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HomeWork 15 2 proofs, Gradient Method by hand

- Proof 1 If the matrix $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, and $b > c$ the A is always not positive definite.

> This is false my counter example

$$\begin{pmatrix} x \\ y \end{pmatrix} \begin{bmatrix} 16 & 4 \\ 0 & 3 \end{bmatrix} (x, y) = \begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} 16x + 4y \\ 3y \end{pmatrix} = 16x^2 + 4xy + 3y^2$$

We complete the square to get

$$16\left(x + \frac{y}{8}\right)^2 - \frac{16y^2}{4 \cdot 16} + 3y^2 = \underbrace{16\left(x + \frac{y}{8}\right)^2 + \frac{11y^2}{4}}$$

We can observe for any values of x and y this will always yield a positive result. By virtue of squaring. Therefore proving this was a false statement!

-> Proof 2 If the matrix $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ is not positive definite. Then $b > c$.

This is false by counter example: $\begin{bmatrix} 1 & 8 \\ 0 & 9 \end{bmatrix}$

we see $b < c$ and

$$(x\ y) \begin{bmatrix} 1 & 8 \\ 0 & 9 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x^2 + 8xy + 9y^2$$ we complete the square to

get $(x + 4y)^2 - 16y^2 + 9y^2$ or $(x + 4y)^2 - 7y^2$

which is a non-positive definite matrix as the values $x = -4$, $y = 1$ yieds a value < 0. Therefore disproving our original theory.

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Homework 15 Gradient Method by hand!

Sudo Code for $k = 0, \ldots, n-1$

$$\alpha_k = \frac{r_k^T r_k}{d_k^T A d_k} \qquad \beta_k = \frac{r_{k+1}^T r_{k+1}}{r_k R_k^T}$$

$$x_{k+1} = x_k + \alpha_k d_k \qquad d_{k+1} = r_{k+1} + \beta_k d_k$$

$$r_{k+1} = r_k - \alpha_k A d_k$$

$$A = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{matrix} 6 \\ 3 \end{matrix}$$

$$x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$d_0 = r_0 = \begin{pmatrix} 6 \\ 3 \end{pmatrix} - A x_0 = \begin{matrix} 6 \\ 3 \end{matrix} - \begin{matrix} 4 \\ 7 \end{matrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

1^{st} itteration

$$\alpha_0 = \frac{r_0^T R_0}{d_0^T A d_0} = \frac{\begin{pmatrix} 2 \\ -4 \end{pmatrix}(2\ -4)}{\begin{pmatrix} 2 \\ -4 \end{pmatrix}\begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}\begin{pmatrix} 2 \\ -4 \end{pmatrix}} = \frac{20}{56} = \frac{5}{14}$$

$$\vec{x}_1 = x_0 + \alpha_0 d_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{5}{14} \end{pmatrix}\begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{matrix} 14/14 \\ 14/14 \end{matrix} - \begin{matrix} \frac{10}{14} \\ -20/14 \end{matrix} = \begin{pmatrix} 4/14 \\ 34/14 \end{pmatrix} \begin{pmatrix} 2/7 \\ 17/7 \end{pmatrix}$$

$$\vec{r}_1 = \begin{pmatrix} 2 \\ -4 \end{pmatrix} - \frac{5}{14}\begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}\begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix} - \frac{5}{14}\begin{pmatrix} -4 \\ -16 \end{pmatrix} \begin{matrix} \frac{28}{14} + \frac{20}{14} \\ -\frac{56}{14} + \frac{80}{14} \end{matrix} = \begin{matrix} \frac{48}{14} \\ \frac{24}{14} \end{matrix} = \begin{pmatrix} \frac{24}{7} \\ -\frac{12}{7} \end{pmatrix}$$

$$\beta_0 = \frac{r_1^T R_1}{r_0^T R_0} = \frac{\left(\frac{24}{7}\right)^2 + \left(\frac{12}{7}\right)^2}{20} \quad \frac{720}{49} \times 1/20 = \frac{36}{49} = \left(\frac{6^2}{7^2}\right)$$

$$d_1 = r_1 + \beta_0 d_0 = \begin{pmatrix} \frac{24}{7} \\ -12/7 \end{pmatrix} + \left(\frac{6^2}{7^2}\right)\begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{matrix} \frac{24}{7} \\ -\frac{12}{7} \end{matrix} + \begin{matrix} \frac{72}{49} \\ -\frac{144}{49} \end{matrix} \begin{pmatrix} \frac{240}{49} \\ -\frac{228}{49} \end{pmatrix}$$

2nd itteratio

$$\alpha_1 = \frac{r_1^T R_1}{d_1 A d_1} = \frac{\left(24/7\right)^2 + \left(12/7\right)^2}{d_1^T A d_1}$$

$$\vec{x}_2 = \begin{matrix} \frac{2}{7} \\ \frac{14}{7} \end{matrix} + \frac{\left(24/7\right)^2 + \left(12/7\right)^2}{d^T A d} d = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$