Eigenvalues and Population Dynamics

Gene Chu, Lucy Liu and Emily Sun

University of Michigan, MATH 214

Introduction

Modeling population dynamics is important in many real-world applications such as studying population growth, decline, and management. Transition matrices allow us to model the movement between two populations of fixed sizes and fixed movement rates. Similarly, Leslie matrices allow us to model the short and long term age group distributions of an isolated population of individuals based on fixed birth and death rates among each age group. By determining the eigenvalues and eigenvectors for these matrices, we can see how the population of different areas or age groups changes over time.

General Overview

Transition matrices are special matrices where all of its columns are distribution vectors. A distribution vector has only positive or zero components and its elements must add up to 1. Typically, transition matrices are used to model the movement of "items" between two or more "places" over one unit of time, where the entries of the transition matrix define the probability that any one "item" will be in a particular "place" in the next time frame. The general form for a transition matrix is

$$P = \begin{bmatrix} P_{1,1} & \dots & P_{1,j} \\ \vdots & \ddots & \vdots \\ P_{i,1} & \dots & P_{i,j} \end{bmatrix}$$

where $P_{i,j}$ represents the probability that an item in place i moves to place j. The sum of every column must equal 1, since each item must move to some place. Therefore, if we have a state vector S_t describing the location of the items at a particular time t and the transition vector P, we can calculate the state of the items at time $t+1$ by multiplying P by S_t, as follows: $S_{t+1} = PS_t$. By raising P to some power k, we can determine the state at any time $t+k$ in the future, or even predict the long-term behavior by letting k go to infinity.

Leslie matrices are used in population ecology to model population dynamics within a particular population (instead of between populations as with transition matrices) by modeling how individuals in the population progress between different age groups given the rates of birth and death in each age group. When modeling with Leslie matrices, we assume an isolated population with unlimited resources and space to grow. Leslie matrices are similar to transition matrices, but the columns do not need to add up to 1. They take on the general form where the entries of the first row are the fecundity for each age group, or the per-capita averages of female offspring surviving to a certain age range born from mothers of a certain age group, f. The last column, except for the first entry, is a column of zeroes. Under the first row, there is a diagonal matrix whose non-zero entries are the fraction of individuals surviving to the next age group, s.

We also need a population vector n_t where n_i gives the number of individuals in the ith age group at time t. This allows us to determine the number of individuals in each age group at time $t+1$ by multiplying the Leslie matrix with the population vector, as shown below:

$$\begin{bmatrix} n_0 \\ n_1 \\ \vdots \\ n_{\omega-1} \end{bmatrix}_{t+1} = \begin{bmatrix} f_0 & f_1 & f_2 & \dots & f_{\omega-2} & f_{\omega-1} \\ s_0 & 0 & 0 & \dots & 0 & 0 \\ 0 & s_1 & 0 & \dots & 0 & 0 \\ 0 & 0 & s_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & s_{\omega-2} & 0 \end{bmatrix} \begin{bmatrix} n_0 \\ n_1 \\ \vdots \\ n_{\omega-1} \end{bmatrix}_t$$

Figure 1: Leslie Matrix Equation

Application of Transition Matrices

One application of transition matrices is to model population movement between two places, assuming a fixed population size and no other factors changing the populations. Using information from the UN database, we determine the immigration and emigration rates between Belgium and the Netherlands in 2017 as follows. The population of Belgium was 11.35 million and the population of the Netherlands was 17.08 million. 150,578 people moved to the Netherlands from Belgium and 59,147 people moved to Belgium from the Netherlands. This means that 98.68% of people in Belgium stayed while 1.32% of people moved to the Netherlands, and 99.65% of people in the Netherlands stayed while 0.35% of people moved to Belgium. Assuming that the percentages of people moving between the two countries remains constant for the foreseeable future, this movement can be represented by the transition matrix below:

$$A = \begin{bmatrix} 0.9868 & 0.0035 \\ 0.0132 & 0.9965 \end{bmatrix}$$

Computations

We can diagonalize this transition matrix by first calculating the eigenvalues using the characteristic equation.

$$det(A - \lambda I_n) = \lambda^2 - 1.983\lambda + 0.983 = 0$$

$$\lambda_1 = 1, \lambda_2 = 0.983$$

From there, we find the eigenvectors by computing the basis of each eigenspace (the span of the kernel of A - λI for each eigenvalue).

$$E_{\lambda 1} = \begin{bmatrix} 0.25 \\ 1 \end{bmatrix}, E_{\lambda 2} = \begin{bmatrix} -1.05 \\ 1 \end{bmatrix}$$

We can then create matrix S whose columns are the eigenvectors and matrix D whose diagonal entries are the eigenvalues. The diagonalized transition matrix is then:

$$A = \begin{bmatrix} 0.25 & -1.05 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.983 \end{bmatrix} \begin{bmatrix} 0.77 & 0.81 \\ -0.77 & 0.19 \end{bmatrix}$$

The populations of Belgium and the Netherlands at time k are represented by uk, where c is the initial population column vector (in millions) times the inverse of S:

$$u_k = \begin{bmatrix} 0.25 & -1.05 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.983 \end{bmatrix}^k \begin{bmatrix} 22.5 \\ -5.4 \end{bmatrix}$$

So for example, the populations of Belgium and the Netherlands at time k would be:

$$u_k = \begin{bmatrix} 0.25 \\ 1 \end{bmatrix} (1)^k (22.5) + \begin{bmatrix} -1.05 \\ 1 \end{bmatrix} (0.983)^k (-5.4)$$

As k goes to an infinitely large number, the second value goes to zero and we can see that in the long term, the populations of the two countries would achieve the following steady state:

$$u_k = \begin{bmatrix} 0.25 \\ 1 \end{bmatrix} (1)^k (22.5) = \begin{bmatrix} 5.62 \\ 22.5 \end{bmatrix}$$

where Belgium has 5.62 million people and the Netherlands has 22.5 million people.

Application of Leslie Matrices

Applying Leslie matrices is similar to transition matrices. As with transition matrices, we multiply powers of Leslie matrices with an initial population state to find the short term population states. We can also look at the dominant eigenvalue and the corresponding eigenvector of the Leslie matrix to find the long term steady-state age distribution of the population (the eigenvector) and the exponential rate the population will grow at once it reaches this state (the eigenvalue).
Using the data from a study done on the life cycles of blue monkeys in Kenya over a period of 29 years, we extracted the necessary data to create a Leslie matrix. From the ages of 0 to 6, we observed the number of individuals that survived to the next age and calculated the fecundity for each age group by multiplying the fraction of individual surviving to the next age group by the average number of children produced by females of that age group:

Age	Individuals present	Fraction survived	Fecundity
0	383	N/A	0.00
1	290	0.75	0.00
2	244	0.84	0.00
3	228	0.93	0.93
4	203	0.89	5.98
5	192	0.94	24.44
6	181	0.94	21.88

Table 1: Population data of blue monkeys in Kenya

Computations

We used this data to determine a Leslie matrix for the blue monkey population.

$$L = \begin{bmatrix} 0 & 0 & 0 & 0.93 & 5.98 & 24.44 & 21.88 \\ 0.75 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.84 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.93 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.89 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.94 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.94 & 0 \end{bmatrix}$$

The initial population vector, n_0, is simply the first column of the table. The population vector at time t, n_t, is then calculated as:

$$n_t = L^t n_0 = \begin{bmatrix} 0 & 0 & 0 & 0.93 & 5.98 & 24.44 & 21.88 \\ 0.75 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.84 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.93 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.89 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.94 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.94 & 0 \end{bmatrix}^t \begin{bmatrix} 383 \\ 290 \\ 244 \\ 228 \\ 203 \\ 192 \\ 181 \end{bmatrix}$$

The only eigenvalue of the Leslie matrix L is 1.707. This value means that once the stable age distribution has been reached, the population undergoes exponential growth at rate 1.707. To find the stable age distribution, we calculate the corresponding eigenvector. This turns out to be the zero vector. Therefore we know that the population of blue monkeys actually ends up dying out in the long term - they are not able to sustain a birth rate greater than the death rate.

Conclusion

Population dynamics are important to understand in real-world applications and linear algebra helps us accurately model population behavior with transition and Leslie matrices. By multiplying the matrices themselves with initial population state vectors, we can see how the populations change in the short term. Eigenvalues are useful in predicting and modeling populations in the long term because we can calculate the eigenvalues and eigenvectors of transition matrices and Leslie matrices to see long-term population behavior (growth or decline) and predict the steady or equilibrium state of the populations.

Works Cited

1. Bronikowski AM, Cords M, Alberts SC, Altmann J, Brockman DK, Fedigan LM, Pusey A, Stoinski T, Strier KB, Morris WF (2016) Female and male life tables for seven wild primate species. Scientific Data 3: 160006. https://doi.org/10.1038/sdata.2016.6
2. Chamberlain, Andrew. "Using Eigenvectors to Find Steady State Population Flows." *Medium*, Medium, 4 Oct. 2017, medium.com/@andrew.chamberlain/using-eigenvectors-to-find-steady-state-population-flows-cd938f124764.
3. Jones, James Holland. "Leslie Matrix I." *Stanford*, Stanford, 2 May 2008, web.stanford.edu/ jhj1/teachingdocs/Jones-Leslie1-050208.pdf.
4. "Leslie Matrix." *Wikipedia*, Wikimedia Foundation, 17 Dec. 2018, en.wikipedia.org/wiki/Leslie$_matrix$.
5. "United Nations Population Division | Department of Economic and Social Affairs." *United Nations*, United Nations, www.un.org/en/development/desa/population/migration/data/estimates2/ estimates17.asp.