Contents

0 Calculus Prerequisite for MATH 311

Students in this Statistics I (Probability) course are expected to be able to use many calculus techniques skillfully. The following is a list of most of the calculus topics used during this course.

1. Find the derivative of any function including the use of product rule and chain rule.
2. Use basic techniques of integration including substitution and integration by parts.
3. Be able to use the Fundamental Theorem of Calculus to do problems such as $\frac{d}{dx}\int_x^4 e^{t^2}dt$
4. Find improper integrals.
5. Use Taylor approximations to functions and recognize sums as Taylor approximations for many basic functions such as e^x.
6. Use standard techniques for finding the sum of an infinite series, including recognizing geometric series.
7. Find partial derivatives of a function of several variables.
8. Compute double integrals including change of variables in double integrals.

You also need to be comfortable working with limits and sets.

This document is for you if you are not familiar with some of these topics. For example, if you cannot solve all the following five questions, you definitely need this review.

Question 1. Express the infinite sum $\sum_{i=1}^{\infty} \frac{x^i}{i!} = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ as a real function.

Question 2. Find $\int xe^{1-x}\, dx$.

Question 3. Find $\int (2-x)^8\, dx$.

Question 4. Find $\int_{-\infty}^{0} xe^x\, dx$

Question 5. Evaluate the double integral

$$\int\int_R x\, dxdy,$$

where R is a triangular region given by $R = \{(x, y) : 0 \le x \le 1, 0 \le y \le 1, y \le 1-x\}$.

1 Product Rule and Chain Rule

Product Rule. Let $f(x)$ and $g(x)$ be two differentiable functions. Then the derivative of the product is given by

$$\frac{d}{dx}\left[f\left(x\right)g\left(x\right)\right]=\left[\frac{d}{dx}f\left(x\right)\right]g\left(x\right)+f\left(x\right)\left[\frac{d}{dx}g\left(x\right)\right].$$

Example 1. What is the derivative of $f(x)=(3x+2)\sqrt{x}$?

Solution.

$$\begin{aligned} f^{'}(x)=&(3x+2)^{'}(\sqrt{x})+(3x+2)(\sqrt{x})^{'}\\ =&(3)\sqrt{x}+(3x+2)\frac{1}{2\sqrt{x}}=3\sqrt{x}+\frac{3x+2}{2\sqrt{x}}. \end{aligned}$$

$\square$

Example 2. What is the derivative of $f(x)=(x+1)(x^2-7x)(e^x)$?

Solution. Think of f as being: $[(x+1)(x^2-7x)][e^x]$

$$\begin{aligned} f^{'}(x)=&[(x+1)(x^2-7x)]^{'}[e^x]+[(x+1)(x^2-7x)][e^x]^{'}\\ =&\Big[(x+1)^{'}(x^2-7x)+(x+1)(x^2-7x)^{'}\Big][e^x]+[(x+1)(x^2-7x)][e^x]\\ =&\Big[(1)(x^2-7x)+(x+1)(2x-7)\Big][e^x]+[(x+1)(x^2-7x)][e^x] \end{aligned}$$

$\square$

Remark. The **Quotient Rule** can be derived from the Product Rule: Let $f(x)=g(x)/h(x)$ be differentiable. Then

$$f^{'}(x)=\frac{g^{'}(x)h(x)-g(x)h^{'}(x)}{h^2(x)}.$$

Chain Rule. If $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable, then

$$F^{'}(x) = f^{'}(g(x)) \cdot g^{'}(x).$$

Remark. The Chain Rule is the most important rule for differentiation in calculus since all other rules are simply consequences of it. For example, the product rule can be derived from the Chain Rule.

Example. What is the derivative of $f(x) = e^{x^2}$?

Solution. Think of f as being: $f(x) = g(h(x))$ with $g(x) = e^x$ and $h(x) = x^2$

$$\begin{aligned}\left[e^{x^2}\right]^{'} =& g^{'}(h(x)) \cdot h^{'}(x) \\ =& e^{h(x)} \cdot (2x) = e^{x^2} \cdot (2x).\end{aligned}$$

$\square$

2 Review of Antiderivatives

Definition. A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Theorem. If F is an antiderivative of f on an interval I , then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

Notation: In the above, we can write

$$\int f(x)\ dx = F(x) + C.$$

We have a list of antiderivatives that must be memorized!!!

$$\begin{aligned}
\frac{d\ x^n}{dx} = nx^{n-1} &\implies \int x^n\ dx = \frac{1}{n+1}x^{n+1} + C\\
\frac{d\ e^x}{dx} = e^x &\implies \int e^x\ dx = e^x + C\\
\frac{d\ a^x}{dx} = a^x \ln a &\implies \int a^x\ dx = \frac{a^x}{\ln a} + C\\
\frac{d\ \ln|x|}{dx} = \frac{1}{x} &\implies \int \frac{dx}{x} = \ln|x| + C\\
\frac{d\ \sin x}{dx} = \cos x &\implies \int \cos x\ dx = \sin x + C\\
\frac{d\ \cos x}{dx} = -\sin x &\implies \int \sin x\ dx = -\cos x + C\\
\frac{d\ \tan^{-1} x}{dx} = \frac{1}{1+x^2} &\implies \int \frac{dx}{1+x^2} = \tan^{-1} x + C
\end{aligned}$$

3 The Definite Integral

Theorem (Properties of the Definite Integral)**.** Suppose $f(x)$ and $g(x)$ are both integrable.

(0) $\int_a^a f(x)\ dx = 0$ (since $\Delta x = 0$)

(1) $\int_a^b c\ dx = c(b-a)$, where c is any constant;

(2) $\int_b^a f(x)\ dx = -\int_a^b f(x)\ dx$, where $a < b$, because Δx changes from $\frac{b-a}{n}$ to $\frac{a-b}{n}$;

(3) $\int_a^b [f(x) \pm g(x)]\ dx = \int_a^b f(x)dx \pm \int_a^b g(x)\ dx$;

(4) $\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx$, where c is any constant.

(5) $\int_a^c f(x)\ dx + \int_c^b f(x)\ dx = \int_a^b f(x)\ dx$, where c is any constant (c need not be between a and b.);

(6) If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\ dx \geq 0$;

(7) If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\ dx \geq \int_a^b g(x)\ dx$;

(8) If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)\ dx \leq M(b-a)$. $\square$

Theorem. If f is continuous on $[a, b]$, then

$$\boxed{\left|\int_a^b f(x)\ dx\right| \leq \int_a^b |f(x)|\ dx.}$$

4 The Fundamental Theorem of Calculus

Theorem 1 (The Fundamental Theorem of Calculus)**.** Suppose f is continuous on $[a, b]$. Then

1. If $g(x) = \int_a^x f(t)dt$, then $g^{'}(x) = f(x)$. It can be written as $\boxed{\dfrac{d}{dx}\displaystyle\int_a^x f(t)dt = f(x)}$.

2. $\int_a^b f(x)dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F^{'} = f$. It can also be written as $\boxed{\displaystyle\int_a^b F^{'}(x)dx = F(b) - F(a)}$.

5 Indefinite Integrals

Definition (Indefinite Integral)**.** The notion of an antiderivative of $f(x)$ was introduced as being a function $F(x)$ such that $F'(x) = f(x)$. Moreover, the Fundamental Theorem of Calculus (Theorem 1) establishes the connections between antiderivatives and definite integrals: $\boxed{\int_a^x f(t)dt}$ is an antiderivative of $f(x)$. It is customary, to denote an antiderivative of $f(x)$ by

$$F(x) + C = \int f(x)\ dx \qquad \text{Here } C \text{ is an arbitrary real number.}$$

and call it an **indefinite integral**.

$$\int f(x)\ dx = F(x) + C \iff \frac{d\ F(x)}{dx} = D_x(F(x)) = f(x).$$

□

Remark. Indefinite integrals are functions but definite integrals with constant lower and upper limits are numbers.

Properties of the indefinite integral

1. $\int \big(f(x) \pm g(x)\big)\ dx = \int f(x)\ dx \pm \int g(x)\ dx.$

2. $\int cf(x)\ dx = c \int f(x)\ dx.$

Remark. When we apply Property 1 we do not add an arbitrary constant for each integral of the right hand side. For example,

$$\int (x^2+1)\ dx = \int x^2\ dx + \int 1\ dx = \frac{x^3}{3} + x + C.$$

□

6 The Substitution Rule

The Substitution Rule is a result of the Chain Rule. Recall that the chain rule is: if $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable, then $\boxed{F^{'}(x) = f^{'}(g(x)) \cdot g^{'}(x)}$.

Remember the concept of a differential: $dx = \Delta x$ for independent variable x and $\boxed{dv = v'dx}$ (for a constant c, $d(cv) = cv'dx$) for dependent variable v.

Theorem (The Substitution Rule). If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))\; g'(x)dx = \int f(u)\; du.$$

Remark. In practice, it is convenient to use $\boxed{du = u'dx}$ for dependent variable u.

Example 1. If $u = 1 - x^4$ then $du = -4x^3dx$ and so we have

$$\begin{aligned}\int x^3(\underbrace{1-x^4}_{=u})\; dx &= -\frac{1}{4}\int (\underbrace{1-x^4}_{=u})\,\underbrace{(-4x^3)\; dx}_{=du}\\ &= -\frac{1}{4}\int u\; du = (-\frac{1}{4})(\frac{u^2}{2}) + C = -\frac{1}{8}(1-x^4)^2 + C.\end{aligned}$$

Or we do the following

$$\begin{aligned}\int x^3(1-x^4)\; dx &= \int (1-x^4)d(\frac{x^4}{4}) = \int (1-x^4)d\left(\frac{-(1-x^4)}{4}\right)\\ &= \frac{-1}{4}\int (1-x^4)d(1-x^4) = \frac{-1}{4}\int u\; du = \cdots\cdots .\end{aligned}$$

□

Example 2.

$$\int \frac{1}{x}\; dx = \begin{cases} \int \frac{1}{x}\; dx = \ln x + C & \text{if } x > 0 \\ \\ \int \frac{1}{x}\; dx = \int \frac{1}{-x}\; d(-x) = \ln(-x) + C & \text{if } x < 0 \end{cases}$$

$$= \ln|x| + C$$

Example 3. Let $u = -2x$. Then $du = -2dx$.

$$\int e^{-2x}\; dx = \int e^u \frac{-1}{2}\; du = \frac{-1}{2}\int e^u\; du = \cdots\cdots$$

Theorem (The Substitution Rule for Definite Integrals). If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\ dx = \int_{g(a)}^{g(b)} f(u)\ du.$$

$\square$

Remark. If we do not explicitly write down the substitution, the lower and upper limits in the integral should not be changed. For example,

$$\int_0^1 x^3(1-x^4)\ dx = \boxed{\frac{-1}{4}\int_0^1 (1-x^4)d(1-x^4)} = \left(\frac{-1}{4}\right)\frac{(1-x^4)^2}{2}\Big|_0^1.$$

$\square$

Example 1. Here we set $u = g(x) = 4 + 3x$ and compute as follows.

$$\begin{aligned}\int_0^7 \sqrt{4+3x}\ dx =& \frac{1}{3}\int_0^7 \sqrt{4+3x}\ d(4+3x) = \frac{1}{3}\int_{g(0)}^{g(7)} u^{\frac{1}{2}}\ du \\ =& \frac{1}{3}\int_4^{25} u^{\frac{1}{2}}\ du = \frac{1}{3}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big|_{u=4}^{u=25} \\ =& \frac{2}{9}[125-8] = \frac{234}{9}.\end{aligned}$$

Example 2. Here we set $u = g(x) = -x^2$ and compute as follows.

$$\begin{aligned}\int_0^1 xe^{-x^2}\ dx &= -\frac{1}{2}\int_0^1 e^{-x^2}\ (-2xdx) = -\frac{1}{2}\int_{g(0)}^{g(1)} e^u\ du \\ &= -\frac{1}{2}\int_0^{-1} e^u\ du = -\frac{1}{2}\left[e^u\right]_0^{-1} = -\frac{1}{2}\left[\frac{1}{e}-1\right] \\ &= \frac{e-1}{2e}.\end{aligned}$$

Theorem (Integrals of Symmetric Functions)**.** Suppose that $f(x)$ is continuous on the symmetric interval $[-a, a]$. Then

(a) If $f(x)$ is an even function, then $\int_{-a}^a f(x)\ dx = 2\int_0^a f(x)\ dx$.

(b) If $f(x)$ is an odd function, then $\int_{-a}^a f(x)\ dx = 0$.

Example 1. The cosine function is an even function and so we have the following.

$$\int_{\frac{-\pi}{3}}^{\frac{\pi}{3}} \cos x \ dx = 2\int_{0}^{\frac{\pi}{3}} \cos x \ dx = 2\sin x\Big|_{x=0}^{x=\frac{\pi}{3}} = 2[\frac{\sqrt{3}}{2}] = \sqrt{3}.$$

Example 2. The sine function is an odd function and so we have the following.

$$\int_{\frac{-\pi}{3}}^{\frac{\pi}{3}} \sin^5 x \ dx = 0.$$

7 Integration by Parts

This is an anti-differentiation technique that follows from the **Product Rule** for derivatives.

$$\begin{aligned}(uv)^{'} &= u^{'}v + uv^{'} \\ &\Longrightarrow \boxed{\int u\ v'dx = uv - \int v\ u'dx} \\ &\Longrightarrow \boxed{\int u\ dv = uv - \int v\ du}.\end{aligned}$$

This is the formula method for **Integration by Parts**. There is a "diagram method" for integration by parts that often makes things easier, particularly, when we have to do repeated applications of Integration by Parts.

(1) Formula Method: $\int u\ dv = uv - \int v\ du.$

(2) Diagram Method:

diff		**int**	
u		v'	
	$\searrow$		Diagonal arrows = products
u'	$(-)\rightarrow$	v	Horizontal arrows = prod.& int.

Remark. $v = \int v^{'}\ dx$ should be easy to find.

Integration by parts is very useful in deriving reduction formulas, i.e. reduce an integrand to a simpler case. However, there are no general rules when integration by parts should be used. Usually integrands involving e^x, $\ln x$, $\sin^{-1} x$, $\cos^{-1} x$, or $\tan^{-1} x$ are done using integration by parts.

Example 1 Find $\int \ln x \, dx$.

Solution.

$$\begin{array}{cccc} \text{diff.} & & \text{int.} & \\ \ln x & & 1 & \\ & \searrow & & \\ \frac{1}{x} & (-) \rightarrow & x & \text{Easy Integration} \end{array}$$

Thus,

$$\int \ln x \, dx = x \ln x - \int 1 \, dx = x \ln x - x + C.$$

$\square$

Remark. If the integration is still hard to get after one step, we can apply the technique of "Integration by Parts" again.

Example 2 Find $\int t^2 e^t \; dt$.

Solution.

$$\begin{array}{ccc}
\text{diff.} & & \text{int.} \\
t^2 & & e^t \\
 & \searrow & \\
2t & (-) \rightarrow & e^t
\end{array}$$

Therefore,

$$\int t^2 e^t \; dt = t^2 e^t - 2 \int t e^t \; dt.$$

We apply the integration by parts to $\int t e^t \; dt$,

$$\int t e^t \; dt = \int t \; de^t = t e^t - e^t \int dt = t e^t - e^t + C.$$

Therefore,

$$\int t^2 e^t \; dt = t^2 e^t - 2(t e^t - e^t) + C = t^2 e^t - 2t e^t + 2e^t + C.$$

Remark. Repeated application of Integration by Parts can be made easier by the Diagram Method, but be careful to the signs.

$$\begin{array}{ccc}
\text{diff.} & & \text{int.} \\
t^2 & & e^t \\
 & \searrow & \\
2t & & e^t \\
 & (-) \searrow & \\
2 & (+) \rightarrow & e^t
\end{array}$$

$\square$

8 Improper Integrals

Definition (**Definition of Improper Integrals of Type I).** The integrals of *unbounded domain* $\int_a^\infty f(x)\,dx$, $\int_{-\infty}^b f(x)\,dx$ and $\int_{-\infty}^\infty f(x)\,dx$ are defined in the following way.

(a) If $\int_a^t f(x)\ dx$ exists for every $t \geq a$ and the limit $\lim_{t\to\infty} \int_a^t f(x)\ dx$ exists and is finite, then the **improper integral** $\int_a^\infty f(x)\ dx$ is said to be **convergent** and is defined to be value of this limit

$$\int_a^\infty f(x)\ dx = \lim_{t\to\infty} \int_a^t f(x)\ dx.$$

(b) If $\int_t^b f(x)\ dx$ exists for every $t \leq b$ and the limit $\lim_{t\to-\infty} \int_t^b f(x)\ dx$ exists and is finite, then the **improper integral** $\int_{-\infty}^b f(x)\ dx$ is said to be **convergent** and is defined to be value of this limit

$$\int_{-\infty}^b f(x)\ dx = \lim_{t\to-\infty} \int_t^b f(x)\ dx.$$

(c) If $\int_a^\infty f(x)\ dx$ and $\int_{-\infty}^a f(x)\ dx$ are convergent, then the **improper integral** $\int_{-\infty}^\infty f(x)\ dx$ is defined to be value of this limit

$$\int_{-\infty}^\infty f(x)\ dx = \int_{-\infty}^a f(x)\ dx + \int_a^\infty f(x)\ dx$$

where a is any real number.

An improper integral that is not convergent, is said to be **divergent**. (Convergence or divergence depends only on the “tail”.

Example 1 Evaluate

$$\int_{-\infty}^{0} xe^x \, dx$$

Solution. First we find an antiderivative by Integration by Parts.

$$\begin{array}{ccc} \text{diff.} & & \text{int.} \\ x & & e^x \\ & \searrow & \\ 1 & (-)\rightarrow & e^x \end{array}$$

Therefore,

$$\begin{aligned} \int_{-\infty}^{0} xe^x \, dx &= \lim_{t\to-\infty} \int_{t}^{0} xe^x \, dx \\ &= \lim_{t\to-\infty} \Big[xe^x - e^x\Big]_t^0 = \lim_{t\to-\infty} \Big[-1 - (te^t - e^t)\Big] \end{aligned}$$

Certainly,

$$\lim_{t\to-\infty} e^t = \lim_{t\to\infty} e^{-t} = \lim_{t\to\infty} \frac{1}{e^t} = 0.$$

We apply L'Hospital Rule to evaluate $\lim\limits_{t\to-\infty} te^t$

$$\lim_{t\to-\infty} te^t = \lim_{t\to-\infty} \frac{t}{e^{-t}} = \lim_{t\to\infty} \frac{-t}{e^t} = \lim_{t\to\infty} \frac{-1}{e^t} = 0.$$

Now we put these parts together to get

$$\int_{-\infty}^{0} xe^x \, dx = \lim_{t\to-\infty} \Big[-1 - (te^t - e^t)\Big] = -1.$$

□

Example 2 - The p -Test for Improper Integrals.

$$\int_1^\infty \frac{1}{x^p}\ dx = \begin{cases} \text{diverges, if } p \leq 1 \\ \text{converges, if } p > 1 \end{cases}$$

Solution.

$$\int_1^t \frac{1}{x^p}\ dx = \begin{cases} \ln t & \text{if } p = 1 \\ \frac{t^{-p+1}}{-p+1} - \frac{1}{-p+1} & \text{if } p \neq 1 \end{cases}$$

Now,

$$\lim_{t\to\infty} \ln t = \infty \implies \int_1^\infty \frac{1}{x^p}\ dx \text{ divergent when } p = 1 \text{ and}$$

$$\lim_{t\to\infty} \frac{1}{1-p}(t^{1-p} - 1) = \begin{cases} \infty & \text{if } p < 1 \\ -\frac{1}{1-p} & \text{if } p > 1. \end{cases}$$

□

Definition (Definition of Improper Integrals of Type II - Discontinuous Integrand). If $f(x)$ has *discontinuous* point on $[a, b]$, we define $\int_a^b$ in what follows.

(a) If f is continuous on $[a, b)$, discontinuous at b, but the limit $\lim_{t \to b^-} \int_a^t f(x)\, dx$ exists and is finite, then the **improper integral** $\int_a^b f(x)\, dx$ is said to be **convergent** and is defined to be value of this limit

$$\int_a^b f(x)\, dx = \lim_{t \to b^-} \int_a^t f(x)\, dx.$$

(b) If f is continuous on $(a, b]$, discontinuous at a, but the limit $\lim_{t \to a^+} \int_a^t f(x)\, dx$ exists and is finite, then the **improper integral** $\int_a^b f(x)\, dx$ is said to be **convergent** and is defined to be value of this limit

$$\int_a^b f(x)\, dx = \lim_{t \to a^+} \int_t^b f(x)\, dx.$$

(c) If f is continuous on $[a, c)$ and on $(c, b]$ with $a < c < b$, discontinuous at c, but the limit $\lim_{t \to c^-} \int_a^t f(x)\, dx$ and $\lim_{t \to c^+} \int_t^b f(x)\, dx$ both exist and are finite, then the **improper integral** $\int_a^b f(x)\, dx$ is defined to be

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

$\square$

Example 1 Find $\int_0^3 \frac{dx}{x-1}$, if convergent.

Solution. Be careful. The following calculation is **NOT valid**.

$$\int_0^3 \frac{dx}{x-1} = \Big[\ln|x-1|\Big]_0^3 = \ln 2 - \ln|-1| = \ln 2.$$

This is an improper integral due to the fact that $\frac{1}{x-1}$ is discontinuous at $c = 1$ with $0 < 1 < 3$. This means that we must check the convergence of both

$$\int_0^1 \frac{dx}{x-1} \qquad \text{and} \qquad \int_1^3 \frac{dx}{x-1}.$$

$$\begin{aligned}
\lim_{t\to 1^-} \int_0^t \frac{dx}{x-1} &= \lim_{t\to 1^-} \int_0^t \frac{dx}{x-1} = \lim_{t\to 1^-} \Big[\ln|x-1|\Big]_0^t \\
&= \lim_{t\to 1^-} \Big[\ln|t-1| - \ln|0-1|\Big] = \lim_{t\to 1^-} \ln(1-t) \\
&= \lim_{s\to 0^+} \ln s = -\infty
\end{aligned}$$

Therefore, $\int_0^3 \frac{dx}{x-1}$ is divergent. □

Example 2 - The p -Test for type II Improper Integrals.

$$\int_0^1 \frac{1}{x^p}\,dx = \begin{cases} \text{diverges, if } p \geq 1 \\ \text{converges, if } p < 1 \end{cases}$$

□

Theorem (Comparison Test for Improper integrals)**.** Suppose that f and g are continuous functions on the interval $[a, \infty)$ with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

(a) If $\int_a^\infty f(x)\, dx$ is convergent, then $\int_a^\infty g(x)\, dx$ is convergent.

(b) If $\int_a^\infty g(x)\, dx$ is divergent, then $\int_a^\infty f(x)\, dx$ is divergent.

Remark.

(1) The convergence depends on the tail only. Thus the results hold as long as $f(x) \geq g(x) \geq 0$ for $x \geq b$, where $b > a$ is a large number.

(2) A similar theorem is true for Type II improper integrals. For example, one version of the comparison test is as follows

□

Theorem (Comparison Test for type II Improper integrals)**.** Suppose that the functions f and g are continuous on the interval $[a, b)$ and discontinuous at b, with $f(x) \geq g(x) \geq 0$ for $x \in [b - \varepsilon, b)$, where ε is a small number.

(a) If $\int_a^b f(x)\, dx$ is convergent, then $\int_a^b g(x)\, dx$ is convergent.

(b) If $\int_a^b g(x)\, dx$ is divergent, then $\int_a^b f(x)\, dx$ is divergent.

□

Example 1. Test $\int_0^\infty \frac{x}{(x^2+2)^2}\,dx$ for convergence or divergence.

Solution. We claim that it is convergent. We only need to check the "tail", i.e. $\int_1^\infty \frac{x}{(x^2+2)^2}\,dx$. First observe that $\frac{x}{(x^2+2)^2} < \frac{1}{x^2}$ on $[1,\infty)$ because

$$x \cdot x^2 = x^3 < x^4 + 4x^2 + 4 = (x^2+2)^2 \implies \frac{x}{(x^2+2)^2} < \frac{1}{x^2}.$$

Now observe that

$$\int_1^\infty \frac{1}{x^2}\,dx \text{ is convergent .}$$

By the p-Test and then by the Comparison Test $\int_1^\infty \frac{x}{(x^2+2)^2}\,dx$ is convergent and hence $\int_0^\infty \frac{x}{(x^2+2)^2}\,dx$ is convergent. We can actually find the value of $\int_0^\infty \frac{x}{(x^2+2)^2}\,dx$

$$\int_0^\infty \frac{x}{(x^2+2)^2}\,dx = \lim_{t\to\infty} \frac{1}{2}\int_0^t \frac{d(x^2+2)}{(x^2+2)^2} = \lim_{t\to\infty} \frac{1}{2}\Big[-(x^2+2)^{-1}\Big]_0^t = \frac{1}{4}. \quad \square$$

Example 2 Test $\int_0^\infty e^{-x^2}\,dx$ for convergence or divergence.

Solution. We need only test the "tail". $\int_1^\infty e^{-x^2}\,dx$ since

$$\int_0^\infty e^{-x^2}\,dx = \underbrace{\int_0^1 e^{-x^2}\,dx}_{\text{finite number}} + \int_1^\infty e^{-x^2}\,dx$$

On the interval $[1,\infty)$ we have $e^{-x^2} = \frac{1}{e^{x^2}} \le \frac{1}{e^x} = e^{-x}$ and so we test $\int_1^\infty e^{-x}\,dx$ and apply the Comparison Test to get convergence.

$$\int_1^\infty e^{-x}\,dx = \lim_{t\to\infty}\int_1^t e^{-x}\,dx = \lim_{t\to\infty}\Big[-e^{-x}\Big]_1^t = \lim_{t\to\infty}\Big[-e^{-t}+e^{-1}\Big]_1^t = \frac{1}{e}$$

This means $\int_1^\infty e^{-x^2}\,dx$ is convergent by the convergence test. $\square$

9 Infinite Sequences and Series

Definition. A **sequence** can be thought of as a list of numbers written in a definite order

$$\{a_n\} = \{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, \dots\}$$

The a_1 is called the first term, a_2 is called the second term, etc. and in general a_n is called the nth term. We are dealing with infinite sequence so that each a_n has a successor a_{n+1}.

Example.

$$\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \underbrace{\frac{n}{n+1}}_{n^{th}\text{ term}}, \dots\right\}$$

Definition. A sequence $\{a_n\}$ has a real number L as a limit, written as

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty$$

provided the limit exists. In this case, we say that $\{a_n\}$ **converges or is convergent.**

Remark.

(1) Rigorous definition of limit: $\lim_{n\to\infty} a_n = L \iff$ for every positive number ϵ, there is a corresponding N_ϵ such that $|L - a_n| < \epsilon$ for all $n \geq N_\epsilon$.

(2) Rigorous definition of $\lim_{n\to\infty} a_n = \infty$: A sequence $\{a_n\}$ has ∞ as a limit, written $\lim_{n\to\infty} a_n = \infty$, provided for every positive number M, there exists a N such that $a_n > M$ for all $n \geq N$. □

Definition. The sequence $\{a_n\}$ is called divergent if $\lim_{n\to\infty} a_n = \infty$ or $-\infty$ or $\lim_{n\to\infty} a_n$ does not exist.

Example. The sequence $\{a_n = (-1)^n\} = \{-1, 1, -1, 1, -1, 1, \dots\}$ is divergent, since the terms of the sequence oscillate between 1 and -1. So the limit $\lim_{n\to\infty}(-1)^n$ does not exist.

An infinite series (or simply a series) is an expression of the form

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = \underbrace{\sum a_n}_{\text{short form}} .$$

We are trying to add all the terms in a sequence $\{a_n\}$. Naturally, when one is adding infinitely many numbers one ask what exactly does this mean.

$$1+2+3+\cdots+n+\cdots > \text{ any finite number?}$$
$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots+\frac{1}{2^n}+\cdots \text{ Is this equal to a finite number?}$$

These are the type of questions we will be asking. To answer these questions we need to develop some mathematical tools.

Definition. Given a series $\sum a_n$, we define the N^{th} **partial sum** to be

$$S_N=\sum_{n=1}^{N} a_n = a_1+a_2+\cdots+a_N.$$

The series $\sum a_n$ is said to be**convergent** provided the sequence of partial sums $\{S_N\}$ converges and in this case, the value of the series is

$$\sum_{i=1}^{\infty} a_n = L \iff \lim_{N\to\infty} S_N = L.$$

The series is said to be **divergent** if it is not convergent, i.e. $\{S_N\}$ is divergent.

□

Example - Harmonic Series Diverges The series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, since

$$\sum_{n=1}^{\infty}\frac{1}{n} = 1+\frac{1}{2}+\underbrace{\frac{1}{3}+\frac{1}{4}}_{>\frac{1}{2}}+\underbrace{\frac{1}{5}\cdots+\frac{1}{8}}_{>\frac{1}{2}}+\underbrace{\frac{1}{9}\cdots+\frac{1}{16}}_{>\frac{1}{2}}+\underbrace{\frac{1}{16}\cdots+\frac{1}{32}}_{>\frac{1}{2}}+\cdots$$

tells us that we are adding infinitely many terms all greater than $\frac{1}{2}$. □

Theorem - Geometric Series Test. A Geometric series is one that can be written in the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a+ar+ar^2+ar^3+\cdots \text{ with } a\neq 0.$$

This series is convergent if $|r|<1$ with

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \text{ when } |r|<1$$

and it is divergent if $|r|>1$. (Notice that we must write the series in the form given in order to find the sum.)

Proof. Let S_N denote the N^{th} partial sum.

$$S_N = \sum_{n=1}^{N} ar^{n-1}$$

We need to study $\lim_{n\to\infty} S_N$. Certainly, if $r = 1$, then $S_N = \sum_{i=1}^{N} ar^{n-1} = \sum_{i=1}^{N} a = Na$ and $\lim_{n\to\infty} S_N = \pm\infty$. Now, we assume that $r \neq 1$ Then

$$\begin{aligned}
S_N &= \sum_{n=1}^{N} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots + ar^{N-1} \\
rS_N &= r\sum_{n=1}^{N} ar^{n-1} = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^N
\end{aligned}$$

$$\begin{aligned}
S_N - rS_N &= (a + ar + ar^2 + ar^3 + \cdots + ar^{N-1}) - (ar + ar^2 + ar^3 + ar^4 + \cdots + ar^N) \\
&= (a - ar^N)
\end{aligned}$$

$$(1-r)S_N = (a - ar^N) \implies S_N = \frac{a - ar^N}{1-r} = \frac{a(1-r^N)}{1-r}$$

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{a(1-r^N)}{1-r} \begin{cases} \frac{a}{1-r} & \text{if } |r| < 1 \\ \text{divergent} & \text{if } |r| > 1 \end{cases}$$

We are using the **Theorem.** The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r. Moreover,

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

□

Theorem - n^{th}- Term Test for Divergence If $\lim\limits_{n\to\infty} a_n$ either does not exist or does not equal 0, then the series

$$\sum_{n=1}^{\infty} a_n \text{ is divergent.}$$

Remark 1. This is a test for divergence only and

$$\lim_{n\to\infty} a_n = 0 \text{ does } \textbf{not} \text{ imply that } \sum_{n=1}^{\infty} a_n \text{ is convergent.}$$

As can be seen from the Harmonic Series:

$$\underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{\text{divergent}} \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} = 0.$$

Remark 2. If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

Recall: The p-Test for Improper Integrals.

$$\int_1^{\infty} \frac{1}{x^p}\, dx = \begin{cases} \text{diverges, if } p \le 1 \\ \text{converges, if } p > 1 \end{cases}$$

Theorem (Integral Test). If the tail of a sequence $\{a_n\}$ can be given by the values of a function $f(x)$ which is continuous, positive, and decreasing on $[a, \infty)$ then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \quad \Longleftrightarrow \quad \int_a^{\infty} f(x)\, dx \text{ converges.}$$

That is:

(i) If $\sum_{n=1}^{\infty} a_n$ is convergent , then $\int_a^{\infty} f(x)\, dx$ is convergent.

(ii) If $\int_a^{\infty} f(x)\, dx$ is converges, then $\sum_{n=1}^{\infty} a_n$ is convergent.

Theorem (p -Test for Series). The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \le 1$.

□

We adopt the convention that $(x-a)^0 = 1$.

Definition. A series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

is called **a power series in** $(x-a)$ **or a power series about** a **or a power series centered at** a **or a power series near** a. The constants c_n are called the coefficients of the series.

A power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ centered at a is often called a **Taylor Series** and when $a = 0$ it is called a **Maclaurin Series** of the function $f(x)$ provided

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n.$$

Definition. The series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ is called the **Taylor Series of** $f(x)$ **centered at** a**.** When $a = 0$, it is often called a **Maclaurin Series**. $\square$

Some functions and their **Maclaurin Series** are (Memorize them):

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 \cdots \qquad \text{with } -1 < x < 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} \cdots \qquad \text{with } -1 < x \leq 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad \text{with } -\infty < x < \infty$$

$$\begin{aligned} (1+x)^k &= \sum_{n=0}^{\infty} \binom{k}{n} x^n \\ &= 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad \text{with } -1 < x < 1 \end{aligned}$$

10 Functions of Several Variables

Recall: A function f is a rule that assigns to each element in a set $D \subseteq \mathbb{R}$ exactly one element, called $f(x)$, in a set $E \subseteq \mathbb{R}$.

Example. $f(x) = x^2, x \in D = [-1, 1]$.

Definition. A **function of two variables**, defined on D in the plane ($\mathbb{R}^2$), is a rule f that assigns to each ordered pair of real numbers (x, y) in D a unique real number denoted by $f(x, y)$. The set D is the **domain** of f and its **range** is the set of values that f takes on D, that is, $\{f(x, y)|(x, y) \in D\}$.

Example. $f(x, y) = x^2 + y^2, D = \{(x, y) : x^2 + y^2 \leq 1\}$.

Note. If D is not specified, we take $D = \{\text{all points for which } f \text{ is meaningful}\}$.

Definition. If f is a function of two variables with domain D, then the **graph** of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$, where $(x, y) \in D$.

Example. $f(x, y) = x^2 + y^2, D = \{(x, y) : x^2 + y^2 \leq 1\}$.

Recall: $f^{'}(x) = \frac{df(x)}{dx} = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$ – Instantaneous rate of change of f with respect to x.

Example. $f(x) = 3x^2 + 2\sin x + 4$ and $f^{'}(x) = 6x + 2\cos$.

Q. How do we differentiate $f(x, y)$?

Definition (Partial Derivatives)**.** The partial derivatives of $f(x,y)$(with respect to x and with respect to y) are the two functions defined, respectively, by

$$\frac{\partial f(x,y)}{\partial x} = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h},$$
$$\frac{\partial f(x,y)}{\partial y} = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$

whenever these limits exist.

Notations. If $z = f(x,y)$,

$$\frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = f_x(x,y) = D_x[f(x,y)] = D_1[f(x,y)],$$
$$\frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = f_y(x,y) = D_y[f(x,y)] = D_2[f(x,y)]$$

Rules.

- To calculate $\frac{\partial f(x,y)}{\partial x}$, simply regard y as a constant and differentiate w.r.t. x;
- To calculate $\frac{\partial f(x,y)}{\partial y}$, simply regard x as a constant and differentiate w.r.t. y.

Example. Compute both the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ of $f(x,y) = x^2 - 4xy + y$.

Solution.

$$\frac{\partial f(x,y)}{\partial x} = 2x - 4y + 0.$$
$$\frac{\partial f(x,y)}{\partial y} = 0 - 4x + 1.$$

□

Recall: The single-variable chain rule expresses the derivative of a composite function $f(g(t))$ in terms of the derivatives of f and g:

$$D_t[f(g(t))] = f^{'}(g(t)) \cdot g^{'}(t).$$

If we write $w = f(x), x = g(t)$, we have

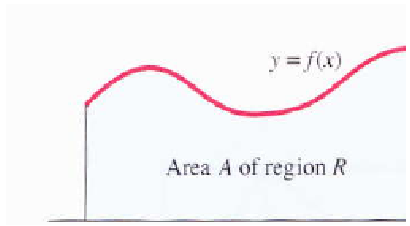
$$\frac{dw}{dt} = \frac{dw}{dx}\cdot\frac{dx}{dt}.$$

Theorem (The Chain Rule)**.** Suppose that $w = f(x,y)$ has continuous first-order partial derivatives and that $x = g(t)$ and $y = h(t)$ are differentiable functions. Then w is a differentiable function of t, and

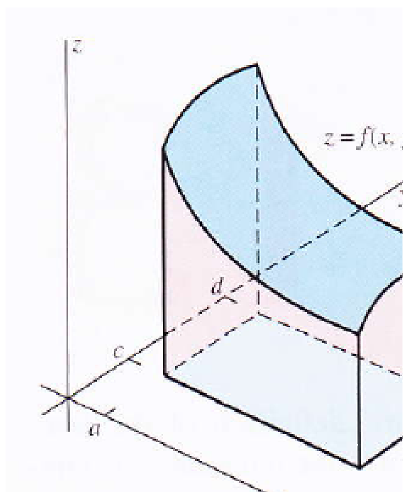
$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$

11 Double Integrals over Rectangles

Recall: Suppose $f(x) \geq 0$,



Now,



Volume of the solid bounded above by the graph $z = f(x, y)$ of the nonnegative function f over the rectangle R in the x-y plane is given by

$$V = \int\int_R f(x, y) dA,$$

where dA represents a differential element of area A.

Theorem (Double Integrals as Iterated Single Integrals)**.** Suppose that $f(x,y)$ is continuous on the rectangle $R:\ [a,b]\times[c,d]$. Then

$$\int\int_R f(x,y)dA = \int_a^b \left(\int_c^d f(x,y)dy\right)dx = \int_c^d \left(\int_a^b f(x,y)dx\right)dy$$

This theorem tells us how to compute a double integral by means of two successive (or iterated) single-variable integrations, each of which we can compute by using the fundamental theorem of calculus (if the function f is sufficiently well behaved on R).

Example. Evaluate the integral

$$\int\int_R (4x^3+6xy^2)dA$$

over the rectangle $R:\ [1,3]\times[-2,1]$.

Solution.

$$\begin{aligned}\int\int_R (4x^3+6xy^2)dA &= \int_1^3 \left[\int_{-2}^1 (4x^3+6xy^2)dy\right]dx\\ &= \int_1^3 \left[4x^3(1+2)+6x\int_{-2}^1 y^2dy\right]dx\\ &= \int_1^3 \left[12x^3+6x\cdot\frac{y^3}{3}\Big|_{-2}^1\right]dx\\ &= \int_1^3 \left[12x^3+18x\right]dx\\ &= \frac{12x^4}{4}\Big|_1^3+\frac{18x^2}{2}\Big|_1^3\\ &= 3(3^4-1)+9(3^2-1)=312.\end{aligned}$$

Or you can do the following

$$\int\int_R (4x^3+6xy^2)dA = \int_{-2}^1 \left[\int_1^3 (4x^3+6xy^2)dx\right]dy = \cdots\cdots$$

$\square$

12 Double Integrals Over More General Regions

The Double Integral of a bounded function f over a more general plane region R can be defined similarly.

Evaluation of Double Integrals

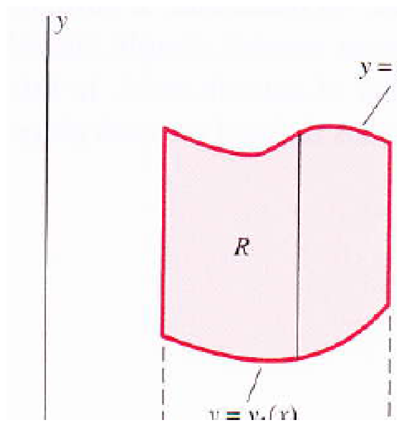


Figure 1: A **vertically simple region** R

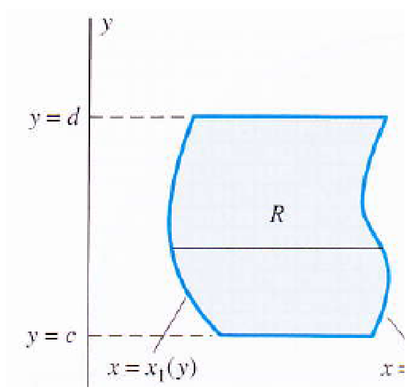


Figure 2: A **horizontally simple region** R

Theorem (Evaluation of Double Integrals)**.** Suppose that $f(x,y)$ is continuous on the region R. If R is the **vertically simple region** given by

$a \le x \le b, y_1(x) \le y \le y_2(x),$ where $y_1(x), y_2(x)$ are continuous functions of x on $[a,b]$,

then

$$\int\int_R f(x,y)\,dA = \int_a^b \left[\int_{y_1(x)}^{y_2(x)} f(x,y)\,dy\right] dx.$$

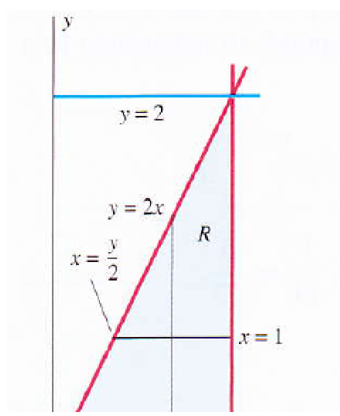
If R is the **horizontally simple region** given by

$c \le y \le d, x_1(y) \le x \le x_2(y),$ where $x_1(y), x_2(y)$ are continuous functions of y on $[c,d]$,

then

$$\int\int_R f(x,y)\,dA = \int_c^d \left[\int_{x_1(y)}^{x_2(y)} f(x,y)\,dx\right] dy.$$

Example. R is given in the following graph. Find $\int\int_R x dA$.



Solution.

Method 1. The integration region R can be written as

$$R = \{(x,y) : 0 \le x \le 1, 0 \le y \le 2x\}.$$

Therefore,

$$\begin{aligned}\int\int_R dA &= \int_0^1 \left[\int_0^{2x} x dy\right] dx \\ &= \int_0^1 x(2x-0)dx = \int_0^1 2x^2 dx = \frac{2x^3}{3}\Big|_0^1 = \frac{2}{3}.\end{aligned}$$

Method 2. The integration region R can be written as

$$R = \{(x, y) : 0 \le y \le 2, \frac{y}{2} \le x \le 1\}.$$

Therefore,

$$\begin{aligned}\int\int_R dA &= \int_0^2 \left[\int_{y/2}^1 x dx\right] dy \\ &= \int_0^2 \left[\frac{x^2}{2}\right]_{y/2}^1 dy = \int_0^2 \left[\frac{1}{2} - \frac{y^2}{8}\right] dy \\ &= \left[\frac{y}{2} - \frac{y^3}{3 \times 8}\right]_0^2 = \frac{1}{2} \times 2 - \frac{2^3}{24} = 1 - \frac{1}{3} = \frac{2}{3}.\end{aligned}$$

$\square$

For more examples, check your Calculus III text or class notes.

Properties of Double Integrals

Let c be a constant and f and g be continuous functions on a region R on which $f(x, y)$ attains a minimum value m and a maximum value M. Let $a(R)$ denote the area of the region R. If all the indicated integrals exist, then:

- $\int\int_R cf(x, y)\ dA = c\int\int_R f(x, y)\ dA$
- $\int\int_R [f(x, y) + g(x, y)]\ dA = \int\int_R f(x, y)\ dA + \int\int_R g(x, y)\ dA$
- $m \cdot a(R) \le \int\int_R f(x, y)\ dA \le M \cdot a(R)$
- $\int\int_R f(x, y)\ dA = \int\int_{R_1} f(x, y)\ dA + \int\int_{R_2} f(x, y)\ dA$, where R_1 and R_2 are simply two non-overlapping regions (with disjoint interiors) with union R.

13 Change of Variables in Double Integrals

Recall:

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))g^{'}(u)\,du,$$

where $a = g(c)$ and $b = g(d)$ (or $c = g^{-1}(a)$ and $d = g^{-1}(b)$). The method of substitution involves a **change of variables** that is tailored to the evaluation of a given integral. In the above, the transformation is $x = g(u)$.

Example. Find $\int_1^2 (1+\frac{1}{2}x)^3\,dx$.

Solution.

Let $u = 1 + \frac{1}{2}x$. Then

$$x = g(u) = 2(u-1) \text{ and } u = g^{-1}(x) = 1 + \frac{1}{2}x.$$

Thus

$$g^{-1}(1) = \frac{3}{2}, \quad g^{-1}(2) = 2.$$

$$\begin{aligned}\int_1^2 (1+\frac{1}{2}x)^3\,dx &= \int_{3/2}^2 g^{'}(u)\,du \\ &= \int_{3/2}^2 u^3 2du = \frac{2u^4}{4}\Big|_{3/2}^2 = \frac{2^4}{2} - \frac{(3/2)^4}{2} = 8 - \frac{81}{32}.\end{aligned}$$

□

Consider a continuously differentiable transformation

$$T : \mathbb{R}^2_{uv} \to \mathbb{R}^2_{xy}.$$

Namely,

$$T : \begin{cases} x = g_1(u,v) \\ y = g_2(u,v) \end{cases}, \quad \frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v} \text{ exist and continuous.}$$

Definition (The Jacobian)**.** The Jacobian of the transformation T is defined by

$$J_T(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \triangleq \frac{\partial(x,y)}{\partial(u,v)} \ (\textbf{notation}).$$

If T is one to one, then the inverse transformation T^{-1} exists. Write

$$T^{-1}: \begin{cases} u = h_1(x, y) \\ v = h_2(x, y) \end{cases}$$

Formula: $\boxed{\dfrac{\partial(x,y)}{\partial(u,v)} \cdot \dfrac{\partial(u,v)}{\partial(x,y)} = 1}$. That is,

$$\boxed{\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}}.$$

Theorem (Change of Variables)**.** Assume $T : \mathbb{R}^2_{uv} \to \mathbb{R}^2_{xy}$, $S(bounded) \longmapsto R(bounded)$ is one-to-one from the interior of S to the interior of R. If $f(x, y)$ is continuous on R, then

$$\int\int_R f(x,y)\, dxdy = \int\int_S f(T(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \, dudv$$

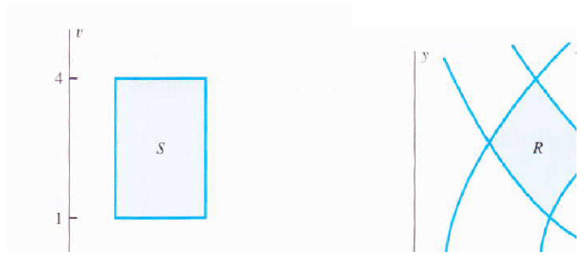
Remark. The theorem of change-of-variables can be generalized to triple integrals and multiple integrals.

Example. Suppose that R is the plane region of unit density that is bounded by the hyperbolas

$$xy = 1, xy = 3 \text{ and } x^2 - y^2 = 1, x^2 - y^2 = 4.$$

Find $\int\int_R (x^2 + y^2)\, dxdy$.

Solution.



The hyperbolas bounding R are u-curves and v-curves if

$$u = xy, \quad v = x^2 - y^2.$$

We can most easily write the integrand $x^2 + y^2$ in terms of u and v by noting that

$$4u^2 + v^2 = 4x^2y^2 + (x^2 - y^2)^2 = (x^2 + y^2)^2.$$

Therefore, $x^2 + y^2 = \sqrt{4u^2 + v^2}$.

Now,

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ 2x & -2y \end{vmatrix} = -2(x^2 + y^2).$$

Therefore,

$$\frac{\partial(x, y)}{\partial(u, v)} = 1 \Big/ \left(\frac{\partial(u, v)}{\partial(x, y)} \right) = \frac{-1}{2(x^2 + y^2)} = \frac{-1}{2\sqrt{4u^2 + v^2}}.$$

We are now ready to apply the change-of-variables theorem, with the regions S and R as shown in the figure.

$$\begin{aligned} \int\int_R (x^2 + y^2)\, dxdy &= \int_1^4 \int_1^3 \sqrt{4u^2 + v^2} \left| \frac{-1}{2\sqrt{4u^2 + v^2}} \right| dudv \\ &= \int_1^4 \int_1^3 \sqrt{4u^2 + v^2} \frac{1}{2\sqrt{4u^2 + v^2}}\, dudv = \int_1^4 \int_1^3 \frac{1}{2}\, dudv = 3. \end{aligned}$$

$\square$