Erik Sundblad

Week 10 Homework 2.3(1-5, 15), 2.4(2, 3, 8)

Sec 2.3

Problem(1) Find norm of A

a)
$$A=\begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix},\quad \|A\|_\infty = 6$$

b)
$$A=\begin{bmatrix}1 & 5 & 1\\ -1 & 2 & -3\\ 1 & -7 & 0\end{bmatrix}\quad \|A\|_\infty = 7$$

Problem (2)

Find infinity Norm and Cond(A)

a)
$$Cond(A) = \|A\|_\infty \times \|A^{-1}\|_\infty$$

$$A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\quad \|A\|_\infty = 7\quad A^{-1}: \left[\begin{array}{cc|cc}1 & 2 & 1 & 0\\ 3 & 4 & 0 & 1\end{array}\right] \sim_{R_2-3R_1}$$

$$\left[\begin{array}{cc|cc}1 & 2 & 1 & 0\\ 0 & -2 & -3 & 1\end{array}\right] \sim_{R_2/-2} \left[\begin{array}{cc|cc}1 & 2 & 1 & 0\\ 0 & 1 & 3/2 & -1/2\end{array}\right] \sim_{R_1-2R_2} \left[\begin{array}{cc|cc}1 & 0 & -2 & 1\\ 0 & 1 & 3/2 & -1/2\end{array}\right]$$

So $A^{-1} = \begin{bmatrix}-2 & 1\\ 3/2 & -1/2\end{bmatrix}$ and $\|A^{-1}\| = 3$ So

$$Cond(A) = 7\times 3 = \boxed{21}$$

Continued

Homework 10 (2.3 (2-5), 15, 2.4 (2, 3, 8))
Problem (2)

b) Find $\|A\|_\infty$ and Cond(A)

$A = \begin{bmatrix} 1 & 2.01 \\ 3 & 6 \end{bmatrix}$ $\|A\|_\infty = 9$ $A^{-1} = \left[\begin{array}{cc|cc} 1 & 2.01 & 1 & 0 \\ 3 & 6 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - 3R_1}$

$\left[\begin{array}{cc|cc} 1 & 2.01 & 1 & 0 \\ 0 & -.97 & -3 & 1 \end{array}\right] \xrightarrow{R_2/-.97} \left[\begin{array}{cc|cc} 1 & 2.01 & 1 & 0 \\ 0 & 1 & \frac{300}{97} & \frac{100}{97} \end{array}\right] \xrightarrow{R_1 - 2.01 R_2}$

$\left[\begin{array}{cc|cc} 1 & 0 & 1-\frac{2.1(300)}{97} & -\frac{2.01(100)}{97} \\ 0 & 1 & \frac{300}{97} & \frac{100}{97} \end{array}\right]$

So $A^{-1} = \begin{bmatrix} 1-\frac{2.1(300)}{97} & -\frac{2.01(100)}{97} \\ \frac{300}{97} & \frac{100}{97} \end{bmatrix}$

and $\|A^{-1}\|_\infty = \left|1 - \frac{2.1(300)}{97}\right| + \left|\frac{2.01(100)}{97}\right|$

Cond(A) =

$9 \times \|A^{-1}\|$

$\approx \boxed{65.59}$

c)

$A = \begin{bmatrix} 6 & 3 \\ 4 & 2 \end{bmatrix}$ $\|A\| = 9$ $A^{-1} = \left[\begin{array}{cc|cc} 6 & 3 & 1 & 0 \\ 4 & 2 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - 4/6 R_1}$

$\left[\begin{array}{cc|cc} 6 & 3 & 1 & 0 \\ 0 & 0 & -4/6 & 1 \end{array}\right]$ So Not invertible!

Homework 10 2.3(3-5, 15), 2.4 (2, 3, 8)

Sec 2.3

Problem 3

Find Forward, Backward error, Error Mag

for system $\begin{bmatrix} 1 & 1 \\ 1.0001 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2.0001 \end{bmatrix}$ $x_1, x_2 = (1, 1)$

a)

Back error $= \dfrac{\|r\|_\infty}{\|b\|_\infty}$ Forward Error $= \dfrac{\|x - x_a\|_\infty}{\|x\|_\infty}$

$\dfrac{\text{Forward}}{\text{Backward}}$ = Magnification

$x_a = (-1, 3)$

Forward $= \dfrac{\left\|\begin{matrix} 1 - -1 \\ 1 - 3 \end{matrix}\right\|_\infty}{1} = \dfrac{2}{1} = 2$

Backward $= \dfrac{2e^{-4}}{\left\|\begin{matrix} 2 \\ 2.0001 \end{matrix}\right\|_\infty} = 1e^{-4}$ $r = b - Ax_a$

$\begin{bmatrix} 2 \\ 2.0001 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 - 1.0001 \end{bmatrix} = \begin{matrix} 0 \\ 2e^{-4} \end{matrix}$

So Magnification equals $2 / 1e^{-4} = \boxed{20,000}$

→ b, c, d, e will all be similar opperations

Homework 10 2.3(4,5,15) 2.4(2,3,8)

4) Find forward/Backward Error and erro mag

$$x_1 + 2x_2 = 3 \quad \text{so} \quad \begin{bmatrix} 1, & 2 \\ 3, & -4 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$$
$$3x_1 - 4x_2 = 7$$

First we find the true x_1, x_2

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 3 & -4 & 7 \end{array}\right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -10 & -8 \end{array}\right] \quad \text{So} \quad \boxed{\begin{array}{l} x_2 = 1/5 \\ x_1 = 13/5 \end{array}}$$

a) approximate $\vec{x} = [1, 1]$

$$\text{forward} = \frac{\|x - x_a\|_\infty}{\|x\|_\infty}, \quad \text{Backward} = \frac{\|r\|_\infty}{\|b\|_\infty} \qquad r = b - A\bar{x}$$

$$\text{Magnificatio} = \frac{\text{forward}}{\text{backward}}$$

$$\text{forward} = \frac{\left\|\begin{pmatrix} 1/5 - 1 \\ 13/5 - 1 \end{pmatrix}\right\|_\infty}{\left\|\begin{pmatrix} 1/5 \\ 13/5 \end{pmatrix}\right\|_\infty} = \frac{\left\|\begin{pmatrix} -4/5 \\ 8/5 \end{pmatrix}\right\|_\infty}{13/5} = \boxed{8/13} \text{ forward}$$

$$r = \begin{pmatrix} 3 \\ 7 \end{pmatrix} - \begin{pmatrix} 1+2 \\ 3-4 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \end{pmatrix} \quad \text{Backward} = \frac{\left\|\begin{pmatrix} 0 \\ 8 \end{pmatrix}\right\|_\infty}{\left\|\begin{pmatrix} 3 \\ 7 \end{pmatrix}\right\|_\infty} = \boxed{8/7} \text{ Backward}$$

$$\text{Magnification} = \frac{8/13}{8/7} = \boxed{7/13}$$

Part b, and c will be very similar!

Homework 10 2.3(5, 15), 2.4(2, 3, 8)

Section 2.3

Problem 5 Find forward error, backward error and error magnification for a given approximate $\vec{x}$ of the system:

$$\left[\begin{array}{cc|c} 1 & -2 & 3 \\ 3 & -4 & 7 \end{array}\right] \quad \text{or} \quad \begin{array}{l} x_1 - 2x_2 = 3 \\ 3x_1 - 4x_2 = 7 \end{array}$$

as $M + x =$

First we find $\vec{x}$ true values.

$$\left[\begin{array}{cc|c} 1 & -2 & 3 \\ 3 & -4 & 7 \end{array}\right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 2 & -2 \end{array}\right] \quad \text{So} \quad \begin{array}{l} x_1 = \\ x_2 = \end{array} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Eq's: Forward $= \dfrac{\|x - x_a\|_\infty}{\|x\|_\infty}$, Backward $= \dfrac{\|r\|_\infty}{\|b\|_\infty}$, Error Mag. $\dfrac{\text{Forward}}{\text{Backward}}$

a) $\vec{x}_a = [-2, -4]$ forward $= \dfrac{\left\|\begin{pmatrix} 1 - -2 \\ -1 - -4 \end{pmatrix}\right\|_\infty}{\left\|\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\|} = \dfrac{3}{1} = \boxed{3}$

$r = b - Ax_a$

So $r = \begin{pmatrix} 3 \\ 7 \end{pmatrix} - \begin{pmatrix} -2 + 8 \\ -6 + 16 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \end{pmatrix}$ Backward $\dfrac{\|\begin{smallmatrix} -3 \\ -3 \end{smallmatrix}\|_\infty}{7} = \boxed{\dfrac{3}{7}}$

Error Magnification $= 3 / \tfrac{3}{7} = \boxed{7}$

Part b, c, d are the same part e on Next page

Homework 10 2.3(5, 15), 2.4(2,3,8)

2.3

Problem 5 Continued

e)

Find the cond(A) when $A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$

$$\text{Cond}(A) = \|A\|_\infty \times \|A^{-1}\|_\infty$$

$$7 \times \|A^{-1}\|_\infty$$

$$\left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 3 & -4 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 2 & -3 & 1 \end{array}\right] \xrightarrow{R_2/2} \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 1 & -3/2 & 1/2 \end{array}\right]$$

$$\xrightarrow{R_1 + 2R_2} \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & -3/2 & 1/2 \end{array}\right] \quad \text{So } A^{-1} = \begin{bmatrix} -2 & 1 \\ -3/2 & 1/2 \end{bmatrix}$$

Then $7 \times \|A^{-1}\|_\infty = 7 \times 3 = \boxed{21} = \text{Cond}(A)$

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15)

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ satisfies $\|A\|_\infty = \dfrac{\|Ax\|_\infty}{\|x\|_\infty}$

HomeWork 10 2.4(2, 3, 8)

2) Find $PA = LU$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & -1 \\ -1 & 1 & -1 \end{bmatrix} \underset{\begin{bmatrix} 0&1&0 \\ 1&0&0 \\ 0&0&1 \end{bmatrix}}{\overset{P_1}{\sim}} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 1 & -1 \end{bmatrix} \overset{R_3 + \frac{1}{2}R_1}{\sim} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ -\frac{1}{2} & \frac{3}{2} & -\frac{3}{2} \end{bmatrix} \overset{R_2 - \frac{1}{2}R_1}{\sim} \begin{bmatrix} 2 & 1 & -1 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} & -\frac{3}{2} \end{bmatrix}$$

$$\underset{\begin{bmatrix} 1&0&0 \\ 0&0&1 \\ 0&1&0 \end{bmatrix}}{\overset{P_2}{\sim}} \begin{bmatrix} 2 & 1 & -1 \\ -\frac{1}{2} & \frac{3}{2} & -\frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \overset{R_3 + \frac{1}{3}R_2}{\sim} \begin{bmatrix} 2 & 1 & -1 \\ -\frac{1}{2} & \frac{3}{2} & -\frac{3}{2} \\ \frac{1}{2} & -\frac{1}{3} & 0 \end{bmatrix}$$

So we then can say $PA = LU$ is

$$\overset{P}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}} \overset{A}{\begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & -1 \\ -1 & 1 & -1 \end{bmatrix}} = \overset{L}{\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{1}{2} & -\frac{1}{3} & 1 \end{bmatrix}} \overset{U}{\begin{bmatrix} 2 & 1 & -1 \\ 0 & \frac{3}{2} & -\frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}}$$

b, c, d will be similar steps

3, 8 Next Page

HomeWork 10 (3,8)

Problem 3

Solve the system by PA=LU Then Solve back Substitution.

a) $\begin{bmatrix} 3 & 7 \\ 6 & 1 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -11 \end{pmatrix}$ So $A = \begin{bmatrix} 3 & 7 \\ 6 & 1 \end{bmatrix} \xrightarrow{P_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \begin{bmatrix} 6 & 1 \\ 3 & 7 \end{bmatrix} \xrightarrow{R_2 - \frac{1}{2}R_1}$

$\begin{bmatrix} 3 & 1 \\ \frac{1}{2} & 6.5 \end{bmatrix}$ So we get $\underset{P}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}\underset{A}{\begin{bmatrix} 3 & 7 \\ 6 & 1 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix}}\underset{U}{\begin{bmatrix} 6 & 1 \\ 0 & \frac{13}{2} \end{bmatrix}}$

Then we back Subtitute.

$LU = Pb$ So

$\underset{L\ \ b=c}{\left[\begin{array}{cc|c} 1 & 0 & -11 \\ \frac{1}{2} & 1 & 1 \end{array}\right]}$ $c_1 = -11$, $c_2 = \frac{13}{2}$ So $\underset{U\ \ c=x}{\left[\begin{array}{cc|c} 6 & 1 & -11 \\ 0 & \frac{13}{2} & \frac{13}{2} \end{array}\right]}$ $\boxed{\begin{matrix} x_1 = -2 \\ x_2 = 1 \end{matrix}}$

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b) $\begin{bmatrix} 3 & 1 & 2 \\ 6 & 3 & 4 \\ 3 & 1 & 5 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$ to Start $A = \begin{bmatrix} 3 & 1 & 2 \\ 6 & 3 & 4 \\ 3 & 1 & 5 \end{bmatrix} \xrightarrow{P} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 6 & 3 & 4 \\ 3 & 1 & 2 \\ 3 & 1 & 5 \end{bmatrix} \xrightarrow{R_3 - \frac{1}{2}R_1} \begin{bmatrix} 6 & 3 & 4 \\ 3 & 1 & 2 \\ \frac{1}{2} & -\frac{1}{2} & 3 \end{bmatrix} \xrightarrow{R_2 - \frac{1}{2}R_1} \begin{bmatrix} 6 & 3 & 4 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 3 \end{bmatrix} \xrightarrow{R_3 - R_2}$ Next Page

HomeWork 10 2.4 (3cont, 8)

Prblem 3 continued

$$\sim \begin{bmatrix} 6 & 3 & 4 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 3 \end{bmatrix} \text{ so } PA=LU= \overset{P}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 3 & 1 & 2 \\ 6 & 3 & 4 \\ 3 & 1 & 5 \end{bmatrix}} = \overset{L}{\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 1 & 1 \end{bmatrix}} \overset{U}{\begin{bmatrix} 6 & 3 & 4 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 3 \end{bmatrix}}$$

We then say $LU=Pb$ So we can back Substitute.

$Lc=Pb$

$$\overset{L \quad b}{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{2} & 1 & 1 & 3 \end{array}\right]} = \overset{c}{\begin{pmatrix} 1 \\ -\frac{1}{2} \\ 3 \end{pmatrix}} \text{ so } \overset{U \quad c}{\left[\begin{array}{ccc|c} 6 & 3 & 4 & 1 \\ 0 & -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 3 & 3 \end{array}\right]} = \overset{x}{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}}$$

Sec 2.4

Problem 8 Find largest Multiplier of $PA=LU$ of

$$A=\begin{bmatrix} 10 & 20 & 1 \\ 1 & 1.99 & 6 \\ 0 & 50 & 1 \end{bmatrix} \underset{R_2-\frac{1}{10}R_1}{\sim} \begin{bmatrix} 10 & 20 & 1 \\ \frac{1}{10} & 0.01 & \frac{59}{10} \\ 0 & 50 & 1 \end{bmatrix} \underset{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}{\overset{P}{\sim}} \begin{bmatrix} 10 & 20 & 1 \\ 0 & 50 & 1 \\ 0 & \frac{1}{100} & \frac{59}{10} \end{bmatrix}$$

$$\overset{3-\frac{1/100}{50}}{\sim} \begin{bmatrix} 10 & 20 & 1 \\ \frac{1}{10} & 50 & 1 \\ 0 & \frac{1}{5000} & 1-\frac{5.9}{5000} \end{bmatrix} \text{ So } \overset{P}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}} \times \overset{A}{\begin{bmatrix} 10 & 20 & 1 \\ 1 & 1.99 & 6 \\ 0 & 50 & 1 \end{bmatrix}} = \overset{L}{\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{10} & 1 & 0 \\ 0 & \frac{1}{5000} & 1 \end{bmatrix}} \overset{U}{\begin{bmatrix} 10 & 20 & 1 \\ 0 & 50 & 1 \\ 0 & 0 & 1-\frac{5.9}{5000} \end{bmatrix}}$$

Our largest Multiplier was $=\frac{1}{10}$!