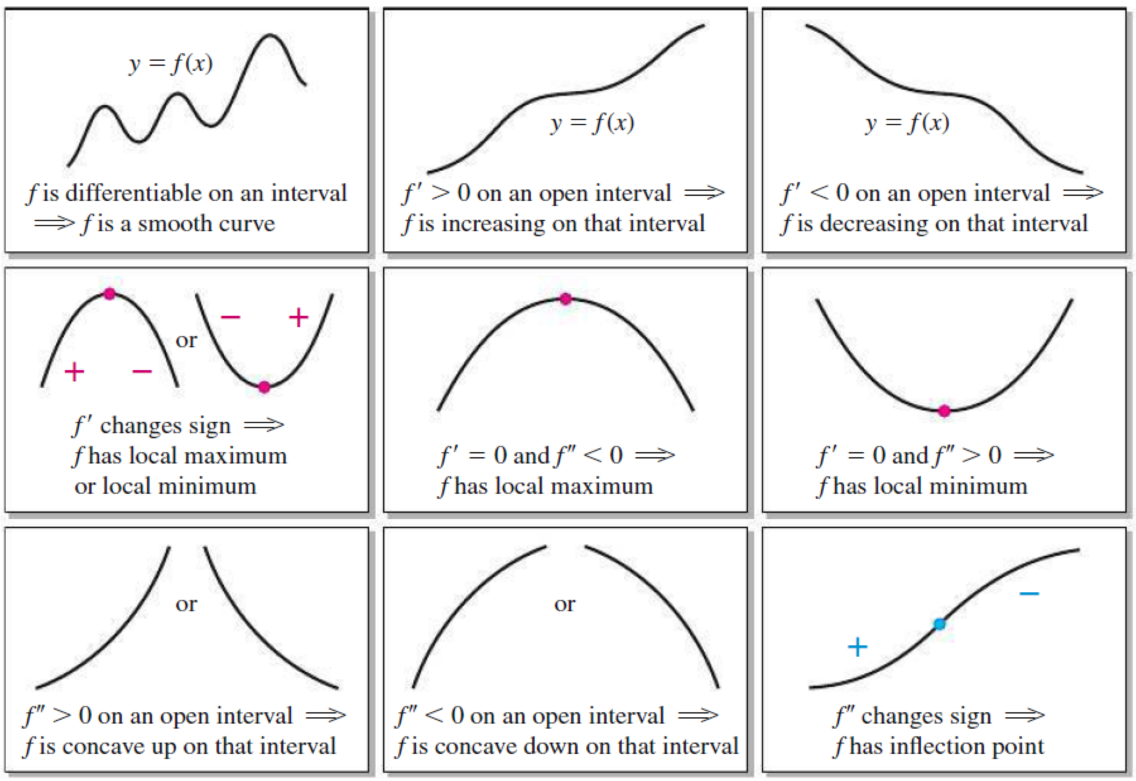


Figure 18: Functions and their derivatives

4.4 Graphing Functions

We have been learning how we can understand the behavior of a function based on its first and second derivatives. While we have been treating the properties of a function separately (increasing and decreasing, concave up and concave down, etc.), we combine them here to produce an accurate graph of the function without plotting lots of extraneous points.

Why bother? Graphing utilities are very accessible, whether on a computer, a hand–held calculator, or a smartphone. These resources are usually very fast and accurate. We will see that our method is not particularly fast – it will require time (but it is not *hard*). So again: why bother?

We are attempting to understand the behavior of a function f based on the information given by its derivatives. While all of a function's derivatives relay information about it, it turns out that "most" of the behavior we care about is explained by f' and f''. Understanding the interactions between the graph of f and f'' and f'' is important. To gain this understanding, one might argue that all that is needed is to look at lots of graphs. This is true to a point, but is somewhat similar to stating that one understands how an engine works after looking only at pictures. It is true that the basic ideas will be conveyed, but "hands–on" access increases understanding.

The following Key Idea summarizes what we have learned so far that is applicable to sketching graphs of functions and gives a framework for putting that information together. It is followed by several examples.

Graphing Guidelines for $y = f(x)$:

To produce an accurate sketch a given function f, consider the following steps.

1. **Identify the domain of f or interval of interest.** Generally, we assume that the domain is the entire real line then find restrictions, such as where a denominator is 0 or where negatives appear under the radical.

2. **Exploit symmetry.** Take advantage of symmetry. For example, is the function even $f(-x) = f(x)$, odd $f(-x) =$, or neither?

3. **Find the first and second derivatives.** They are needed to determine extreme values, concavity, inflection points, and intervals of increase and decrease. Computing derivatives—particularly second derivatives—may not be practical, so some functions may need to be graphed without complete derivative information.

4. **Find the critical points and possible inflection points of f.** Determine points at which $f^{'}(x) = 0$ or $f^{'}$ is undefined. Determine points at which $f^{''}(x) = 0$ or $f^{''}$ is undefined.

5. **Find intervals on which the function is increasing/decreasing and concave up/down.** The first derivative determines the intervals of increase and decrease. The second derivative determines the intervals on which the function is concave up or concave down.

6. **Identify extreme values and inflection points.** Use either the First or Second Derivative Test to classify the critical points. Both x- and y-coordinates of maxima, minima, and inflection points are needed for graphing.

7. **Locate all asymptotes and determine end behavior.** Vertical asymptotes often occur at zeros of denominators. Horizontal asymptotes require examining limits $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$; these limits determine end behavior.

8. **Find the intercepts.** The y-intercept of the graph is found by setting $x = 0$. The x-intercepts are found by setting $y = 0$; they are the real zeros (or roots) of f (those values of x that satisfy $f(x) = 0$).

9. Create a number line that includes all critical points, possible points of inflection, and locations of vertical asymptotes. For each interval created, determine whether f is increasing or decreasing, concave up or down.

10. **Choose an appropriate graphing window and plot a graph.** Use the results of the previous steps to graph the function. If you use graphing software,

check for consistency with your analytical work. Is your graph *complete* — that is, does it show all the essential details of the function?

Example Use the graphing guidelines to graph $f(x) = \dfrac{x^3}{3} - 400x$ on its domain.

Solution.

$\square$

Example Analyze the function $f(x) = e^{-x^2}$ and draw its graph.

Solution.

$\square$

4.5 Optimization Problems

In Section 4.1 we learned about extreme values – the largest and smallest values a function attains on an interval. We motivated our interest in such values by discussing how it made sense to want to know the highest/lowest values of a stock, or the fastest/slowest an object was moving. In this section we apply the concepts of extreme values to solve "word problems," i.e., problems stated in terms of situations that require us to create the appropriate mathematical framework in which to solve the problem.

We start with a classic example which is followed by a discussion of the topic of optimization.

Example. Optimization: perimeter and area: A man has 100 feet of fencing, a large yard, and a small dog. He wants to create a rectangular enclosure for his dog with the fencing that provides the maximal area. What dimensions provide the maximal area?

One can likely guess the correct answer – that is great. We will proceed to show how calculus can provide this answer in a context that proves this answer is correct.

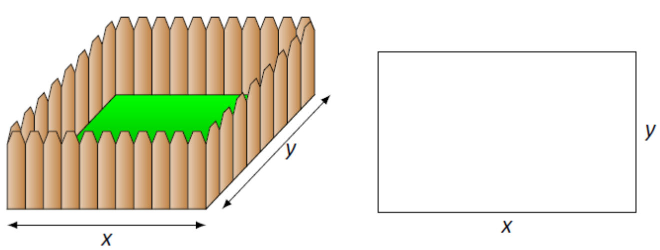


Figure 19: A sketch of the enclosure in the example

It helps to make a sketch of the situation. Our enclosure is sketched twice in Figure 19, either with green grass and nice fence boards or as a simple rectangle. Either way, drawing a rectangle forces us to realize that we need to know the dimensions of this rectangle so we can create an area function – after all, we are trying to maximize the area.

We let x and y denote the lengths of the sides of the rectangle. Clearly,

$$\text{Area} = xy.$$

We do not yet know how to handle functions with 2 variables; we need to reduce this down to a single variable. We know more about the situation: the man has 100 feet of fencing. By knowing the perimeter of the rectangle must be 100, we can create another equation:

$$\text{Perimeter} = 100 = 2x + 2y.$$

We now have 2 equations and 2 unknowns. In the latter equation, we solve for y:

$$y = 50 - x.$$

Now substitute this expression for y in the area equation:

$$\text{Area} = A(x) = x(50 - x).$$

Note we now have an equation of one variable; we can truly call the Area a function of x.

This function only makes sense when $0 \leq x \leq 50$, otherwise we get negative values of area. So we find the extreme values of $A(x)$ on the interval $[0, 50]$.

To find the critical points, we take the derivative of $A(x)$ and set it equal to 0, then solve for x.

$$\begin{aligned} A(x) &= x(50 - x) \\ &= 50x - x^2; \\ A'(x) &= 50 - 2x. \end{aligned}$$

We solve $50 - 2x = 0$ to find $x = 25$; this is the only critical point. We evaluate $A(x)$ at the endpoints of our interval and at this critical point to find the extreme values; in this case, all we care about is the maximum.

Clearly $A(0) = 0$ and $A(50) = 0$, whereas $A(25) = 625\text{ft}^2$. This is the maximum. Since we earlier found $y = 50 - x$, we find that y is also 25. Thus the dimensions of the rectangular enclosure with perimeter of 100 ft. with maximum area is a square, with sides of length 25 ft.

This example is very simplistic and a bit contrived. (After all, most people create a design then buy fencing to meet their needs, and not buy fencing and plan later.) But it models well the necessary process: create equations that describe a situation, reduce an equation to a single variable, then find the needed extreme value.

"In real life," problems are much more complex. The equations are often *not* reducible to a single variable (hence multi–variable calculus is needed) and the equations themselves may be difficult to form. Understanding the principles here will provide a good foundation for the mathematics you will likely encounter later.

We outline here the basic process of solving these optimization problems.

Guidelines for Optimization Problems

1. Understand the problem. Clearly identify what quantity is to be maximized or minimized. Make a sketch if helpful.

2. Identify the **objective function** (the function to be optimized). Write it in terms of the variables of the problem.

3. Use the constraint(s) to eliminate all but one independent variable of the objective function.

4. Identify the **domain** of this function, keeping in mind the context of the problem.

5. Find the extreme values of this function on the determined domain. If necessary, check the endpoints.

6. Identify the values of all relevant quantities of the problem. Find the absolute maximum or minimum value of the objective function in the determined domain.

Example Airline regulations Suppose an airline policy states that all baggage must be box-shaped with a sum of length, width, and height not exceeding 64 in. What are the dimensions and volume of a square-based box with the greatest volume under these conditions?

Solution.

$\square$

Example Ladder over the fence An 8-foot-tall fence runs parallel to the side of a house 3 feet away. What is the length of the shortest ladder that clears the fence and reaches the house? Assume that the vertical wall of the house and the horizontal ground have infinite extent.

Solution.

□

4.6 Differentials and Linear Approximation

Recall that the derivative of a function f can be used to find the slopes of lines tangent to the graph of f. At $x = a$, the tangent line to the graph of f has equation

$$y = f^{'}(a)(x - a) + f(a).$$

The tangent line can be used to find good approximations of $f(x)$ for values of x near c.

For instance, we can approximate $\sin 1.1$ using the tangent line to the graph of $f(x) = \sin x$ at $x = \pi/3 \approx 1.05$. Recall that $\sin(\pi/3) = \sqrt{3}/2 \approx 0.866$, and $\cos(\pi/3) = 1/2$. Thus the tangent line to $f(x) = \sin x$ at $x = \pi/3$ is:

$$\ell(x) = \frac{1}{2}(x - \pi/3) + 0.866.$$

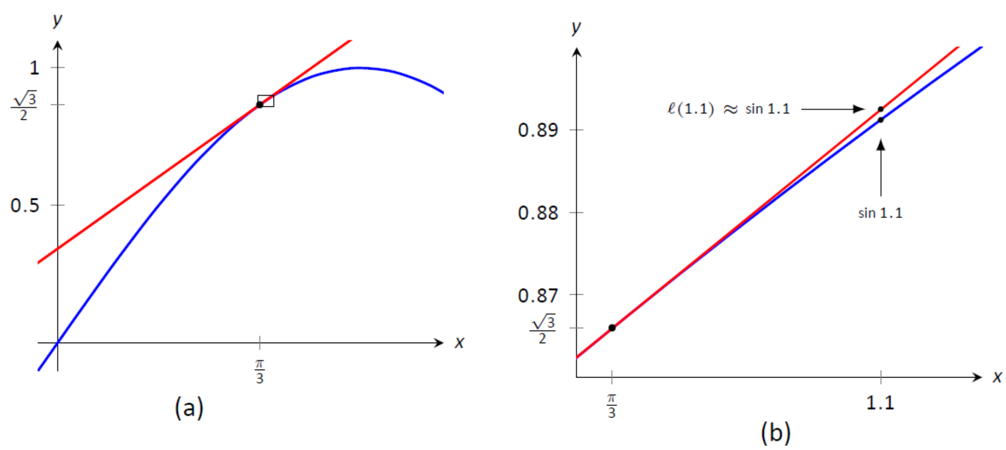


Figure 20: Graphing $f(x) = \sin x$ and its tangent line at $x = \pi/3$ in order to estimate $\sin 1.1$

In Figure 20(a), we see a graph of $f(x) = \sin x$ graphed along with its tangent line at $x = \pi/3$. The small rectangle shows the region that is displayed in Figure 20(b). In this figure, we see how we are approximating $\sin 1.1$ with the tangent line, evaluated at 1.1. Together, the two figures show how close these values are.

Using this line to approximate $\sin 1.1$, we have:

$$\begin{aligned}\ell(1.1) &= \frac{1}{2}(1.1 - \pi/3) + 0.866\\ &= \frac{1}{2}(0.053) + 0.866 = 0.8925.\end{aligned}$$

(How good of an approximation this is?)

The tangent line $y = f^{'}(a)(x-a) + f(a)$ represents a new function ℓ that we call the **linear approximation** to f at the point a. If f and $f^{'}$ are easy to evaluate at a, then the value of f at **points near** a is easily approximated using the linear approximation ℓ. That is,

$$f(x) \approx \ell(x) = f(a) + f^{'}(a)(x-a).$$

This approximation improves as x approaches a.

Definition (Linear Approximation to f at a). Suppose f is differentiable on an interval I containing the point a. The linear approximation to f at a is the linear function

$$\ell(x) = f(a) + f^{'}(a)(x-a), \quad \text{for } x \text{ in } I.$$

Example Linear approximations and errors

a. Find the linear approximation to $f(x) = \sqrt{x}$ at $x = 1$ and use it to approximate $\sqrt{1.1}$.

b. Use linear approximation to estimate the value of $\sqrt{0.1}$.

Solution.

$\square$

Linear Approximation and Concavity

Additional insight into linear approximation is gained by bringing concavity into the picture. Figure 21 (a) shows the graph of a function f and its linear approximation (tangent line) at the point $(a, f(a))$. In this particular case, f is concave up on an interval containing a, and the graph of L lies below the graph of f near a. As a result, the linear approximation evaluated at a point near a is less than the exact value of f at that point. In other words, the linear approximation underestimates values of f near a.

The contrasting case is shown in Figure 21 (b), where we see graphs of f and L when f is concave down on an interval containing a. Now the graph of L lies above the graph of f, which means the linear approximation overestimates values of f near a.

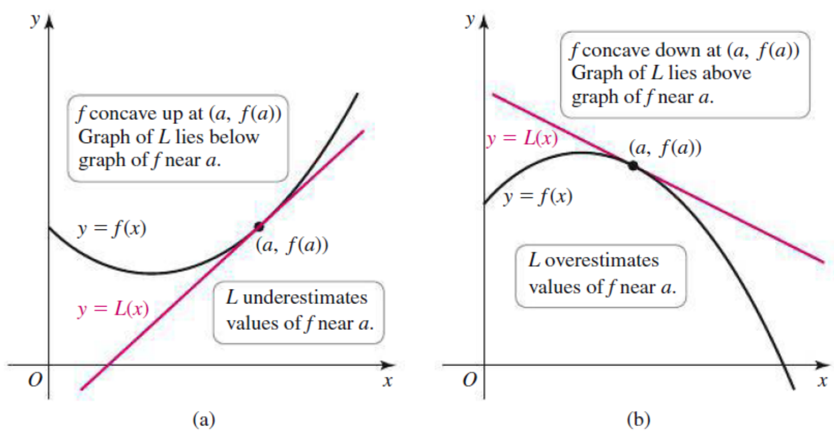


Figure 21: Concavity and Linear Approximations

We can make another observation related to the degree of concavity (also called curvature). A large value of $|f''(a)|$ (large curvature) means that near $(a, f(a))$, the slope of the curve changes rapidly and the graph of f separates quickly from the tangent line. A small value of $|f''(a)|$ (small curvature) means the slope of the curve changes slowly and the curve is relatively flat near $(a, f(a))$; therefore, the curve remains close to the tangent line. As a result, absolute errors in linear approximation are larger when $|f''(a)|$ is large.

Example Linear approximation and concavity

a. Find the linear approximation to $f(x) = x^{1/3}$ at $x = 1$ and $x = 27$.

b. Use the linear approximations of part (a) to approximate $\sqrt[3]{2}$ and $\sqrt[3]{26}$.

c. Are the approximations in part (b) overestimates or underestimates?

d. Compute the error in the approximations of part (b). Which error is greater? Explain.

Solution.

$\square$

In Section 3.1, we explored the meaning and use of the derivative. We now revisit some of those ideas.

Linear approximation says that a function f can be approximated as

$$f(x) \approx f(a) + f^{'}(a)(x-a),$$

where a is fixed and x is a **nearby point**. We first rewrite this expression as

$$\underbrace{f(x)-f(a)}_{\Delta y} \approx f^{'}(a)\underbrace{(x-a)}_{\Delta x}.$$

The factor $\Delta x = x - a$ $(x = \Delta x + a)$ is the change in the x-coordinate between a and a nearby point x. Similarly, $f(x)-f(a) = f(\Delta x + a) - f(a)$ is the corresponding change in the y-coordinate. So we write this approximation as

$$\boxed{\Delta y \approx f^{'}(a)\Delta x.} \tag{4.1}$$

We now introduce two new variables, dx and dy in the context of a formal definition.

Definition (Differentials of x and y)**.** Let $y = f(x)$ be differentiable. The **differential of** x, denoted dx, is any nonzero real number (usually taken to be a small number). The **differential of** y, denoted dy, is

$$\boxed{dy = f^{'}(x)dx.}$$

We can solve for $f^{'}(x)$ in the above equation: $f^{'}(x) = dy/dx$. This states that the derivative of f with respect to x is the differential of y divided by the differential of x; this is **not** the alternate notation for the derivative, $\frac{dy}{dx}$. This latter notation was chosen because of the fraction–like qualities of the derivative, but again, it is one symbol and **not a fraction**.

It is helpful to organize our new concepts and notations in one place.

Differential Notation

Let $y = f(x)$ be a differentiable function.

1. Let Δx represent a small, nonzero change in x value: from a to $a + \Delta x$.

2. Let dx represent a small, nonzero change in x value (i.e., $\Delta x = dx$).

3. Let Δy be the change in y value as x changes by Δx; hence

$$\Delta y = f(x + \Delta x) - f(x).$$

4. Let $dy = f^{'}(x)dx$ which, by Equation (4.1), is an *approximation* of the change in y value as x changes by Δx: $\boxed{\Delta y \approx f^{'}(a)\Delta x}$ and thus $dy \approx \Delta y$.

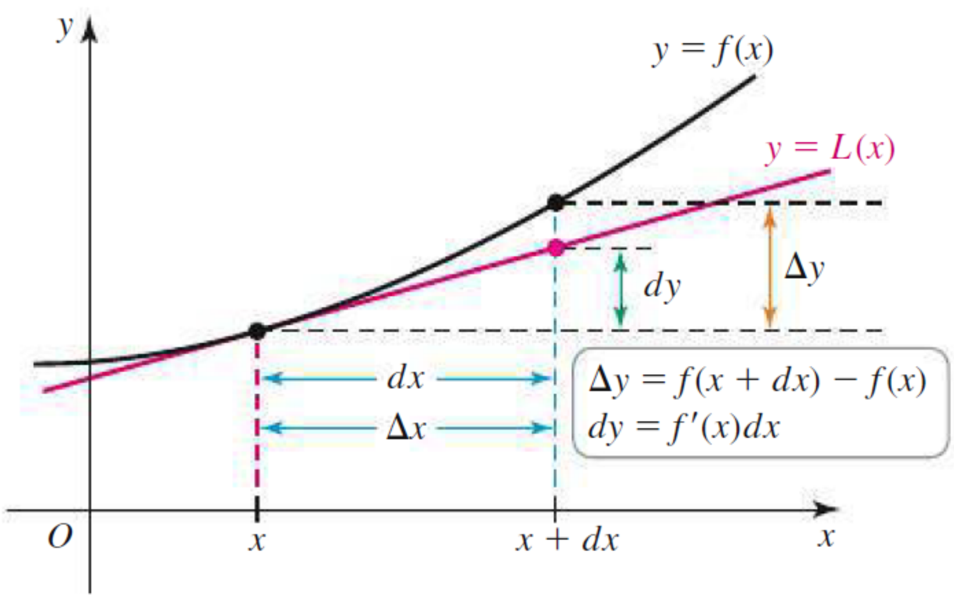


Figure 22: Differentials of x and y.

What is the value of differentials? Like many mathematical concepts, differentials provide both practical and theoretical benefits.

Example. Estimating changes with linear approximations.

a. Approximate the change in $y = f(x) = x^9 - 2x + 1$ when x changes from 1.00 to 1.05.

b. Approximate the change in the surface area of a spherical hot-air balloon when the radius decreases from 4m to 3.9m.

Solution.

$\square$

Example. In each of the following, find the differential dy.

1. $y = \sin x$ 2. $y = e^x(x^2+2)$ 3. $y = \sqrt{x^2+3x-1}$

Solution.

1. $y = \sin x$: As $f(x) = \sin x$, $f'(x) = \cos x$. Thus
$$dy = \cos(x)dx.$$

2. $y = e^x(x^2+2)$: Let $f(x) = e^x(x^2+2)$. We need $f'(x)$, requiring the Product Rule.

   We have $f'(x) = e^x(x^2+2) + 2xe^x$, so
$$dy = \big(e^x(x^2+2) + 2xe^x\big)dx.$$

3. $y = \sqrt{x^2+3x-1}$: Let $f(x) = \sqrt{x^2+3x-1}$; we need $f'(x)$, requiring the Chain Rule.

   We have $f'(x) = \dfrac{1}{2}(x^2+3x-1)^{-\frac{1}{2}}(2x+3) = \dfrac{2x+3}{2\sqrt{x^2+3x-1}}$. Thus
$$dy = \frac{(2x+3)dx}{2\sqrt{x^2+3x-1}}.$$

$\square$

Example. Use the notation of differentials to write the approximate change in $f(x) = 3\cos^2 x$ given a small change dx.

Solution.

$\square$

4.7 L'Hôpital's Rule

The study of **limits in Chapter 2** was thorough but not exhaustive. Some limits, called **indeterminate forms**, cannot generally be evaluated using the techniques presented in Chapter 2. These limits tend to be the more interesting limits that arise in practice.

Our treatment of limits exposed us to the notion of "0/0", an indeterminate form. If $\lim\limits_{x\to c} f(x) = 0$ and $\lim\limits_{x\to c} g(x) = 0$, we do not conclude that $\lim\limits_{x\to c} f(x)/g(x)$ is $0/0$; rather, we use $0/0$ **as notation to describe the fact that both the numerator and denominator approach 0**. The expression $0/0$ has no numeric value; other work must be done to evaluate the limit.

Other indeterminate forms exist; they are: ∞/∞, $0\cdot\infty$, $\infty-\infty$, 0^0, 1^∞ and ∞^0. Just as "0/0" does not mean "divide 0 by 0," the expression "∞/∞" does not mean "divide infinity by infinity." Instead, it means "a quantity is growing without bound and is being divided by another quantity that is growing without bound." We cannot determine from such a statement what value, if any, results in the limit. Likewise, "$0\cdot\infty$" does not mean "multiply zero by infinity." Instead, it means "one quantity is shrinking to zero, and is being multiplied by a quantity that is growing without bound." We cannot determine from such a description what the result of such a limit will be.

This section introduces L'Hôpital's Rule, a powerful method of resolving limits that produce the indeterminate forms $0/0$ and ∞/∞. We'll also show how algebraic manipulation can be used to convert other indeterminate expressions into one of these two forms so that our new rule can be applied.

Theorem (L'Hôpital's Rule: 0/0)**.** Let $\lim\limits_{x\to c} f(x) = 0$ and $\lim\limits_{x\to c} g(x) = 0$, where f and g are differentiable functions on an open interval I containing c, and $g'(x) \neq 0$ on I except possibly at c. Then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

Please read the proof of a special case ($g'(c) \neq 0$) in the textbook.

We demonstrate the use of L'Hôpital's Rule in the following examples; we will often use "LHR" as an abbreviation of "L'Hôpital's Rule."

ExampleUsing L'Hôpital's Rule Evaluate the following limits, using L'Hôpital's Rule as needed.

1. $\lim_{x\to 0} \dfrac{\sin x}{x}$
2. $\lim_{x\to 1} \dfrac{x^3+x^2-2x}{x-1}$
3. $\lim_{x\to 0} \dfrac{x^2}{1-\cos x}$
4. $\lim_{x\to 0} \dfrac{\sqrt{9+3x}-3}{x}$

Solution.

$\square$

L'Hôpital's Rule requires evaluating $\lim_{x\to c} \frac{f'(x)}{g'(x)}$. It may happen that this second limit is another indeterminate form to which L'Hôpital's Rule may again be applied. Let's see more examples.

Example. Evaluate the following limits.

a. $\lim_{x\to 0} \frac{e^x - x - 1}{x^2}$ **b.** $\lim_{x\to 2} \frac{x^3 - 3x^2 + 4}{x^4 - 4x^3 + 7x^2 - 12x + 12}$

Solution.

□

Note that at each step where L'Hôpital's Rule was applied, it was *needed*: the initial limit returned the indeterminate form of "0/0." If the initial limit returns, for example, 1/2, then L'Hôpital's Rule does not apply.

The following theorem extends our initial version of L'Hôpital's Rule in two ways. It allows the technique to be applied to the indeterminate form ∞/∞ and to limits where x approaches $\pm\infty$.

Theorem (L'Hôpital's Rule: ∞/∞)**.** 1. Let $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$, where f and g are differentiable on an open interval I containing a. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

2. Let f and g be differentiable functions on the open interval (a, ∞) for some value a, where $g'(x) \neq 0$ on (a, ∞) and $\lim_{x\to\infty} f(x)/g(x)$ returns either "0/0" or "∞/∞". Then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

A similar statement can be made for limits where x approaches $-\infty$.

Example Evaluate the following limits.

a. $\lim_{x\to\infty} \dfrac{4x^3 - 6x^2 + 1}{2x^3 - 10x + 3}$ **b.** $\lim_{x\to\frac{\pi}{2}^-} \dfrac{1 + \tan x}{\sec x}$

Solution.

□

Indeterminate Forms $0 \cdot \infty$ and $\infty - \infty$

L'Hôpital's Rule can only be applied to ratios of functions. When faced with an indeterminate form such as $0 \cdot \infty$ or $\infty - \infty$, we can sometimes **apply algebra to rewrite the limit** so that L'Hôpital's Rule can be applied. We demonstrate the general idea in the next two examples.

Example Evaluate $\lim_{x\to\infty} x^2 \sin\left(\dfrac{1}{4x^2}\right)$

Solution.

$\square$

Example Evaluate $\lim_{x\to\infty} \left(x - \sqrt{x^2 - 3x}\right)$

Solution.

$\square$

Indeterminate Forms 0^0, 1^∞ and ∞^0

When faced with an indeterminate form that involves a power, it often helps to employ the natural logarithmic function. The following Key Idea expresses the concept, which is followed by an example that demonstrates its use.

Evaluating Limits Involving Indeterminate Forms 0^0, 1^∞ and ∞^0

$$\text{If } \lim_{x\to c} \ln\big(f(x)\big) = L, \quad \text{then } \lim_{x\to c} f(x) = \lim_{x\to c} e^{\ln(f(x))} = e^L.$$

Remark. If L is ∞ or $-\infty$, then $\lim_{x\to c} f(x) = \infty$ or $\lim_{x\to c} f(x) = 0$, respectively.

Example Evaluate the following limits.

a. $\lim_{x\to 0^+} x^x$ **b.** $\lim_{x\to\infty} \left(1+\dfrac{1}{x}\right)^x$

Solution.

$\square$

Growth Rates of Functions

An important use of L'Hôpital's Rule is to compare the growth rates of functions. Here are two questions—one practical and one theoretical—that demonstrate the importance of comparative growth rates of functions.

Suppose the functions f and g both approach infinity as $x \to \infty$. Although the values of both functions become arbitrarily large as the values of x become sufficiently large, sometimes one function is growing more quickly than the other. For example, $f(x) = x^2$ and $g(x) = x^3$ both approach infinity as $x \to \infty$. However, as shown in the following table, the values of x^3 are growing much faster than the values of x^2.

x	10	100	1,000	10,000
$f(x) = x^2$	100	10,000	1,000,000	10,000,000
$g(x) = x^3$	1,000	1,000,000	1,000,000,000	1,000,000,000,000

In fact,

$$\lim_{x\to\infty} \frac{x^3}{x^2} = \lim_{x\to\infty} x = \infty \quad \text{or} \quad \lim_{x\to\infty} \frac{x^2}{x^3} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

As a result, we say x^3 is growing more rapidly than x^2 as $x \to \infty$.

Another example, $f(x) = x^2$ and $g(x) = 3x^2 + 4x + 1$. Although the values of $g(x)$ are always greater than the values of $f(x)$ for $x > 0$, each value of $g(x)$ is roughly three times the corresponding value of $f(x)$ as $x \to \infty$, as shown in the following table.

x	10	100	1,000	10,000
$f(x) = x^2$	100	10,000	1,000,000	10,000,000
$g(x) = 3x^2 + 4x + 1$	341	30,401	3,004,001	300,040,001

In fact,

$$\lim_{x\to\infty} = \lim_{x\to\infty} \frac{x^2}{3x^2 + 4x + 1} = \frac{1}{3}.$$

In this case, we say that x^2 and $3x^2 + 4x + 1$ are growing at the same rate as $x \to \infty$.

Definition (Growth Rates of Functions (as $x \to \infty$)). Suppose f and g are functions with $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$. Then f grows faster than g as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty \quad \text{or equivalently} \quad \lim_{x\to\infty} \frac{g(x)}{f(x)} = 0.$$

The functions f and g have comparable growth rates if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = M,$$

where $0 < M < \infty$ (M is nonzero and finite).

Example Powers of x vs. powers of $\ln x$ Compare the growth rates as $x \to \infty$ of the following pairs of functions.

a. $f(x) = \ln x$ and $g(x) = x^p$, where $p > 0$.

b. $f(x) = \ln^q x$ and $g(x) = x^p$, where $p > 0$ and $q > 0$.

Solution.

□

Example Powers of x vs. exponentials Compare the rates of growth of $f(x) = x^p$ and $g(x) = e^x$ as $x \to \infty$, where p is a positive real number.

Solution.

$\square$

Theorem (Ranking Growth Rates as $x \to \infty$)**.** Let $f << g$ mean that g grows faster than f as $x \to \infty$. With positive real numbers p, q, r, and s and $b > 1$,

$$\ln^q x << x^p << x^p \ln^r x << x^{p+s} << b^x << x^x.$$

4.8 Newton's Method

Solving equations is one of the most important things we do in mathematics, yet we are surprisingly limited in what we can solve analytically. For instance, equations as simple as $x^5 + x + 1 = 0$ or $\cos x = x$ cannot be solved by algebraic methods in terms of familiar functions. Fortunately, there are methods that can give us *approximate* solutions to equations like these. These methods can usually give an approximation correct to as many decimal places as we like. This section focuses on one technique called Newton's Method.

Newton's Method is built around tangent lines. The main idea is that if x is sufficiently close to a root of $f(x)$, then the tangent line to the graph at $(x, f(x))$ will cross the x-axis at a point closer to the root than x.

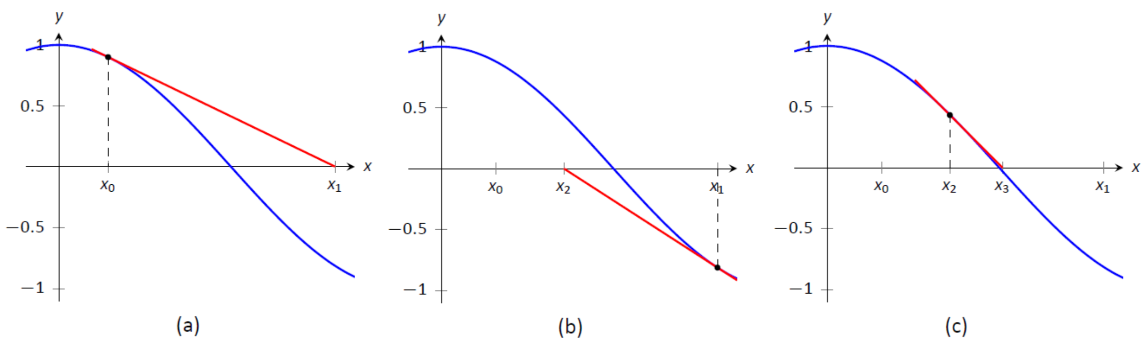


Figure 23: Demonstrating the geometric concept behind Newton's Method.

We start Newton's Method with an initial guess about roughly where the root is. Call this x_0. (See Figure 23(a).) Draw the tangent line to the graph at $(x_0, f(x_0))$ and see where it meets the x-axis. Call this point x_1. Then repeat the process – draw the tangent line to the graph at $(x_1, f(x_1))$ and see where it meets the x-axis. (See Figure 23(b).) Call this point x_2. Repeat the process again to get x_3, x_4, etc. This sequence of points will often converge rather quickly to a root of f.

We can use this *geometric* process to create an *algebraic* process. Let's look at how we found x_1. We started with the tangent line to the graph at $(x_0, f(x_0))$. The slope of this tangent line is $f'(x_0)$ and the equation of the line is

$$y = f'(x_0)(x - x_0) + f(x_0).$$

This line crosses the x-axis when $y = 0$, and the x–value where it crosses is what we called x_1. So let $y = 0$ and replace x with x_1, giving the equation:

$$0 = f'(x_0)(x_1 - x_0) + f(x_0).$$

Now solve for x_1:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Since we repeat the same geometric process to find x_2 from x_1, we have

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

In general, given an approximation x_n, we can find the next approximation, x_{n+1} as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

We summarize this process as follows.

Newton's Method for Approximating Roots of $f(x) = 0$

Let f be a differentiable function on an interval I with a root in I. To approximate the value of the root, accurate to d decimal places:

1. Choose a value x_0 as an initial approximation of the root. (This is often done by looking at a graph of f.)

2. Create successive approximations iteratively; given an approximation x_n, compute the next approximation x_{n+1} as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

3. Stop the iterations when successive approximations do not differ in the first d places after the decimal point. Here d is the convergence standard you set. For example, if you choose $d = 3$, then successive approximations differ no more than 0.001.

Example Approximate the roots of $f(x) = x^3 - 5x + 1$ using seven steps of Newton's method. Use $x_0 = -3, x_0 = 1$, and $x_0 = 4$ as initial approximations.

Solution.

$\square$

Example Find the points at which the curves $y = \cos x$ and $y = x$ intersect.

Solution.

$\square$

Example Find the x-coordinate of the first local maximum and the first local minimum of the function $f(x) = e^{-x} \sin 2x$ on the interval $(0, \infty)$.

Solution.

$\square$

Remark. It is also possible for Newton's Method to not converge while each successive approximation is well defined (See Example 4 in the textbook). **There is no "fix" to this problem**; Newton's Method simply will not work and another method must be used.

While Newton's Method does not always work, it does work "most of the time," and it is generally very fast. Once the approximations get close to the root, Newton's Method can as much as double the number of correct decimal places with each successive approximation. A course in Numerical Analysis will introduce the reader to more iterative root finding methods, as well as give greater detail about the strengths and weaknesses of Newton's Method.

4.9 Antiderivatives

Given a function $y = f(x)$, a *differential equation* is one that incorporates y, x, and the derivatives of y. For instance, a simple differential equation is:

$$y' = 2x.$$

Solving a differential equation amounts to finding a function y that satisfies the given equation. Take a moment and consider that equation; can you find a function y such that $y' = 2x$?

Can you find another? And yet another?

Hopefully one was able to come up with at least one solution: $y = x^2$. "Finding another" may have seemed impossible until one realizes that a function like $y = x^2+1$ also has a derivative of $2x$. Once that discovery is made, finding "yet another" is not difficult; the function $y = x^2+123,456,789$ also has a derivative of $2x$. The differential equation $y' = 2x$ has many solutions. This leads us to some definitions.

Definition (Antiderivatives and Indefinite Integrals)**.** Let a function $f(x)$ be given. An **antiderivative** of $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$.
The set of all antiderivatives of $f(x)$ is the **indefinite integral of** f, denoted by

$$\int f(x)\ dx.$$

Knowing one antiderivative of f allows us to find infinitely more, simply by adding a constant. Not only does this give us *more* antiderivatives, it gives us *all* of them.

Theorem (Antiderivative Forms)**.** Let $F(x)$ be any antiderivatives of $f(x)$ on an interval I. Then all the antiderivatives of f on I have the form

$$F(x) + C,$$

where C is an arbitrary constant.

Remark. Every time an indefinite integral sign $\int$ appears, it is followed by a function called the **integrand**, which in turn is followed by the differential dx. For

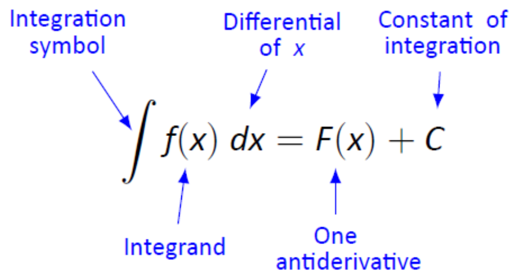


Figure 24: Understanding the indefinite integral notation.

now, dx simply means that x is the independent variable, or the variable of integration. The notation $\int f(x)\, dx$ represents all the antiderivatives of f.

Figure 24 shows the typical notation of the indefinite integral. The integration symbol, $\int$, is in reality an "elongated S," representing "take the sum." We will later see how *sums* and *antiderivatives* are related. The $\int$ symbol and the differential dx are not "bookends" with a function sandwiched in between; rather, the symbol $\int$ means "find all antiderivatives of what follows," and the function $f(x)$ and dx are multiplied together; the dx does not "just sit there."

Example Determine the following indefinite integrals

a. $\int 3x^3\, dx$ **b.** $\int \dfrac{1}{1+x^2}\, dx$ **c.** $\int \sin t\, dt$

Solution.

$\square$

Theorem (Power Rule for Indefinite Integrals)**.**

$$\int x^p\, dx = \frac{x^{p+1}}{p+1} + C,$$

where $p \neq -1$ is a real number and C is an arbitrary constant.

Theorem (Constant Multiple and Sum Rules)**.**

$$\textbf{Constant Multiple Rule:} \int cf(x)\, dx = c\int f(x)\, dx, \text{ for real numbers } c$$

$$\textbf{Sum Rule:} \int (f(x)+g(x))\, dx = \int f(x)\, dx + \int g(x)\, dx$$

Example Determine the following indefinite integrals

a. $\int (3x^5 + 2 - 5\sqrt{x})\, dx$ **b.** $\int \left(\frac{4x^{19} - 5x^{-8}}{x^2} \right) dx$ **c.** $\int (z^2 + 1)(2z - 5)\, dt$

Solution.

$\square$

We restate a list of derivatives here to stress the relationship between derivatives and antiderivatives. This list will also be useful as a glossary of common antiderivatives as we learn (D_x denotes $\frac{d}{dx}$).

$$
\begin{aligned}
D_x(x^n) = nx^{n-1} &\implies \int x^n\ dx = \frac{1}{n+1}x^{n+1} + C, \quad n \neq -1 \\
D_x(e^x) = e^x &\implies \int e^x\ dx = e^x + C \\
D_x(a^x) = a^x \ln a &\implies \int a^x\ dx = \frac{a^x}{\ln a} + C \\
D_x(\ln|x|) = \frac{1}{x} &\implies \int \frac{dx}{x} = \ln|x| + C \\
D_x(\sin x) = \cos x &\implies \int \cos x\ dx = \sin x + C \\
D_x(\cos x) = -\sin x &\implies \int \sin x\ dx = -\cos x + C \\
D_x(\tan x) = \sec^2 x &\implies \int \sec^2 x\ dx = \tan x + C \\
D_x(\sec x) = \sec x \tan x &\implies \int \sec x \tan x\ dx = \sec x + C \\
D_x(\cot x) = -\csc^2 x &\implies \int \csc^2\ dx = -\cot x + C \\
D_x(\csc x) = -\csc x \cot x &\implies \int \csc x \cot x\ dx = -\csc x + C \\
D_x(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} &\implies \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C \\
D_x(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}} &\implies \int \frac{-dx}{\sqrt{1-x^2}} = \cos^{-1} x + C = -\sin^{-1} x + C \\
D_x(\tan^{-1} x) = \frac{1}{1+x^2} &\implies \int \frac{dx}{1+x^2} = \tan^{-1} x + C \\
D_x(\cot^{-1} x) = \frac{-1}{1+x^2} &\implies \int \frac{-dx}{1+x^2} = \cot^{-1} x + C = -\tan^{-1} x + C \\
D_x(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}} &\implies \int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}|x| + C \\
D_x(\csc^{-1} x) = \frac{-1}{x\sqrt{x^2-1}} &\implies \int \frac{-dx}{x\sqrt{x^2-1}} = \csc^{-1}|x| + C = -\sec^{-1}|x| + C \\
D_x(\ln|\sec x|) = \tan x &\implies \int \tan x\ dx = \ln|\sec x| + C \\
D_x(\ln|\sin x|) = \cot x &\implies \int \cot x\ dx = \ln|\sin x| + C
\end{aligned}
$$

Example Determine the following indefinite integrals

a. $\int \sin(3x)\, dx$ **b.** $\int \sec ax \tan ax\, dx$, where $a \neq 0$ is a real number

Solution.

$\square$

Example Determine the following indefinite integrals

a. $\int \sec^2 3x\, dx$ **b.** $\int \cos \frac{x}{2}\, dx$

Solution.

$\square$

Example Determine the following indefinite integrals

a. $\int e^{ax}\, dx$ **b.** $\int \frac{1}{a^2 + x^2}\, dx$

Solution.

$\square$

Example Determine the following indefinite integrals

a. $\int e^{-10t}\, dt$ **b.** $\int \frac{4}{\sqrt{9 - x^2}}\, dx$ **c.** $\int \frac{1}{\sqrt{16x^2 + 1}}\, dx$

Solution.

$\square$

Initial Value Problems

An equation involving an unknown function and its derivatives is called a **differential equation**. The equation

$$\frac{dy}{dx} = f(x) \tag{4.2}$$

is a simple example of a differential equation. Solving this equation means finding a function y with a derivative f. Therefore, the solutions of Equation 4.2 are the antiderivatives of f. If F is one antiderivative of f, every function of the form $y = F(x) + C$ is a solution of that differential equation. For example, the solutions of

$$\frac{dy}{dx} = 6x^2$$

are given by

$$y = \int 6x^2 \, dx = 2x^3 + C.$$

Sometimes we are interested in determining whether a particular solution curve passes through a certain point (x_0, y_0) —that is, $y(x_0) = y_0$. The problem of finding a function y that satisfies a differential equation

$$\frac{dy}{dx} = f(x) \tag{4.3}$$

with the additional condition

$$y(x_0) = y_0 \tag{4.4}$$

is an example of an initial-value problem. The condition $y(x_0) = y_0$ is known as an initial condition. For example, looking for a function y that satisfies the differential equation

$$\frac{dy}{dx} = 6x^2$$

and the initial condition

$$y(1) = 5.$$

is an example of an initial-value problem. Since the solutions of the differential equation are $y = 2x^3 + C$, to find a function y that also satisfies the initial condition, we need to find C such that $y(1) = 2(1)^3 + C = 5$. From this equation, we see that $C = 3$, and we conclude that $y = 2x^3 + 3$ is the solution of this initial-value problem as shown in Figure 25.

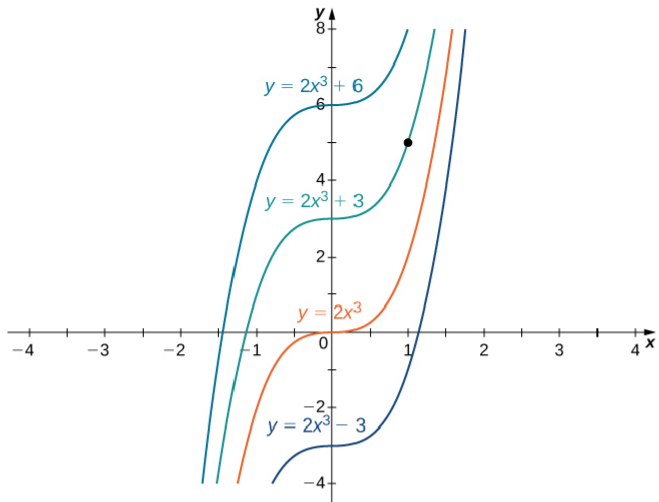


Figure 25: Some of the solution curves of the differential equation $\frac{dy}{dx} = 6x^2$ are displayed. The function $y = 2x^3 + 3$ satisfies the differential equation and the initial condition $y(1) = 5$.

Example Solve the initial value problem $f'(x) = x^2 - 2x$ with $f(1) = \frac{1}{3}$.

Solution.

$\square$

Initial Value Problems for Velocity and Position

Suppose an object moves along a line with a (known) velocity $v(t)$, for $t \geq 0$. Then its position is found by solving the initial value problem

$$s^{'}(t) = v(t), s(0) = s_0, \text{ where } s_0 \text{ is the (known) initial position.}$$

If the (known) acceleration of the object $a(t)$ is given, then its velocity is found by solving the initial value problem

$$v^{'}(t) = a(t), v(0) = v_0, \text{ where } v_0 \text{ is the (known) initial velocity.}$$

Example Runner A begins at the point $s(0) = 0$ and runs with velocity $v(t) = 2t$. Runner B begins with a head start at the point $S(0) = 8$ and runs with velocity $V(t) = 2$. Find the positions of the runners for $t \geq 0$ and determine who is ahead at $t = 6$ time units.

Solution.

$\square$

Example Neglecting air resistance, the motion of an object moving vertically near Earth's surface is determined by the acceleration due to gravity, which is approximately $9.8m/s^2$. Suppose a stone is thrown vertically upward at $t = 0$ with a velocity of $40m/s$ from the edge of a cliff that is $100m$ above a river.

a. Find the velocity $v(t)$ of the object, for $t \geq 0$.

b. Find the position $s(t)$ of the object, for $t \geq 0$.

c. Find the maximum height of the object above the river.

d. With what speed does the object strike the river?

Solution.

$\square$