§12.1 Functions of Two Variables

The Euclidean 3-space. The Euclidean 3-space denoted by $\mathbb{R}^3$ is the set $\{(x, y, z)|x, y, z \in \mathbb{R}\}$. To specify the location of a point in $\mathbb{R}^3$ geometrically, we use a **right-handed** rectangular coordinate system, in which three mutually perpendicular coordinate axes meet at the origin. It is common to use the x and y axes to represent the horizontal coordinate plane and the z-axis for the vertical height.

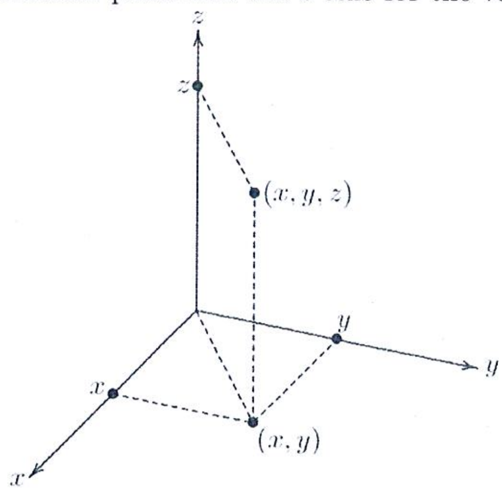


Figure 12.1: A right-handed coordinate system

Example 4:(SageMath) Describe the position of the points with coordinates (1, 2, 3) and (0, 0,-1).

i) 1 unit along the x-axis, 2 units in the direction parallel to the y-axis and 3 units up in the direction parallel to the z-axis.

ii) 1 unit in the negative z-direction

Recall: A function f is a rule that assigns to each element in a set $D \subseteq \mathbb{R}$ exactly one element, called $f(x)$, in a set $E \subseteq \mathbb{R}$. For a real-valued function $f : D \to \mathbb{R}$ defined on a subset D of $\mathbb{R}$, the graph of f consists of all the points $(x, f(x))$ in the xy-plane.

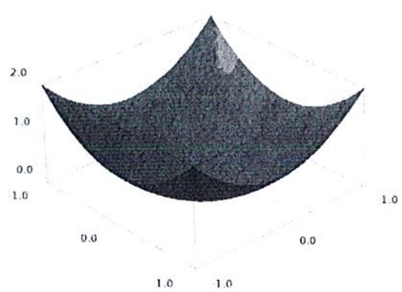
Example.(SageMath) $f(x) = x^2, x \in D = [-1, 1]$.

Definition. *A **function of two variables**, defined on D in the plane ($\mathbb{R}^2$), is a rule f that assigns to each ordered pair of real numbers (x, y) in D a unique real number denoted by $f(x, y)$. The set D is the **domain** of f and its **range** is the set of values that f takes on D, that is, $\{f(x, y)|(x, y) \in D\}$.*

*A **function of three variables**, defined on D in the space ($\mathbb{R}^3$), is a rule f that assigns to each point (x, y, z) in D a unique real number denoted by $f(x, y, z)$. The set D is the **domain** of f and its **range** is the set of values that f takes on D, that is, $\{f(x, y, z)|(x, y, z) \in D\}$.*

Example.(SageMath) $f(x, y) = x^2 + y^2$, $\cancel{D = \{(x, y) : x^2 + y^2 \le 1}\}$.

$D = \{(x,y), -1 \le x, y \le 1\}$



Note. If D is not specified, we take $D = \{\text{all points for which } f \text{ is meaningful}\}$.
Example. $f(x, y, z) = x^2 + y^2 - z$, $D = \mathbb{R}^3$.

Example: State the largest possible domain of definition of the given function f.

(a) $f(x, y) = \frac{1}{x-y}$.

$x - y \neq 0 \Rightarrow x \neq y$

$\Rightarrow D = \{(x, y) \mid (x, y) \in \mathbb{R}^2,\ x \neq y\}$

(b) $f(x, y) = \sqrt{2x} + \sqrt[3]{3y}$.

$2x \geq 0 \Rightarrow x \geq 0$

$\Rightarrow D = \{(x, y) \mid x \geq 0\}$

(c) $f(x, y) = \ln(x^2 - y^2 - 1)$.

$x^2 - y^2 - 1 > 0 \Rightarrow x^2 - y^2 > 1$

$D = \{(x, y) \mid x^2 - y^2 > 1\}$

(d) $f(x, y, z) = \frac{1}{\sqrt{z-x^2-y^2}}$.

$z - x^2 - y^2 > 0 \Rightarrow z > x^2 + y^2$

$D = \{(x, y, z) \mid (x, y, z) \in \mathbb{R}^3 \text{ and } z > x^2 + y^2\}$

Algebraic Examples: Formulas

Example 3: A cylinder with closed ends has radius r and height h. If its volume is V and its surface area is A, find formulas for the functions $V = f(r, h)$ and $A = g(r, h)$.

$V = f(r,h) =$ Area of base. Height $= \pi r^2 h$

$A = g(r,h) = 2$ Area of base + Area of side $= 2\pi r^2 + 2\pi rh$

Graphing Equations in 3-Space

Example 6.(SageMath) What do the graphs of the equations $z = 0$, $z = 3$, and $z = -1$ look like? These are equations of planes parallel to the xy-plane.

Example 8. You are 2 units below the xy-plane and in the yz-plane.What are your coordinates?

$z = -2$ $\quad$ $x = 0$

$(0, y, -2)$

Example 9. You are standing at the point $(4, 5, 2)$, looking at the point $(0.5, 0, 3)$. Are you looking up or down?

z-coordinate 2 versus 3. we are looking up

Distance Between Two Points

The distance $|P_1P_2|$ between two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is given by

$$Distance = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

Example 11. Find the distance between $(1, 2, 1)$ and $(-3, 1, 2)$.

$$d = \sqrt{(1+3)^2 + (2-1)^2 + (1-2)^2} = \sqrt{16+1+1} = \sqrt{18} = 3\sqrt{2}$$

Example 12. Find an expression for the distance from the origin to the point (x, y, z).

$$d = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = \sqrt{x^2+y^2+z^2}$$

Example 13. Find an equation for a sphere of radius 1 with center at the origin.

The sphere consists of all points (x, y, z) whose distance from the origin is 1. $\Rightarrow x^2+y^2+z^2=1$

§12.2 GRAPHS AND SURFACES

Definition. *If f is a function of two variables with domain D, then the* **graph** *of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$, where $(x, y) \in D$.*

Example: Sketch the graph $f(x, y) = x^2 + y^2$.

(see page 2)

Example 1: Let $f(x, y) = x^2 + y^2$. Describe in words the graphs of the following functions:

(a) $g(x, y) = x^2 + y^2 + 3$

(b) $h(x, y) = 5 - x^2 - y^2$

(c) $k(x, y) = x^2 + (y - 1)^2$

(a) Shifting $f(x,y)$ up by 3 units

(b) reflecting $f(x,y)$ over the xy-plane and shift up by 5 units

(c) Shift right by 1 unit in the direction of y-axis

Example 2:(SageMath) Describe the graph of $G(x, y) = e^{-(x^2+y^2)}$. What symmetry does it have?

i) $G(x,y) > 0$, it is above the xy-plane.

ii) $x^2+y^2=0$ when $(x=0, y=0)$ and $G(0,0)=1$ which is the maximum of $G(x,y)$;

When (x,y) moves further from the origin, $G(x,y) \downarrow \to 0$

The surface is asymptotic to the xy-plane

iii) Consider $x^2+y^2=r^2 \Rightarrow G(x,y) = e^{-r^2}$

$G(x,y)$ is the same at all points on the circle.

$G(x,y)$ has circular symmetry.

Cross-Sections and the Graph of a Function

Definition. *For a function $f(x, y)$, the function we get by holding x fixed and letting y vary is called a cross-section of f with x fixed. The graph of the cross-section of $f(x, y)$ with $x = c$ is the curve, or cross-section, we get by intersecting the graph of f with the plane $x = c$.*

We define a cross-section of f with y fixed similarly.

Example. For example, the cross-section of $f(x, y) = x^2 + y^2$ with $x = 2$ is $f(2, y) = 4 + y^2$. The graph of this cross-section is the curve we get by intersecting the graph of f with the plane perpendicular to the x-axis at $x = 2$. (See Figure 12.18.)

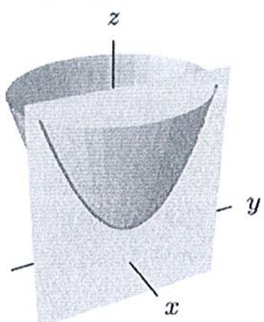


Figure 12.18: Cross-section of the surface $z = f(x, y)$ by the plane $x = 2$

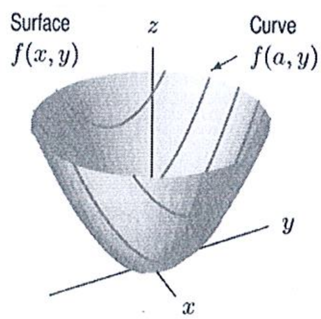


Figure 12.19: The curves $z = f(a, y)$ with a constant: cross-sections with x fixed

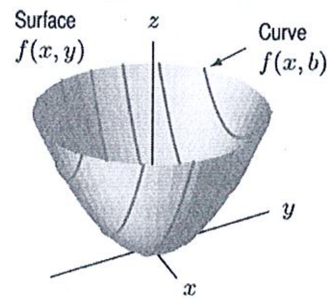


Figure 12.20: The curves $z = f(x, b)$ with b constant: cross-sections with y fixed

i) when x is fixed $x = c$, $z = c^2 + y^2$

ii) when y is fixed $y = c$, $z = x^2 + c^2$

Example 3.(SageMath) Describe the cross-sections of the function $g(x,y) = x^2 - y^2$ with y fixed and then with x fixed. Use these cross-sections to describe the shape of the graph of g. (See Figure 12.21.)

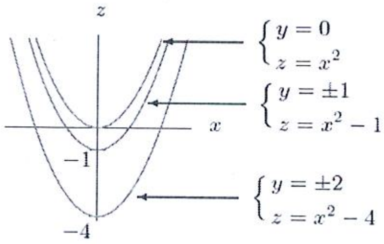


Figure 12.21: Cross-sections of $g(x,y) = x^2 - y^2$ with y fixed

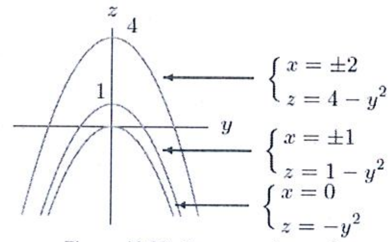


Figure 12.22: Cross-sections of $g(x,y) = x^2 - y^2$ with x fixed

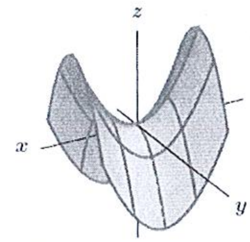


Figure 12.23: Graph of $g(x,y) = x^2 - y^2$ showing cross sections

i) When y is fixed, $y = b$

$$z = g(x,b) = x^2 - b^2$$

$(x^2 - z - b^2 = 0)$ which is a parabola

ii) when x is fixed, $x = a$

$$z = g(a,y) = a^2 - y^2$$

which is a parabola open downward in the zy-plane

Linear Functions

Linear functions are central to single-variable calculus; they are equally important in multivariable calculus. You may be able to guess the shape of the graph of a linear function of two variables. (Its a plane.) Lets look at an example.

Example 4. (SageMath) Describe the graph of $f(x, y) = 1 + x\cancel{y}. -y$

Solution The plane $x = a$ is vertical and parallel to the yz-plane. Thus, the cross-section with $x = a$ is the line $z = 1 + \cancel{a}y$ which slopes downward in the y-direction. Similarly, the plane $y = b$ is parallel to the xz-plane. Thus, the cross-section with $y = b$ is the line $z = 1 + \cancel{x}b$ which slopes upward in the x-direction. Since all the cross-sections are lines, you might expect the graph to be a flat plane, sloping down in the y-direction and up in the x-direction. This is indeed the case.

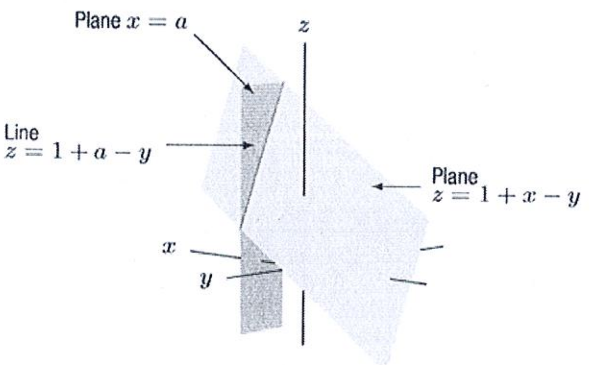


Figure 12.24: Graph of the plane $z = 1 + x - y$ showing cross-section with $x = a$

When One Variable is Missing: Cylinders

Example [illegible]. (SageMath) Graph the equation $z = x^2$ in 3-space.

i) y is missing $\Rightarrow$ cross-sections with y-fixed are all the same

ii) cross-sections with x-fixed are horizontal lines (in planes parallel to the zy-plane)

It is a trough-shaped surface.

Example 5. (SageMath) Graph the equation $x^2 + y^2 = 1$ in 3-space.

z is missing, $\Rightarrow$ crosssections with z-fixed are circles $x^2 + y^2 = 1$ $\Rightarrow$ The surface is a cylinder.

§12.3 CONTOUR DIAGRAMS

Definition. *The* ***Contour lines*** *or* ***level curves*** *of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant in the range of f.*

That is, Contour lines, or level curves, are obtained from a surface by slicing it with horizontal planes. A contour diagram is a collection of level curves labeled with function values.

Example 3.(SageMath) Find equations for the contours of $f(x, y) = x^2 + y^2$ and draw a contour diagram for f. Relate the contour diagram to the graph of f.

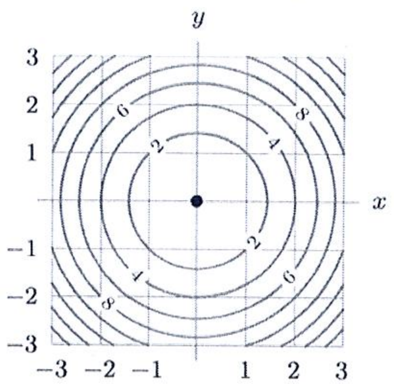


Figure 12.39: Contour diagram for $f(x, y) = x^2 + y^2$ (even values of c only)

bowl-shaped surface.

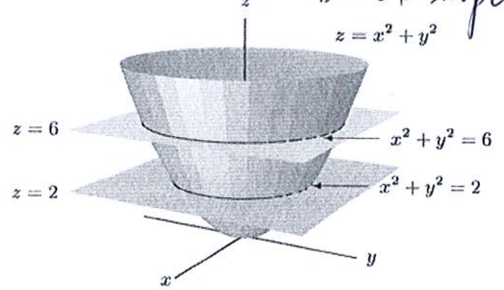


Figure 12.40: The graph of $f(x, y) = x^2 + y^2$

$x^2 + y^2 = c, \quad c \geq 0$ $\quad \begin{cases} c = 1, 2, 3, 4 \\ r = \sqrt{c} = 1, \sqrt{2}, \sqrt{3}, 2 \end{cases}$

which is a circle of radius $\sqrt{c}$. (more closely packed)

f gets steeper as we move further from the origin.

Example 4. Draw a contour diagram for $f(x, y) = \sqrt{x^2 + y^2}$ and relate it to the graph of f.

$c = \sqrt{x^2+y^2} \Rightarrow x^2+y^2 = c^2 \quad (c \geq 0)$

$c \nearrow 1 \Rightarrow$ radius $r \nearrow 1$ $\quad \begin{cases} c = 1, 2, 3, 4 \ldots \\ r = c = 1, 2, 3, 4 \ldots \end{cases}$

$\Rightarrow$ The contours are equally spaced concentric circles.

It is a cone rather than a bowl.

Example 5. Draw a contour diagram for $f(x, y) = 2x + 3y + 1$.

$c = 2x + 3y + 1 \Rightarrow y = -\frac{2}{3}x + (c-1)/3$ which are parallel lines with slope $-2/3$ $(c \nearrow 3,\ y \nearrow 1)$

§12.4 LINEAR FUNCTIONS

Linear functions played a central role in one-variable calculus because many one-variable functions have graphs that look like a line when we zoom in. In two-variable calculus, a linear function is one whose graph is a plane.

What Makes a Plane Flat? What makes the graph of the function $z = f(x, y)$ a plane? Linear functions of one variable have straight line graphs because they have constant slope. On a plane, the situation is a bit more complicated. If we walk around on a tilted plane, the slope is not always the same: it depends on the direction in which we walk. However, at every point on the plane, the slope is the same as long as we choose the same direction. If we walk parallel to the x-axis, we always find ourselves walking up or down with the same slope; the same is true if we walk parallel to the y-axis. In other words, the slope ratios $\Delta z/\Delta x$ (with y fixed) and $\Delta z/\Delta y$ (with x fixed) are each constant.

Theorem. *If a plane has slope m in the x-direction, has slope n in the y-direction, and passes through the point (x_0, y_0, z_0), then its equation is*

$$z = z_0 + m(x - x_0) + n(y - y_0).$$

This plane is the graph of the linear function

$$f(x, y) = z_0 + m(x - x_0) + n(y - y_0).$$

If we write $c = z_0 - mx_0 - ny_0$, then we can write $f(x, y)$ in the equivalent form

$$f(x, y) = c + mx + ny.$$

Note. Just as in 2-space a line is determined by two points, so in 3-space a plane is determined by **three points**, provided they do not lie on a line.

Example.(SageMath) Draw the graph of $z = 5 + 2x\cancel{y} - y$.

Example 2. Find the equation of the plane passing through the points $(1, 0, 1)$, $(1, -1, 3)$, and $(3, 0, -1)$.

$$\begin{cases} m = \Delta z / \Delta x = \dfrac{1-(-1)}{1-3} = -1 & \text{using } (1,0,1) \text{ and } (3,0,-1) \\ n = \dfrac{\Delta z}{\Delta y} = \dfrac{1-3}{0-(-1)} = -2 & \text{using } (1,0,1) \text{ and } (1,-1,3) \end{cases}$$

$z = -x - 2y + c$ and $c = 2$ using the point $(1, 0, 1)$

$\Rightarrow z = 2 - x - 2y$

Linear Functions from a Numerical Point of View

A linear function can be recognized from its table by the following features:

- Each row and each column is linear.
- All the rows have the same slope.
- All the columns have the same slope (although the slope of the rows and the slope of the columns are generally different).

Example 3. The table contains values of a linear function. Fill in the blank and give a formula for the function.

$x \backslash y$	1.5	2.0
2	0.5	1.5
3	−0.5	? 0.5

$0.5 \searrow -0.5$ as x $2 \nearrow 3$

$\Rightarrow ? = 1.5 - 1 = 0.5$

$m = \dfrac{0.5-(-0.5)}{2-3} = \dfrac{1}{-1} = -1$ and

$n = \dfrac{0.5-1.5}{1.5-2} = \dfrac{-1}{-0.5} = 2$

$\Rightarrow z = -x + 2y + c$

use the point $(2, 2, 1.5) \Rightarrow c = -0.5$

$\Rightarrow z = -0.5 - x + 2y$

What Does the Contour Diagram of a Linear Function Look Like?

Example 4.(SageMath) Find the equation of the linear function whose contour diagram is in Figure 12.66.

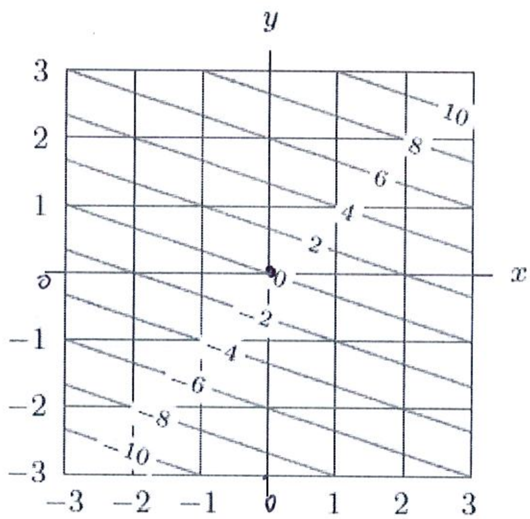


Figure 12.66: Contour map of linear function $f(x, y)$

$$m = \frac{-10-(-8)}{-1-1} = \frac{-2}{-2} = +1$$

$$n = \frac{-6-0}{-1-1} = \frac{-6}{-2} = 3$$

$$\Rightarrow z = x + 3y + c$$

Use the point $(0, 0, 0) \Rightarrow c = 0$

$$\Rightarrow f(x, y) = x + 3y$$

Note. Notice also that the contours are evenly spaced. Thus, no matter which contour we are on, a fixed increase in one of the variables causes the same increase in the value of the function.

§12.5 FUNCTION OF THREE VARIABLES

In applications of calculus, functions of any number of variables can arise. One difficulty with functions of more than two variables is that it is hard to visualize them since the graph of a function of three variables would be a solid in 4-space. Since we cant easily visualize 4-space, we wont use the graphs of functions of three variables. On the other hand, it is possible to draw contour diagrams for functions of three variables, only now the **contours are surfaces** in 3-space.

Definition (Level Surface). *A* ***level surface****, or level set of a function of three variables, $f(x, y, z)$, is a surface of the form $f(x, y, z) = c$, where c is a constant. The function f can be represented by the family of level surfaces obtained by allowing c to vary.*

Example 1. The temperature, in °C, at a point (x, y, z) is given by $T = f(x, y, z) = x^2 + y^2 + z^2$. What do the level surfaces of the function f look like and what do they mean in terms of temperature? (SageMath)

Solution.

The level surface corresponding to $T = 100$ is

$$x^2 + y^2 + z^2 = 100$$

which is a sphere with radius $r = \sqrt{100} = 10$

The temperature is a constant on each sphere.

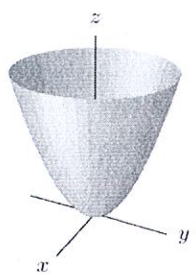
The level surfaces become more closely spaced as we move farther from the origin.

$$\begin{cases} T = 1, 2, 3, 4 \ldots \\ r = 1, \sqrt{2}, \sqrt{3}, 2 \ldots \end{cases}$$

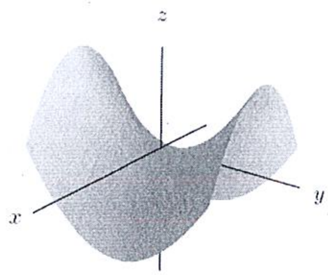
□

A Catalog of Surfaces

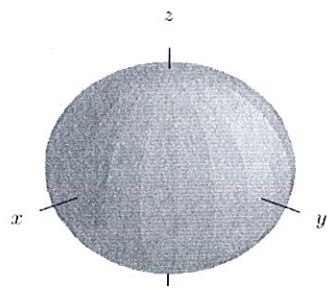
For later reference, here is a small catalog of the surfaces we have encountered.



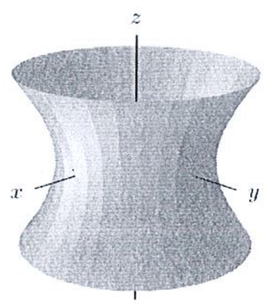
Elliptical paraboloid $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$



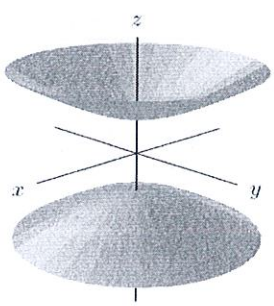
Hyperbolic paraboloid $z = -\frac{x^2}{a^2} + \frac{y^2}{b^2}$



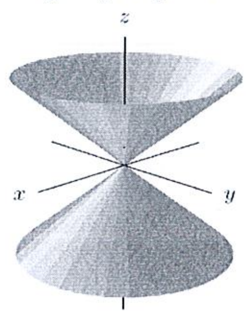
Ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$



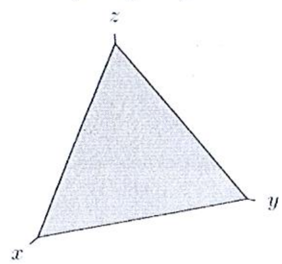
Hyperboloid of one sheet $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$



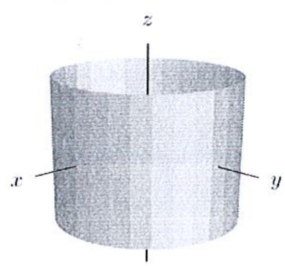
Hyperboloid of two sheets $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$



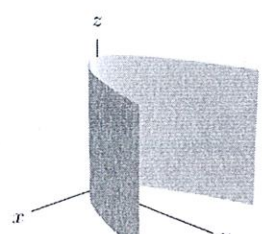
Cone $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$



Plane $ax + by + cz = d$



Cylindrical surface $x^2 + y^2 = a^2$



Parabolic cylinder $y = ax^2$

(These are viewed as equations in three variables x, y, and z)

How Surfaces Can Represent Functions of Two Variables and Functions of Three Variables

You may have noticed that we have used surfaces to represent functions in two different ways.

- First, we used a **single surface** to represent a two-variable function $f(x, y)$.
- Second, we used **a family of level surfaces** to represent a three-variable function $g(x, y, z)$. These level surfaces have equation $g(x, y, z) = c$.

A single surface that is the graph of a two-variable function $f(x, y)$ can be thought of as one member of the family of level surfaces representing the three-variable function

$$g(x, y, z) = f(x, y) - z.$$

The graph of f is the level surface $g = 0$.

Conversely, a single level surface $g(x, y, z) = c$ can be regarded as the graph of a function $f(x, y)$ if it is possible to solve for z.

§12.6 LIMITS AND CONTINUITY of Functions of Two Variables

Definition. *Let f be a function of two variables whose domain D includes points arbitrarily close to (a,b). Then we say that the* **limit of f(x,y) as (x, y) approaches (a, b) is L** *and we write*

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if } (x,y) \in D \text{ and } 0 < \sqrt{(x-a)^2+(y-b)^2} < \delta \text{ then } |f(x,y)-L| < \epsilon.$$

Remark. The definition says that the distance between $f(x,y)$ and L can be made arbitrarily small by making the distance from (x,y) to (a,b) sufficiently small (but not 0).

Examples:

(1) $\lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2} = 0.$

(2) $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} \overset{Let\ z=x^2+y^2}{=} \lim_{z\to 0} \frac{\sin z}{z} = 1.$

Definition. *A function f of two variables is called continuous at (a,b) if*

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

We say f is continuous on the domain D if f is continuous at every point in D.

Example: $\sqrt{x^2+y^2}$ is continuous at $(0,0)$.

Theorem (Limit Laws). *Suppose that c is a constant and the limits $\lim_{(x,y)\to(a,b)} f(x,y) = L$ and $\lim_{(x,y)\to(a,b)} g(x,y) = M$ exist, then*

1. $\lim_{(x,y)\to(a,b)} [f(x,y) \pm g(x,y)] = L \pm M$ *(Sum law)*
2. $\lim_{(x,y)\to(a,b)} [f(x,y)g(x,y)] = LM$ *(Product law)*
3. $\lim_{(x,y)\to(a,b)} cf(x,y) = cL$
4. $\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}$ *if $M \neq 0$ (Quotient law)*

Example: Show that $\lim_{(x,y)\to(a,b)} xy = ab$.

proof. $\lim_{(x,y)\to(a,b)} xy = [\lim_{(x,y)\to(a,b)} x] \cdot [\lim_{(x,y)\to(a,b)} y] = ab$

by the product law. □

Remark. In general, let $P(x,y) = \sum c_{ij}x^iy^j$, where c_{ij} are constants, i,j are non-negative integers and the sum involves finite sums. We have

$$\lim_{(x,y)\to(a,b)} P(x,y) \overset{\text{Sum law}}{=} \sum \lim_{(x,y)\to(a,b)} c_{ij}x^iy^j \overset{\text{Product law}}{=} \sum c_{ij}\left(\lim_{x\to a} x^i\right)\left(\lim_{y\to b} y^j\right)$$
$$= \sum c_{ij}a^ib^j = P(a,b)$$

That is, $\boxed{\text{Every polynomial in two (or more) variables is an everywhere continuous function.}}$

Examples:

(1) $\lim_{(x,y)\to(-1,2)} 2x^4y^2 - 7xy + 5 = 2(-1)^4 2^2 - 7\times(-1)(2) + 5 = 27$

(2) $\lim_{(x,y)\to(1,-2)} 3x^2 - 4xy + 5y^2 = 3 - 4(-2) + 5(-2)^2 = 31$

Theorem. *Any composition of continuous multivariate function is also a continuous function.*

Remark. Any finite combination involving sums, products, and compositions of the familiar elementary function is continuous, except possibly at points where the formula for the function is meaningless.

Example. Show that $f(x,y) = \begin{cases} \frac{\sin(x^2+y^2)}{x^2+y^2}, & \text{unless x=y=0;} \\ 1, & \text{if x=y=0.} \end{cases}$ is continuous everywhere in $\mathbb{R}^2$.

proof. First, $f(x,y)$ is continuous everywhere in $\mathbb{R}^2$ except possibly at the origin $(0,0)$.

But $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = 1 = f(0,0)$

$\Rightarrow f(x,y)$ is continuous at $(0,0)$ as well. □

Example. Evaluate $\lim_{(x,y)\to(1,2)} \left(e^{xy}\sin\frac{\pi y}{4} + xy\ln\sqrt{y-x}\right)$.

Solution. $f(x,y)$ is continuous at any point with $y>x$

$\Rightarrow \lim_{(x,y)\to(1,2)} f(x,y) = f(1,2) = e^2\sin(\frac{\pi}{2}) + 2\ln\sqrt{1} = e^2$

□

Example. Evaluate $\lim_{(x,y)\to(2,-1)} \ln\left(\frac{1+x+2y}{3y^2-x}\right)$.

Solution.

At $(2,-1)$ $f(x,y) = \ln\left(\frac{1+2-2}{3-2}\right) = \ln(1) = 0$

$f(x,y)$ is meaningful at $(2,-1)$ and hence continuous at $(2,-1)$

$\Rightarrow \lim_{(x,y)\to(2,-1)} f(x,y) = f(2,-1) = \ln(1) = 0$ □

Example. Show that $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$.

Solution. Let $(x,y) = (r\cos\theta, r\sin\theta)$ polar coordinates

$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \frac{r^2\cos\theta\sin\theta}{r} = \lim_{r\to 0} r\cos\theta\sin\theta = 0$

since $|\cos\theta\sin\theta| \le 1$ for all θ. □

Example. Show that $\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = 0$.

Solution. Use $(x,y) = (r\cos\theta, r\sin\theta)$

$\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = \lim_{r\to 0} \frac{r^3(\cos^3\theta-\sin^3\theta)}{r^2(\sin^2\theta+\cos^2\theta)} = \lim_{r\to 0} r\frac{\cos^3\theta-\sin^3\theta}{1}$

$= 0$ since $|\cos^3\theta - \sin^3\theta| \le 2$ for all θ □

Remark. The polar coordinate substitution method likely works if the numerator has higher degree than the denominator.

Example. Show that $\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2-y^2}$ does not exist.

Remark. The idea is to show that the function $f(x,y)$ approaches different values as $(x,y) \to (0,0)$ from different directions.

Solution. Let $y = mx$. $f(x,y) = \frac{2x(mx)}{x^2 - m^2x^2} = \frac{2m}{1-m^2}$ (if $x \neq 0$)

$\Rightarrow f(x,y) \to \frac{2(\frac{1}{2})}{1-(\frac{1}{2})^2} = \frac{4}{3}$ as $(x,y) \to (0,0)$ along $y = \frac{1}{2}x$

$f(x,y) \longrightarrow \frac{2(-\frac{1}{2})}{1-(-\frac{1}{2})^2} = \frac{-4}{3}$ as $(x,y) \to (0,0)$ along $y = \frac{-1}{2}x$

□

Example. Show that $\lim_{(x,y)\to(0,0)} \frac{x^2-2y^2}{x^2+y^2}$ does not exist.

Solution. let $y = mx$, $f(x,y) = \frac{x^2 - 2(m^2x^2)}{x^2 + m^2x^2} = \frac{1-2m^2}{1+m^2}$ if $x \neq 0$

$m = 1$, $\Rightarrow f(x,y) \to -\frac{1}{2}$ as $(x,y) \to (0,0)$ along $y = x$

$m = 2 \Rightarrow f(x,y) \to -\frac{7}{5}$ as $(x,y) \to (0,0)$ along $y = 2x$.

□

Remark.

(1) The method of substitution $y = mx$ likely works to prove that a limit of $f(x,y)$ as $(x,y) \to (0,0)$ does not exist if the numerator and the denominator are of the same degree.

(2) However, this method cannot be used to prove that a limit exists. To show $\lim_{(x,y)\to(a,b)} f(x,y) = L$ exists, $f(x,y)$ must approaches L for any and every mode of approach of $(x,y) \to (a,b)$.

§13.1 Displacement Vectors

Definition. *The* ***displacement vector*** *from one point to another is an arrow with its tail at the first point and its tip at the second. The* **magnitude** *(or length) of the displacement vector is the distance between the points and is represented by the length of the arrow. The direction of the displacement vector is the direction of the arrow.*

Remarks.

- Vectors are written with an arrow over them, $\vec{v}$, to distinguish them from s-calars.
- Displacement vectors which point in the same direction and have the same magnitude are considered to be the same, even if they do not coincide.
- We use the notation $\overrightarrow{PQ}$ to denote the displacement vector from a point P to a point Q. The magnitude, or length, of a vector $\vec{v}$ is written$\|\vec{v}\|$.

Theorem (Addition of Displacement Vectors). *The sum, $\vec{v} + \vec{w}$, of two vectors $\vec{v}$ and $\vec{v}$ is the combined displacement resulting from first applying $\vec{v}$ and then $\vec{w}$. (See Figure 13.3.) The sum $\vec{v} + \vec{w}$ gives the same displacement.*

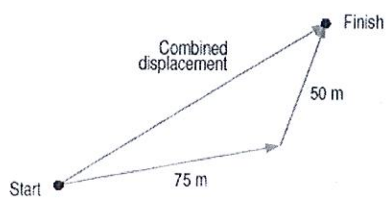


Figure 13.2: Sum of displacements of robots on Mars

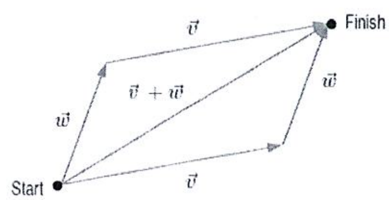


Figure 13.3: The sum $\vec{v} + \vec{w} = \vec{w} + \vec{v}$

Theorem (Subtraction of Displacement Vectors). *The difference, $\overrightarrow{w} - \overrightarrow{v}$, is the displacement vector that, when added to $\overrightarrow{v}$, gives $\overrightarrow{w}$. That is, $\overrightarrow{w} = \overrightarrow{v} + (\overrightarrow{w} - \overrightarrow{v})$. (See Figure 13.4.)*

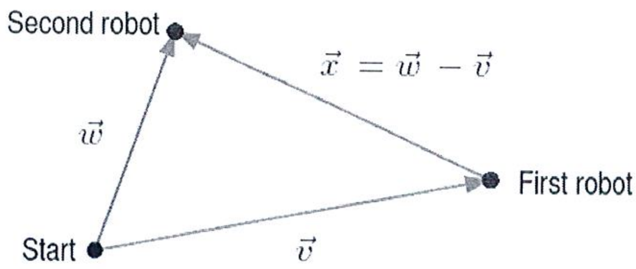


Figure 13.4: The difference $\vec{w} - \vec{v}$

Remark. The zero vector, $\overrightarrow{0}$, is a displacement vector with zero length. It has no direction.

Theorem (Scalar Multiplication of Displacement Vectors). *If λ is a scalar and $\overrightarrow{v}$ is a displacement vector, the scalar multiple of $\overrightarrow{v}$ by λ, written $\lambda\overrightarrow{v}$, is the displacement vector with the following properties:*

- *The displacement vector $\lambda\overrightarrow{v}$ is parallel to $\overrightarrow{v}$, pointing in the same direction if $\lambda > 0$ and in the opposite direction if $\lambda < 0$.*
- *The magnitude of $\lambda\overrightarrow{v}$ is λ times the magnitude of $\overrightarrow{v}$, that is, $\|\lambda\overrightarrow{v}\| = |\lambda|\ \|\overrightarrow{v}\|$.*

Remark. The zero vector, , Two vectors $\overrightarrow{v}$ and $\overrightarrow{w}$ are **parallel** if one is a scalar multiple of the other, that is, if $\overrightarrow{w} = \lambda\overrightarrow{v}$, for some scalar λ.

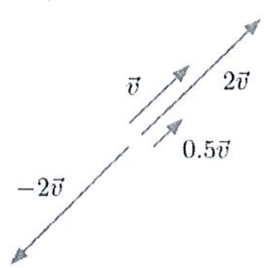


Figure 13.5: Scalar multiples of the vector $\vec{v}$

Components of Displacement Vectors: The Vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$

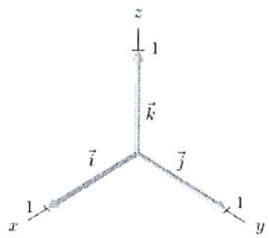


Figure 13.8: The vectors $\vec{i}$, $\vec{j}$ and $\vec{k}$ in 3-space

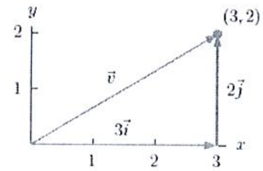


Figure 13.9: We resolve $\vec{v}$ into components by writing $\vec{v} = 3\vec{i} + 2\vec{j}$

Any displacement in 3-space or the plane can be expressed as a combination of displacements in the coordinate directions. For example, Figure 13.9 shows that the displacement vector $\vec{v}$ from the origin to the point $(3, 2)$ can be written as a sum of displacement vectors along the $x-$ and $y-$axes:

$$\vec{v} = 3\vec{i} + 2\vec{j}$$

This is called resolving $\vec{v}$ into components. In general:

Theorem (Resolving a displacement vector). *We resolve $\vec{v}$ into components by writing $\vec{v}$ in the form*

$$\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k},$$

where v_1, v_2, v_3 are scalars. We call $v_1\vec{i}$, $v_2\vec{j}$, and $v_3\vec{k}$ the components of $\vec{v}$.

Remark. Another common notation for the vector is $\vec{v} = \langle v_1, v_2, v_3 \rangle$.

Example 2. Resolve the displacement vector, $\vec{v}$, from the point $P_1 = (2, 4, 10)$ to the point $P_2 = (3, 7, 6)$ into components.

$\overrightarrow{P_1P_2} = (3-2)\vec{i} + (7-4)\vec{j} + (6-10)\vec{k} = \langle 1, 3, -4 \rangle$

Example 3. Decide whether the vector $\vec{v} = 2\vec{i} + 3\vec{j} + 5\vec{k}$ is parallel to each of the following vectors: $\vec{w} = 4\vec{i} + 6\vec{j} + 10\vec{k}$, $\vec{a} = -\vec{i} - 1.5\vec{j} - 2.5\vec{k}$, $\vec{b} = 4\vec{i} + 6\vec{j} + 9\vec{k}$.

$\vec{w} = 2\vec{v}$, $\vec{a} = -0.5\vec{v}$; $\vec{b} \nparallel \vec{v}$

Theorem (Components of Displacement Vectors). *The displacement vector from the point $P_1 = (x_1, y_1, z_1)$ to the point $P_2 = (x_2, y_2, z_2)$ is given in components by*

$$\overrightarrow{P_1P_2} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}$$

A displacement vector whose tail is at the origin is called a **position vector**. Thus, any point (x_0, y_0, z_0) in space has associated with it the position vector $\vec{r_0} = x_0\,\vec{i} + y_0\,\vec{j} + z_0\,\vec{k}$. (See Figure 13.11.) In general, a position vector gives the displacement of a point from the origin.

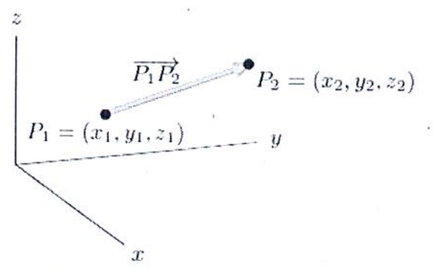


Figure 13.10: The displacement vector $\overrightarrow{P_1P_2} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}$

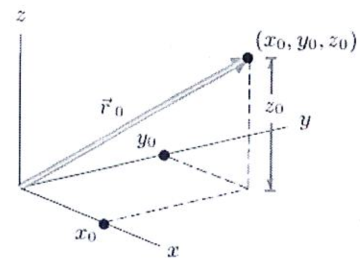


Figure 13.11: The position vector $\vec{r}_0 = x_0\vec{i} + y_0\vec{j} + z_0\vec{k}$

Remark. The coordinates of the finish point of a position vector $x_0\,\vec{i} + y_0\,\vec{j} + z_0\,\vec{k}$ is (x_0, y_0, z_0).

The Magnitude of a Vector in Components

For a vector $\vec{v} = v_1\,\vec{i} + v_2\,\vec{j} + v_3\,\vec{k}$, we have

$$\boxed{\|\vec{v}\| = \text{Length of the arrow} = \sqrt{v_1^2 + v_2^2 + v_3^3}.}$$

Theorem (Addition and Scalar Multiplication of Vectors in Components). *Suppose the vectors $\vec{v}$ and $\vec{w}$ are given in components:*

$$\vec{v} = v_1\,\vec{i} + v_2\,\vec{j} + v_3\,\vec{k} \quad \textit{and} \quad \vec{w} = w_1\,\vec{i} + w_2\,\vec{j} + w_3\,\vec{k}.$$

Then,

$$\boxed{\vec{v} + \vec{w} = (v_1 + w_1)\,\vec{i} + (v_2 + w_2)\,\vec{j} + (v_3 + w_3)\,\vec{k},}$$

and

$$\boxed{\lambda\vec{v} = \lambda v_1\,\vec{i} + \lambda v_2\,\vec{j} + \lambda v_3\,\vec{k}.}$$

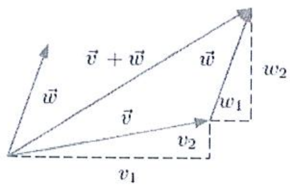


Figure 13.13: Sum $\vec{v}+\vec{w}$ in components

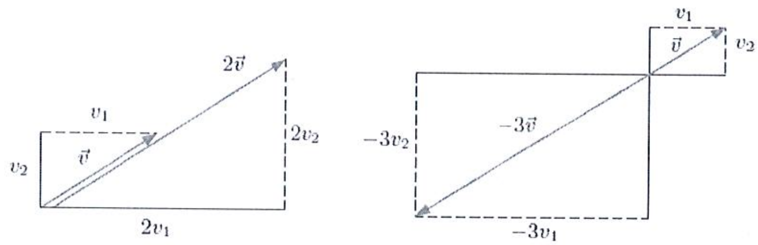


Figure 13.14: Scalar multiples of vectors showing $\vec{v}$, $2\vec{v}$, and $-3\vec{v}$

Resolving a 2-dimensional Vector.

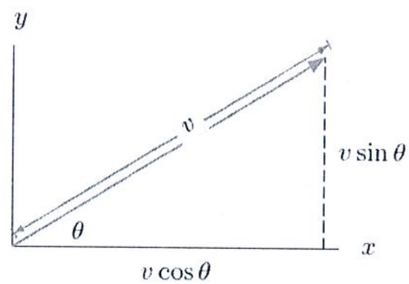


Figure 13.15: Resolving a vector: $\vec{v} = (v\cos\theta)\vec{i} + (v\sin\theta)\vec{j}$

Example 4. Resolve $\vec{v}$ into components if $\|\vec{v}\| = 2$ and $\theta = \pi/6$.

Sol. $\vec{v} = \|\vec{v}\|\cos\theta\vec{i} + \|\vec{v}\|\sin\theta\vec{j}$

$= 2\cos\left(\frac{\pi}{6}\right)\vec{i} + 2\sin\left(\frac{\pi}{6}\right)\vec{j}$

$= 2\times\frac{\sqrt{3}}{2}\vec{i} + 2\cdot\frac{1}{2}\vec{j}$

$= \langle\sqrt{3}, 1\rangle$

Unit Vectors. A unit vector is a vector whose magnitude is 1. A unit vector in the direction of any non-zero vector $\vec{v}$ is

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}.$$

Example 5. Find a unit vector, $\vec{u}$, in the direction of the vector $\vec{v} = \vec{i} + 3\vec{j}$.

$$\|\vec{v}\| = \sqrt{1^2+3^2} = \sqrt{10}$$

$$\Rightarrow \vec{u} = \frac{\vec{v}}{\sqrt{10}} = \langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \rangle \simeq \langle 0.32, 0.95 \rangle$$

Example 6. Find a unit vector at the point (x, y, z) that points radially outward away from the origin. let $\vec{v} = \langle x, y, z \rangle$

$$\|\vec{v}\| = \sqrt{x^2+y^2+z^2}$$

$$\Rightarrow \vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{x^2+y^2+z^2}} \langle x, y, z \rangle$$

Properties of Vectors. For any vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ and any scalars α and β,

Commutativity $\vec{v} + \vec{w} = \vec{w} + \vec{v}$

Associativity $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

$\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v}$

Distributivity $(\alpha + \beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}$

$\alpha(\vec{v} + \vec{w}) = \alpha\vec{v} + \alpha\vec{w}$

Identity $1\vec{v} = \vec{v}$

$0\vec{v} = \vec{0}$

$\vec{v} + \vec{0} = \vec{v}$

$\vec{w} + (-1)\vec{v} = \vec{w} - \vec{v}$

§13.3 THE DOT PRODUCT

Definition (DOT PRODUCT). *Let* $\overrightarrow{v} = \langle v_1, v_2, v_3\rangle$ *and* $\overrightarrow{w} = \langle w_1, w_2, w_3\rangle$. *The* ***dot product*** *or* ***scalar product*** *of* $\overrightarrow{v}$ *and* $\overrightarrow{w}$ *is the number* $\overrightarrow{v}\cdot\overrightarrow{w} = v_1w_1 + v_2w_2 + v_3w_3$.

Example. Let $\overrightarrow{v} = \langle 1, 2, 3\rangle$ and $\overrightarrow{w} = \langle -1, 0, -1\rangle$. Find $\overrightarrow{v}\cdot\overrightarrow{w}$. (sageMath)

$$\vec{v}\cdot\vec{w} = (1)\times(-1) + (2)\times(0) + (3)\times(-1)$$
$$= -1 + 0 - 3 = -4$$

Theorem. *If* θ *is the angle between the vectors* $\overrightarrow{v}$ *and* $\overrightarrow{w}$, *then* $\overrightarrow{v}\cdot\overrightarrow{w} = \|\overrightarrow{v}\|\|\overrightarrow{w}\|\cos\theta$, $0 \le \theta \le \pi$.

proof.

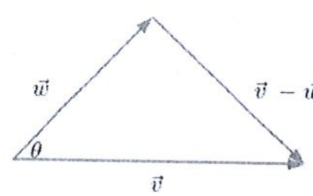


Figure 13.28: Triangle used in the justification of $\|\vec{v}\|\|\vec{w}\|\cos\theta = v_1w_1 + v_2w_2 + v_3w_3$

By the law of cosine
$$\|\vec{v}-\vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\|\vec{w}\|\cos\theta$$

Now $\|\vec{v}-\vec{w}\|^2 = (\vec{v}-\vec{w})\cdot(\vec{v}-\vec{w}) = \|\vec{v}\|^2 - 2\vec{v}\cdot\vec{w} + \|\vec{w}\|^2$

Therefore, $\vec{v}\cdot\vec{w} = \|\vec{v}\|\cdot\|\vec{w}\|\cos\theta$

or $\cos\theta = \dfrac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\cdot\|\vec{w}\|}$ □

Normal Vectors and the Equation of a Plane.

A **normal vector** to a plane is a vector that is perpendicular to the plane, that is, it is perpendicular to every displacement vector between any two points in the plane.

Theorem (equation of a plane). *The equation of the plane with normal vector* $\vec{n} = \langle a, b, c\rangle$ *and containing the point* $P_0 = (x_0, y_0, z_0)$ *is*

Proof. Let $P(x, y, z)$ be any point on the plane.

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Remark. Letting $d = ax_0 + by_0 + cz_0$ (a constant), we can write the equation of the plane in the form

$$ax + by + cz = d.$$

Example 5.

(a) Find the equation of the plane perpendicular to $\vec{n} = \langle -1, 3, 2\rangle$ and passing through the point $(1, 0, 4)$.

(b) Find a vector parallel to the plane.

(a) $-(x-1) + 3(y-0) + 2(z-4) = 0$ or $-x + 3y + 2z = 7$

(b) Any vector perpendicular to $\vec{n}$ is also parallel to the plane.
For example, $\vec{v} = \langle 3, -1, 0\rangle$

Example 6. Find a normal vector to the plane with equation (a) $x - y + 2z = 5$ (b) $z = 0.5x + 1.2y$.

(a) $\vec{n} = \langle 1, -1, 2\rangle$

(b) $\vec{n} = \langle 0.5, 1.2, -1\rangle$ since the plane can be written as $0.5x + 1.2y - z = 0$.

Resolving a Vector into Components: Projections

In Section 13.1, we resolved a vector into components parallel to the axes. Now we see how to resolve a vector, $\vec{v}$, into components, called $\vec{v}_{parallel}$ and $\vec{v}_{perp}$, which are parallel and perpendicular, respectively, to a given non-zero vector, $\vec{u}$. (See Figure 13.31.)

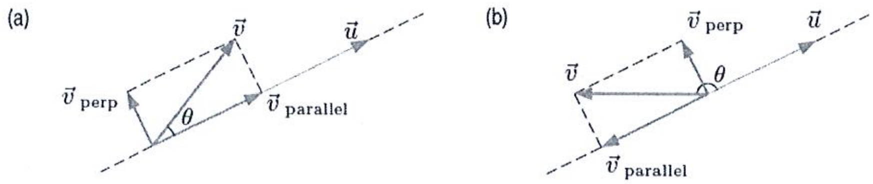


Figure 13.31: Resolving $\vec{v}$ into components parallel and perpendicular to $\vec{u}$
(a) $0 < \theta < \pi/2$ (b) $\pi/2 < \theta < \pi$

The projection of $\vec{v}$ on $\vec{u}$, written $\vec{v}_{parallel}$, measures (in some sense) how much the vector $\vec{v}$ is aligned with the vector $\vec{u}$. The length of $\vec{v}_{parallel}$ is the length of the shadow cast by $\vec{v}$ on a line in the direction of $\vec{u}$.

Theorem (Projection of $\vec{v}$ on the Line in the Direction of the Unit Vector $\vec{u}$). *If $\vec{v}_{parallel}$ and $\vec{v}_{perp}$ are components of $\vec{v}$ that are parallel and perpendicular, respectively, to a unit vector $\vec{u}$, then*
Projection of $\vec{v}$ onto $\vec{u}$ $= \vec{v}_{parallel} = (\vec{v}\cdot\vec{u})\vec{u}$ *provided* $\|\vec{u}\| = 1$
and $\vec{v} = \vec{v}_{parallel} + \vec{v}_{perp}$ *so* $\vec{v}_{perp} = \vec{v} - \vec{v}_{parallel}$. suppose $0 \leq \theta \leq \frac{\pi}{2}$

proof. $\|\vec{v}_{parallel}\| = \|\vec{v}\| \cos\theta = \vec{v}\cdot\vec{u}$ since $\|\vec{u}\|=1$

$$\left(\text{Recall } \cos\theta = \frac{\vec{v}\cdot\vec{u}}{\|\vec{v}\| \times \|\vec{u}\|}\right)$$

$$\Rightarrow \vec{v}_{parallel} = (\|\vec{v}\| \cos\theta)\,\vec{u} = (\vec{v}\cdot\vec{u})\,\vec{u}$$

If $\frac{\pi}{2} < \theta \leq \pi$, the formula still hods (see Fig 13.31(b))

Furthermore, $\vec{v}_{perp} = \vec{v} - \vec{v}_{parallel}$ □

Example 8. Figure 13.32 shows the force the wind exerts on the sail of a sailboat. Find the component of the force in the direction in which the sailboat is traveling.

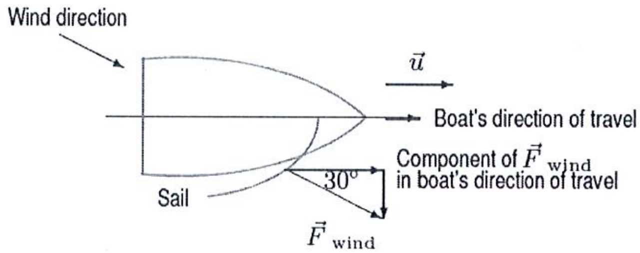


Figure 13.32: Wind moving a sailboat

Let $\vec{u}$ be a unit vector in the direction of travel.

The component of the force of the wind in the direction of $\vec{u}$ is

$$\vec{F}_{parallel} = (\vec{F}\cdot\vec{u})\vec{u} = (\|\vec{F}\|\cos 30^\circ)\vec{u}$$

$$= \frac{\sqrt{3}}{2}\|\vec{F}\|\vec{u}$$

$$\approx 86.6\% \ \|\vec{F}\|\vec{u}$$

which is about 86.6% of the total force.

§13.4 THE CROSS PRODUCT

The Area of a Parallelogram Consider the parallelogram formed by the vectors $\vec{v}$ and $\vec{w}$ with an angle of θ between them. Then Figure 13.35 shows

$$Area\ of\ parallelogram = \text{Base} \cdot Height = \|\vec{v}\|\|\vec{w}\|\sin\theta.$$

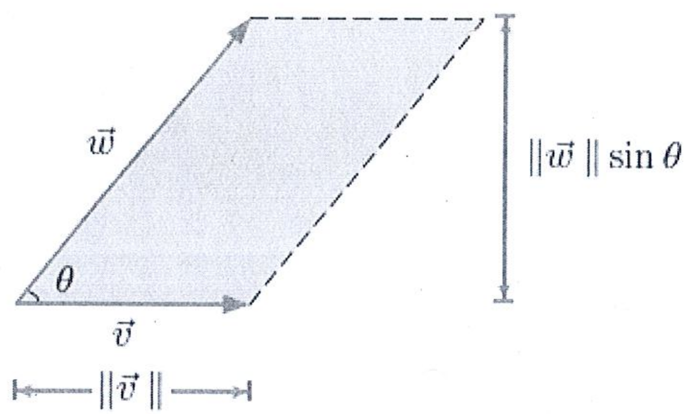


Figure 13.35: Parallelogram formed by $\vec{v}$ and $\vec{w}$ has Area $= \|\vec{v}\|\|\vec{w}\|\sin\theta$

We define the direction of a cross-product using the right-hand rule: **The right-hand rule**: Place $\vec{v}$ and $\vec{w}$ so that their tails coincide and curl the fingers of your right hand through the smaller of the two angles from $\vec{v}$ to $\vec{w}$; your thumb points in the direction of the normal vector, $\vec{n}$.

Definition (cross product). *Let $\vec{v} = \langle v_1, v_2, v_3\rangle$ and $\vec{w} = \langle w_1, w_2, w_3\rangle$. The **cross product** or **vector product** of $\vec{v}$ and $\vec{w}$ is*

$$\begin{aligned}\vec{v} \times \vec{w} &= \langle v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1\rangle \\ &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\ &= \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}\vec{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}\vec{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}\vec{k}\end{aligned}$$

Theorem. *If θ is the angle between $\vec{v}$ and $\vec{w}$, $0 \le \theta \le \pi$, then*

$$\|\vec{v} \times \vec{w}\| = \|\vec{v}\|\|\vec{w}\| \sin\theta.$$

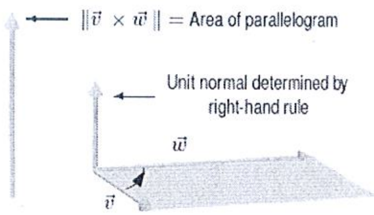


Figure 13.36: Area of parallelogram = $\|\vec{v} \times \vec{w}\|$

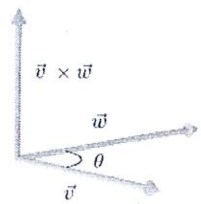


Figure 13.37: The cross product $\vec{v} \times \vec{w}$

Remarks.

- $\vec{i} \times \vec{j} = \vec{k}$, $\vec{j} \times \vec{k} = \vec{i}$, $\vec{k} \times \vec{i} = \vec{j}$.
- $\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = \vec{0}$.

(sageMath)

Example 3. Find the cross product of $\vec{v} = \langle 2, 1, -2 \rangle$ and $\vec{w} = \langle 3, 0, 1 \rangle$ and check that the cross product is perpendicular to both $\vec{v}$ and $\vec{w}$.

Sol. $\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -2 \\ 3 & 0 & 1 \end{vmatrix} = \vec{i}\begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & -2 \\ 3 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix}$

$= i - 8\vec{j} - 3\vec{k}$

i) $\vec{v} \cdot (\vec{v} \times \vec{w}) = \langle 2, 1, -2 \rangle \cdot \langle 1, -8, -3 \rangle = 2 - 8 + 6 = 0$

ii) $\vec{w} \cdot (\vec{v} \times \vec{w}) = \langle 3, 0, 1 \rangle \cdot \langle 1, -8, -3 \rangle = 3 + 0 - 3 = 0$

Theorem. *$\vec{v}$ and $\vec{w}$ are parallel if and only if $\vec{v} \times \vec{w} = \vec{0}$.*

Theorem (Properties of the Cross Product). *1.* $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$.

2. $(\lambda \vec{v}) \times \vec{w} = \lambda(\vec{v} \times \vec{w}) = \vec{v} \cdot (\lambda \vec{w})$.

3. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$.

4. $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$.

5. $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

6. $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$.

Remark

- In fact, $\vec{a} \cdot (\vec{b} \times \vec{c})$ is the sign volume of the parallelepiped determined by $\vec{a}$, $\vec{b}$ and $\vec{c}$. See the next two pages.

Areas and Volumes Using the Cross Product and Determinants

Theorem (Area of a parallelogram). *Area of a parallelogram with edges* $\vec{v} = \langle v_1, v_2, v_3 \rangle$ *and* $\vec{w} = \langle w_1, w_2, w_3 \rangle$ *is given by*

$$Area = \|\vec{v} \times \vec{w}\|, \quad where\ \vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Example 3. Find the area of the parallelogram with edges $\vec{v} = \langle 2, 1, -3 \rangle$ and $\vec{w} = \langle 1, 3, 2 \rangle$.

Sol. $\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -3 \\ 1 & 3 & 2 \end{vmatrix}$

$= \begin{vmatrix} 1 & -3 \\ 3 & 2 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} \vec{k}$

$= 11\vec{i} - 7\vec{j} + 5\vec{k}$

$\Rightarrow Area = \|\vec{v} \times \vec{w}\| = \sqrt{11^2 + 7^2 + 5^2} = \sqrt{195}$

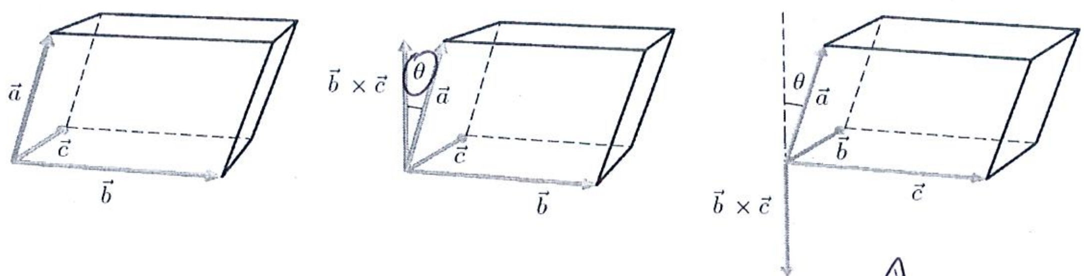
Volume of a Parallelepiped

Theorem (Volume of a Parallelepiped). *Volume of a parallelepiped with edges $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$ is given by*

$$Volumn = |(\overrightarrow{b} \times \overrightarrow{c}) \cdot \overrightarrow{a}| = |\overrightarrow{a} \cdot (\overrightarrow{b} \times \overrightarrow{c})| = |(\overrightarrow{a} \times \overrightarrow{b}) \cdot \overrightarrow{c}|$$

$$= Absolute\ value\ of\ the\ determinant \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Area of base of parallelepiped = $\|\vec{b} \times \vec{c}\|$.



Volume of a parallelepiped

The vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are called a right-handed set

The vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are called a left-handed set

Height $= \|\vec{a}\| \cos\theta$

Volumn $= \|\vec{b} \times \vec{c}\| \cdot \|\vec{a}\| \cos\theta$

$= (\vec{b} \times \vec{c}) \cdot \vec{a}$

$\cos(\pi - \theta) = -\cos\theta$

Corollary The vectors $\overrightarrow{a}$, $\overrightarrow{b}$ and $\overrightarrow{c}$ all lie on a plane if and only if $\overrightarrow{a} \cdot (\overrightarrow{b} \times \overrightarrow{c}) = 0$.

The Equation of a Plane Through Three Points

A plane can also be determined by three points on it (provided they do not lie on the same line). In that case we can find an equation of the plane by first determining two vectors in the plane and then finding a **normal vector** using the cross product, as in the following example.

Example 4. Find an equation of the plane containing the points $P = (1, 3, 0), Q = (3, 4, -3)$, and $R = (3, 6, 2)$.

Sol. $\overrightarrow{PQ} = \langle 3-1, 4-3, -3-0 \rangle = \langle 2, 1, -3 \rangle$

$\overrightarrow{PR} = \langle 3-1, 6-3, 2-0 \rangle = \langle 2, 3, 2 \rangle$

$\Rightarrow$ A normal vector

$$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -3 \\ 2 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -3 \\ 3 & 2 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & -3 \\ 2 & 2 \end{vmatrix} \vec{j}$$

$$+ \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} \vec{k} = \langle 11, -10, 4 \rangle$$

choose the point $P_0 = P(1, 3, 0)$ on the plane, the equation of the plane is $\vec{n} \cdot \overrightarrow{P_0P} = 0$

or $\langle 11, -10, 4 \rangle \cdot \langle x-1, y-3, z \rangle = 0$

or $11(x-1) - 10(y-3) + 4(z-0) = 0$

which simplifies to $11x - 10y + 4z = -19$.

§14.1 The Partial Derivative

Recall: $f'(x) = \frac{df(x)}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ – Instantaneous rate of change of f with respect to x.

Example. $f(x) = 3x^2 + 2\sin x + 4$ and $f'(x) = 6x + 2\cos x$.

Q. How do we differentiate $f(x, y)$?

Definition (Partial Derivatives). *The partial derivatives of $f(x, y)$ (with respect to x and with respect to y) are the two functions defined, respectively, by*

$$\frac{\partial f(x,y)}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h},$$
$$\frac{\partial f(x,y)}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

whenever these limits exist.

Notations. If $z = f(x, y)$,

$$\frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = f_x(x,y) = D_x[f(x,y)] = D_1[f(x,y)],$$
$$\frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = f_y(x,y) = D_y[f(x,y)] = D_2[f(x,y)],$$
$$f_x(a,b) = \frac{\partial f}{\partial x}|_{(a,b)},\ f_y(a,b) = \frac{\partial f}{\partial y}|_{(a,b)}$$

Rule.

- To calculate $\frac{\partial f(x,y)}{\partial x}$, simply regard y as a constant and differentiate w.r.t. x;
- To calculate $\frac{\partial f(x,y)}{\partial y}$, simply regard x as a constant and differentiate w.r.t. y.

Example. Compute both the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ of $f(x, y) = x^2 - 4xy + y$.

Solution. $\frac{\partial f}{\partial x} = 2x - 4y$, $\frac{\partial f}{\partial y} = -4x + 1$

□

Example.(SageMath) Compute both the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ of $f(x, y) = x \sin y + e^{xy} \cos y$.

Solution.

$$\frac{\partial f}{\partial x} = \sin y + y e^{xy} \cos y$$

$$\frac{\partial f}{\partial y} = x \cos y + x e^{xy} \cos y + e^{xy}(-\sin y)$$

□

Example. Compute the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ of $f(x, y, z) = x^2 - 16yz + z^2$.

Solution.

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = -16z, \quad \frac{\partial f}{\partial z} = -16y + 2z$$

□

Estimating Partial Derivatives from a Contour Diagram.

Example 4. Figure 14.6 shows the contour diagram for the temperature $H(x, t)$ (in ^{o}C) in a room as a function of distance x (in meters) from a heater and time t (in minutes) after the heater has been turned on. What are the signs of $H_x(10, 20)$ and $H_t(10, 20)$? Estimate these partial derivatives and explain the answers in practical terms.

$H(10, 20) \simeq 25$

i) $x \uparrow$ $H(x, 20) \downarrow$, $H_x(10, 20) < 0$

ii) $t \uparrow$. $H(10, t) \uparrow$ $H_t(10, 20) > 0$

$H_x(10, 20) \simeq \frac{20-25}{14} \simeq -0.36$

$H_t(10, 20) \simeq \frac{30-25}{32} \simeq 0.16$

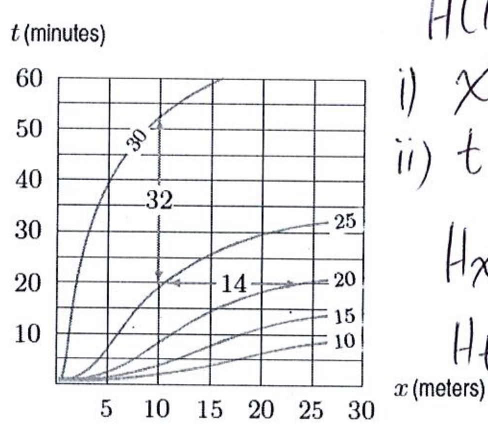


Figure 14.6: Temperature in a heated room: Heater at $x = 0$ is turned on at $t = 0$

§14.2 INTERPRETATION OF PARTIAL DERIVATIVES

Computing partial derivatives algebraically we can use all the differentiation formulas from one-variable calculus to find partial derivatives.

Example 1. Let $f(x,y) = \dfrac{x^2}{y+1}$. Find $f_x(3,2)$ algebraically.

Solution. $f_x = \dfrac{2x}{y+1} \Rightarrow f_x(3,2) = \dfrac{2 \times 3}{2+1} = 2$

□

Example 2. Compute the partial derivatives with respect to x and with respect to y for the following functions.

(a) $f(x,y) = y^2 e^{3x}$ (b)$z = (3xy+2x)^5$ (c) $g(x,y) = e^{x+3y}\sin(xy)$

Solution.

(a) $f_x = 3y^2e^{3x}$, $f_y = 2e^{3x}y$

(b) $z_x = 5(3xy+2x)^4(3y+2)$

$z_y = 5(3xy+2x)^4\, 3x = 15x(3xy+2x)^4$

(c) $g_x = e^{x+3y}\sin(xy) + e^{x+3y}\cos(xy)\cdot y$

$g_y = 3e^{x+3y}\sin(xy) + e^{x+3y}\cos(xy)\cdot x$ □

Example 3. Find all the partial derivatives of $f(x,y,z) = \dfrac{x^2y^3}{z}$.

Solution. $f_x = \dfrac{2xy^3}{z}$, $f_y = 3x^2y^2/z$

$f_z = x^2y^3\left(-\dfrac{1}{z^2}\right) = \dfrac{-x^2y^3}{z^2}$

□

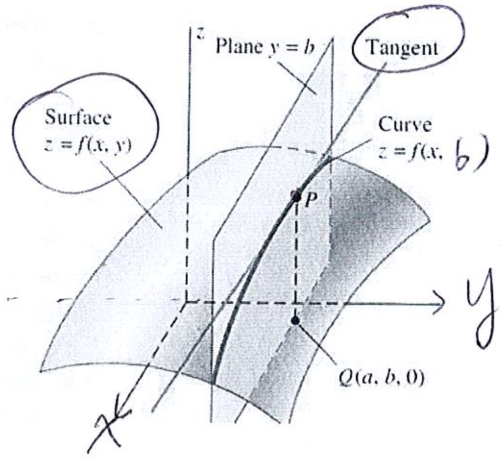
Geometric Interpretation of Partial Derivatives.

Recall that if $y = f(x)$, $f'(a)$ is the slope of the tangent line through the point $(a, f(a))$.

Q. What are the analogues for $f_x(a,b)$ and $f_y(a,b)$ respectively? Consider $f_x(a,b)$ only for illustration. Fix $y = b$, then $z = f(x,b)$ is an x-curve on the surface $z = f(x,y)$.

$$\frac{\partial z}{\partial x}|_{x=a} = \lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h} = f_x(a,b).$$

That is, **Geometric Interpretation of f_x:** $f_x(a,b)$ is the slope of the line tangent at $P(a,b,f(a,b))$ to the x-curve $z = f(x,b)$ through P on the surface $z = f(x,y)$.



$y = b$ fixed

Figure 14.1: An $x - curve$ and its tangent line at P

$P(a, b, f(a,b))$

$P'(a+1, b, f_x^P(a+1-a)+f(a,b$

PP'

And the tangent line has direction vector $v_1 =< 1, 0, f_x(a,b) >$.

Similarly, fix $\cancel{y}\,x = a$, then $z = f(a,y)$ is an y-curve on the surface $z = f(x,y)$.

$$\frac{\partial z}{\partial y}|_{y=b} = \lim_{h\to 0}\frac{f(a,b+h)-f(a,b)}{h} = f_y(a,b).$$

Thus, **Geometric Interpretation of f_y:** $f_y(a,b)$ is the slope of the line tangent at $P(a,b,f(a,b))$ to the y-curve $z = f(a,y)$ through P on the surface $z = f(x,y)$. And the tangent line has direction vector $v_2 =< 0, 1, f_y(a,b) >$.

Let n be the normal vector of the plane tangent to the surface $z = f(x, y)$ at $P(a, b, f(a, b))$. Then,

$$\begin{aligned} n =& v_1 \times v_2 = \begin{vmatrix} i & j & k \\ 1 & 0 & f_x(a,b) \\ 0 & 1 & f_y(a,b) \end{vmatrix} \\ =& - f_x(a,b)i - f_y(a,b)j + k \end{aligned}$$

and the equation of the plane is

$$-f_x(a,b)(x-a) - f_y(a,b)(y-b) + (z - f(a,b)) = 0.$$

That is, $\boxed{z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)}$ which is the equation of the plane tangent to the surface $z = f(x, y)$ at point $P(a, b, f(a, b))$. The plane is called the **tangent plane** to the surface at the point.

Remark. If the surface $z = f(x, y)$ is re-written as $\boxed{g(x, y, z) = f(x, y) - z} = 0$, then the equation of the tangent plane at the point (a, b, c) $(c = f(a, b))$ is

$$\boxed{g_x(a,b,c)(x-a) + g_y(a,b,c)(y-b) + g_z(a,b,c)(z-c) = 0.}$$

$g_z(a,b,c) = -1$

Example 14.3.1. (SageMath) Write an equation of the plane tangent to the surface $z = x^2 + y^2$ at the point $P(3, 4, 25)$.

Solution. Let $g(x, y, z) = x^2 + y^2 - z$

$$\begin{cases} g_x = 2x & \Rightarrow g_x(3,4,25) = 6 \\ g_y = 2y & \Rightarrow g_y(3,4,25) = 8 \\ g_z = -1 & \Rightarrow g_z(3,4,25) = -1 \end{cases}$$

$\Rightarrow$ Equation of the tangent plane is

$6(x-3) + 8(y-4) - (z-25) = 0.$

□

§14.3 LOCAL LINEARITY AND THE DIFFERENTIAL

Recall that the **Tangent Plane to the Surface** $z = f(x,y)$ at the Point $P(a, b, f(a,b))$ is $\boxed{z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}$

Tangent Plane Approximation to $f(x,y)$ **for** (x,y) **Near the Point** (a,b)**:** $\boxed{f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).}$ Figure 14.25 shows the tangent plane approximation graphically.

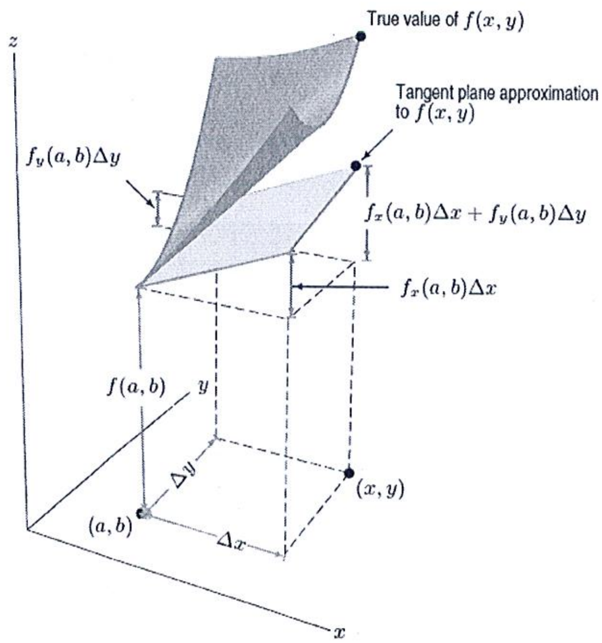


Figure 14.25: Local linearization: Approximating $f(x,y)$ by the z-value from the tangent plane

In general,

$$f(x+\Delta x, y+\Delta y) - f(x,y) \approx f_x(x,y)\Delta x + f_y(x,y)\Delta y,$$

That is, $\Delta f \approx df = f_x(x,y)\Delta x + f_y(x,y)\Delta y$.

Example. Use $f(P)$ and df to estimate $f(Q)$, where $f(x,y) = x^2 + 3xy - 2y^2$, $P(3,5)$ and $Q(3.2, 4.9)$. $f(3,5) = 3^2 + 3 \times 3 \times 5 - 2 \times 5^2 = 9 + 45 - 50 = 4$ $\boxed{f(Q) \approx 9.26}$

Solution. $f_x = 2x + 3y \Rightarrow f_x(3,5) = 2\times 3 + 3\times 5 = 21$

$f_y = 3x - 4y \Rightarrow f_y(3,5) = 3\times 3 - 4\times 5 = -11$

$\Delta x = 3.2 - 3 = 0.2$ and $\Delta y = 4.9 - 5 = -0.1$

$\Rightarrow f(3.2, 4.9) \approx f(3,5) + 21 \times 0.2 - 11 \times (-0.1) = 4 + 4.2 + 1.1 = 9.3$

Example 2. Find the local linearization of $f(x, y) = x^2 + y^2$ at the point $(3, 4)$. Estimate $f(2.9, 4.2)$ and $f(2, 2)$ using the linearization and compare your answers to the true values.

Solution.

$f(3,4) = 3^2 + 4^2 = 25$

$f'_x = 2x \Rightarrow f_x(3,4) = 6; \; f_y = 2y \Rightarrow f_y(3,4) = 8$

i) $\Delta x = 2.9 - 3 = -0.1, \; \Delta y = 4.2 - 4 = 0.2$

$\Rightarrow f(2.9, 4.2) \simeq 25 + 6 \times (-0.1) + 8 \times 0.2 = 26$

ii) $\Delta x = 2 - 3 = -1, \; \Delta y = 2 - 4 = -2$

$f(2,2) \simeq 25 + 6 \times (-1) + 8 \times (-2) = 3$

□

Example. Estimate $\sqrt{2(2.02)^3 + (2.97)^2}$.

Solution. consider $f(x,y) = \sqrt{2x^3 + y^2}$, $P(a,b) = (2,3)$

$f(2,3) = \sqrt{2 \times 2^3 + 3^2} = 5$

$f_x = \dfrac{\cancel{2} \times 3x^2}{\cancel{2}\sqrt{2x^3 + y^2}} \Rightarrow f_x(2,3) = \dfrac{12}{5}$

$f_y = \dfrac{\cancel{2}y}{\cancel{2}\sqrt{2x^3 + y^2}} \Rightarrow f_y(2,3) = \dfrac{3}{5}$

$\Delta x = 2.02 - 2 = 0.02, \quad \Delta y = 2.97 - 3 = -0.03$

□

$\Rightarrow \sqrt{2(2.02)^3 + (2.97)^2} \simeq 5 + \frac{12}{5}(0.02) + \frac{3}{5}(-0.03)$

$= 5 + 0.03 = 5.03$

Functions of Three or More Variables.

Increments and differentials of functions of more than two variables are defined similarly. A function $w = f(x, y, z)$ has increment

$$\Delta w = \Delta f = f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z)$$

and differential

$$dw = df \approx \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z.$$

That is,

$$\boxed{dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz,}$$

if we write w for f, dx for Δx, dy for Δy and dz for Δz.

Example. x, y, z are supposed to be 100mm but may be in error by 1mm. Use differentials to estimate the maximum resulting error in volumn.

Solution. $V = xyz$

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = yz\,dx + xz\,dy + xy\,dz$$

$x = y = z = 100$ and $dx = \pm 1$, $dy = \pm 1$, $dz = \pm 1$

$\Rightarrow dV = 100^2(\pm 1) + 100^2(\pm 1) + 100^2(\pm 1)$

$\Rightarrow$ maximum error is $3 \times 100^2 = 30000\ mm^3 = 30\ cm^3$

□

Remark. In general, for $f(x_1, x_2, \dots, x_n)$ we have

$$f(x_1+\Delta x_1, x_2+\Delta x_2, \dots, x_n+\Delta x_n) - f(x_1, x_2, \dots, x_n) \approx f_{(x_1)}\Delta x_1 + f_{(x_2)}\Delta x_2 + \dots + f_{(x_n)}\Delta x_n.$$

Set $\overrightarrow{x} = < x_1, x_2, \dots, x_n >$, $\overrightarrow{h} = < \Delta x_1, \Delta x_2, \dots, \Delta x_n >$ and $\nabla f(x) = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right\rangle$.

Then,

$$df = \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2 + \dots + \frac{\partial f}{\partial x_n}\Delta x_n = \nabla f \cdot \overrightarrow{h}$$

gradient

or $\Delta f \approx \nabla f \cdot \overrightarrow{h}$. That is,

$$\boxed{f(\overrightarrow{x} + \overrightarrow{h}) \approx f(\overrightarrow{x}) + \nabla f \cdot \overrightarrow{h}}.$$

Theorem (Linear Approximation). $f(\vec{a}+\vec{h}) \approx f(\vec{a}) + \nabla f(\vec{a}) \cdot \vec{h}$ *is a "good" approximation if partial derivatives of f are continuous near $\vec{x} = \vec{a}$ and $\vec{h}$ is small.*

Example 4. Compute the differentials of the following functions.
(a) $f(x, y) = x^2 e^{5y}$ (b) $z = x\sin(xy)$ (c) $f(x, y) = x\cos(2x)$.

Solution. (a) $f_x = 2xe^{5y}$ and $f_y = 5x^2e^{5y}$

$$df = 2xe^{5y}dx + 5x^2e^{5y}dy$$

(b) $df = [\sin(xy) + xy\cos(xy)]dx + x^2\cos(xy)dy$

(c) $df = [\cos(2x) - 2x\sin(2x)]dx + 0dy$

□

Example. Use differentials to approximate $(\sqrt{26})(\sqrt[3]{28})(\sqrt[4]{17})$.

Solution. consider $f(x,y,z) = \sqrt{x}\sqrt[3]{y}\sqrt[4]{z}$ and $P(25,27,16)$

$\Rightarrow f(25,27,16) = \sqrt{25}\sqrt[3]{27}\sqrt[4]{16} = 5\times3\times2 = 30$

$f_x = \frac{1}{2\sqrt{x}}\sqrt[3]{y}\sqrt[4]{z}$

$f_y = \frac{1}{3}\frac{1}{y^{\frac{2}{3}}}\sqrt{x}\sqrt[4]{z}$

$f_z = \frac{1}{4}\sqrt{x}\sqrt[3]{y}\frac{1}{z^{\frac{3}{4}}}$

$\Rightarrow \begin{cases} f_x(25,27,16) = \frac{1}{2}\times\frac{1}{5}\times3\times2 = \frac{3}{5} \\ f_y(25,27,16) = \frac{1}{3}\cdot\frac{1}{9}\cdot5\cdot2 = \frac{10}{27} \\ f_z(25,27,16) = \frac{1}{4}\cdot5\cdot3\cdot\frac{1}{8} = \frac{15}{32} \end{cases}$

$\Rightarrow \sqrt{26}\sqrt[3]{28}\sqrt[4]{17} = f(26,28,17) \simeq 30 + \frac{3}{5}(26-25) + \frac{10}{27}(28-27) + \frac{15}{32}(17-16)$

$\simeq 30 + 0.6 + \frac{10}{27} + \frac{15}{32} \simeq 31.439$

(True value $= 31.44$)

□

§14.4 GRADIENTS AND DIRECTIONAL DERIVATIVES IN THE PLANE

Definition (Directional Derivative). *The directional derivative of the function f at the point (a, b) in the direction of the unit vector $\vec{u} = \langle u_1, u_2 \rangle$ is*

$$f_{\vec{u}}(a, b) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h}$$

provided that this limit exists.

Remarks.

(1) If $\vec{v}$ is not a unit vector, we define $f_{\vec{u}}(a, b)$, where $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|}$.

(2) $f_{\vec{u}}(a, b)$ is the instantaneous rate of change of f with respect to the change in the point $P(a, b)$ in the direction of $\vec{u}$.

(3) If $\vec{u} = \vec{i}$, $f_{\vec{u}}(a, b) = f_x(a, b)$; If $\vec{u} = \vec{j}$, $f_{\vec{u}}(a, b) = f_y(a, b)$.

Example 3.(SageMath) Calculate the directional derivative of $f(x, y) = x^2 + y^2$ at $(1, 0)$ in the direction of the vector $\vec{i} + \vec{j}$.

Solution. consider the unit vector $\vec{u} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

$$f_{\vec{u}}(1,0) = \lim_{h \to 0} \frac{f(1+\frac{h}{\sqrt{2}}, 0+\frac{h}{\sqrt{2}}) - f(1,0)}{h}$$

$$= \lim_{h \to 0} \frac{(1+\frac{h}{\sqrt{2}})^2 + (\frac{h}{\sqrt{2}})^2 - 1}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{2}h + h^2}{h} = \lim_{h \to 0} (\sqrt{2} + h) = \sqrt{2}.$$

□

Computing Directional Derivatives from Partial Derivatives

Definition (Gradient Vector). *The Gradient Vector of a differentiable function f at the point (a,b) is*

$$grad\ f(a,b) \overset{\text{or}}{=} \nabla f(a,b) = f_x(a,b)\overrightarrow{i} + f_y(a,b)\overrightarrow{j} = \langle f_x(a,b), f_y(a,b)\rangle$$

Theorem (Calculation of Directional Derivatives). *If the real-valued function f is differentiable at $P(a,b)$, and $\overrightarrow{u}$ is a unit vector then the directional derivative $f_{\overrightarrow{u}}(a,b)$ exists and is given by*

$$f_{\overrightarrow{u}}(a,b) = \nabla f(a,b)\cdot\overrightarrow{u},$$

where $\nabla f(a,b) = \left\langle \frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b)\right\rangle$ is the gradient (vector) of f.

Remark. $f_{\overrightarrow{u}}(a,b)$ makes sense even for $\overrightarrow{u}$ with $\|\overrightarrow{u}\| \neq 1$, and can be calculated by $\nabla f(a,b)\cdot\overrightarrow{u}$ as well.

Example 5. Find the gradient vector of $f(x,y) = x + e^y$ at the point $(1,1)$.

Solution.

$$\nabla f = \langle f_x, f_y\rangle = \langle 1, e^y\rangle$$

$$\Rightarrow \nabla f(1,1) = \langle 1, e^1\rangle = \langle 1, e\rangle$$

$$= \vec{i} + e\vec{j}$$

Recall $\vec{v}\cdot\vec{w} = \|\vec{v}\|\,\|\vec{w}\|\cos\theta$ □

Interpretation of the Gradient Vector:

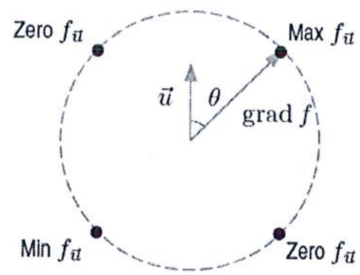


Figure 14.31: Values of the directional derivative at different angles to the gradient

Theorem. *$f_{\vec{u}}(P)$ attains its maximum value $|\nabla f(P)|$ when $\theta = 0$, i.e. $\vec{u} = \frac{\nabla f}{|\nabla f|}$; and $f_{\vec{u}}(P)$ attains its minimum value $-|\nabla f(P)|$ when $\theta = \pi$, i.e. $\vec{u} = -\frac{\nabla f}{|\nabla f|}$. Therefore, ∇f points in the direction in which f increases the most rapidly and $-\nabla f$ points in the direction in which f decreases the most rapidly.*

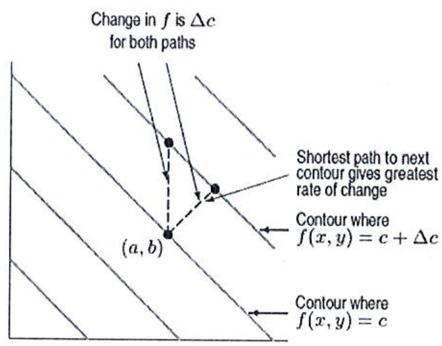


Figure 14.32: Close-up view of the contours around (a, b), showing the gradient is perpendicular to the contours

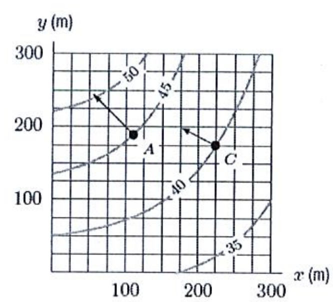


Figure 14.33: A temperature map showing directions and relative magnitudes of two gradient vectors

Geometric Properties of the Gradient Vector in the Plane: If f is a differentiable function at the point (a, b) and $\nabla f(a, b) \neq \vec{0}$, then:

- The direction of $\nabla f(a, b)$ is
  - Perpendicular to the contour of f through (a, b);
  - In the direction of the maximum rate of increase of f.
- The magnitude of the gradient vector, $\|\nabla f(a, b)\|$ is
  - The maximum rate of change of f at that point

– Large when the contours are close together and small when they are far apart.

Example 6. Explain why the gradient vectors at points A and C in Figure 14.33 have the direction and the relative magnitudes they do.

Solution.

i) The gradient points directly toward warmer temp

ii) The magnitude of the gradient vector measures the rate of change

The contours are closer together at A so it has larger magnitude

□

Example 7. Use the gradient to find the directional derivative of $f(x,y) = x + e^y$ at the point $(1,1)$ in the direction of the vectors $\vec{i} - \vec{j}$, $\vec{i} + 2\vec{j}$, and $\vec{i} + 3\vec{j}$.

Solution. $\nabla f(1,1) = \langle 1, e \rangle$ by Example 5.

i) A unit vector in the direction of $\vec{i} - \vec{j}$ is $\vec{s} = \langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \rangle$

$\Rightarrow f_{\vec{s}}(1,1) = \langle 1, e \rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \rangle = \frac{1-e}{\sqrt{2}} \simeq -1.215$

ii) A unit vector in the direction of $\vec{i} + 2\vec{j}$ is $\vec{v} = \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$

$\Rightarrow f_{\vec{v}}(1,1) = \langle 1, e \rangle \cdot \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle = \frac{1+2e}{\sqrt{5}} \simeq 2.879$

iii) A unit vector in the direction of $\vec{i} + 3\vec{j}$ is $\vec{w} = \langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \rangle$

$\Rightarrow f_{\vec{w}}(1,1) = \langle 1, e \rangle \cdot \langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \rangle = \frac{1+3e}{\sqrt{10}} \simeq 2.895$

□

§14.5 GRADIENTS AND DIRECTIONAL DERIVATIVES IN SPACE

All concepts in last section can be generalized to $n+1$ dimensional space.

Definition (Gradient Vector). *The Gradient Vector of a differentiable function $f(x_1,\dots,x_n)$ is*

$$grad\ f(x_1,\dots,x_n) \overset{\text{or}}{=} \nabla f(x_1,\dots,x_n) = \langle f_{x_1}, f_{x_2},\dots,f_{x_n}\rangle.$$

Thus, the Gradient Vector of $f(x_1,\dots,x_n)$ at $P(a_1,\dots,a_n)$ is

$$\langle f_{x_1}(a_1,\dots,a_n),\dots,f_{x_n}(a_1,\dots,a_n)\rangle.$$

Definition (Directional Derivative). *The directional derivative of the function $f(x_1,x_2,\dots,x_n)$ at the point $P(a_1,a_2,\dots,a_n)$ in the direction of the unit vector $\overrightarrow{u}$ is*

$$grad\ f(a_1,a_2,\dots,a_n) \overset{\text{or}}{=} \nabla f(a_1,a_2,\dots,a_n) = \langle f_{x_1}(a_1,a_2,\dots,a_n),\dots,f_{x_n}(a_1,a_2,\dots,a_n)\rangle$$

provided that this limit exists.

Theorem (Calculation of Directional Derivatives). *If the real-valued function $f(x_1,x_2,\dots,x_n)$ is differentiable at $P(a_1,a_2,\dots,a_n)$, and $\overrightarrow{u}$ is a unit vector then the directional derivative $f_{\overrightarrow{u}}(a_1,a_2,\dots,a_n)$ exists and is given by*

$$f_{\overrightarrow{u}}(a_1,a_2,\dots,a_n) = \nabla f(a_1,a_2,\dots,a_n)\cdot\overrightarrow{u},$$

where $\nabla f(a_1,a_2,\dots,a_n) = \left\langle \frac{\partial f}{\partial x_1}(a_1,a_2,\dots,a_n),\dots,\frac{\partial f}{\partial x_n}(a_1,a_2,\dots,a_n)\right\rangle$ is the gradient (vector) of f.

Geometric Properties of the Gradient Vector in the Space: If f is a differentiable function at the point (a,b,c) and $\nabla f(a,b,c)\neq\overrightarrow{0}$, then:

$$f_{\overrightarrow{u}}(a,b,c) = \nabla f(a,b,c)\cdot\overrightarrow{u}.$$

And

- the direction of $\nabla f(a,b,c)$ is
  - perpendicular to the level surface of f through (a,b,c);
  - in the direction of the maximum rate of increase of f

- The magnitude of the gradient vector, $\|\nabla f(a,b,c)\|$ is the maximum rate of change of f at the point (a,b,c).

Example 1. Find the directional derivative of $f(x,y,z) = xy + z$ at the point $(-1,0,1)$ in the direction of the vector $\vec{v} = \langle 2,0,1\rangle$.

Solution. Unit vector $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \langle \frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}} \rangle$

$\nabla f = \langle y, x, 1 \rangle \Rightarrow \nabla f(-1,0,1) = \langle 0, -1, 1 \rangle$

$\Rightarrow f_{\vec{u}}(-1,0,1) = \langle \frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}} \rangle \cdot \langle 0,-1,1 \rangle = \frac{1}{\sqrt{5}}$

□

Example 2. Let $f(x,y,z) = x^2 + y^2$ and $g(x,y,z) = -x^2 - y^2 - z^2$. What can we say about the direction of the following vectors?

(a)$\nabla f(0,1,1)$ (b) $\nabla f(1,0,1)$ (c) $\nabla g(0,1,1)$ (d) $\nabla g(1,0,1)$

Solution.

i) $z=1$, $x^2+y^2=1$ is a level surface of f and contains both $(0,1,1)$ and $(1,0,1)$. All gradient vectors are horizontal since f does not change in the z-direction. They are perpendicular to the cylinder and point outward since $f \uparrow$ as $|x|$ and $|y| \uparrow$

ii) $(0,1,1)$, $(1,0,1)$ are on the same level surface $\overbrace{-x^2-y^2-z^2}^{g(x,y,z)} = -2$ (which is the sphere $x^2+y^2+z^2=2$). The gradient vectors point inward since as $|x|, |y|, |z| \uparrow$, $g(x,y,z) \downarrow$.

□

Theorem (Implicit Function Theorem). *Suppose that the function $F(x, y, z)$ is continuously differentiable near the point (x_0, y_0, z_0) at which $F(x_0, y_0, z_0) = 0$ and $F_z(x_0, y_0, z_0) \neq 0$. Then there exists a function $z = f(x, y)$ such that $f(x_0, y_0) = z_0$ and $F(x, y, f(x, y)) = 0$ for (x, y) near (x_0, y_0). Moreover, $z = f(x, y)$ is uniquely defined and has continuous first order partial derivatives for (x, y) near (x_0, y_0).*

The Gradient Vector as a Normal Vector

$F(x, y, z)$ is continuous differentiable (all the first order partial derivatives exist and are continuous). Assume that at $P(x_0, y_0, z_0)$, $F(x_0, y_0, z_0) = 0$ and $\nabla F(P) \neq 0$. That is, $< \frac{\partial F}{\partial x}(P), \frac{\partial F}{\partial y}(P), \frac{\partial F}{\partial z}(P) > \neq 0$. Say, $\frac{\partial F}{\partial z}(P) \neq 0$. The Implicit Function Theorem $\Rightarrow F(x, y, z)$ defines a surface $z = f(x, y)$ near (x_0, y_0) (Note that if $\frac{\partial F}{\partial x}(P) \neq 0$, then $F(x, y, z)$ defines a surface $x = f(y, z)$).

Now $r(t)$ is a differential curve on the surface with $r(t_0) = (x_0, y_0, z_0)$ and $r'(t_0) \neq 0$. Then $F(r(t)) = F(x(t), y(t), z(t)) = 0 \Rightarrow \frac{dF(r(t))}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = \nabla f \cdot r'(t) = 0$.

Theorem (Gradient Vector as Normal Vector). *Suppose that $F(x, y, z)$ is continuously differentiable and let $P_0(x_0, y_0, z_0)$ be a point of the graph of the equation $F(x, y, z) = 0$ at which $\nabla F(P_0) \neq 0$. If $r(t)$ is a differentiable curve on this surface with $r(t_0) = \langle x_0, y_0, z_0 \rangle$ and $r'(t_0) \neq 0$, then*

$$\nabla F(P_0) \cdot r'(t_0) = 0.$$

Thus $\nabla F(P_0)$ is perpendicular to the tangent vector $r'(t_0)$, as indicated in the following figure.

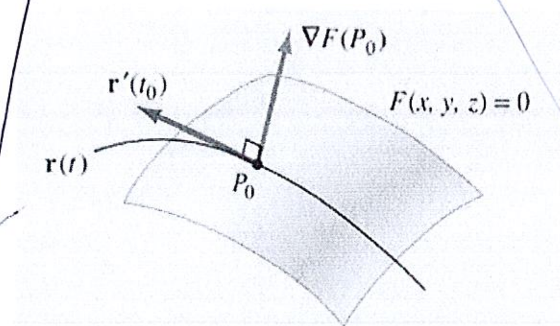


Figure 14.2: The gradient vector ∇F is perpendicular to the tangent vector.

Remark. (1)The tangent plane has equation

$$\boxed{F_x(P_0)(x-x_0)+F_y(P_0)(y-y_0)+F_z(P_0)(z-z_0)=0}.$$

(2) For a surface $z=f(x,y)$, we may write $F(x,y,z)=f(x,y)-z$. Therefore, the surface $z=f(x,y)$ coincides with the surface $F(x,y,z)=0$. Since $F_x=f_x, F_y=f_y, F_z=-1$, we have the tangent plane to $z=f(x,y)$ at (x_0,y_0,z_0) is given by

$$f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)=z-f(x_0,y_0)$$

which is the same as the one we deduced in §14.2.

Example 5. Find the equation of the tangent plane to the sphere $x^2+y^2+z^2=14$ at the point $(1,2,3)$. $F(x,y,z)=x^2+y^2+z^2-14=0$

Solution.

$\nabla F=\langle 2x,2y,2z\rangle$ and $\nabla F(1,2,3)=\langle 2,4,6\rangle$

$\Rightarrow$ The equation of the tangent plan is

$\langle 2,4,6\rangle\cdot\langle x-1,\ y-2,\ z-3\rangle=0$

or $\overset{1}{\cancel{2}}(x-1)+\overset{2}{\cancel{4}}(y-2)+\overset{3}{\cancel{6}}(z-3)=0$

or $x+2y+3z=14$

□

Caution: Scale on the Axis and the Geometric Interpretation of the Gradient. When we interpreted the gradient of a function geometrically, we tacitly assumed that the units and scales along the x and y axes were the same. If the scales are not the same, the gradient vector may not look perpendicular to the contours.

§14.6 THE CHAIN RULE

Recall: The single-variable chain rule expresses the derivative of a composite function $f(g(t))$ in terms of the derivatives of f and g:

$$D_t[f(g(t))] = f'(g(t)) \cdot g'(t).$$

If we write $w = f(x), x = g(t)$, we have

$$\frac{dw}{dt} = \frac{dw}{dx} \cdot \frac{dx}{dt}.$$

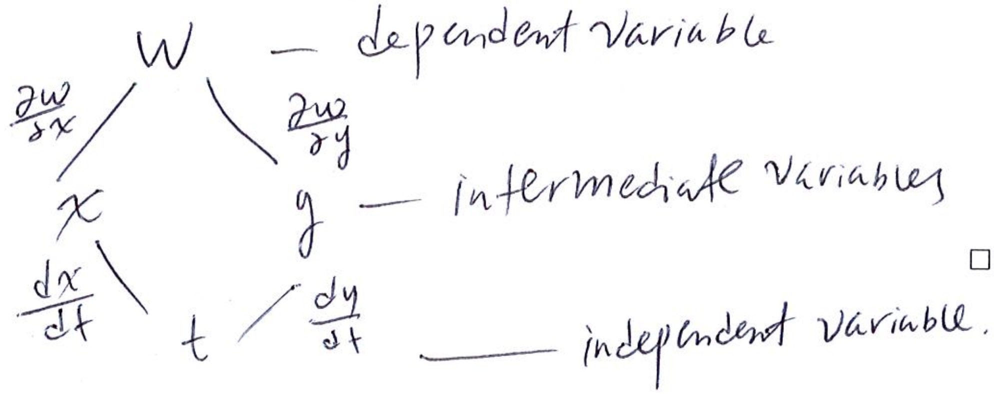
Theorem (The Chain Rule). *Suppose that $w = f(x, y)$ has continuous first-order partial derivatives and that $x = g(t)$ and $y = h(t)$ are differentiable functions. Then w is a differentiable function of t, and*

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$

Proof. We have the linear approximation $\Delta w \approx \frac{\partial w}{\partial x}\Delta x + \frac{\partial w}{\partial y}\Delta y$.

$$\Rightarrow \frac{\Delta w}{\Delta t} \simeq \frac{\partial w}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial w}{\partial y}\frac{\Delta y}{\Delta t}$$

$$\Delta t \to 0 \Rightarrow \frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt}$$



□

Example 2. (SageMath) Suppose that $z = f(x, y) = x \sin y$, where $x = t^2$ and $y = 2t + 1$. Let $z = g(t)$. Compute $g'(t)$ directly and using the chain rule.

Solution.

Method 1. $z = g(t) = f(t^2, 2t+1) = t^2 \sin(2t+1)$

$$\Rightarrow g'(t) = 2t \sin(2t+1) + 2t^2 \cos(2t+1)$$

Method 2. $$\frac{dz}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= \sin y \cdot 2t + x \cos y \cdot 2$$

$$= 2t \sin(2t+1) + 2t^2 \cos(2t+1)$$

□

Theorem (The General Chain Rule). *Suppose that w is a function of the variables $x_1, x_2, \ldots, x_m$ and that each of these is a function of the variables $t_1, t_2, \ldots, t_n$. If all these functions have continuous first-order partial derivatives, then*

$$\frac{\partial w}{\partial t_i} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial w}{\partial x_m}\frac{\partial x_m}{\partial t_i}$$

for each i, $1 \leq i \leq n$.

Example 4. Let $w = x^2 e^y, x = 4u$, and $y = 3u^2 - 2v$. Compute $\partial w/\partial u$ and $\partial w/\partial v$ using the chain rule.

Solution. $$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} = 2xe^y \cdot 4 + x^2 e^y \cdot 6u$$

$$= (8x + 6x^2 u)e^y = (32u + 96u^3)e^{3u^2-2v}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} = 2xe^y \cdot 0 + x^2 e^y(-2)$$

□

$$= -2x^2 e^y = -32u^2 e^{3u^2-2v}$$

Example 5. A quantity z can be expressed either as a function of x and y, so that $z = f(x, y)$, or as a function of u and v, so that $z = g(u, v)$. The two coordinate systems are related by

$$x = u + v, y = u - v.$$

(a) Use the chain rule to express $\partial z/\partial u$ and $\partial z/\partial v$ in terms of $\partial z/\partial x$ and $\partial z/\partial y$.

(b) Solve the equations in part (a) for $\partial z/\partial x$ and $\partial z/\partial y$.

(c) Show that the expressions we get in part (b) are the same as we get by expressing u and v in terms of x and y and using the chain rule.

Solution.

(a) $$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}$$

(b) $$\begin{cases} \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \\ \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} \end{cases} \Rightarrow \begin{cases} \frac{\partial z}{\partial x} = \frac{1}{2}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) \\ \frac{\partial z}{\partial y} = \frac{1}{2}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) \end{cases}$$

(c) $$\begin{cases} x = u + v \\ y = u - v \end{cases} \Rightarrow \begin{cases} u = \frac{1}{2}(x + y) \\ v = \frac{1}{2}(x - y) \end{cases}$$

$$\Rightarrow \begin{cases} \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{1}{2}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) \\ \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{1}{2}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) \end{cases} \square$$

Example. $w = u^2 - v^2 + \sin x + e^y, u = x + y, v = 2x - y$. Find $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$.

Solution. $w = w(u, v, x, y)$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} = 2u - 2v(2) + \cos x$$

$$= 2(x+y) - 4(2x-y) + \cos x = -6x + \cos x + 6y$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial y} = 2u + (-2v)(-1) + e^y$$

$$= 2(x+y) + 2(2x-y) + e^y = 6x + e^y$$

□

Implicit Partial Differentiation

Sometimes we need to investigate a function $z = g(x, y)$ that is not defined explicitly by a formula giving z in terms of x and y, but instead is defined implicitly by an equation of the form $F(x, y, z(x, y)) = 0$.

Example. $x^3 + y^3 + z^3 = xyz, z = f(x, y), F(x, y, z) = x^3 + y^3 + z^3 - xyz = 0$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Solution. i) Diff wrt x $\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = (3x^2 - yz) + (3z^2 - xy)\frac{\partial z}{\partial x} = 0$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{3x^2 - yz}{xy - 3z^2}$$

ii) Diff wrt y. $\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = (3y^2 - xz) + (3z^2 - xy)\frac{\partial z}{\partial y} = 0$

$$\Rightarrow \frac{\partial z}{\partial y} = \frac{3y^2 - xz}{xy - 3z^2}$$

□

Question: Given $F(x, y, z) = 0$, how do we know there indeed exists $z = f(x, y)$?

Theorem (Implicit Function Theorem). *Suppose that the function $F(x, y, z)$ is continuously differentiable near the point (x_0, y_0, z_0) at which $F(x_0, y_0, z_0) = 0$ and $F_z(x_0, y_0, z_0) \neq 0$. Then there exists a function $z = f(x, y)$ such that $f(x_0, y_0) = z_0$ and $F(x, y, f(x, y)) = 0$ for (x, y) near (x_0, y_0). Moreover, $z = f(x, y)$ is uniquely defined and has continuous first order partial derivatives for (x, y) near (x_0, y_0).*

§14.7 SECOND-ORDER PARTIAL DERIVATIVES

Since the partial derivatives of a function are themselves functions, we can differentiate them, giving second-order partial derivatives. A function $z = f(x, y)$ has two first-order partial derivatives, f_x and f_y, and four second-order partial derivatives.

The Second-Order Partial Derivatives of $z = f(x, y)$

$$\frac{\partial^2 z}{\partial x^2} = f_{xx} = (f_x)_x, \qquad \frac{\partial^2 z}{\partial x \partial y} = f_{yx} = (f_y)_x$$

$$\frac{\partial^2 z}{\partial y \partial x} = f_{xy} = (f_x)_y, \qquad \frac{\partial^2 z}{\partial x^2} = f_{yy} = (f_y)_y$$

Example 1. $f(x, y) = xy^2 + 3x^2 e^y$, compute f_x, f_y, f_{xy} and f_{yx}.

Solution. $f_x = y^2 + 6xe^y = \begin{cases} f_{xx} = 6e^y \\ f_{xy} = 2y + 6xe^y \end{cases}$

$f_y = 2xy + 3x^2 e^y$

$\Rightarrow f_{yx} = 2y + 6xe^y = f_{xy}$

□

Theorem (Theorem 14.1: Equality of Mixed Partial Derivatives). *If f_{xy} and f_{yx} are continuous at (a, b), an interior point of their domain, then*

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Remark. This is true for higher-order partial derivatives. For example, $f_{xxy} = f_{xyx} = f_{yxx}$ if they are all continuous. More generally, the order in which the differentiations are performed is unimportant as long as all derivatives involved are continuous.

Linear and Quadratic Approximations.

For a function of one variable, local linearity tells us that the best linear approximation is the degree-1 Taylor polynomial

$$f(x) \approx f(a) + f'(a)(x-a) \quad \textit{for } x \textit{ near } a.$$

A better approximation to $f(x)$ is given by the degree-2 Taylor polynomial:

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 \quad \textit{for } x \textit{ near } a.$$

Theorem (Taylor Polynomial of Degree 1 Approximating $f(x,y)$ for (x,y) near (a,b)). *If f has continuous first-order partial derivatives, then*

$$f(x,y) \approx L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

Theorem (Taylor Polynomial of Degree 2 Approximating $f(x,y)$ for (x,y) near (a,b)). *If f has continuous second-order partial derivatives, then*

$$\begin{aligned} f(x,y) \approx Q(x,y) = & f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ & + \frac{f_{xx}(a,b)}{2}(x-a)^2 + f_{xy}(a,b)(x-a)(y-b) + \frac{f_{yy}(a,b)}{2}(y-b)^2. \end{aligned}$$

Example 4. Let $f(x,y) = \cos(2x+y) + 3\sin(x+y)$.

(a) Compute the linear and quadratic Taylor polynomials, L and Q, approximating f near $(0,0)$.

(b) Explain why the contour plots of L and Q for $-1 \le x \le 1$, $-1 \le y \le 1$ look the way they do.

Solution.

$f_x = -2\sin(2x+y) + 3\cos(x+y) \Rightarrow f_x(0,0) = 3$

$f_{xx} = -4\cos(2x+y) - 3\sin(x+y) \Rightarrow f_{xx}(0,0) = -4$

$f_{xy} = -2\cos(2x+y) - 3\sin(x+y) \Rightarrow f_{xy}(0,0) = -2$

$f_y = -\sin(2x+y) + 3\cos(x+y) \Rightarrow f_y(0,0) = 3$

$f_{yy} = -\cos(2x+y) - 3\sin(x+y) \Rightarrow f_{yy}(0,0) = -1$

Example 5. Find the Taylor polynomial of degree 2 at the point $(1,2)$ for the function $f(x,y)=\frac{1}{xy}$.

Solution.

$$\frac{1}{xy} \approx Q(x,y) = \frac{1}{2} - \frac{1}{2}(x-1) - \frac{1}{4}(y-2) + \frac{1}{2}(x-1)^2 + \frac{1}{4}(x-1)(y-2) + \frac{\frac{1}{4}}{2}(y-2)^2$$

$$= \frac{1}{2} - \frac{x-1}{2} - \frac{y-2}{4} + \frac{(x-1)^2}{2} + \frac{(x-1)(y-2)}{4} + \frac{(y-2)^2}{8}.$$

□

Example 4 (continued)

$$f(x,y) \approx L(x,y) = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0)$$
$$= 1 + 3x + 3y$$

$$f(x,y) \approx Q(x,y) = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) + \frac{f_{xx}(0,0)}{2}(x-0)^2 + f_{xy}(0,0)(x-0)(y-0) + \frac{f_{yy}(0,0)}{2}(y-0)^2$$

$$= 1 + 3x + 3y - 2x^2 - 2xy - \frac{1}{2}y^2$$

§14.8 DIFFERENTIABILITY

Recall that in Section 14.3 we gave an informal introduction to the concept of differentiability. We called a function $f(x, y)$ *differentiable* at a point (a, b) if it is well approximated by a linear function near (a, b). This section focuses on the precise meaning of the phrase well approximated. By looking at examples, we shall see that local linearity requires the existence of partial derivatives, but they do not tell the whole story. In particular, existence of partial derivatives at a point is not sufficient to guarantee local linearity at that point. We begin by discussing the relation between continuity and differentiability.

Differentiability for Functions of Two Variables

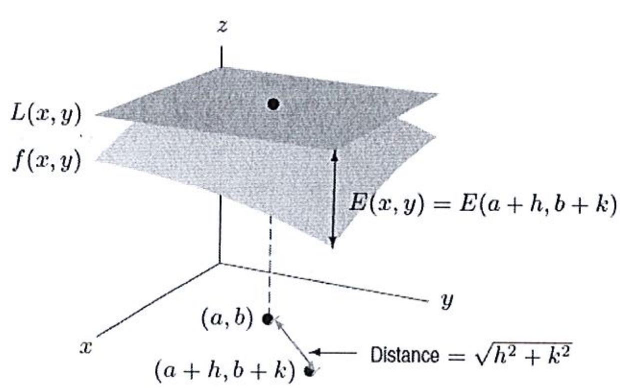
Definition (differentiability of a function). *A function $f(x, y)$ is* ***differentiable at the point*** (a, b) *if there is a linear function*$L(x, y) = f(a, b) + m(x-a) + n(y-b)$ *such that if the error $E(x, y)$ is defined by*

$$f(x, y) = L(x, y) + E(x, y),$$

and if $h = x-a, k = y-b$, then the relative error $\dfrac{E(a + h, b + k)}{\sqrt{h^2 + k^2}}$

$$\lim_{h \to 0, k \to 0} \frac{E(a + h, b + k)}{\sqrt{h^2 + k^2}} = 0.$$

The function f is differentiable on a region R if it is differentiable at each point of R. The function $L(x, y)$ is called the local linearization of $f(x, y)$ near (a, b).



Graph of function $z = f(x, y)$ and its local linearization $z = L(x, y)$ near the point (a, b)

Partial Derivatives and Differentiability

Example 1. Show that if f is a differentiable function with local linearization $L(x,y) = f(a,b) + m(x-a) + n(y-b)$, then $m = f_x(a,b)$ and $n = f_y(a,b)$.

Proof. Suppose $h \to 0$ and $k = 0$. We know that

$$0 = \lim_{h \to 0} \frac{E(a+h, b+0)}{\sqrt{h^2+0^2}} = \lim_{h\to 0} \frac{f(a+h,b) - L(a+h,b)}{h}$$

$$= \lim_{h\to 0} \frac{f(a+h,b) - f(a,b) - mh}{h} = \lim_{h\to 0}\left(\frac{f(a+h,b)-f(a,b)}{h}\right) - m$$

$$= f_x(a,b) - m \Rightarrow m = f_x(a,b)$$

$n = f_y(a,b)$ is found in a similar way.

□

Example 3. Consider the function $f(x,y) = x^{1/3}y^{1/3}$. Show that the partial derivatives $f_x(0,0)$ and $f_y(0,0)$ exist, but that f is not differentiable at $(0,0)$.

Proof. $f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0$

Similarly, $f_y(0,0) = 0$

If there exists a linear approximation near $(0,0)$, $L(x,y) = 0$

Now $E(x,y) = f(x,y) - L(x,y) = f(x,y)$

$\lim_{\substack{h\to 0\\ k\to 0}} \frac{h^{\frac{1}{3}} k^{\frac{1}{3}}}{\sqrt{h^2+k^2}}$ exists?

Consider $k = h \to 0$, $\lim_{h\to 0} \frac{h^{\frac{1}{3}} h^{\frac{1}{3}}}{\sqrt{h^2+h^2}} = \lim_{h\to 0}\frac{h^{\frac{2}{3}}}{h\sqrt{2}} = \lim_{h\to 0} \frac{1}{h^{\frac{1}{3}}\sqrt{2}}$

□

which does not exist $\Rightarrow f(x,y)$ is not differentiable at $(0,0)$

Remark.

- If a function is differentiable at a point, then both partial derivatives exist there.
- Having both partial derivatives at a point does not guarantee that a function is differentiable there.

Example 4. Suppose that f is the function of two variables defined by

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, (x,y) \neq (0,0) \\ 0, (x,y) = (0,0). \end{cases}$$

It can be shown that $f(x,y)$ is not continuous at the origin. Show that the partial derivatives $f_x(0,0)$ and $f_y(0,0)$ exist. Could f be differentiable at $(0,0)$?

Solution. $f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \left(\frac{1}{h} \cdot \frac{0}{h^2+0^2} \right) = \lim_{h \to 0} \frac{0}{h} = 0$

Similarly $f_y(0,0) = 0$

But f cannot be differentiable at $(0,0)$ since it is not continuous there.

□

Remark.

- If a function is differentiable at a point, then it is continuous there.
- Having both partial derivatives at a point does not guarantee that a function is continuous

Theorem (Continuity of Partial Derivatives Implies Differentiability). *If the partial derivatives, f_x and f_y, of a function f exist and are continuous on a small disk centered at the point (a, b), then f is differentiable at (a, b).*

Example 5. Show that the function $f(x, y) = \ln(x^2 + y^2)$ is differentiable everywhere in its domain.

Solution. $f_x = \dfrac{2x}{x^2+y^2}$ and $f_y = \dfrac{2y}{x^2+y^2}$

are continuous at all points except $(0,0)$

$\Rightarrow f(x,y)$ is differentiable everywhere in its domain.

□

§15.1 Critical Points: Local Extrema and Saddle Points

Consider $f(x, y) : D \to \mathbb{R}, D \subseteq \mathbb{R}^2$.

Definition. *$f(a, b)$ is called a* ***local maximum value*** *of $f(x, y)$ provided that it is the absolute maximum value of f on some disk D that is centered at (a, b) and lies wholly within the domain R.*
Similarly, $f(a, b)$ is called a ***local minimum value*** *of $f(x, y)$ provided that it is the absolute minimum value of f on some disk D that is centered at (a, b) and lies wholly within the domain R.*

Remark. A disk centred at (x_0, y_0) with radius r has equation(inequality)

$$(x - x_0)^2 + (y - y_0)^2 \leq r^2.$$

Theorem (Thm1. **Necessary** Conditions for Local Extrema)**.** *Suppose that $f(x, y)$ attains a local maximum value or a local minimum value at the point (a, b) and that both $f_x(a, b)$ and $f_y(a, b)$ exist. Then,*

$$f_x(a, b) = f_y(a, b) = 0.$$

Recall parallel results for single variable functions.

Proof. Suppose $f(a, b)$ is a local maximum. Then $G(x) \triangleq f(x, b)$ and $H(y) \triangleq f(a, y)$ have local maximum at $x = a$ and $y = b$ respectively. Therefore, $G'(a) = f_x(a, b) = 0$. Similarly, $H'(b) = f_y(a, b) = 0$. □

Remark. The **necessary** condition $f_x(a, b) = f_y(a, b) = 0$ is NOT a sufficient condition for a local extrema. For example, consider $f(x, y) = y^2 - x^2$. $f_x = -2x$, $f_y = 2y$. So $f_x(0, 0) = f_y(0, 0) = 0$. But on the line $y = 2x$, $f(x, y) = (2x)^2 - x^2 = 3x^2$ has a minimum at $(0, 0)$. Whereas on the line $x = 2y$, $f(x, y) = y^2 - (2y)^2 = -3y^2$ has a maximum at $(0, 0)$. Thus, f can have neither a local maximum nor a local minimum at $(0, 0)$.

Remark. Recall that the plane tangent to the surface $z = f(x, y)$ at (a, b) is

$$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

1. The tangent plane is **horizontal** if and only if $f_x(a,b) = 0 = f_y(a,b)$.

2. By Theorem 1, the tangent plane at a local extremum (maximum or minimum points) is horizontal.

Example. Find all the points on the surface

$$z = x^4 + y^3 - 3y$$

at which the tangent plane is horizontal.

Solution.

$z = f(x, y)$

$f_x = 4x^3$ and $f_y = 3y^2 - 3$

$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = \pm 1 \end{cases}$

□

The points are $(0, 1, -2)$ and $(0, -1, 2)$

Definition (critical points). *Points where the gradient is either $\vec{0}$ or undefined (NOT both f_x and f_y exist) are called* ***critical points*** *of the function.*

Example 1. Find the critical points of $f(x, y) = x^2 - 2x + y^2 - 4y + 5$.

Solution. Let $\begin{cases} f_x = 2x - 2 = 0 \\ f_y = 2y - 4 = 0 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 2 \end{cases}$

$\Rightarrow (1, 2, \cancel{0})$ is the only one critical point.

□

Example 2. Find and analyze any critical points of $f(x, y) = \sqrt[-]{x^2 + y^2}$.

Solution. Let $\begin{cases} f_x = -x / \sqrt{x^2+y^2} = 0 \\ f_y = -y / \sqrt{x^2+y^2} = 0 \end{cases}$ not possible

But they are undefined at $(0, 0) \Rightarrow (0, 0, \cancel{0})$ is a critical point. Note that $f(x, y) \le 0 \Rightarrow$ f has a local and global maximum at $(0, 0)$

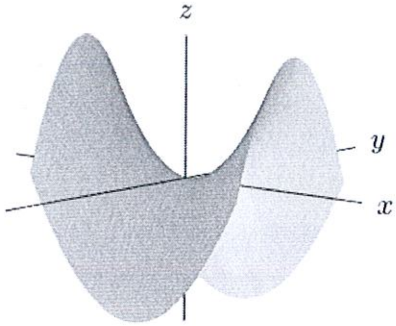
□

Example 3. Find and analyze any critical points of $g(x, y) = x^2 - y^2$.

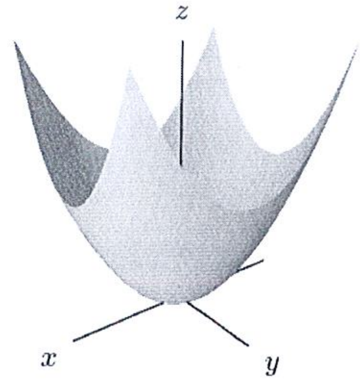
Solution. Let $\begin{cases} g_x = 2x = 0 \\ g_y = -2y = 0 \end{cases} \Rightarrow (0, 0)$ is the only critial point.

Near the origin, g takes on both positive and negative values $\Rightarrow (0, 0, 0)$ is neither a local maximum nor a local minimum

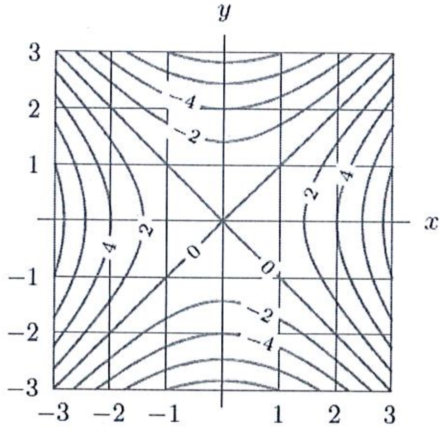
□



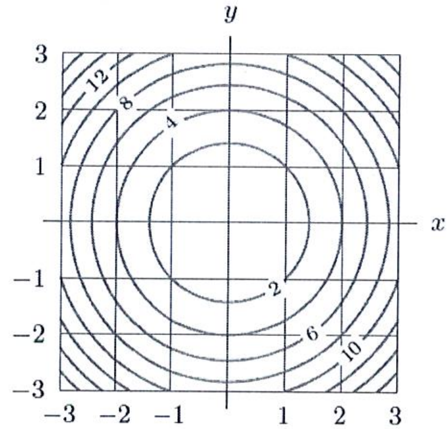
Graph of
$g(x, y) = x^2 - y^2$, showing
saddle shape at the origin



Graph of $h(x, y) = x^2 + y^2$, showing
minimum at the origin



Contours of $g(x, y) = x^2 - y^2$,
showing a saddle shape at the origin



Contours of $h(x, y) = x^2 + y^2$,
showing a local minimum at the origin

Example 4. Find the local extrema of the function $f(x, y) = 8y^3 + 12x^2 - 24xy$.

Solution. Let $\begin{cases} f_x = 24x - 24y = 0 \\ f_y = 24y^2 - 24x = 0 \end{cases} \Rightarrow \begin{cases} x = y \\ x = y^2 \end{cases}$

$\Rightarrow$ The solutions are $(0,0)$, $(1,1)$

Look at the contour plot:

i) f has a local minimum at $(1,1)$

ii) $f(0,0) = 0$ and f takes both positive and negative values nearby $\Rightarrow$ $(0,0)$ is neither a local minimum nor a local maximum. □

Classifying Critical Points We can see whether a critical point of a function, f, is a local maximum, local minimum, or neither by looking at the contour diagram. There is also an analytic method for making this distinction.

Theorem (Second-Derivative Test for Functions of Two Variables). *Suppose* (x_0, y_0) *is a point where* $grad f(x_0, y_0) = \vec{0}$. *Let*

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2 .$$

- *If* $D > 0$ *and* $f_{xx}(x_0, y_0) > 0$, *then* f *has a local minimum at* (x_0, y_0).
- *If* $D > 0$ *and* $f_{xx}(x_0, y_0) < 0$, *then* f *has a local maximum at* (x_0, y_0).
- *If* $D < 0$, *then* f *has a saddle point at* (x_0, y_0).
- *If* $D = 0$, *anything can happen:* f *can have a local maximum, or a local minimum, or a saddle point, or none of these, at* (x_0, y_0).

The Shape of the Graph of $f(x, y) = ax^2 + bxy + cy2$

Example 5. Find and analyze the local extrema of the function $f(x, y) = x^2 + xy + y^2$.

Solution. Let $\begin{cases} f_x = 2x + y = 0 \\ f_y = x + 2y = 0 \end{cases}$ $\Rightarrow (0, 0)$ is the only critical point

And $f(0, 0) = 0$

Now $f_{xx} = 2$ and $f_{xx}(0,0) = 2$, $f_{yy} = 2$ and $f_{yy}(0,0) = 2$

$f_{xy} = 1$ and $f_{xy}(0,0) = 1$

$\Rightarrow D = f_{xx}(0,0) f_{yy}(0,0) - (f_{xy}(0,0))^2 = 2 \times 2 - 1 = 3 > 0$

and $f_{xx}(0,0) = 2 > 0$

$\Rightarrow f$ has a local minimum at $(0,0)$ $\square$

Example 6. (SageMath) Find the local maxima, minima, and saddle points of $f(x, y) = \frac{1}{2}x^2 + 3y^3 + 9y^2 - 3xy + 9y - 9x$.

Solution. Let $\begin{cases} f_x = x - 3y - 9 = 0 \Rightarrow x = 3y + 9 \\ f_y = 9y^2 + 18y - 3x + 9 = 0 \quad ② \end{cases}$

Replace x in ② by $x = 3y + 9$. $9y^2 + 9y - 18 = 0 \Rightarrow \begin{cases} y = -2 \\ y = 1 \end{cases}$

$(3, -2)$, and $(12, 1)$ are the critical points.

$D(x, y) = f_{xx} f_{yy} - f_{xy}^2 = 1 \times (18y + 18) - (-3)^2 = 18y + 9$

i) $D(3, -2) < 0 \Rightarrow (3, -2)$ is the saddle point of f.

ii) $D(12, 1) = 18 + 9 > 0$ and $f_{xx}(12, 1) > 0 \Rightarrow (12, 1)$ is a local minimum.

□

Example 7. Classify the critical points of $f(x, y) = x^4 + y^4$, and $g(x, y) = -x^4 - y^4$, and $h(x, y) = x^4 - y^4$.

Solution. Let $\begin{cases} f_x = 4x^3 = 0 \\ f_y = 4y^3 = 0, \end{cases}$ $\begin{cases} g_x = -4x^3 = 0 \\ g_y = -4y^3 = 0 \end{cases}$ $\begin{cases} h_x = 4x^3 = 0 \\ h_y = -4y^3 = 0 \end{cases}$

$\Rightarrow$ Each has a critical point $(0, 0)$

Furthermore, the second derivatives are 0 $\Rightarrow D = 0$

i) $f(0,0) = 0$ and $f(x, y) \geq 0 \Rightarrow (0, 0)$ is a local minimum of f

ii) $g(0,0) = 0$ and $g(x, y) \leq 0 \Rightarrow (0, 0)$ is a local maximum of g

iii) $h(0,0) = 0$, $\begin{cases} x = 2y, & h(x, y) = 15y^4 \text{ local min} \\ y = 2x, & h(x, y) = -15x^4 \text{ local max} \end{cases}$

□

$\Rightarrow (0, 0)$ is a saddle point of $h(x, y)$.

§15.2 Optimization: Finding Global Extrema

Consider $f(x,y) : D \to \mathbb{R}, D \subseteq \mathbb{R}^2$.

Definition (Global maximum and global minimum). *For function f,*

- ***Global maximum value*** *M of f: largest value attained by f in D. That is,*

$$f(x,y) \leq M = f(a,b) \quad \textit{for all points } (x,y) \textit{ of } D;$$

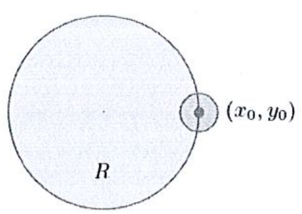
- ***Global minimum value*** *m of f: smallest value attained by f in D. That is,*

$$f(x,y) \geq m = f(c,d) \quad \textit{for all points } (x,y) \textit{ of } D.$$

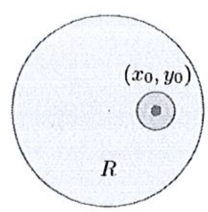
Definition. *The point (a,b) of R called an **interior point** of R provided that some circular disk centered at (a,b) lies wholly within R.*

- A **closed region** is one which contains its boundary;
- A **bounded region** is one which does not stretch to infinity in any direction.

A region R in 2-space is bounded if the distance between every point (x,y) in R and the origin is less than some constant K. Closed and bounded regions in 3-space are defined in the same way.



Boundary point (x_0, y_0) of R



Interior point (x_0, y_0) of R

Not all functions have a global maximum or minimum: it depends on the function and the region.

Recall: If f is continuous on $[a,b]$, then f attains its absolute maximum and minimum values on $[a,b]$.

Theorem (Thm2. Existence of Extreme Values(Thm 15.1)). *Let R consists of the points on and within a simple closed curve C in the plane. Suppose that the function f is continuous on this region, R. Then f attains the global maximum value at some point (a, b) on R and attains an global minimum value at some point (c, d) on R.*

Remark: simple

Remark: If f is not continuous or the region R is not closed and bounded, there is no guarantee that f achieves a global maximum or global minimum on R.

Theorem (Thm3. Types of Absolute Extrema). *Suppose that f* **is continuous on the plane region R consisting of the points on and within a simple closed curve C.** *If $f(a, b)$ is either the global maximum or the global minimum value of $f(x, y)$ on R, then (a, b) is either*

1. *an interior point of D at which*
$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0\,,$$
2. *an interior point of R at where not both partial derivatives exist, or*
3. *a point of the boundary curve C of R.*

Remark. An interior point (a, b) where either condition 1 or condition 2 in Theorem 3 holds is a **critical point** of f.

Method of finding largest /smallest value of $f(x, y)$:
Assume that f is continuous on R enclosed by a simple curve C.

1. First locate the interior critical points. Find points at which either $\begin{cases} f_x = 0, \\ f_y = 0, \end{cases}$ or NOT both f_x and f_y exist.
2. Next find the possible extreme values of f on the boundary curve C: the technique depends on the nature of C.
3. Finally compare the values of f at the points found in steps 1 and 2.

Example. Find the maximum and minimum values attained by the function

$$f(x, y) = \sqrt{x^2 + y^2}$$

defined on $D = \{(x, y) : x^2 + y^2 \leq 1\}$.

Solution. ① let $\begin{cases} f_x = x/\sqrt{x^2+y^2} = 0 \\ f_y = y/\sqrt{x^2+y^2} = 0 \end{cases}$ both fail to exist at $(0,0)$

$\Rightarrow$ The only critical point is $(0,0)$ and $f(0,0) = 0$

② At the boundary $C: \{(x,y) : x^2+y^2=1\}$ $f(x,y) = \sqrt{1} = 1$

③ From ① & ②, f attains minimum value 0 at $(0,0)$ and maximum value at each and every point of the boundary circle. □

Example. Find the maximum and minimum values attained by the function

$$f(x, y) = x^2 + y^2 - x$$

at points of the square region D with vertices at $(-1, -1), (1, -1), (1, 1), (-1, 1)$.

Solution. ① let $\begin{cases} f_x = 2x - 1 = 0 \\ f_y = 2y = 0 \end{cases}$ $\Rightarrow \left(\frac{1}{2}, 0\right)$ is the only critical point.

and $f\left(\frac{1}{2}, 0\right) = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$

② Along $y = \pm 1$, $f(x,y) = x^2 + 1 - x = x^2 - x + 1$, $-1 \leq x \leq 1$

and thus $\begin{cases} f \text{ attains the min at } x = \frac{1}{2} \text{ and } f\left(\frac{1}{2}, \pm 1\right) = \frac{3}{4} \\ f \text{ attains the max at } x = -1 \text{ and } f(-1, \pm 1) = 3 \end{cases}$

③ Along $x = -1$, $f(x,y) = y^2 + 2$, $-1 \leq y \leq 1$ and thus □

f attains the $\begin{cases} \text{max } (\pm 1)^2 + 2 = 3 \text{ at } (-1, \pm 1) \\ \text{min } \quad 0 + 2 = 2 \text{ at } (-1, 0) \end{cases}$

Example. Find the maximum and minimum values attained by the function

$$f(x,y) = xy^2$$

on the region $D = \{(x,y) : x^2 + y^2 \le 3\}$.

Solution.

① Let $\begin{cases} f_x = y^2 = 0 \\ f_y = 2xy = 0 \end{cases}$ $\Rightarrow$ $(x, 0)$ is a critical point for any $x \in [-\sqrt{3}, \sqrt{3}]$

And $f(x, 0) = 0$

② Along the boundary circle $x^2 + y^2 = 3$,

$f(x,y) = x[3 - x^2] = -x^3 + 3x \triangleq f_1(x)$, $-\sqrt{3} \le x \le \sqrt{3}$

$f_1'(x) = -3x^2 + 3 = 3(1 - x^2)$, $f_1'(x) = 0 \Rightarrow x = \pm 1$

$\Rightarrow$ $\begin{cases} y = \pm\sqrt{2} \\ x = \pm 1 \end{cases}$, $\begin{cases} y = 0 \\ x = \pm\sqrt{3} \end{cases}$ are possible values of x when $f_1(x)$ attains its extrema

And $f(1, \pm\sqrt{2}) = 2$, $f(-1, \pm\sqrt{2}) = -2$, $f(\pm\sqrt{3}, 0) = 0$

□

From ① and ②, we conclude that the maximum value of $f(x,y)$ on D is $f(1, \pm\sqrt{2}) = 2$ and the minimum is $f(-1, \pm\sqrt{2}) = -2$.

Example 4. Investigate the global maxima and minima of the following functions:

(a) $h(x, y) = 1 + x^2 + y^2$ on the disk $x^2 + y^2 \leq 1$.

(b) $f(x, y) = x^2 2x + y^2 4y + 5$ on the xy-plane.

(c) $g(x, y) = x^2 y^2$ on the xy-plane.

Solution.

let $\begin{cases} f_x = -2x/(x^2+y^2)^2 \\ f_y = -2y/(x^2+y^2)^2 \end{cases}$ both fail to exist at $(0,0)$

$\Rightarrow$ The only critical point is $(0,0)$, but it is not in the domain of $f(x,y)$.

At the boundary $C = \{(x,y) : x^2+y^2 = 1\}$,

$f(x,y) = \frac{1}{1} = 1$

Note that the domain is bounded, but not closed. □

Example 6. Does the function $f = \dfrac{1}{x^2 + y^2}$ have a global maximum or minimum on the region R given by $0 < x^2 + y^2 \leq 1$? *Solution.*

As $(x,y) \to (0,0)$, $f(x,y) \to \infty$

So $f(x,y)$ has no global maximum.

□

Finding Highest and Lowest Points of Surfaces.

(a) Here the surface $z = f(x, y)$ may be defined on an **unbounded** region R.

(b) A surface opens downward has a highest point; A surface opens upward has a lowest point.

Example. $z = 6x - 8y - x^2 - y^2$ opens downward. Find the highest point.

Solution. Let $\begin{cases} z_x = 6 - 2x = 0 \\ z_y = -8 - 2y = 0 \end{cases} \Rightarrow \begin{cases} x = 3 \\ y = -4 \end{cases}$

$(3, -4)$ is the only critical point and

$f(3,-4) = 6\times 3 - 8(-4) - 3^2 - (-4)^2 = 18 + 32 - 9 - 16 = 25$

Thus, the highest point on the surface is $(3, -4, 25)$.

□

Example. $z = 3x^4 + 4x^3 + 6y^4 - 16y^3 + 12y^2$ opens upward. Find the lowest point.

Solution. Let $\begin{cases} z_x = 12x^3 + 12x^2 = 0 \\ z_y = 24y^3 + 24y = 0 \end{cases} \Rightarrow \begin{cases} x^2(x+1) = 0 \\ y(y^2+1) = 0 \end{cases}$

$\Rightarrow (0, 0)$ and $(-1, 0)$ are two critical points

$f(0,0) = 0, \quad f(-1, 0) = 3 - 4 = -1$

Thus, $(-1, 0, -1)$ is the lowest point on the surface of f.

□

Applied Maximum-Minimum Problems. Method:

1. Set up the formula f.
2. Restrict f to a bounded plane region which is enclosed by a simple curve and the desired extreme value occurs.
3. Apply the above method of finding global extrema of f.

Example. Find the maximum possible volume of a rectangular box if the sum of the lengths of its 12 edges is 6 meters.

Solution.

Step 1. (set up the formula) Let the length, width and height of the box be x, y, z meters, respectively

Then $4(x+y+z)=6$ or $x+y+z=\frac{3}{2}$, $V = x\cdot y\left(\frac{3}{2}-x-y\right)$

Step 2. (restrict the function on a domain that satisfies Thm3, and $\cdots$) $x+y \le \frac{3}{2}$

The max Volumn must be attained at some point in the interior of $D=\{(x,y) \mid x\ge 0, y\ge 0, x+y\le\frac{3}{2}\}$

let $\begin{cases} V_x = \frac{3}{2}y - 2xy - y^2 = 0 \\ V_y = \frac{3}{2}x - x^2 - 2xy = 0 \end{cases} \Rightarrow \begin{cases} \left(\frac{3}{2}-2x-y\right)y=0 \\ \left(\frac{3}{2}-x-2y\right)x=0 \end{cases}$

$\Rightarrow$ either $y=0$ or $\frac{3}{2}-2x-y=0$ and either $x=0$ or $\frac{3}{2}-x-2y=0$

Step 3. (conclusion)

$\Rightarrow \begin{cases} y=0 \\ x=0 \end{cases}$ or $\begin{cases} y=0 \\ \frac{3}{2}-x-2y=0 \end{cases}$ or $\begin{cases} x=0 \\ \frac{3}{2}-2x-y=0 \end{cases}$ or $\begin{cases} \frac{3}{2}-2x-y=0 \\ \frac{3}{2}-x-2y=0 \end{cases}$

$\Rightarrow \begin{cases} x=0 \\ y=0 \end{cases}$ or $\begin{cases} x=\frac{3}{2} \\ y=0 \end{cases}$ or $\begin{cases} x=0 \\ y=\frac{3}{2} \end{cases}$ or $\begin{cases} x=\frac{1}{2} \\ y=\frac{1}{2} \end{cases}$

Thus, the max volumn is attained when $x=y=z=\frac{1}{2}$.

$\square$

Example. A rectangular box is to be closed with a volume of 48 ft^3. The material for its top and bottom costs 3 dollar per ft^2 and the material for its four sides costs 4 dollar per ft^2. Find the dimensions that minimize the total coast of the material needed to construct the box.

Solution. Let x, y, z be the length, width and height, respectively.

$\Rightarrow$ cost $C = 3 \times 2 \times xy + 2 \times 4 \times xz + 2 \times 4 \times yz$

$= 6xy + 8xz + 8yz$ subject to

$V = xyz = 48 \Rightarrow z = 48/xy$

Thus $C = 6xy + 8x \cdot \frac{48}{xy} + 8y \cdot \frac{48}{xy} = 6xy + \frac{384}{y} + \frac{384}{x}$

Let $\begin{cases} C_x = 6y - \frac{384}{x^2} = 0 \\ C_y = 6x - \frac{384}{y^2} = 0 \end{cases} \Rightarrow 6xy = \frac{384}{x} = \frac{384}{y} \Rightarrow x = y$

Substitute $x = y$ into the second equation $6y - \frac{384}{y^2} = 0$

$\Rightarrow y^3 = \frac{384}{6} = 64 \Rightarrow y = x = 4$

$\Rightarrow (4, 4)$ is a critical point

$\Rightarrow$ And the minimal cost $C = 6 \times 4 \times 4 + \frac{384}{4} + \frac{384}{4} = 288$

The dimensions are $x = 4$, $y = 4$, $z = \frac{48}{4 \times 4} = 3$ □

§15.3 CONSTRAINED OPTIMIZATION: LAGRANGE MULTIPLIERS

Lagrange Multipliers are a way of solving optimization problems where the domain is restricted, usually to that of another function. For example: find the maximum point on the intersection of a cylinder with a plane.

Definition. *Suppose P_0 is a point satisfying the constraint $g(x, y) = 0$.*

- *f has a **local maximum** at P_0 **subject to the constraint** if $f(P_0) \geq f(P)$ for all points P near P_0 satisfying the constraint.*
- *f has a **global maximum** at P_0 subject to the constraint if $f(P_0) \geq f(P)$ for all points P satisfying the constraint.*

Theorem (Lagrange Multiplier (with one constraint)). *Let $f(x, y)$ and $g(x, y)$ be continuously differentiable functions. If the maximum (or minimum) value of $f(x, y)$ subject to the constraint*

$$g(x, y) = 0$$

occurs at a point P where $\nabla g(P) \neq 0$, then

$$\nabla f(P) = \lambda \nabla g(P)$$

for some constant λ.

The number λ is called the **Lagrange multiplier**.

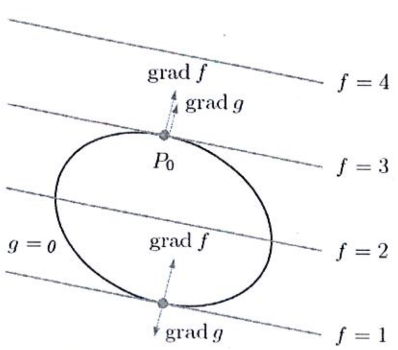


Figure 15.28: Maximum and minimum values of $f(x, y)$ on $g(x, y) = 0$ are at points where grad f is parallel to grad g

Lagrange Multipliers Method

Constrained optimization problems are frequently solved using a **Lagrangian function**, L. For example, to optimize $f(x,y)$ subject to the constraint $g(x,y)=0$, we use the Lagrangian function

$$L(x,y,\lambda)=f(x,y)-\lambda g(x,y).$$

To see how the function L is used, compute the partial derivatives of L w.r.t x,y and λ and solve the following system of equations to get the extreme point (x_0,y_0) and the corresponding $\lambda=\lambda_0$ value.

Step 1. Identify the quantity $z=f(x,y)$ to be maximized or minimized, subject to the constraint $g(x,y)=0$.

Step 2. List the following three equation and solve for x,y and λ (the associated values of λ are called **Lagrange multipliers**):

$$\begin{aligned} g(x,y) &=0, \\ f_x(x,y) &=\lambda g_x(x,y), \\ f_y(x,y) &=\lambda g_y(x,y). \end{aligned}$$

Then the local extremum points are found among the solutions of these equations.

Note. The maximum or minimum (or both) of f may occur at a point where $g_x(x,y)=0=g_y(x,y)$. The Lagrange multiplies method may fail to locate these exceptional points.

Example. Find the points of the rectangular hyperbola $xy = 1$ that are closest to the origin $(0, 0)$. We need to minimize $d = \sqrt{x^2+y^2}$, where

Solution. $P(x, y)$ is on the curve $xy = 1$

Let $f(x,y) = d^2 = x^2+y^2$ and $g(x,y) = xy-1 = 0$

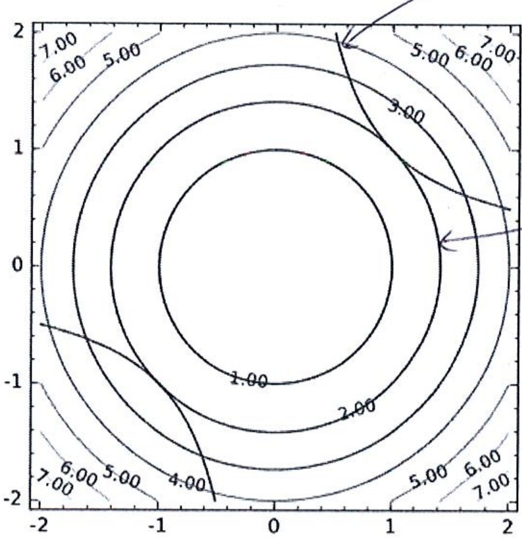
$$\begin{cases} 2x = \lambda y \\ 2y = \lambda x \\ xy - 1 = 0 \end{cases} \Rightarrow \begin{cases} 2x^2 = \lambda xy = 2y^2 \\ xy = 1 \end{cases} \Rightarrow x = y$$

$\Rightarrow$ x and y have the same sign

$x^2 = 1 \Rightarrow (x=1, y=1)$ or $(x=-1, y=-1)$ □

Remark: This example illustrates an interesting geometric interpretation of the Lagrange Multipliers Theorem. We see in the following figure the constraint curve $g(x, y) = 0$ together with typical level curves of the function $f(x, y)$. Because the gradient vectors ∇f and ∇g are normal to the level curves of the functions f and g, respectively. It follows that the curves $f(x, y) = M$ and $g(x, y) = 0$ are tangent to one another at the point P where the two gradient vectors are collinear and f attains its maximum (or minimum) value M. In effect, the Lagrange multiplier criterion serves to select, from among the level curves of f, the one that is tangent to the constraint curve at P.

$xy = 1$

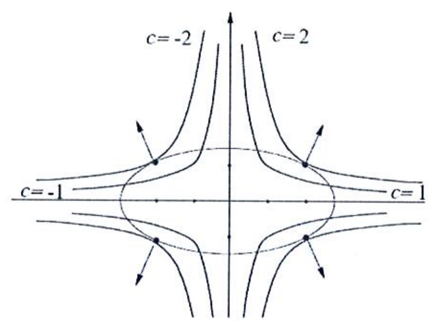


$x^2 + y^2 = 2$

Example.(SageMath) Find the extremal values of the function $f(x, y) = xy$ subject to the constraint

$$g(x, y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0.$$

Solution.



$$\begin{cases} y = \lambda \frac{x}{4} & ① \\ x = \lambda y & ② \end{cases} \Rightarrow \lambda^2 = 4 \Rightarrow \lambda = \pm 2$$

$x^2 + 4y^2 - 8 = 0$ ③ and $x = \pm 2y$

$(\pm 2y)^2 + 4y^2 - 8 = 0 \Rightarrow 8y^2 = 8 \Rightarrow y = \pm 1$

and $x = \pm 2$

$\Rightarrow$ There are four extermal points $(2,1)$, $(2,-1)$, $(-2,-1)$, $(-2,1)$

i) The maximum value 2 is attained at points $(2,1)$, $(-2,-1)$

ii) The min value -2 is attained at points $(2,-1)$, $(-2,1)$.

□

Example 1. Find the maximum and minimum values of $x+y$ on the circle $x^2+y^2 = 4$.

Solution.

$f(x,y) = x+y$, $g(x,y) = x^2+y^2-4$

$$\begin{cases} 1 = \lambda \cdot 2x \\ 1 = \lambda \cdot 2y \\ x^2+y^2 = 4 \end{cases} \qquad \lambda \cdot 2x = \lambda \cdot 2y \Rightarrow x = y$$

$(y^2) + y^2 = 4 \Rightarrow 2y^2 = 4 \Rightarrow y^2 = 2$

$\Rightarrow$ extremal points $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$

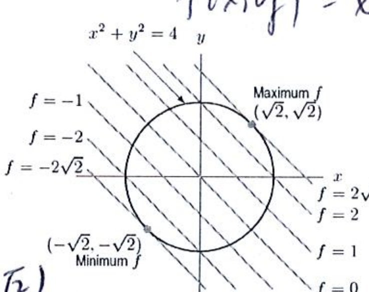


Figure 15.29: Maximum and minimum values of $f(x, y) = x + y$ on the circle $x^2 + y^2 = 4$ are at points where contours of f are tangent to the circle

i) The max value $2\sqrt{2}$ of f is attained at $(\sqrt{2}, \sqrt{2})$

ii) The min value $-2\sqrt{2}$ of f is attained at $(-\sqrt{2}, -\sqrt{2})$

Optimization with Inequality Constraints

Strategy for Optimizing $f(x, y)$ Subject to the Constraint $g(x, y) \leq 0$

- Find all points in the region $g(x, y) < 0$ where grad f is zero or undefined.
- Use Lagrange multipliers to find the local extrema of f on the boundary $g(x, y) = 0$.
- Evaluate f at the points found in the previous two steps and compare the values.

$f(x,y) = (x-1)^2 + (y-2)^2$

Example 2. Find the maximum and minimum values of ~~$x+y$~~ on the circle $x^2 + y^2 \leq 45$.

Solution.

i) let $\begin{cases} f_x = 2(x-1) = 0 \\ f_y = 2(y-2) = 0 \end{cases} \Rightarrow (1, 2)$ is a critical point in the interior of the region since $1^2 + 2^2 < 45$.

ii) let $\begin{cases} \cancel{2}(x-1) = \lambda \cdot \cancel{2}x \\ \cancel{2}(y-2) = \lambda \cdot \cancel{2}y \\ x^2 + y^2 = 45 \end{cases}$ Note that $x \neq 0$, $-2 = 0$ otherwise. Similarly $y \neq 0$.

$$\Rightarrow \frac{x-1}{x} = \frac{y-2}{y} \Rightarrow y = 2x$$

$$x^2 + (2x)^2 = 45 \Rightarrow 5x^2 = 45 \Rightarrow x^2 = 9$$

$$\Rightarrow x = \pm 3, \quad y = \pm 6$$

$\begin{cases} f(1,2) = 0 \\ f(3,6) = 20 \\ f(-3,-6) = 80 \end{cases} \Rightarrow$ The min of f is at $(1,2)$ and the max of f is at $(-3,-6)$

Lagrange Multipliers in Three Dimensions

Now suppose that $f(x, y, z)$ and $g(x, y, z)$ are continuously, differentiable functions and that we want to find the points on the surface

$$g(x, y, z) = 0$$

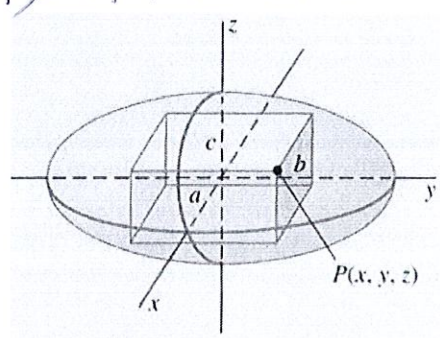
at which the function $f(x, y, z)$ attains its maximum and minimum values.

To find the possible locations of the extrema of f subject to the constraint g we can attempt to solve simultaneously the four equations

$$\begin{aligned} g(x, y, z) &= 0, \\ f_x(x, y, z) &= \lambda g_x(x, y, z), \\ f_y(x, y, z) &= \lambda g_y(x, y, z), \\ f_z(x, y, z) &= \lambda g_z(x, y, z). \end{aligned}$$

Example. Find the maximum volume of a rectangular box inscribed in the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ with its faces parallel to the coordinate planes.

Solution. Let $P(x, y, z)$ be the vertex of the box that lies in the first octant.



$$\begin{cases} V(x,y,z) = 8xyz \\ g(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \end{cases}$$

Let $\begin{cases} 8yz = \lambda \cdot 2x/a^2 \\ 8xz = \lambda \cdot 2y/b^2 \\ 8xy = \lambda \cdot 2z/c^2 \end{cases} \Rightarrow 8xyz = 2\lambda \frac{x^2}{a^2} = 2\lambda \frac{y^2}{b^2} = 2\lambda \frac{z^2}{c^2}$

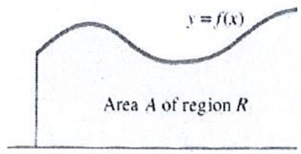
$\Rightarrow x^2/a^2 = y^2/b^2 = z^2/c^2$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$\Rightarrow \frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} = \frac{1}{3} \Rightarrow x = \frac{a}{\sqrt{3}},\ y = \frac{b}{\sqrt{3}},\ z = \frac{c}{\sqrt{3}}$ □

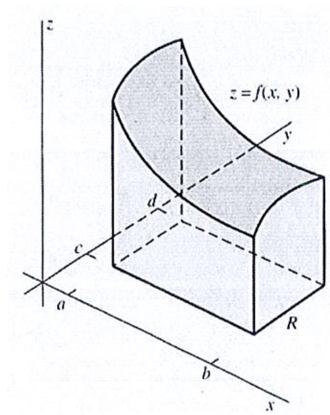
And $V_{max} = 8 \cdot \frac{abc}{3\sqrt{3}} = \frac{8}{3\sqrt{3}} abc$

§16.1 The Definite Integral of a Function of Two Variables

Recall: Suppose $f(x) \geq 0$,



Now,



Volume of the solid bounded above by the graph $z = f(x, y)$ of the nonnegative function f over the rectangle R in the $x - y$ plane is given by

$$V = \int\int_R f(x, y)dA,$$

where dA represents a differential element of area A.

Definition (Double Integral). ***Partition*** *$\mathcal{P}$ of R into sub-rectangles $R_1, \ldots, R_k$ determined by the points*

$$a = x_0 < x_1 < x_2 < \cdots < x_m = b$$

and

$$c = y_0 < y_1 < y_2 < \cdots < y_n = d.$$

Next, we choose an arbitrary point (x_i^*, y_i^*) of the ith rectangle R_i for each i, $1 \leq i \leq k$. The collection of points $S = \{(x_i^*, y_i^*) | 1 \leq i \leq k\}$ is called a **selection** *for the partition $\mathcal{P} = \{R_i | 1 \leq i \leq k\}$. As a measure of the* size *of the rectangles of the partition $\mathcal{P}$, we* *define its* **norm** $|\mathcal{P}|$ *to be the maximum of the lengths of the diagonals of the rectangles* $\{R_i\}$*. Thus, the volume of the rectangular column with base* R_i *and height* $f(x_i^*, y_i^*)$ *is given by* $V_i = f(x_i^*, y_i^*)\Delta A_i$*, where* ΔA_i *denotes the area of* R_i*. The volume of the solid is then approximated by the* **Riemann sum** *(the sum of* V_i*)*

$$\sum_{i=1}^{k} f(x_i^*, y_i^*)\Delta A_i.$$

We therefore define the **double integral** *of the rectangle* R *to be*

$$V = \int\int_R f(x,y)dA \triangleq \lim_{|\mathcal{P}|\to 0} \sum_{i=1}^{k} f(x_i^*, y_i^*)\Delta A_i$$

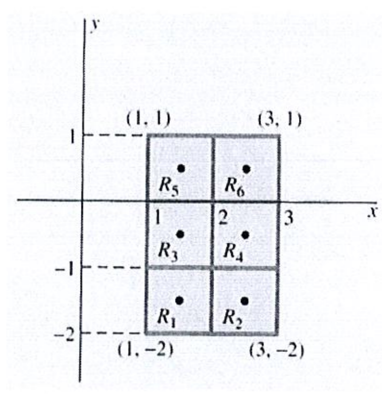
provided that the limit exists.

Remark. (1) The limit exists if $f(x,y)$ is continuous on R.
(2) The limit (if exists) does not depend on $\mathcal{P}$ or the choice of S. For example, one partition could be $x_i = a + i\frac{b-a}{m}, i = 0, \ldots, m$ and $y_j = c + j\frac{d-c}{n}, j = 0, \ldots, n$. And $|\mathcal{P}| \to 0 \Leftrightarrow m \to \infty$ and $n \to \infty$.

Example. Approximate the value of the integral

$$\int\int_R (4x^3 + 6xy^2)dA$$

over the rectangle $R:\ [1,3]\times[-2,1]$ by calculating the Riemann sum for the partition illustrated by the following graph with (x_i^*, y_i^*) selected as the center of R_i.



Solution.

$\Delta A_i = 1$, $(x_1^*, y_1^*) = (1.5, -1.5)$, $(x_2^*, y_2^*) = (2.5, -1.5)$

$(x_3^*, y_3^*) = (1.5, -0.5)$, $(x_4^*, y_4^*) = (2.5, -0.5)$

$(x_5^*, y_5^*) = (1.5, 0.5)$ $(x_6^*, y_6^*) = (2.5, 0.5)$

With $f(x,y) = 4x^3 + 6xy^2$, the Riemann Sum is

$$\sum_{i=1}^{6} f(x_i^*, y_i^*)\,\Delta A_i = \sum_{i=1}^{6} f(x_i^*, y_i^*) = 4\sum_{i=1}^{6} {x_i^*}^3 + 6\sum_{i=1}^{6} x_i^* {y_i^*}^2$$

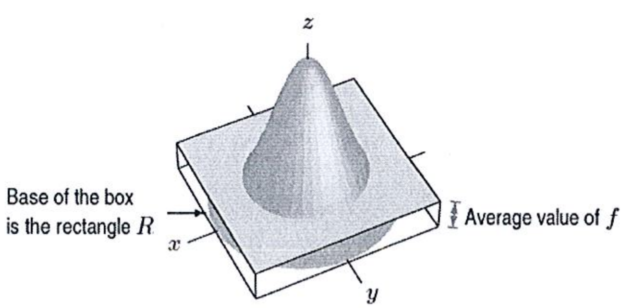
$$= 4\times(1.5^3 + 2.5^3 + 1.5^3 + 2.5^3 + 1.5^3 + 2.5^3) + 6\left[1.5^3 + 2.5\times1.5^2 + 1.5\times0.5^2 + 2.5\times0.5^2 + 1.5\times0.5^2 + 2.5\times2.5^2\right]$$

$$= \cdots = 294 \Rightarrow \iint_R (4x^3 + 6xy^2)\,dA \simeq 294$$

□

Interpretation of the Double Integral as Area and Average Value:

If x, y, z represent length and f is positive, then
Average value of f on region R $= \dfrac{1}{\text{Area of R}} \int_R f\, dA.$



§16.2 Iterated Integrals

Theorem (Double Integrals as Iterated Single Integrals). *Suppose that $f(x, y)$ is continuous on the rectangle $R: [a, b] \times [c, d]$. Then*

$$\int\int_R f(x,y)dA = \int_a^b \left(\int_c^d f(x,y)dy\right) dx = \int_c^d \left(\int_a^b f(x,y)dx\right) dy$$

Remark.

- This theorem tells us how to compute a double integral by means of two successive (or iterated) single-variable integrations, each of which we can compute by using the fundamental theorem of calculus (if the function f is sufficiently well behaved on R).
- For any function we are likely to meet, it does not matter in which order we integrate over a rectangular region R; we get the same value for the double integral either way.

Example. Evaluate the integral

$$\int\int_R (4x^3 + 6xy^2)dA$$

over the rectangle $R: [1, 3] \times [-2, 1]$.

Solution. $\iint_R (4x^3+6xy^2)dA = \int_1^3 \left[\int_{-2}^1 (4x^3+6xy^2)dy\right] dx$

$= \int_1^3 \left[4x^3y + 6x\frac{y^3}{3}\right]_{y=-2}^{1} dx = \int_1^3 [(4x^3+2x)-(-8x^3-16x)]dx$

$= \int_1^3 [12x^3+18x]dx = \left[\frac{12}{4}x^4 + \frac{18}{2}x^2\right]_1^3$

$= [3(3^4)+9\times 3^2] - [3+9] = [243+81] - [12] = 312.$

□

Example 1.(SageMath) Evaluate $\int_0^{\pi/2}\int_0^{\pi/2}(\cos x \sin y)\, dydx$.

Solution. $= \int_0^{\frac{\pi}{2}} \cos x(-\cos y)\Big|_{y=0}^{\frac{\pi}{2}} dx = -\int_0^{\frac{\pi}{2}} \cos x\,(0-1)\,dx$

$= \int_0^{\frac{\pi}{2}} \cos x\, dx = \sin x\Big|_0^{\frac{\pi}{2}} = 1.$

□

Example 2. Evaluate $\int_0^1\int_{-2}^2 x^2 e^y\, dxdy$.

Solution. $= \int_0^1 e^y \frac{x^3}{3}\Big|_{x=-2}^{2} dy = \int_0^1 e^y \frac{1}{3}(8++8)\,dy$

$= \frac{16}{3}\int_0^1 e^y dy = \frac{16}{3} e^y\Big|_0^1 = \frac{16}{3}(e-1).$

□

Example 3. Evaluate $\int_0^{\pi/2}\int_0^{\pi/2}(y-1)\cos x\, dxdy$.

Solution. $= \int_0^{\frac{\pi}{2}} (y-1)\sin x\Big|_{x=0}^{\frac{\pi}{2}} dy$

$= \int_0^{\frac{\pi}{2}} (y-1)(1-0)\,dy = \left(\frac{y^2}{2} - y\right)\Big|_0^{\frac{\pi}{2}}$

$= \frac{1}{2}\left(\frac{\pi^2}{4}\right) - \frac{\pi}{2} = \frac{\pi^2}{8} - \frac{\pi}{2}$

□

Example 4. Evaluate $\int_1^e \int_1^e \frac{1}{xy}\, dydx$.

Solution.

$$= \int_1^e \frac{1}{x}\left[\int_1^e \frac{1}{y} dy\right] dx = \int_1^e \frac{1}{x} \ln y \Big|_{y=1}^{e} dx$$

$$= \int_1^e \frac{1}{x}(1-0)\, dx = \ln x \Big|_1^e = 1-0 = 1$$

□

Example 5. Evaluate $\int_1^2 \int_1^3 \left(\frac{x}{y} + \frac{y}{x}\right) dxdy$.

Solution.

$$= \int_1^2 \left[\frac{x^2}{2y} + y\ln x\right]_{x=1}^{3} dy$$

$$= \int_1^2 \left[\left(\frac{9}{2y} + y\ln 3\right) - \left(\frac{1}{2y} + 0\right)\right] dy$$

$$= \int_1^2 \left(\frac{8}{2y} + y\ln 3\right) dy$$

$$= \left[4\ln y + \frac{\ln 3}{2} y^2\right]_1^2$$

$$= 4\ln 2 + 2\ln 3 - \left(0 + \frac{\ln 3}{2}\right)$$

$$= 4\ln 2 + \frac{3\ln 3}{2}$$

□

Double Integrals Over More General Regions

The Double Integral of a bounded function f over a more general plane region R can be defined similarly as that in the Theorem.

Evaluation of Double Integrals

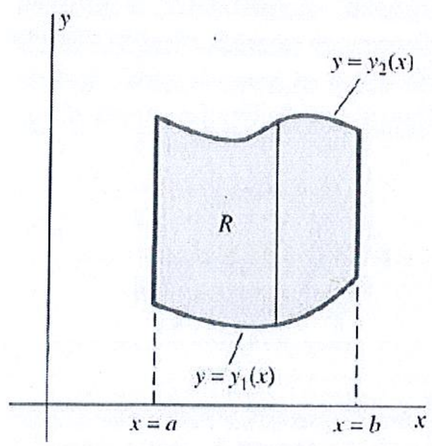


Figure 16.1: A vertically simple region R

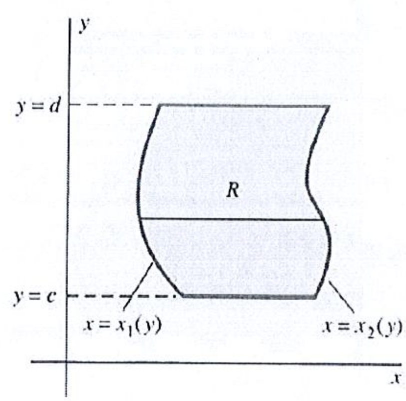


Figure 16.2: A horizontally simple region R

Theorem (Evaluation of Double Integrals). *Suppose that $f(x,y)$ is continuous on the region R. If R is the vertically simple region given by*

$a \le x \le b, y_1(x) \le y \le y_2(x)$, *where $y_1(x), y_2(x)$ are continuous functions of x on $[a,b]$,*

then

$$\int\int_R f(x,y)\,dA = \int_a^b \int_{y_1(x)}^{y_2(x)} f(x,y)\,dydx.$$

If R is the horizontally simple region given by

$c \le y \le d, x_1(y) \le x \le x_2(y)$, *where $x_1(y), x_2(y)$ are continuous functions of y on $[c,d]$,*

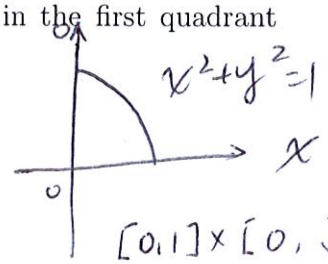
then

$$\int\int_R f(x,y)\,dA = \int_c^d \int_{x_1(y)}^{x_2(y)} f(x,y)\,dxdy.$$

Limits on Iterated Integrals:

- The limits on the outer integral must be constants.
- The limits on the inner integral can involve only the variable in the outer integral. For example, if the inner integral is with respect to x, its limits can be functions of y.

Example. Evaluate $\int\int_R (x+y)\,dA$, where R is the region in the first quadrant bounded by the unit circle and the coordinate axes.



$[0,1]\times[0,\sqrt{1-x^2}]$

Solution.

$$= \int_0^1 \left[\int_0^{\sqrt{1-x^2}} (x+y)dy\right]dx$$

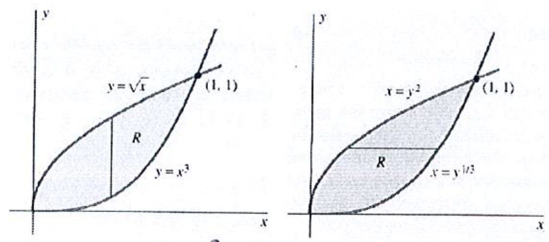
$$= \int_0^1 \left[xy + \frac{y^2}{2}\right]_{y=0}^{\sqrt{1-x^2}} dx = \int_0^1 \left(x\sqrt{1-x^2} + \frac{1-x^2}{2}\right)dx$$

$$= \int_0^1 x\sqrt{1-x^2}\,dx + \int_0^1 \frac{1-x^2}{2}\,dx$$

$$= \int_0^1 -\frac{1}{2}(1-x^2)^{\frac{1}{2}} d(1-x^2) + \left(\frac{x}{2} - \frac{x^3}{6}\right)\Big|_0^1 = -\frac{1}{2}\frac{(1-x^2)^{\frac{3}{2}}}{\frac{3}{2}}\Big|_0^1 + \left(\frac{1}{2}-\frac{1}{6}\right) \qquad \square$$

$$= -\frac{1}{3}(0-1) + \frac{2}{6} = \frac{2}{3}$$

Example. Evaluate $\int\int_R xy^2\, dA$, where R is the first-quadrant region bounded by the two curves $y=\sqrt{x}$ and $y=x^3$.



Method 1: $R=[0,1]\times[x^3,\sqrt{x}]$

Solution.

$$\iint_R xy^2 dA = \int_0^1\int_{x^3}^{\sqrt{x}} xy^2\,dy\,dx = \int_0^1 [\frac{xy^3}{3}]_{y=x^3}^{\sqrt{x}} dx$$

$$=\int_0^1\left(\frac13 x^{\frac52} - \frac13 x^{10}\right)dx = \left[\frac13\cdot\frac27\cdot x^{\frac72} - \frac13\frac1{11}x^{11}\right]_0^1 = \frac{2}{21}-\frac1{33}=\frac{5}{77}$$

Method 2. $R: [y^2, y^{\frac13}]\times[0,1]$

$$\iint_R xy^2 dA = \int_0^1\left[\int_{y^2}^{y^{\frac13}} xy^2\,dx\right] dy = \int_0^1 \left[\frac{x^2y^2}{2}\right]_{x=y^2}^{y^{\frac13}} dy$$

$$=\int_0^1\left(\frac12 y^{\frac83} - \frac{y^6}{2}\right)dy = \left[\frac12\cdot\frac3{11}y^{\frac{11}{3}} - \frac1{14}y^7\right]_0^1 = \frac3{22}-\frac1{14}=\frac5{77}$$

□

Example. Evaluate $\int_0^2\int_{y/2}^1 dxdy$.

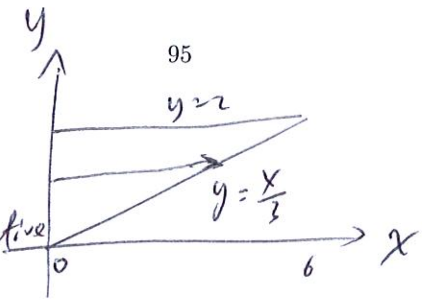
Solution.

$$=\int_0^2\left(1-\frac y2\right)dy = \left(y-\frac{y^2}{4}\right)_0^2$$

$$=\left(2-\frac{2^2}{4}\right) = 2-1=1$$

□

Example 7.(SageMath) Evaluate $\int_0^6 \int_{x/3}^2 x\sqrt{y^3+1}\,\cancel{dx}\,dy\,dx$



Solution. $R = [0,6]\times[{}^{2}x/3, 2]$

Method 1. $\sqrt{y^3+1}$ has no elementary antiderivative

Method 2. Write $R = [0, 3y]\times[0,2]$

$$\iint_R x\sqrt{y^3+1}\,dA = \int_0^2\int_0^{3y} x\sqrt{y^3+1}\,dx\,dy = \int_0^2\left[\sqrt{y^3+1}\,\frac{x^2}{2}\Big|_{x=0}^{3y}\right]dy$$

$$= \int_0^2 \sqrt{y^3+1}\,\frac{9y^2}{2}\,dy = \int_0^2 \frac{9}{2}\cdot\frac{1}{3}(y^3+1)^{\frac{1}{2}}\,d(y^3+1)$$

$$= \frac{3}{2}\cdot\frac{(y^3+1)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^2 = (8+1)^{\frac{3}{2}} - 1 = 27-1 = 26$$

□

Properties of Double Integrals

Let c be a constant and f and g be continuous functions on a region R on which $f(x,y)$ attains a minimum value m and a maximum value M. Let $a(R)$ denote the area of the region R. If all the indicated integrals exist, then:

- $\int\int_R cf(x,y)\,dA = s\int\int_R f(x,y)\,dA$
- $\int\int_R [f(x,y)+g(x,y)]\,dA = \int\int_R f(x,y)\,dA + \int\int_R g(x,y)\,dA$
- $m\cdot a(R) \le \int\int_R f(x,y)\,dA \le M\cdot a(R)$
- $\int\int_R f(x,y)\,dA = \int\int_{R_1} f(x,y)\,dA + \int\int_{R_2} f(x,y)\,dA$, where R_1 and R_2 are simply two non-overlapping regions (with disjoint interiors) with union R

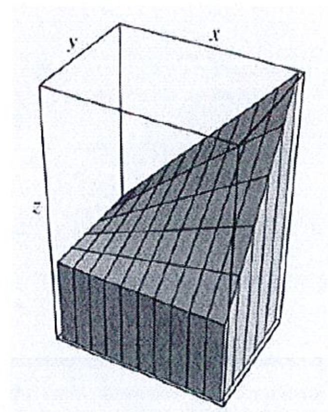
Geometric interpretations of double integrals:

(1) **Volume below** $z = f(x,y)$: Suppose that the nonnegative(≥ 0) function f is continuous on the bounded plane region R. Then the volume V of the solid that

Example. The rectangle R in the xy-plane consists of those points (x, y) for which $0 \le x \le 2$ and $0 \le y \le 1$. Find the volume V of the solid that lies below the surface $z = 1 + xy$ and above R.

Solution.

$$V = \iint_R z\,dA = \int_0^2 \int_0^1 (1+xy)\,dy\,dx$$



$$= \int_0^2 \left(y + \frac{xy^2}{2} \right)\Big|_{y=0}^{1} dx$$

$$= \int_0^2 \left(1 + \frac{x}{2}\right) dx$$

$$= \left(x + \frac{1}{4}x^2\right)\Big|_0^2 = 3$$

□

One Application of Double Integrals

Consider a plane region R that corresponds to a thin plate or lamina of uniform (or constant) density $\delta(x, y)$ - a continuous function. Then the **mass** m of the lamina is

$$m = \int\int_R \delta(x, y)\, dA.$$

Example 4. Find the mass M of a metal plate R bounded by $y = x$ and $y = x^2$, with density given by $\delta(x, y) = 1 + xy$ kg/meter2. (See Figure 16.17.)

Solution.

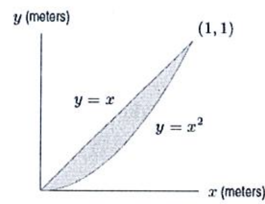


Figure 16.17: A metal plate with density $\delta(x, y)$

$$M = \iint_R \delta(x, y)\, dA$$

where $R = [0, 1] \times [x^2, x]$

$$\Rightarrow M = \int_0^1 \int_{x^2}^{x} (1 + xy)\, dy\, dx$$

$$= \int_0^1 \left(y + \frac{xy^2}{2}\right)\Big|_{y=x^2}^{x} dx$$

$$= \int_0^1 \left[\left(x + \frac{x^3}{2}\right) - \left(x^2 + \frac{x^5}{2}\right)\right] dx$$

$$= \left[\frac{x^2}{2} + \frac{x^4}{8} - \frac{x^3}{3} - \frac{x^6}{12}\right]_0^1$$

$$= \frac{1}{2} + \frac{1}{8} - \frac{1}{3} - \frac{1}{12} = \frac{5}{24} \simeq 0.2083$$

§16.3 TRIPLE INTEGRALS

The definition of the triple integral is the three-dimensional version of the definition of the double integral. Let $f(x, y, z)$ be continuous on the bounded space region W. And suppose that W lies inside the rectangular block R determined by the inequalities

$$a \leq x \leq b, c \leq y \leq d, p \leq z \leq q.$$

The triple integral defined by limit of Riemann sums is

$$\iiint_W f(x, y, z)\, dV,$$

where dV is differentia element of volume.

Mass m of solid body W with density $\delta(x, y, z)$ is

$$m = \iiint_W \delta(x, y, z)\, dV.$$

Volume V of solid W (with density $\delta(x, y, z) \equiv 1$ is

$$V = \iiint_W dV.$$

Note. We compute triple integrals by means of iterated integrals.

Limits on Triple Integrals

- The limits for the outer integral are constants.
- The limits for the middle integral can involve only one variable (that in the outer integral).
- The limits for the inner integral can involve two variables (those on the two outer integrals).

(i)**Case 1:** If W is a **rectangular block**, we can integrate in any order.

Example 1. (SageMath) A cube C has sides of length 4 cm and is made of a material of variable density. If one corner is at the origin and the adjacent corners are on the positive x, y, and z axes, then the density at the point (x, y, z) is $\delta(x, y, z) = 1 + xyz$ gm/cm^3. Find the mass of the cube.

Solution.

$f(x, y, z) = 1 + xyz$ and $W = \{(x, y, z) | 0 \leq x \leq 4, 0 \leq y \leq 4, 0 \leq z \leq 4\}$.

$$M = \iiint_W \delta(x,y,z)\,dV = \int_0^4 \int_0^4 \underline{\int_0^4 (1+xyz)\,dx}\,dy\,dz$$

$$= \int_0^4 \int_0^4 \left(x + \frac{1}{2} yz x^2\right)\Big|_{x=0}^{4} dy\,dz$$

$$= \int_0^4 \underline{\int_0^4 (4 + 8yz)\,dy}\,dz$$

$$= \int_0^4 (4y + 4y^2 z)\Big|_{y=0}^{y=4} dz = \int_0^4 (16 + 64z)\,dz$$

$$= (16z + 32z^2)\Big|_0^4 = 16 \times 4 + 32 \times 16 = 576$$

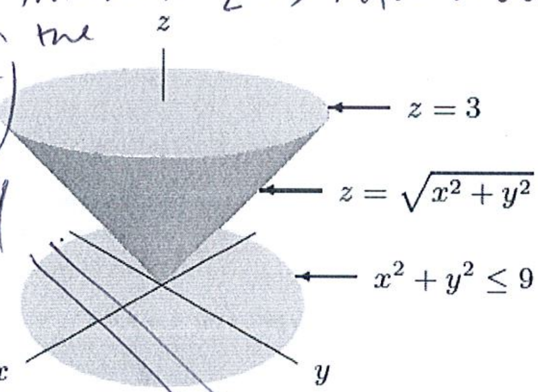
$\square$

(ii) **Case 2:** W is z-simple: $z_1(x,y) \le z \le z_2(x,y)$, $(x,y) \in R$, where R is the vertical projection of W into xy-plane. Then

$$\iiint_W f(x,y,z)\,dV = \iint_R \left[\int_{z_1(x,y)}^{z_2(x,y)} dz\right] dA, \quad dA = dxdy \text{ or } dA = dydx.$$

Example 3.(SageMath) Set up an iterated integral to compute the mass of the solid cone bounded by $z = \sqrt{x^2+y^2}$ and $z = 3$, if the density is given by $\delta(x,y,z) = z$.

Solution. The cone $z = \sqrt{x^2+y^2}$ intersects $z=3$ in the circle $x^2+y^2=9$
$\Rightarrow$ there is a stack for all (x,y) in the region $x^2+y^2 \le 9$ ($r=3$ for the circle)



Lining up the stacks parallel to the y-axis gives a slice from $y = -\sqrt{9-x^2}$ to $y = \sqrt{9-x^2}$ for each fixed x. ($-3 \le x \le 3$)

$\Rightarrow W = \{(x,y,z) \mid -3 \le x \le 3,\ -\sqrt{9-x^2} \le y \le \sqrt{9-x^2},\ \sqrt{x^2+y^2} \le z \le 3\}$

$$\Rightarrow M = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{\sqrt{x^2+y^2}}^{3} z\,dz\,dy\,dx$$

(Note. Setting up the limits on the two outer integrals is just like setting up the limits for a double integral over the region $x^2+y^2 \le 9$)

□

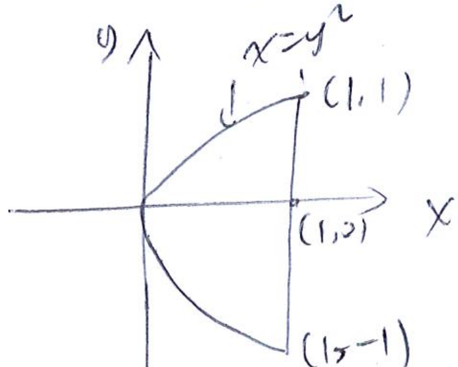
(iii) **Case 3:** W is y-simple: $y_1(x,z) \le y \le y_2(x,z)$, $(x,z) \in R$, where R is the vertical projection of T into xz-plane. Then

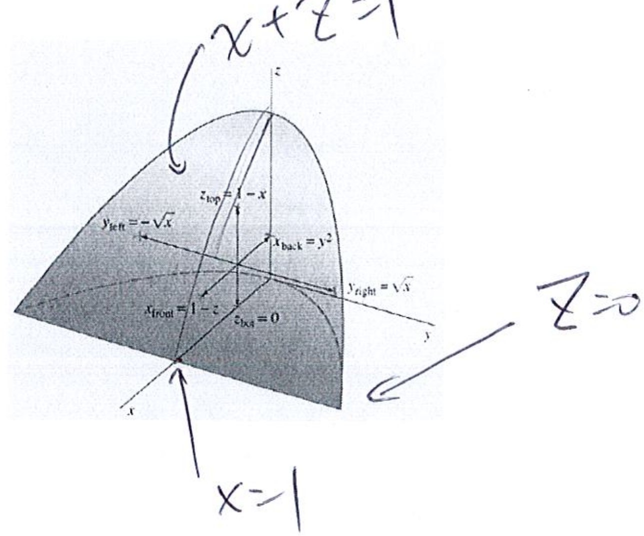
$$\iiint_W f(x,y,z)\,dV = \iint_R \left[\int_{y_1(x,z)}^{y_2(x,z)} \,dy\right] dA, \quad dA = dxdz \text{ or } dA = dzdx.$$

Example. Compute by triple integration the volume of the region T that is bounded by the parabolic cylinder $x = y^2$ and the planes $z = 0$ and $x + z = 1$.

Solution.

$0 \le z \le 1-x$





$y^2 \le x \le 1$

$-1 \le y \le 1$

$$\Rightarrow V = \int_{-1}^{1}\int_{y^2}^{1}\int_0^{1-x} dz\,dx\,dy = 2\int_0^1\int_{y^2}^1 (1-x)\,dx\,dy$$

$$= 2\int_0^1 \left[x - \frac{x^2}{2}\right]_{x=y^2}^{1} dy = 2\int_0^1 \left(\frac{1}{2} - y^2 + \frac{1}{2}y^4\right) dy$$

$$= \cdots = \frac{8}{15}.$$ □

or $-\sqrt{x} \le y \le \sqrt{x}$

$0 \le x \le 1$

$0 \le z \le 1-x$

(iv)**Case 4:** W is x-simple: $x_1(y,z) \le x \le x_2(y,z)$, $(y,z) \in R$, where R is the vertical projection of T into yz-plane. Then

$$\iiint_W f(x,y,z)\,dV = \iint_R \left[\int_{x_1(y,z)}^{x_2(y,z)} dx\right] dA, \quad dA = dydz \text{ or } dA = dzdy.$$

Example. Find the volume of the oblique segment of a paraboloid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = y + 2$.

Method 1 $x^2+y^2 \le z \le y+2$

Circular disk: $x^2+y^2=y+2$

Solution.

□

§16.4 Double Integrals in Polar Coordinates

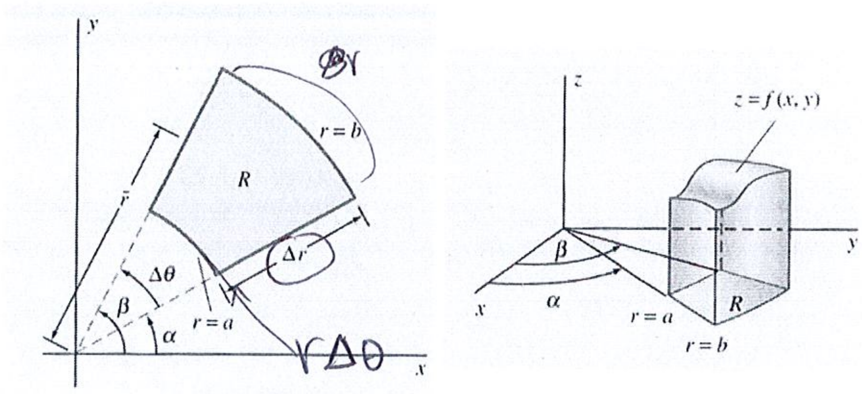


Figure 16.3: A polar rectangle R: $a \le r \le b, \alpha \le \theta \le \beta$ and a solid region whose base is the polar rectangle R

$$\int\int_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\boxed{?}\,drd\theta$$

$$= \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r\,drd\theta,$$

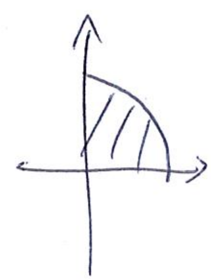
since $dA = dr \cdot rd\theta = rdrd\theta$.

Example.

$$\int\int_R (x+y)\,dA = \int_0^1 \int_0^{\sqrt{1-x^2}} (x+y)\,dydx = \frac{2}{3}.$$

Find the integral using polar coordinates.

Solution. $R:\ x^2+y^2 \le 1$, 1st quadrant $\Rightarrow R:\ 0 \le r \le 1,\ 0 \le \theta \le \frac{\pi}{2}$



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

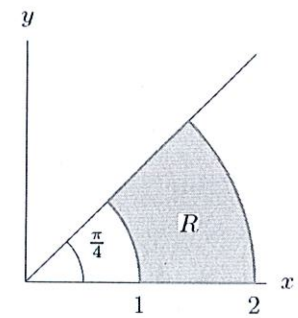
$$\int\int_R (x+y)\,dA = \int_0^1 \int_0^{2\pi} r(\cos\theta + \sin\theta)\, r\,drd\theta$$

$$= \int_0^1 r^2 (\sin\theta - \cos\theta)\Big|_0^{\frac{\pi}{2}} dr = \int_0^1 r^2\left((1-0) - (0-1)\right) dr$$

$$= 2\int_0^1 r^2 dr = \frac{2}{3} r^3\Big|_0^1 = \frac{2}{3}$$

□

Example 2.(SageMath) Compute the integral of $f(x, y) = 1/(x^2 + y^2)^{3/2}$ over the region R shown in Figure 16.36



$1 \le r \le 2$

$0 \le \theta \le \frac{\pi}{4}$

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

Figure 16.4: Figure 16.36: Integrate f over the polar region

Solution.

First, $f(x,y) = \frac{1}{(r^2)^{\frac{3}{2}}} = \frac{1}{r^3}$

$$\iint_R f\,dA = \int_0^{\pi/4}\int_1^2 \frac{1}{r^3}\, r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left(\int_1^2 r^{-2}\,dr\right) d\theta = \int_0^{\frac{\pi}{4}} -\frac{1}{r}\Big|_{r=1}^{r=2} d\theta$$

$$= \int_0^{\pi/4} \frac{1}{2}\, d\theta = \frac{\pi}{8}$$

$\square$

More General Polar-Coordinate Regions

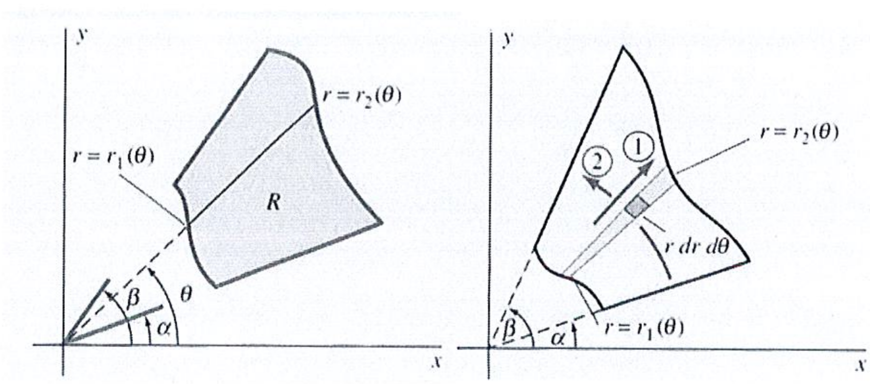


Figure 16.5: A radially simple region R: $\alpha \le \theta \le \beta$, $r_1(\theta) \le r \le r_2(\theta)$; Integrating first with respect to r and then with respect to θ.

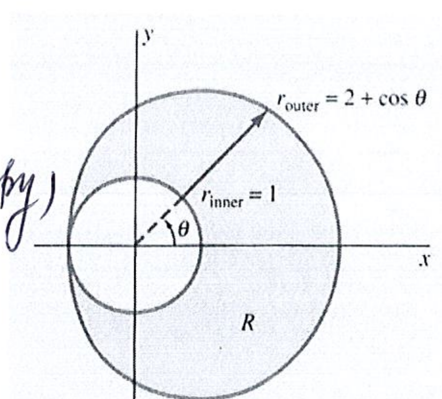
$$\int\int_R f(x, y)\, dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta) r\; dr d\theta.$$

And the area of R is given by

$$A = \int\int_R dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} r\; dr d\theta.$$

Example. The region R is bounded on the inside by the circle $r = 1$ and on the outside by the limacon $r = 2 + \cos\theta$. Find the area of the region.

Solution.



$$A = \int_\alpha^\beta \int_{r_{inner}}^{r_{outer}} r\, dr\, d\theta$$

$$= 2\int_0^\pi \int_1^{2+\cos\theta} r\, dr\, d\theta \quad (\text{by symmetry})$$

$$= 2\int_0^\pi \frac{1}{2}\left[(2+\cos\theta)^2 - 1^2\right] d\theta$$

$$= \int_0^\pi (3 + 4\cos\theta + \cos^2\theta)\, d\theta$$

$$= \int_0^\pi \left(3 + 4\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta = \int_0^\pi \left(3 + \frac{1}{2}\right) d\theta = \frac{7\pi}{2}$$

§16.5 Triple Integrals in Cylindrical and Spherical Coordinates

Cylindrical Coordinates: (r, θ, z)

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

$dA = r dr d\theta$

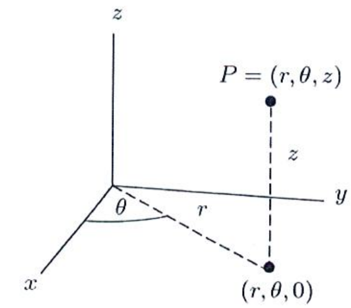


Figure 16.6: Cylindrical coordinates: (r, θ, z)

Suppose that $f(x, y, z)$ is a continuous function defined on the z-simple region T

$$z_1(x, y) \leq z \leq z_2(x, y), \quad (x, y) \in R,$$

where R is the projection of T into the xy-plane, as usual.

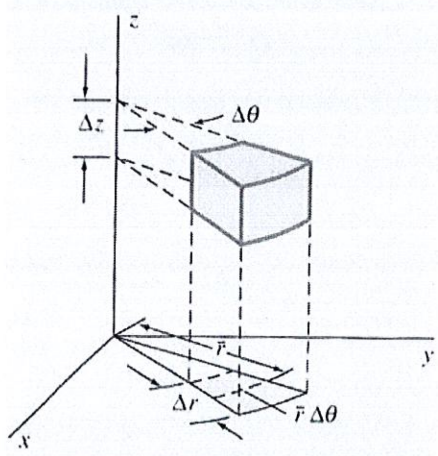
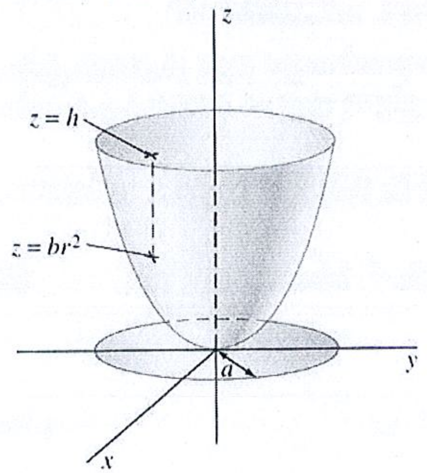


Figure 16.7: The volume of the cylindrical block is $\Delta V = \Delta r \cdot \bar{r}\Delta\theta \cdot \Delta z = \bar{r}\Delta z\Delta r\Delta\theta$

$$\int\int\int_W f(x,y,z)\,dV = \int\int_R \underbrace{\left[\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz\right]}_{F(x,y)} dA$$
$$= \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} F(r\cos\theta, r\sin\theta)\, r dr d\theta.$$

Example. Find the volume of the solid T that is bounded by the paraboloid $z = b(x^2+y^2)$ $(b>0)$ and the plane: $z = h$ $(h>0)$.



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

Solution. $z = b(x^2+y^2) = br^2$ and $z = h$

$br^2 = h \Rightarrow r = \sqrt{h/b}$ which is the radius of the circle over which the solid lies.

$$\Rightarrow V = \iiint_W dV = \int_0^{2\pi}\int_0^{\sqrt{h/b}}\int_{br^2}^{h} r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{h/b}} (hr - br^3)\,dr\,d\theta = 2\pi\left[\frac{1}{2}h\left(\sqrt{\frac{h}{b}}\right)^2 - \frac{1}{4}b\left(\sqrt{\frac{h}{b}}\right)^4\right]$$

$$= \frac{\pi h^2}{2b}$$

□

Example 1.(SageMath) Describe in cylindrical coordinates a wedge of cheese cut from a cylinder 4 cm high and 6 cm in radius; this wedge subtends an angle of /6 at the center. (See Figure 16.44.)

Find the mass of the wedge if its density is 1.2 grams/cm^3.

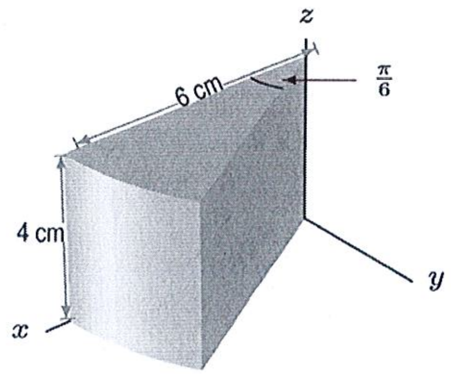


Figure 16.8: A wedge of cheese

Solution. $W: 0 \le r \le 6, \quad 0 \le z \le 4, \quad 0 \le \theta \le \frac{\pi}{6}$

$$M = \iiint_W 1.2\, dV = \int_0^4 \int_0^{\pi/6} \int_0^6 1.2 r\, dr\, d\theta\, dz$$

$$= \int_0^4 \int_0^{\pi/6} 0.6 r^2 \Big|_0^6 d\theta\, dz = 21.6 \int_0^4 \int_0^{\pi/6} d\theta\, dz$$

$$= 21.6 \cdot \frac{\pi}{6} \int_0^4 dz = 21.6 \times 4 \times \frac{\pi}{6} \simeq 45.23 g.$$

□

Spherical Coordinate Integrals

When the boundary surfaces of the region W of integration are spheres, cones, or other surfaces with simple descriptions in spherical coordinates, it is generally advantageous to transform a triple integral over W into spherical coordinates.

Relation between Cartesian and Spherical Coordinates

$$\begin{aligned} x &= \rho \sin\phi \cos\theta \\ y &= \rho \sin\phi \sin\theta \;, \quad 0 \le \phi \le \pi, 0 \le \theta \le 2\pi. \\ z &= \rho \cos\phi \end{aligned}$$

Also, $\rho^2 = x^2 + y^2 + z^2$.

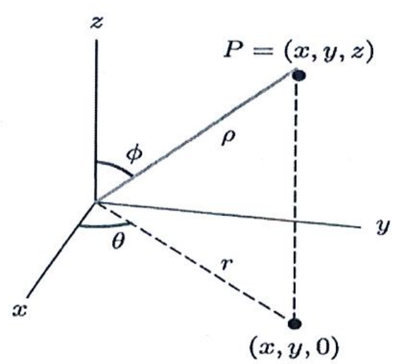


Figure 16.9: Spherical coordinates: (ρ, ϕ, θ)

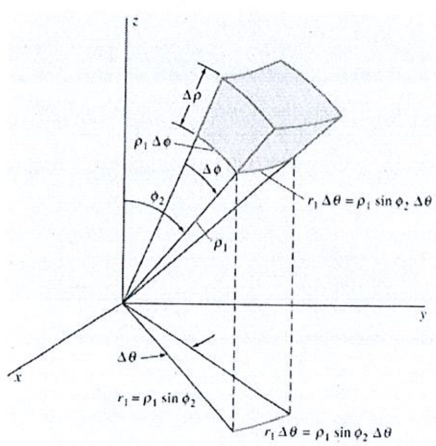


Figure 16.10: The volume of the spherical block is approximately $r_1\Delta\theta \cdot \rho_1\Delta\phi \cdot \Delta\rho = \rho_1^2 \sin\phi_2 \Delta\rho\Delta\phi\Delta\theta$.

$$W : \theta_1 \le \theta \le \theta_2, \phi_1 \le \phi \le \phi_2, \rho_1(\phi, \theta) \le \rho \le \rho_2(\phi, \theta) \quad \text{centrally simple}$$

$$\iiint_T f(x,y,z)\,dV = \int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2}\int_{\rho_1(\phi,\theta)}^{\rho_2(\phi,\theta)} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta.$$

Example 4. Show that the volume of a solid ball with radius $\rho = a$ is $\frac{4}{3}\pi a^3$.

Proof. $0 \le \rho \le a, \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi$

$V = \iiint_T 1\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi} \frac{1}{3}a^3 \sin\phi\, d\phi\, d\theta$

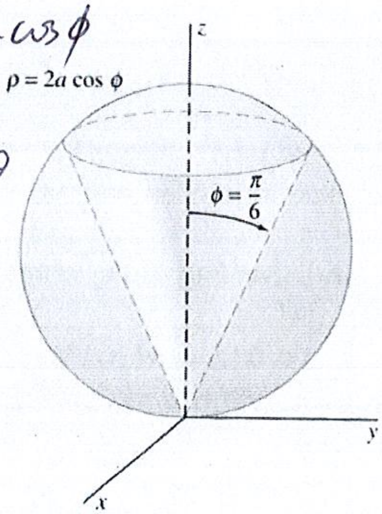
$= \frac{1}{3}a^3 \int_0^{2\pi} (-\cos\phi)\Big|_0^{\pi} d\theta = \frac{2}{3}a^3 \int_0^{2\pi} d\theta = \frac{4\pi a^3}{3}$

□

Example.(SageMath) Find the volume of the uniform "ice-cream cone" C that is bounded by the cone $\phi = \pi/6$ and the sphere $\rho = 2a\cos\phi$ of radius a.

Solution.

$0 \le \theta \le 2\pi, \quad 0 \le \phi \le \frac{\pi}{6}, \quad 0 \le \rho \le 2a\cos\phi$



$V = \int_0^{2\pi}\int_0^{\pi/6}\int_0^{2a\cos\phi} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

$= \frac{8a^3}{3}\int_0^{2\pi}\int_0^{\pi/6} \cos^3\phi \sin\phi\, d\phi\, d\theta$

$= \frac{16}{3}$

§16.6 APPLICATIONS OF INTEGRATION TO SURFACE AREA AND PROBABILITY

Please take MATH 311: Statistics II to see the application of multivariate integration to probability.

For the application to surface area, some is covered by §21.3: of the textbook.

Let f be a differentiable function of 2 variables defined on a domain D. We wish to find the surface area of the graph of f over D. It is simply equal to

$$\int\int_D dS.$$

Therefore we need to express the differential of the surface area dS in terms of the differential dA of the domain. To do so, take any point $P'(x,y)$ in D and let P be the corresponding point on the graph of f. Consider an increment dx along the x-direction and an increment dy along the y-direction at the point $P'(x,y)$. Thus $dA = |dxdy|$. These increments sweep out an increment of surface area on the surface at P. The differential dS of this area at P is given by the corresponding area on the tangent plane to the surface at P.

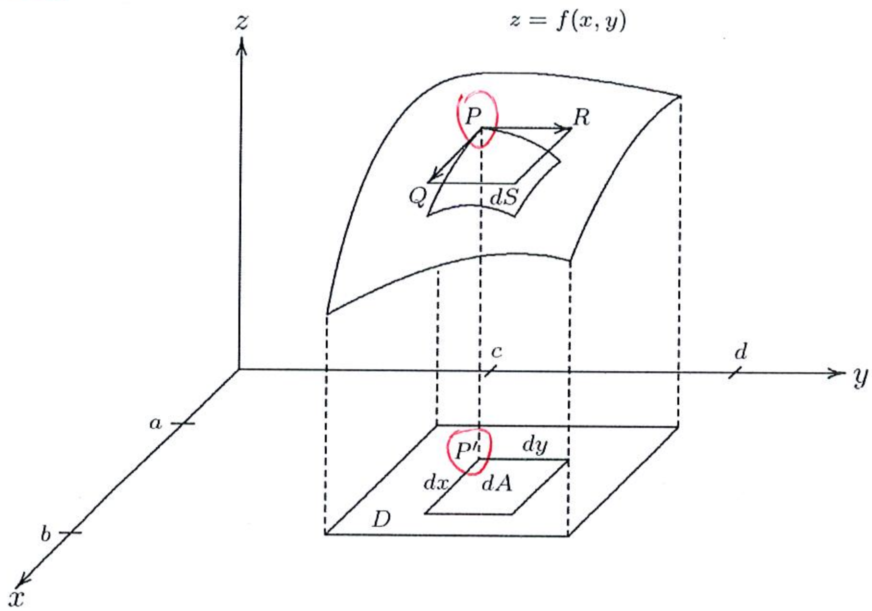


Figure 16.11: Surface Area

Let $\overrightarrow{PQ}$ be the vector on the tangent plane at P with x-component dx, and $\overrightarrow{PR}$ the vector with y-component dy. Thus, $\overrightarrow{PQ} = \langle dx, 0, f_x(x,y)dx\rangle$ and $\overrightarrow{PR} = \langle 0, dy, f_y(x,y)dy\rangle$. Recall (Chapter 13) that the area of the parallelogram spanned by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is the magnitude of the cross product $\overrightarrow{PQ} \times \overrightarrow{PR}$. ?

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ dx & 0 & f_x dx \\ 0 & dy & f_y dy \end{vmatrix} = \langle -f_x, -f_y, 1\rangle dxdy.$$

Therefore, $dS = |\langle -f_x, -f_y, 1\rangle dxdy| = \sqrt{f_x^2 + f_y^2 + 1}$. Consequently,

$$surfacearea = \int\int_D \sqrt{f_x^2 + f_y^2 + 1}\, dA.$$

Example. Find the area of the hemisphere $z = \sqrt{1 - x^2 - y^2}$.

Solution. $f_x = \dfrac{-x}{\sqrt{1-x^2-y^2}}, \quad f_y = \dfrac{-y}{\sqrt{1-x^2-y^2}}$

$$\text{Area} = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{\frac{x^2}{1-x^2-y^2} + \frac{y^2}{1-x^2-y^2} + 1}\; dy\, dx$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{\frac{1}{1-r^2}}\; r\,dr\,d\theta \quad (\text{polar coordinates})$$

$$= \int_0^{2\pi} 1\, d\theta = 2\pi \qquad \lim_{a\to 1^-}\int_0^a \frac{1}{\sqrt{1-r^2}}\, r\,dr = \lim_{a\to 1^-} -\sqrt{1-a^2} + 1 = 1$$ □

Example. Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

Solution. The paraboloid lies above the circular disk

$$D = \{(r,\theta) \mid 0 \le \theta \le 2\pi,\ 0 \le r \le 3\}$$

$$\text{Surface area} = \iint_D \sqrt{1 + f_x^2 + f_y^2}\, dA = \iint_D \sqrt{1 + 4(x^2+y^2)}\, dA$$

$$= \int_0^{2\pi}\int_0^3 \sqrt{1+4r^2}\; r\,dr\,d\theta \quad (\text{polar coordinates})$$

$$= \frac{\pi}{6}\left(37\sqrt{37} - 1\right)$$ □

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du \quad (u = g(x))$$

$$= \int_a^b f[g(x)]\,d[g(x)]$$

§21.2 Change of Variables in Multiple Integrals

Recall:

$$\cancel{\int_a^b f(x)\,dx = \int_{g(a)}^{g(b)} f(g(u))g'(u)\,du.} = \int_{g(a)}^{g(b)} f(g(u))\,d[g(u)]$$

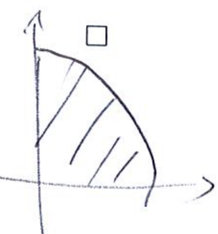
Example. Find $\int_1^2 (1+\frac{1}{2}x)^3\,dx$.

Solution. $= 2\int_1^2 (1+\frac{1}{2}x)^3\,d[1+\frac{1}{2}x]$

$$= 2\cdot\frac{1}{4}(1+\tfrac{1}{2}x)^4\Big|_1^2 = \frac{1}{2}[2^4 - 1.5^4]$$ □

Example. Find $\int_0^1 \int_0^{\sqrt{1-y^2}} \frac{1}{1+x^2+y^2}\,dxdy$.

Solution. $\int_0^{\pi/2}\int_0^1 \frac{1}{1+r^2}\,r\,dr\,d\theta$ $\begin{cases} x = r\cos\theta \\ y = r\sin\theta. \end{cases}$ $0 \le r \le 1$, $0 \le \theta \le \frac{\pi}{2}$



$$= \int_0^{\pi/2}\left[\frac{1}{2}\int_0^1 \frac{1}{1+r^2}\,d(1+r^2)\right]d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln 2\,d\theta = \frac{\pi\ln 2}{4}$$ □

Example. Find $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} (x^2+y^2+z^2)\,dzdydx$.

Solution. $= \int_0^{\pi/2}\int_0^1\int_0^{\sqrt{1-r^2}} (r^2+z^2)\,r\,dz\,dr\,d\theta$ $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z. \end{cases}$ $0 \le r \le 1$, $0 \le \theta \le \frac{\pi}{2}$

$$= \int_0^{\pi/2}\int_0^1 r\left[r^2\sqrt{1-r^2} + \frac{1}{3}(1-r^2)^{\frac{3}{2}}\right]dr\,d\theta$$

or spherical coordinate □

General Change of Coordinates

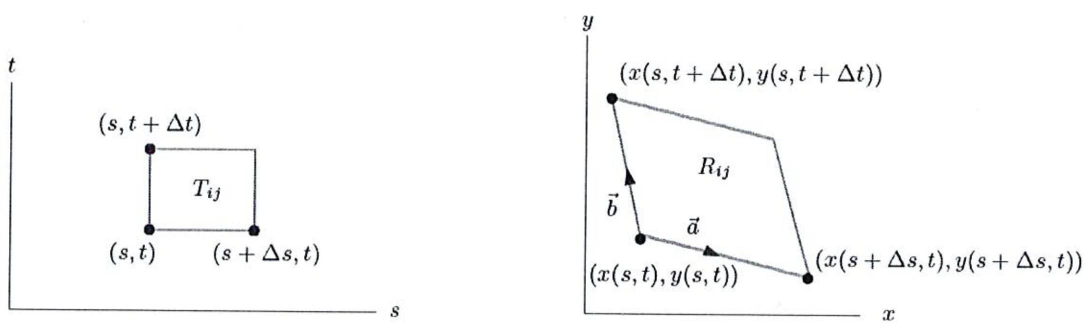


Figure 21.19: A small rectangle T_{ij} in the st-plane and the corresponding region R_{ij} of the xy-plane

We now consider a general change of coordinates, where x, y coordinates are related to s, t coordinates by the differentiable functions

$$x = x(s,t), y = y(s,t).$$

We divide T into small rectangles T_{ij} with sides of length Δs and Δt (See Figure 21.19). The corresponding piece R_{ij} of the xy-plane is a quadrilateral with curved sides. If we choose Δs and Δt small, then by local linearity of $x(s,t)$ and $y(s,t)$, we know R_{ij} is approximately a parallelogram. Recall that the area of the parallelogram with sides $\overrightarrow{a}$ and $\overrightarrow{b}$ is $\|\overrightarrow{a} \times \overrightarrow{b}\|$. Thus, we need to find the sides of R_{ij} as vectors

The side of R_{ij} corresponding to the bottom side of T_{ij} has endpoints $(x(s,t), y(s,t))$ and $(x(s+\Delta s, t), y(s+\Delta s, t))$, so in vector form that side is

$$\overrightarrow{a} = (x(s+\Delta s, t)x(s,t))\overrightarrow{i} + (y(s+\Delta s, t)y(s,t))\overrightarrow{j} \approx \left(\frac{\partial x}{\partial s}\Delta s\right)\overrightarrow{i} + \left(\frac{\partial y}{\partial s}\Delta s\right)\overrightarrow{j}.$$

Similarly, the side of R_{ij} corresponding to the left edge of T_{ij} is given by

$$\overrightarrow{b} \approx \left(\frac{\partial x}{\partial t}\Delta t\right)\overrightarrow{i} + \left(\frac{\partial y}{\partial t}\Delta t\right)\overrightarrow{j}.$$

Computing the cross product, we get

$$\text{Area } R_{ij} = \|\overrightarrow{a} \times \overrightarrow{b}\| \approx \left|\frac{\partial x}{\partial st} \cdot \frac{\partial y}{\partial t} - \frac{\partial x}{\partial t} \cdot \frac{\partial y}{\partial s}\right| \Delta s \Delta t.$$

(1). Change of Variables in Double Integrals

Now we consider a continuously differentiable transformation

$$T : \mathbb{R}^2_{st} \to \mathbb{R}^2_{xy}.$$

Namely,

$$T : \begin{cases} x = f(s,t) \\ y = g(s,t) \end{cases}, \quad \frac{\partial x}{\partial s}, \frac{\partial x}{\partial t}, \frac{\partial y}{\partial s}, \frac{\partial y}{\partial t} \text{ exist and continuous.}$$

Definition (The Jacobian). *The Jacobian of the transformation T is defined by*

$$J_T(s,t) = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix} \triangleq \frac{\partial(x,y)}{\partial(s,t)} \ \textit{(notation)}.$$

If T is one to one, then the inverse transformation T^{-1} exists. Write

$$T^{-1}:\begin{cases} s = h(x,y) \\ t = k(x,y) \end{cases}.$$

Formula: $\boxed{\dfrac{\partial(x,y)}{\partial(s,t)} \cdot \dfrac{\partial(s,t)}{\partial(x,y)} = 1}$.

Theorem (Change of Variables). *Assume $T : \mathbb{R}^2_{st} \to \mathbb{R}^2_{xy}$, $S(bounded) \longmapsto R(bounded)$ is one-to-one from the interior of S to the interior of R. If $f(x,y)$ is continuous on R, then*

$$\int\int_R f(x,y)\,dxdy = \int\int_S f(T(s,t)) \left|\frac{\partial(x,y)}{\partial(s,t)}\right| dsdt$$

Example 1. Suppose that the transformation T from the $r\theta$-plane to the xy-plane is determined by the polar equations

$$T:\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases},$$

then the Jacobian of T is?

Solution.

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r$$

□

Example 2. Find the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Solution. let $\begin{cases} x = as \\ y = bt \end{cases}$ $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \longrightarrow s^2 + t^2 = 1$ (denoted by T)

In the s-t plane, $J = \cancel{\dfrac{\partial(s,t)}{\partial(x,y)}} = \dfrac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$

$\Rightarrow$ Area of xy-ellipse $= \iint_R 1\,dA = \iint_T ab\,dsdt = ab \cdot \pi 1^2$
$= ab\pi$.

□

(2). Change of Variables in Triple Integrals

Consider a continuously differentiable transformation

$$T : \mathbb{R}^3_{uvw} \to \mathbb{R}^3_{xyz}, \quad R \longmapsto S.$$

Then the Jacobian of T is the determinant

$$J_T(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \triangleq \frac{\partial(x,y,z)}{\partial(u,v,w)} \text{ (notation).}$$

Now $T(R) = S$, then (under assumptions equivalent to those stated in the proceeding Theorem) the change-of-variables formula for triple integrals is

$$\int\!\!\int\!\!\int_R f(x,y,z)\, dxdydz = \int\!\!\int\!\!\int_S f(T(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dudvdw.$$

Example. Let T be the spherical-coordinate transformation given by

$$T : \begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = r\cos\phi \end{cases}.$$

Then the Jacobian of T is?

Solution.

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$$

$$= \cdots = \rho^2 \sin\phi$$

□

Example 3. Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Solution. Let $\begin{cases} x = as \\ y = bt \\ z = cu \end{cases}$ ellipsoid $\longrightarrow$ Sl-u-sphere $s^2 + t^2 + u^2 = 1$.

$$J = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc \implies \text{Volume} = abc \cdot \text{Volum of sphere}$$

$$= abc \cdot \frac{4}{3}\pi 1^3$$

§17.1 Parameterized Curves

A curve in the plane may be parameterized by a pair of equations of the form $x = f(t), y = g(t)$. As the parameter t changes, the point (x, y) traces out the curve. In this section we find **parametric equations** for curves in three dimensions, and we see how to write parametric equations using position vectors.

Definition (parametric curve). *A* ***parametric curve*** *C in the plane is a pair of functions*

$$x = f(t), y = g(t),$$

that give x and y as continuous functions of the real number t (the parameter) in some interval I. The two equations are called the ***parametric equations*** *of the curve.*

Remarks.

- A given curve may have several different parametrizations.
- A Parametric Equations in three dimensions is defined similarly. That is,

Definition (parametric curve). *A* ***parametric curve*** *C in the three-dimensional space is*

$$x = f(t), y = g(t), z = h(t)$$

that give x, y and z as continuous functions of the real number t (the parameter) in some interval I. The three equations are called the ***parametric equations*** *of the curve.*

Example 1. Find parametric equations for the curve $y = x^2$ in the xy-plane.

Solution.

i) $x = t,\ y = t^2,\quad -\infty < t < \infty$

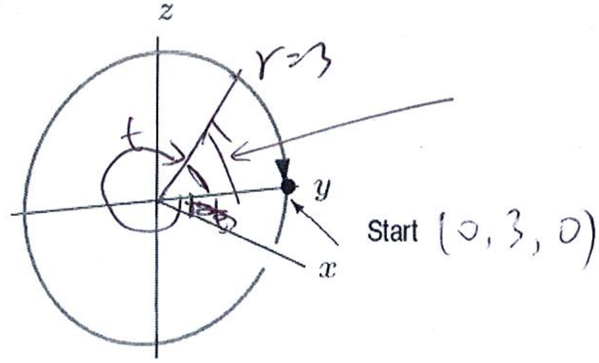
ii) $x = t,\ y = t^2,\ z = 0$

$\square$

Example 2. Find parametric equations for a particle that starts at $(0, 3, 0)$ and moves around a circle as shown in Figure 17.1.

Solution.

$$\begin{cases} x = 0 \text{ for all } t \\ y = 3\cos t \\ z = -3\sin t \end{cases}$$



?

Figure 17.1: Circle of radius 3 in the yz-plane, centered at origin

□

(sagemath)

Example 3. Describe in words the motion given parametrically by

$$x = \cos t, y = \sin t, z = t.$$

Solution. x and y-coordinates give circular motion in the xy-plane while the z-coordinate increase steadily. The curve is called a helix.

$\square$

Example 4. Find parametric equations for the line parallel to the vector $2\vec{i} + 3\vec{j} + 4\vec{k}$ and through the point $(1, 5, 7)$.

Solution.

$P(x, y, z)$ P_0 $(1, 5\ 7)$

$\rightarrow \langle 2, 3, 4 \rangle$

$\overrightarrow{P_0P} = t\langle 2, 3, 4 \rangle$

$t \in (-\infty, +\infty)$

$\Rightarrow \langle x-1, y-5, z-7 \rangle = t\langle 2, 3, 4 \rangle$

$$\Rightarrow \begin{cases} x - 1 = 2t \\ y - 5 = 3t \\ z - 7 = 4t \end{cases} \Rightarrow \begin{cases} x = 1 + 2t \\ y = 5 + 3t \\ z = 7 + 4t \end{cases}$$

$\square$

Theorem (Parametric Equations of a Line). ***Parametric Equations of a Line*** *through the point* (x_0, y_0, z_0) *and parallel to the vector* $\langle a, b, c \rangle = a\vec{i} + b\vec{j} + c\vec{k}$ *are*

$\boxed{x = x_0 + at, y = y_0 + bt, z = z_0 + ct.}$

Example 5. (a) Describe in words the curve given by the parametric equations $x = 3 + t, y = 2t, z = 1 - t$.

(b) Find parametric equations for the line through the points $P(1, 2, -1)$ and $Q(3, 3, 4)$.

Solution.

(a) The curve is a line through the point $(3, 0, 1)$ and parallel to the vector $\vec{i} + 2\vec{j} - \vec{k}$.

(b) $\overrightarrow{PQ} = \langle 3-1, 3-2, 4-(-1) \rangle = \langle 2, 1, 5 \rangle = 2\vec{i} + \vec{j} + 5\vec{k}$

Using the point P, the equations are

$x = 1 + 2t, \quad y = 2 + t, \quad z = -1 + 5t$ $\square$

or use the point Q, the equations are

$x = 3 + 2t, \quad y = 3 + t, \quad z = 4 + 5t.$

Using Position Vectors to Write Parameterized Curves as Vector-Valued

A point in the plane with coordinates (x, y) can be represented by the position vector $\langle x, y\rangle = \vec{r} = x\vec{i} + y\vec{j}$.

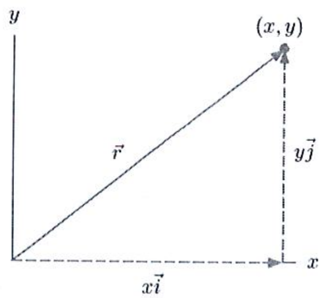


Figure 17.3: Position vector $\vec{r}$ for the point (x, y)

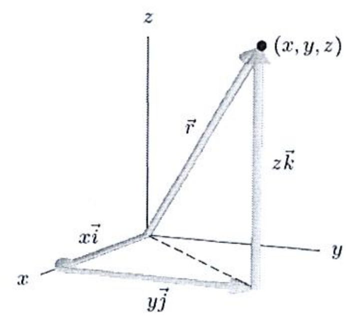


Figure 17.4: Position vector $\vec{r}$ for the point (x, y, z)

Definition (vector-valued function). *A vector-valued function $\vec{r}(t)$ is a function whose domain is a set of real numbers and whose range is a set of vectors. In other word,*

$$\vec{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k},$$

where f, g, h are called the **component** *functions of $\vec{r}$.*

For example, the circular motion in the plane $x = \cos t, y = \sin t$ can be written as $\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j}$; The helix in 3-space $x = \cos t, y = \sin t, z = t$ can be written as $\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + t\vec{k}$

Example 6. Use vectors to give a parameterization for the circle of radius $\frac{1}{2}$ centered at the point $(-1, 2)$. Figure 17.7

Solution.

For the circle of $r=1$ centered at $(0,0)$ $\vec{r_1}(t) = \langle \cos t, \sin t\rangle$, $0 \le t \le 2\pi$

The point $(-1,2)$ has position vector $\vec{r_0} = \langle -1, 2\rangle$

The position vector $\vec{r}(t)$ on the circle of $r=\frac{1}{2}$ centered at $(-1,2)$

is found by adding $\frac{1}{2}\vec{r_1}$ to $\vec{r_0}$

$\Rightarrow \vec{r}(t) = \vec{r_0} + \frac{1}{2}\vec{r_1}(t) = \langle -1, 2\rangle + \frac{1}{2}\langle \cos t, \sin t\rangle$

$= \langle -1 + \frac{1}{2}\cos t,\ 2 + \frac{1}{2}\sin t\rangle$

or $x = -1 + \frac{1}{2}\cos t$, $y = 2 + \frac{1}{2}\sin t$, $0 \le t \le 2\pi$

Parametric Equation of a Line

Theorem (Parametric Equation of a Line in Vector Form). *The line through the point with position vector* $\vec{r_0} = x_0\vec{i} + y_0\vec{j} + z_0\vec{k}$ *in the direction of the vector* $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$ *has parametric equation*

$$\vec{r}(t) = \vec{r_0} + t\vec{v}.$$

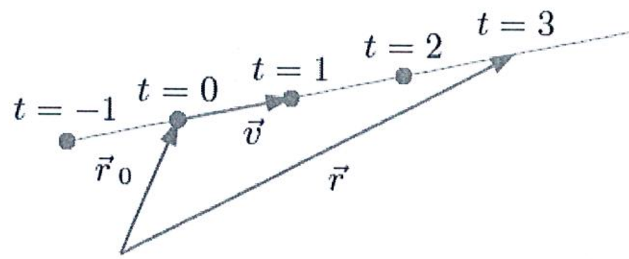


Figure 17.8: The line $\vec{r}(t) = \vec{r}_0 + t\vec{v}$

Example 4. (SageMath) Find parametric equations for the line parallel to the vector $2\vec{i} + 3\vec{j} + 4\vec{k}$ and through the point $(1, 5, 7)$.

Solution.

$\vec{v} = \langle 2, 3, 4\rangle$ and $\vec{r_0} = \langle 1, 5, 7\rangle$

$\Rightarrow \vec{r}(t) = \vec{r_0} + t\vec{v} = \langle 1, 5, 7\rangle + t\langle 2, 3, 4\rangle$

$= \langle 1+2t, 5+3t, 7+4t\rangle$

or $x = 1+2t$, $y = 5+3t$, $z = 7+4t$ □

Example 7. (a) Find parametric equations for the line passing through the points $(2, -1, 3)$ and $(-1, 5, 4)$.

(b) Represent the line segment from $(2, -1, 3)$ to $(-1, 5, 4)$ parametrically.

Solution. (a) $\vec{v} = \langle 2, -1, 3 \rangle - \langle -1, 5, 4 \rangle = \langle 3, -6, -1 \rangle$

$\Rightarrow \vec{r}(t) = \langle 2, -1, 3 \rangle + t\langle 3, -6, -1 \rangle, \ -\infty < t < \infty$

(b) i) $t = 0$ corresponds to the point $(2, -1, 3)$

ii) $t = -1$ corresponds to the point $(-1, 5, 4)$

$\Rightarrow$ The parameterization of the segment is

$\vec{r}(t) = \langle 2, -1, 3 \rangle + t\langle 3, -6, -1 \rangle, \ -1 \le t \le 0,$ □

Intersection of Curves and Surfaces

Parametric equations for a curve enable us to find where a curve intersects a surface.

Example 8. Find the points at which the line $x = t, y = 2t, z = 1 + t$ pierces the sphere of radius 10 centered at the origin.

Solution. The equation of the sphere is $x^2 + y^2 + z^2 = 10^2$

Substitute the parametric equations of the line into the equation of the sphere, gives $t^2 + 4t^2 + (1+t)^2 = 100$ or

$6t^2 + 2t - 99 = 0$ ($\Leftrightarrow t = -4.23, t = 3.90$)

$\Rightarrow$ The line cuts the sphere at the two points □

i) $(x, y, z) = (-4.23, 2\times(-4.23), 1 - 4.23) = (-4.23, -8.46, -3.23)$

ii) $(x, y, z) = (3.90, 2\times 3.90, 1 + 3.90) = (3.90, 7.80, 4.90)$

Example 9. Two particles move through space, with equations $\vec{r_1}(t) = t\,\vec{i} + (1+2t)\,\vec{j} + (3-2t)\,\vec{k}$ and $\vec{r_2}(t) = (-2-2t)\,\vec{i} + (1-2t)\,\vec{j} + (1+t)\,\vec{k}$. Do the particles ever collide? Do their paths cross?

Solution. i) check if they pass the same point at the same time.

Let $\vec{r_1}(t) = \vec{r_2}(t)$. $\Rightarrow t = -2-2t,\ 1+2t = 1-2t,\ 3-2t = 1+t$

$\Rightarrow t = -2/3,\ t = 0,\ t = 2/3$ respectively $\Rightarrow$ They don't collide.

ii) check if they pass a same point at time t_1 and t_2, respectively

$t_1 = -2-2t_2,\quad 1+2t_1 = 1-2t_2,\quad 3-2t_1 = 1+t_2$

For the 1st two equations $\begin{cases} t_1 = 2 \\ t_2 = -2 \end{cases}$ which satisfies the 3rd equation. ANS: Yes.

Example 10. Are the lines $x = -1+t, y = 1+2t, z = 5-t$ and $x = 2+2t, y = 4+t, z = 3+t$ parallel? Do they intersect?

Solution.

i) $\vec{r_1}(t) = \langle -1, 1, 5\rangle + t\langle 1, 2, -1\rangle$

$\vec{r_2}(t) = \langle 2, 4, 3\rangle + t\langle 2, 1, 1\rangle$

The direction vectors are not multiples of each other so, the lines are not parallel.

ii) $-1+t_1 = 2+t_2,\quad 1+2t_1 = 4+t_2,\quad 5-t_1 = 3+t_2$

The 1st two equations give $t_1 = 1,\ t_2 = -1$ which do not satisfy the 3rd equation, so the lines do not intersect.

§17.2 MOTION, VELOCITY, AND ACCELERATION

In this section we see how to find the vector quantities of velocity and acceleration from a parametric equation for the motion of an object. The velocity of a moving particle can be represented by the velocity vector.

Definition (velocity vector). *The* ***velocity vector*** *of a moving object is a vector $\vec{v}$ such that:*

- *The magnitude of $\vec{v}$ is the speed of the object.*
- *The direction of $\vec{v}$ is the direction of motion.*

Thus, the speed of the object is $\|\vec{v}\|$ and the velocity vector is tangent to the objects path.

Calculus with vector-valued functions

A **vector-valued function** $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ is a function of one variablethat is, there is only one input value. However, the output values are now three-dimensional vectors instead of simply numbers. It is natural to wonder if there is a corresponding notion of derivative for vector-valued functions.

One way to approach the question of the derivative for vector-valued functions is to write down an expression that is analogous to the derivative we already understand, and see if we can make sense of it. This gives us

$$\begin{aligned}
\vec{r}\,'(t) &= \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{\langle f(t+\Delta t) - f(t), g(t+\Delta t) - g(t), h(t+\Delta t) - h(t)\rangle}{\Delta t} \\
&= \lim_{\Delta t \to 0} \left\langle \frac{f(t+\Delta t) - f(t)}{\Delta t}, \frac{g(t+\Delta t) - g(t)}{\Delta t}, \frac{h(t+\Delta t) - h(t)}{\Delta t} \right\rangle \\
&= \left\langle f'(t), g'(t), h'(t), \right\rangle,
\end{aligned}$$

which is **Componentwise Differentiation**.

The limiting vector $\left\langle f'(t), g'(t), h'(t), \right\rangle$ will (usually) be a good, non-zero vector that is tangent to the curve.

What about the length of this vector? Consider the length of one of the vectors that approaches the tangent vector:

$$\left|\frac{\vec{r}(t+\Delta t)-\vec{r}(t)}{\Delta t}\right| = \frac{|\vec{r}(t+\Delta t)-\vec{r}(t)|}{|\Delta t|}.$$

So by performing an obvious calculation to get something that looks like the derivative of $\vec{r}(t)$, we get precisely what we would want from such a derivative: the vector $\vec{r}'(t)$ points in the direction of travel of the object and its length tells us the speed of travel. In the case that t is time, then, we call $\vec{v}(t) = \vec{r}'(t)$ ~~is usefulit is~~ a vector tangent to the curve.

Definition (velocity vector). *The velocity vector, $\vec{v}(t)$, of a moving object with position vector $\vec{r}(t)$ at time t is $\vec{r}'(t)$.*

Exercise 10. find the velocity $\vec{v}(t)$ and speed $\|\vec{v}(t)\|$ for the particle whose equations of motion are given by $x = (t-1)^2, y = 2, z = 2t^3 - 3t^2$. Find any times at which the particle stops. $\vec{r}(t) = \langle (t-1)^2, 2, 2t^3-3t^2 \rangle$

Solution.
$\vec{v}(t) = \vec{r}'(t) = \langle 2(t-1), 0, 6t^2-6t \rangle$

$\|\vec{v}(t)\| = \sqrt{4(t-1)^2 + 36t^2(t-1)^2} = |t-1|\sqrt{4+36t^2}$

when $t=1$, $\vec{v}(t) = 0$.

□

Example 3. Find the tangent line at the point $(1,1,2)$ to the curve defined by the parametric equation

$$\vec{r}(t) = t^2\vec{i} + t^3\vec{j} + 2t\vec{k}.$$

Solution. $\vec{r}'(t) = \langle 2t, 3t^2, 2 \rangle$ which is the direction vector of the tangent line.

$\Rightarrow \vec{r}(t) = \langle 1,1,2 \rangle + t\langle 2t, 3t^2, 2 \rangle$

□

The Acceleration Vector

Just as the velocity of a particle moving in 2-space or 3-space is a vector quantity, so is the rate of change of the velocity of the particle, namely its acceleration.

Definition (acceleration vector). *The acceleration vector of an object moving with velocity $\vec{v}(t)$ at time t is $\vec{a}(t) = \vec{r}''(t)$.*

Exercise 6.(SageMath) Find the velocity and acceleration vectors for the particle whose equations of motion are given by $x = 3\cos(t^2), y = 3\sin(t^2), z = t^2$.

Solution. $\vec{r}(t) = \langle 3\cos(t^2), 3\sin t^2, t^2 \rangle$

$\vec{v}(t) = \vec{r}'(t) = \langle -6t\sin(t^2), 6t\cos(t^2), 2t \rangle$

$\vec{a}(t) = \vec{v}'(t) = \langle -6\sin t^2 - 12t^2\cos t^2, 6\cos(t^2) - 12t^2\sin(t^2), 2 \rangle$

□

Motion in a Circle and Along a Line

We now consider the velocity and acceleration vectors for two basic motions: uniform motion around a circle, and motion along a straight line.

Uniform Circular Motion: For a particle whose motion is described by

$$\vec{r}(t) = R\cos(\omega t)\vec{i} + R\sin(\omega t)\vec{j}$$

- Motion is in a circle of radius R with period $2\pi/|\omega|$.
- Velocity, $\vec{v}$, is tangent to the circle and speed is constant $\|\vec{v}\| = |\omega|R$.
- Acceleration, $\vec{a}$, points toward the center of the circle with $\|\vec{a}\| = \|\vec{v}\|^2/R$.

Example 5. Consider the motion given by the vector equation

$$\vec{r}(t) = 2\vec{i} + 6\vec{j} + (t^3 + t)(4\vec{i} + 3\vec{j} + \vec{k}).$$

Show that this is straight-line motion in the direction of the vector $4\vec{i} + 3\vec{j} + \vec{k}$ and relate the acceleration vector to the velocity vector.

Solution. $\vec{v}(t) = \vec{r}'(t) = (3t^2+1)\langle 4, 3, 1\rangle$ which is always positive and $\|\vec{v}(t)\| = 3t^2+1\sqrt{4^2+3^2+1^2} = \sqrt{26}(3t^2+1)$

$\Rightarrow$ The speed is decreasing until $t=0$ then start increasing

$\vec{a}(t) = \vec{v}'(t) = 6t\langle 4, 3, 1\rangle \begin{cases} >0 & \text{if } t>0 \text{ same direction as } \vec{v} \\ <0 & \text{if } t<0 \text{ oppositie direction to } \vec{v} \end{cases}$

$\Rightarrow$ This is straight-line motion in the diretion of $\langle 4, 3, 1\rangle$

□

Motion in a Straight Line: For a particle whose motion is described by

$$\vec{r}(t) = \vec{r_0} + f(t)\vec{v}$$

- Motion is along a straight line through the point with position vector $\vec{r_0}$ parallel to $\vec{v}$.
- Velocity, $\vec{v}$, and acceleration, $\vec{a}$, are parallel to the line.

The Length of a Curve

The speed of a particle is the magnitude of its velocity vector:

$$\text{Speed} = \|\vec{v}(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

Thus,

$$\text{Distance traveled} = \int_a^b \|\vec{v}(t)\|\, dt.$$

Theorem (The Length of a Curve). *If the curve C is given parametrically for $a \leq t \leq b$ by smooth functions and if the velocity vector $\vec{v}(t)$ is not $\vec{0}$ for $a < t < b$, then*

$$\textit{Length of } C = \int_a^b \|\vec{v}(t)\|\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

Example 6.(SageMath) Find the circumference of the ellipse given by the parametric equations

$$x = 2\cos t, y = \sin t, 0 \leq t \leq 2\pi.$$

Solution.

Circumference $= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$= \int_0^{2\pi} \sqrt{(-2\sin t)^2 + \cos^2(t)}\, dt = \int_0^{2\pi} \sqrt{4\sin^2 t + \cos^2(t)}\, dt$

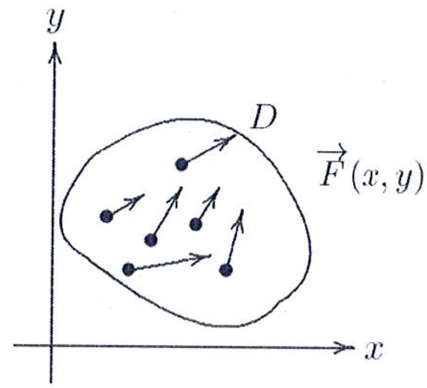
$= \int_0^{2\pi} \sqrt{3\sin^2 t + 1}\, dt \simeq 9.69$ (calculator)

$\square$

§17.3 VECTOR FIELDS

A vector field is a function that assigns a vector to each point in the plane or in 3-space.

Definition (Vector Field). *A vector field in 2-space is a function $\vec{F}(x, y)$ whose value at a point (x, y) is a 2- dimensional vector.*
Similarly, a vector field in 3-space is a function $\vec{F}(x, y, z)$ whose values are 3-dimensional vectors.



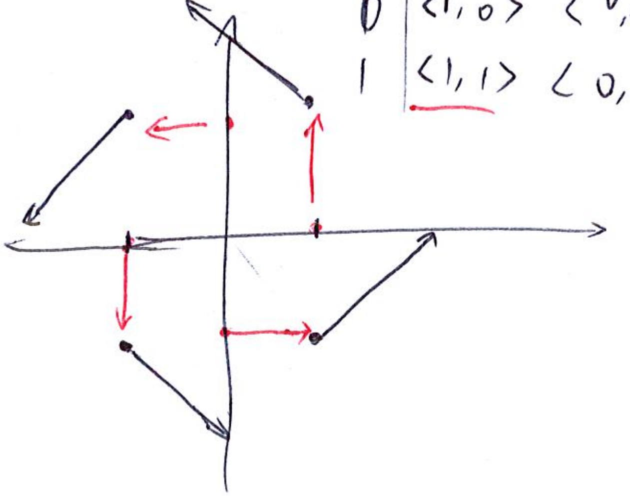
Notice the arrow over the function, $\vec{F}$, indicating that its value is a vector, not a scalar. We may write $\vec{F}(x, y)$ in terms of its component functions. That is $\vec{F}(x, y) = P(x, y)\vec{i} + Q(x, y)\vec{j} = \langle P(x, y), Q(x, y)\rangle$, or simply $\vec{F} = P\vec{i} + Q\vec{j}$.

Example 1.(SageMath) Sketch the vector field in 2-space given by $\vec{F}(x, y) = -y\vec{i} + x\vec{j}$.

Solution.

$x \backslash y$	-1	0	1
-1	$\langle 1, -1\rangle$	$\langle 0, -1\rangle$	$\langle -1, -1\rangle$
0	$\langle 1, 0\rangle$	$\langle 0, 0\rangle$	$\langle -1, 0\rangle$
1	$\langle 1, 1\rangle$	$\langle 0, 1\rangle$	$\langle -1, 1\rangle$

$\|F(x,y)\| = \sqrt{y^2 + x^2}$

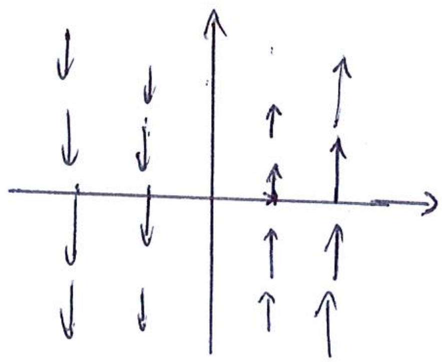


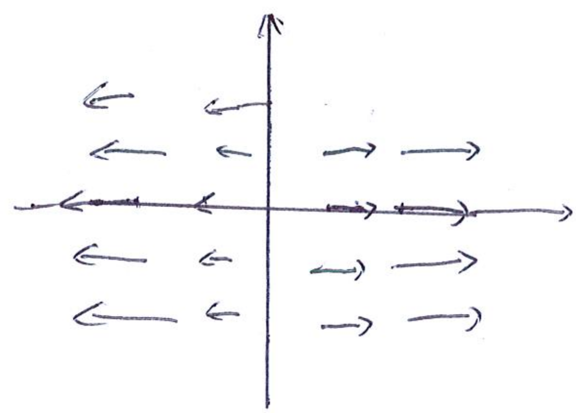
□

Example 2. Sketch the vector fields in 2-space given by
(a) $\vec{F}(x,y) = x\,\vec{j}$ (b) $\vec{G}(x,y) = x\,\vec{i}$.

Solution.
(a) $\vec{F}(x,y) = \langle 0, x \rangle$ (b) $\vec{G}(x,y) = \langle x, 0 \rangle$

$\|\vec{F}(x,y)\| = |x|$ $\|\vec{G}(x,y)\| = |x|$





□

Example 3.(SageMath) Describe the vector field in 3-space given by $\vec{F}(x,y,z) = \langle x, y, z \rangle$.

Solution. and $\|\vec{F}(x,y,z)\| = \sqrt{x^2+y^2+z^2}$

At point (x,y,z), position vector $\langle x,y,z \rangle$ has its tail at (x,y,z).

□

Gradient Vector Fields.

The gradient of a scalar function f is a function that assigns a vector to each point, and is therefore a vector field. It is called the **gradient field** of f. Many vector fields in physics are gradient fields.

Example 5. Sketch the gradient field of the functions in Figures 17.26-17.28.

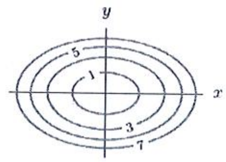


Figure 17.26: The contour map of $f(x, y) = x^2 + 2y^2$

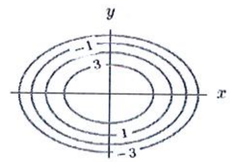


Figure 17.27: The contour map of $g(x, y) = 5 - x^2 - 2y^2$

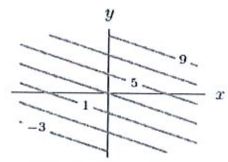


Figure 17.28: The contour map of $h(x, y) = x + 2y + 3$

Solution.

$\nabla f = \langle 2x, 4y \rangle$ $\quad \nabla g = \langle -2x, -4y \rangle$ $\quad \nabla h = \langle 1, 2 \rangle$

For a function $f(x,y)$, the ∇f at a point is perpendicular to the contours in the direction of increasing f and its magnitude is the rate of change in that direction. The rate of change is large when the contours are close together and small when they are far apart.

□

§17.4 THE FLOW OF A VECTOR FIELD

Optional. Please read if you are interested.

§18.1 The Idea of a Line Integral

The single integral $\int_a^b f(x)\, dx$ might be described as an integral along the x-axis. we now define integrals along curves in space (or in the plane). Such integrals are called line integrals (although the phrase "curve integrals" might be more appropriate).

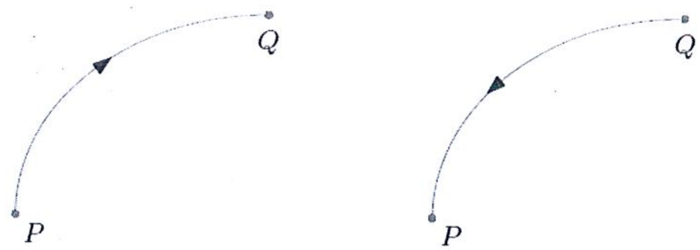


Figure 18.1: A curve with two different orientations represented by arrowheads

A curve is said to be **oriented** if we have chosen a direction of travel on it.

Consider a curve C in the space: $x = x(t), y = y(t), z = z(t)$. We assume C is a smooth curve, meaning that $\vec{r}'(t) \neq 0$, and $\vec{r}'(t)$ is continuous for all t. Let $f(x, y, z)$ be a continuous function defined in a domain containing C. $f(x, y, z)$ might be considered as the density of the curve/wire at the point (x, y, z). To define the line integral of f along C, we subdivide arc from $\vec{r}(a)$ to $\vec{r}(b)$ into n small arcs of length $\Delta s_i, i = 1, \ldots, n$. Pick an arbitrary point (x_i^*, y_i^*, z_i^*) inside the ith small arc and form the Riemann sum $\sum_{i=1}^n f(x_i^*, y_i^*, z_i^*)\Delta s_i$. The line integral of f along C is the limit of this Riemann sum.

Definition. *The line integral of f along C is defined to be*

$$\int_C f(x, y, z)\, ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta s_i. \tag{18.1}$$

We can pull back the integral to an integral in terms of t using the parametrization $\vec{r}$. Recall that from last Chapter the arc length differential is given by $ds = \|r'(t)\||dt|$. Thus,

$$\int_C f(x, y, z)\, ds = \int_C f(\vec{r})\|r'(t)\|dt = \int_a^b f(x(t), y(t), z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

Note that since $a \le t \le b$, $|dt| = dt$.

Thus we may evaluate the line integral $\int_C f(x,y,z)\ ds$ by expressing everything in terms of the parameter t including the symbolic arc-length element

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt.$$

Remark. A curve C that lies in the xy-plane may be regarded as a space curve for which z and $z'(t)$ are 0. In this case we simply suppress the variable and write

$$\int_C f(x,y)\ ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt.$$

Example.(SageMath) Evaluate $\int_C (2 + x^2 y)\ ds$, where C is the upper half of the unit circle traversed in the counterclockwise sense.

Solution. We paramet[e]rize C by $x = \cos t$, $y = \sin t$, $t \in [0, \pi]$

Thus,

$$\int_C (2+x^2y)\, ds = \int_0^\pi (2 + \cos^2 t \sin t)\sqrt{\sin^2 t + \cos^2 t}\, dt$$

$$= \int_0^\pi (2 + \cos^2 t \sin t)\, dt = 2\pi + (-1)\int_0^\pi \cos^2(t)\, d\cos t$$

$$= 2\pi - \frac{\cos^3 t}{3}\Big|_0^\pi = 2\pi - \left[\frac{(-1)^3}{3} - \frac{1^3}{3}\right] = 2\pi + \frac{2}{3}.$$

□

Line Integrals with Respect to Coordinate Variables

We obtain a different kind of line integral by replacing Δs_i in Equation 18.1 with

$$\Delta x_i = x(t_i) - x(t_{i-1}) = x'(t_i^*)\Delta t$$

The line integral of f along C with respect to x is defined to be

$$\int_C f(x,y,z)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta x_i.$$

Thus,

$$\int_C f(x,y,z)\,dx = \int_a^b f(x(t),y(t),z(t))x'(t)\,dt.$$

Similarly,

$$\int_C f(x,y,z)\,dy = \int_a^b f(x(t),y(t),z(t))y'(t)\,dt$$

$$\int_C f(x,y,z)\,dz = \int_a^b f(x(t),y(t),z(t))z'(t)\,dt.$$

These three integrals typically occur together. If P, Q and R are continuous functions of the variables x, y, and z then we write (indeed, define)

$$\int_C P\,dx + Q\,dy + R\,dz = \int_C P\,dx + \int_C Q\,dy + \int_C R\,dz \tag{18.2}$$

Remarks

(1) Although it would be natural enough to write $\int_C (P\,dx + Q\,dy + R\,dz)$ on the left hand side in Equation 18.2, the parentheses are customarily omitted.

(2)These line integrals are evaluated by expressing x, y, z, dx, dy and dz in terms of t as determined by a suitable parametrization of the curve C. The result is an ordinary single-variable integral.

Example. Evaluate the line integral

$$\int_C y\,dx + z\,dy + x\,dz$$

$$\begin{cases} dx = dt \\ dy = 2t\,dt \\ dz = 3t^2\,dt \end{cases}$$

where C is the parametric curve $x = t, y = t^2, z = t^3, 0 \le t \le 1$.

$$\Rightarrow \int_C y\,dx + z\,dy + x\,dz = \int_0^1 t^2 dt + t^3 2t\,dt + t(3t^2 dt)$$

$$= \int_0^1 (t^2 + 3t^3 + 2t^4)\,dt = \left[\frac{1}{3}t^3 + \frac{3}{4}t^4 + \frac{2}{5}t^5\right]_0^1 = \frac{89}{60}$$

Line Integrals of Vector Fields

Consider a vector field $\overrightarrow{F}$ and an oriented curve C.

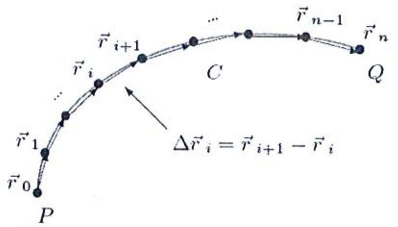


Figure 18.2: The curve C, oriented from P to Q, approximated by straight line segments represented by displacement vectors $\Delta\vec{r}_i = \vec{r}_{i+1} - \vec{r}_i$

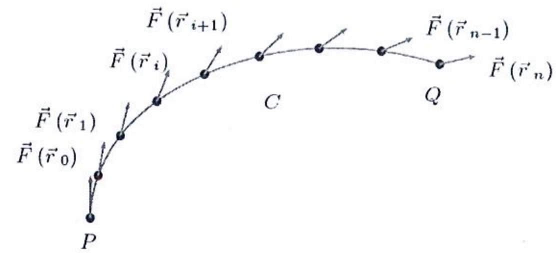


Figure 18.3: The vector field $\vec{F}$ evaluated at the points with position vector $\vec{r}_i$ on the curve C oriented from P to Q

Since the **dot product** can be used to measure to what extent two vectors point in the same or opposing directions, we form the dot product $\boxed{\overrightarrow{F}(\overrightarrow{r}_i) \cdot \Delta\overrightarrow{r}_i}$ for each point with position vector $\overrightarrow{r}_i$ on C .

Definition (Line Integrals of Vector Fields). *Let $\overrightarrow{F}$ be a continuous vector field defined on a domain containing a smooth oriented curve C given by a vector function $\overrightarrow{r}(t), t \in [a,b]$. The line integral of $\overrightarrow{F}$ along the oriented curve C is*

$$\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = \lim_{\|\Delta\overrightarrow{r}_i\| \to 0} \sum_{i=0}^{n-1} \overrightarrow{F}(\overrightarrow{r}_i) \cdot \Delta\overrightarrow{r}_i = \int_a^b \overrightarrow{F}(\overrightarrow{r}(t)) \cdot \overrightarrow{r}'(t)\, dt.$$

Example 2. The vector field $\overrightarrow{F}$ and the oriented curves C_1, C_2, C_3, C_4 are shown in Figure 18.5. The curves C_1 and C_3 are the same length. Which of the line integrals $\int_{C_i} \overrightarrow{F} \cdot d\overrightarrow{r}$, for $i = 1, 2, 3, 4$, are positive? Which are negative? Arrange these line integrals in ascending order.

Solution.

$\int_{C_i} \vec{F} \cdot d\vec{r}$, $i=1,2,3$ are positive, since $\vec{F} \cdot \Delta\vec{r}$ are positive

($\vec{F}$ and $\Delta\vec{r}$ are approximately parallel and in the same direction.

i) $\int_{C_1} \vec{F} \cdot d\vec{r} < \int_{C_3} \vec{F} \cdot d\vec{r}$

since the vector field along C_1 is smaller.

ii) $\int_{C_3} \vec{F} \cdot d\vec{r} < \int_{C_2} \vec{F} \cdot d\vec{r}$

since the curve C_2 is longer

$\Rightarrow \int_{C_4} \vec{F} \cdot d\vec{r} < \int_{C_1} \vec{F} \cdot d\vec{r} < \int_{C_3} \vec{F} \cdot d\vec{r} < \int_{C_2} \vec{F} \cdot d\vec{r}$

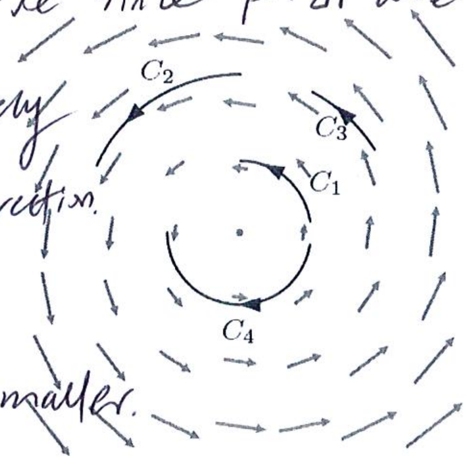


Figure 18.5: Vector field and paths C_1, C_2, C_3, C_4

□

Two Applications of the Line Integral

1. Work

Recall (from Section 13.3) that if a constant force $\overrightarrow{F}$ acts on an object while it moves along a straight line through a displacement $\overrightarrow{d}$, the work done by the force on the object is

$$Work done = \overrightarrow{F} \cdot \overrightarrow{d}.$$

$$\boxed{\text{Work done by force } \overrightarrow{F}(\overrightarrow{r}) \text{ along curve } C = \int_C \overrightarrow{F} \cdot d\overrightarrow{r}.}$$

2. Circulation

Definition (Circulation). *If C is an oriented closed curve, the line integral of a vector field $\overrightarrow{F}$ around C is called the* ***circulation*** *of $\overrightarrow{F}$ around C.*

Circulation is a measure of the net tendency of the vector field to point around the curve C.

Properties of Line Integrals

Line integrals share some basic properties with ordinary one-variable integrals:

For a scalar constant λ, vector fields $\overrightarrow{F}$ and $\overrightarrow{G}$, and oriented curves C, C_1, and C_2,

1. $\int_{-C} \overrightarrow{F} \cdot d\overrightarrow{r} = -\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$
2. $\int_C \lambda \overrightarrow{F} \cdot d\overrightarrow{r} = \lambda \int_C \overrightarrow{F} \cdot d\overrightarrow{r}$
3. $\int_C (\overrightarrow{F} + \overrightarrow{G}) \cdot d\overrightarrow{r} = \int_C \overrightarrow{F} \cdot d\overrightarrow{r} + \int_C \overrightarrow{G} \cdot d\overrightarrow{r}$
4. $\int_{C_1+C_2} \overrightarrow{F} \cdot d\overrightarrow{r} = \int_{C_1} \overrightarrow{F} \cdot d\overrightarrow{r} + \int_{C_2} \overrightarrow{F} \cdot d\overrightarrow{r}$

§18.2 COMPUTING LINE INTEGRALS OVER PARAMETERIZED CURVES

Recall that

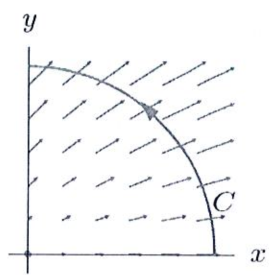
> If $\vec{r}(t), t \in [a,b]$ is a smooth parameterization of an oriented curve C and $\vec{F}$ is a continuous vector the curve C, then
> $$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt.$$

Example 1. Compute $\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F} = (x+y)\vec{i} + y\vec{j}$, and C is the quarter unit circle, oriented counterclockwise as shown in Figure 18.21.

Solution.

$\vec{F} = \langle x+y, y\rangle$

$C: \vec{r}(t) = \langle \cos t, \sin t\rangle, \quad 0 \le t \le \frac{\pi}{2}$



and $r'(t) = \langle -\sin t, \cos t\rangle$

Figure 18.21: The vector field $\vec{F} = (x+y)\vec{i} + y\vec{j}$ and the quarter circle C

$$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = \int_0^{\frac{\pi}{2}} \langle \cos t + \sin t, \sin t\rangle \cdot \langle -\sin t, \cos t\rangle\, dt$$

$$= \int_0^{\pi/2} (-\cos t \sin t - \sin^2 t + \sin t \cos t)\, dt$$

$$= \int_0^{\pi/2} -\sin^2 t\, dt = -\int_0^{\frac{\pi}{2}} \frac{1-\cos 2t}{2}\, dt$$

$$= -\frac{\pi}{4} + \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos 2t\, dt = \frac{-\pi}{4} \approx -0.7854$$

□

Example 3.(SageMath) Let C be the closed curve consisting of the upper half-circle of radius 1 and the line forming its diameter along the x-axis, oriented counterclockwise. (See Figure 18.24.) Find $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = -y\,\vec{i} + x\,\vec{j}$.

Solution. $C_1: \langle \cos t, \sin t \rangle, \quad 0 \le t \le \pi$

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^{\pi} \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle\, dt$$

$$= \int_0^{\pi} (\sin^2 t + \cos^2 t)\, dt = \int_0^{\pi} 1\, dt = \pi$$

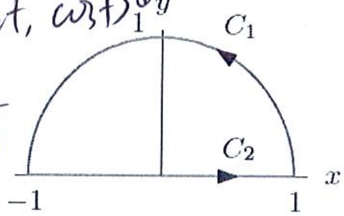


Figure 18.24: The curve $C = C_1 + C_2$ for Example 3

For $\int_{C_2} \vec{F} \cdot d\vec{r}$, $\quad \vec{F} = \langle 0, x \rangle$ and $C_2 = \langle x, 0 \rangle$

$\Rightarrow \vec{F} \cdot \Delta r = 0$

$$\Rightarrow \int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} = \pi + 0 = \pi.$$

□

Notation: $\int_C \vec{F} \cdot d\vec{r}$ **in the component form**

Suppose

$$\vec{F}(x, y, z) = P(x, y, z)\,\vec{i} + Q(x, y, z)\,\vec{j} + R(x, y, z)\,\vec{k},$$

and

$$C : \vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}, \quad t \in [a, b].$$

Then,

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}'(t)\,dt\\ &= \int_a^b \langle P(\vec{r}(t)), Q(\vec{r}(t)), R(\vec{r}(t))\rangle\cdot\langle x'(t), y'(t), z'(t)\rangle\,dt\\ &= \int_a^b P(\vec{r}(t))x'(t)\,dt + \int_a^b Q(\vec{r}(t))y'(t)\,dt + \int_a^b R(\vec{r}(t))z'(t)\,dt\\ &= \int_C P\,dx + Q\,dy + R\,dz\end{aligned}$$

That is, we have the following result.

Theorem 18.1 (Equivalent Line Integrals). *Suppose that the vector field* $\vec{F}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$ *has continuous component functions. Then*

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz}$$

Sometimes, it is helpful to think of $\vec{F}\cdot d\vec{r}$ as $\langle P, Q, R\rangle\cdot\langle dx, dy, dz\rangle = Pdx + Qdy + Rdz$.

Example 5. Evaluate $\int_C xydx - y^2dy$ where C is the line segment from $(0,0)$ to $(2,6)$.

Note: $\vec{F} = \langle xy, -y^2\rangle$

Solution.

Parameterize C by $x = t, y = 3t$ $(\vec{r} = 0 + t\langle 2,6\rangle)$

$0 \le t \le 2$.

$\Rightarrow dx = dt,\ dy = 3dt$

$\int_C xydx - y^2dy = \int_0^2 (t\times 3t)dt - (3t)^2(3dt)$

$= \int_0^2 (-24t^2)dt = -24\cdot\frac{t^3}{3}\Big|_0^2 = -64.$

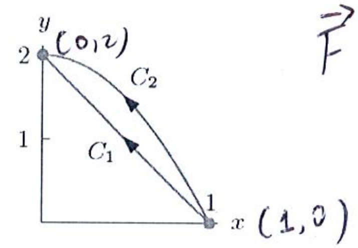
□

Example 2. Consider the vector field $\vec{F} = x\vec{i} + y\vec{j}$. Suppose C_1 is the line segment joining (1,0) to (0,2) and C_2 is a part of a parabola with its vertex at (0,2), joining the same points in the same order. Find a scalar function f with grad$f = \vec{F}$. Hence, find an easy way to calculate the line integrals, and explain how we could have expected them to be the same.

Solution. Let $f(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$. Then

$\vec{F} = \langle x, y \rangle$

$\vec{F} = \langle f_x, f_y \rangle$.



Find the line integral of $\vec{F} = x\vec{i} + y\vec{j}$ over the curves C_1 and C_2

Since $\vec{F} = \nabla f$, we have

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_1} \nabla f \cdot d\vec{r} = f(0,2) - f(1,0)$$

$$= \frac{2^2}{2} - \frac{1}{2} = 2 - \frac{1}{2} = \frac{3}{2}$$

$\square$

§18.3 The Fundamental Theorem for Line Integrals and Independence of Path

Lets recall the fundamental theorem for Calculus:

$$\int_a^b F'(x)dx = F(b) - F(a).$$

It has the following generalization in terms of line integrals:

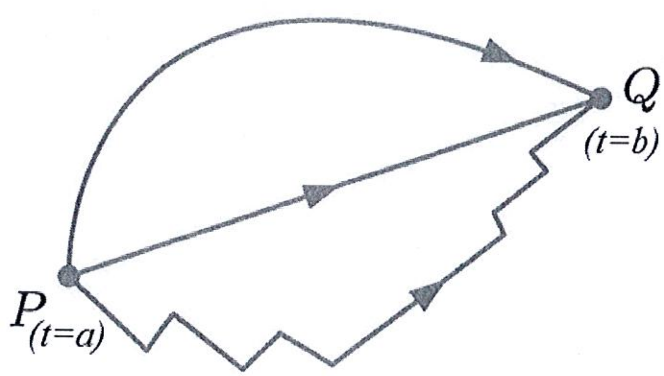
Theorem (Theorem 18.1: The Fundamental Theorem of Calculus for Line Integrals). *Let C be a smooth curve given by $\vec{r}(t), t \in [a, b]$. Let f be a function of 2 or 3 variables whose gradient ∇f (grad f) is continuous on the path C. Then*

$$\int_C \nabla f \cdot d\vec{r}(t) = f(\vec{r}(b)) - f(\vec{r}(a)).$$

Proof.

$$\begin{aligned}
\int_C \nabla f \cdot d\vec{r}(t) &= \int_C \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)dt \\
&= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right) dt \\
&= \int_a^b \frac{d}{dt} f(\vec{r}(t)) \qquad \text{by chain rule} \\
&= f(\vec{r}(b)) - f(\vec{r}(a)). \qquad \text{by Fundamental Theorem of Calculus}
\end{aligned}$$

□



Independence of Path.

Definition. *A vector field $\overrightarrow{F}$ is said to be* **path-independent,** *or* **conservative,** *if for any two points P and Q, the line integral $\int_C \nabla F \cdot d\overrightarrow{r}$ has the same value along any piecewise smooth path C from P to Q lying in the domain of $\overrightarrow{F}$.*

If, on the other hand, the line integral $\int_C \nabla F \cdot d\overrightarrow{r}$ does depend on the path C joining P to Q, then $\overrightarrow{F}$ is said to be a **path-dependent** *vector field.*

Theorem. *If $\overrightarrow{F}$ is a continuous gradient vector field, then $\overrightarrow{F}$ is pathindependent.*

Theorem (Theorem 18.2: Path-independent Fields Are Gradient Fields). *If $\overrightarrow{F}$ is a continuous path-independent vector field on an open region R, then $\overrightarrow{F} = grad f$ for some f defined on R.*

How to Construct f from $\overrightarrow{F}$?(proof) First, notice that there are many different choices for f, since we can add a constant to f without changing ∇f. If we pick a fixed starting point P, then by adding or subtracting a constant to f, we can ensure that $f(P) = 0$. For any other point Q, we define $f(Q)$ by the formula

$$f(Q) = \int_C \overrightarrow{F} \cdot d\overrightarrow{r}, \quad \text{where } C \text{ is any path from } P \text{ to } Q.$$

Combining Theorems 18.1 and 18.2, we have

Theorem. *A continuous vector field $\overrightarrow{F}$ defined on an open region is path-independent if and only if $\overrightarrow{F}$ is a gradient vector field.*

Example 3. Show that the vector field $\overrightarrow{F}(x, y) = y\cos x\,\overrightarrow{i} + (\sin x + y)\,\overrightarrow{j}$ is path-independent.

Proof. Suppose there is an f s.t. $\vec{F} = \nabla f$.

or $\frac{\partial f}{\partial x} = y\cos x$ and $\frac{\partial f}{\partial y} = \sin x + y$

$\Rightarrow \begin{cases} f(x,y) = y\sin x + C(y), \text{ where } C(y) \text{ is a function of } y \text{ only.} \\ \text{and } \frac{\partial f}{\partial y} = \sin x + C'(y), \text{ where } C'(y) = y \end{cases}$

Thus, $f(x,y) = y\sin x + \frac{y^2}{2} + C$, where C is a constant $\square$

$\Rightarrow \vec{F}$ is path-independent

§18.4 PATH-DEPENDENT VECTOR FIELDS AND GREENS THEOREM

How to Tell If a Vector Field Is Path-Dependent Using Line Integrals?

Definition (closed curve). *A curve is called* ***closed*** *if its terminal point coincides with its initial point.*

Theorem. *A vector field is path-independent if and only if* $\boxed{\int_C \vec{F} \cdot d\vec{r} = 0}$ *for every closed curve* C.

Hence, to see if a field is path-dependent, we look for a closed path with nonzero circulation. For instance, the vector field in Example 1 has nonzero circulation around a circle around the origin, showing it is path-dependent.

Example 2. Does the vector field $\vec{F} = 2xy\,\vec{i} + xy\,\vec{j}$ have a potential function? If so, find it. Suppose there is an f s.t. $\vec{F} = \nabla f$: $\frac{\partial f}{\partial x} = 2xy$, $\frac{\partial f}{\partial y} = xy$

Solution.

$\Rightarrow f(x,y) = x^2y + C(y)$, where $C(y)$ is a function of y

$\frac{\partial f}{\partial y} = x^2 + C'(y) = xy \Rightarrow C'(y) = xy - x^2$

which is not a function of y only. ANS. No. □

Is there an easier way to see that a vector field has no potential function, other than by trying to find the potential function and failing? The answer is yes.

Theorem. *Let* $\vec{F}(x,y) = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j}$ *be a vector field on* $D \subset \mathbb{R}^2$*, where* P *and* Q *have continuous partial derivatives in* D*. If* $\vec{F}$ *is conservative, then*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Example 3. Show that $\vec{F} = 2xy\,\vec{i} + xy\,\vec{j}$ cannot be a gradient vector field.

Solution. Let $P = 2xy$, $Q = xy$

$\frac{\partial P}{\partial y} = 2x \neq \frac{\partial Q}{\partial x} = y$

□

Definition (Curl). *Let $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ be a vector field on $\mathbb{R}^3$. The **curl** of $\vec{F}$ is defined by*

$$curl\vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

The curl of a vector field $\vec{F}$ is a vector field which measures the rotational effect of $\vec{F}$. You can read **Chapter 20** for the geometric meaning of curl $\vec{F}$. **Remark.** If $\vec{F} = P\vec{i} + Q\vec{j}$, then curl $\vec{F} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$.

Greens Theorem gives the relationship between a line integral along a simple closed curve C on the plane and the double integral over the plane region D that C bounds.

Definition (Simple curve). *A **simple curve** is a curve which does not intersect itself.*

Theorem (Theorem 18.3: Greens Theorem). *Suppose C is a piecewise smooth simple closed curve that is the boundary of a region R in the plane and oriented so that the region is on the left as we move around the curve. See Figure 18.44. Suppose $\vec{F} = P\vec{i} + Q\vec{j}$ is a smooth vector field on an open region containing R and C. Then*

$$\int_C \vec{F} \cdot d\vec{r} = \int_C Pdx + Qdy = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

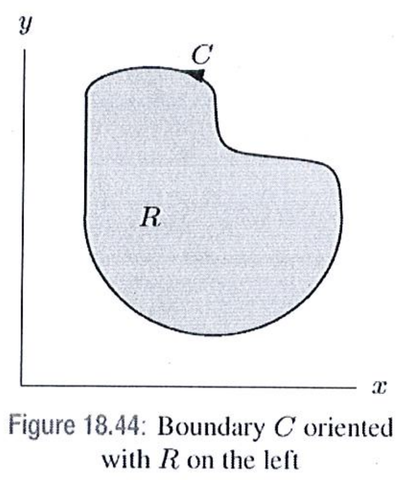


Figure 18.44: Boundary C oriented with R on the left

Example 4.(SageMath) Use Greens Theorem to evaluate $\int_C (y^2\vec{i} + x\vec{j}) \cdot d\vec{r}$, where C is the counterclockwise path around the perimeter of the rectangle $0 \le x \le 2, 0 \le y \le 3$.

$P = y^2, \; Q = x$

Solution.

$$\int_C \langle y^2, x\rangle \cdot d\vec{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$= \int_0^3 \int_0^2 (1 - 2y)\,dx\,dy = -12.$$

□

The Curl Test for Vector Fields in 2-Space:

Suppose $\vec{F} = P\vec{i} + Q\vec{j}$ is a vector field with continuous partial derivatives such that

- The domain of $\vec{F}$ has the property that every closed curve in it encircles a region that lies entirely within the domain. In particular, the domain of $\vec{F}$ has no holes.
- $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 0$

Then $\vec{F}$ is path-independent, so $\vec{F}$ is a gradient field and has a potential function.

The Curl Test for Vector Fields in 3-Space:

Suppose $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is a vector field on 3-space with continuous partial derivatives such that

- The domain of $\vec{F}$ has the property that every closed curve in it can be contracted to a point in a smooth way, staying at all times within the domain.
- curl $\vec{F} = 0$

Then $\vec{F}$ is path-independent, so $\vec{F}$ is a gradient field and has a potential function.

Example 5. Let $\overrightarrow{F}$ be the vector field given by $\overrightarrow{F}(x,y) = \dfrac{-y\overrightarrow{i} + x\overrightarrow{j}}{x^2+y^2}$.

(a) Calculate $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$, where $P = -y/(x^2+y^2)$ and $Q = x/(x^2+y^2)$. Does the curl test imply that $\overrightarrow{F}$ is path-independent?

(b) Calculate $\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$, where C is the unit circle centered at the origin and oriented counterclockwise. Is $\overrightarrow{F}$ a path-independent vector field?

(c) Explain why the answers to parts (a) and (b) do not contradict Greens Theorem.

Solution.

(a) $$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{-1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial Q}{\partial x}$$

But $\vec{F}$ is undefined at $(0,0)$. $\Rightarrow$ The curl Test doesn't apply.

(b) $\vec{r}(t) = \langle \cos t, \sin t \rangle$, $0 \le t \le 2\pi$ $\Rightarrow$ $\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle \, dt$$

$$= \int_0^{2\pi} dt = 2\pi \neq 0 \Rightarrow \vec{F} \text{ is not path-independent.}$$

(see Figure 18.49)

($\vec{F} = \nabla\left(\tan^{-1}\left(\frac{y}{x}\right)\right)$ and $\tan^{-1}(y/x)$ is θ from polar coordinates $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. θ increas by 2π each time we wind once around the origin counterclockwise)

(c) The domain of $\vec{F}$ is the "punctured plane". That is, $\vec{F}$ is not defined at the origin.

Example 7. Decide if the following vector fields are path-independent and whether or not the curl test applies.

(a) $\vec{F} = \dfrac{x\vec{i} + y\vec{j} + z\vec{k}}{(x^2 + y^2 + z^2)^{3/2}}$.

(b) $\vec{G} = \dfrac{-y\vec{i} + x\vec{j}}{x^2 + y^2} + z^2\vec{k}$.

Solution.

(a) Let $f = \dfrac{-1}{(x^2+y^2+z^2)^{\frac{1}{2}}}$. Then $\vec{F} = \nabla f$

Thus, $\vec{F}$ is a gradient field and therefore path-independent

(calculations show that $\text{curl}\vec{F} = 0$)

The domain of $\vec{F}$ is all $\mathbb{R}^3$ except the origin and any closed curve in the domain can be pulled to a point without leaving the domain. Thus the curl Test applies.

(b) The domain of G is $\{(x,y,z) \mid x \neq 0, y \neq 0\}$ which is the $\mathbb{R}^3$ minus the z-axis. $\Rightarrow$ The curl test does not apply.

We choose a closed curve C and calculate $\int_C \vec{G} \cdot d\vec{r}$.

Let C be the circle $x^2+y^2=1$, $z=0$ traversed counterclockwise when viewed from the positive z-axis.

$\vec{r}(t) = \langle \cos t, \sin t \rangle$, $0 \le t \le 2\pi$, $\Rightarrow r'(t) = \langle -\sin t, \cos t \rangle$

$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle \, dt = \int_0^{2\pi} dt = 2\pi$

$\Rightarrow \vec{F}$ is not path-independent.

§19.3 CURL AND THE DIVERGENCE OF A VECTOR FIELD

Divergence and curl are two measurements of vector fields that are very useful in a variety of applications. Both are most easily understood by thinking of the vector field as representing a flow of a liquid or gas; that is, each vector in the vector field should be interpreted as a velocity vector. Roughly speaking, divergence measures the tendency of the fluid to collect or disperse at a point, and curl measures the tendency of the fluid to swirl around the point. Divergence is a scalar, that is, a single number, while curl is itself a vector. The magnitude of the curl measures how much the fluid is swirling, the direction indicates the axis around which it tends to swirl. These ideas are somewhat subtle in practice, and are beyond the scope of this course. You can find additional information on the web, for example at `http://mathinsight.org/curl_idea` and `http://mathinsight.org/divergence_idea`.

Recall that if $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is a vector field on $\mathbb{R}^3$. The **curl** of $\vec{F}$ is defined by

$$\operatorname{curl}\vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}.$$

We have used the del operator ∇. We let

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}.$$

That is, we regard ∇ as a 3-dimensional vector consisting of the **operators of partial differentiations** with respect to x, y, z.

We can multiple ∇ by a scalar function (on the right), take the dot product with a function, or the cross product with a vector field. For example, we may regard the gradient of a function f as being the scalar multiplication of ∇ and f. That is,

$$\text{grad } f = \nabla f = \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}.$$

The curl of a vector field $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ can be regarded as the cross product between ∇ and $\overrightarrow{F}$.

$$curl\ \overrightarrow{F} = \nabla \times \overrightarrow{F} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$
$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\overrightarrow{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\overrightarrow{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\overrightarrow{k}.$$

Example. Let $\overrightarrow{F}(x, y, z) = xz\overrightarrow{i} + xyz\overrightarrow{j} - y^2\overrightarrow{k}$. Find curl $\overrightarrow{F}$.

Solution. P Q R

$$\text{curl } \overrightarrow{F} = (-2y - xy)\overrightarrow{i} + (x - 0)\overrightarrow{j} + (yz - 0)\overrightarrow{k}$$

$$= \langle -2y - xy,\ x,\ yz \rangle.$$

□

Definition. *Let $\overrightarrow{F} = P\overrightarrow{i} + Q\overrightarrow{j} + R\overrightarrow{k}$ be a vector field on $\mathbb{R}^3$. The divergence of $\overrightarrow{F}$ is defined by*

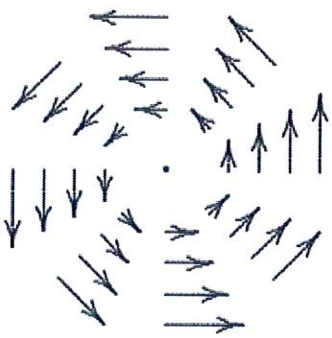
$$div\ \overrightarrow{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$
$$= \left(\overrightarrow{i}\frac{\partial}{\partial x} + \overrightarrow{j}\frac{\partial}{\partial y} + \overrightarrow{k}\frac{\partial}{\partial z}\right) \cdot (P\overrightarrow{i} + Q\overrightarrow{j} + R\overrightarrow{k})$$
$$= \nabla \cdot \overrightarrow{F}.$$

Here are two simple but useful facts about divergence and curl.

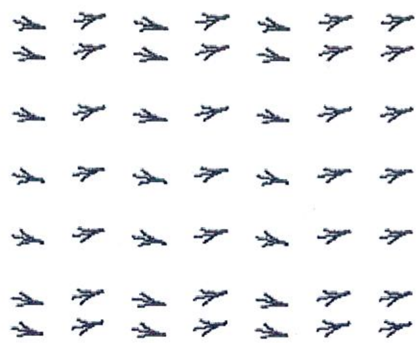
Theorem. $\nabla \cdot (\underbrace{\nabla \times \overrightarrow{F}}_{\text{curl } \overrightarrow{F}}) = 0$. *In words, this says that the divergence of the curl is zero.*

Theorem. $\nabla \times (\nabla f) = 0$. *That is, the curl of a gradient is the zero vector. Recalling that gradients are conservative vector fields, this says that the curl of a conservative vector field is the zero vector. Under suitable conditions, it is also true that if the curl of $\overrightarrow{F}$ is 0 then $\overrightarrow{F}$ is conservative.*

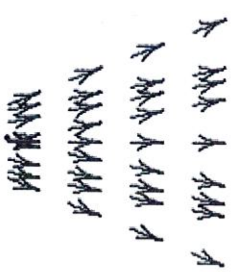
For a velocity vector field $\overrightarrow{F}$, div $\overrightarrow{F}$ measures the amount of flow radiating at a point. If the flow is uniform and without compression or expansion, then div $\overrightarrow{F} = 0$. Thus, if div $\overrightarrow{F} = 0$, we say that $\overrightarrow{F}$ is **incompressible**. Whereas curl $\overrightarrow{F}$ measures the rotational effect of the vector field $\overrightarrow{F}$. Therefore, if curl $\overrightarrow{F} = 0$, then we say that $\overrightarrow{F}$ is **irrotational**.



curl $\mathbf{F} \neq 0$



curl $\mathbf{F} = 0$
irrotational



div $\mathbf{F} \neq 0$

div $\mathbf{F} = 0$
incompressible

Another differential operator occurs when we compute the divergence of a gradient vector field ∇f. If f is a function of three variables, we have

$$\text{div } (\nabla f) = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

We abbreviate this expression as $\nabla^2 f$. The operator $\nabla^2 = \nabla \cdot \nabla$ is called the **Laplace**

operator because of its relation to Laplaces equation:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.$$

We can also apply the Laplace operator ∇^2 to a vector field $\overrightarrow{F} = P\overrightarrow{i} + Q\overrightarrow{j} + R\overrightarrow{k}$ in terms of its components:

$$\nabla^2 \overrightarrow{F} = \nabla^2 P\overrightarrow{i} + \nabla^2 Q\overrightarrow{j} + \nabla^2 R\overrightarrow{k}.$$