Contents

5.1 Antiderivatives or Indefinite Integrals

The topics covered in this section are from section 4.9.

5.1.1 Review: Definition and Properties of Indefinite Integrals

In Calculus I, we spent considerable time considering the derivatives of a function and their applications and discussed, “the other direction,” antiderivatives. That is, given a function $f(x)$, we discussed functions $F(x)$ such that $F^{'}(x) = f(x)$. We first review this topic.

Given a function $y = f(x)$, a *differential equation* is one that incorporates y, x, and the derivatives of y. For instance, a simple differential equation is:

$$y^{'} = 2x.$$

Solving a differential equation amounts to finding a function y that satisfies the given equation. Take a moment and consider that equation; can you find a function y such that $y^{'} = 2x$?

Can you find another? And yet another?

Hopefully one was able to come up with at least one solution: $y = x^2$. “Finding another” may have seemed impossible until one realizes that a function like $y = x^2+1$

also has a derivative of $2x$. Once that discovery is made, finding "yet another" is not difficult; the function $y = x^2 + 123,456,789$ also has a derivative of $2x$. The differential equation $y' = 2x$ has many solutions. This leads us to some definitions.

Definition (Antiderivatives and Indefinite Integrals)**.** Let a function $f(x)$ be given. An **antiderivative** of $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$.
The set of all antiderivatives of $f(x)$ is the **indefinite integral of** f, denoted by

$$\int f(x)\ dx.$$

Knowing one antiderivative of f allows us to find infinitely more, simply by adding a constant. Not only does this give us *more* antiderivatives, it gives us *all* of them.

Theorem (Antiderivative Forms)**.** Let $F(x)$ be any antiderivatives of $f(x)$ on an interval I. Then all the antiderivatives of f on I have the form

$$F(x) + C,$$

where C is an **arbitrary constant**.

Remark. Every time an indefinite integral sign $\int$ appears, it is followed by a function called the **integrand**, which in turn is followed by the differential dx. For now, dx simply means that x is the independent variable, or the variable of integration. The notation $\int f(x)\ dx$ represents all the antiderivatives of f.

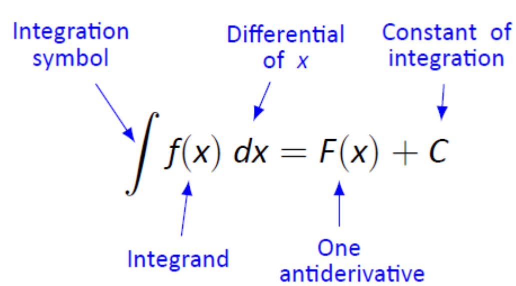


Figure 1: Understanding the indefinite integral notation.

Figure 1 shows the typical notation of the indefinite integral. The integration symbol, $\int$, is in reality an "elongated S," representing "take the sum." We will later see how *sums* and *antiderivatives* are related. The $\int$ symbol and the differential dx are not "bookends" with a function sandwiched in between; rather, the symbol $\int$ means "find all antiderivatives of what follows," and the function $f(x)$ and dx are multiplied together; the dx does not "just sit there."

Example Determine the following indefinite integrals

a. $\int 3x^3\,dx$ **b.** $\int \frac{1}{1+x^2}\,dx$ **c.** $\int \sin t\,dt$

Solution.

□

Theorem (Power Rule for Indefinite Integrals)**.**

$$\int x^p\,dx = \frac{x^{p+1}}{p+1} + C,$$

where $p \neq -1$ is a real number and C is an arbitrary constant.

Theorem (Constant Multiple and Sum Rules)**.**

$$\textbf{Constant Multiple Rule: } \int cf(x)\,dx = c\int f(x)\,dx, \text{ for real numbers } c$$

$$\textbf{Sum Rule: } \int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$$

We restate a list of derivatives here to stress the relationship between derivatives and antiderivatives. This list will also be useful as a glossary of common antiderivatives as we learn (D_x denotes $\frac{d}{dx}$).

$$D_x(x^n) = nx^{n-1} \implies \int x^n\ dx = \frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$$

$$D_x(e^x) = e^x \implies \int e^x\ dx = e^x + C$$

$$D_x(a^x) = a^x \ln a \implies \int a^x\ dx = \frac{a^x}{\ln a} + C$$

$$D_x(\ln|x|) = \frac{1}{x} \implies \int \frac{dx}{x} = \ln|x| + C$$

$$D_x(\sin x) = \cos x \implies \int \cos x\ dx = \sin x + C$$

$$D_x(\cos x) = -\sin x \implies \int \sin x\ dx = -\cos x + C$$

$$D_x(\tan x) = \sec^2 x \implies \int \sec^2 x\ dx = \tan x + C$$

$$D_x(\sec x) = \sec x \tan x \implies \int \sec x \tan x\ dx = \sec x + C$$

$$D_x(\cot x) = -\csc^2 x \implies \int \csc^2\ dx = -\cot x + C$$

$$D_x(\csc x) = -\csc x \cot x \implies \int \csc x \cot x\ dx = -\csc x + C$$

$$D_x(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \implies \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$$

$$D_x(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}} \implies \int \frac{-dx}{\sqrt{1-x^2}} = \cos^{-1} x + C = -\sin^{-1} x + C$$

$$D_x(\tan^{-1} x) = \frac{1}{1+x^2} \implies \int \frac{dx}{1+x^2} = \tan^{-1} x + C$$

$$D_x(\cot^{-1} x) = \frac{-1}{1+x^2} \implies \int \frac{-dx}{1+x^2} = \cot^{-1} x + C = -\tan^{-1} x + C$$

$$D_x(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}} \implies \int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}|x| + C$$

$$D_x(\csc^{-1} x) = \frac{-1}{x\sqrt{x^2-1}} \implies \int \frac{-dx}{x\sqrt{x^2-1}} = \csc^{-1}|x| + C = -\sec^{-1}|x| + C$$

$$D_x(\ln|\sec x|) = \tan x \implies \int \tan x\ dx = \ln|\sec x| + C$$

$$D_x(\ln|\sin x|) = \cot x \implies \int \cot x\ dx = \ln|\sin x| + C$$

Example Determine the following indefinite integrals

a. $\int \sin(3x)\,dx$ **b.** $\int \sec ax \tan ax\,dx$, where $a \neq 0$ is a real number

Solution.

$\square$

Example Determine the following indefinite integrals

a. $\int \sec^2 3x\,dx$ **b.** $\int \cos \frac{x}{2}\,dx$

Solution.

$\square$

5.1.2 Review: Initial Value Problems

An equation involving an unknown function and its derivatives is called a **differential equation**. The equation

$$\frac{dy}{dx} = f(x) \tag{5.1}$$

is a simple example of a differential equation. Solving this equation means finding a function y with a derivative f. Therefore, the solutions of Equation 5.1 are the antiderivatives of f. If F is one antiderivative of f, every function of the form $y = F(x) + C$ is a solution of that differential equation. For example, the solutions of

$$\frac{dy}{dx} = 6x^2$$

are given by

$$y = \int 6x^2\, dx = 2x^3 + C.$$

Sometimes we are interested in determining whether a particular solution curve passes through a certain point (x_0, y_0) —that is, $y(x_0) = y_0$. The problem of finding a function y that satisfies a differential equation

$$\frac{dy}{dx} = f(x) \tag{5.2}$$

with the additional condition

$$y(x_0) = y_0 \tag{5.3}$$

is an example of an initial-value problem. The condition $y(x_0) = y_0$ is known as an initial condition. For example, looking for a function y that satisfies the differential equation

$$\frac{dy}{dx} = 6x^2$$

and the initial condition

$$y(1) = 5.$$

is an example of an initial-value problem. Since the solutions of the differential equation are $y = 2x^3 + C$, to find a function y that also satisfies the initial condition, we need to find C such that $y(1) = 2(1)^3 + C = 5$. From this equation, we see that $C = 3$, and we conclude that $y = 2x^3 + 3$ is the solution of this initial-value problem as shown in Figure 2.

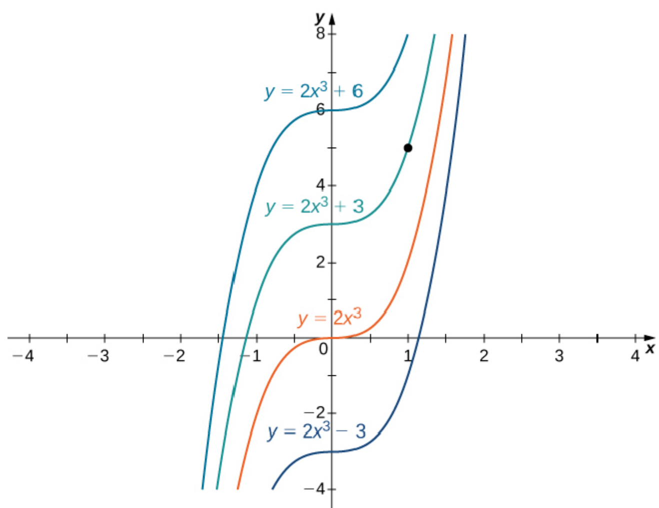


Figure 2: Some of the solution curves of the differential equation $\frac{dy}{dx} = 6x^2$ are displayed. The function $y = 2x^3 + 3$ satisfies the differential equation and the initial condition $y(1) = 5$.

Example Solve the initial value problem $f'(x) = x^2 - 2x$ with $f(1) = \frac{1}{3}$.

Solution.

$\square$

Initial Value Problems for Velocity and Position

Suppose an object moves along a line with a (known) velocity $v(t)$, for $t \geq 0$. Then its position is found by solving the initial value problem

$$s'(t) = v(t), s(0) = s_0, \text{ where } s_0 \text{ is the (known) initial position.}$$

If the (known) acceleration of the object $a(t)$ is given, then its velocity is found by solving the initial value problem

$$v'(t) = a(t), v(0) = v_0, \text{ where } v_0 \text{ is the (known) initial velocity.}$$

Example Runner A begins at the point $s(0) = 0$ and runs with velocity $v(t) = 2t$. Runner B begins with a head start at the point $S(0) = 8$ and runs with velocity $V(t) = 2$. Find the positions of the runners for $t \geq 0$ and determine who is ahead at $t = 6$ time units.

Solution.

$\square$

Example Neglecting air resistance, the motion of an object moving vertically near Earth's surface is determined by the acceleration due to gravity, which is approximately $9.8m/s^2$. Suppose a stone is thrown vertically upward at $t = 0$ with a velocity of $40m/s$ from the edge of a cliff that is $100m$ above a river.

a. Find the velocity $v(t)$ of the object, for $t \geq 0$.

b. Find the position $s(t)$ of the object, for $t \geq 0$.

c. Find the maximum height of the object above the river.

d. With what speed does the object strike the river?

Solution.

$\square$

5.1.3 Review: Power Rule for Indefinite Integrals

Recall that the power rule for indefinite integrals is

$$\boxed{\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1.}$$

Let's see some examples in this section.

Example 1. Find

$$\int \left(x^2 + \sqrt[3]{x} - \sqrt{x}\right) \, dx.$$

Example 2. Find

$$\int \left(x^{100} + \frac{\sqrt{x} - x^{3/2}}{x}\right) \, dx.$$

Example 3. Find

$$\int \left(x^{10} + \sqrt{x} + \frac{1}{\sqrt{x}} \right) \, dx.$$

Example 4. Find

$$\int \left(x^{-3} + \frac{x - x^2}{\sqrt{x}} \right) \, dx.$$

5.2 Definite Integrals

5.2.1 Review of $\sum$ notation and some formulas

Note: The definition of a definite integral involves the use of sigma ($\sum$) notation to describe a sum.

Sigma Notation. If f is a function defined on a finite set $\{k, k+1, \ldots, m-1, m\}$ then

$$\sum_{i=k}^{m} f(i) = f(k) + f(k+1) + \cdots + f(m-1) + f(m).$$

□

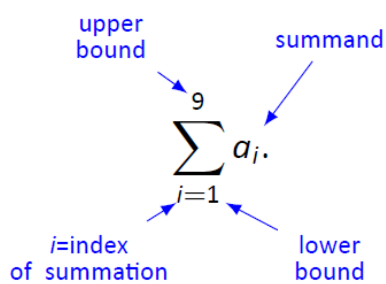


Figure 3: Understanding summation notation.

The upper case sigma represents the term "sum". The index of summation above is i; any symbol can be used. By convention, the index takes on only the integer values between (and including) the lower and upper bounds.

Examples:

$$\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30,$$

$$\sum_{i=-1}^{7} i = -1 + 0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 = 27.$$

□

Remark 1. One should notice that in $\sum_{i=k}^{m} f(i)$ the i is simply an **indexing** variable and so

$$\sum_{i=k}^{m} f(i) = \sum_{j=k}^{m} f(j).$$

Remark 2. One can do a change of variable in the sigma notation.

$$\begin{aligned}\sum_{i=3}^{6}(i-2)^2 =&(3-2)^2 + (4-2)^2 + (5-2)^2 + (6-2)^2\\ =&1^2 + 2^2 + 3^2 + 4^2 = \sum_{i=1}^{4} i^2\end{aligned}$$

$$\therefore \quad \sum_{i=3}^{6}(i-2)^2 = \sum_{j=1}^{4} j^2 \text{ where } j = i-2.$$

Example. Write $2^3 + \cdots + n^3$ in sigma notation starting with $j = 0$.

Solution.

$$2^3 + \cdots + n^3 = \sum_{i=2}^{n} i^3 = \sum_{j=0}^{n-2}(j+2)^3, \text{ where } j = i-2. \qquad \square$$

Some Basic Properties and Formulas.

1. $\sum_{i=1}^{n} c = nc$

2. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$

3. $\sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

4. $\boxed{\sum_{i=1}^{n} i = \frac{n(n+1)}{2}}$

5. $\boxed{\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}}$

6. $\boxed{\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2}$

Proof. To establish Part **1**, we need only think of c as the function $f(x) = c$ and then

$$\sum_{i=1}^{n} c = \sum_{i=1}^{n} f(i) = f(1) + \cdots + f(n) = c + \cdots + c = nc.$$

Part **2** is the distributive law and Part **3** follows by simply writing out the terms and then using the commutative law and the associative law to collect terms

For Part **4**, set $S = \sum_{i=1}^{n} i$, write it out forwards and backward and add to get:

$$\begin{aligned} S =& 1 + 2 + 3 + \cdots + (n-1) + n \\ S =& n + (n-1) + \cdots + 3 + 2 + 1 \\ \text{Add} \quad 2S =& (n+1) + (n+1) + \cdots + (n+1) = n(n+1) \\ \text{Therefore} \quad S = \sum_{i=1}^{n} i =& \frac{n(n+1)}{2}. \end{aligned}$$

(*Read the following proof if you are interested.*)

We prove **5** and omit the proof of **6** which is similar. Set $S = \sum_{i=1}^{n} i^2$. Observe first that

$$\begin{aligned} &\sum_{i=1}^{n} [(1+i)^3 - i^3] \\ &= (2^3 - 1^3) + (3^3 - 2^3) + \cdots + (n^3 - (n-1)^3) + ((n+1)^3 - n^3) \\ &= (n+1)^3 - 1 = n^3 + 3n^2 + 3n \end{aligned}$$

and on the other hand

$$\begin{aligned} \sum_{i=1}^{n} [(1+i)^3 - i^3] &= \sum_{i=1}^{n} [1 + 3i + 3i^2 + i^3 - i^3] \\ &= \sum_{i=1}^{n} [1 + 3i + 3i^2] \\ &= \sum_{i=1}^{n} 1 + 3\sum_{i=1}^{n} i + 3\sum_{i=1}^{n} i^2 \\ &= n + 3\frac{n(n+1)}{2} + 3S = \frac{3}{2}n^2 + \frac{5}{2}n + 3S. \end{aligned}$$

Therefore,

$$n^3 + 3n^2 + 3n = \frac{3}{2}n^2 + \frac{5}{2}n + 3S$$

and so we have

$$\begin{aligned} S =& \frac{1}{3}[n^3 + 3n^2 + 3n - (\frac{3}{2}n^2 + \frac{5}{2}n)] \\ =& \frac{1}{3}[n^3 + \frac{3}{2}n^2 + \frac{1}{2}n] = \frac{2n^3 + 3n^2 + n}{6} = \frac{n(n+1)(2n+1)}{6}. \end{aligned}$$

$\square$

5.2.2 Riemann Sums

A fundamental calculus technique is to first answer a given problem with an **approximation**, then **refine the approximation** to make it better, then use limits in the refining process to find the exact answer. That is what we will do here.

Consider the region given in Figure 4, which is the area under $y = 4x - x^2$ on $[0, 4]$. What is the signed area of this region?

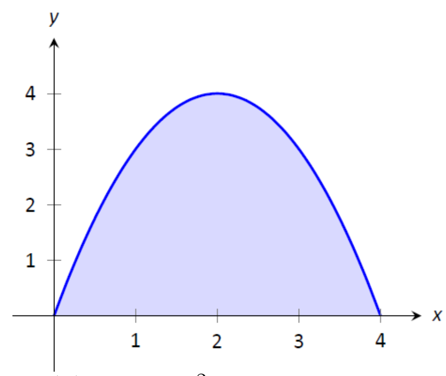


Figure 4: A graph of $f(x) = 4x - x^2$. What is the area of the shaded region?

We start by approximating. We can surround the region with a rectangle with height and width of 4 and find the area is approximately 16 square units. This is obviously an *over–approximation*; we are including area in the rectangle that is not under the parabola.

We have an approximation of the area, using one rectangle. How can we refine our approximation to make it better? The key to this section is this answer: *use more rectangles.*

Let's use 4 rectangles with an equal width of 1. This *partitions* the interval $[0, 4]$ into 4 *subintervals*, $[0, 1]$, $[1, 2]$, $[2, 3]$ and $[3, 4]$. On each subinterval we will draw a rectangle.

There are three common ways to determine the height of these rectangles: the **Left Hand Rule**, the **Right Hand Rule**, and the **Midpoint Rule**.

For these, check this web app I developed `https://xuemaozhang.shinyapps.io/riemannsum`, and jump to the definition of **partion** and **Riemann Sum**.

The **Left Hand Rule**(LHR) says to evaluate the function at the left–hand endpoint of the subinterval and make the rectangle that height. In Figure 5 (a), the rectangles drawn have height determined by the Left Hand Rule (LHR); the heights are $f(0), f(1), f(2), f(3)$.

The **Right Hand Rule** (RHR) says the opposite: on each subinterval, evaluate the function at the right endpoint and make the rectangle that height. In Figure 5 (b), the rectangles drawn are drawn using $f(1), f(2), f(3), f(4)$ as their heights.

The **Midpoint Rule** (MPR) says that on each subinterval, evaluate the function at the midpoint and make the rectangle that height. Figure 5 (c), the rectangles drawn were made using the Midpoint Rule (MPR), with heights of $f(0.5), f(1.5), f(2.5), f(3.5)$.

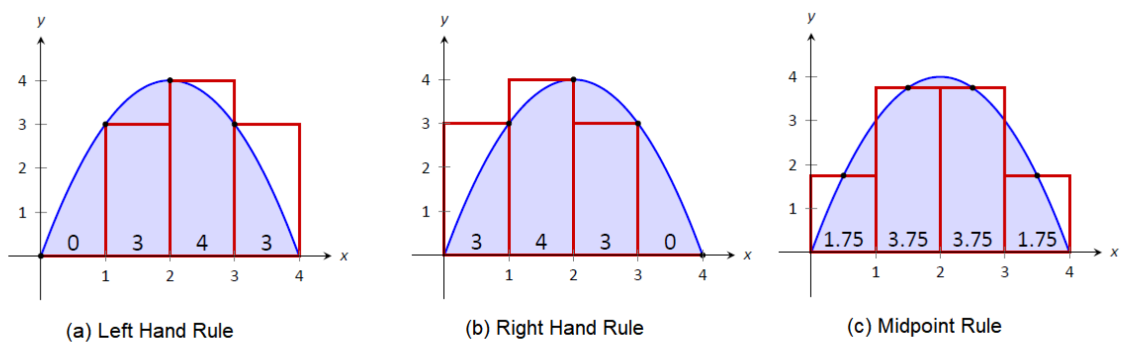


Figure 5: Approximating the area of the shaded region under $f(x) = 4x - x^2$ using the Left Hand Rule, Left Hand Rule, and Midpoint Rule, using 4 equally spaced subintervals

We now calculate the three approximations. We break the interval $[0, 4]$ into four subintervals as before.

(1) In Figure 5(a) we see 4 rectangles drawn on $f(x) = 4x - x^2$ using the Left Hand Rule.

Note how in the first subinterval, $[0, 1]$, the rectangle has height $f(0) = 0$. We add up the areas of each rectangle (height$\times$ width) for our Left Hand Rule approximation:

$$f(0) \cdot 1 + f(1) \cdot 1 + f(2) \cdot 1 + f(3) \cdot 1 =$$

$$0 + 3 + 4 + 3 = 10.$$

(2) Figure 5(b) shows 4 rectangles drawn under f using the Right Hand Rule; note how the $[3, 4]$ subinterval has a rectangle of height 0.

In this example, these rectangle seem to be the mirror image of those found in part (a) of the Figure. This is because of the symmetry of our shaded region. Our approximation gives the same answer as before, though calculated a different way:

$$\begin{aligned} &f(1)\cdot 1 + f(2)\cdot 1 + f(3)\cdot 1 + f(4)\cdot 1 = \\ &\qquad\qquad 3 + 4 + 3 + 0 = 10. \end{aligned}$$

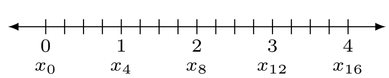
(3) Figure 5(c) shows 4 rectangles drawn under f using the Midpoint Rule.

This gives an approximation of $\int_0^4 (4x - x^2)\, dx$ as:

$$\begin{aligned} &f(0.5)\cdot 1 + f(1.5)\cdot 1 + f(2.5)\cdot 1 + f(3.5)\cdot 1 = \\ &\qquad\qquad 1.75 + 3.75 + 3.75 + 1.75 = 11. \end{aligned}$$

Our three methods provide two approximations: 10 and 11.

It is hard to tell at this moment which is a better approximation: 10 or 11? We can continue to refine our approximation by using more rectangles. We will approximate area using 16 equally spaced subintervals and the Right Hand Rule. Before doing so, it will pay to do some careful preparation.



The figure shows a number line of $[0, 4]$ divided, or *partitioned*, into 16 equally spaced subintervals. We denote 0 as x_0; we have marked the values of x_4, x_8, x_{12} and x_{16}. We could mark them all, but the figure would get crowded. While it is easy to figure that $x_9 = 2.25$, in general, we want a method of determining the value of x_i without consulting the figure. Consider:

$$x_i = \underbrace{x_0}_{\textit{starting value}} + i \overbrace{\Delta x}_{\textit{subinterval size}}, i = 1, \ldots, 16.$$

So $x_9 = x_0 + 9(4/16) = 2.25$.

If we had partitioned $[0, 4]$ into 100 equally spaced subintervals, each subinterval would have length $\Delta x = 4/100 = 0.04$. We could compute x_{32} as

$$x_{32} = x_0 + 32(4/100) = 1.28.$$

(That was far faster than creating a sketch first.)

Given any subdivision of $[0, 4]$, the first subinterval is $[x_0, x_1]$; the second is $[x_1, x_2]$; the i^{th} subinterval is $[x_{i-1}, x_i]$.

When using the Left Hand Rule, the height of the i^{th} rectangle will be $f(x_{i-1})$.

When using the Right Hand Rule, the height of the i^{th} rectangle will be $f(x_i)$.

When using the Midpoint Rule, the height of the i^{th} rectangle will be $f\left(\dfrac{x_{i-1}+x_i}{2}\right)$.

Thus the approximation with 16 equally spaced subintervals can be expressed as follows, where $\Delta x = 4/16 = 1/4$:

Left Hand Rule: $\displaystyle\sum_{i=1}^{16} f(x_{i-1})\Delta x = \sum_{i=1}^{16} f\Big(x_0 + (i-1)\Delta x\Big)\Delta x$

Right Hand Rule: $\displaystyle\sum_{i=1}^{16} f(x_i)\Delta x = \sum_{i=1}^{16} f\Big(x_0 + i\Delta x\Big)\Delta x$

Midpoint Rule: $\displaystyle\sum_{i=1}^{16} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x = \sum_{i=1}^{16} f\Big(x_0 + (i-\frac{1}{2})\Delta x\Big)\Delta x$

Example. Approximate the area of the region under $\int_0^4 (4x - x^2)\, dx$ on $[0, 4]$ using the Right Hand Rule and summation formulas with 16 and 1000 equally spaced intervals.

Solution. Using the formula derived before, using 16 equally spaced intervals and the Right Hand Rule, we can approximate the area as

$$\sum_{i=1}^{16} f(x_i)\Delta x.$$

We have $\Delta x = 4/16 = 0.25$. Note that

$$x_i = 0 + i\Delta x$$

Using the summation formulas, we have the approximation

$$\begin{aligned}
\sum_{i=1}^{16} f(x_i)\Delta x &= \sum_{i=1}^{16} f(i\Delta x)\Delta x \\
&= \sum_{i=1}^{16} \big(4i\Delta x - (i\Delta x)^2\big)\Delta x \\
&= \sum_{i=1}^{16}(4i\Delta x^2 - i^2\Delta x^3) \\
&= (4\Delta x^2)\sum_{i=1}^{16} i - \Delta x^3 \sum_{i=1}^{16} i^2 \qquad (5.4) \\
&= (4\Delta x^2)\frac{16\cdot 17}{2} - \Delta x^3\frac{16(17)(33)}{6} \qquad (\Delta x = 0.25) \\
&= 10.625
\end{aligned}$$

We were able to sum up the areas of 16 rectangles with very little computation. In Figure 6 the function and the 16 rectangles are graphed. While some rectangles over–approximate the area, other under–approximate the area (by about the same amount). Thus our approximate area of 10.625 is likely a fairly good approximation.

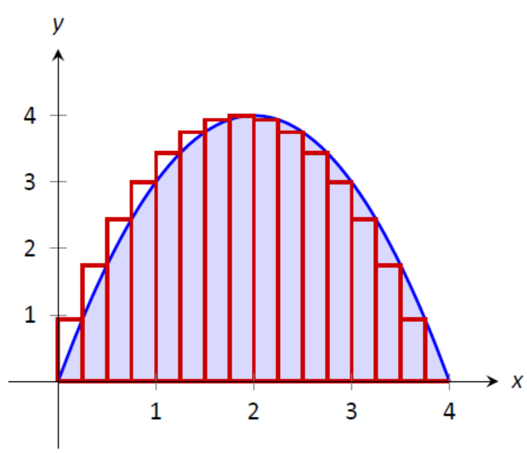


Figure 6: Approximation with the Right Hand Rule and 16 evenly spaced subintervals.

Notice Equation (5.4); by changing the 16's to 1,000's (and appropriately changing the value of Δx), we can use that equation to sum up 1000 rectangles!

We do so here, skipping from the original summand to the equivalent of Equation (5.4) to save space. Note that $\Delta x = 4/1000 = 0.004$.

$$\begin{aligned}
\sum_{i=1}^{1000} f(x_i)\Delta x &= (4\Delta x^2)\sum_{i=1}^{1000} i - \Delta x^3 \sum_{i=1}^{1000} i^2 \\
&= (4\Delta x^2)\frac{1000\cdot 1001}{2} - \Delta x^3 \frac{1000(1001)(2001)}{6} \\
&= 10.666656
\end{aligned}$$

Using many, many rectangles, we have a likely good approximation.

Before the above example, we stated what the summations for the Left Hand, Right Hand and Midpoint Rules looked like. Each had the same basic structure, which was:

1. each rectangle has the same width, which we referred to as Δx, and
2. each rectangle's height is determined by evaluating f at a particular point in each subinterval. For instance, the Left Hand Rule states that each rectangle's height is determined by evaluating f at the left hand endpoint of the subinterval the rectangle lives on.

One could partition an interval $[a, b]$ with subintervals that do not have the same size. We refer to the length of the i th subinterval as Δx_i. Also, one could determine each rectangle's height by evaluating f at *any* point c_i in the i th subinterval. Thus the height of the i th subinterval would be $f(c_i)$, and the area of the i th rectangle would be $f(c_i)\Delta x_i$. These ideas are formally defined below.

Definition (Partition)**.** A **partition** of a closed interval $[a, b]$ is a set of numbers x_0, x_1, x_2, $\ldots$ x_n where

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b.$$

The length of the $i^{\text{ th}}$ subinterval, $[x_{i-1}, x_i]$, is $\Delta x_i = x_i - x_{i-1}$. If $[a, b]$ is partitioned into subintervals of equal length, we let Δx represent the length of each subinterval. The **size of the partition**, denoted $||\Delta x||$, is the length of the largest subinterval of the partition.

Summations of rectangles with area $f(c_i)\Delta x_i$ are named after mathematician Georg Friedrich Bernhard Riemann, as given in the following definition.

Definition (Riemann Sum)**.** Let f be defined on a closed interval $[a, b]$, let P :

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$$

be a partition of $[a, b]$ and let x_i^* denote any value in the $i^{\text{ th}}$ subinterval.

The sum

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

is called a **Riemann sum** of f on $[a, b]$.

Remark. "Usually" Riemann sums are calculated using one of the three methods we have introduced. The uniformity of construction makes computations easier. Before working on an example, let's summarize some of what we have learned in a convenient way.

Riemann Sum Concepts: Consider $\sum_{i=1}^{n} f(x_i^*)\Delta x_i$, the approximation of the area of the region under $f(x) \geq 0$ on $[a, b]$.

1. When the n subintervals have equal length, $\Delta x_i = \Delta x = \dfrac{b-a}{n}$.
2. The $i^{\text{ th}}$ term of an equally spaced partition is $x_i = a + i\Delta x$. (Thus $x_0 = a$ and $x_n = b$.)

3. The Left Hand Rule summation is: $\sum_{i=1}^{n} f(x_{i-1})\Delta x = \sum_{i=1}^{n} f\Big(x_0 + (i-1)\Delta x\Big)\Delta x.$

4. The Right Hand Rule summation is: $\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} f\Big(x_0 + i\Delta x\Big)\Delta x.$

5. The Midpoint Rule summation is:

$$\sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x = \sum_{i=1}^{n} f\Big(x_0 + (i - \frac{1}{2})\Delta x\Big)\Delta x.$$

6. We can choose any point between x_{i-1} and x_i besides x_{i-1}, x_i and $\frac{x_{i-1}+x_i}{2}$ to construct a Riemann sum.

Example Evaluate the left, right, and midpoint Riemann sums for $f(x) = x^3 + 1$ between $a = 0$ and $b = 2$ using $n = 50$ subintervals. Make a conjecture about the exact area of the region under the curve.

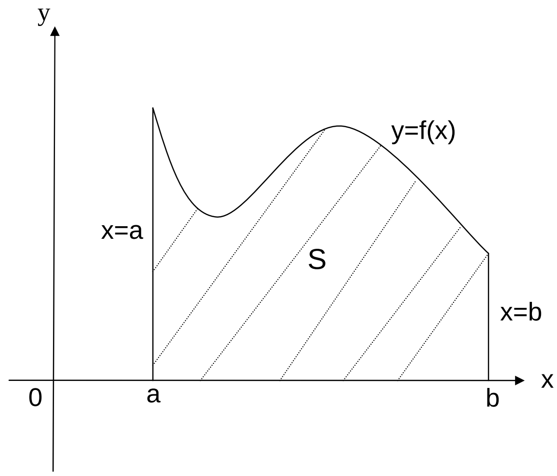
Solution.

See Python code in Riemann_Sum.ipynb.

□

5.2.3 Definite Integrals

Recall **The Area Problem.** Assume that $y = f(x) \geq 0$ on interval $[a, b]$. Find the area bounded by the curves $x = a$, $x = b$, $y = 0$ and $y = f(x)$. This problem is commonly referred to as "to find the area under the curve".



Basic Idea of the Solution. We assume that

• $f(x)$ is continuous on the interval $[a, b]$, (This is not necessary but it is convenient since it allows us to use **subintervals of the same length** and use any choice for **sample points**.)

• x denotes the independent variable and f denotes the function.

Example. Let $y = f(x) = 4x - x^2$ be defined on $[0, 4]$. Find the area under the curve. This is the example in the last sub section.

I - Approximate solution. An approximate solution is obtained by using a **Riemann sum** as outlined below. See the web app as well `https://xuemaozhang.shinyapps.io/riemannsum/`.

Step 1. The interval $[a, b]$ (in the example $a = 0, b = 4$) is partitioned into n **subintervals of equal length**, where n is a finite number, for example, $n = 4, 10, 50$. Now, the length of the interval is $\Delta x = \frac{b-a}{n}$. The end points of our intervals are

$x_i = a + i(\frac{b-a}{n}) = a + i\Delta x.$

$$a = x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n = b$$

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n].$$

In the example, if we choose $n = 4$, then we have 4 intervals of equal length of 1:

$$[0, 1], [1, 2], [2, 3], [3, 4].$$

Step 2. We assume that over a given subinterval $[x_i, x_{i+1}]$ the function is a constant with value $f(x_i^*)$ for some **sample point** $x_i^* \in [x_{i-1}, x_i]$, where x_i^* can be chosen to be left endpoint x_{i-1}, right endpoint x_i or midpoint $(x_{i-1} + x_i)/2$.

Step 3. Now

$$f(x_i^*)\Delta x = \begin{cases} \text{the area of a rectangle over } [x_{i-1}, x_i] \text{ with height } f(x_i^*) \\ \text{ OR} \\ \text{distance traveled over time } [x_{i-1}, x_i] \text{ when speed is } f(x_i^*) \end{cases}$$

Step 4. We now have an approximation to the solution that we seek given in the form of a **Riemann Sum**.

$$\sum_{i=1}^{n} f(x_i^*)\Delta x \cong \begin{cases} \text{area as a sum of area of a rectangles} \\ \text{ OR} \\ \text{distance as a sum of distance over time } [x_{i-1}, x_i] \end{cases}$$

In the example, if we choose right endpoints as sample points

$$R_4 = \sum_{i=1}^{4} f(x_i) \cdot 1 = f(1) \cdot 1 + f(2) \cdot 1 + f(3) \cdot 1 + f(4) \cdot 1 = 3 + 4 + 3 + 0 = 10;$$

If we choose left endpoints as sample points

$$L_4 = \sum_{i=1}^{4} f(x_{i-1}) \cdot 1 = f(0) \cdot 1 + f(1) \cdot 1 + f(2) \cdot 1 + f(3) \cdot 1 = 0 + 3 + 4 + 3 = 10;$$

If we choose midpoints as sample points (let $c_i = \frac{x_{i-1}+x_i}{2}$)

$$M_4 = \sum_{i=1}^{4} f(c_i) \cdot 1 = f(0.5) \cdot 1 + f(1.5) \cdot 1 + f(2.5) \cdot 1 + f(3.5) \cdot 1 = 1.75 + 3.75 + 3.75 + 1.75 = 11.$$

II - Exact Solution. To pass from an approximation to the exact value, the idea is that as we select smaller and **smaller subintervals**, the approximations become better and better.

$$S = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \begin{cases} \text{area in the area problem} \\ \text{OR} \\ \text{distance in the distance problem} \end{cases}$$

In the example, we split the interval $[0, 4]$ into n subintervals

$$\left[0, \frac{4}{n}\right], \left[\frac{4}{n}, \frac{8}{n}\right], \cdots, \left[4 - \frac{4}{n}, 4\right].$$

If we choose right endpoints $\frac{4i}{n}, i = 1, \ldots, n$ as samples points, then we have Riemann Sum

$$\begin{aligned} R_n =& \sum_{i=1}^{n} f(\frac{4i}{n})\frac{4}{n} = \sum_{i=1}^{n} \left(4\frac{4i}{n} - (\frac{4i}{n})^2\right)\frac{4}{n} = 4\left(\frac{4}{n}\right)^2 \sum_{i=1}^{n} i - \left(\frac{4}{n}\right)^3 \sum_{i=1}^{n} i^2 \\ =& \frac{64}{n^2}\frac{n(n+1)}{2} - \frac{64}{n^3}\frac{n(n+1)(2n+1)}{6} = \frac{64}{n}\frac{(n+1)}{2} - \frac{64}{n^2}\frac{(n+1)(2n+1)}{6} \end{aligned}$$

Thus we have

$$\begin{aligned} \lim_{n\to\infty} R_n =& \frac{64}{2}\lim_{n\to\infty}\frac{(n+1)}{n} - \frac{64}{6}\lim_{n\to\infty}\frac{(n+1)(2n+1)}{n^2} \\ =& 32 - \frac{64}{6}\cdot 2 = 32 - \frac{64}{3} = \frac{32}{3}. \end{aligned}$$

The definite integral generalizes the concept of the area under a curve. We lift the requirements that $f(x)$ be continuous and nonnegative, and define the definite integral as follows.

Definition 5.1 (Definite integral)**.** Let $f(x)$ be **a function** defined for $a \leq x \leq b$.

1. Partition $[a, b]$ into n subintervals of width $\Delta x_i, i = 1, \ldots, n$.
2. Let $x_0 = a$, $x_1 = x_0 + \Delta x_1$, ..., $x_i = x_{i-1} + \Delta x_i$ for $i = 1, \ldots, n$, thus the subintervals are $[a, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$, where $x_n = b$.
3. Choose sample points x_i^* between x_{i-1} and x_i.

Then the definite integral of $f(x)$ over the interval $[a, b]$ is the limit of the following Riemann sum provided that this limit exists. If it does exist, we say that f is **integrable** on $[a, b]$. This is denoted by

$$\boxed{\int_a^b f(x)dx = \lim_{\Delta \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i,}$$

where $\Delta = max\{\Delta x_1, \ldots, \Delta x_n\}$ with $\Delta x_i = x_i - x_{i-1}$.

Remark 1. In the above, if $f(x)$ is **continuous** then the subintervals can be **equally spaced**, then $\Delta x = \Delta x_i = \frac{b-a}{n}, i = 1, \ldots, n$. And the definition of $\int_a^b f(x)dx$ for a continuous function $f(x)$ is

$$\boxed{\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.}$$

In this course, we consider **continuous functions only**.

Remark 2. The sample point x_i^* is **any point** in the ith interval $i = 1, 2, \ldots, n$. Especially, x_i^* can be chosen to be

(1) left endpoints $x_i^* = x_{i-1} = a + (i-1)\frac{b-a}{n}$.

(2) right endpoints $x_i^* = x_i = a + (i)\frac{b-a}{n}$.

(3) midpoints.

$$\begin{aligned} x_i^* = \bar{x}_i = x_{i-1} + \frac{1}{2}(x_i - x_{i-1}) &= \frac{x_{i-1} + x_i}{2} \\ = a + (i-1)\frac{b-a}{n} + \frac{1}{2}\frac{b-a}{n} &= a + (2i-1)\frac{b-a}{2n}. \end{aligned}$$

□

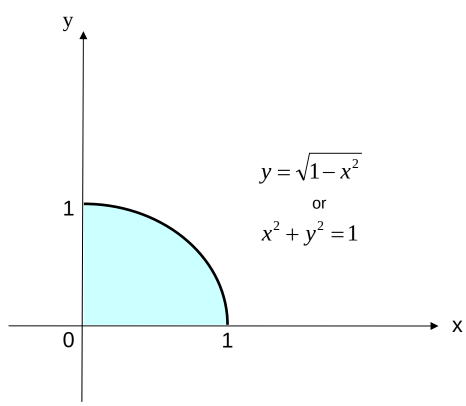
Evaluating Definite Integrals

Most of the functions encountered in this text are integrable (see Exercise 83 for an exception). In fact, if f is continuous on $[a, b]$ or if f is bounded on $[a, b]$ with a finite number of discontinuities, then f is integrable on $[a, b]$. The proof of this result goes beyond the scope of this text.

Theorem (Integrable Functions). If f is **continuous** on $[a, b]$, or if f is bounded and has only a finite number of jump discontinuities (or **piecewise continuous**), then f is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)dx$ exists. □

Example: Evaluating definite integrals using geometry Evaluate $\int_0^1 \sqrt{1-x^2}\,dx$ by interpreting it as an area.

Solution. Set $f(x) = \sqrt{1-x^2}$. If we sketch the graph of $y = f(x)$ over the interval $[0, 1]$,



we see that we have a curve corresponding to the quarter of a circle of radius 1 and so the area of the region is $A = \frac{1}{4}\pi(1)^2 = \frac{\pi}{4}$.

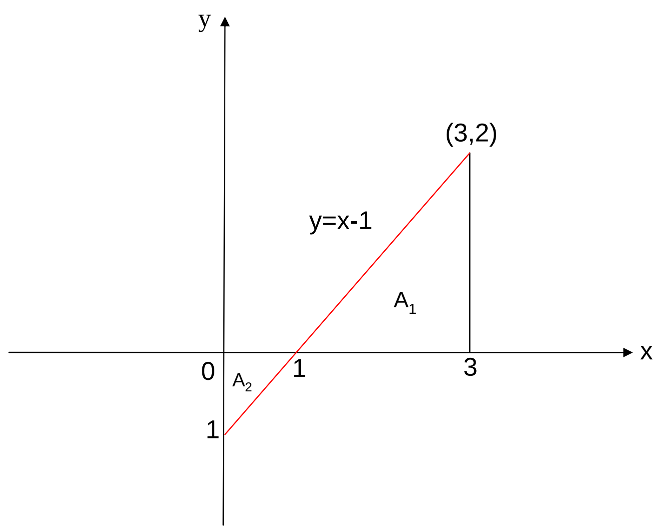
$$\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}.$$

□

Remark. The discussion so far has been for non-negative valued functions. If $f(x)$ can take both positive and negative values, then $\int_a^b f(x)\ dx$ is equal to the area up minus the area down. See the following example.

Example: Area up minus area down Evaluate $\int_0^3 (x-1)\ dx$ by interpreting it as an area.

Solution. Set $f(x) = x - 1$. If we sketch the graph of $y = f(x)$ over the interval $[0, 3]$ we see that we have straight line forming the hypotenuse of two triangles: one under the interval $[0, 1]$ and the other over $[1, 3]$.



Therefore,

$$\int_0^3 (x-1)\ dx = \underbrace{\int_0^1 (x-1)\ dx}_{=-\text{Area down}} + \underbrace{\int_1^3 (x-1)\ dx}_{=\text{Area up}} = -\frac{1}{2}(1)(1) + \frac{1}{2}(2)(2) = \frac{3}{2}$$

Note. The point is that if $x \in [0, 1]$, then the function $f(x)$ is negative and if $x \in [1, 3]$, then the function $f(x)$ is positive and so the Riemann sum splits into two parts

$$\sum_{i=1}^{n} f(x_i^*)\Delta x = \underbrace{\sum_{i=1}^{k} f(x_i^*)\Delta x}_{\leq 0 \text{ over } [0,1]} + \underbrace{\sum_{i=k+1}^{n} f(x_i^*)\Delta x}_{\geq 0 \text{ over } [1,3]}$$

Example: Evaluating definite integrals using geometry

a. $\int_2^4 (2x+3)\,dx$ **b.** $\int_1^6 (2x-6)\,dx$ **c.** $\int_3^4 \sqrt{1-(x-3)^2}\,dx$

Solution.

$\square$

Properties of Definite Integrals

Theorem (Properties of the Definite Integral)**.** Suppose $f(x)$ and $g(x)$ are both integrable.

(0) $\int_a^a f(x)\,dx = 0$ (since $\Delta x = 0$)

(1) $\int_a^b c\,dx = c(b-a)$, where c is any constant;

(2) $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$, where $a < b$, because Δx changes from $\frac{b-a}{n}$ to $\frac{a-b}{n}$;

(3) $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)dx \pm \int_a^b g(x)\,dx$;

(4) $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant.

(5) $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$, where c is any constant (c need not be between a and b.);

(6) If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$;

(7) If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$;

(8) If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$.

(9) The function $|f(x)|$ is also integrable on $[a, b]$, and $\int_a^b |f(x)|\,dx$ is the sum of the areas of the regions bounded by the graph f and the x-axis on $[a, b]$. □

Hint of proof: Write an integral in terms of the limit of a sum. See details in the textbook.

Theorem. If f is continuous on $[a, b]$, then

$$\boxed{\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.}$$

Example. Properties of integrals Assume that $\int_0^5 f(x)\,dx = 3$ and $\int_0^7 f(x)\,dx = -10$. Evaluate the following integrals, if possible.

a. $\int_0^7 2f(x)\,dx =$ **b.** $\int_5^0 f(x)\,dx =$ **c.** $\int_5^7 f(x)\,dx =$ **d.** $\int_7^0 6f(x)\,dx =$

e. $\int_0^7 |f(x)|\,dx =$

Solution.

$\square$

5.3 Fundamental Theorem of Calculus

The results in this section simplify the computation of a definite integral greatly. We begin by defining **functions in terms of definite integrals**. This may seem to be a strange way to define a function, but these functions are common place in applications such as in physics, chemistry and statistics.

Let $f(t)$ be a continuous function defined on $[a, b]$. The definite integral $\int_a^b f(x)\ dx$ is the "area under f" on $[a, b]$. We can turn this concept into a function by letting the upper (or lower) bound vary.

Let $F(x) = \int_a^x f(t)\ dt$. It computes the area under f on $[a, x]$ as illustrated in Figure 7. We can study this function using our knowledge of the definite integral. For instance, $F(a) = 0$ since $\int_a^a f(t)\ dt = 0$. If $f(t) > 0$, then $F(x) > 0$ when $x > a$ (consider the figure for a visual understanding). $F(x)$ is called an Area Function in the textbook.

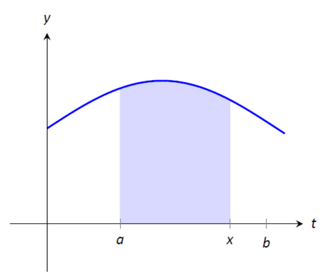


Figure 7: The area of the shaded region is $F(x) = \int_a^x f(t)\ dt$.

We can also apply calculus ideas to $F(x)$; in particular, we can compute its derivative. While this may seem like an innocuous thing to do, it has far–reaching implications, as demonstrated by the fact that the result is given as an important theorem.

Theorem (The Fundamental Theorem of Calculus - Part 1). If $f(x)$ is continuous on the interval $[a, b]$, then the function $F(x)$ defined by

$$F(x) = \int_a^x f(t)\ dt$$

is continuous on $[a, b]$ and differentiable on (a, b) and

$$F'(x) = \frac{d\ F(x)}{dx} = \frac{d}{dx} \int_a^x f(t)\ dt = f(x).$$

$\square$

Remark. The FTC(Fundamental Theorem of Calculus) Part 1 requires that the lower limit is a constant and the upper limit is the variable with respect to which we are computing the derivative.

There are times when these conditions are not met and yet we can apply FTC in conjunction with the Chain Rule. This requires **rewriting the integral** using the Properties of a Definite Integral (5.2.3) (see Example 2 - 3 in what follows). □

Proof. Assume that $a < x < b$. Use Properties 2 and 5 from the list of Properties of the Definite Integrals to get

$$\begin{aligned}
F'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{\int_a^{x+h} f(t)\ dx - \int_a^x f(t)\ dt}{h} \\
&= \lim_{h\to 0} \frac{\int_a^{x+h} f(t)\ dt + \int_x^a f(t)\ dt}{h} = \lim_{h\to 0} \frac{\int_x^a f(t)\ dt + \int_a^{x+h} f(t)\ dt}{h} \\
&= \lim_{h\to 0} \frac{\int_x^{x+h} f(t)\ dt}{h}.
\end{aligned}$$

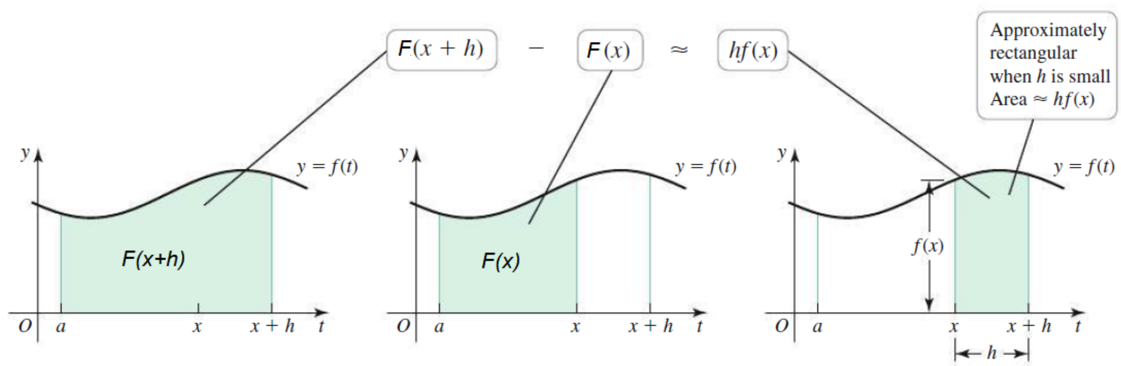


Figure 8: Key idea of the proof of the Fundamental Theorem of Calculus

Let $m = f(x + h_m)$ be the **absolute minimum** of $f(x)$ on the interval between x and $x + h$ so that h_m between 0 and h and $M = f(x + h_M)$ be the **absolute maximum** of $f(x)$ on the interval between x and $x + h$ with h_M between 0 and h. Then using Property 8 of Properties of Definite Integrals (5.2.3) we have

$\boxed{mh \leq \int_x^{x+h} f(t)\ dt \leq Mh}$ and so we have

$$\begin{aligned} f(x) = \underbrace{\lim_{h\to 0} f(x+h_m)}_{h_m\to 0 \text{ as } h\to 0} = \lim_{h\to 0} m = \lim_{h\to 0} \frac{mh}{h} \\ \leq F'(x) = \lim_{h\to 0} \frac{\int_x^{x+h} f(t)\ dt}{h} \\ \leq \lim_{h\to 0} \frac{Mh}{h} = \lim_{h\to 0} M = \underbrace{\lim_{h\to 0} f(x+h_M)}_{h_M\to 0 \text{ as } h\to 0} = f(x) \end{aligned}$$

Thus, $F'(x) = f(x)$ on (a, b).

Recall that differentiability implies continuity and so $F(x)$ is continuous on the open interval (a, b). Thus, we need only show that $F(x)$ is right continuous at a and left continuous at b. We show right continuity at a only.

First observe that $\mathbf{F(a) = 0}$. Let $m = f(c_m)$ with c_m in $[a, x]$ be the absolute minimum of $f(x)$ on $[a, x]$ and $M = f(c_M)$ with c_M in $[a, x]$ be the absolute maximum of $f(x)$ on $[a, x]$.

$$\begin{aligned} f(a)(0) &= \lim_{x\to a^+} f(c_m)(x-a) \\ &\leq \lim_{x\to a^+} F(x) = \lim_{x\to a^+} \int_a^x f(t)\ dt \\ &\leq \lim_{x\to a^+} f(c_M)(x-a) = f(a)(0) \end{aligned}$$

Thus, $F(a) = \lim_{x\to a} F(x)$. □

Example 1. Notation: $D_x = \frac{d}{dx}$.

$$D_x\left(\int_{-1}^x \sqrt[3]{t^3+t-1}\ dt\right) = \frac{d}{dx}\int_{-1}^x \sqrt[3]{t^3+t-1}\ dt = \sqrt[3]{x^3+x-1}.$$

□

Example 2. Find **(a)** $D_x(\int_1^x \sin^2 t\, dt)$ **(b)** $D_x(\int_x^5 \sqrt{t^2+1}\, dt)$ **(c)** $D_x(\int_0^{x^2} \cos t^2\, dt)$

Solution.

$\square$

Example 3. Find $D_x(\int_{\cos x}^{5x} \cos(u^2)\ du)$.

Solution.

$$\begin{aligned}
D_x\left(\int_{\cos x}^{5x} \cos(u^2)\ du\right) &= -\underbrace{D_x\left(\int_0^{\cos x} \cos(u^2)\ du\right)}_{v=\cos x} + \underbrace{D_x\left(\int_0^{5x} \cos(u^2)\ du\right)}_{w=5x} \\
&= -D_x\left(\int_0^{v} \cos(u^2) du\right) + D_x\left(\int_0^{w} \cos(u^2)\ du\right) \\
&= -D_v\left(\int_0^{v} \cos(u^2) du\right)(D_x(v)) + D_w\left(\int_0^{w} \cos(u^2)\ du\right)(D_x(w)) \\
&= -(\cos v^2)D_x(\cos x) + \cos(w^2)D_x(5x) \\
&= -(\cos(\cos^2 x))(-\sin x) + (\cos(5x)^2)(5).
\end{aligned}$$

$\square$.

Using the properties of definite integrals, we know

$$\int_a^b f(t)\ dt = \int_a^c f(t)\ dt + \int_c^b f(t)\ dt = -\int_c^a f(t)\ dt + \int_c^b f(t)\ dt$$
$$= -F(a) + F(b) = F(b) - F(a).$$

We now see how indefinite integrals and definite integrals are related: we can **evaluate a definite integral using antiderivatives!** This is the second part of the Fundamental Theorem of Calculus.

Theorem (The Fundamental Theorem of Calculus - Part 2). If $f(x)$ is continuous on the interval $[a, b]$ then

$$\int_a^b f(x)\ dx = F(b) - F(a) = F(x)\Big|_a^b,$$

where $F(x)$ is any antiderivative of f, i.e. $F'(x) = f(x)$. □

Proof. Let $g(x) = \int_a^x f(t)\ dt$. Then

$$g(b) - g(a) = \int_a^b f(x)\ dx - \int_a^a f(x)\ dx = \int_a^b f(t)\ dt - 0 = \int_a^b f(t)\ dt.$$

If $F(x)$ is any other antiderivative of $f(x)$ then $F'(x) = f(x) = g'(x)$ and so $F(x)$ and $g(x)$ differ by a constant, i.e. $F(x) = g(x) + C$ for some constant C. Thus,

$$F(b) - F(a) = g(b) - g(a) = \int_a^b f(x)\ dx.$$

□

Remark 1. Notice the notation $\Big|$, it is customary and convenient to denote the difference $F(b) - F(a)$ by $F(x)\Big|_a^b$

Remark 2. We must be careful. FTC does not apply to functions $f(x)$ which are not continuous, such as

$$\int_{-1}^1 \frac{1}{x^4}\ dx.$$

because the integrand is not continuous on $[-1, 1]$. Actually, one can show that this integral does not exist (In Calculus II you will study improper internal and see that this is such an integral).

Remark 3. When working with the absolute value function, we must find where the integrand is positive and where it is negative. That is, we must write an absolute value function as a piece-wise function.

Example Evaluate the following definite integrals using the Fundamental Theorem of Calculus, Part 2.

a. $\displaystyle\int_0^{10} (60x - 6x^2)\, dx$ **b.** $\displaystyle\int_0^{2\pi} 3\sin x\, dx$ **c.** $\displaystyle\int_{1/16}^{1/4} \frac{\sqrt{t}-1}{t}\, dt$

Solution.

□

- Please see *Integration.ipynb* for Python code.

In summary, in this section we have learned the FTC (summary of Theorem 5.3 and Theorem 5.3):

Theorem (The Fundamental Theorem of Calculus)**.** Suppose f is continuous on $[a, b]$. Then

1. If $F(x) = \int_a^x f(t)dt$, then $F'(x) = f(x)$. It can be written as $\boxed{\dfrac{d}{dx}\displaystyle\int_a^x f(t)dt = f(x)}$.

2. $\int_a^b f(x)dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$. It can also be written as $\boxed{\displaystyle\int_a^b F'(x)dx = F(b) - F(a)}$.

Contents

5.1 §5.5 Substitution Rule

We start the first integration technique using this example: $\int \cos x dx = \sin x + C$, then $\int \cos 2x dx =?$.

The Substitution Rule is a result of the Chain Rule. Recall that the chain rule is: if $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable, then $\boxed{F^{'}(x) = f^{'}(g(x)) \cdot g^{'}(x)}$.

Remember the concept of a differential: $dx = \Delta x$ for independent variable x and $\boxed{dy = y'dx}$ since $y' = \frac{dy}{dx}$ (for a constant c, $d(cy) = cy'dx$) for dependent variable y. If $u = g(x)$ then u is a dependent function and so $du = u'dx = g'(x)dx$ since $\frac{du}{dx} = u' = g'(x)$. Or we can write

$$\mathbf{d[g(x)] = g'(x)dx}.$$

5.1.1 Substitution Rule for Indefinite Integrals

Theorem (Thm 5.6: Substitution Rule for Indefinite Integrals). If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))\ g'(x)dx = \int f(g(x))\ d[g(x)] = \int f(u)\ du.$$

Remark. In practice, sometimes it is convenient to use $\boxed{\mathbf{g'(x)dx = d[g(x)]}}$ without explicitly writing down the substitution $u = g(x)$.

Proof. (read the proof after class) Suppose that $F(x)$ is an antiderivative of $f(x)$, i.e.

$$F(x) + C = \int f(x)\ dx.$$

Let $u = g(x)$. Then, we have

$$\begin{aligned} &D_x(F(x)) = F'(x) = f(x) \text{ or equivalently} \\ &D_u(F(u)) = F'(u) = f(u) \text{ or equivalently} \\ &\int f(u)\ du = F(u) + C \end{aligned}$$

By the Chain Rule, we have

$$D_x(F(g(x))) = \underbrace{D_u(F(u))}_{=f(u)}\underbrace{D_x(u)}_{=\frac{du}{dx}} = f(u)D_x(u) = f(g(x))g'(x)$$

and so

$$\begin{aligned} \int f(g(x))g'(x)dx =& \int D_x(F(g(x)))\ dx = F(g(x)) + C \\ =& F(u) + C = \int f(u)\ du \end{aligned}$$

$\square$

Some Examples in the Textbook:

Example 1.(perfect substitutions)

(a) $\int 2(2x+1)^3 \ dx$

Solution.

$\square$

(b) $\int 10e^{10x} \ dx$

Solution.

$\square$

Example 2.(Introducing a constant)

(a) $\int x^4(x^5+6)^9\ dx$

Solution.

If $u = x^5 + 6$ then $du = 5x^4dx$ and so we have

$$\int x^4(\underbrace{x^5+6}_{=u})^9\ dx = \frac{1}{5}\int(\underbrace{x^5+6}_{=u})^9\ \underbrace{(5x^4)\ dx}_{=du}$$
$$= \frac{1}{5}\int u^9\ du = \frac{1}{5}(\frac{u^10}{10}) + C = \frac{1}{50}(x^5+6)^{10} + C.$$

Or we do the following without writing the substitution u explicitly.

$$\int x^4(x^5+6)^9\ dx = \int (x^5+6)^9 d(\frac{x^5}{5}) = \int (x^5+6)^9 d\left(\frac{x^5+6}{5}\right)$$
$$=\frac{1}{5}\int (x^5+6)^9 d(x^5+6) = \frac{1}{50}(x^5+6)^{10}.$$

□

(b) $\int \cos^3 x \sin x\ dx$

Solution.

□

Example 3.(more than one substitution)

$\int \frac{x}{\sqrt{x+1}}\ dx$

Solution.

□

Example 4.(integration of $\int f(ax)\ dx$)

(a) $\int e^{ax}\ dx$

Solution.

$\square$

(b) $\int b^x\ dx, b > 0, b \neq 1$

Solution.

$\square$

(c) $\int \frac{1}{\sqrt{a^2 - x^2}}\ dx, a > 0$

Solution.

$\square$

More Examples:

Example 1. If $u = x^4 + 2$, $du = d(x^4+2) = u'dx = 4x^3dx$.

$$\begin{aligned}\int x^3\cos(x^4+2)\ dx =& \int \cos(x^4+2)\ \frac{4x^3}{4}dx\\ =&\frac{1}{4}\int \underbrace{\cos(x^4+2)}_{\cos u}\ \underbrace{4x^3dx}_{=du} = \frac{1}{4}\int \cos u\ du = \frac{1}{4}\sin u + C\\ =&\frac{1}{4}\sin(x^4+2)+C.\end{aligned}$$

Or

$$\boxed{\int x^3\cos(x^4+2)\ dx = \int \cos(x^4+2)\ d\left(\frac{x^4+2}{4}\right) = \frac{1}{4}\int \cos u\ du = \cdots\cdots} \qquad \square$$

Example 2.

$$\begin{aligned}\int \sec^2(3\theta)\ d\theta = \frac{1}{3}\int \sec^2(3\theta)\ 3d\theta = \frac{1}{3}\int \sec^2 u\ du\\ = \frac{1}{3}\tan u + C = \frac{1}{3}\tan(3\theta) + C \quad (\text{we let } u = 3\theta).\end{aligned}$$

Example 3.

$$\int \frac{1}{x}\ dx = \begin{cases} \int \frac{1}{x}\ dx = \ln x + C & \text{if } x > 0 \\ \\ \int \frac{1}{x}\ dx = \int \frac{1}{-x}\ d(-x) = \ln(-x) + C & \text{if } x < 0 \end{cases}$$

$$= \ln|x| + C$$

Example 4 Manipulate with the integrand before substitution.

$$\begin{aligned}\int \tan x\ dx = \int \frac{\sin x}{\cos x}\ dx = \int \frac{-d(\cos x)}{\cos x} = -\int \frac{d(\cos x)}{\cos x}\\ = -\ln|\cos x| + C = \ln|\cos x|^{-1} + C = \ln|\cos x^{-1}| + C\\ = \ln|\sec x| + C.\end{aligned}$$

Example 5. Manipulate with the integrand before substitution (try $\int \csc x\ dx$ as an exercise).

$$\int \sec x \; dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} \; dx$$
$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \; dx = \ln|\sec x + \tan x| + C.$$

Example 6. $u = \tan^{-1} x \implies du = \frac{dx}{1+x^2}$

$$\int \frac{\tan^{-1} x}{x^2+1} \; dx = \int u \; du = \cdots\cdots$$

Example 7. Let $u = e^x$. Then $du = e^x dx$ and $e^{2x} = u^2$.

$$\int \frac{e^x}{e^{2x}+1} \; dx = \int \frac{du}{1+u^2} = \cdots\cdots$$

Example 8. $u = x^2$. Then $du = 2xdx$.

$$\int \frac{x}{1+x^4} \; dx = \frac{1}{2} \int \frac{du}{1+u^2} = \cdots\cdots$$

Example 9. $u = \tan\theta \implies du = \sec^2 \theta d\theta$

$$\int \tan^2 \theta \sec^2 \theta \; d\theta = \int u^2 \; du = \cdots\cdots$$

Example 10. $u = \frac{\pi}{x} \implies du = -\pi x^{-2} \; dx$

$$\int \frac{\cos(\frac{\pi}{x})}{x^2} \; dx = -\frac{1}{\pi} \int \cos(\frac{\pi}{x})(-\pi x^{-2} \; dx) = -\frac{1}{\pi} \int \cos u \; du = \cdots\cdots$$

Example 11. Find $\int \frac{dx}{a^2+x^2}$, $a \neq 0$.

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a^2} \int \frac{dx}{1+(\frac{x}{a})^2} = \frac{1}{a} \int \frac{d(\frac{x}{a})}{1+(\frac{x}{a})^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C.$$

Example 12. Find $\frac{dx}{\sqrt{a^2-x^2}}$, $a > 0$.

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{1}{a} \frac{dx}{\sqrt{1-(\frac{x}{a})^2}} = \int \frac{d(\frac{x}{a})}{\sqrt{1-(\frac{x}{a})^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C.$$

□

5.1.2 Substitution Rule for Definite Integrals

Theorem (Thm 5.7: The Substitution Rule for Definite Integrals). If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\ dx = \int_{g(a)}^{g(b)} f(u)\ du.$$

$\square$

Remark 1. If we do not explicitly write down the substitution, we do not change the lower limit and the upper limit. For example,

$$\int_0^1 x^3(1-x^4)\ dx = \boxed{\frac{-1}{4}\int_0^1 (1-x^4)d(1-x^4)} = \left(\frac{-1}{4}\right)\frac{(1-x^4)^2}{2}\Big|_0^1.$$

Remark 2. We can avoid changing the lower limit and the upper limit. For example,

$$u = 1-x^4, \quad \int x^3(1-x^4)\ dx = \boxed{\frac{-u^2}{8}} \text{Thus, } \int_0^1 x^3(1-x^4)\ dx = \frac{-u^2}{8}\Big|_{x=0}^{x=1}.$$

$\square$

Proof. This proof amounts to finding the value of both the left hand side and the right hand side and showing that they are equal.

For the LHS: Let $F(x)$ be an antiderivative of $f(x)$. Then, by the Substitution Rule for Indefinite Integrals, $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$, and so by Part 2 of the FTC, we have

$$\int_a^b f(g(x))g'(x)\ dx = F(g(b)) - F(g(a))$$

For the RHS: Since $F(x)$ be an antiderivative of $f(x)$, $F(u)$ is an antiderivative for $f(u)$ and so by FTC Part 2, we have

$$\int_{g(a)}^{g(b)} f(u)\ du = F(g(b)) - F(g(a))$$

The result now follows. $\square$

Some examples:

Example. $\int_0^7 \sqrt{4+3x}\ dx$

Here we set $u = g(x) = 4 + 3x$ and compute as follows.

$$\begin{aligned}\int_0^7 \sqrt{4+3x}\ dx =& \frac{1}{3}\int_0^7 \sqrt{4+3x}\ d(4+3x) = \frac{1}{3}\int_{g(0)}^{g(7)} u^{\frac{1}{2}}\ du\\ =& \frac{1}{3}\int_4^{25} u^{\frac{1}{2}}\ du = \frac{1}{3}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big|_{u=4}^{u=25}\\ =& \frac{2}{9}[125-8] = \frac{234}{9}.\end{aligned}$$

Example. $\int_0^1 xe^{-x^2}\ dx$

Here we set $u = g(x) = -x^2$ and compute as follows.

$$\begin{aligned}\int_0^1 xe^{-x^2}\ dx &= -\frac{1}{2}\int_0^1 e^{-x^2}\ (-2xdx) = -\frac{1}{2}\int_{g(0)}^{g(1)} e^u\ du\\ &= -\frac{1}{2}\int_0^{-1} e^u\ du = -\frac{1}{2}\left[e^u\right]_0^{-1} = -\frac{1}{2}\left[\frac{1}{e}-1\right]\\ &= \frac{e-1}{2e}.\end{aligned}$$

Example. $\int_0^1 \frac{\sin^{-1}x}{\sqrt{1-x^2}}\ dx$

Here we set $u = g(x) = \sin^{-1}x$ and compute as follows.

$$\begin{aligned}\int_0^1 \frac{\sin^{-1}x}{\sqrt{1-x^2}}\ dx &= \int_0^1 \sin^{-1}x(\frac{1}{\sqrt{1-x^2}}\ dx)\\ &= \int_{g(0)}^{g(1)} u\ du = [\frac{u^2}{2}]_0^{\sin^{-1}1}\\ &= \frac{(\sin^{-1}1)^2}{2} = \frac{1}{2}\left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{8}.\end{aligned}$$

Example 5. Evaluate the following integrals.

(a) $\int_0^2 \dfrac{1}{(x+3)^3}\ dx$

Solution.

$\square$

(b) $\int_2^3 \dfrac{x^2}{x^3-7}\ dx$

Solution.

$\square$

(c) $\int_0^{\pi/2} \sin^4 x \cos x\ dx$

Solution.

$\square$

Example 6. Evaluate $\int_0^{\pi/2} \cos^2 x\ dx$

Solution.

$\square$

5.2 §5.4 Working with Integrals: More Theorems

With the Fundamental Theorem of Calculus in hand, we may begin an investigation of integration and its applications. In this section, we discuss the role of symmetry in integrals, we use the slice-and-sum strategy to define the average value of a function, and we explore a theoretical result called the Mean Value Theorem for Integrals.

Integrating Even and Odd Functions

Theorem (Integrals of Symmetric Functions)**.** Suppose that $f(x)$ is continuous on the symmetric interval $[-a, a]$. Then

(a) If $f(x)$ is an even function, then $\int_{-a}^{a} f(x)\ dx = 2\int_0^a f(x)\ dx$.

(b) If $f(x)$ is an odd function, then $\int_{-a}^{a} f(x)\ dx = 0$.

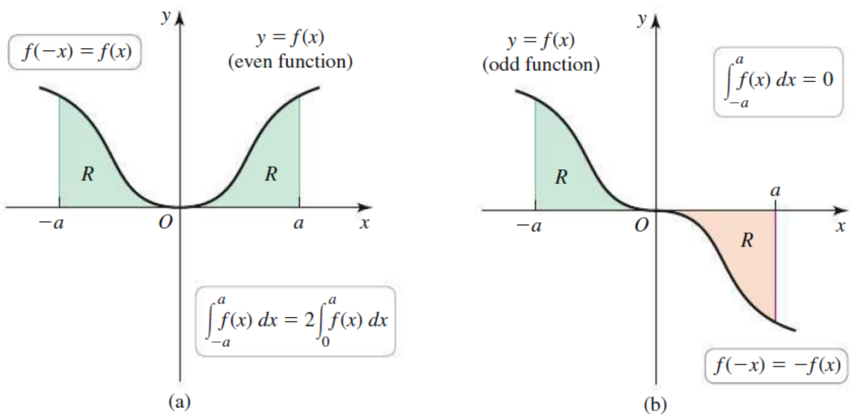


Figure 1: Integrals of Even and Odd Functions

Proof. We split the integral in two

$$\int_{-a}^{a} f(x)\ dx = \int_{-a}^{0} f(x)\ dx + \int_0^a f(x)\ dx = -\int_0^{-a} f(x)\ dx + \int_0^a f(x)\ dx.$$

For the first part, we use the substitution $u = -x$, then

$$-\int_0^{-a} f(x)\ dx = -\int_0^a f(-u)\ (-du) = \int_0^a f(-u)\ du.$$

Therefore,

$$\int_{-a}^{a} f(x)\ dx = \int_0^a f(-v)\ dv + \int_0^a f(v)\ dv.$$

(a) If f is even, then $f(-v) = f(v)$, thus,

$$\int_{-a}^{a} f(x)\ dx = \int_{0}^{a} f(v)\ dv + \int_{0}^{a} f(v)\ dv = 2\int_{0}^{a} f(v)\ dv.$$

(b) If f is odd, then $f(-v) = -f(v)$, thus,

$$\int_{-a}^{a} f(x)\ dx = -\int_{0}^{a} f(v)\ dv + \int_{0}^{a} f(v)\ dv = 0.$$

□

Example. The cosine function is an even function and so we have the following.

$$\int_{\frac{-\pi}{3}}^{\frac{\pi}{3}} \cos x\ dx = 2\int_{0}^{\frac{\pi}{3}} \cos x\ dx = 2\sin x\Big|_{x=0}^{x=\frac{\pi}{3}} = 2[\frac{\sqrt{3}}{2}] = \sqrt{3}.$$

Example. The sine function is an odd function and so we have the following.

$$\int_{\frac{-\pi}{3}}^{\frac{\pi}{3}} \sin^5 x\ dx = 0.$$

Example Evaluate the following integrals using symmetry arguments.

a. $\displaystyle\int_{-2}^{2} (x^4 - 3x^3)\ dx$ **b.** $\displaystyle\int_{-\pi/2}^{\pi/2} (\cos x - 4\sin^3 x)\ dx$

Solution.

□

Mean Value Theorem for Integrals

Recall that the average of numbers $y_1, \ldots, y_n$ is simply the sum divided by the number of values n

$$\text{Average value of } y = \frac{y_1 + y_2 + \cdots + y_n}{n} = \frac{\sum_{i=1}^n y_i}{n}.$$

Now suppose that we have a continuous function of time t, say temperature $T(t)$, what would it mean to say the average temperature over the period of time, say from $t = a$ to $t = b$. One would probably accept an approximation obtained as follows.

(i) Partitioning the time period into n intervals of equal length

$$\Delta t = \frac{b-a}{n}$$

with endpoints

$$t_0 = a < \cdots < t_i = a + i\frac{b-a}{n} < \cdots < t_n = a + n\frac{b-a}{n} = b.$$

(ii) Selecting a sample temperature in each time interval $T(t_i^*)$ with

$$t_{i-1} \le t_i^* \le t_i.$$

(iii) Compute an approximation

$$\text{Average Temperature } \approx \frac{T(t_1^*) + \cdots + T(t_n^*)}{n} = \Big(T(t_1^*) + \cdots + T(t_n^*)\Big)\frac{1}{n}.$$

Since

$$\frac{1}{n} = \frac{\Delta t}{b-a},$$

we find

$$\begin{aligned}\text{Average Temperature } &\approx \Big(T(t_1^*) + \cdots + T(t_n^*)\Big)\frac{1}{n} \\ &= \Big(T(t_1^*) + \cdots + T(t_n^*)\Big)\frac{\Delta t}{b-a} \\ &= \frac{T(t_1^*)\Delta t + \cdots + T(t_n^*)\Delta t}{b-a}.\end{aligned}$$

Notice that the preceding procedure conforms to the definition of the Definite integral. Thus if the function $T(t)$ for the temperature is known, we could get an accurate value for the average temperature.

$$\begin{aligned}\text{Average Temperature} &= \lim_{n\to\infty}\left(\frac{T(t_1^*)\Delta t + \cdots + T(t_n^*)\Delta t}{b-a}\right)\\ &= \frac{\lim_{n\to\infty}\sum_{i=1}^{n} T(t_i^*)\Delta t}{b-a}\\ &= \frac{1}{b-a}\int_a^b T(t)\ dt.\end{aligned}$$

Definition (Average Value of a Function f)**.** If f is a continuous function on the interval $[a,b]$, then the average value of f on $[a,b]$ is defined to be

$$\boxed{\bar{f}\ or\ f_{ave} = \frac{1}{b-a}\int_a^b f(x)\ dx.}$$

□

Recall - The Mean Value Theorem for Derivatives. If g is continuous on $[a,b]$ and differentiable on (a,b), then there is some c in $[a,b]$ such that

$$g'(c) = \frac{g(b)-g(a)}{b-a}.$$

Theorem (**Thm 5.5: The Mean Value Theorem for Integrals**)**.** If f is a continuous function on the interval $[a,b]$ and f_{ave} is the average value of f on $[a,b]$, then there is some c in $[a,b]$ such that

$$\boxed{f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)\ dx.}$$

Proof. By the FTC,

$$g(x) = \int_a^x f(t)\ dt \implies g'(x) = f(x)$$

and by the Mean Value Theorem for Derivatives, there is some c in $[a,b]$ such that

$$f(c) = g'(c) = \frac{g(b)-g(a)}{b-a} = \frac{\int_a^b f(t)\ dt - \int_a^a f(t)\ dt)}{b-a} = \frac{1}{b-a}\int_a^b f(x)\ dx.$$

□

Example 1 Let $f(x) = \sqrt{x}$.

(a) Find f_{ave} the average value of f on the interval $[0, 4]$.

(b) Find c between 0 and 4 such that $f(c) = f_{ave}$.

(c) Sketch the graph of $y = f(x)$ and a rectangle that has the same area as the area under the curve.

Solution.

(a)

$$f_{ave} = \frac{1}{4-0}\int_0^4 \sqrt{x}\ dx = \frac{1}{4}\int_0^4 x^{\frac{1}{2}}\ dx = \frac{1}{4}\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^4 = \frac{4}{3}.$$

(b) $f(c) = f_{ave} \iff \sqrt{c} = \frac{4}{3} \iff c = \frac{16}{9} \approx 1.75.$

(c) **Hint:** $\int_a^b f(x)dx = (b-a)f(c)$. □

Example 2 Let $f(x) = 2 + 6x - 3x^2$. Find the number b such that f_{ave} the average value of $f(x) = 2 + 6x - 3x^2$ on the interval $[0, b]$ is equal to 3.

Solution.

$$3 = f_{ave} = \frac{1}{b-0}\int_0^b (2 + 6x - 3x^2)\ dx = \frac{1}{b}\left[2x + 3x^2 - x^3\right]_0^b$$

$$\iff 3b = 2b + 3b^2 - b^3 \iff b = 3b^2 - b^3 \iff 1 = 3b - b^2$$

$$\iff b^2 - 3b + 1 = 0 \iff b = \frac{3 \pm \sqrt{(-3)^2 - 4}}{2} = \frac{3 \pm \sqrt{5}}{2}$$

We may select b to be either of the values given by $\frac{3\pm\sqrt{5}}{2}$. □

Example 3 Find the point(s) on the interval $(0, 1)$ at which $f(x) = 2x(1 - x)$ equals its average value on $[0, 1]$.

Solution.

□

Contents

In this brief chapter, we revisit the exponential and logarithmic functions.

7.1 §7.1 Logarithmic and Exponential Functions Revisited

Exponents

Here we go over the definition of x^y when x and y are arbitrary real numbers, with $x > 0$.

For any real number x and any positive integer $n = 1, 2, 3, \ldots$ one defines

$$x^n = \overbrace{x \cdot x \cdot \cdots \cdot x}^{n \text{ times}}$$

and, if $x \neq 0$,

$$x^{-n} = \frac{1}{x^n}.$$

One defines $x^0 = 1$ for any $x \neq 0$.

To define $x^{p/q}$ for a general fraction $\frac{p}{q}$ one must assume that the number x is positive. One then defines

$$x^{p/q} = \sqrt[q]{x^p}.$$

It is shown in precalculus texts that the **power functions** satisfy the following properties:

$$\boxed{(xy)^a = x^a y^a, \quad x^a x^b = x^{a+b}, \quad \frac{x^a}{x^b} = x^{a-b}, \quad \left(x^a\right)^b = x^{ab}}$$

provided a and b are fractions. And these properties still hold if a and b are real numbers (not necessarily fractions.) We won't go through the proofs here.

Now instead of considering x^a as a function of x we can pick a positive number a and consider the **exponential function**

$$\boxed{f(x) = a^x.}$$

This function is defined for all real numbers x (as long as the base a is positive.).

The exponential function

$$\boxed{f(x) = e^x}$$

with base e (Euler's constant) is so prevalent in the sciences that it is often referred to as the **exponential function** or **the natural exponential function**. Recall that

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

and

$$\frac{de^x}{dx} = e^x.$$

Logarithmic Functions

The **logarithm function**, written $\log_a x$, is the inverse of the exponential function a^x. The function

$$f(x) = log_a x,$$

is called the **logarithmic function with base** a.

Theorem (Properties of Logs)**.** The following properties can be derived from the exponential function.

1. $\log_a(a^x) = x$
2. $a^{\log_a x} = x, \quad x > 0$
3. $\log_a(x^p) = p \log_a x, \quad x > 0$
4. $\log_a(A) = \dfrac{\log_b(A)}{\log_b(a)}$
5. $\log_a(B) + \log_a(C) = \log_a(BC)$
6. $\log_a(B) - \log_a(C) = \log_a(\frac{B}{C})$

The logarithm with base e is called the **Natural Logarithm**, and is written

$$\ln x = \log_e x.$$

Thus we have

$$\boxed{e^{\ln x} = x \qquad \ln e^x = x}$$

where the second formula holds for all real numbers x but the first one only makes sense for $x > 0$.

For any positive number a we have $a = e^{\ln a}$, and also

$$a^x = e^{x \ln a}.$$

By the chain rule you then get

$$\boxed{\frac{da^x}{dx} = a^x \ln a.}$$

Derivatives of Logarithms Since the natural logarithm is the inverse function of $f(x) = e^x$ we can find its derivative by implicit differentiation. Here is the computation (which you are supposed to be able to do yourself).

The function $f(x) = \log_a x$ satisfies

$$a^{f(x)} = x$$

Differentiate both sides, and use the chain rule on the left,

$$(\ln a)a^{f(x)} f'(x) = 1.$$

Then solve for $f'(x)$ to get

$$f'(x) = \frac{1}{(\ln a)a^{f(x)}}.$$

Finally we remember that $a^{f(x)} = x$ which gives us the derivative of a^x

$$\frac{da^x}{dx} = \frac{1}{x \ln a}.$$

In particular, the natural logarithm has a very simple derivative, namely, since $\ln e = 1$ we have

$$\boxed{\frac{d \ln x}{dx} = \frac{1}{x}.}$$

Therefore,

$$\boxed{\ln x = \int_1^x \frac{1}{t}\, dt, \qquad x > 0.}$$

Theorem. More Properties of Logs and Exponentials:

- $\dfrac{d}{dx}(\ln|x|) = \dfrac{1}{x}, \quad x \neq 0$
- $\dfrac{d}{dx}(\ln|u(x)|) = \dfrac{u^{'}(x)}{u(x)}, \quad u(x) \neq 0$
- $\int \frac{1}{x}\ dx = \ln|x| + C$
- $\frac{d}{dx}(e^{u(x)}) = e^{u(x)}u^{'}(x)$
- $\int e^x\ dx = e^x + C$

Example 1. Evaluate $\int_0^4 \dfrac{x}{x^2+9}\ dx$.

Solution.

$\square$

Example 2. Evaluate $\int \dfrac{e^x}{1+e^x}\ dx$.

Solution.

$\square$

Theorem (Derivatives and Integrals with Other Bases)**.** Let $b > 0$ and $b \neq 1$. Then

$$\frac{d}{dx}(\log_b |u(x)|) = \frac{1}{\ln b}\frac{u'(x)}{u(x)}, \quad u(x) \neq 0$$

$$\frac{d}{dx}(b^{u(x)}) = (\ln b)b^{u(x)}u'(x)$$

$$\boxed{\int b^x \, dx = \frac{1}{\ln b}b^x + C.}$$

Example 3. Evaluate the following integrals.

a. $\int x3^{x^2} \, dx$.

Solution.

$\square$

b. $\int_1^4 \frac{6^{-\sqrt{x}}}{\sqrt{x}} \, dx$.

Solution.

$\square$

Theorem (General Power Rule)**.** For any real number p,

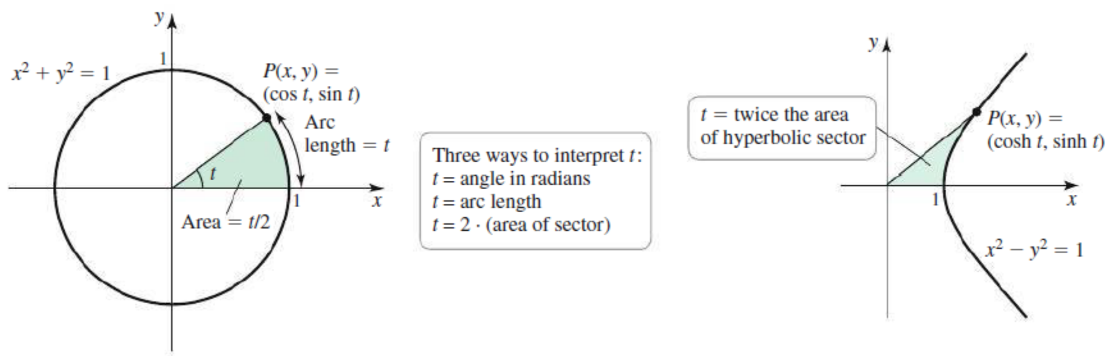
$$\boxed{\frac{d}{dx}(x^p) = px^{p-1} \quad \textit{and} \quad \frac{d}{dx}(u^p(x)) = pu^{p-1}(x)u'(x)}$$

Example 4 - Logarithmic Differentiation. Evaluate the derivative of $f(x) = x^{2x}$.

Solution.

7.3 §7.3 Hyperbolic Functions

In this section, we introduce a new family of functions called the hyperbolic functions, which are closely related to both trigonometric functions and exponential functions. Hyperbolic functions find widespread use in applied problems in fluid dynamics, projectile motion, architecture, and electrical engineering, to name just a few areas. Hyperbolic functions are also important in the development of many theoretical results in mathematics.



Definition.

Hyperbolic cosine

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Hyperbolic sine

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Hyperbolic tangent

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Hyperbolic cotangent

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Hyperbolic secant

$$sechx = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

Hyperbolic cosecant

$$cschx = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

Please read the textbook about the Calculus of these Hyperbolic functions if you are interested.

Contents

Chapter 8 involves various integration techniques, i.e. techniques for finding indefinite integrals. **The only way to get reasonably good at finding indefinite integrals is by doing a lot of examples.**

8.1 §8.1 Basic Approaches

Before plunging into new integration techniques, we devote this section to two practical goals. The first is to review what you learned about the **substitution** method in Section 5.5. The other is to introduce several basic **simplifying procedures** that are worth keeping in mind for any integral that you might be working on.

To simplify the integration procedure, we need to memorize some typical integrals. The following is the list of antiderivatives that **must be memorized**.

Note: D_x denotes $\frac{d}{dx}$.

$$D_x(x^{n+1}) = (n+1)x^n \implies \int x^n \ dx = \frac{1}{n+1}x^{n+1} + C, n \neq -1$$

$$D_x(e^x) = e^x \implies \int e^x \ dx = e^x + C$$

$$D_x(a^x) = a^x \ln a \implies \int a^x \ dx = \frac{a^x}{\ln a} + C$$

$$D_x(\ln|x|) = \frac{1}{x} \implies \int \frac{dx}{x} = \ln|x| + C$$

$$D_x(\sin x) = \cos x \implies \int \cos x \ dx = \sin x + C$$

$$D_x(\cos x) = -\sin x \implies \int \sin x \ dx = -\cos x + C$$

$$D_x(\tan x) = \sec^2 x \implies \int \sec^2 x \ dx = \tan x + C$$

$$D_x(\sec x) = \sec x \tan x \implies \int \sec x \tan x \ dx = \sec x + C$$

$$D_x(\cot x) = -\csc^2 x \implies \int \csc^2 \ dx = -\cot x + C$$

$$D_x(\csc x) = -\csc x \cot x \implies \int \csc x \cot x \ dx = -\csc x + C$$

$$D_x(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \implies \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$$

$$D_x(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}} \implies \int \frac{-dx}{\sqrt{1-x^2}} = \cos^{-1} x + C = -\sin^{-1} x + C$$

$$D_x(\tan^{-1} x) = \frac{1}{1+x^2} \implies \int \frac{dx}{1+x^2} = \tan^{-1} x + C$$

$$D_x(\cot^{-1} x) = \frac{-1}{1+x^2} \implies \int \frac{-dx}{1+x^2} = \cot^{-1} x + C = -\tan^{-1} x + C$$

$$D_x(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}} \implies \int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}|x| + C$$

$$D_x(\csc^{-1} x) = \frac{-1}{x\sqrt{x^2-1}} \implies \int \frac{-dx}{x\sqrt{x^2-1}} = \csc^{-1}|x| + C = -\sec^{-1}|x| + C$$

$$D_x(\ln|\sec x|) = \tan x \implies \int \tan x \ dx = \ln|\sec x| + C$$

$$D_x(\ln|\sin x|) = \cot x \implies \int \cot x \ dx = \ln|\sin x| + C$$

$$D_x(\ln|\sec x + \tan x|) = \sec x \implies \int \sec x \ dx = \ln|\sec x + \tan x| + C$$

$$D_x(-\ln|\csc x + \cot x|) = \csc x \implies \int \csc x \ dx = -\ln|\csc x + \cot x| + C$$

Example 1.(Substitution review) Evaluate $\int \tan ax \ dx, a \neq 0$

Solution.

$\square$

Example 2. (Multiplication by 1) Derive the integral formula

$$\int \sec x \ dx = \ln|\sec x + \tan x| + C$$

Solution. We derived this before.

$\square$

Example 3. (Subtle substitution) Evaluate $\int \frac{dx}{e^x + e^{-x}}$.

Solution.

$\square$

Example 4. (Split up fractions) Evaluate $\int \frac{\cos x + \sin^3 x}{\sec x} \ dx$.

Solution.

$\square$

Example 5. (Division with rational functions) Evaluate $\int \frac{x^2 + 2x - 1}{x + 4}\, dx$.

Solution.

$\square$

Example 6. (Complete the square) Evaluate $\int \frac{dx}{\sqrt{-x^2 - 8x - 7}}$.

Solution.

$\square$

8.2 §8.2 Integration by Parts

This is an anti-differentiation technique that follows from the **Product Rule** for derivatives.

$$
\begin{aligned}
(uv)' &= u'v + uv' \\
&\Longrightarrow \boxed{\int u\ v'dx = uv - \int v\ u'dx} \\
&\Longrightarrow \boxed{\int u\ dv = uv - \int v\ du}.
\end{aligned}
$$

This is the formula method for **Integration by Parts**. There is a "diagram method" for integration by parts that often makes things easier, particularly, when we have to do repeated applications of Integration by Parts.

(1) **Formula Method**: $\boxed{\int u\ dv = uv - \int v\ du.}$

(2) **Diagram Method**:

diff.		int.	
u		v'	
	$\searrow$		Diagonal arrows = products
u'	$(-) \rightarrow$	v	Horizontal arrows = prod.& int.

Remark 1. $v = \int v'\ dx$ should be easy to find.

Remark 2. Integration by parts is very useful in deriving reduction formulas, i.e. reduce an integrand to a simpler case. However, there are no general rules when integration by parts should be used. Usually integrands involving $\ln x$, $\sin^{-1} x$, $\cos^{-1} x$, or $\tan^{-1} x$ are done using integration by parts.

In the following, we will run through the examples in the textbook using the **Diagram Method**.

Example 1. Find $\int xe^x \ dx$.

Solution.

$$\begin{array}{ccc} \text{diff.} & & \text{int.} \\ x & & e^x \\ & \searrow & \\ 1 & (-) \rightarrow & e^x \end{array}$$

Therefore,

$$\int xe^x \ dx = xe^x - \int e^x \ dx = xe^x - e^x + C.$$

Example 2. Find $\int x \sin x \ dx$.

Solution. $\int x \sin x \ dx$

$$\begin{array}{ccc} \text{diff.} & & \text{int.} \\ u & & v' \\ & \searrow & \\ u' & (-) \rightarrow & v \end{array} = \begin{array}{ccc} \text{diff.} & & \text{int.} \\ x & & \sin x \\ & \searrow & \\ 1 & (-) \rightarrow & -\cos x \end{array}$$

Therefore,

$$\begin{aligned} \int x \sin x \ dx &= -x \cos x - \int -\cos \ dx \\ &= -x \cos x + \int \cos x \ dx = -x \cos x + \sin x + C. \end{aligned}$$

$\square$

Remark. If the integration is still hard to get after one step, we can apply the technique of "Integration by Parts" again.

Example 3.

a. Find $\int x^2 e^x \, dx$.

Solution.

$$\begin{array}{ccc} \text{diff.} & & \text{int.} \\ x^2 & & e^x \\ & \searrow & \\ 2x & (-)\rightarrow & e^x \end{array}$$

Therefore,

$$\int x^2 e^x \, dx = x^2 e^x - 2\int x e^x \, dx.$$

We apply the integration by parts to $\int x e^x \, dx$,

$$\int x e^x \, dx = \int x \, de^x = x e^x - \int e^x dx = x e^x - e^x + C.$$

Therefore,

$$\int x^2 e^x \, dx = x^2 e^x - 2(x e^x - e^x) + C = x^2 e^x - 2x e^x + 2e^x + C.$$

Remark. Repeated application of Integration by Parts can be made easier by the Diagram Method, but be careful to the signs.

$$\begin{array}{ccc} \text{diff.} & & \text{int.} \\ x^2 & & e^x \\ & \searrow & \\ 2x & & e^x \\ & (-)\searrow & \\ 2 & (+)\rightarrow & e^x \end{array}$$

$\square$

b. How would you evaluate $\int x^n e^x \, dx$, where n is a positive integer?

Solution.

$$\begin{array}{ccc} \text{diff.} & & \text{int.} \\ x^n & & e^x \\ & \searrow & \\ nx^{n-1} & (-)\rightarrow & e^x \end{array}$$

Therefore,

$$\int x^n e^x \; dx = x^n e^x - n \int x^{n-1} e^x \; dx.$$

We see that integration by parts reduces the power of the variable in the integrand.

Remark. Sometimes we simply go back to the original integration if we apply the integration by parts over and over. This can be easily checked by the diagram method. We must be careful because it might seem that we are going in circles BUT we are not. The following is an example of this.

Example 4. Find $\int e^{2x} \sin x \; dx$.

Solution.

$$\begin{array}{ccc}
\text{diff.} & & \text{int.} \\
\sin x & & e^{2x} \\
& \searrow & \\
\cos x & & \frac{1}{2}e^{2x} \\
& (-) \searrow & \\
-\sin x & (+) \rightarrow & \frac{1}{4}e^{2x}
\end{array}$$

Now, we have

$$\begin{aligned}
\int e^{2x} \sin x \; dx &= \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x - \frac{1}{4}\int e^{2x}\sin x \; dx \\
\Longrightarrow \frac{5}{4}\int e^{2x} \sin x \; dx &= \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x \\
\Longrightarrow \int e^{2x} \sin x \; dx &= \frac{e^{2x}}{5}\Big[2\sin x - \cos x\Big] + C.
\end{aligned}$$

$\square$

Theorem (**Integration by Parts for Definite Integrals**). Let u and v be differentiable. Then $\boxed{\int_a^b u(x)v^{'}(x)\ dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)u^{'}(x)\ dx.}$

Remark. Or we can first find the anti-derivative of $u(x)v^{'}(x)$ and then find the definite integral.

Example. Find $\int_4^9 \frac{\ln y}{\sqrt{y}}\ dy$.

Solution.

$$\int \frac{\ln y}{\sqrt{y}}\ dy = 2\sqrt{y}\ln y - 2\int y^{-1/2}\ dy = 2\sqrt{y}\ln y - 4\sqrt{y} + C.$$

diff.		int.
$\ln y$		$y^{-\frac{1}{2}}$
	$\searrow$	
$\frac{1}{y}$	$(-) \rightarrow$	$2\sqrt{y}$

Therefore,

$$\int_4^9 \frac{\ln y}{\sqrt{y}}\ dy = (2\sqrt{y}\ln y - 4\sqrt{y})\Big|_4^9$$

□

More examples

Example. Find $\int \ln x\ dx$.

Solution.

diff.		int.	
$\ln x$		1	
	$\searrow$		
$\frac{1}{x}$	$(-) \rightarrow$	x	Easy Integration

Thus,

$$\boxed{\int \ln x\ dx = x\ln x - \int 1\ dx = x\ln x - x + C}.$$

□

Example. Find $\int (\ln x)^2\ dx$.

Solution.

Example. Find $\int x^3 \tan^{-1} x \, dx$.

Solution. $\int x^3 \tan^{-1} x \, dx$

$$\begin{array}{ccc} \text{diff.} & & \text{int.} \\ \tan^{-1} x & & x^3 \\ & \searrow & \\ \frac{1}{1+x^2} & (-) \rightarrow & \frac{x^4}{4} \end{array}$$

Now, by long division, we have

$$\frac{x^4}{1+x^2} = x^2 - 1 + \frac{1}{1+x^2}.$$

Thus,

$$\begin{aligned} \int x^3 \tan^{-1} x \, dx &= \frac{x^4}{4} \tan^{-1} x - \frac{1}{4} \int \frac{x^4}{1+x^2} \, dx \\ &= \frac{x^4}{4} \tan^{-1} x - \frac{1}{4} \int \left(x^2 - 1 + \frac{1}{1+x^2} \right) dx \\ &= \frac{x^4}{4} \tan^{-1} x - \frac{1}{4} \left[\frac{x^3}{3} - x + \tan^{-1} x \right] + C. \end{aligned}$$

□

Example. Find $\int \sin^{-1} x \, dx$.

Solution. Think of this integral as $\int (\sin^{-1} x)(1) \, dx$ and stop the diagram after one step.

$$\begin{array}{ccc} \text{diff.} & & \text{int.} \\ \sin^{-1} x & & 1 \\ & \searrow & \\ \frac{1}{\sqrt{1-x^2}} & (-) \rightarrow & x \end{array}$$

Now, put things together with integration by Substitution to get

$$\begin{aligned} \int \sin^{-1} x \, dx &= x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} \, dx \text{ (Set } u = 1 - x^2) \\ &= x \sin^{-1} x + \frac{1}{2} \int \frac{d(1-x^2)}{\sqrt{1-x^2}} = x \sin^{-1} x + \frac{1}{2} \int u^{-\frac{1}{2}} \, du \\ &= x \sin^{-1} x + \frac{1}{2}(2) u^{\frac{1}{2}} + C = x \sin^{-1} x + \sqrt{1-x^2} + C. \end{aligned}$$

Example. Find $\int s2^s\ ds$.

Solution. Recall that $\int 2^s\ ds = 2^s/\ln 2$.

$$\int s2^s\ ds = \frac{1}{\ln 2}s2^s - \frac{1}{\ln 2}\int 2^s\ ds = \frac{1}{\ln 2}s2^s - \frac{1}{(\ln 2)^2}2^s.$$

$$\begin{array}{ccc} \text{diff.} & & \text{int.} \\ s & & e^{2^s} \\ & \searrow & \\ 1 & (-)\rightarrow & 2^s/\ln 2 \end{array}$$

□

Example. Find $\int x^2 \cos mx\ dx$.

Solution.

$$\begin{aligned}\int x^2 \cos mx\ dx =& \frac{x^2 \sin mx}{m} + \frac{2x\cos mx}{m^2} - \frac{2}{m^2}\int \cos mx\ dx \\ =& \frac{x^2 \sin mx}{m} + \frac{2x\cos mx}{m^2} - \frac{2\sin mx}{m^3} + C.\end{aligned}$$

$$\begin{array}{ccc} \text{diff.} & & \text{int.} \\ x^2 & & \cos mx \\ & \searrow & \\ 2x & & \sin mx/m \\ & (-)\searrow & \\ 2 & (+)\rightarrow & -\cos mx/m^2 \end{array}$$

□

8.3 §8.3 Trigonometric Integrals

At the moment, our inventory of integrals involving trigonometric functions is rather limited. The goal of this section is to develop techniques for evaluating integrals involving trigonometric functions. We must memorize the trigonometric identities:

TR1. $\sin^2 x + \cos^2 x = 1$.

TR2. $\tan^2 x + 1 = \sec^2 x$.

TR3. $\cot^2 x + 1 = \csc^2 x$.

TR4. $\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x$.

TR5. $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$.

TR6. $\sin x \sin y = \frac{1}{2}[\cos(x - y) - \cos(x + y)]$.

TR7. $\sin^2 x = \frac{1}{2}[1 - \cos(2x)]$, since $\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$.

TR8. $\cos x \cos y = \frac{1}{2}[\cos(x - y) + \cos(x + y)]$.

TR9. $\cos^2 x = \frac{1}{2}[1 + \cos(2x)]$.

TR10. $\sin x \cos y = \frac{1}{2}[\sin(x - y) + \sin(x + y)]$.

TR11. $\sin x \cos x = \frac{1}{2}[\sin(2x)]$, since $\sin(2x) = \sin(x + x) = 2\sin x \cos x$.

Example. Find $\int \sin^6 x \cos^3 x \, dx$.

Solution. Since the exponent on the cosine function is odd we keep one of these to construct our differential and rewrite the others as powers of the sine function.

$$\sin^6 x \cos^3 x \, dx = \sin^6 \cos^2 x(\cos x \, dx) = \sin^6 x(1 - \sin^2 x)d(\sin x).$$

Let $u = \sin x$. Then

$$\int \sin^6 x \cos^3 x \, dx = \int u^6(1-u^2) \, du = \int (u^6-u^8) \, du = \frac{u^7}{7}-\frac{u^9}{9} = \frac{\sin^7 x}{7}-\frac{\sin^9 x}{9}+C.$$

This example gives us a glimpse at how to tackle these integrals.

Let $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$. Then integrals involving trigonometric functions can sometimes be classified as follows:

Integrand	**Restriction on** m	**Restriction on** n	**Procedure**
$\sin mx \sin nx$	$\neq 0$	$\neq 0$	Use TR6
$\cos mx \cos nx$	$\neq 0$	$\neq 0$	Use TR8
$\sin mx \cos nx$	$\neq 0$	$\neq 0$	Use TR10
$\sin^m x \cos^n x$	Odd in $\mathbb{Z}_{>0}$	None	$\underbrace{-\sin^{m-1} x}_{-(1-\cos^2 x)^{\frac{m-1}{2}}} \cos^n x \ d(\cos x)$
$\sin^m x \cos^n x$	None	Odd in $\mathbb{Z}_{>0}$	$\sin^m x \underbrace{\cos^{n-1} x}_{(1-\sin^2 x)^{\frac{n-1}{2}}} \ d(\sin x)$
$\sin^m x \cos^n x$	Even in $\mathbb{Z}_{\geq 0}$	Even in $\mathbb{Z}_{\geq 0}$	Use TR7 & TR9 $\rightarrow \cos(2x)$
$\tan^m x \sec^n x$	None	Even in $\mathbb{Z}_{>0}$	Use: $\sec^n x \ dx =$ $(\tan^2 x + 1)^{\frac{n-2}{2}} \ d(\tan x)$
$\tan^m x \sec^n x$	in $\mathbb{Z}_{\geq 2}$	$= 0$	Reduce power using: $\tan^m x = \tan^{m-2} x(\sec^2 x - 1)$ $= \tan^{m-2} x \ d(\tan x) - \tan^{m-2} x = \cdots$
$\tan^m x \sec^n x$	$= 0$	Odd in $\mathbb{Z}_{>0}$	If $n = 1,$ use foumula. If not, use repeated int. by parts.
$\tan^m x \sec^n x$	Odd in $\mathbb{Z}_{>0}$	Odd in $\mathbb{Z}_{>0}$	Simplify by using: $\tan^m x \sec^n x$ $= \tan^{m-1} x \sec^{n-1} x \ d\sec x$ $= (\sec^2 x - 1)^{\frac{m-1}{2}} \sec^{n-1} x \ d(\sec x)$
$\tan^m x \sec^n x$	Even in $\mathbb{Z}_{>0}$	Odd in $\mathbb{Z}_{>0}$	Use: $\tan^m x = (\sec^2 x - 1)^{\frac{m}{2}}$
$\cot^m x \csc^n x$			Use: Similar to $\tan^m x \sec^n x$

Example 1.

a. $\int \cos^5 x \; dx$

Solution.

$\square$

b. $\int \sin^4 x \; dx$

Solution.

$\square$

Example 2.

a. $\int \sin^4 x \cos^2 x \; dx$

Solution.

$\square$

b. $\int \sin^3 x(\cos x)^{-2}\ dx$

Solution.

$\square$

Example. $\int \cos^2 x\ dx$.

Solution. The Outline says that the integral $\int \sin^m x \cos^n x\ dx$ with m and n both even is done using TR7 and TR9.

$$\begin{aligned}\int \cos^2 x\ dx &= \int \frac{1}{2}(1+\cos 2x)\ dx \\ &= \frac{x}{2} + \frac{1}{2}\int \cos 2x\ dx = \frac{x}{2} + \frac{1}{4}\sin 2x + C\end{aligned}$$

$\square$

Remark. Try $\int \cos^4 x\ dx$ and $\int \cos^6 x\ dx$.

Example. $\int \cos \pi x \cos 4\pi x \; dx$.

Solution. We first simplify the integrand by substitution and then use Identity TR8: $\cos A \cos B = \frac{1}{2}\big[\cos(A-B) + \cos(A+B)\big]$.

$$\int \cos \pi x \cos 4\pi x \; dx = \frac{1}{\pi}\int \cos \pi x \cos 4\pi x \; d(\pi x) = \frac{1}{\pi}\int \underbrace{\cos u}_{\cos A}\underbrace{\cos 4u}_{\cos B} \; du$$
$$= \frac{1}{\pi}\int \frac{1}{2}[\underbrace{\cos(1-4)u}_{\cos(A-B)} + \underbrace{\cos(1+4)u}_{\cos(A+B)}] \; du.$$

Therefore,

$$\begin{aligned}
\int \cos \pi x \cos 4\pi x \; dx &= \frac{1}{2\pi}\int [\cos(-3u) + \cos 5u] \; du \\
&= \frac{1}{2\pi}\Big[\int \cos(3u) \; du + \int \cos 5u \; du\Big] \\
&= \frac{1}{2\pi}\Big[\frac{1}{3}\int \cos 3u \; d(3u) + \frac{1}{5}\int \cos 5u \; d(5u)\Big] \\
&= \frac{1}{2\pi}\Big[\frac{1}{3}\sin 3u + \frac{1}{5}\sin 5u\Big] + C = \frac{1}{10\pi}\sin 5u + \frac{1}{6\pi}\sin 3u + C \\
&= \frac{1}{10\pi}\sin 5\pi x + \frac{1}{6\pi}\sin 3\pi x + C.
\end{aligned}$$

□

Example (reading). The Outline says that the integral $\int \tan^m x \sec^n x \; dx$ with $m = 0$ and n odd is done by repeated use of Integration by Parts. As a special case we consider

$$\int \sec^3 x \; dx.$$

Solution. This requires Integration by Parts. In order to apply Integration by Parts, we rewrite the integrand

$$\int \sec^3 x \; dx = \int \sec x \sec^2 x \; dx.$$

Again, we apply Integration by Parts.

$$\begin{array}{ccc} \text{diff.} & & \text{int.} \\ \sec x & & \sec^2 x \\ & \searrow & \\ \sec x \tan x & (-) \rightarrow & \tan x \end{array}$$

$$\begin{aligned} \int \sec^3 x \ dx &= \sec x \tan x - \int \sec x \tan^2 x \ dx = \sec x \tan x - \int \sec x(\sec^2 x - 1) \ dx \\ &= \sec x \tan x - \int (\sec^3 x - \sec x) \ dx \\ &= \sec x \tan x - \int \sec^3 x \ dx + \int \sec x \ dx. \end{aligned}$$

Therefore,

$$\begin{aligned} \int \sec^3 x \ dx &= \frac{1}{2}\big(\sec x \tan x + \int \sec x \ dx\big) \\ &= \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C. \end{aligned}$$

□

Example 4. Evaluate the following integrals.

a. $\int \tan^3 x \sec^4 x \ dx$

Solution.

□

b. $\int \tan^2 x \sec x \ dx$

Solution.

$\square$

Reduction Formulas

Some reduction formulas have been developed to ease the trigonometric integration workload. These formulas are not required to be memorized.

Assume that n is a positive integer.

1. $\displaystyle\int \sin^n x \ dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \ dx$
2. $\displaystyle\int \cos^n x \ dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \ dx$
3. $\displaystyle\int \tan^n x \ dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \ dx$
4. $\displaystyle\int \sec^n x \ dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \ dx, \ n \neq 1$

Example 3. Evaluate $\int \tan^4 dx$.

Method 1 Sol.

$\square$

Method 2 (using reduction formula).

$\square$

Example. $\int \cot^5 x \sin^4 x \ dx$.

Solution.

$$\begin{aligned}\int \cot^5 x \sin^4 x \ dx &= \int \frac{\cos^5 x}{\sin^5 x} \sin^4 x \ dx = \int \frac{\cos^5 x}{\sin x} \ dx \\ &= \int \frac{(\cos^2 x)^2}{\sin x} \cos x \ dx = \int \frac{(1-\sin^2 x)^2}{\sin x} \cos x \ dx.\end{aligned}$$

Set $u = \sin x$ then $du = \cos x \ dx$. Therefore,

$$\begin{aligned}\int \cot^5 x \sin^4 x \ dx &= \int \frac{(1-u^2)^2}{u} \ du = \int \frac{1-2u^2+u^4}{u} \ du \\ &= \int \frac{1}{u} \ du - \int 2u \ du + \int u^3 \ du \\ &= \ln|u| - u^2 + \frac{u^4}{4} + C \\ &= \ln|\sin x| - \sin^2 x + \frac{1}{4}\sin^4 x + C.\end{aligned}$$

$\square$

8.4 §8.4 Trigonometric Substitutions

Integrals involving a **radical** (integer power sometimes) of certain forms are often solved by a trigonometric (inverse) substitution:

$$\int f(g(x))g^{'}(x)\,dx = \int f(u)\,du \overset{\text{Let } x=g(t)}{\overbrace{\Longrightarrow}} \boxed{\int f(x)\,dx = \int f(g(t))g^{'}(t)\,dt}$$

When a constant $c > 0$ and the integrand involves one of

(i) $\sqrt{c^2 - x^2}$. Substitute: $x = c\sin\theta$. Then

$$\sqrt{c^2 - x^2} = \sqrt{c^2(1 - \sin^2\theta)} = c|\cos\theta| = c\cos\theta \ \text{ for } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.$$

(ii) $\sqrt{x^2 + c^2}$. Substitute: $x = c\tan\theta$. Then

$$\sqrt{x^2 + c^2} = \sqrt{c^2(\tan^2\theta + 1)} = c\sec\theta \ \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

(iii) $\sqrt{x^2 - c^2}$. Substitute: $x = c\sec\theta$. Then

$$\sqrt{x^2 - c^2} = \sqrt{c^2(\sec^2\theta - 1)} = c\tan\theta \ \text{ for } 0 \le \theta < \frac{\pi}{2} \text{ or } \pi \le \theta < \frac{3\pi}{2}.$$

Example 1. Verify that the area of a circle of radius r is πr^2.

Solution.

□

Example 2.(similar to Example 1) Evaluate $\int \frac{dx}{(16-x^2)^{3/2}}$.

Solution.

$\square$

Example 3. Evaluate $\int_0^2 \sqrt{1+4x^2}\,dx$.

Solution.

$\square$

Example 4.(similar to Example 3) Evaluate $\int \dfrac{1}{(1+x^2)^2}\, dx$.

Solution.

$\square$

Example 5. (similar to Example 3) Evaluate $\int \frac{1}{\sqrt{x^2+4}}\, dx$.

Solution.

$\square$

Exercise 19. Evaluate $\int \frac{dx}{\sqrt{x^2-81}}\, dx$.

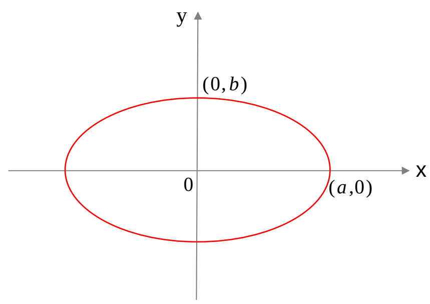
Solution.

$\square$

Example (reading). Show that the area enclosed by the following ellipse is πab

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Solution.



$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}=\frac{a^2-x^2}{a^2} \implies y=\pm\frac{b}{a}\sqrt{a^2-x^2}$$

The ellipse can be thought of as four parts of equal area and so we have

$$\begin{aligned}
\text{Area } &= 4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx. && \begin{cases}\text{Set } x=a\sin\theta \\ dx=a\cos\theta\ d\theta\end{cases}\\
&= 4\int_0^{\frac{\pi}{2}} \frac{b}{a}\sqrt{a^2-a^2\sin^2\theta}\ (a\cos\theta\ d\theta) && \begin{cases}\text{Since } \theta=\sin^{-1}\frac{x}{a} \\ \theta=0 \text{ when } x=0 \\ \theta=\frac{\pi}{2} \text{ when } x=a\end{cases}\\
&= 4\int_0^{\frac{\pi}{2}} ab\sqrt{1-\sin^2\theta}\cos\theta\ d\theta\\
&= 4\int_0^{\frac{\pi}{2}} (ab)\cos^2\theta\ d\theta=(4ab)\int_0^{\frac{\pi}{2}}\frac{1}{2}\big(1+\cos(2\theta)\big)\ d\theta\\
&= 2ab\Big[\int_0^{\frac{\pi}{2}} d\theta+\frac{1}{2}\int_0^{\frac{\pi}{2}}\cos(2\theta)\ d(2\theta)\Big]\\
&= (2ab)\Big[\theta+\frac{1}{2}\sin(2\theta)\Big]_0^{\frac{\pi}{2}}=(2ab)\Big[\frac{\pi}{2}-0+\frac{1}{2}(\sin\pi-\sin 0)\Big]\\
&= (2ab)\frac{\pi}{2}=\pi ab.
\end{aligned}$$

□

Sometimes the form of the expression under the radical does not fit into a standard form. But we can still apply the substitution directly. However, we need "**to Complete the Squares**" before we use the substitution.

Completing the Squares. To find the roots of $ax^2 + bx + c$ when $a \neq 0$, we first Complete the Squares

$$\begin{aligned} ax^2 + bx + c &= a(x^2 + \frac{b}{a}x + \frac{c}{a}) \\ &= a(x^2 + \frac{b}{a}x + (\frac{b}{2a})^2 - (\frac{b}{2a})^2 + \frac{c}{a}) \\ &= a\Big[(x + \frac{\mathbf{b}}{\mathbf{2a}})^2 - (\frac{b^2 - 4ac}{4a^2})\Big]. \end{aligned}$$

When $ax^2 + bx + c$ is under a radical as in $\sqrt{ax^2 + bx + c}$ then we need to study three cases to simplify the expression $\sqrt{ax^2 + bx + c}$.

$$\begin{aligned} \sqrt{ax^2 + bx + c} &= \sqrt{a\Big[(x + \frac{b}{2a})^2 - (\frac{b^2 - 4ac}{4a^2})\Big]} \\ &= \begin{cases} \sqrt{u^2 - d^2} & \text{if } a > 0 \ \& \ \left(\frac{b^2-4ac}{4a^2}\right) > 0 \\ \sqrt{u^2 + d^2} & \text{if } a > 0 \ \& \ \left(\frac{b^2-4ac}{4a^2}\right) < 0 \\ \sqrt{d^2 - u^2} & \text{if } a < 0 \ \& \ \left(\frac{b^2-4ac}{4a^2}\right) > 0 \end{cases}, \end{aligned}$$

where $u = x + \frac{b}{2a}$ is the variable in substitution and d is a constant.

(i) $\sqrt{ax^2 + bx + c} = \sqrt{u^2 - d^2}$. Substitute: $u = d\sec\theta$.

(ii) $\sqrt{ax^2 + bx + c} = \sqrt{u^2 + d^2}$. Substitute: $u = d\tan\theta$.

(iii) $\sqrt{ax^2 + bx + c} = \sqrt{d^2 - u^2}$. Substitute: $u = d\sin\theta$.

Example 6. Evaluate $\int_1^4 \dfrac{\sqrt{x^2+4x-5}}{x+2}\,dx$.

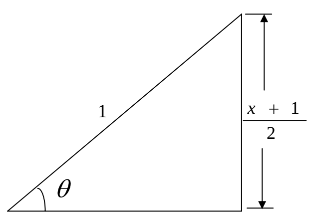
Solution.

$\square$

More examples:

Example. Find $\int \frac{x}{\sqrt{3-2x-x^2}}\,dx$.

Solution. $3-2x-x^2 = 2^2-(x+1)^2$. Now, set $x+1 = 2\sin\theta$. Notice that $\sin\theta = \frac{x+1}{2}$, so set up a triangle that describes this substitution.



Replace the various terms in the integrand of $\int \frac{x}{\sqrt{3-2x-x^2}}\,dx$ by the their equivalent expressions given by

$$x = 2\sin\theta - 1,\ dx = 2\cos\theta\ d\theta,$$
$$\sqrt{3-2x-x^2} = \sqrt{2^2-(x+1)^2} = \sqrt{2^2-2^2\sin^2\theta} = 2\sqrt{1-\sin^2\theta} = 2\cos\theta$$

Now,

$$\int \frac{x}{\sqrt{3-2x-x^2}}\,dx = \int \frac{2\sin\theta-1}{2\cos\theta}(2\cos\theta\ d\theta) = \int (2\sin\theta - 1)\ d\theta$$
$$= -2\cos\theta - \theta + C = -2\frac{\sqrt{2^2-(x+1)^2}}{2} - \sin^{-1}\left(\frac{x+1}{2}\right) + C.$$

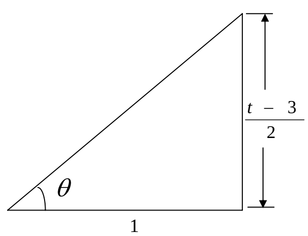
□

Example (reading). Find $\int \frac{dt}{\sqrt{t^2-6t+13}}\ dt$.

Solution. First, we complete the squares

$$\begin{aligned} t^2 - 6t + 13 &= t^2 - 6t + 3^2 - 3^3 + 13 = (t-3)^2 + 4 \\ &= (t-3)^2 + 4 = (t-3)^2 + 2^2 = u^2 + c^2. \\ \sqrt{t^2 - 6t + 13} &= \sqrt{(t-3)^2 + 2^2} \\ &= \sqrt{u^2 + c^2} \implies \text{ Substitution } t - 3 = u = c\tan\theta = 2\tan\theta. \end{aligned}$$

Now set $t - 3 = 2\tan\theta$ ($\tan\theta = \frac{t-3}{2}$)and set up a triangle that describes this substitution.



Replace the various terms in the integrand of $\int \frac{dt}{\sqrt{t^2-6t+13}}$ by the their equivalent expressions given by

$$\begin{aligned} &t = 3 + 2\tan\theta, dt = 2\sec^2\theta\ d\theta, \\ &\sqrt{t^2 - 6t + 13} = \sqrt{(t-3)^2 + 2^2} = \sqrt{2^2\tan^2\theta + 2^2} = 2\sqrt{\tan^2\theta + 1} = 2\sec\theta. \end{aligned}$$

Now,

$$\begin{aligned} \int \frac{dt}{\sqrt{t^2 - 6t + 13}} = \int \frac{2\sec^2\theta\ d\theta}{2\sec\theta} &= \int \sec\theta\ d\theta \\ &= \ln|\sec\theta + \tan\theta| + C \\ &= \ln|\frac{\sqrt{(t-3)^2 + 2^2}}{2} + \frac{t-3}{2}| + C \\ &= \ln|\sqrt{(t-3)^2 + 2^2} + (t-3)| + C. \end{aligned}$$

□

8.5 §8.5 Integration of Rational Functions by Partial Fractions

The idea behind partial fractions is to "undo" or reverse the addition of rational functions (that is, decomposition of the rational functions). For example,

$$\frac{3x}{x^2+2x-8} \overset{\text{method of partial fractions}}{=\!=\!=\!=\!=} \frac{1}{x-2}+\frac{2}{x+4}.$$

A rational function is the ratio of two polynomials. A **polynomial** is an expression of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are constants. Such a polynomial is said to have degree n,

$$\deg P(x) = n, \quad \text{provided } a_n \neq 0.$$

A **rational function** is a function that can be expressed as the ratio of two polynomials.

$$f(x) = \frac{P(x)}{Q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}.$$

The Method of Partial Fractions consists of three steps to simplify the integrand and then do the integration after this is done.

Step 1. If $\deg P(x) < \deg Q(x)$, go to step 2; If $\deg P(x) \geq \deg Q(x)$, we divide $Q(x)$ by $P(x)$ (long division) and represent $f(x)$ as

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)} \text{ where } R(x) = 0 \text{ or } \deg R(x) < \deg Q(x).$$

Step 2. Factor $Q(x)$ as far as possible (there are two types of factors: linear and quadratic),

$$Q(x) = (\alpha_1 x + \beta_1)^{\epsilon_1} \cdots (\alpha_k x + \beta_k)^{\epsilon_k} (a_1 x^2 + b_1 x + c_1)^{e_1} \cdots (a_\ell x^2 + b_\ell x + c_\ell)^{e_\ell},$$

where each quadratic polynomial has no real roots, i.e.

$$a_i x^2 + b_i x + c_i \text{ has the property that } b_i^2 - 4a_i c_i < 0.$$

Step 3. Using this factorization of (x), express $\frac{R(x)}{Q(x)}$ as the sum of terms of the form

$$\frac{A_{i1}}{(\alpha_i x+\beta_i)}+\cdots+\frac{A_{i\epsilon_i}}{(\alpha_i x+\beta_i)^{\epsilon_i}} \text{ for } i=1,\ldots,k \tag{$\star$}$$

plus terms of the form

$$\frac{A_{i1}x+B_{i1}}{(a_i x^2+b_i x+c_i)}+\cdots+\frac{A_{ie_i}x+B_{ie_i}}{(a_i x^2+b_i x+c_i)^{e_1}} \text{ for } i=1,\ldots,\ell \tag{$\star\star$}$$

The expressions in ($\star$) are easy to integrate by substitution but those in ($\star\star$) are more complicated and might involve completing the squares.

SUMMARY **Partial Fraction Decompositions**

Let $f(x) = p(x)/q(x)$ be a proper rational function in reduced form. Assume the denominator q has been factored completely over the real numbers and m is a positive integer.

1. **Simple linear factor** A factor $x - r$ in the denominator requires the partial fraction $\dfrac{A}{x-r}$.
2. **Repeated linear factor** A factor $(x-r)^m$ with $m > 1$ in the denominator requires the partial fractions
$$\frac{A_1}{(x-r)}+\frac{A_2}{(x-r)^2}+\frac{A_3}{(x-r)^3}+\cdots+\frac{A_m}{(x-r)^m}.$$
3. **Simple irreducible quadratic factor** An irreducible factor $ax^2 + bx + c$ in the denominator requires the partial fraction
$$\frac{Ax+B}{ax^2+bx+c}.$$
4. **Repeated irreducible quadratic factor** (See Exercises 83–86.) An irreducible factor $(ax^2+bx+c)^m$ with $m > 1$ in the denominator requires the partial fractions
$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_mx+B_m}{(ax^2+bx+c)^m}.$$

Example (reading). Find

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\, dx.$$

Solution. Step 1. Divide the denominator by the numerator (use long division) to get

$$\frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} = x + 1 + \frac{4x}{x^3 - x^2 - x + 1}$$

Step 2. Factor the denominator. This often involve <u>trial and error</u> while looking for a root.

x	$x^3 - x^2 - x + 1$
0	1
1	0
-1	0

$\implies (x-1)(x+1)$ divides $x^3 - x^2 - x + 1$.

Divide $x^3 - x^2 - x + 1$ by $x^2 - 1$.

$$\frac{x^3 - x^2 - x + 1}{x^2 - 1} = x - 1 \implies x^3 - x^2 - x + 1 = (x-1)^2(x+1).$$

Step 3. Now we must represent $\frac{4x}{x^3 - x^2 - x - 1}$ in the form of $(\star)$.

$$\frac{4x}{x^3 - x^2 - x + 1} = \frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

Multiply through by the denominator on the left hand side to get

$$\frac{4x}{(x-1)^2(x+1)}(x-1)^2(x+1) = \frac{A}{x-1}(x-1)^2(x+1) + \frac{B}{(x-1)^2}(x-1)^2(x+1) + \frac{C}{x+1}(x-1)^2(x+1)$$

and so we must solve

$$(\bullet) \qquad 4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

for A, B, and C. Since $(\bullet)$ is valid for all possible value of x we substitute values in for x if this is helpful.

$$x = 1 \implies 4(1) = A((1)-1)((1)+1) + B((1)+1) + C((1)-1)^2 = 2B$$

$$\implies 4 = 2B \implies B = 2$$

Put this into ($\bullet$) and simplify to get

$$2x - 2 = 2(x-1) = A(x-1)(x+1) + C(x-1)^2$$

$$(\bullet\bullet) \qquad 2 = A(x+1) + C(x-1)$$

$$x = 1 \implies 2 = 2A \implies A = 1$$

$$x = -1 \implies 2 = -2C \implies C = -1$$

Or we can expand the right hand side of ($\bullet$) and compare the corresponding coefficients. Thus,

$$\frac{4x}{x^3 - x^2 - x + 1} = \frac{4x}{(x-1)^2(x+1)} = \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1}$$

and

$$\begin{aligned}\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\, dx &= \int \left[x + 1 + \frac{4x}{x^3 - x^2 - x + 1}\right] dx \\ &= \int \left[x + 1 + \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1}\right] dx \\ &= \frac{x^2}{2} + x + \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + K,\end{aligned}$$

where K is any constant. $\square$

Example 1. Evaluate $\int f(x)\ dx = \int \dfrac{3x^2 + 7x - 2}{x^3 - x^2 - 2x}\ dx$

Solution.

$\square$

Example 3.(repeated linear factors) Evaluate $\int f(x)\ dx = \int \dfrac{5x^2 - 3x + 2}{x^3 - 2x^2}\ dx$

Solution.

$\square$

Example 5.(irreducible quadrtic factors) Evaluate $\int f(x)\,dx = \int \frac{7x^2 - 13x + 13}{(x-2)(x^2-2x+3)}\,dx$

Solution.

$\square$

8.6 §8.6 Integration Strategies

The only way to get to the point that you can do indefinite integrals is by experience. **DO PROBLEMS**. After you have done **a lot** of problems then you will see that the process follows these guideline.

1. **Simplify the integrand.** This might involve algebraic manipulation or the use of trigonometric identities which makes the method of integration obvious.
2. **Substitution.** See if there is a substitution that either puts the integrand in a standard form or which makes the method of integration obvious.
3. **Determine the form of the integrand.**
   (a) Rational Function
   (b) Trigonometric Function
   (c) Contains Radical
   (d) Integration by Parts (Does the integrand contain and inverse trigonometric function or ln?)

Remark. Some Standard Simplifying Substitutions.

(a) Integrals involving $\sqrt[n]{ax+b}$. Substitute: $u = \sqrt[n]{ax+b}$ or $u = ax+b$.

(b) Integrals involving $\sqrt{ax^2+bx+c}$. Complete the squares to get one of

  (i) $\sqrt{ax^2+bx+c} = \sqrt{p^2-u^2}$. Substitute: $u = p\sin\theta$
  (ii) $\sqrt{ax^2+bx+c} = \sqrt{u^2-p^2}$. Substitute: $u = p\sec\theta$
  (iii) $\sqrt{ax^2+bx+c} = \sqrt{u^2+p^2}$. Substitute: $u = p\tan\theta$

(c) Transform rational functions of trigonometric functions into rational functions in variable u by substituting: $u = \tan\frac{x}{2}$ so that (**Exercise 8.4.86**)

$$\sin x = \frac{2u}{1+u^2}, \qquad \cos x = \frac{1-u^2}{1+u^2}, \qquad dx = \frac{2du}{1+u^2}.$$

Example 1. Evaluate $\int \frac{\sin x + 1}{\cos^2 x}\ dx$

Solution.

$\square$

Example 2. Evaluate $\int \frac{x^2}{\sqrt{4 - x^6}}$.

Solution.

$\square$

Example (Reading). Find

$$\int \frac{dx}{2\sqrt{x+3}+x}$$

Solution. Using the idea in Part (a). Set $u = \sqrt{x+3}$. Then

$$u = \sqrt{x+3} \implies u^2 = x+3 \implies x = u^2 - 3 \implies dx = 2udu.$$

$$\int \frac{dx}{2\sqrt{x+3}+x} = \int \frac{2udu}{2u+u^2-3} = \int \frac{2udu}{u^2+2u-3} = \int \frac{2udu}{(u+3)(u-1)}.$$

Apply the Method of Partial Fractions.

$$\frac{2u}{(u+3)(u-1)} = \frac{A}{(u+3)} + \frac{B}{(u-1)} \implies 2u = A(u-1) + B(u+3)$$

$$\implies A = \frac{3}{2}, B = \frac{1}{2}$$

$$\int \frac{2udu}{(u+3)(u-1)} = \frac{3}{2}\int \frac{du}{u+3} + \frac{1}{2}\int \frac{du}{u-1} = \frac{3}{2}\ln|u+3| + \frac{1}{2}\ln|u-1| + C$$

$$= \frac{3}{2}\ln|\sqrt{x+3}+3| + \frac{1}{2}\ln|\sqrt{x+3}-1| + C.$$

□

Example 3. Evaluate the following integrals.

a. $\int \dfrac{4-3x^2}{x(x^2-4)}\, dx$

Solution.

□

b. (Reading) $\int xe^{\sqrt{1+x^2}}\, dx$

c. $\int \ln(1+x^2)\ dx$

Solution.

$\square$

Example 4. Evaluate $\int xe^{\sqrt{x}}\ dx$

Solution.

$\square$

8.7 §8.7 Other Methods of Integration

Omit - Using Tables and Computer Algebra Systems such as Mathematica and Sage-Math. We used **Python** in this course.

8.8 §8.9 Improper Integrals

Definition (Definition of Improper Integrals over Infinite Intervals). The integrals of *unbounded domain* $\int_a^\infty f(x)\,dx$, $\int_{-\infty}^b f(x)\,dx$ and $\int_{-\infty}^\infty f(x)\,dx$ are defined in the following way.

(a) If $\int_a^t f(x)\,dx$ exists for every $t \geq a$ and the limit $\lim\limits_{t\to\infty} \int_a^t f(x)\,dx$ exists and is finite, then the **improper integral** $\int_a^\infty f(x)\,dx$ is said to be **convergent** and is defined to be value of this limit

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx.$$

(b) If $\int_t^b f(x)\,dx$ exists for every $t \leq b$ and the limit $\lim\limits_{t\to-\infty} \int_t^b f(x)\,dx$ exists and is finite, then the **improper integral** $\int_{-\infty}^b f(x)\,dx$ is said to be **convergent** and is defined to be value of this limit

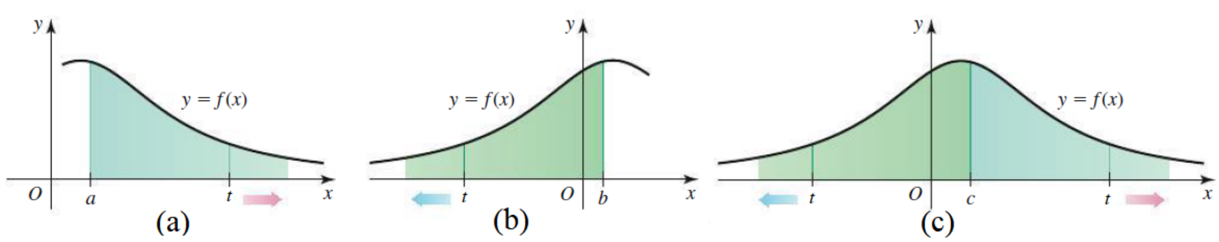
$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx.$$

(c) If $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then the **improper integral** $\int_{-\infty}^\infty f(x)\,dx$ is defined to be value of this limit

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

where a is any real number.

An improper integral that is not convergent, is said to be **divergent**. (Convergence or divergence depends only on the "tail".



Remark. To simplify the calculation of the definite integral $\int_a^t f(x)\,dx$ or $\int_t^b f(x)\,dx$, we can find the indefinite integral $\int f(x)\,dx$ first.

To check how to use Python to calculate improper integrals, see **Integration.ipynb**.

Example 1.

a. $\int_0^\infty e^{-x}\,dx$

Solution.

$\square$

b. $\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$.

Solution.

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx &= \lim_{t\to-\infty}\int_t^0 \frac{1}{1+x^2}\,dx + \lim_{t\to\infty}\int_0^t \frac{1}{1+x^2}\,dx \\
&= \lim_{t\to-\infty}\Big[\tan^{-1}x\Big]_t^0 + \lim_{t\to\infty}\Big[\tan^{-1}x\Big]_0^t \\
&= \lim_{t\to-\infty}\Big[\tan^{-1}0 - \tan^{-1}t\Big] + \lim_{t\to\infty}\Big[\tan^{-1}t - \tan^{-1}0\Big] \\
&= \Big[0 - (-\frac{\pi}{2})\Big] + \Big[\frac{\pi}{2} - 0\Big] = \pi.
\end{aligned}$$

$\square$

Example 2 - The p-Test for Improper Integrals.

$$\boxed{\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \text{diverges, if } p \le 1 \\ \text{converges, if } p > 1 \end{cases}}$$

Proof.

$$\int_1^t \frac{1}{x^p}\,dx = \begin{cases} \ln t & \text{if } p = 1 \\ \frac{t^{-p+1}}{-p+1} - \frac{1}{-p+1} & \text{if } p \neq 1 \end{cases}$$

Now,

$$\lim_{t\to\infty} \ln t = \infty \implies \int_1^\infty \frac{1}{x^p}\,dx \text{ divergent when } p = 1 \text{ and}$$

$$\lim_{t\to\infty} \frac{1}{1-p}(t^{1-p}-1) = \begin{cases} \infty & \text{if } p < 1 \\ -\frac{1}{1-p} & \text{if } p > 1. \end{cases} \qquad \square$$

Example. Evaluate

$$\int_{-\infty}^0 xe^x\,dx$$

Solution. First we find an antiderivative by Integration by Parts.

$$\begin{array}{ccc} \text{diff.} & & \text{int.} \\ x & & e^x \\ & \searrow & \\ 1 & (-)\rightarrow & e^x \end{array}$$

Therefore,

$$\begin{aligned} \int_{-\infty}^0 xe^x\,dx &= \lim_{t\to-\infty} \int_t^0 xe^x\,dx \\ &= \lim_{t\to-\infty} \Big[xe^x - e^x\Big]_t^0 = \lim_{t\to-\infty} \Big[-1-(te^t-e^t)\Big] \end{aligned}$$

Certainly,

$$\lim_{t\to-\infty} e^t = \lim_{t\to\infty} e^{-t} = \lim_{t\to\infty} \frac{1}{e^t} = 0.$$

We apply L'Hospital Rule to evaluate $\lim\limits_{t\to-\infty} te^t$

$$\lim_{t\to-\infty} te^t = \lim_{t\to-\infty} \frac{t}{e^{-t}} = \lim_{t\to\infty} \frac{-t}{e^t} = \lim_{t\to\infty} \frac{-1}{e^t} = 0.$$

Now we put these parts together to get

$$\int_{-\infty}^0 xe^x\,dx = \lim_{t\to-\infty} \Big[-1-(te^t-e^t)\Big] = -1.$$

$\square$

Definition (**Definition of Improper Integrals of Unbounded(Discontinuous) Integrand).** If $f(x)$ has *discontinuous* point on $[a, b]$, we define $\int_a^b$ in what follows.

(a) If f is continuous on $(a, b]$, discontinuous at a, but the limit $\lim\limits_{t\to a^+} \int_t^b f(x)\,dx$ exists and is finite, then the **improper integral** $\int_a^b f(x)\,dx$ is said to be **convergent** and is defined to be value of this limit

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx.$$

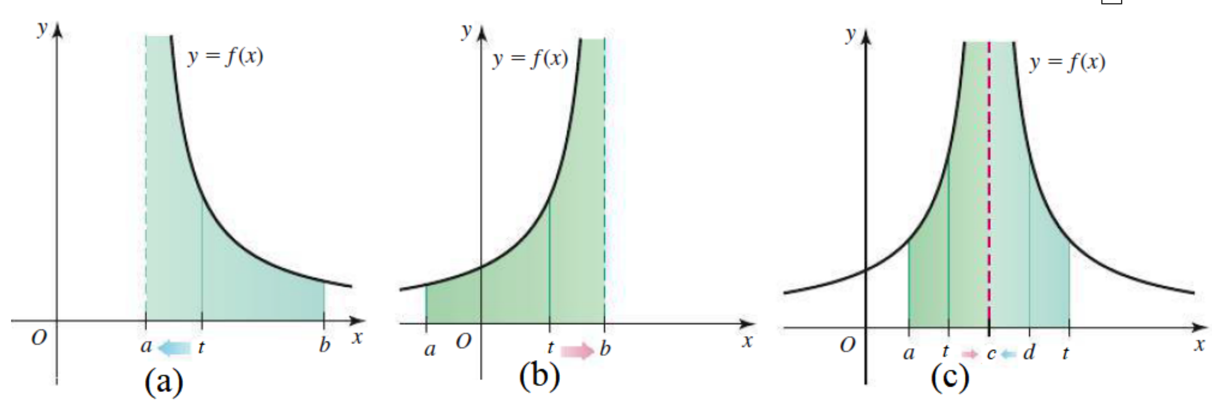
(b) If f is continuous on $[a, b)$, discontinuous at b, but the limit $\lim\limits_{t\to b^-} \int_a^t f(x)\,dx$ exists and is finite, then the **improper integral** $\int_a^b f(x)\,dx$ is said to be **convergent** and is defined to be value of this limit

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx.$$

(c) If f is continuous on $[a, c)$ and on $(c, b]$ with $a < c < b$, discontinuous at c, but the limit $\lim\limits_{t\to c^-} \int_a^t f(x)\,dx$ and $\lim\limits_{t\to c^+} \int_t^b f(x)\,dx$ both exist and are finite, then the **improper integral** $\int_a^b f(x)\,dx$ is defined to be

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

□



Example 4. Find the area of the region R between the graph of $f(x) = \frac{1}{\sqrt{9-x^2}}$ and the x-axis on the interval $(-3, 3)$ (if it exists).

Solution.

$\square$

Example 5. Evaluate $\int_1^{10} \frac{1}{(x-2)^{1/3}}\, dx$

Solution.

$\square$

The p-test for Improper Integrals of Unbounded Integrand

$$\boxed{\int_0^1 \frac{1}{x^p}\,dx = \begin{cases} \text{diverges, if } p \geq 1 \\ \text{converges, if } p < 1 \end{cases}}$$

Proof. Exercise 94

Example. Find $\int_0^3 \frac{dx}{x-1}$, if convergent.

Solution. Be careful. The following calculation is **NOT valid**.

$$\int_0^3 \frac{dx}{x-1} = \Big[\ln|x-1|\Big]_0^3 = \ln 2 - \ln|-1| = \ln 2.$$

This is an improper integral due to the fact that $\frac{1}{x-1}$ is discontinuous at $c = 1$ with $0 < 1 < 3$. This means that we must check the convergence of both

$$\int_0^1 \frac{dx}{x-1} \qquad \text{and} \qquad \int_1^3 \frac{dx}{x-1}.$$

$$\begin{aligned}
\lim_{t\to 1^-}\int_0^t \frac{dx}{x-1} &= \lim_{t\to 1^-}\int_0^t \frac{dx}{x-1} = \lim_{t\to 1^-}\Big[\ln|x-1|\Big]_0^t \\
&= \lim_{t\to 1^-}\Big[\ln|t-1| - \ln|0-1|\Big] = \lim_{t\to 1^-}\ln(1-t) \\
&= \lim_{s\to 0^+}\ln s = -\infty
\end{aligned}$$

Therefore, $\int_0^3 \frac{dx}{x-1}$ is divergent. $\square$

The Comparison Test

There are some cases in which we cannot determine whether a given improper integral converges, simply because it is impossible to compute an anti-derivative.

Theorem (Thm 8.2: Comparison Test for Type I Improper integrals)**.** Suppose that f and g are continuous functions on the interval $[a, \infty)$ with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

(a) If $\int_a^\infty f(x)\, dx$ is convergent, then $\int_a^\infty g(x)\, dx$ is convergent.

(b) If $\int_a^\infty g(x)\, dx$ is divergent, then $\int_a^\infty f(x)\, dx$ is divergent.

Theorem (Limit Comparison Theorem - Type I)**.** Let $f(x)$ and $g(x)$ be two **positive continuous** functions on $[a, \infty)$. Assume that the limit

$$L = \lim_{x\to\infty} \frac{f(x)}{g(x)}$$

exists and finite. That is, $0 < L < \infty$. Then,

$$\int_a^\infty f(x)\, dx \text{ converges } \Leftrightarrow \int_a^\infty g(x)\, dx \text{ converges}$$

Remark. Similar theorems are true for the other two cases of improper integrals over infinite intervals.

Example 7. Determine whether the following integrals converge.

a. $\int_0^\infty e^{-x^2}\, dx$

Solution. We need only test the "tail". $\int_1^\infty e^{-x^2}\, dx$ since

$$\int_0^\infty e^{-x^2}\, dx = \underbrace{\int_0^1 e^{-x^2}\, dx}_{\text{finite number}} + \int_1^\infty e^{-x^2}\, dx$$

On the interval $[1, \infty)$ we have $e^{-x^2} = \frac{1}{e^{x^2}} \leq \frac{1}{e^x} = e^{-x}$ and so we test $\int_1^\infty e^{-x}\, dx$ and apply the Comparison Test to get convergence.

$$\int_1^\infty e^{-x}\, dx = \lim_{t\to\infty} \int_1^t e^{-x}\, dx = \lim_{t\to\infty} \Big[- e^{-x} \Big]_1^t = \lim_{t\to\infty} \Big[- e^{-t} + e^{-1} \Big]_1^t = \frac{1}{e}$$

This means $\int_1^\infty e^{-x^2}\, dx$ is convergent by the convergence test. $\square$

b. $\int_1^\infty \dfrac{1}{\sqrt[3]{x^2 - 0.5}}\, dx$

Solution.

$\square$

More examples

Example.

Test $\int_0^\infty \frac{x}{(x^2+2)^2}\, dx$ for convergence or divergence.

Solution. We claim that it is convergent. We only need to check the "tail", i.e. $\int_1^\infty \frac{x}{(x^2+2)^2}\, dx$. First observe that $\frac{x}{(x^2+2)^2} < \frac{1}{x^2}$ on $[1, \infty)$ because

$$x \cdot x^2 = x^3 < x^4 + 4x^2 + 4 = (x^2+2)^2 \implies \frac{x}{(x^2+2)^2} < \frac{1}{x^2}.$$

Now observe that

$$\int_1^\infty \frac{1}{x^2}\, dx \text{ is convergent }.$$

By the p-Test and then by the Comparison Test $\int_1^\infty \frac{x}{(x^2+2)^2}\, dx$ is convergent and hence $\int_0^\infty \frac{x}{(x^2+2)^2}\, dx$ is convergent. We can actually find the value of $\int_0^\infty \frac{x}{(x^2+2)^2}\, dx$

$$\int_0^\infty \frac{x}{(x^2+2)^2}\, dx = \lim_{t\to\infty} \frac{1}{2} \int_0^t \frac{d(x^2+2)}{(x^2+2)^2} = \lim_{t\to\infty} \frac{1}{2} \Big[-(x^2+2)^{-1} \Big]_0^t = \frac{1}{4}.$$

$\square$

Example. Test $\int_0^\infty \frac{x\tan^{-1}x}{(x^2+1)^2}\,dx$ for convergence or divergence.

Solution. Notice that we are not actually asked to evaluate this integral but just determine convergence or divergence.

Method I: We apply the Test for Convergence. First observe that $\tan^{-1}x \le \frac{\pi}{2} < 2$. Therefore,

$$\frac{x\tan^{-1}x}{(x^2+1)^2} \le \frac{2x}{(x^2+1)^2} \text{ and so}$$

$$\text{if } \int_0^\infty \frac{2x}{(x^2+1)^2}\,dx \text{ converges then } \int_0^\infty \frac{x\tan^{-1}x}{(x^2+1)^2}\,dx \text{ converges.}$$

Now,

$$\int_0^\infty \frac{2x}{(x^2+1)^2}\,dx = \int_0^\infty \frac{d(x^2+1)}{(x^2+1)^2} = \int_1^\infty \frac{1}{u^2}\,du \text{ (Convergent by the } p\text{-Test)}$$

$$\therefore \int_0^\infty \frac{x\tan^{-1}x}{(x^2+1)^2}\,dx \text{ (Convergent by the Comparison Test.)}$$

Method II Evaluate the improper integral... □

Theorem (Comparison Test for Type II Improper integrals)**.** Suppose that the two function $f(x) \ge g(x) \ge 0$ on the interval $(a, b]$, but $f(x)$ and $g(x)$ are discontinuous (unbounded) at a.

(a) If $\int_a^b f(x)\,dx$ is convergent, then $\int_a^b g(x)\,dx$ is convergent.

(b) If $\int_a^b g(x)\,dx$ is divergent, then $\int_a^b f(x)\,dx$ is divergent.

Theorem (Limit Comparison Theorem - Type II)**.** Let $f(x)$ and $g(x)$ be two **positive continuous** functions on $(a, b]$, but $f(x)$ and $g(x)$ are discontinuous (unbounded) at a. Assume that the limit

$$L = \lim_{x\to a^+} \frac{f(x)}{g(x)}$$

exits and finite. That is, $0 < L < \infty$. Then,

$$\int_a^b f(x)\,dx \text{ converges } \Leftrightarrow \int_a^b g(x)\,dx \text{ converges}$$

Remark. Similar theorems are true for the other two cases of improper integrals of unbounded integrand.

Exercise 8.9.85. Determine whether the following integral converges or diverges.

$$\int_0^1 \frac{dx}{\sqrt{x^{1/3}+x}}$$

Solution.

Exercise 8.9.86. Determine whether the following integral converges or diverges.

$$\int_0^1 \frac{1+\sin x}{x^5}\, dx$$

Solution.

8.9 §8.8 Numerical Integration

The two primary cases when we want to use approximate **numerical integration** are: i) when the antiderivative is too difficult or impossible to find, for example $\int e^{x^2}\ dx$ and $\int \sqrt{1+x^3}\ dx$; ii) when we are involved in a mathematical model in which the formula of the function is unknown.

Because numerical methods do not typically produce exact results, we should be concerned about the accuracy of approximations, which leads to the ideas of **absolute** and **relative error**.

Definition (Error of Approximation)**.** Suppose c is a computed numerical solution to a problem having an exact solution x. There are two common measures of the error in c as an approximation to x:

$$\textbf{absoluate error} = |c - x|$$

and

$$\textbf{relative error} = \frac{|c-x|}{|x|} \quad (if\ x \neq 0).$$

Now, the only method is to find a numerical approximation for the definite integral

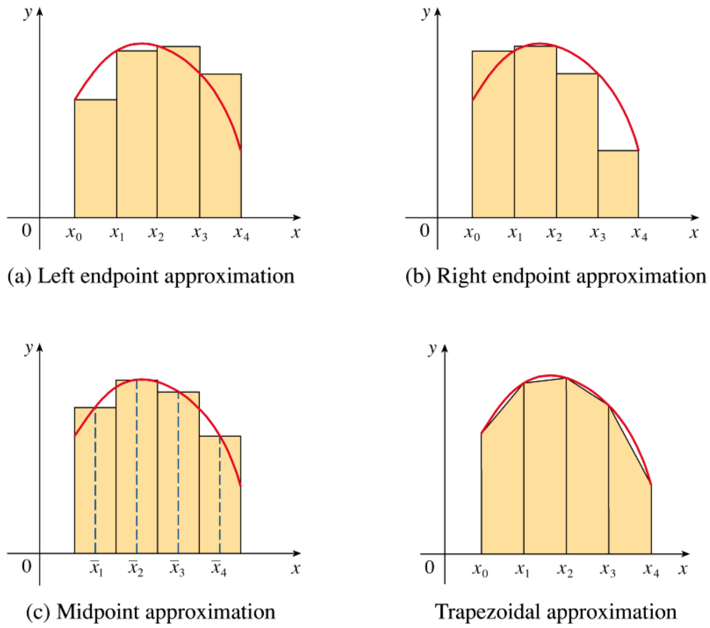
$$\int_a^b f(x)\ dx$$

by partitioning the interval $[a, b]$ into n subintervals of equal length

$$a = x_0 < x_1 = a + \frac{b-a}{n} < \cdots < x_i = a + i\frac{b-a}{n} < \cdots < x_n = a + n\frac{b-a}{n} = b.$$

Note that

$$\Delta x = \frac{b-a}{n} \quad \text{and} \quad x_i = a + i\frac{b-a}{n}.$$



Integral approximations

We consider three types of Approximate Integration.

(1). A Riemann Sum Approximation - Midpoint Rule,

(2). Trapezoidal Rule, and

(3). Simpson's Rule.

MIDPOINT RULE.

$$\boxed{M_n = \int_a^b f(x)\, dx \approx \Delta x\big[f(\bar{x}_1) + \cdots + f(\bar{x}_{n-1}) + f(\bar{x}_n)\big]},$$

where $\bar{x}_i = \frac{x_{i-1}+x_i}{2}$ is the midpoint of the subinterval. □

Moreover, if $|f''(x)| \le K$ on $[a, b]$ and E_M is the error of the Midpoint Approximation, then

$$|E_M| = \left| \int_a^b f(x)\, dx - M_n \right| \le \frac{K(b-a)^3}{24n^2}.$$

□

A trapezoid is a quadrilateral with one pair of opposite side parallel. In our case, two sides will be parallel and perpendicular to the base. If such a trapezoid has base of width Δx and parallel side of length $f(x_{i-1})$ and $f(x_i)$, then

$$\text{Area of Trapezoid } = \Delta x \frac{f(x_{i-1}) + f(x_i)}{2}.$$

The integral is approximated by

$$\sum_{i=1}^{n} \Delta x \frac{f(x_{i-1}) + f(x_i)}{2} = \frac{\Delta x}{2}\big[f(x_0) + 2f(x_1 - 1) + \cdots + 2f(x_{n-1}) + f(x_n)\big].$$

TRAPEZOIDAL RULE.

$$\boxed{T_n = \int_a^b f(x)\ dx \approx \frac{\Delta x}{2}\big[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\big].}$$

Moreover, if $|f''(x)| \leq K$ on $[a, b]$ and E_T is the error of the approximation, then

$$|E_T| = \Big|\int_a^b f(x)\ dx - T_n\Big| \leq \frac{K(b-a)^3}{12n^2}.$$

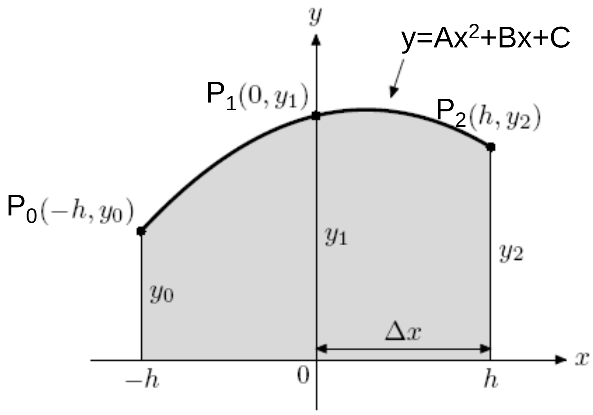
□

Simpson's Rule is somewhat more involved. On subintervals of length $2\Delta x$, we approximate the curve with parabolas and use the sum of the integrals of these approximating parabolas to approximate the integral. This requires that the number of parts n of our partition is **even** $\underline{n = 2k}$. On the interval $[x_{2(\ell)}, x_{2\ell+2}]$ the approximating parabola is required to pass through the points

$$P_0 = \big(x_{2\ell}, f(x_{2\ell})\big), \quad P_1 = \big(x_{2\ell+1}, f(x_{2\ell+1})\big), \quad P_2 = \big(x_{2\ell+2}, f(x_{2\ell+2})\big)$$

To get an approximation on the interval $x_{2\ell} \leq x_{2\ell+1} \leq x_{2\ell+2}$, we simplify things by shifting the graph so that $x_{2\ell} = -h$, $x_{2\ell+1} = 0$ and $x_{2\ell+2} = h$. Then

$$P_0 = \big(-h, f(-h)\big), \quad P_1 = \big(0, f(0)\big), \quad P_2 = \big(h, f(h)\big)$$



Now, our approximating polynomial on $[-h, h]$ has the form $Ax^2 + Bx + C$ and so the approximating integral is

$$\begin{aligned}\int_{-h}^{h} (Ax^2 + Bx + C)\ dx &= \left[A\frac{x^3}{3} + B\frac{x^2}{2} + Cx\right]_{-h}^{h} \\ &= \left[A\frac{h^3}{3} + B\frac{h^2}{2} + Ch\right] - \left[A\frac{-h^3}{3} + B\frac{h^2}{2} - Ch\right] \\ &= 2A\frac{h^3}{3} + 2Ch = \frac{h}{3}(2Ah^2 + 6C).\end{aligned}$$

Now, we want to represent this integral using the values $y_0 = f(-h)$, $y_1 = f(0)$, $y_2 = f(h)$ and so we solve the system for the expression $2Ah^2 + 6C$.

$$\begin{aligned} y_0 &= f(-h) = Ah^2 - Bh + C \\ y_1 &= f(0) = C \qquad \Longrightarrow\ y_0 + y_2 = 2Ah^2 + 2C \\ y_2 &= f(h) = Ah^2 + Bh + C \end{aligned}$$

$$\begin{aligned} &\Longrightarrow\ y_0 + 4y_1 + y_2 = 2Ah^2 + 6C \\ &\Longrightarrow\ \int_{-h}^{h} (Ax^2 + Bx + C)\ dx = \frac{h}{3}[y_0 + 4y_1 + y_2]. \end{aligned}$$

Finally, since the shift that we made to move the interval $[x_{2\ell}, x_{2\ell+2}]$ to the interval $[-h, h]$ does not affect the height of the graph at corresponding points and the value of the approximating integral depends only on the values of $y_0 = f(x_{2\ell})$, $y_1 = f(x_{2\ell+1})$

and $y_2 = f(x_{2\ell+2})$, we see that

$$\int_{x_{2\ell}}^{x_{2\ell+2}} \underbrace{Ax^2 + Bx + C}_{\text{Approx. Parabola}} \quad dx = \frac{h}{3}[f(x_{2\ell}) + 4f(x_{2\ell+1}) + f(x_{2\ell+2})].$$

Note that here $n = 2k$ and $h = \Delta x = \frac{b-a}{n} = \frac{b-a}{2k}$ and so

$$\int_{x_{2\ell}}^{x_{2\ell+2}} \underbrace{Ax^2 + Bx + C}_{\text{Approx. Parabola}} \quad dx = \frac{\Delta x}{3}[f(x_{2\ell}) + 4f(x_{2\ell+1}) + f(x_{2\ell+2})].$$

Simpson's Rule is an approximation obtained by adding together all the approximations over the k intervals of length $2\Delta x$:

$$[a, a + 2\Delta x], \cdots, [a + 2(k-1)\Delta x, a + 2k\Delta x]$$

SIMPSON'S RULE. Partition $[a, b]$ into $n = 2k$ subintervals using the end points

$$x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \cdots, x_n = a + 2k\Delta x = b.$$

Then

$$S_n = \int_a^b f(x)\ dx \approx \frac{\Delta x}{3}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots$$
$$+ 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\Big].$$

□

Moreover, if $|f^{(4)}(x)| \le K$ for all x in $[a, b]$ and E_S is the error of the Simpson's Rule approximation, then

$$|E_S| = \Big| \int_a^b f(x)\ dx - S_n \Big| \le \frac{K(b-a)^5}{180n^4}.$$

□

Example. Set $n = 10$ and use each of the Midpoint Rule, Trapezoidal Rule and Simpson's Rule to approximate

$$I = \int_0^1 e^{x^2}\ dx.$$

Also, use the bound on the error term in each case to get an estimate of the accuracy of the approximation.

Solution. First we bound the errors. In the cases of the Midpoint Approximation and the Trapezoidal Rule Approximation, we need to bound the second derivative on $[0,1]$ with K and in the case of the Simpson's Rule Approximation, we need to bound the fourth derivative on $[0,1]$ with $\hat{K}$.

$$\begin{aligned}
f(x) &= e^{x^2} \\
f'(x) &= 2xe^{x^2} \\
f''(x) &= (2+4x^2)e^{x^2} \\
f^{(3)}(x) &= (12x+8x^3)e^{x^2} > 0 \text{ on } (0,1] \implies f''(x) \text{ incr. on } [0.1] \\
&\implies K = f''(1) = (6)e = 16.3096 \\
f^{(4)} &= (12+48x^2+16x^4)e^{x^2} \\
f^{(5)}(x) &= (120x+112x^2+32x^5)e^{x^2} > 0 \text{ on } (0,1] \implies f^{(4)}(x) \text{ incr. on } [0.1] \\
&\implies \hat{K} = f^{(4)}(1) = (76)e = 206.589
\end{aligned}$$

Now,

	Error Bound Formula	Error Bound Value
Midpoint Rule	$\frac{K(b-a)^3}{24*n^2}$	$\frac{(16.3096)(1-0)^3}{24*10^2} = .00679$
Trapezoidal Rule	$\frac{K(b-a)^3}{12*n^2}$	$\frac{(16.3096)(1-0)^3}{12*10^2} = .013591$
Simpson's Rule Rule	$\frac{\hat{K}(b-a)^5}{180*n^4}$	$\frac{(206.589)(1-0)^5}{180*10^4} = .00011477$

Computing the values needed in the approximations is done easily using a spread sheet such as Excel. We used this software for the following values.

i	x_i	$\bar{x}_i$	Midpt. Approx.	Trap. Approx.	Simp. Approx.
0	0			$f(x_0) = 1$	$f(x_0) = 1$
1	.1	.05	$f(\bar{x}_1) = 1.0025$	$2f(x_1) = 2.0201$	$4f(x_1) = 4.0402$
2	.2	.15	$f(\bar{x}_2) = 1.022755$	$2f(x_2) = 2.0816$	$2f(x_2) = 2.0816$
3	.3	.25	$f(\bar{x}_3) = 1.06449$	$2f(x_3) = 2.1883$	$4f(x_3) = 4.37669$
4	.4	.35	$f(\bar{x}_4) = 1.13031$	$2f(x_4) = 2.347$	$2f(x_4) = 2.347$
5	.5	.45	$f(\bar{x}_5) = 1.2244$	$2f(x_5) = 2.568$	$4f(x_5) = 5.1361$
6	.6	.55	$f(\bar{x}_6) = 1.3532$	$2f(x_6) = 2.8666$	$2f(x_6) = 2.8666$
7	.7	.65	$f(\bar{x}_7) = 1.52577$	$2f(x_7) = 3.2646$	$4f(x_7) = 6.5292$
8	.8	.75	$f(\bar{x}_8) = 1.755$	$2f(x_8) = 3.7929$	$2f(x_8) = 3.7929$
9	.9	..85	$f(\bar{x}_9) = 2.05957$	$2f(x_9) = 4.4958$	$4f(x_9) = 8.9916$
10	1	.95	$f(\bar{x}_{10}) = 2.4657$	$f(x_{10}) = 2.718$	$f(x_{10}) = 2.718$
			$\Delta x(Sum)$	$\frac{\Delta x}{2}(Sum)$	$\frac{\Delta x}{3}(Sum)$
			$= 1.46039$	$= 1.4672$	$= 1.4627$

It is interesting to note how close these values are and how different are the error bounds.

Contents

Question. What is a definite integral? Is it just an area as previously discussed? We shall see that it can be many more things.

Chapter 6 involves explanations via graphs. You are strongly encouraged to study these diagrams that the professional drawers have prepared for you in the textbook. They are well prepared with shading and accuracy. You are encouraged to use a TI-83/84 calculator to draw any functions in an exam.

6.1 §6.1 The Net change Theorem

This section is left for your own **reading**.

Theorem (The Net change Theorem)**.** The integral of the rate of change is the net change:

$$\int_a^b F'(x)\ dx = F(b) - F(a).$$

Example. If $f(x)$ is the slope of a trail at a distance x from the start, what does $\int_3^5 f(x)\ dx$ represent?

Solution. Suppose that the elevation of the trail at x is $e(x)$. Then the slope of the trail at x is $e'(x) = f(x)$.

By the Net Change Theorem, $\int_3^5 f(x)\ dx = \int_3^5 e'(x)\ dx = e(5) - e(3)$.

□

The following lists some other examples.

• If $V(t)$ is the volume of water in a reservoir at time t, then $V'(t)$ is the rate at which the water is flowing into/out of the reservoir at time t. Thus,

$$\int_{t_1}^{t_2} V'(t)\ dt = V(t_2) - V(t_1)$$

is the net change in the amount of water in the reservoir between time t_1 and t_2.

• If $[C](t)$ is the concentration of a product due to a chemical reaction at time t, then the rate of reaction is $D_x([C](t))$ and so

$$\int_{t_1}^{t_2} D_x([C](t)\ dt = [C](t_2) - [C](t_1)$$

is the net change in the concentration of C from time t_1 to t_2.

• If the mass of a rod measured from the left end to a point x is given by $m(x)$, then the linear density is $\rho(x) = m'(x)$. So

$$\int_a^b \rho(x)\ dx = m(b) - m(a)$$

is the mass of the rod between the points $x = a$ and $x = b$.

• If the rate of growth of a population at time t is $dn(t)/dt$, then

$$\int_{t_1}^{t_2} D_x(n(t))\ dt = n(t_2) - n(t_1)$$

is the net change in the population over time the time period $[t_1, t_2]$.

• If $C(x)$ is the cost of producing x units of a commodity, then $C'(x)$ is the marginal cost. So

$$\int_{t_1}^{t_2} C'(x)\ dx = C(t_2) - C(t_1)$$

is the increase in cost when production is increased from t_1 units to t_2 units.

If an object moves along a straight line with an nondecreasing position function $s(t)$, measured from left to right, then its velocity is $\boxed{v(t) = s'(t) \geq 0}$. So

$$\int_{t_1}^{t_2} v(t)\ dt = s(t_2) - s(t_1) = \text{ total distance traveled.}$$

Definition. Here, $s(t_2) - s(t_1)$ is generally called the **displacement** of the object between $t = t_1$ and $t = t_2$.

When we do not assume $v(t) = s'(t) \geq 0$, then

$\int_{t_1}^{t_2} v(t)\, dt = s(t_2) - s(t_1) = \boxed{\text{net change}}$ of position or the **displacement of the object**

and the total **distance travelled** is computed as

$$\int_a^b |v(x)|\, dx = \boxed{\text{total distance}} \text{ traveled.}$$

Theorem (Theorem 6.1 Position from Velocity)**.** Given the velocity $v(t)$ of an object moving along a line and its initial position $s(0)$, the position function of the object for future times $t \geq 0$ is

$$s(t) = s(0) + \int_0^t v(x)\, dx.$$

Example 2 (Position from velocity). A block hangs at rest from a massless spring at the origin ($s = 0$). At $t = 0$, the block is pulled downward 1/4 m to its initial position $s(0) = -1/4$ and released (Figure 6.4). Its velocity (in m/s) is given by $v(t) = 0.25 \sin t$, for $t \geq 0$. Assume that the upward direction is positive.

a. Find the position of the block, for $t \geq 0$.

b. Graph the position function, for $0 \leq t \leq 3\pi$.

c. When does the block move through the origin for the first time?

d. When does the block reach its highest point for the first time and what is its position at that time? When does the block return to its lowest point?

Solution.

□

• Given the acceleration $a(t)$ of an object moving along a line with velocity $v(t)$, $\boxed{a(t) = v^{'}(t)}$.

$$\int_{t_1}^{t_2} a(t)\ dt = v(t_2) - v(t_1) = \boxed{\text{net change}} \text{ of velocity}$$

Theorem (Theorem 6.2 Velocity from Acceleration)**.** Given the acceleration $a(t)$ of an object moving along a line and its initial velocity $v(0)$, the velocity of the object for future times $t \geq 0$ is

$$v(t) = v(0) + \int_0^t a(x)\ dx.$$

□

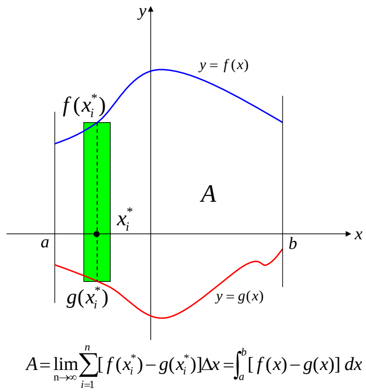
Example 4 (Motion in a gravitational field). An artillery shell is fired directly upward with an initial velocity of $300m/s$ from a point $30m$ above the ground. Assume that only the force of gravity acts on the shell and it produces an acceleration of $9.8m/s^2$. Find the velocity of the shell while it is in the air.

Solution.

□

6.2 §6.2 Regions Between Curves

Again, we rely on the slice-and-sum strategy (Section 5.2) for finding areas by Riemann sums.



Theorem. The area A bounded by $x = a$, $x = b$ with $a < b$ and the curves $y = f(x)$ and $y = g(x)$ is given by

$$\text{Area} \ = \int_a^b |f(x) - g(x)| \ dx.$$

Remark 1. The key for **manual calculation** is to **sketch the graphs of the curves** involved so that we can determine the upper curve and the lower curve and **intersection(s)** of the curves between $x = a$ and $x = b$. If the curves **cross** each other, we must set up more than one integral to compute the area. For example, we want to find the area bounded by $x = a$, $x = b$ and the curves $y = f(x)$ and $y = g(x)$. Suppose further that $f(x) \geq g(x)$ on $[a, c]$ and that $g(x) \geq f(x)$ on $[c, b]$. Then the diagram in this case is something like the figure in Example 5 and the required integrals are

$$\begin{aligned}\text{Area} &= \int_a^c \Big(\text{upper curve} - \text{lower curve}\Big) \ dx + \int_c^b \Big(\text{upper curve} - \text{lower curve}\Big) \ dx \\ &= \int_a^c [f(x) - g(x)] \ dx + \int_c^b [g(x) - f(x)] \ dx. \quad \square\end{aligned}$$

Remark 2. With technology, Area $= \int_a^b |f(x) - g(x)| \ dx$ regardless the postions of $f(x)$ and $g(x)$. Technologies will detect the intersections between $x = a$ and $x = b$ for us.

Remark 3. Try different functions using this web app `https://xuemaozhang.shinyapps.io/Area/`.

Example 1. Find the area of the region bounded by the graphs of $f(x) = 5 - x^2$ and $g(x) = x^2 - 3$.

Solution.

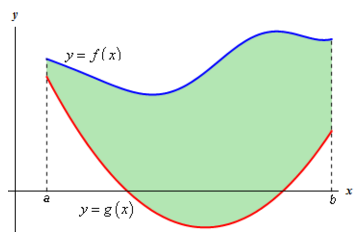
$\square$

Example 2. Find the area of the region bounded by the graphs of $f(x) = -x^2 + 3x + 6$ and $g(x) = |2x|$.

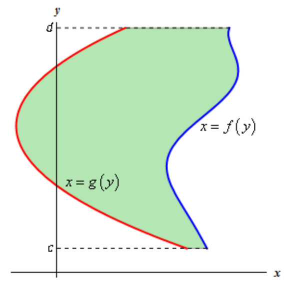
Solution.

$\square$

Remark. Sometimes it is better to integrate with respect to the y variable.



$$A = \int_a^b [f(x) - g(x)]dx$$



$$A = \int_c^d [f(y) - g(y)]dy$$

Theorem. The area A bounded by $y = c$, $y = d$ with $c < d$ and the curves $x = f(y)$ and $x = g(y)$ is given by

$$\text{Area } = \int_c^d |f(y) - g(y)| \, dy.$$

Remark. If $f(y) \geq g(y)$ for all y in $[c, d]$ (that is, $f(y)$ is to the right of $g(y)$), then the integrand becomes $f(y) - g(y)$.

Example 3. Find the area of the region R bounded by the graphs of $y = x^3, y = x + 6$, and the x-axis.

Solution.

□

Example 4. Find the area of the region R in the first quadrant bounded by the curves $y = x^{2/3}$ and $y = x - 4$.

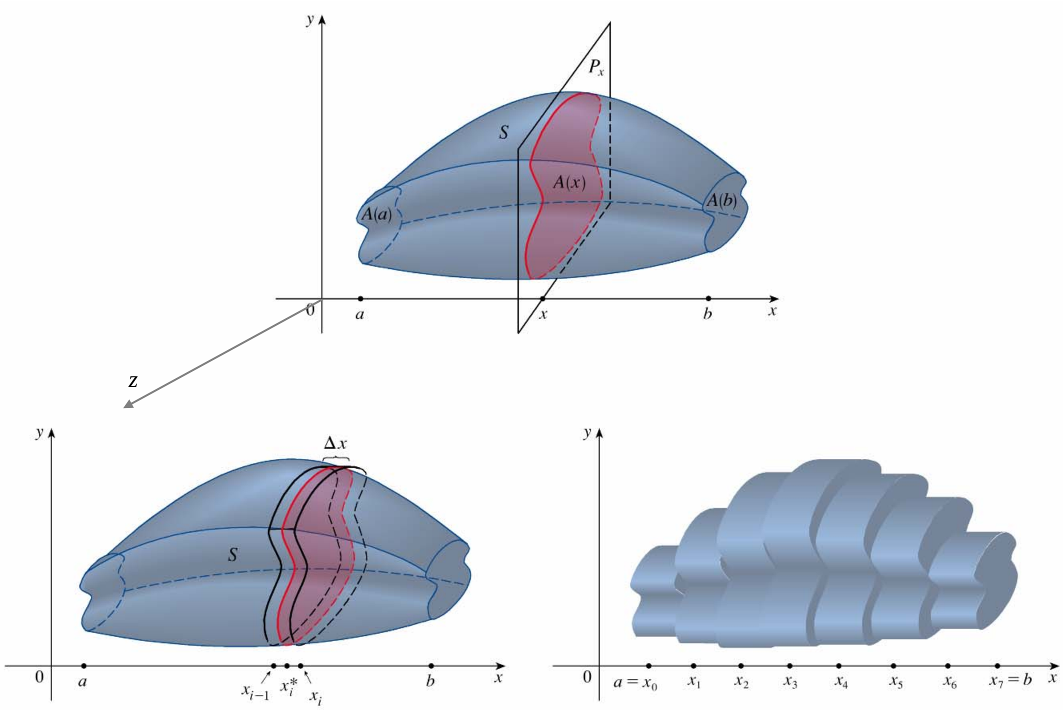
Solution.

$\square$

6.3 §6.3 Volume by Slicing

We have seen that integration is used to compute the area of two-dimensional regions bounded by curves. Integrals are also used to find the volume of three-dimensional regions (or solids). Once again, the **slice-and-sum** method is the key to solving these problems.

In this section, we first analyze the problem of finding the volume of a solid that extends in the x-direction from $x = a$ to $x = b$. For each value of x between a and b we know the cross sectional area is given by $A(x)$. Study illustrations pp. 425-426 (see the following two figures as well).



A cross-section and “slabs” of a solid *S*

Recall that the volume of a cylinder is given by

$$\text{Volume} = Ah = \text{Area of base } \times \text{ Height} .$$

For example, for a circular cylinder $A = \pi r^2$.

We see that the volume of one of our cross sectional slabs is given by

$$\boxed{\text{Volume of the } i^{th} \text{ slab } = A(x_i^*)\Delta x.}$$

Theorem (General Slicing Method)**.** Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x (the plane through x and perpendicular to the x-axis) is $A(x)$, then the **volume** of S is given by

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\, dx.$$

Remark. The key is to find the cross sectional area $A(x)$ at x. Both the **Disc Method** and the **Washer Method** are examples of methods involving a known cross sectional area.

Example 1.[Volume of a "parabolic cube"] Let R be the region in the first quadrant bounded by the coordinate axes and the curve $y = 1 - x^2$. A solid has a base R, and cross sections through the solid perpendicular to the base and parallel to the y-axis are **squares** (Figure 6.25a). Find the volume of the solid.

Solution.

□

The Disk Method

The volume of a $\boxed{\textbf{disc}}$ of height Δx and radius R is

$$\text{Volume of Disc } = \pi R^2 \Delta x.$$

We now consider a specific type of solid known as a **solid of revolution**.

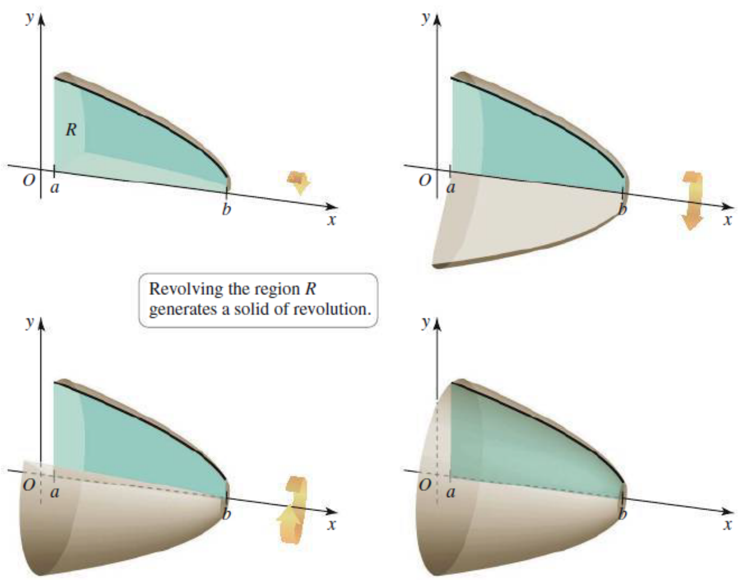


Illustration Choose a function and use the geogebra web app `https://www.geogebra.org/m/fd9kfvrh` for illustration.

Theorem (Disk Method about the x-Axis)**.** Let f be continuous with $f(x) \geq 0$ on the interval $[a, b]$. If the region R bounded by the graph of f, the x-axis, and the lines $x = a$ and $x = b$ is revolved about the x-axis, the **volume** of the resulting solid of revolution is is given by

$$V = \int_a^b A(x)\, dx = \int_a^b \pi[f(x)]^2\, dx.$$

Example 3. Let R be the region bounded by the curve $f(x) = (x+1)^2$, the x-axis, and the lines $x = 0$ and $x = 2$ (Figure 6.30a). Find the volume of the solid of revolution obtained by revolving R about the x-axis.

Solution.

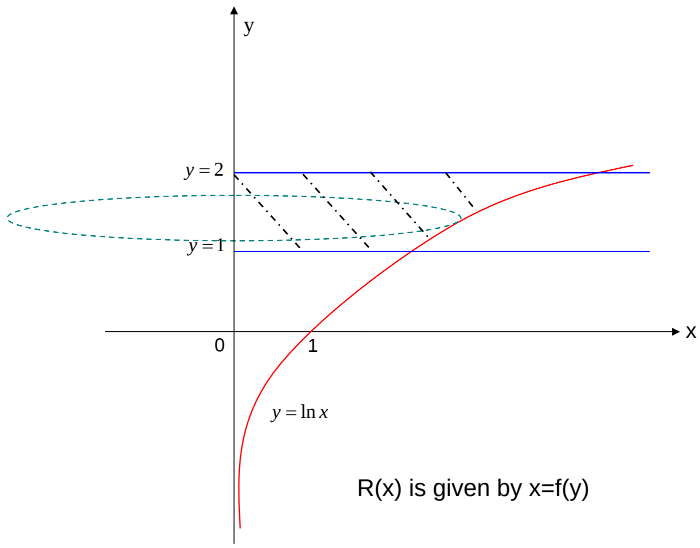
$\square$

What we learned about revolving regions about the x-axis applies to revolving regions about the y-axis.

Theorem (Disk Method about the y-Axis)**.** Let $x = g(y)$ be continuous with $g(y) \geq 0$ on the interval $[c, d]$. If the region R bounded by the graph of g, the y-axis, and the lines $y = c$ and $y = d$ is revolved about the y-axis, the **volume** of the resulting solid of revolution is is given by

$$V = \int_c^d A(y)\, dy = \int_c^d \pi[g(y)]^2\, dy.$$

Example Find the volume of the solid obtained by rotating the region bounded by the curves $y = \ln x$, $y = 1$, $y = 2$ and $x = 0$ about the y-axis.



Solution. The graphs of all these curves are well known. It is clear from the diagram that we want to integrate wrt y. Also,

$$y = \ln x \iff x = e^y$$

and for each fixed y the radius of our disc is $x = e^y$ and so the cross sectional area is $A(y) = \pi(e^y)^2 = \pi e^2 y$.

$$\begin{aligned}
\text{Volume } &= \int_1^2 \pi e^{2y}\, dy \\
&= \frac{\pi}{2}\int_1^2 e^{2y}\, 2d(y) = \frac{\pi}{2}\int_1^2 e^{2y}\, d(2y) \\
&= \frac{\pi}{2}\left[e^{2y}\right]_1^2 = \frac{\pi}{2}(e^4 - e^2)
\end{aligned}$$

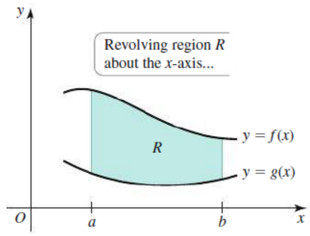
□

Washer Method

The volume of a $\boxed{\textbf{washer}}$ of height Δx and inside radius r and outside radius R is

$$\text{Volume of Washer } = \pi R^2 \Delta x - \pi r^2 \Delta x = \pi(R^2 - r^2)\Delta x,$$

where R and r are functions of x. □



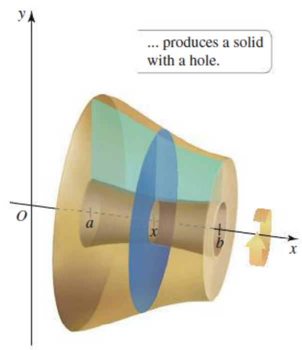


Illustration Choose a function and use the geogebra web app `https://www.geogebra.org/m/fd9kfvrh` for illustration.

Theorem (Washer Method about the x-Axis)**.** Let f and g be continuous functions with $f(x) \geq g(x) \geq 0$ on $[a, b]$. Let R be the region bounded by $y = f(x), y = g(x)$, and the lines $x = a$ and $x = b$. When R is revolved about the x-axis, the volume of the resulting solid of revolution is

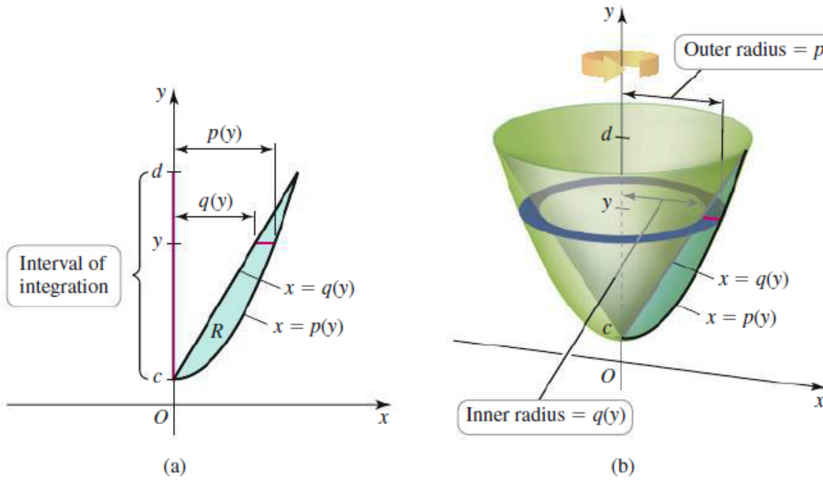
$$V = \int_a^b A(x)\, dx = \int_a^b \pi \left\{ [f(x)]^2 - [g(x)]^2 \right\}\, dx.$$

Example 4. The region R is bounded by the graphs of $f(x) = \sqrt{x}$ and $g(x) = x^2$ between $x = 0$ and $x = 1$. What is the volume of the solid that results when R is revolved about the x-axis?

Solution.

□

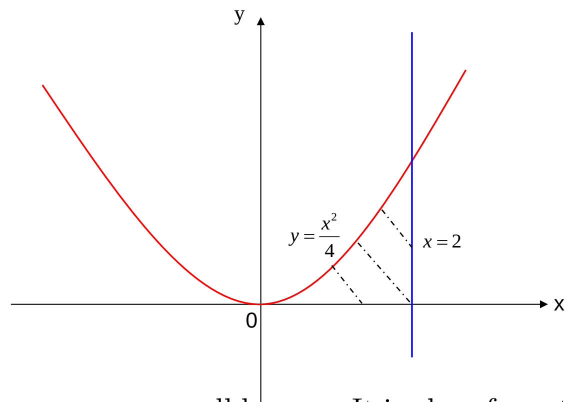
Again, what we learned about revolving regions about the x-axis applies to revolving regions about the y-axis.



Theorem (Washer Method about the y-Axis)**.** Let $p(y)$ and $q(y)$ be continuous functions with $p(y) \geq q(y) \geq 0$ on $[c, d]$. Let R be the region bounded by $x = p(y), x = q(y)$, and the lines $y = c$ and $y = d$. When R is revolved about the y-axis, the volume of the resulting solid of revolution is

$$V = \int_c^d A(y)\ dy = \int_c^d \pi \left\{ [p(y)]^2 - [q(y)]^2 \right\}\ dy.$$

Example. Find the volume of the solid obtained by rotating the region bounded by the curves $y = \frac{1}{4}x^2$, $x = 2$ and $y = 0$ about the y-axis. (What if the region is rotating about the x-axis?)



Solution. The graphs of all these curves are well known. It is clear from the diagram that we want to integrate wrt y. Also,

$$y = \frac{1}{4}x^2 \iff x = 2\sqrt{y}$$

and for each fixed y the inside radius $x = 2\sqrt{y}$, the outside radius is $x = 2$ and so the cross sectional area is $A(y) = \pi(2^2 - (2\sqrt{y})^2) = \pi(4 - 4y)$.

$$\text{Volume } = \int_0^1 \underbrace{\pi(4 - 4y)}_{=A(y)}\ dy = \pi\Big[4y - 2y^2\Big]_0^1 = 2\pi.$$

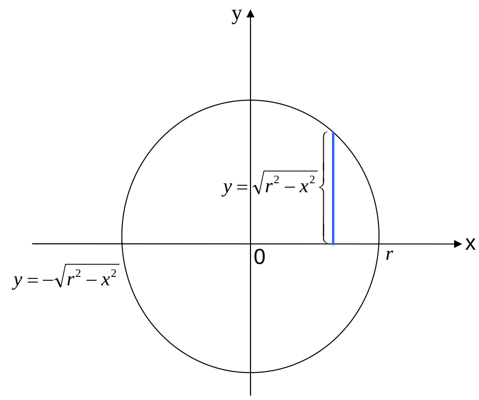
Example 5. Let R be the region in the first quadrant bounded by the graphs of $x = y^3$ and $x = 4y$. Which is greater, the volume of the solid generated when R is revolved about the x-axis or the y-axis?

Solution.

$\square$

More examples

Example. Find a formula for the volume of a sphere of radius r.



Solution. A sphere of radius r can be thought of as being the result of rotating the curve of a semi-circle $y = \sqrt{r^2 - x^2}$ about the x-axis.
For each fixed x between $x = -r$ and $x = r$, the cross section is a circle of radius $y = \sqrt{r^2 - x^2}$ and the area of this circle is given by

$$A(x) = \pi(\sqrt{r^2 - x^2})^2$$

and so the volume is

$$\begin{aligned} \text{Volume} &= \int_{-r}^{r} \pi(\sqrt{r^2 - x^2})^2 \, dx \\ &= \pi \int_{-r}^{r} (r^2 - x^2) \, dx = \Big[r^2 x - \frac{1}{3}x^3 \Big]_{-r}^{r} \\ &= \pi \Big[[r^3 - \frac{1}{3}r^3] - [r^2(-r) - \frac{1}{3}(-r)^3] \Big] = \pi \Big[2r^3 - \frac{2}{3}r^3 \Big] \\ &= \frac{4}{3}\pi r^3. \end{aligned}$$

Remark. A slight modification of this Example allows us to find the volume of the cap of a sphere.

Example (Reading). Find a formula for the volume of the cap of a sphere of radius r and height h.

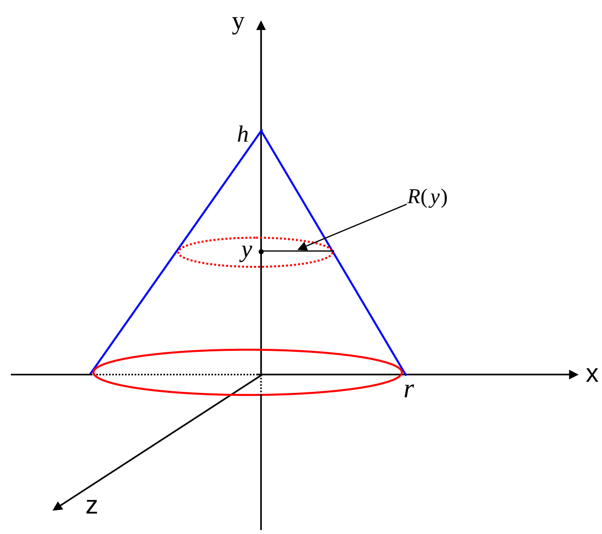
Solution.

$$\begin{aligned}\text{Volume} &= \int_{r-h}^{r} \pi(\sqrt{r^2 - x^2})^2 \, dx = \int_{r-h}^{r} \pi(r^2 - x^2) \, dx \\ &= \pi\left[r^2 x - \frac{1}{3}x^3\right]_{r-h}^{r} = \frac{\pi}{3}(3rh^2 - h^3).\end{aligned}$$

□

Remark. These examples involved either the disc method or the washer method. However, recall that the general method of finding volume by using known cross sectional is more general than this.

Example (Reading). Find a formula for the volume of a right circular cone whose base is a circle of radius r and those height is h.



Solution. We set up a coordinate system so that the base of the cone is on the xz-plan and the y-axis goes straight through the center. We integrate wrt y. By similar triangles, we see that as y varies from $y = 0$ to $y = h$, the radius $R(y)$ of the cross section circle at height y is found as follows

$$\frac{h}{r} = \frac{h-y}{R(y)} \Longrightarrow R(y) = \frac{r(h-y)}{h}$$

The cross sectional area is given by

$$A(y) = \pi R(y)^2 = \pi \left(\frac{r(h-y)}{h}\right)^2$$

and

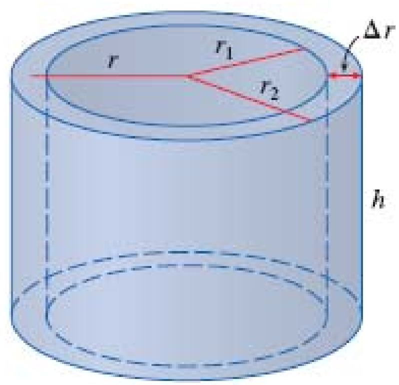
$$\begin{aligned}
\text{Volume } &= \int_0^h \pi \left(\frac{r(h-y)}{h}\right)^2 \, dy = \frac{\pi r^2}{h^2} \int_0^h (y^2 - 2hy + h^2) \, dy \\
&= \frac{\pi r^2}{h^2} \left[\frac{y^3}{3} - hy^2 + h^2 y\right]_0^h = \frac{\pi r^2}{h^2} \left[\frac{h^3}{3} - h^3 + h^3\right] \\
&= \frac{\pi r^2 h}{3}.
\end{aligned}$$

□

6.4 §6.4 Volumes By Cylindrical Shells

Several volume problems that we have seen are those of finding the volume of rotation of a region in the xy-plane about an axis. Sometimes the solution is simplified by the Method of Shells. This method uses the difference between the volumes of two concentric cylinders.

Find the volume of the cylinder shell with (1) $r_1 =$ inside radius; (2) $r_2 =$ outside radius; (3) $h =$ height; (4) Can no top or bottom.



Then, Volume of Shell

$$\begin{aligned}
&= V_2 - V_1 = \pi r_2^2 h - \pi r_1^2 h = \pi(r_2^2 - r_1^2)h \\
&= \pi h(r_2 + r_1)(r_2 - r_1) \text{ (sample point must be between } r_1 \text{ and } r_2) \\
&= 2\pi \frac{r_2 + r_1}{2} h(r_2 - r_1) \qquad \left(r_1 \le \frac{r_2 + r_1}{2} \le r_2\right) \\
&= 2\pi r h \Delta r, \text{ where } r \text{ is the average radius} \\
&= [\text{circumference}][\text{height}][\text{thickness}].
\end{aligned}$$

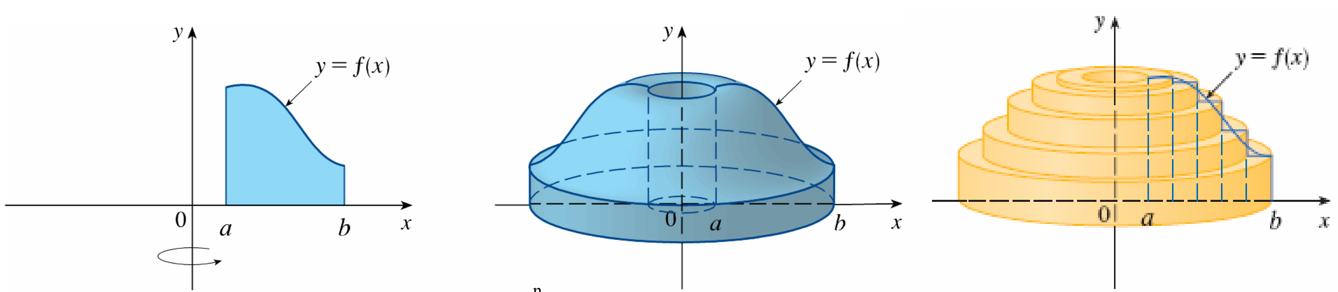
When we represent the height h as a function of r, we get

$$\text{Volume of Shell } = 2\pi r h(r)\ \Delta r.$$

Illustration Choose a function and use the geogebra web app `https://www.geogebra.org/m/ggfye7dj` for illustration of the Cylindrical Shell method.

Theorem. The volume of the solid in the following figure, obtained by rotating the region under the curve $y = f(x)$ (from a to b) about the y-axis, is

$$V = \int_a^b 2\pi x f(x)\, dx, \qquad \text{where } \ 0 \le a < b.$$



$$V = \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi \overline{x}_i f(\overline{x}_i) \Delta_x = \int_a^b 2\pi x f(x) dx$$

Example (reading). Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.

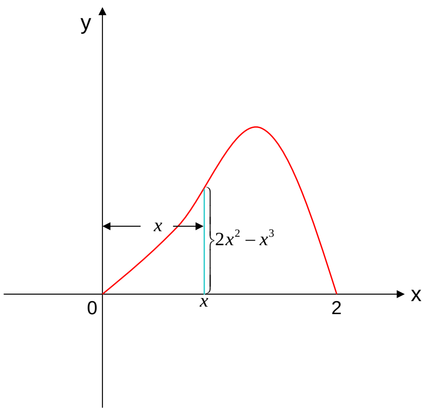
Solution. For the graph, we focus on the function $y = 2x^2 - x^3$:

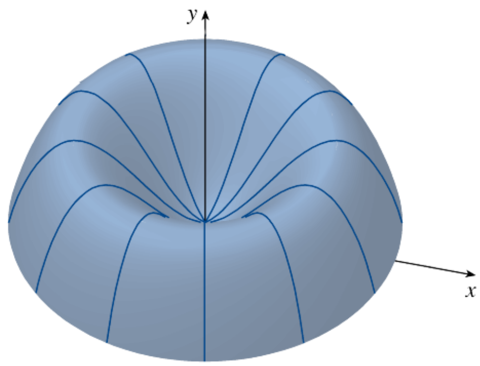
(1) Where is $y = 2x^2 - x^3$ neg. and pos.? $0 = 2x^2 - x^3 \implies 0 = x^2(2 - x)$;

(2) Where is $y = 2x^2 - x^3$ inc. and dec.? $0 = y' = x(4 - 3x) \implies x = 0, 4/3$;

(3) Where is $y = 2x^2 - x^3$ CU and CD? $y'' = 4 - 6x \implies$ CU if $x < 2/3$ and CD if $x > 2/3$.

With this information in hand we sketch the graph of $y = 2x^2 - x^3$.





Computer-generated picture of the solid

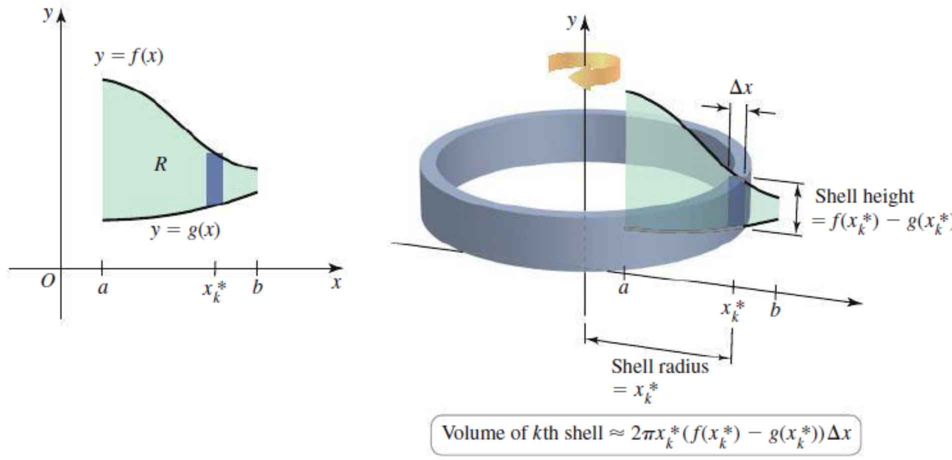
The intersection points with the x-axis are $(0, 0)$ and $(2, 0)$. We can see that the shell in the graph has radius x, circumference $2\pi x$ and height $f(x) = 2x^2 - x^3$. Thus, by the shell method, the volume is

$$V = \int_0^2 2\pi x(2x^2 - x^3)\, dx = 2\pi \int_0^2 x(2x^2 - x^3)\, dx$$
$$= 2\pi \left[\frac{x^4}{2} - \frac{x^5}{5}\right]_0^2 = 2\pi(8 - 32/5) = \frac{16}{5}\pi.$$

□

Q. Find the volume of the solid obtained when the region is rotated about the x-axis.

We can generalize this method to the case that the region R is bounded by two curves, $y = f(x)$ and $y = g(x)$, where $f(x) \geq g(x)$ on $[a, b]$.



Theorem (Volumn by the Shell Method)**.** Let f and g be continuous functions with $f(x) \geq g(x)$ on $[a, b]$. If R is the region bounded by the curves $y = f(x)$ and $y = g(x)$ between the lines $x = a$ and $x = b$, the volume of the solid generated when R is revolved about the y-axis, is

$$V = \int_a^b 2\pi x[f(x) - g(x)]\, dx, \qquad \text{where } \ 0 \leq a < b.$$

Remark 1. If $g(x) = 0$, the theorem is reduced to the previous result.

Remark 2. We can derive the result when a region R is revolved about the x-axis.

Remark 3. We can derive the result when a region R is revolved about other lines.

Example 1. (A sine bowl) Let R be the region bounded by the graph of $f(x) = sin(x^2)$, the x-axis, and the vertical line $x = \sqrt{\pi/2}$ (Figure 6.44). Find the volume of the solid generated when R is revolved about the y-axis.

Solution.

$\square$

Example 2. Let R be the region in the first quadrant bounded by the graph of $y = \sqrt{x-2}$ and the line $y = 2$. Find the volume of the solid generated when R is revolved about the x-axis.

Solution.

$\square$

Example 3. A cylindrical hole with radius r is drilled symmetrically through the center of a sphere with radius a, where $0 \leq r \leq a$. What is the volume of the remaining material?

Solution.

$\square$

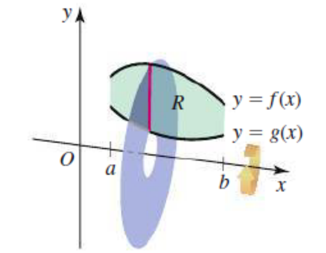
Summary:

- The disk method is just a special case of the washer method. For **solids of revolution**, either method can be used. In practice, one method usually produces an integral that is easier to evaluate than the other method.

- The method of cylindrical shells is a second way of computing volumes of **solids of revolution**. It frequently leads to simpler computations than does the method of cross sections (disc method or washer method).

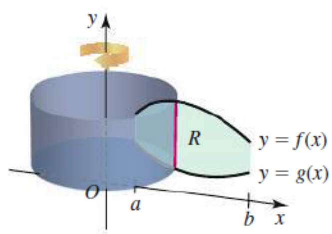
SUMMARY Disk / Washer and Shell Methods

Integration with respect to x



Disk / washer method about the x-axis
Disks / washers are *perpendicular* to the x-axis.

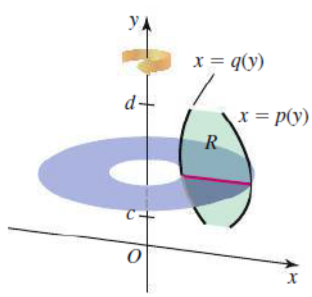
$$\int_a^b \pi(f(x)^2 - g(x)^2)\,dx$$



Shell method about the y-axis
Shells are *parallel* to the y-axis.

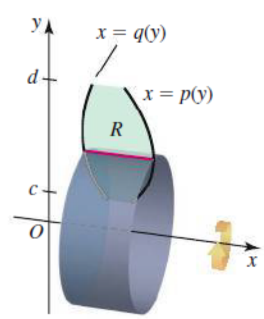
$$\int_a^b 2\pi x(f(x) - g(x))\,dx$$

Integration with respect to y



Disk/washer method about the y-axis
Disks / washers are *perpendicular* to the y-axis.

$$\int_c^d \pi(p(y)^2 - q(y)^2)\,dy$$



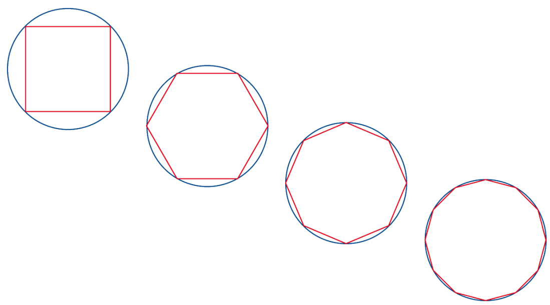
Shell method about the x-axis
Shells are *parallel* to the x-axis.

$$\int_c^d 2\pi y(p(y) - q(y))\,dy$$

6.5 §6.5 The length of Curves

This section is left for your own **reading**.

As in all applications of definite integrals we have seen, the key is to obtain an appropriate Riemann Sum approximation. We treat the case of a function $y = f(x)$ having a continuous derivative $f'(x)$ on the interval $[a, b]$.



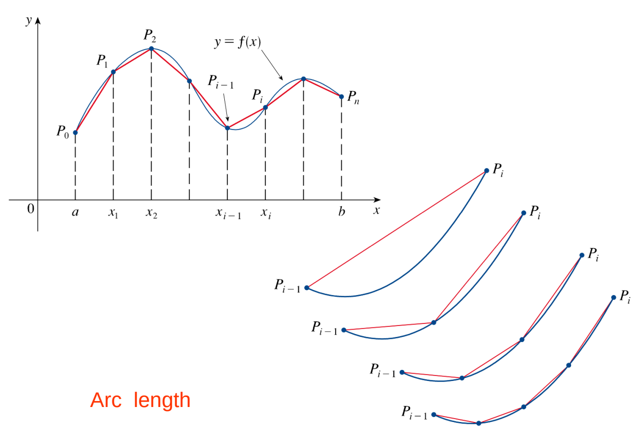
Circumference is the limit of lengths of inscribed polygons

Derivation:

Let $y = f(x)$ be a function with continuous derivative $f'(x)$ on the interval $[a, b]$. To find the Riemann Sum we use the following outline.

1. Partition the interval $[a, b]$ into n equal parts of length $\Delta x = \frac{b-a}{n}$:

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$



2. Form a polygonal path to approximate the graph of the curve $y = f(x)$ by connecting the points by straight line segments.

$$\begin{aligned} P_0 &= (x_0, f(x_0)) \\ P_1 &= (x_1, f(x_1)) \\ &\vdots \\ P_{n-1} &= (x_{n-1}, f(x_{n-1})) \\ P_n &= (x_n, f(x_n)) \end{aligned}$$

3. Find an expression for the distance $|P_{i-1}P_i|$ from P_{i-1} to P_i.

$$\begin{aligned} |P_{i-1}P_i| &= \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} \\ &= \sqrt{(\Delta x)^2 + (f(x_i) - f(x_{i-1}))^2}. \end{aligned}$$

4. Apply the Mean Value Theorem for Derivatives to f on the interval $[x_{i-1}, x_i]$ to get

$$f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1}) = f'(x_i^*)\Delta x$$

for some x_i^* between x_{i-1} and x_i. Thus we have

$$\begin{aligned} |P_{i-1}P_i| &= \sqrt{(\Delta x)^2 + (f'(x_i^*)\Delta x)^2} = \sqrt{(\Delta x)^2 + (f'(x_i^*)\Delta x)^2} \\ &= \sqrt{1 + (f'(x_i^*))^2}\Delta x. \end{aligned}$$

5. We now have

$$\text{Arc Length } = \int_a^b \sqrt{1 + (f'(x))^2}\, dx = \lim_{n\to\infty} \sum_{i-1}^{n} \sqrt{1 + (f'(x_i^*))^2}\Delta x.$$

Since $f'(x)$ is assumed to be continuous, $\sqrt{1 + [f'(x)]^2}$ is continuous and this definite integral has a finite value, i.e. $\int_a^b \sqrt{1 + [f'(x)]^2}\, dx$ exists.

$\square$

Theorem (The Arc Length Formula)**.** If f' is continuous on $[a, b]$, then the length of the curve $y = f(x)$, $a \leq x \leq b$, is given by

$$\boxed{L = \int_a^b \sqrt{1 + [f'(x)]^2}dx.}$$

Remark. If the curve is given by the equation $x = g(y), c \leq y \leq d$ and $g'(y)$ is continuous, then the length of the curve is given by

$$\boxed{L = \int_c^d \sqrt{1 + [g'(y)]^2}dy.}$$

Example 1. Find the length of the curve $f(x) = x^{3/2}$ between $x = 0$ and $x = 4$ (Figure 6.56).

Solution.

$\square$

Example 2. Find the length of the curve $f(x) = 2e^x + \frac{1}{8}e^{-x}$ on the interval $[0, \ln 2]$.

Solution.

$\square$

Example 3. Confirm that the circumference of a circle of radius r is $2\pi r$.

Solution.

$\square$

Example 4. Find the length of the curve $f(x) = x^2$ on the interval $[0, 2]$.

Solution.

$\square$

Example 5.[Arc length for $x = g(y)$] Find the length of the curve $y = f(x) = x^{2/3}$ between $x = 0$ and $x = 8$ (Figure 6.58).

Solution.

$\square$

Example 6.[Arc length for $x = g(y)$] Find the length of the curve $y = f(x) = \ln(x + \sqrt{x^2 - 1})$ on the interval $[1, \sqrt{2}]$ (Figure 6.59).

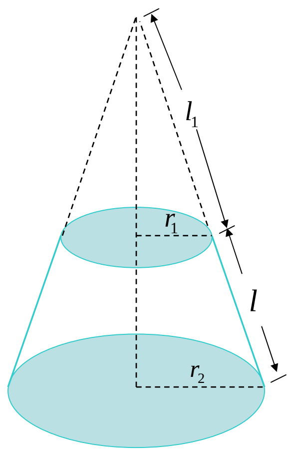
Solution.

$\square$

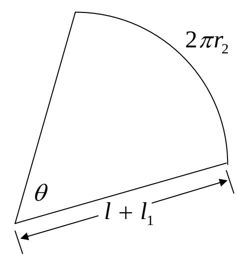
6.6 §6.6 Surface Area

This section is left for your own **reading**.

In this section, we find the surface area of a **surface of revolution** when the curve $y = f(x)$, $a \le x \le b$, is rotated about the x-axis. We approximate such a surface with frustums of a cone. Visualize a right circular cone of slant height $\ell_1 + \ell$ having base radius r_2 and cut the top off of the cone by removing a smaller cone of slant height ℓ_1. The remaining geometric object is a frustum.



Frustum from a cone



Surface area of the cone

Homework: Please read the following derivation. We first find a suitable representation of the surface area of a frustum in terms of slant height ℓ and average radius $\frac{r_1+r_2}{2}$.

1. First observe that the surface area of a frustum is the difference of the surface areas of two cones.

2. Take our larger cone and cut it straight up the side from the base to the point making a cut of length $\ell_1 + \ell$, and lay it flat to obtain a sector of a circle whose radius is $\ell_1 + \ell$ the slant height of the cone. Observe that the circular perimeter is $2\pi r_2$, the perimeter of the base circle of our cone.

3. Observe that the area of the cone equals the area of this sector and use a geometric ratio to express the area of this sector of a circle in terms of the slant

height $\ell_1 + \ell$ and the radius r_2.

Therefore,

$$\begin{aligned}
\text{Area of Cone } &= \text{Area of Sector of Circle} \\
&= \text{Fraction of Circle} \times \text{Area of Circle} \\
&= (\frac{\theta}{2\pi})(\pi(\ell_1+\ell)^2) = \frac{1}{2}(\ell_1+\ell)^2\theta \\
&= \frac{1}{2}(\ell_1+\ell)^2\big(2\pi\frac{r_2}{\ell_1+\ell}\big) \left\{ \tfrac{\theta}{2\pi} = \tfrac{2\pi r_2}{2\pi(\ell+\ell_1)} \qquad \text{Geo. Ratio} \right. \\
&= \pi(\ell_1+\ell)r_2 \\
&= \pi \times \text{Slant Height of Cone} \times \text{Radius of Base circle}
\end{aligned}$$

4. Use similar triangles to express the surface area of a frustum in terms of slant height ℓ and average radius $\frac{r_1+r_2}{2}$.

$$\begin{aligned}
\frac{\ell_1}{r_1} = \frac{\ell_1+\ell}{r_2} &\implies \ell_1 r_2 = (\ell_1+\ell)r_1 \\
&\implies (r_2-r_1)\ell_1 = r_1\ell
\end{aligned}$$

$$\begin{aligned}
\text{Area of Frustum } &= \text{Area of Big Cone} - \text{Area o Small Cone} \\
&= \pi(\ell+\ell_1)r_2 - \pi\ell_1 r_1 \\
&= \pi r_1\ell + \pi\ell r_2 = \pi\ell(r_1+r_2) \\
&= \pi\ell\big(2\frac{r_1+r_2}{2}\big) = 2\pi\ell\Big(\underbrace{\frac{r_1+r_2}{2}}_{\text{ave. radius}} \Big) \\
&= 2\pi\ell r = 2 \times \pi \times \text{slant ht.} \times \text{ave. radius}
\end{aligned}$$

It is now time to bring in the partition of our interval $[a, b]$ and revise the formula

$$\boxed{\text{Area of Frustum } = 2\pi r\ell}$$

using some of the same ideas that were used in arriving at our formula for arc length.

1. Partition the interval $[a, b]$ into n equal parts -

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

with each part having length $\Delta x = \frac{b-a}{n}$.

2. Form a polygonal path approximate the graph of the curve $y = f(x)$ by connecting the points by straight line segments.

$$\begin{aligned}
P_0 &= (x_0, f(x_0)) \\
P_1 &= (x_1, f(x_1)) \\
&\vdots \\
P_{n-1} &= (x_{n-1}, f(x_{n-1}) \\
P_n &= (x_n, f(x_n))
\end{aligned}$$

Each line segment $P_{i-1}P_i$ generates a frustum when rotated.

3. Apply the Mean Value Theorem for Derivatives to $f(x_i) - f(x_{i-1})$ to get a formula for the slant height

$$\begin{aligned}
\text{Slant Height} =& |P_{i-1}P_i| \\
&= \sqrt{(\Delta x)^2 + (f'(x_i^*)\Delta x)^2} \\
&= \sqrt{1 + (f'(x_i^*))^2}\Delta x
\end{aligned}$$

We now have that the surface area of a surface of revolution of the curve $y = f(x) \geq 0$, $a \leq x \leq b$, with continuous derivative is given by

$$\begin{aligned}
\text{Surface Area } &= \lim_{n\to\infty} \sum_{i=1}^{n} \text{Surface area of the } i^{th} \text{ Frustum} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1 + (f'(x_i^*))^2}\Delta x \\
&= \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\ dx = \int_a^b 2\pi y\sqrt{1 + (y')^2}\ dx
\end{aligned}$$

Theorem (Area of a Surface of Revolution). Suppose the function f is nonnegative and has a continues derivative. Then the **surface area** of the surface obtained by rotating the curve $y = f(x), a \le x \le b$, about the x-axis is

$$\boxed{S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}dx = \int_a^b 2\pi y \; ds,}$$

where $ds = \sqrt{1 + (f'(x))^2} \; dx$.

Remark 1. Let's see how we have the second formula. Note that if we define an arc length function by

$$s(x) = \int_a^x \sqrt{1 + (f'(t))^2} \; dt$$

then by the Fundamental Theorem of Calculus Part I, $ds = \sqrt{1 + (f'(x))^2} \; dx$ and so

$$\boxed{\text{Surface Area } = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2} \; dx = \int_a^b 2\pi y \; ds.}$$

Remark 2. If the curve is given by the equation $x = g(y), c \le y \le d$, then the formula for the surface area (rotating the curve about the y-axis) is given by

$$\boxed{S = \int_c^d 2\pi x\sqrt{1 + [g'(y)]^2}dy = \int_c^d 2\pi g(y)\sqrt{1 + [g'(y)]^2}dy.}$$

Example 1. The graph of $f(x) = 2\sqrt{x}$ on the interval $[1, 3]$ is revolved about the x-axis. What is the area of the surface generated (Figure 6.65)?

Solution.

$\square$

Example 2. Find the surface area of a sphere of radius r.

Solution.

□

Remark. Then we can find the surface area of a **spherical zone**.

Example 3. The curved surface of a funnel is generated by revolving the graph of $y = f(x) = x^3 + \frac{1}{12x}$ on the interval $[1, 2]$ about the x-axis (Figure 6.67). Find the surface area.

Solution.

□

Example 4. Consider the function $y = \ln\left(\dfrac{x + \sqrt{x^2 - 1}}{2}\right)$. Find the area of the surface generated when the part of the curve between the points $(\frac{5}{4}, 0)$ and $(\frac{17}{8}, \ln 2)$ is revolved about the y-axis (Figure 6.68).

Solution.

$\square$

More examples

Example.(reading) Find the surface area of the surface obtained when

$$y = 1 - x^2, \qquad 0 \le x \le 1$$

is rotated about the y-axis.

Solution. First, we find x as a function of y and the value of the radical $\sqrt{1+(\frac{dx}{dy})^2}$.

$$y = 1 - x^2 \implies x = f(y) = \sqrt{1-y} \text{ since } x \ge 0$$

$$\begin{aligned}\sqrt{1+(\frac{dx}{dy})^2} &= \sqrt{1+(-\frac{1}{2}\frac{1}{\sqrt{1-y}})^2}\\ &= \sqrt{1+\frac{1}{4(1-y)}}\end{aligned}$$

since $\frac{dx}{dy}$ is non-positive $x = f(y)$ is defined on the interval $0 \le y \le 1$ and

$$\begin{aligned}f(y)\sqrt{1+(f'(y))^2} &= \sqrt{1-y}\sqrt{1+(\frac{dx}{dy})^2}\\ &= \sqrt{1-y}\sqrt{1+\frac{1}{4(1-y)}}\\ &= \sqrt{1-y+\frac{1}{4}} = \sqrt{\frac{5}{4}-y} = \frac{1}{2}\sqrt{5-y}\end{aligned}$$

$$\begin{aligned}\text{Surface Area} &= \int_0^1 2\pi f(y)\sqrt{1+(f'(y))^2}\, dy\\ &= \int_0^1 2\pi\frac{1}{2}\sqrt{5-y}\, dy = \Big[-\frac{2\pi}{3}(5-y)^{\frac{3}{2}}\Big]_0^1\\ &= \frac{2\pi}{3}\Big((5)^{\frac{3}{2}} - (4)^{\frac{3}{2}}\Big).\end{aligned}$$

□

Example.(reading) Show that surface area of **Gabriel's Horn** is infinite while its volume is finite. Gabriel's Horn is defined to be that surface obtained when

$$y = \frac{1}{x}, \qquad 1 \le x$$

is rotated about the x-axis.

Solution. First, we find the value of the radical $\sqrt{1+(y')^2}$.

$$y = x^{-1} \implies y' = -x^{-2} \implies \sqrt{1+(y')^2} = \sqrt{1+(-x^{-2})^2} = \sqrt{1+x^{-4}}$$

Now, we want to show that

$$\int_1^\infty 2\pi x^{-1}\sqrt{1+x^{-4}}\ dx \text{ is divergent}$$

$$\text{or equivalently that } \int_1^\infty x^{-1}\sqrt{1+x^{-4}}\ dx \text{ is divergent.}$$

We do this by the Comparison Test. Notice that

$$x^{-1}\sqrt{1+x^{-4}} > \frac{1}{x}$$

Now,

$$\int_1^\infty \frac{dx}{x} \text{ is divergent (p-Test with } p = 1)$$

$$\therefore \int_1^\infty x^{-1}\sqrt{1+x^{-4}}\ dx \text{ is divergent (Comparison Test)}$$

Finally,

$$\begin{aligned}\text{Volume of Gabriel's Horn} &= \int_1^\infty \pi\left(\frac{1}{x}\right)^2\ dx = \lim_{t\to\infty}\int_1^t \pi\left(\frac{1}{x}\right)^2\ dx \\ &= \lim_{t\to\infty}\Big[-\pi x^{-1}\Big]_1^t = \lim_{t\to\infty}\Big[-\pi t^{-1}+\pi\Big]_1^t = \pi.\end{aligned}$$

Contents

Chapter 10, for the most part, is dedicated to the study of convergence and divergence of infinite series. Since this involves the study of the convergence and divergence of the sequence of **partial sums**. We begin with the study of sequences.

10.2 §10.2 Sequences

Definition. A **sequence** can be thought of as a list of numbers written in a definite order

$$\{a_n\} = \{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, \dots\}$$

The a_1 is called the first **term**, a_2 is called the second term, etc. and in general a_n is called the nth term.

The subscript n in a_n is called an **index**, and it indicates the order of terms in the sequence. The choice of a starting index is arbitrary, but sequences usually begin with $n = 0$ or $n = 1$.

We are dealing with infinite sequence so that each a_n has a successor a_{n+1}.

Examples.

A sequence may be defined with an explicit formula of the form $a_n = f(n)$, for $n = 1, 2, 3, \dots$. See (a), (b) and (c).

(a)

$$\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \underbrace{\frac{n}{n+1}}_{n^{th}\text{ term}}, \ldots\right\}$$

(b) Although a sequence is a list of numbers, **the sequence does not have to start at** $n = 1$.

$$\left\{\sqrt{n-3}\right\}_{n=3}^{\infty} = \left\{0, 1, \sqrt{2}, \sqrt{3} \ldots, \sqrt{n-3}, \ldots\right\}$$

(c)

$$\left\{\cos\frac{n\pi}{6}\right\}_{n=0}^{\infty} = \left\{1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, \ldots, \underbrace{\cos\frac{n\pi}{6}}_{n^{th}\text{ term}}, \ldots\right\}$$

(d) Sometimes sequences are defined **recursively**. For example, $a_{n+1} = f(a_n)$.

Fibonacci sequence is one such example

$$f_1 = 1 \text{ and } f_2 = 1$$

$$f_n = f_{n-1} + f_{n-2},\ n \geq 3$$

$$\{f_n\}_{n=1}^{\infty} = \{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$$

To generate a term in this sequence, we need to know the two terms just before it. What number comes after 21 in the Fibonacci sequence?

Definition (**Limit** of a Sequence)**.** A sequence $\{a_n\}$ has a real number L as a limit, written as

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty$$

provided the limit exists. In this case, we say that $\{a_n\}$ **converges or is convergent.** Otherwise, we say that the sequence sequence **diverges** or is **divergent**.

Remark.(Rigorous definition of limit) $\lim_{n\to\infty} a_n = L \iff$ for every positive number ϵ, there is a corresponding N_ϵ such that $|L - a_n| < \epsilon$ for all $n \geq N_\epsilon$. □

Example. The sequence $\{a_n = (-1)^n\} = \{-1, 1, -1, 1, -1, 1, \dots\}$ is divergent, since the terms of the sequence oscillate between 1 and -1. So the limit $\lim_{n\to\infty}(-1)^n$ does not exist.

The following theorem is obvious.

Theorem. If $\{a_n\}$ is a convergent sequence and $\lim\limits_{n\to\infty} a_n = L$, then

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} a_{n+1}.$$

Proof. $\lim\limits_{n\to\infty} a_n = L \iff$ the values of the sequence get closer and closer to L as n get larger and larger $\iff \lim_{n\to\infty} a_{n+1} = L$. $\square$

The following result allows us for example to use **L'Hospital's Rule** if helpful.

Theorem (Thm 10.1 Function Value Theorem) If one can associate a function $f(x)$ with the sequence $\{a_n\}$ in the following sense

$$a_n = f(n),$$

then

$$\lim_{x\to\infty} f(x) = L \implies \lim_{n\to\infty} a_n = L.$$

$\square$

Remark. The sequence above $a_n = f(n)$ is a function of n. It is not continuous.

Example. If $r > 0$, then

$$\lim_{x\to\infty} \frac{1}{x^r} = 0 \implies \lim_{n\to\infty} \frac{1}{n^r} = 0$$

since $\frac{1}{n^r} = f(n)$, where $f(x) = \frac{1}{x^r}$.

Example. Find

$$\lim_{n\to\infty} \frac{\ln n}{n}.$$

Solution. Note if we define $f(x) = \frac{\ln x}{x}$, then $f(n) = \frac{\ln n}{n}$ and so we need only find L where $\lim\limits_{x\to\infty} f(x) = L$ and use the Function Value Theorem. We do this using L'Hospital's Rule.

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$$

$$\therefore \quad \lim_{x\to\infty} \frac{\ln n}{n} = 0.$$

Remark. For Python code dealing with sequences, see **Sequence.ipynb**.

Recall the following four typical limits in Calculus I.

$$\begin{aligned}\lim_{x\to 0}\frac{\sin x}{x} &=1\\ \lim_{x\to\infty} x^{\frac{1}{x}} &=1\\ \lim_{x\to\infty}(1+\frac{1}{x})^x &=e\\ \lim_{x\to\infty}(1-\frac{1}{x})^x &=\frac{1}{e}\end{aligned}$$

Because of the correspondence between limits of sequences and limits of functions at infinity, we have the following properties that are analogous to those of functions in Chapter 2.

Limit Laws for Sequences: Let $\{a_n\}$ and $\{b_n\}$ be two convergent sequences and c be any constant. Then

$$\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n \quad \text{and} \quad \lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty}(a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\cdot\left(\lim_{n\to\infty} b_n\right)$$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if } \lim_{n\to\infty} b_n \neq 0$$

$$\lim_{n\to\infty} a_n^p = \left(\lim_{n\to\infty} a_n\right)^p \quad \text{if } p>0 \text{ and } a_n>0.$$

□

Example 1. Determine whether the sequence is convergent or divergent. Find the limit if it is convergent.

a. $a_n = \dfrac{3n^3}{n^3+1}$

Solution. We set up a function to use the Function Value Theorem. Set $f(x) = \frac{3x^3}{x^3+1}$ and we can use L'hospital's Rule to find $\lim\limits_{x\to\infty}\frac{3x^3}{x^3+1}$

$$\lim_{x\to\infty}\frac{3x^3}{x^3+1} = \lim_{x\to\infty}\frac{9x^2}{3x^2} = \lim_{x\to\infty}\frac{9}{3} = \lim_{x\to\infty} 3 = 3 \implies \lim_{x\to\infty}\frac{3n^3}{n^3+1} = \lim_{x\to\infty} a_n = 3.$$

□

b. $b_n = \left(\dfrac{n+5}{n}\right)^n$

Solution.

$\square$

c. $c_n = n^{1/n}$

Solution.

$\square$

Terminology for Sequences

Definition. • A sequence $\{a_n\}$ is called **increasing** if $a_k < a_{k+1}$ for all $k \geq 1$, i.e.

$$a_1 < a_2 < a_3 < \cdots$$

• A sequence $\{a_n\}$ is called **nondecreasing** if $a_k \leq a_{k+1}$ for all $k \geq 1$, i.e.

$$a_1 \leq a_2 \leq a_3 \leq \cdots$$

• A sequence $\{a_n\}$ is called **decreasing** if $a_k > a_{k+1}$ for all $k \geq 1$, i.e.

$$a_1 > a_2 > a_3 > \cdots$$

• A sequence $\{a_n\}$ is called **nonincreasing** if $a_k \geq a_{k+1}$ for all $k \geq 1$, i.e.

$$a_1 \geq a_2 \geq a_3 \geq \cdots$$

• A sequence $\{a_n\}$ is called **monotonic** if it is either nonincreasing or nondecreasing (it moves in one direction).

Example. Show that the sequence $\{\frac{n}{n^2+1}\}$ is decreasing.

Solution.

Method I. Set $f(x) = \frac{x}{x^2+1}$. It suffices to show that $f(x)$ is decreasing.

$$f'(x) = \frac{x^2+1-x(2x)}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2} < 0 \text{ for all } x > 1.$$

Method II.

$$\begin{aligned} \frac{n}{n^2+1} > \frac{n+1}{(n+1)^2+1} &\iff n[(n+1)^2+1] > (n+1)(n^2+1) \\ &\iff n^3+2n^2+2n > n^3+n^2+n+1 \\ &\iff n^2+n > 1 \end{aligned}$$

The last inequality is true for all $n \geq 1$ and so the first inequality is true for all $n \geq 1$. Therefore, $\{a_n = \frac{n}{n^2+1}\}$ is decreasing.

Definition. • A sequence $\{a_n\}$ is **bounded above** if there is some number M such that

$$a_n \leq M \qquad \text{for all } n.$$

• A sequence $\{a_n\}$ is **bounded below** if there is some number m such that

$$m \leq a_n \qquad \text{for all } n.$$

• If the sequence $\{a_n\}$ is both bounded above and bounded below, then the sequence $\{a_n\}$ is said to be **bounded**.

Monotonic Sequence Theorem. Every bounded monotonic sequence is convergent.

□

Example 84 (Reading) Suppose (show) that the sequence given recursively by

$$a_0 = 1 \quad \text{and} \quad a_{n+1} = \frac{1}{2}a_n + 2 \quad \text{for } n = 0, 1, 2, 3, \ldots$$

is monotonic and bounded. Find $\lim_{n\to\infty} a_n$.

Theorem (The Squeeze Theorem). If $\{a_n\}$ and $\{c_n\}$ are two convergent sequences with the same limit, $\lim\limits_{n\to\infty} a_n = L = \lim\limits_{n\to\infty} c_n$, and $a_n \le b_n \le c_n$ for $n > n_0$, then

$$\lim_{n\to\infty} b_n = L.$$

$\square$

Example 4. Find the limit of the sequence $a_n = \dfrac{\cos n}{n^2+1}$.

Solution.

$\square$

$\boxed{\textbf{Theorem.} \quad \text{If } \lim\limits_{n\to\infty} |a_n| = 0, \text{ then } \lim\limits_{n\to\infty} a_n = 0.}$

Remark. This Theorem does NOT say that if $\{|a_n|\}$ is convergent, then $\{a_n\}$ is convergent. $\square$

Example. Determine whether the following sequence is convergent or divergent.

$$a_n = \frac{(-1)^n n^3}{n^3+2n^2+1}.$$

Solution. Notice

$$|a_n| = \left|\frac{(-1)^n n^3}{n^3+2n^2+1}\right| = \frac{n^3}{n^3+2n^2+1}$$

and we can use repeated applications of L'Hospital's Rule

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n^3}{n^3+2n^2+1} = \lim_{n\to\infty} \frac{3n^2}{3n^2+4n} = \lim_{n\to\infty} \frac{6n}{6n+4} = \lim_{n\to\infty} \frac{6}{6} = 1.$$

Therefore, $\{|a_n|\}$ is convergent. But

$$\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} \frac{(-1)^{2n}((2n)^3}{(2n)^3+2(2n)^2+1} = 1,$$

$$\lim_{n\to\infty} a_{2n+1} = \lim_{n\to\infty} \frac{(-1)^{2n+1}((2n+1)^3}{(2n+1)^3+2(2n+1)^2+1}$$

$$= \lim_{n\to\infty} (-1)\frac{((2n+1)^3}{(2n+1)^3+2(2n+1)^2+1} = -1.$$

Since the even terms converge to 1 and the odd terms converge to -1, there is no limit. Therefore, $\{a_n\}$ is divergent. That is, **absolute convergence does not mean convergence**. $\square$

Example. Determine wether the following sequence is convergent or divergent.

$$a_n = \frac{\sin 2n}{1+\sqrt{n}}.$$

Solution. We use The Squeeze Theorem to show the convergence of the absolute sequence $\{|a_n|\}$ and then use the above Theorem to get the convergence of $\{a_n\}$.

$$0 \le \left|\frac{\sin 2n}{1+\sqrt{n}}\right| \le \frac{1}{1+\sqrt{n}} \text{ and } \lim_{n\to\infty} \frac{1}{1+\sqrt{n}} = 0$$

$$\therefore \quad \{b_n = \left|\frac{\sin 2n}{1+\sqrt{n}}\right|\} \text{ is convergence with limit} = 0 \text{ by Squeeze Thm.}$$

$$\therefore \quad \{a_n = \frac{\sin 2n}{1+\sqrt{n}}\} \text{ is convergent to 0.}$$

$\square$

Theorem - Continuity and Limits: If $\lim_{n\to\infty} a_n = L$ and $f(x)$ is continuous at L, then $\lim_{n\to\infty} f(a_n) = f(L) = f(\lim_{n\to\infty} a_n)$.

Example. Determine whether the following sequence is convergent or divergent.

$$a_n = (2^{3n+1})^{\frac{1}{n}}.$$

Solution.

$$a_n = (2^{3n+1})^{\frac{1}{n}} = (2^{3n}2)^{\frac{1}{n}} = (2^{3n})^{\frac{1}{n}}(2)^{\frac{1}{n}} = 8(2)^{\frac{1}{n}}$$

Since 2^x is continuous at $x = 0$, we see that

$$\lim_{n\to\infty} a_n = 8(2)^{\lim_{n\to\infty} \frac{1}{n}} = 8 \cdot 2^0 = 8.$$

Example. Determine the limit of $\{a_n = \cos \frac{\pi}{n}\}$.

Solution. Recall that $f(x) = \cos x$ is continuous on the real line. Also, notice that

$$\lim_{n\to\infty} \frac{\pi}{n} = 0 = L \qquad \therefore \lim_{n\to\infty} \cos \frac{\pi}{n} = \lim_{n\to\infty} f(\frac{\pi}{n}) = f(L) = \cos 0 = 1.$$

$\square$

Geometric Sequences

Theorem.(**Geometric Sequences**) The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ ($\{r^n\}$ is monotonic if $r > 0$ and oscillates if $r < 0$) and divergent for all other values of r. Moreover,

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

□

Example 3. Determine the limits of the following sequences.

a. $a_n = 5(0.6)^n - \dfrac{1}{3^n}$ **b.** $b_n = \dfrac{2n^2 + n}{2^n(3n^2 - 4)}$

Solution.

□

Growth Rates of Sequences

The results in Chapter 4 (section 4.7 L'Hôpital's Rule) about the relative growth rates of functions are applied to sequences.

Theorem (Growth Rates of Sequences)**.** The following sequences are ordered according to increasing growth rates as $n \to \infty$; that is, $\{a_n\}$ appears before $\{b_n\}$ in the list, then $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\lim_{n\to\infty} \frac{b_n}{a_n} = \infty$;

$$\{\ln^q n\} << \{n^p\} << \{n^p \ln^r n\} << \{n^{p+s}\} << \{b^n\} << \{n!\} << \{n^n\}$$

The ordering applies for positive real numbers p, q, r, s, and $b > 1$.

Example 6. Compare growth rates of sequences to determine whether the following sequences converge.

a. $\left\{\dfrac{\ln n^{10}}{0.00001n}\right\}$ **b.** $\left\{\dfrac{n^8 \ln n}{n^{8.001}}\right\}$ **c.** $\left\{\dfrac{n!}{10^n}\right\}$

Solution.

$\square$

10.3 §10.3 Infinite Series

An infinite series (or simply a series) is an expression of the form

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = \underbrace{\sum a_n}_{\text{short form}} .$$

We are trying to **add all the terms in a sequence** $\{a_n\}$. Naturally, when one is adding infinitely many numbers one ask what exactly does this mean.

$$1 + 2 + 3 + \cdots + n + \cdots > \text{ any finite number?}$$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots \text{ Is this equal to a finite number?}$$

These are the type of questions we will be asking. To answer these questions we need to develop some mathematical tools.

Definition (Infinite Series)**.** Given a series $\sum a_n$, we define the N^{th} **partial sum** to be

$$S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + \cdots + a_N, \quad N = 1, 2, \ldots .$$

The series $\sum a_n$ is said to be **convergent** provided the *sequence of partial sums* $\{S_N\}$ converges and in this case, the value of the series is

$$\sum_{i=1}^{\infty} a_n = L \iff \lim_{N\to\infty} S_N = L.$$

The series is said to be **divergent** if it is not convergent, i.e. $\{S_N\}$ is divergent.

Theorem - Harmonic Series Diverges The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, since

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{>\frac{1}{2}} + \underbrace{\frac{1}{5} \cdots + \frac{1}{8}}_{>\frac{1}{2}} + \underbrace{\frac{1}{9} \cdots + \frac{1}{16}}_{>\frac{1}{2}} + \underbrace{\frac{1}{16} \cdots + \frac{1}{32}}_{>\frac{1}{2}} + \cdots$$

tells us that we are adding infinitely many terms all greater than $\frac{1}{2}$.

Remark See Part I of Python code **Series.ipynb** for checking convergence or divergence of a series.

Theorem - Geometric Series Test. A Geometric series is one that can be written in the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots \text{ with } a \neq 0.$$

This series is convergent if $|r| < 1$ with

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \text{ when } |r| < 1$$

and it is divergent if $|r| \geq 1$. (Notice that we must write the series in the form given in order to find the sum.)

Proof (Reading). Let S_N denote the N^{th} partial sum.

$$S_N = \sum_{n=1}^{N} ar^{n-1}$$

We need to study $\lim_{n\to\infty} S_N$. Certainly, if $r = 1$, then $S_N = \sum_{i=1}^{N} ar^{n-1} = \sum_{i=1}^{N} a = Na$ and $\lim_{n\to\infty} S_N = \pm\infty$. Now, we assume that $r \neq 1$ Then

$$\begin{aligned}
S_N &= \sum_{n=1}^{N} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots + ar^{N-1} \\
rS_N &= r\sum_{n=1}^{N} ar^{n-1} = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^N \\
S_N - rS_N &= (a + ar + ar^2 + ar^3 + \cdots + ar^{N-1}) - (ar + ar^2 + ar^3 + ar^4 + \cdots + ar^N) \\
&= (a - ar^N) \\
(1-r)S_N &= (a - ar^N) \implies S_N = \frac{a - ar^N}{1-r} = \frac{a(1-r^N)}{1-r}
\end{aligned}$$

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{a(1-r^N)}{1-r} \begin{cases} \frac{a}{1-r} & \text{if } |r| < 1 \\ \text{divergent} & \text{if } |r| \geq 1 \end{cases}$$

We are using the **Theorem.** The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r. Moreover,

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

□

Example 1. Evaluate the following geometric series or state that the series diverges.

a. $\sum_{k=0}^{\infty} 1.1^k$ **b.** $\sum_{k=0}^{\infty} e^{-k}$ **c.** $\sum_{k=2}^{\infty} 3(-0.75)^k$

Solution.

$\square$

Example 2. Write $1.0\overline{35} = 1.0353535\cdots$ as a geometric series and express its value as a fraction.

Solution.

$\square$

More examples.

Example (Reading). Determine if the geometric series

$$\sum_{n=1}^{\infty} \frac{10^n}{(-9)^{n-1}}$$

is convergent or divergent and if convergent find its sum.

Solution. First, we write in the form

$$\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=1}^{\infty} 10\frac{10^{n-1}}{(-9)^{n-1}} = \sum_{n=1}^{\infty}(10)\left(\underbrace{\frac{-10}{9}}_{=r}\right)^{n-1}$$

so that we can apply the Geometric Series Test. $r = \frac{-10}{9}$ has absolute value greater than 1 and so the series is divergent. □

Example (Reading). Determine if the geometric series

$$\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$$

is convergent or divergent and if convergent find its sum.

Solution. First, we write in the form

$$\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=1}^{\infty} \left[\frac{1}{4} \cdot \frac{(-3)^{n-1}}{4^{n-1}}\right] = \sum_{i=1}^{\infty} \left[\frac{1}{4}\left(-\frac{3}{4}\right)^{n-1}\right]$$

so that we can apply the Geometric Series Test. Since $r = \frac{-3}{4}$ has absolute value less than 1 this series converges and

$$\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n} = \frac{a}{1-r} = \frac{\frac{1}{4}}{1-\frac{-3}{4}} = \frac{1}{4}\frac{1}{\frac{4+3}{4}} = \frac{1}{7}.$$

□

Telescoping Series

With geometric series, we carried out the entire evaluation process by finding a formula for the sequence of partial sums and evaluating the limit of the sequence. Not many infinite series can be subjected to this sort of analysis. With another class of series, called **telescoping series**, it can also be done. Here are some examples.

Example 3. Evaluate the following series.

a. $\sum_{k=1}^{\infty}\left(\cos\frac{1}{k}-\cos\frac{1}{k+1}\right)$ **b.** $\sum_{k=3}^{\infty}\frac{1}{(k-2)(k-1)}$

Solution.

$\square$

Theorem (Limit Laws for Series). Let $\sum a_n$ and $\sum b_n$ be two series.

(i) If $\sum a_n$ is convergent, then $\sum ca_n = c\sum a_n$;

(ii) If $\sum a_n$ and $\sum b_n$ are convergent, $\sum(a_n \pm b_n)$ is convergent and $\sum(a_n \pm b_n) = \sum a_n \pm \sum b_n$.

(iii) If $\sum a_n$ is convergent and $\sum b_n$ is divergent $\Rightarrow \sum(a_n + b_n)$ is divergent.

(iv) If both $\sum(a_n + b_n)$ and $\sum a_n$ are convergent, then so is $\sum b_n$.

These follow immediately from the corresponding laws for sequence. □

Remark 1. If $\sum a_n$ is divergent, $\sum ca_n$ is divergent for any $c \neq 0$.

Remark 2. If M is a positive integer, then $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=M}^{\infty} a_n$ either both converge or both diverge.

Remark 3. $\sum a_n$ and $\sum b_n$ are both divergent $\not\Rightarrow \sum(a_n + b_n)$ is divergent.

Example 4. Evaluate the following series.

a. $\sum\limits_{k=2}^{\infty}\left(\dfrac{1}{3^k} - \dfrac{2}{3^{k+2}}\right)$ **b.** $\sum\limits_{k=1}^{\infty}\left(5(\dfrac{2}{3})^k - \frac{7^{k-1}}{6^k}\right)$

Solution.

□

Example. Determine if the series

$$\sum_{n=1}^{\infty} \frac{1+3^{n-1}}{2^n}$$

is convergent or divergent and if convergent find its sum.

Solution. We first recognize this as being the sum of two geometric series. One of which is convergent and the other is divergent.

$$\sum_{n=1}^{\infty} \frac{1+3^{n-1}}{2^n} = \sum_{n=1}^{\infty} \frac{1}{2} \cdot \frac{1+3^{n-1}}{2^{n-1}} = \underbrace{\sum_{n=1}^{\infty} \frac{1}{2} \cdot \frac{1}{2^{n-1}}}_{\text{convergent}} + \underbrace{\sum_{n=1}^{\infty} \frac{1}{2} \cdot \frac{3^{n-1}}{2^{n-1}}}_{\text{divergent}}$$

$$\therefore \qquad \sum_{n=1}^{\infty} \frac{1+3^{n-1}}{2^n} \text{ is divergent.}$$

Example. Determine if the series

$$\sum_{n=1}^{\infty} \left[\frac{3}{5^n} + \frac{2}{n}\right]$$

is convergent or divergent and if convergent find its sum.

Solution. We first notice recognize this series as the sum of a convergent and a divergent series

$$\sum_{n=1}^{\infty} \left[\frac{3}{5^n} + \frac{2}{n}\right] = \underbrace{\sum_{n=1}^{\infty} \left[\frac{3}{5} \cdot \frac{1}{5^{n-1}}\right]}_{\text{convergent}} + 2 \underbrace{\sum_{n=1}^{\infty} \left[\frac{1}{n}\right]}_{\text{divergent}}$$

$$\therefore \qquad \sum_{n=1}^{\infty} \left[\frac{3}{5^n} + \frac{2}{n}\right] \text{ is divergent.}$$

Example. Be careful. Recall the convergent telescoping series example done above.

$$\underbrace{\sum_{n=1}^{\infty} \frac{1}{n(n+1)}}_{\text{convergent}} = \sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{1}{n+1}\right] = \underbrace{\sum_{n=1}^{\infty} \left[\frac{1}{n}\right]}_{\text{divergent}} + \underbrace{\sum_{n=1}^{\infty} \left[\frac{-1}{n+1}\right]}_{\text{divergent}}$$

Remark. This is what we mentioned before, the sum/difference of two divergent series may not be divergent.

10.4 §10.4 The Divergence and Integral Tests

The Divergence Test

Theorem - n^{th}- Term Test for Divergence If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim\limits_{n\to\infty} a_n = 0$.

- If $\lim\limits_{n\to\infty} a_n$ either does not exist or does not equal 0, then the series

$$\sum_{n=1}^{\infty} a_n \text{ is divergent.}$$

Proof. We prove the contrapositive. That is, we assume that the series converges to value S and show that $\sum_{n=1}^{\infty} a_n$ exits and equals 0.

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \Big[a_1 + \cdots + a_n - (a_1 + \cdots + a_{n-1})\Big] \\
&= \lim_{n\to\infty} \Big[\sum_{i=1}^{n} a_i - \sum_{i=1}^{n-1} a_i\Big] = \lim_{n\to\infty} \Big[S_n - S_{n-1}\Big] \\
&= \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0.
\end{aligned}$$

Remark. This is a **test for divergence only** and

$$\lim_{n\to\infty} a_n = 0 \text{ does } \textbf{not} \text{ imply that } \sum_{n=1}^{\infty} a_n \text{ is convergent.}$$

As can be seen from the Harmonic Series:

$$\underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{\text{divergent}} \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} = 0.$$

□

Example. Show that

$$\sum_{n=1}^{\infty} \ln(1+\frac{1}{n})$$

is divergent but $\lim\limits_{n\to\infty} \ln(1+\frac{1}{n}) = 0$

Solution. The function $\ln x$ is continuous at $x = 1$, and $\lim_{n\to\infty} 1+\frac{1}{n} = 1$ so by Continuity and Limits, we have

$$\lim_{n\to\infty} \ln(1+\frac{1}{n}) = \ln(\lim_{n\to\infty}(1+\frac{1}{n}))) = \ln 1 = 0$$

Finally, we have a **telescoping divergent series** as can be seen from:

$$\begin{aligned}
\ln(1+\frac{1}{n}) &= \ln(\frac{n+1}{n}) = \ln(n+1) - \ln n \\
S_N &= \sum_{n=1}^{N} \ln(1+\frac{1}{n}) = \sum_{n=1}^{N} \Big[\ln(n+1) - \ln n\Big] \\
&= \Big[\ln(2) - \ln 1\Big] + \Big[\ln(3) - \ln 2\Big] + \Big[\ln(4) - \ln 3\Big] \\
&\quad + \cdots + \Big[\ln(N) - \ln(N-1)\Big] + \Big[\ln(N+1) - \ln N\Big] \\
&= \ln(N+1)
\end{aligned}$$

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \ln(N+1) = \infty \therefore \text{ The series } \textstyle\sum_{n=1}^{\infty} \ln(1+\frac{1}{n}) \text{ is divergent.}$$

□

Example 1. Determine whether the following series diverge or state that the Divergence Test is inconclusive

a. $\sum\limits_{k=0}^{\infty} \dfrac{k}{k+1}$ **b.** $\sum\limits_{k=1}^{\infty} \dfrac{1+3^k}{2^k}$ **c.** $\sum\limits_{k=1}^{\infty} \dfrac{1}{k}$ **d.** $\sum\limits_{k=1}^{\infty} \frac{1}{k^2}$

Solution.

□

The Integral Test

In this subsection we relate the convergence of series with the convergence of improper integrals. The p-Test for improper integrals is useful here.

Recall: The p-Test for Improper Integrals.

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \text{diverges, if } p \le 1 \\ \text{converges, if } p > 1 \end{cases}$$

Theorem (Integral Test). If the tail of a sequence $\{a_n\}$ can be given by the values of a function $f(x)$ which is continuous, positive, and decreasing on $[a, \infty)$ then

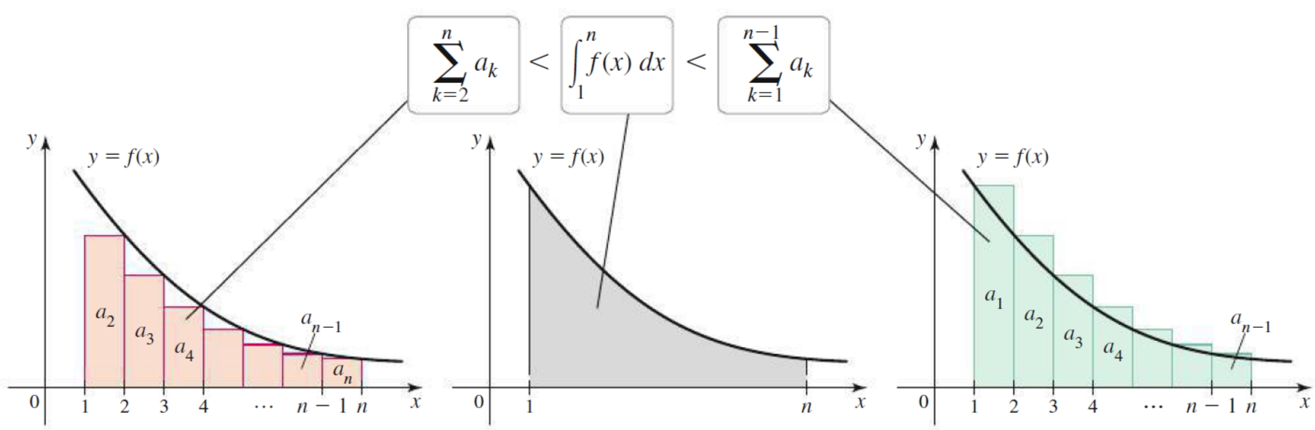
$$\sum_{n=1}^\infty a_n \text{ converges} \iff \int_a^\infty f(x)\,dx \text{ converges.}$$

That is:

(i) If $\sum_{n=1}^\infty a_n$ is convergent , then $\int_a^\infty f(x)\,dx$ is convergent.

(ii) If $\int_a^\infty f(x)\,dx$ converges, then $\sum_{n=1}^\infty a_n$ is convergent.

$$\sum_{k=2}^n a_k < \int_1^n f(x)\,dx < \sum_{k=1}^{n-1} a_k.$$



We explain why this is true using the following Example.

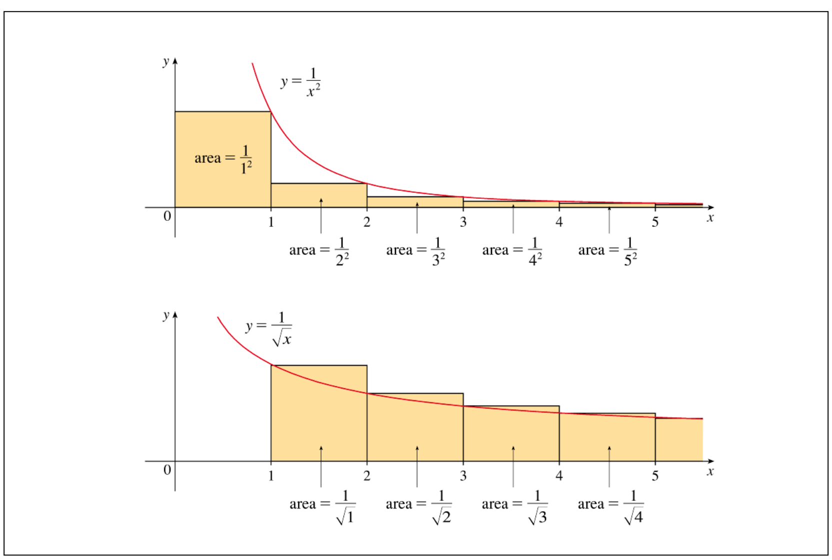
Example. Let us consider two series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ and the corresponding functions $f(x) = \frac{1}{x^2}$ and $g(x) = \frac{1}{\sqrt{x}}$. Both of $f(x)$ and $g(x)$ are continuous, positive, and decreasing on $[1, \infty)$. Moreover, by the p-Test for Improper Integrals we have

$$\int_1^{\infty} \frac{1}{x^2}\, dx \text{ is convergent, } (p = 2 > 1) \text{ while}$$
$$\int_1^{\infty} \frac{1}{\sqrt{x}}\, dx \text{ is divergent } (p = \tfrac{1}{2} < 1).$$

Therefore, by the Integral Test,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ is convergent, but}$$
$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ is divergent.}$$

To see what is going on here, we graph $y = f(x)$ and $y = g(x)$ and interpret the sequence on the graph of its corresponding function.



Comparing series with integrals

Example 2. (Applying the Integral Test) Determine whether the following series converge.

a. $\sum_{k=1}^{\infty} \frac{k}{k^2+1}$ **b.** $\sum_{k=3}^{\infty} \frac{1}{\sqrt{2k-5}}$ **c.** $\sum_{n=1}^{\infty} \frac{1}{k^2+4}$

Solution.

□

In general, the Integral Test combined with The p-Test for Improper Integrals give us:

Theorem (p -Test for Series). The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

□

Example 3. (Using the p-series test) Determine whether the following series converge or diverge.

a. $\sum_{k=1}^{\infty} k^{-3}$ **b.** $\sum_{k=3}^{\infty} \frac{1}{\sqrt[4]{k^3}}$ **c.** $\sum_{k=4}^{\infty} \frac{1}{(k-1)^2}$

Solution.

$\square$

Example. Determine whether the series

$$1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \cdots$$

is convergent or divergent.

Solution.

$$1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}} \quad \text{and} \quad p = \frac{3}{2} \implies \text{convergence.}$$

$\square$

Example. (Reading) Determine the values of p for which the series

$$\sum_{n=3}^{\infty} \frac{1}{n \ln n[\ln(\ln n)]^p}$$

is converges.

Solution. Set $u = \ln(\ln x)$ and $f(u) = u^{-p}$. Then $f(x) > 0$ provided $u > 0$ which is true when $x \geq 3$. Moreover,

$$f'(u) = (-p)u^{-p-1} < 0 \text{ when } x \geq 3.$$

Thus, $f(u)$ is continuous, positive and decreasing when $x \geq 3$. Further, $du = \frac{1}{\ln x} \cdot \frac{1}{x} dx$.

$$\int_3^{\infty} \frac{1}{x \ln x[\ln(\ln x)]^p}\, dx = \int_{\ln(\ln 3)}^{\infty} \frac{1}{u^p}\, du \text{ convergent} \iff p > 1$$

Therefore,

$$\sum_{n=3}^{\infty} \frac{1}{n \ln n[\ln(\ln n)]^p} \text{ is convergent} \iff p > 1$$

$\square$

Estimating the Value of Infinite Series

This sub-section is left for your own reading.

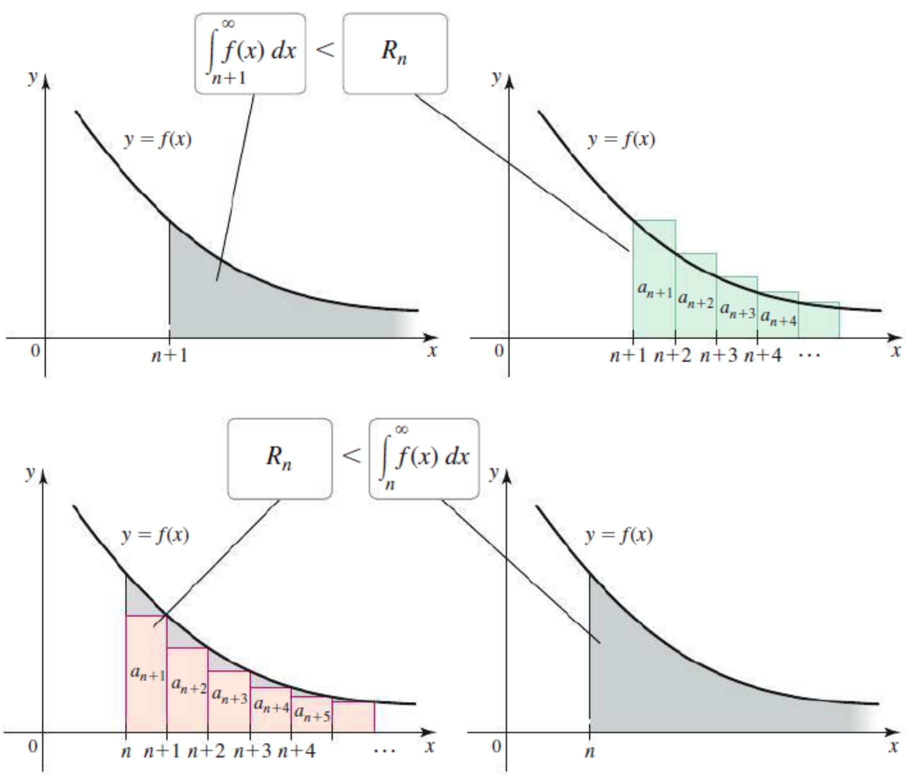
The Integral Test is powerful in its own right, but it comes with an added bonus. It can be used to estimate the value of a convergent series with positive terms. We define the **remainder** to be the error in approximating a convergent series by the sum of its first n terms; that is,

$$R_n = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

Theorem (Thm 10.13 Remainder Estimate for the Integral Test). Suppose that $f(k) = a_k$, where f is **continuous, positive and decreasing** for $x \geq n$ and $S = \sum_{k=1}^{\infty} a_k$ is convergent. Then, if $S_n = \sum_{k=1}^{n} a_k$ is used to approximate S we find the error $R_n = S - S_n$ in this estimate is remainder R_n and we have

$$\int_{n+1}^{\infty} f(x)\, dx < S - S_n < \int_{n}^{\infty} f(x)\, dx$$
$$\Longrightarrow S_n + \int_{n+1}^{\infty} f(x)\, dx < S < S_n + \int_{n}^{\infty} f(x)\, dx.$$

$\square$



Example 4. (Approximating a p-series)

a. How many terms of the convergent p-series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ must be summed to obtain an approximation that is within 10^{-3} of the exact value of the series?

b. Find an approximation to the series using 50 terms of the series.

Solution.

$\square$

10.5 §10.5 Comparison Tests

As in the case of convergence and divergence of improper integrals there is a Comparison Test for convergence and divergence of series.

Theorem. Suppose that $\sum a_n$ and $\sum b_n$ are series of positive terms, i.e. $a_n > 0$ and $b_n > 0$. Then

(i) $\sum b_n$ convergent and $a_n \leq b_n \implies \sum a_n$ convergent, and

(ii) $\sum b_n$ divergent and $a_n \geq b_n \implies \sum a_n$ divergent.

Proof. (Reading) Let

$$S_N = \sum_{n=1}^{N} a_n, \qquad T_N = \sum_{n=1}^{N} b_n, \qquad T = \sum_{n=1}^{\infty} b_n$$

(i) Since $a_n > 0$ and $b_n > 0$, the sequence $\{S_N\}$ and $\{T_N\}$ are increasing sequences.

$$\begin{aligned} a_n \leq b_n &\implies S_N \leq T_N \leq T \\ &\implies \{S_N\} \text{ is monotonic and bounded by } T \\ &\implies \{S_N\} \text{ converges } \implies \sum a_n \text{ converges.} \end{aligned}$$

(ii) Since $a_n > 0$ and $b_n > 0$, the sequence $\{S_N\}$ and $\{T_N\}$ are increasing sequences. Now, $\{T_N\}$ is increasing and divergent and hence $\{T_N\}$ goes to infinity as N goes to infinity.

$$a_n \geq b_n \implies S_N \geq T_N \to \infty \implies S_N \to \infty \implies \sum a_n \text{ is divergent.}$$

□

Remark. To Apply the Comparison Test, we need the convergence or divergence of some series to compare with. Here are two examples that we have seen.

- $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.
- $\sum ar^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \geq 1$.

The problem in using the Comparison Test is to come up with a known series to compare with. This to a large extent requires experience. **Do the problems!**

Example. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{3n+2}{n(n+1)}$$

is convergent or divergent.

Solution. We use the Comparison Test.

$$\underbrace{\frac{3n+2}{n(n+1)}}_{>0} = \frac{3+2/n}{n+1} > \frac{3}{n+1} > \underbrace{\frac{3}{2n}}_{>0}$$

Now, the Harmonic Series is divergent and so we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n} \text{ divergent } &\implies \frac{3}{2}\sum_{n=1}^{\infty} \frac{1}{n} \text{ divergent} \\ &\implies \sum_{n=1}^{\infty} \frac{3n+2}{n(n+1)} \text{ divergent.} \end{aligned}$$

□

Example 1. (Using the Comparison Test) Determine whether the following series converge.

a. $\sum_{k=1}^{\infty} \frac{k^3}{2k^4-1}$ **b.** $\sum_{k=2}^{\infty} \frac{\ln k}{k^3}$

Solution.

□

Remark. Sometimes it is difficult to get the inequality to work for the Comparison Test and it is easier to use the Limit Comparison Test. For example, $\sum\limits_{n=1}^{\infty} \frac{n^3}{2n^4+1}$

Theorem - The Limit Comparison Test. Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series of positive terms, i.e. $a_n > 0$ and $b_n > 0$.

1. If C is a finite positive number ($0 < C < \infty$) and
$$\lim_{n\to\infty} \frac{a_n}{b_n} = C > 0$$
then either both series, $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, converge or both series diverge.
2. If $C = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges
3. If $C = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof (Case 1). (Reading) Let m and M be such that $m < C < M$ and since $\frac{a_n}{b_n}$ is close to C we have

$$m < C < M \implies m < \frac{a_n}{b_n} < M \implies mb_n < a_n < Mb_n$$

for all large values of n, i.e. there exists some K such that $mb_n < a_n < Mb_n$ for all $n \geq K$. Also, by the Comparison Test we have

$$\begin{aligned} \sum_{n=1}^{\infty} b_n \text{ convergent} &\implies \sum_{n=1}^{\infty} Mb_n \text{ convergent} \implies \sum_{n=1}^{\infty} a_n \text{ convergent} \\ \sum_{n=1}^{\infty} b_n \text{ divergent} &\implies \sum_{n=1}^{\infty} mb_n \text{ divergent} \implies \sum_{n=1}^{\infty} a_n \text{ divergent} \end{aligned}$$

□

Example. Determine whether the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{2n+3}$$

is convergent or divergent.

Solution. We looking at such a series ask yourself what is the dominant part of the term. In this case, the 2 doesn't do much and the 3 doesn't do much and so we think to compare with the divergent Harmonic Series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{2n+3}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{2n+3} = \frac{1}{2} \text{ by L'H}$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{2n+3} \text{ is divergent since } \sum_{n=1}^{\infty} \frac{1}{n} \text{ is.}$$

□

Example. Determine wether the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n!}{n^n}$$

is convergent or divergent.

Solution. By the p -Test $\sum_{n=1}^{\infty} \frac{2}{n^2}$ is convergent and for $n \geq 3$

$$\frac{n!}{n^n} = \left(\frac{1}{n}\right)\left(\frac{2}{n}\right)\left(\frac{3}{n}\right)\left(\frac{4}{n}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{n}{n}\right) < \frac{1}{n}\frac{2}{n}\cdot 1\cdots 1 = \frac{2}{n^2}$$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n!}{n^n} \text{ is convergent.}$$

□

Example. Determine whether the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}} = \sum_{n=1}^{\infty} \frac{1}{n \cdot n^{\frac{1}{n}}}$$

is convergent or divergent.

Solution. Because

$$\lim_{n\to\infty} \ln(n^{\frac{1}{n}}) = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1}{n} = 0 \text{ by L'H}$$

$$\therefore \lim_{n\to\infty} n^{\frac{1}{n}} = \lim_{n\to\infty} e^{\ln(n^{\frac{1}{n}})} = e^{\lim_{n\to\infty} \ln(n^{\frac{1}{n}})} = e^0 = 1$$

seems to be telling us that maybe we should ignore $\frac{1}{n^{\frac{1}{n}}}$ and apply the limit Comparison Test using the divergent Harmonic Series $\sum \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n \cdot n^{\frac{1}{n}}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n \cdot n^{\frac{1}{n}}} = \lim_{n\to\infty} \frac{1}{n^{\frac{1}{n}}} = 1$$

Therefore,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}} = \sum_{n=1}^{\infty} \frac{1}{n \cdot n^{\frac{1}{n}}}$$

diverges by the limit Comparison Test. □

Example 2. (Using the Limit Comparison Test) Determine whether the following series converge.

a. $\sum_{k=1}^{\infty} \frac{5k^4 - 2k^2 + 3}{2k^6 - k + 5}$ **b.** $\sum_{k=1}^{\infty} \frac{\ln k}{k^2}$

Solution.

□

10.6 §10.6 Alternating Series Test

For the most part we have been testing series of positive terms for convergence or divergence but now we force the series terms to alternate between positive and negative. It turns out that this test is easy to apply from the point of view that we do not have to compute an improper integral nor do we have to come up with some series to compare it with.

Definition. If $\{a_n\}$ is a sequence of **positive** terms, then the series

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \cdots$$

or

$$\sum_{n=1}^{\infty}(-1)^{n}a_n = -a_1 + a_2 - a_3 + a_4 - a_5 + a_6 - \cdots$$

is called an **alternate series**. □

Theorem - Alternating Series Test. The alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges provided

(i) $a_n \geq a_{n+1} > 0$ for all n (nonincreasing), and

(ii) $\lim\limits_{n\to\infty} a_n = 0$.

(Since we need only test the **Tail** for convergence or divergence, the question is from some point on do the terms **alternate in sign**?; is the corresponding sequence decreasing or nonincreasing and does the limit of this sequence equal 0?.)

Proof (Reading). We first consider the subsequence of the partial sums corresponding to even N, i.e. $N = 2K$. We show $a_1 > S_{2K+2} > S_{2K}$.

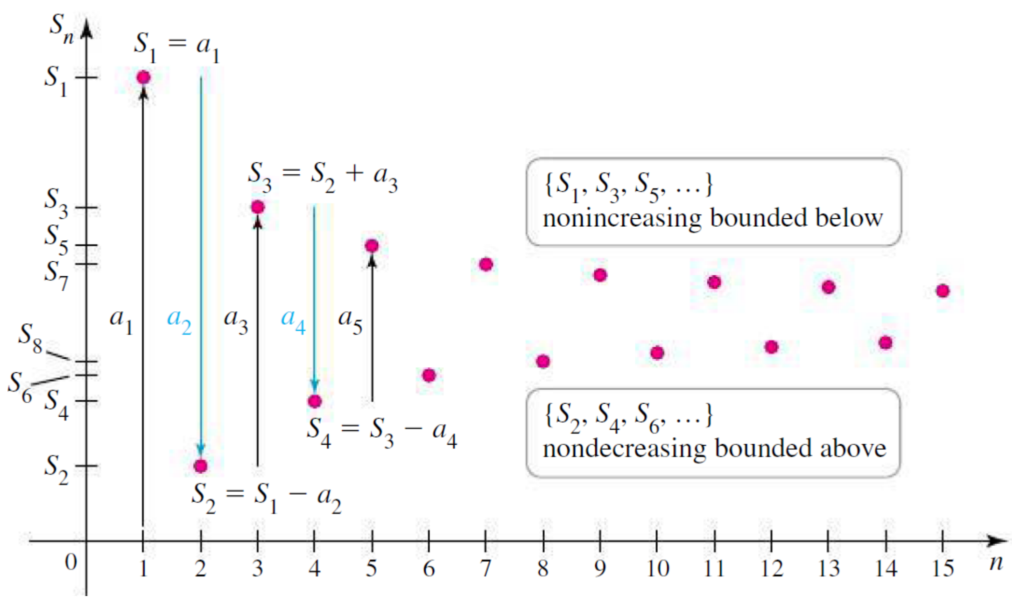
$$S_{2K} = a_1 - \underbrace{(a_2 - a_3)}_{\geq 0} - \underbrace{(a_4 - a_5)}_{\geq 0} - \cdots - \underbrace{(a_{2K-2} - a_{2K-1})}_{\geq 0} - \underbrace{a_{2K}}_{>0} < a_1$$

$$S_{2K+2} = a_1 - \underbrace{(a_2 - a_3)}_{\geq 0} - \cdots - \underbrace{(a_{2K-2} - a_{2K-1})} - \underbrace{(a_{2K} - a_{2K+1})}_{\geq 0} - \underbrace{a_{2K+2}}_{>0}$$

$$= S_{2K} + \underbrace{(a_{2K+1} - a_{2K+2})}_{\geq 0}$$

$\therefore \quad a_1 > S_{2K+2} > S_{2K}$ and $\{S_{2K}\}$ is bounded monotonic increasing.

$\therefore \quad \{S_{2K}\}$ has a limit. $\quad \lim_{K\to\infty} S_{2K} = S$



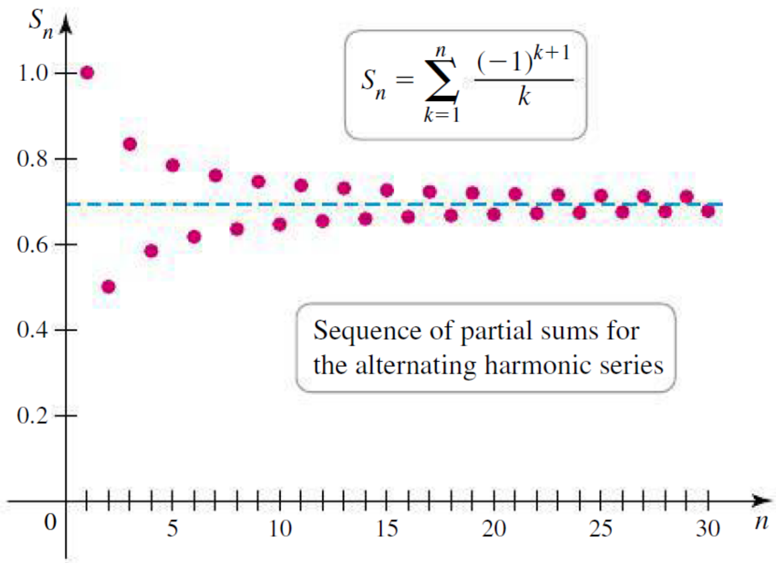
Now,

$$\lim_{K\to\infty} S_{2K+1} = \lim_{K\to\infty} [S_{2K} + a_{2K+1}] = \lim_{K\to\infty} [S_{2K}] + \lim_{K\to\infty} [a_{2K+1}] = S + 0 = S$$

What all this means is that both the odd partial sums and the even partial sums get closer and closer to S as N goes to infinity. Therefore. all the partial sums get closer and closer to S as N goes to infinity and S is the limit of the partial sums. $\square$

Recall that the Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Theorem (Alternating Harmonic Series)**.** The alternating Harmonic Series $\sum_{n+1}^{\infty} \frac{(-1)^{n=1}}{n}$ converges.



Proof.

□

Example 1. Determine whether the following series converge or diverge.

a. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

Solution.

□

b.$2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \cdots$

Solution.

$\square$

c. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n}$

Solution. We want to apply the Alternating Series Test to show convergence. The question is from some point on: do the terms alternate in sign; is the corresponding sequence decreasing; and does the limit of this sequence equal 0.

Certainly, the terms in the sequence $\{(-1)^{n-1}\frac{\ln n}{n}\}_{n=2}^{\infty}$ alternate in sign.

$$D_x(\frac{\ln x}{x}) = \frac{1-\ln x}{x^2} < 0 \text{ for } x > e \implies \frac{\ln n}{n} \text{ is decr. for } n > 3, \text{ and}$$

$$\lim_{x\to\infty} \frac{\ln x}{x} \overset{L'H}{=} \lim_{x\to\infty} \frac{1}{x} = 0 \implies \lim_{n\to\infty} \frac{\ln n}{n} = 0.$$

$$\sum_{n=2}^{\infty} (-1)^{n+1} \frac{\ln n}{n} \text{ is convergent.}$$

$\square$

Example. Test the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{\ln(n+4)}$$

for convergence or divergence.

Solution. Since we have an alternating series we need only show

(i) $\frac{1}{\ln(n+4)} \geq \frac{1}{\ln(n+5)} > 0$ for all n, and

(ii) $\lim\limits_{n\to\infty} \frac{1}{\ln(n+4)} = 0$.

For (i), we first see that since $n+5 > 1$, $\ln(n+5) > 0$ and hence $\frac{1}{\ln(n+5)} > 0$.

$$\frac{1}{\ln(n+4)} \geq \frac{1}{\ln(n+5)} \iff \underbrace{\ln(n+5) \geq \ln(n+4)}_{\text{true since ln is an incr. func.}}$$

(ii) Since $\lim\limits_{n\to\infty} \ln(n+4) = \infty$ we know that $\lim\limits_{n\to\infty} \frac{1}{\ln(n+4)} = 0$. Therefore, by the Alternating Series Test

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{\ln(n+4)} \text{ converges.}$$

$\square$

Example. Test the series

$$\sum_{n=1}^{\infty}(-1)^n \cos\frac{\pi}{n}$$

for convergence or divergence.

Solution. It is very important to be sure that the Test that you want to apply actually applies. The Alternating Series Test is a Test for Convergence only. For $n \geq 3$, the sequence $\{(-1)^n \cos\frac{\pi}{n}\}_{n=3}^{\infty}$ does alternate sign but the Alternating Series Test does not apply because

$$\lim_{n\to\infty}\cos\frac{\pi}{n} = \cos\left(\lim_{n\to\infty}\frac{\pi}{n}\right) = \cos 0 = 1.$$

[Recall - n^{th}**- Term Test for Divergence.**

$$\lim_{n\to\infty} a_n \neq 0 \implies \sum_{n=1}^{\infty} a_n \text{ is divergent.}$$]

Now,

$$\lim_{n\to\infty}[(-1)^n \cos\frac{\pi}{n}] \neq 0 \iff \lim_{n\to\infty}\left|(-1)^n \cos\frac{\pi}{n}\right| \neq 0$$

$$\iff \underbrace{\lim_{n\to\infty}\cos\frac{\pi}{n}}_{=1} \neq 0 \therefore \quad \sum_{n=1}^{\infty}(-1)^n \cos\frac{\pi}{n} \text{ is divergent.}$$

$\square$

Example. Test the series

$$\sum_{n=1}^{\infty}\left(-\frac{n}{5}\right)^n$$

for convergence or divergence.

Solution. This series diverges by the n-Term Test, since $\lim_{n\to\infty}\left(-\frac{n}{5}\right)^n \neq 0$ because

$$\lim_{n\to\infty}\left|-\frac{n}{5}\right|^n = \lim_{n\to\infty}\left(\frac{n}{5}\right)^n = \lim_{n\to\infty}\frac{n^n}{5^n} = \infty$$

$\square$

Remainders in Alternating Series (Reading)

Theorem - Alternate Series Estimate of Remainder If $S = \sum_{n=1}^{\infty}(-1)^{n+1}a_n$ is convergent alternating series with

(i) $a_n \geq a_{n+1} > 0$ for all n, and

(ii) $\lim_{n\to\infty} a_n = 0$,

and $S_N = \sum_{n=1}^{N}(-1)^{n+1}a_n$, then the absolute remainder $|R_N| = |S - S_N|$ is bounded by a_{N+1}, ie.

$$|R_N| = |S - S_N| \leq a_{N+1}.$$

In other words, the magnitude of the remainder is less than or equal to the magnitude of the first neglected term.

Proof. This result can be obtained by rereading the proof of the Alternating Series Test but we omit the proof.

Example. Find the N so that $|R_N| = |S - S_N| < .01$ where

$$S = \sum_{n=1}^{\infty}(-1)^{n-1}ne^{-n}.$$

Solution. Set $a_n = \dfrac{n}{e^n}$. It is easy to show that the alternating series is convergent. Then, since $a_6 = .01287$ and $a_7 = .006383$ we see that $N = 7$ is required for the accuracy desired. $\square$

Absolute and Conditional Convergence

Notice that our tests often test the convergence of a series with positive terms. If our series $\sum a_n$ does not have positive terms, then we can use these tests to study the series $\sum |a_n|$.

Definition (Absolute and Conditional Convergence)**.** (1) A series $\sum a_n$ is said to be **absolutely convergent** provided $\sum |a_n|$ is convergent.

(2) A series $\sum a_n$ is said to be **conditionally convergent** provided $\sum a_n$ is convergent but $\sum |a_n|$ is divergent.

Example. The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

is convergent by the Alternating Series Test because the signs alternate and

(i) $\frac{1}{n} \geq \frac{1}{n+1} > 0$ for all $n = 1, 2, \cdots$, and

(ii) $\lim_{n\to\infty} \frac{1}{n} = 0$

BUT the series of absolute values terms is divergent because it is the Harmonic Series

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}.$$

So $\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n}$ is conditionally convergent. □

Theorem. (i) Absolute convergence $\Longrightarrow$ convergence, BUT

(ii) Convergence does not imply absolute convergence.

Proof.(read) Part (ii) is justified by the preceding Example. Part (i) follows by applying the Comparison Test.

$$0 \leq a_n + |a_n| \leq 2|a_n| \text{ and } \sum |a_n| \text{ convergent } \Longrightarrow \sum (a_n + |a_n|) \text{ convergent.}$$

Now,

$$\sum a_n = \sum (a_n + |a_n|) - \sum |a_n| \text{ is convergent.}$$

□

Example 3. (Absolute and conditional convergence) Determine whether the following series diverge, converge absolutely, or converge conditionally.

a. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ **b.** $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n^3}}$ **c.** $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ **d.** $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+1}$

Solution.

$\square$

Example. Test the series

$$\sum_{n=1}^{\infty} \frac{\sin 4n}{4^n}$$

for convergence

Solution. We test the series $\sum_{n=1}^{\infty} \frac{\sin 4n}{4^n}$ for absolute convergence by comparing it with a geometric series.

$$0 < |a_n| = \left|\frac{\sin 4n}{4^n}\right| \leq \frac{1}{4^n}$$

$$\sum_{n=1}^{\infty} \frac{1}{4^n} \text{ is a convergent geometric series since } r = \frac{1}{4}$$

$$\therefore \qquad \sum_{n=1}^{\infty} \frac{\sin 4n}{4^n} \text{ is absulutely convergent and hence convergent.}$$

□

10.7 §10.7 The Ratio and Root Tests

The **Ratio Test** and **Root Test** are very useful in determining whether a given series is absolutely convergent.

Theorem - The Ratio Test (for Absolute Convergence) Let $\sum_{n=1}^{\infty} a_n$ be an infinite series.

(i)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1 \implies \sum_{n=1}^{\infty} a_n \text{ is absolutely convergent and hence convergent.}$$

(ii)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1 \implies \sum_{n=1}^{\infty} a_n \text{is divergent.}$$

(iii)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1 \implies \sum_{n=1}^{\infty} a_n \text{ the test fails, i.e. no conclusion.}$$

Proof (Reading). (i) Assume $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$ Then we may select an r such that $L < r < 1$ and there is some there is some M such that

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < r < 1 &\implies \frac{|a_{n+1}|}{|a_n|} < r \text{ for all } n \geq M \\
&\implies |a_{n+1}| < r|a_n| \text{ for all } n \geq M \\
&\implies |a_{M+1}| < r|a_M|, |a_{M+2}| < r|a_{M+1}| < r^2|a_M|, \ldots \\
&\implies |a_{M+k}| < r^k|a_M|.
\end{aligned}$$

Now, we compare the series $\sum_{n=1}^{\infty} |a_{M+n}|$ to the geometric series

$$\sum_{n=1}^{\infty} |a_M| r^n \text{ convergent geometric series.}$$

$\therefore$, $\sum_{n=1}^{\infty} |a_{M+n}|$ is convergent.

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$, then there is some r with $L > r > 1$ and some m such that

$$|a_{n+1}| > r|a_n| \text{ for all } n \geq m$$

Continue to follow the proof of (ii) to finish the proof of (ii).

(iii) The series $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$ both satisfy $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$ but one is divergent and the other is convergent. $\square$

Example 1. Use the Ratio Test to determine whether the following series converge.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{n!}$ **b.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^n}{n!}$ **c.** $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} e^{-n}(n^2+4)$

Solution.

$\square$

Example. Test the series

$$\sum_{n=1}^{\infty} \frac{n!}{100^n}$$

for absolutely convergence, conditionally convergent or divergence.

Solution. Since the series has positive terms, it is either absolutely convergent or divergent. We use the Limit Ratio Test.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\frac{(n+1)!}{100^{n+1}}}{\frac{n!}{100^n}} \\
&= \lim_{n\to\infty}\frac{(n+1)!}{100^{n+1}}\frac{100^n}{n!} = \lim_{n\to\infty}\frac{(n+1)}{100} = \infty > 1 \\
\therefore \quad & \sum_{n=1}^{\infty}\frac{n!}{100^n} \text{ diverges.}
\end{aligned}$$

□

The **Root Test** is similar to the Ratio Test and is well suited to series whose general term involves powers.

Theorem - Root Test (for Absolute Convergence) Let $\sum_{n=1}^{\infty} a_n$ be an infinite series.

(i)

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = L < 1 \implies \sum_{n=1}^{\infty} a_n \text{ is absolutely convergent}$$

(and hence convergent).

(ii)

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = L > 1 \text{ or } \lim_{n\to\infty}\sqrt[n]{|a_n|} = \infty \implies \sum_{n=1}^{\infty} a_n \text{is divergent.}$$

(iii)

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = 1 \implies \sum_{n=1}^{\infty} a_n \text{ the test fails, i.e. no conclusion.}$$

Proof. The proof is similar to the Ratio Test and we omit it. □

Example 2. Use the Root Test to determine whether the following series converge.

a. $\sum_{k=1}^{\infty}\left(\frac{3-4k^2}{7k^2+6}\right)^k$ **b.** $\sum_{k=1}^{\infty}\frac{(-2)^k}{k^{10}}$

Solution.

$\square$

Example. Test the series

$$\sum_{n=1}^{\infty}\frac{(-2)^n}{n^n}$$

for absolutely convergence, conditionally convergent or divergence.

Solution. We test the series $\sum_{n=1}^{\infty}\frac{(-2)^n}{n^n}$ for absolute convergence or divergence by The Root Test.

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\sqrt[n]{\left|\frac{(-2)^n}{n^n}\right|}=\lim_{n\to\infty}\sqrt[n]{\left(\frac{2}{n}\right)^n}=\lim_{n\to\infty}\frac{2}{n}=0<1$$

$$\sum_{n=1}^{\infty}\frac{(-2)^n}{n^n}\text{ is absolutely convergent and hence convergent.}$$

Example. Test the series

$$\sum_{n=1}^{\infty}\left(\frac{-2n}{n+1}\right)^n$$

for absolutely convergence, conditionally convergence, or divergence.

Solution. We test the series $\sum_{n=1}^{\infty}\left(\frac{-2n}{n+1}\right)^n$ for absolute convergence or divergence by The Root Test.

$$\begin{aligned}\lim_{n\to\infty}\sqrt[n]{|a_n|} &= \lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{-2n}{n+1}\right)^n\right|} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{2n}{n+1}\right)^n}\\ &= \lim_{n\to\infty}\frac{2n}{n+1} = 2 > 1\end{aligned}$$

$$\therefore \qquad \sum_{n=1}^{\infty}\left(\frac{-2n}{n+1}\right)^n \text{ is divergent by part (ii) of the Root Test.}$$

10.8 §10.8 Choosing a Convergence Test

We have discussed several different tests for convergence and/or divergence. We provide you with a summary of the Tests that we have studied. Your author suggests a strategy to help you but nothing is as helpful as experience. Do the problems!

Strategy. Classify the series according to form.

1. Does the series have the form of a p-series
$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{convergent if } p > 1 \\ \text{Divergent if } p \leq 1. \end{cases}$$
2. Does the series have the form of a geometric series
$$\sum_{n=1}^{\infty} ar^n \begin{cases} \text{convergent if } |r| < 1 \\ \text{Divergent if } |r| \geq 1. \end{cases}$$
3. Does the series have the form 'similar' to a p-series or a geometric series? If so, think of using the Comparison Test or Limit Comparison Test.
4. Does a quick application of the n^{th}-Term Test do the trick?
5. Do we have an Alternating Series?
6. If the series involves a factorial then a Ratio Test may help?
7. If a_n has the form b_n^n, think of the Root Test.
8. Does a_n have the form $a_n = f(n)$ for some continuous, positive,decreasing function with $\int_1^\infty f(x)\, dx$ easily computed? If so think of the Integral Test.

□

Summary of Series Tests

Test	Series	Convergent	Divergent	Comment
n^{th}–Term	$S=\sum_{n=0}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	
Geometric	$S=\sum_{n=0}^{\infty} ar^n$	$\lvert r\rvert<1$	$\lvert r\rvert \geq 1$	$S=\frac{a}{1-r}$
Telescoping	$\sum_{n=1}^{\infty}(b_n-b_{n+1})$	$\lim_{n\to\infty} b_n = L$		$S=b_1-L$
p-Series	$\sum_{n=1}^{\infty}\frac{1}{n^p}$	$p>1$	$p\leq 1$	
Alternating	$S=\sum_{n=1}^{\infty}(-1)^n a_n$	$0<a_{n+1}\leq a_n$ $\lim_{n\to\infty} a_n = 0$		Remainder $\lvert R_N\rvert \leq a_{N+1}$
Integ. f cont. pos. & decr.	$S=\sum_{n=1}^{\infty} a_n$ $a_n=f(n)\geq 0$	$\int_1^{\infty} f(x)dx$ Convergent	$\int_1^{\infty} f(x)dx$ Divergent	$\int_{N+1}^{\infty} f(x)dx < R_N$ $< \int_N^{\infty} f(x)dx$
Root	$S=\sum_{n=0}^{\infty} a_n$	$\lim_{n\to\infty}\sqrt[n]{\lvert a_n\rvert}<1$	$\lim_{n\to\infty}\sqrt[n]{\lvert a_n\rvert}>1$	
Ratio	$S=\sum_{n=0}^{\infty} a_n$	$\lim_{n\to\infty}\frac{\lvert a_{n+1}\rvert}{\lvert a_n\rvert}<1$	$\lim_{n\to\infty}\frac{\lvert a_{n+1}\rvert}{\lvert a_n\rvert}>1$	
Comparison	$S=\sum_{n=0}^{\infty} a_n$	$0<a_n\leq b_n$ $\sum_{n=0}^{\infty} b_n$ Conv.	$0<b_n\leq a_n$ $\sum_{n=0}^{\infty} b_n$ Div.	
Lim. Comp. $(a_n, b_n>0)$	$S=\sum_{n=0}^{\infty} a_n$	$\lim_{n\to\infty}\frac{a_n}{b_n}=$ $L\geq 0 (L<\infty)$ $\sum_{n=0}^{\infty} b_n$ Conv.	$\lim_{n\to\infty}\frac{a_n}{b_n}=$ $L>0 (or\ \infty)$ $\sum_{n=0}^{\infty} b_n$ Div.	

$\square$

Test the series in the textbook for absolute convergence, conditional convergence, or divergence.

Example 1. $\sum_{k=1}^{\infty} \frac{2^k + \cos(\pi k)\sqrt{k}}{3^{k+1}}$.

Solution.

$\square$

Example 2. $\sum_{k=1}^{\infty} \left(1 - \frac{10}{k}\right)^k$

Solution.

$\square$

Example 3. $\sum_{k=4}^{\infty} \frac{1}{\sqrt[4]{k^2 - 6k + 9}}$

Solution.

$\square$

Example 4. $\sum_{k=1}^{\infty} k^2 e^{-2k}$

Solution.

$\square$

Example 5. $\sum_{k=2}^{\infty} \sqrt[3]{\dfrac{k^2-1}{k^4+4}}$

Solution.

$\square$

Contents

Power series provide a way to represent familiar functions and to define new functions. For this reason, power series—like sets and functions—are among the most fundamental entities in mathematics. We adopt the convention that $(x-a)^0 = 1$.

11.2 §11.2 Power Series

In this section, we define Power Series and Taylor Series. Our focus in this class is Taylor Series to be discussed in the next two sections. Most examples in this section will be skipped. You can read the textbook if you'd like to have more examples about general power series.

Definition (Power Series)**.** A series of the form

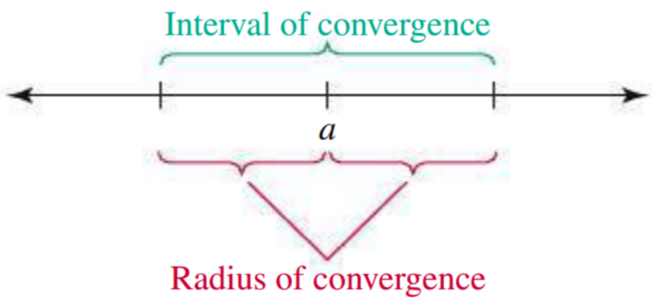
$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

is called **a power series in** $(x-a)$ **or a power series about** a **or a power series centered at** a **or a power series near** a.

• The constants c_n are called the **coefficients** of the series.

• The **radius of convergence** of the power series, denoted R, is the distance from the center of the series to the (Figure 11.14) boundary of the interval of convergence.

Remark. If the series is convergent, it is a function of x. The set of values of x for which the series converges is its **interval of convergence**.



Example: The **Geometric Series** $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, |x| < 1$ is a convergent power series.

Definition (Taylor Series)**.** A power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ centered at a is often called a **Taylor Series** of the function $f(x)$ provided

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n.$$

When $a = 0$ it is called a **Maclaurin Series** of $f(x)$.

We'll discuss Taylor Series in next section.

How do we determine the interval of convergence for a given power series? This can be determined using the **Ratio Test** or the **Root Test**.

Example. Find the domain of the Bessel function of order 0 which is given by

$$J_0(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2^{2n}(n!)^2}$$

Solution. Set $a_n = (-1)^n \frac{x^n}{2^{2n}(n!)^2}$.

$$\begin{aligned}\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}\frac{x^{n+1}}{2^{2(n+1)}((n+1)!)^2}}{(-1)^n \frac{x^n}{2^{2n}(n!)^2}} \right| \\ &= \lim_{n\to\infty} \left| \frac{x}{4(n+1)^2} \right| = 0 < 1 \text{ for all values of } x\end{aligned}$$

Thus, by the Ratio Test the domain of $J_0(x)$ is the real line. $\square$

Theorem - Radius of convergence of Power Series. For any power series, $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only three possibilities for the radius of convergence.

1. $R = 0$ The series converges for $x = a$ only.
2. $R = \infty$ The series converges on the entire real line.
3. $R > 0$ There is some positive real number R such that the series converges when $|x-a| < R$ and divergent when $|x-a| > R$. Note:

$$\begin{aligned} |x-a| < R &\implies -R < x-a < R \\ &\implies a-R < x < a+R \\ &\implies \text{ convergence on the interval } (a-R, a+R) \end{aligned}$$

   In this last case, the series may or may not converge at $x = a - R$ and at $x = a+R$. In order to find the **interval of convergence**, we must test the **end points** of the interval for convergence or divergence. The interval of convergence will be one of $(a-R, a+R)$ or $[a-R, a+R)$ or $(a-R, a+R]$ or $[a-R, a+R]$.

□

Remark: One normally, determines the radius of convergence by using the Ratio Test or Root Test. The basic idea is (for example, using ratio test)

1. Force the limit

$$\lim_{n\to\infty} \frac{|c_{n+1}(x-a)^{n+1}|}{|c_n(x-a)^n|} = \lim_{n\to\infty} \frac{|c_{n+1}(x-a)|}{|c_n|} = |x-a| \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|}$$

   to be less than one if possible.

2. Use

$$|x-a| \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} < 1 \implies R = \frac{1}{\lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|}}$$

   to solve for the radius R of convergence.

Similarly, the power series converges if $\rho < 1$ when Root Test is used. Solve $\rho < 1$ for the radius R of convergence.

Example 1. Find the interval and radius of convergence for each power series.

a. $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ **b.** $\sum_{k=0}^{\infty} \frac{(-1)^k (x-2)^k}{4^k}$ **c.** $\sum_{k=1}^{\infty} k! x^k$

Solution.

$\square$

Example 2. Use the Ratio Test to find the radius and interval of convergence of $\sum_{k=1}^{\infty} \frac{(x-2)^k}{\sqrt{k}}$.

Solution.

$\square$

Example. Find the radius of convergence and the interval of convergence of the series

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n+1}$$

Solution. We have convergence provided

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left|\frac{x^{n+1}}{n+2}\right| \left|\frac{n+1}{x^n}\right| = |x| \lim_{n\to\infty} \left|\frac{n+1}{n+2}\right| = |x| < 1.$$

Therefore, if $|x| < 1$, we have convergence, i.e. the radius of convergence is $R = 1$. Now we must test the end points, namely $x = \pm 1$. First we test the right hand end point $x = 1$. In this case the series becomes

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1} \text{ conv. by Alternating Series Test.}$$

Next, we test the left hand end point $x = -1$. In this case, the series becomes

$$\sum_{n=0}^{\infty} (-1)^n \frac{(-1)^n}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1} \text{ div. by the Comparison Test, i.e. } \sum_{n=0}^{\infty} \frac{1}{n} \text{ div.}$$

Putting this together we find that the interval of convergence is: $(-1, 1]$. $\square$

11.3 §11.3 Taylor Series

In the preceding section we were able to find power series representations for a certain restricted class of functions. Here we investigate more general problems: Which functions have power series representations? How can we find such representations?

11.3.1 Taylor Series of a Function

Linear and Quadratic Approximation

Recall that if a function $f(x)$ is differentiable at a point a, then it can be approximated near a by its tangent line, which is the linear approximation to f at the point a. The linear approximation at a is given by

$$\boxed{f(x) \approx p_1(x) = f(a) + f^{'}(a)(x-a).}$$

This approximation matches f in value and in slope at a. Linear approximation works well if f has a fairly constant slope near a. However, if f has a lot of curvature near a, then the tangent line may not provide an accurate approximation.

To remedy this situation, we create a quadratic approximating polynomial by adding one new term to the linear polynomial.

$$f(x) \approx p_2(x) = f(a) + f^{'}(a)(x-a) + c_2(x-a)^2,$$

where the unknown c_2 must be determined. We require that the polynomial $p_2(x)$ agree with f in value, slope, and concavity at a; that is, $p_2(x)$ must satisfy the matching conditions

$$p_2(a) = f(a), \quad p_2^{'}(a) = f^{'}(a), \quad p_2^{''}(a) = f^{''}(a).$$

Now, it is easy to see $p_2(a) = f(a)$ and

$$p_2^{'}(x) = f^{'}(a) + 2c_2(x-a)$$

which satisfies $p^{'}(a) = f^{'}(a)$ for any c_2. Further,

$$p_2^{''}(a) = 2c_2 = f^{''}(a).$$

It follows that $c_2 = \frac{1}{2} f^{''}(a)$.

Therefore, the quadratic approximation to $f(x)$ at a is given by

$$\boxed{f(x) \approx p_2(x) = f(a) + f^{'}(a)(x-a) + \frac{f^{''}(a)}{2}(x-a)^2.}$$

What we wish to investigate now is the relationship between the function $f(x)$ and the coefficients c_n. The key to doing this is that we can differentiate a power series term by term within its radius of convergence (see results in § 11.4; we apply the result without proof). Toward this end we compute the first few derivatives of $f(x)$.

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} c_n (x-a)^n \implies f(a) = c_0 \implies c_0 = \frac{f(a)}{0!} \\
f'(x) &= \sum_{n=0}^{\infty} n c_n (x-a)^{n-1} = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1} \implies f'(a) = c_1 \implies c_1 = \frac{f'(a)}{1!} \\
f''(x) &= \sum_{n=0}^{\infty} n(n-1) c_n (x-a)^{n-2} = \sum_{n=2}^{\infty} n(n-1) c_n (x-a)^{n-2} \\
&\implies f''(a) = (2 \cdot 1) c_2 \implies c_2 = \frac{f''(a)}{2!} \\
f^{(3)}(x) &= \sum_{n=0}^{\infty} n(n-1)(n-2) c_n (x-a)^{(n-3)} = \sum_{n=3}^{\infty} n(n-1)(n-2) c_n (x-a)^{n-3} \\
&\implies f^{(3)}(a) = (3 \cdot 2 \cdot 1) c_3 \implies c_3 = \frac{f^{(3)}(a)}{3!} \\
&\vdots \\
f^{(n)}(x) &= \cdots \qquad (\textbf{Continue in this fashion}) \implies c_n = \frac{f^{(n)}(a)}{n!}.
\end{aligned}$$

Therefore, if $f(x)$ has a power series representation, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

within the radius of convergence.

Remark: If $f(x)$ has a **power series representation** or **Taylor series** representation, we can use the first n terms to approximate the function, nth order approximation to the function $f(x)$. Check the web app `https://xuemaozhang.shinyapps.io/TaylorApproximation/`

Definition (Taylor/Maclaurin Series for a Function). The series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ is called the **Taylor Series of** $f(x)$ **centered at** a. When $a = 0$, it is often called a **Maclaurin Series**.

Some functions and their Maclaurin Series are:

Table 11.5 (page 740 in the textbook)

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 \cdots \qquad \text{with } |x| < 1$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 \cdots \qquad \text{with } |x| < 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} \cdots \qquad \text{with } -1 < x \le 1$$

$$-\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} \cdots \qquad \text{with } -1 \le x < 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad \text{with } R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad \text{with } R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad \text{with } R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad \text{with } |x| \le 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

$$= 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad \text{with } |x| < 1$$

□

Remark See Part II of Python code **Series.ipynb** for representing a function as a series.

Example. Some of these we have already done. We show the work for the expansion of $f(x) = \sin x$.

$$\begin{aligned}
f(x) = \sin x &\implies f(0) = \sin 0 \implies \frac{f(0)}{0!} = 0 \\
f'(x) = \cos x &\implies f'(0) = \cos 0 \implies \frac{f'(0)}{1!} = \frac{1}{1!} = \frac{(-1)^0}{(2\cdot 0+1)!} \\
f''(x) = -\sin x &\implies f''(0) = -\sin 0 \implies \frac{f''(0)}{2!} = \frac{0}{2!} \\
f^{(3)}(x) = -\cos x &\implies f^{(3)}(0) = -\cos 0 \implies \frac{f^{(3)}(0)}{3!} = \frac{-1}{3!} = \frac{(-1)^1}{(2\cdot 1+1)!} \\
f^{(4)}(x) = \sin x &\implies f^{(4)}(0) = \sin 0 \implies \frac{f^{(4)}(0)}{4!} = \frac{0}{4!} \\
f^{(5)}(x) = \cos x &\implies f^{(5)}(0) = \cos 0 \implies \frac{f^{(5)}(0)}{5!} = \frac{1}{5!} = \frac{(-1)^2}{(2\cdot 2+1)!}
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\sin x &= \frac{(-1)^0}{(2\cdot 0+1)!}x^{2\cdot 0+1} + \frac{(-1)^1}{(2\cdot 1+1)!}x^{2\cdot 1+1} + \frac{(-1)^2}{(2\cdot 2+1)!}x^{2\cdot 2+1} + \cdots \\
&= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}
\end{aligned}$$

$\square$

Example. The expansion for e^x is obvious since the n^{th} derivative of e^x is e^x and so if we set $f(x) = e^x$ we find

$$e^x = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty}\frac{e^0}{n!}x^n = \sum_{n=0}^{\infty}\frac{1}{n!}x^n.$$

$\square$

The Binomial Series

We know from algebra that if n is a positive integer, then $(1+x)^n$ is a polynomial of degree n. In fact,

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n,$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, k \geq 1, \qquad \binom{n}{0} = 1,$$

are called **binomial coefficients**.

Definition (Binomial Coefficients)**.** For a **real number** p and integers $k \geq 1$,

$$\binom{p}{k} = \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!}, k \geq 1, \qquad \binom{p}{0} = 1.$$

Theorem (Binomial Series)**.** For real numbers $p \neq 0$, the Taylor series for $f(x) = (1+x)^p$ centered at 0 is the **binomial series**

$$(1+x)^p = \sum_{k=0}^{\infty} \binom{p}{k} x^k.$$

The series converges for $|x| < 1$ (and possibly at the endpoints, depending on p). If p is a nonnegative integer, the series terminates and results in a polynomial of degree p.

Proof.

$\square$

Example 5(Reading). Consider the function $f(x) = \sqrt{1+x}$.

a. Find the first four terms of the binomial series for f centered at 0.

b. Approximate $\sqrt{1.15}$ to three decimal places. Assume the series for f converges to f on its interval of convergence, which is $[-1, 1]$.

Solution.

$\square$

Example 6 (Reading). Consider the functions

$$f(x) = \sqrt[3]{1+x} \quad and \quad g(x) = \sqrt[3]{c+x}, c > 0.$$

a. Find the first four terms of the binomial series for f centered at 0.

Solution.

$\square$

b. Use part (a) to find the first four terms of the binomial series for g centered at 0.

Solution.

$\square$

c. Use part (b) to approximate $\sqrt[3]{23}$ and $\sqrt[3]{29}$. Assume the series for g converges to g on its interval of convergence.

Solution.

$\square$

11.3.2 Convergence of Taylor Series

This sub-section is left as a **reading** material.

Convergence of Taylor Series

Caution: It might happen that $f(x)$ has a convergent Maclaurin/Taylor Series but that the series does not converge to $f(x)$. For example,

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

A discussion of this is beyond this course!

Remark. Because of this Caution we must find some way of determining when the Taylor Series actually converges to the given function. There are two questions here.

1. Do the sequence of partial sums converge? These partial sums are called the N^{th}**-degree Taylor polynomials centered at** a and denoted by $T_N(x)$
$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$$
2. If the Taylor polynomials do converge, then is their limit equal to $f(x)$?

Let
$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$$
be the N^{th}-degree Taylor polynomials centered at a. The **remainder** R_N is defined by
$$R_N(x) = f(x) - T_N(x).$$

Theorem (Convergence of Taylor Series)**.** Let f have derivatives of all orders on an open interval I containing a. Then

1. the remainder is
$$R_N(x) = \frac{f^{N+1}(c)}{(N+1)!}(x-a)^{N+1} \text{ where c is betwen } x \text{ and } a$$

and thus

$$|R_N(x)| \leq \frac{|f^{N+1}(c)|}{(N+1)!}|x-a|^{N+1} \text{ (where c is betwen } x \text{ and } a)$$

is bounded as follows

$$|R_N(x)| \leq \frac{M}{(N+1)!}|x-a|^{N+1} \text{ for all } |x-a| < R,$$

provided $|f^{N+1}(x)| \leq M$ whenever $|x-a| < R$ and

2. moreover, the Taylor Series converges to $f(x)$,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \text{ for } |x-a| < R$$

if and only if

$$\lim_{N\to\infty} |R_N(x)| = 0 \text{ for } |x-a| < R.$$

□

Example. When $f(x) = \sin x$, we see that $|f^{N+1}(x)| \leq 1$ for every real number x since these derivatives are either $\cos x$ or $\sin x$ and so the remainder is bounded as follows

$$|R_N(x)| \leq \frac{1}{(N+1)!}|x-a|^{N+1} \text{ for all } -\infty < x < \infty,$$

We know that $\lim_{N\to\infty} |R_N(x)| = 0$ if $\sum_{N=0}^{\infty} |R_N(x)|$ converges. Apply the Ratio Test to the series

$$\sum_{N=0}^{\infty} \frac{1}{(N+1)!}|x-a|^{N+1} \text{ to get convergence.}$$

$$\lim_{N\to\infty} \frac{|x-a|^{N+1}}{(N+1)!} \frac{N!}{|x-a|^N} = \lim_{n\to\infty} \frac{|x-a|}{N+1} = 0$$

$$\therefore \quad \sum_{N=0}^{\infty} \frac{1}{(N+1)!}|x-a|^{N+1} \text{ converges.}$$

Now, the Comparison Test, tells us that

$$|R_N(x)| \leq \frac{1}{(N+1)!}|x-a|^{N+1} \implies \sum_{N=0}^{\infty} |R_N(x)| \text{ is convergent}$$

$$\therefore \lim_{N\to\infty} |R_N(x)| = 0.$$

Or we show that $\lim_{N\to\infty}|R_N(x)| = 0$ by comparing growth rates.

Now, we have shown that $\sin x$ is equal to its Taylor series expansion centered at a and in particular to it Maclaurin Series. □

Example 7. Show that the Maclaurin series for $f(x) = e^x$ converges to $f(x)$, for $-\infty < x < \infty$.

Solution.

□

11.3.3 Combining Power Series

A power series defines a function on its interval of convergence. When power series are combined algebraically, new functions are defined. The following theorem, stated without proof, gives three common ways to combine power series.

Theorem (Combining Power Series)**.** Suppose the power series $\sum c_k x^k$ and $\sum d_k x^k$ converge to $f(x)$ and $g(x)$, respectively, on an interval I.

1. **Sum and difference:** The power series $\sum(c_k \pm d_k)x^k$converges to $f(x) \pm g(x)$ on I.

2. **Multiplication by a power:** Suppose m is an integer such that $k+m \geq 0$ for all terms of the power series $x^m \sum c_k x^k = \sum c_k x^{k+m}$. This series converges to $x^m f(x)$ for all $x \neq 0$ in I. When $x=0$, the series converges to $\lim_{x\to 0} x^m f(x)$.

3. **Composition:** If $h(x) = bx^m$, where m is a positive integer and b is a nonzero real number, the power series $\sum c_k (h(x))^k$ converges to the composite function $f(h(x))$, for all x such that $h(x)$ is in I.

Remark See Part III of Python code **Series.ipynb** for combining power series

For example, from our work with Geometric Series we know that

$$\sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} x^{n-1} = \frac{1}{1-x} \text{ provided } |x|<1$$

and so the function $f(x) = \frac{1}{1-x}$ has a power series representation when $|x|<1$. We can modify this argument to get

$$\begin{aligned}\frac{1}{2+x} &= \frac{1}{2(1+\frac{x}{2})} = \frac{1}{2}\left(\frac{1}{1-\left(\frac{-x}{2}\right)}\right)\\ &= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{-x}{2}\right)^n = \sum_{n=0}^{\infty}\left[(-1)^n\frac{x^n}{2^{n+1}}\right] = \sum_{n=0}^{\infty}\left[\frac{(-1)^n}{2^{n+1}}x^n\right] \quad \left(\text{ if } |\frac{x}{2}|<1\right)\end{aligned}$$

□

Example 3. Given the geometric series

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \qquad |x| < 1,$$

find the power series and interval of convergence for the following functions.

a. $\dfrac{x^5}{1-x}$ **b.** $\dfrac{1}{1-2x}$ **c.** $\dfrac{1}{1+x^2}$

Solution.

$\square$

Example. Find the power series representation of

$$f(x) = \frac{x}{2x^2+1}$$

and find its interval of convergence.

Solution. We manipulate our expression so that it is the sum of a geometric series, i.e. so that it has the form $\frac{a}{1-r}$.

$$\frac{x}{2x^2+1} = \frac{x}{1-(-2x^2)} = \sum_{n=1}^{\infty} x(-2x^2)^{n-1} = \sum_{n=1}^{\infty} (-2)^{n-1} x^{2n-1}$$

provided $|-2x^2| = 2x^2 < 1$. Also, the Geometric Series Test implies divergence, if $2x^2 = 1$.

$$|-2x^2| = 2x^2 < 1 \implies x^2 < \frac{1}{2} \implies |x| < \frac{1}{\sqrt{2}} = \text{ rad. of conv.}$$
$$\implies -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}.$$

We do not need to test the end points because we know that there is divergence at the end points, i.e. the interval of convergence is $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. $\square$

Example. Use the Maclaurin Series of $(1+x)^k$ to obtain the Maclaurin Series for

$$f(x) = \frac{x^2}{\sqrt{2+x}}.$$

Solution. We use

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad R = 1$$

after writing our expression in the proper form for this.

$$\frac{x^2}{\sqrt{2+x}} = x^2(2+x)^{-\frac{1}{2}} = x^2\big(2(1+\frac{x}{2})\big)^{-\frac{1}{2}} = \frac{1}{\sqrt{2}}x^2(1+\frac{x}{2})^{-\frac{1}{2}}$$

$$\begin{aligned}
\frac{1}{\sqrt{2}}x^2(1+\frac{x}{2})^{-\frac{1}{2}} &= \frac{1}{\sqrt{2}}x^2\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(\frac{x}{2})^n \\
&= \frac{1}{\sqrt{2}}x^2\Big(1 + (-\frac{1}{2})(\frac{x}{2}) + \frac{(-\frac{1}{2})((-\frac{1}{2})-1)}{2!}(\frac{x}{2})^2 \\
&\qquad + \frac{(-\frac{1}{2})((-\frac{1}{2})-1)((-\frac{1}{2})-2)}{3!}(\frac{x}{2})^3 + \cdots\Big) \\
&= \frac{1}{\sqrt{2}}x^2\Big(1 - \frac{1}{2\cdot 2}x + \frac{3}{2^2\cdot 2!\cdot 2^2}x^2 - \frac{3\cdot 5}{2^3\cdot 3!\cdot 2^3}x^3 \\
&\qquad + \frac{3\cdot 5\cdot 7}{2^4\cdot 4!\cdot 2^4}x^4 + \cdots\Big) \\
&= \frac{1}{\sqrt{2}}x^2 - \frac{1}{2^2\sqrt{2}}x^3 + \frac{3}{2^4\cdot\sqrt{2}\cdot 2!}x^4 \\
&\qquad - \frac{3\cdot 5}{2^6\cdot\sqrt{2}\cdot 3!}x^5 + \frac{3\cdot 5\cdot 7}{2^8\cdot\sqrt{2}\cdot 4!}x^6 + \cdots
\end{aligned}$$

□

11.4 §11.4 Working with Taylor Series

The goal of this final section is to illustrate additional techniques associated with power series. As you will see, power series cover the entire landscape of calculus from limits and derivatives to integrals and approximation. We present five different topics that you can explore selectively.

Remark See Part III of Python code **Series.ipynb** for working with series.

Limits by Taylor Series

An important use of Taylor series is evaluating limits.

Theorem (Limit of a Series)**.** Suppose that the function $f(x)$ has a power series representation

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

with nonzero radius of convergence R. Then, on $(-R, R)$

$$\lim_{x\to x_0} f(x) = \sum_{n=0}^{\infty} \lim_{x\to x_0} a_n x^n, \quad for \quad x_0 \in (-R, R)$$

Remark. The theorem can be applied to a Taylor series of $f(x)$ as well.

Example 1. (A limit by Taylor series) Evaluate $\lim_{x\to 0} \frac{x^2 + 2\cos x - 2}{3x^4}$.

Solution.

$\square$

Example 2. (A limit by Taylor series) Evaluate $\lim_{x\to\infty} 6x^5 sin\frac{1}{x} - 6x^4 + x^2$.

Solution.

$\square$

Differentiating Power Series

Theorem (Termwise Differentiation)**.** Suppose that the function $f(x)$ has a power series representation

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

with nonzero radius of convergence R. Then f is differentiable on $(-R, R)$ and

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Remark. The theorem can be applied to a Taylor series of $f(x)$ as well.

Example 3. (Power series for derivatives) Differentiate the Maclaurin series for $f(x) = \sin x$ to verify that $\sin' x = \cos x$.

Solution.

□

Integrating Power Series

Theorem (Termwise Integration)**.** Suppose that the function $f(x)$ has a power series representation

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

with nonzero radius of convergence R. Then for each x in $(-R, R)$

$$\int_0^x f(t) = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}$$

Remark. The theorem can be applied to a Taylor series of $f(x)$ as well.

Example 4 in §11.2. Consider the geometric series

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \qquad |x| < 1.$$

a. Differentiate this series term by term to find the power series for f' and identify the function it represents.

Solution.

□

b. Integrate this series term by term and identify the function it represents.

Solution.

$\square$

Example 5 in §11.2. Find power series representations centered at 0 for the following functions and give their intervals of convergence.

a. $\tan^{-1} x$

Solution. For $|x| < 1$ we have

$$\begin{aligned}\tan^{-1} x &= \int \frac{1}{1+x^2}\,dx = \int \frac{1}{1-(-x^2)}\,dx = \int (\sum_{n=1}^{\infty} (-x^2)^{n-1})\,dx \\ &= \int (\sum_{n=1}^{\infty} (-1)^{n-1} x^{2(n-1)}\,dx = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{(2n-1)}}{2n-1} + C\end{aligned}$$

To determine the value of C, we set $x = 0$ and find

$$0 = \tan^{-1} 0 = C$$

Thus,

$$\tan^{-1} x = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{(2n-1)}}{2n-1} \text{ provided } -1 < x < 1.$$

$\square$

b. $\ln\left(\dfrac{1+x}{1-x}\right)$

Solution.

$\square$

Example 5. (Approximating a definite integral) Approximate the value of the integral $\int_0^1 e^{-x^2}$ with an error no greater than 5×10^{-4}.

Solution.

$\square$

Representing Real Numbers

When values of x are substituted into a convergent power series, the result may be a series representation of a familiar real number. The following example illustrates some techniques.

Example 6. Evaluating infinite series

a. Use the Maclaurin series for $f(x) = tan^{-1}x$ to evaluate $\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$.

Solution.

$\square$

b. Let $f(x) = (e^x - 1)/x$ for $x \neq 0$, and $f(0) = 1$. Use the Maclaurin series for f to evaluate $f^{'}(1)$ and $\sum_{k=1}^{\infty} \frac{k}{(k+1)!}$.

Solution.

$\square$

Representing Functions as Power Series

Power series have a fundamental role in mathematics in defining functions and providing alternative representations of familiar functions. We have seen a lot of examples already.

Example 7. Identify the function represented by the power series $\sum_{k=0}^{\infty} \frac{(1-2x)^k}{k!}$ and give its interval of convergence.

Solution.

$\square$

Example 8.(Mystery series) Determine the interval of convergence of the power series $\sum_{k=1}^{\infty} \frac{(-1)^k k}{4^k} x^{2k}$ and find the function it represents on this interval.

Solution.

$\square$