Beam Loading

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第 1 章　Static Beam Loading

1.1　Cavity の基礎

1.1.1　Power dissipation

$$P_{diss} = \frac{1}{2} \cdot \frac{V_{cav}^2}{R} = \frac{V_{cav}^2}{R_{sh}} \tag{1.1}$$

1.1.2　Shut impedance

$$R = \frac{1}{2} \cdot R_{sh} \tag{1.2}$$

空洞の入力カップラーの変圧比を $1 : n$ とすると (Fig. 1.1)、

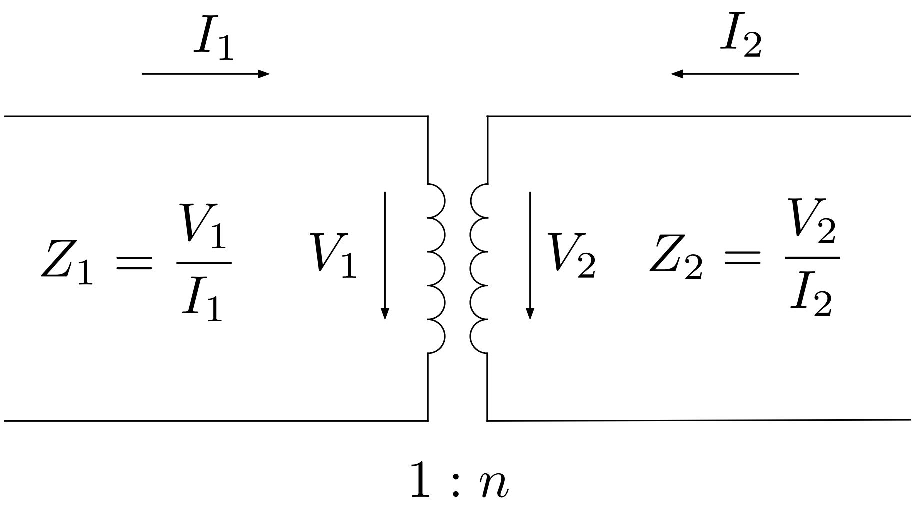

Fig. 1.1　理想的なトランスによる入力カップラー.

$$V_2 = N \cdot V_1,\ I_2 = \frac{1}{N} \cdot I_1 \tag{1.3}$$

したがって、

$$Z_2 = n^2 \cdot Z_1 \tag{1.4}$$

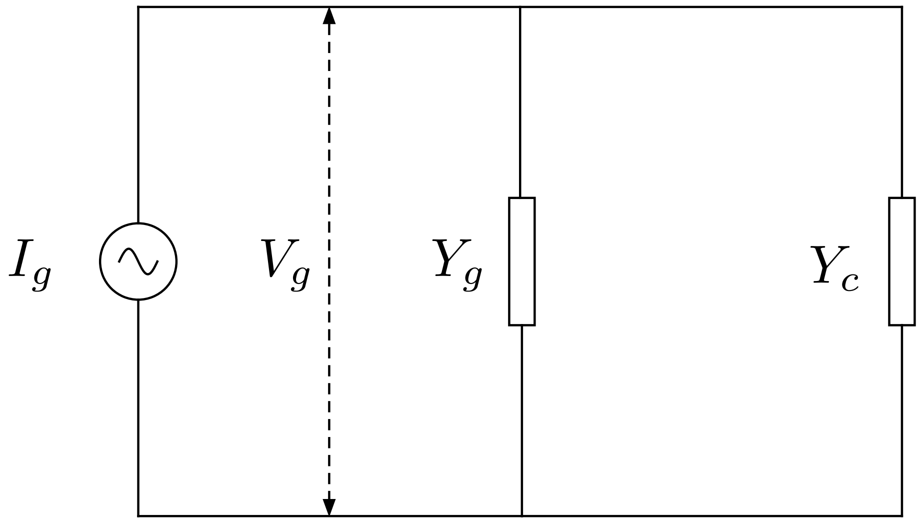

Fig. 1.2　理想的なトランスによる入力カップラー.

1.1.3　Available Power

$$P_{diss} = \frac{1}{2} Y_c V_g^2 \tag{1.5}$$

$$V_g = \frac{I_g}{Y_L} = \frac{I_g}{Y_g + Y_c} \tag{1.6}$$

$$P_{diss} = \frac{1}{2} \frac{Y_c}{(Y_g + Y_c)^2} I_g^2 \tag{1.7}$$

$$\begin{aligned} \frac{\partial P_{diss}}{\partial Y_c} &= \frac{1}{2} \frac{(Y_g + Y_c)^2 - 2Y_c(Y_g + Y_c)}{(Y_g + Y_c)^4} I_g^2 \\ &= \frac{1}{2} \frac{Y_g - Y_c}{(Y_g + Y_c)^3} I_g^2 \end{aligned} \tag{1.8}$$

$\partial P_{diss}/\partial Y_c = 0$ より、$Y_g = Y_c$ の時、P_{diss} は最大になり

$$P_{diss}^{max} = \frac{1}{8} Y_g I_g^2 = \frac{1}{8} \frac{R}{\beta} I_g^2 \equiv P_g \tag{1.9}$$

1.2　ビーム負荷付き空洞の RCL 等価回路

1.2.1　Cavity parameters

共振周波数

$$\omega_0 = \frac{1}{LC} \tag{1.10}$$

Quality factor

$$Q = 2\pi \frac{\text{stored energy in cavity}}{\text{dissipated energy per cycle}} = \frac{\omega_0 W}{P_{diss}} \tag{1.11}$$

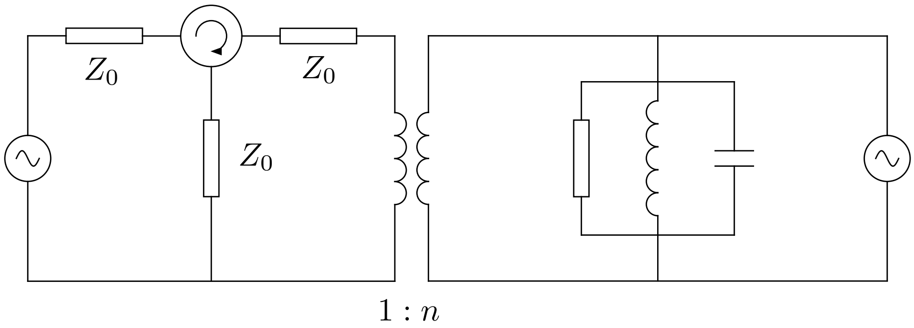

Fig. 1.3　理想的なトランスによる入力カップラー.

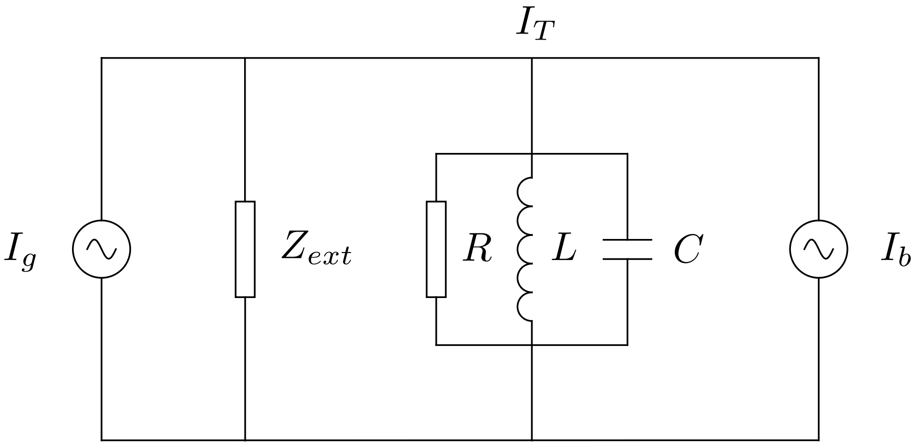

Fig. 1.4　等価回路

Unloaded quality factor

$$Q_0 = \omega_0 \frac{(1/2)CV^2}{V^2/(2R)} = \omega_0 RC \tag{1.12}$$

External quality factor

$$Q_{ext} = 2\pi \frac{\text{stored energy in cavity}}{\text{dissipated energy in external devices per cycle}} = \frac{\omega_0 W}{P_{ext}} \tag{1.13}$$

Loaded quality factor

$$Q_L = 2\pi \frac{\text{stored energy in cavity}}{\text{total energy per cycle}} = \frac{\omega_0 W}{P_{tot}} \tag{1.14}$$

ここで、

$$P_{tot} = P_{diss} + P_{ext} \tag{1.15}$$

したがって、

$$\frac{1}{Q_L} = \frac{1}{Q_0} + \frac{1}{Q_{ext}} \tag{1.16}$$

$$\frac{1}{R_L} = \frac{1}{R} + \frac{1}{Z_{ext}} \tag{1.17}$$

Coupling factor β

$$\beta = \frac{P_{ext}}{P_{cav}} = \frac{Q_0}{Q_{ext}} = \frac{R}{Z_{ext}} = \frac{R}{n^2 Z_0} \tag{1.18}$$

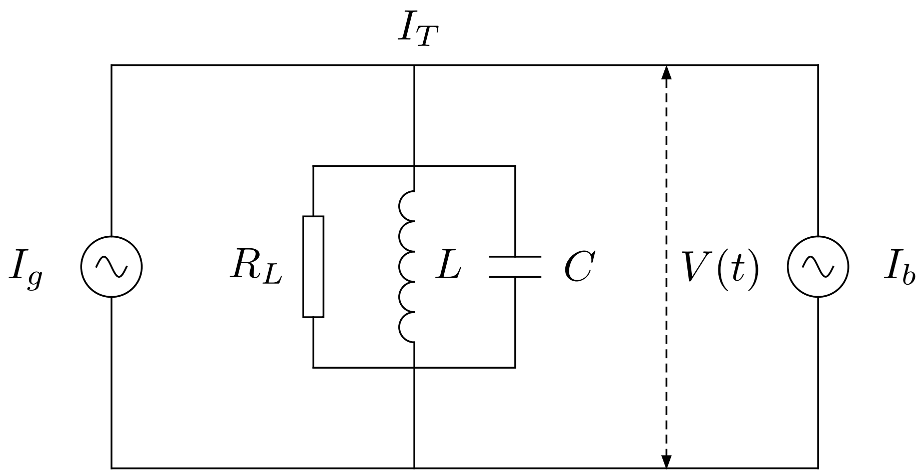

Fig. 1.5　等価回路 2

$$\ddot{V}(t) + \frac{1}{R_L C}\dot{V}(t) + \frac{1}{LC}V(t) = \frac{1}{C}\dot{I}(t) \tag{1.19}$$

Phasor

$$V(t) = \tilde{V}e^{j\omega_c t},\ I(t) = \tilde{I}e^{j\omega_c t} \tag{1.20}$$

1.3 Equations for cavity voltage

$$
\begin{aligned}
\boldsymbol{V}_c(t) &= \tilde{V}(t)e^{j\omega_c t}, \qquad \tilde{V}(t) = \hat{V}(1 + a_v(t))e^{j(\phi_v(t)+\psi_v)} \\
\boldsymbol{I}_b(t) &= \tilde{I}_b(t)e^{j\omega_c t}, \qquad \tilde{I}_b(t) = \hat{I}_b(1 + a_b(t))e^{j(\phi_b(t)+\psi_b)} \\
\boldsymbol{I}_g(t) &= \tilde{I}_g(t)e^{j\omega_c t}, \qquad \tilde{I}_g(t) = \hat{I}_g(1 + a_g(t))e^{j(\phi_g(t)+\psi_g)} \\
\boldsymbol{I}_t(t) &= \boldsymbol{I}_g(t) + \boldsymbol{I}_b(t) = \tilde{I}_t(t)e^{j\omega_c t}
\end{aligned}
\tag{1.21}
$$

$$
\ddot{\boldsymbol{V}}_c(t) + 2\sigma\dot{\boldsymbol{V}}_c(t) + \omega_0^2\boldsymbol{V}_c(t) = 2\sigma R\dot{\boldsymbol{I}}_t(t) \tag{1.22}
$$

1.4 Pedersen Model

1.4.1 Modulation Transfer Function

振幅変調 (AM) や位相変調 (PM) された正弦波信号を伝達関数 $H(s)$ を持つシステムを介した場合、出力信号も一般に振幅変調と位相変調の両方を受けることになる。このようなシステムの完全な特性評価には、システムの伝達関数 $H(s)$ から導き出すことができる次の 4 つの異なる変調伝達関数 (Modulation Transfer Function) を求める必要がある。(Fig.1.6)

1. 入力の振幅変調が出力の振幅を変調する伝達関数：$G_{aa}(s)$
2. 入力の位相変調が出力の位相を変調する伝達関数：$G_{pp}(s)$
3. 入力の振幅変調が出力の位相を変調する伝達関数：$G_{ap}(s)$
4. 入力の位相変量が出力の振幅を変調する伝達関数：$G_{pa}(s)$

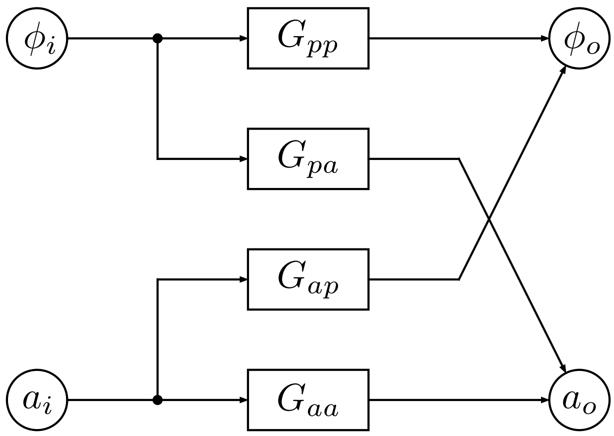

Fig. 1.6 Modulation Transfer Function.

変調の振幅が十分小さい場合、変調の伝送は線形であり、次のようにして求めることができる。

最初に、入力信号の振幅のみが $a_i(t)$ で変調された場合を考える。このとき、入力信号 $x(t)$ は次のようにあらわせる。

$$x(t) = \mathrm{Re}\left\{\hat{X}(1 + a_i(t))e^{j\omega_c t}\right\}$$
$$= \hat{X}\left\{\cos\omega_c t + a_i(t)\cos\omega_c t\right\} \tag{1.23}$$

出力信号 $y(t)$ は $a_i(t)$ によって振幅だけでなく位相も変調され、

$$y(t) = \mathrm{Re}\left\{\hat{Y}(1 + a_{o,a}(t))e^{j(\omega_c t + \phi_0 + \phi_{o,a}(t))}\right\} \tag{1.24}$$

と表すことができる。この時、変調伝達関数 $G_{aa}(s),\ G_{ap}(s)$ は、次のように定義される。

$$G_{aa}(s) = \frac{\hat{a}_{o,a}(s)}{\hat{a}_i(s)}, \quad G_{ap}(s) = \frac{\hat{\phi}_{o,a}(s)}{\hat{a}_i(s)} \tag{1.25}$$

ここで、

$$\begin{aligned}
\mathcal{L}[\cos\omega_c t] &= \frac{1}{2}\mathcal{L}[(e^{j\omega_c t} + e^{-j\omega_c t})] \\
&= \frac{1}{2}\left(\int_0^\infty e^{j\omega_c t}e^{-st}dt + \int_0^\infty e^{-j\omega_c t}e^{-st}dt\right) \\
&= \frac{1}{2}\left(\int_0^\infty e^{-(s-j\omega_c)t}dt + \int_0^\infty e^{-(s+j\omega_c)t}dt\right) \\
&= \frac{1}{2}\left(\left[\frac{e^{-(s-j\omega_c)t}}{-(s-j\omega_c)}\right]_0^\infty + \left[\frac{e^{-(s+j\omega_c)t}}{-(s+j\omega_c)}\right]_0^\infty\right) \\
&= \frac{1}{2}\left(\frac{1}{s-j\omega_c} + \frac{1}{s+j\omega_c}\right) = \frac{s}{s^2+\omega_c^2}
\end{aligned} \tag{1.26}$$

また、$\mathcal{L}\{a_i(t)\} = \hat{a}_i(s)$ とすると、

$$\begin{aligned}
\mathcal{L}[a_i(t)\cos\omega_c t] &= \frac{1}{2}\mathcal{L}[(a_i(t)e^{j\omega_c t} + a_i(t)e^{-j\omega_c t})] \\
&= \frac{1}{2}\left(\int_0^\infty a_i(t)e^{j\omega_c t}e^{-st}dt + \int_0^\infty a_i(t)e^{-j\omega_c t}e^{-st}dt\right) \\
&= \frac{1}{2}\left(\int_0^\infty a_i(t)e^{-(s-j\omega_c)t}dt + \int_0^\infty a_i(t)e^{-(s+j\omega_c)t}dt\right) \\
&= \frac{1}{2}\{\hat{a}_i(s-j\omega_c) + \hat{a}_i(s+j\omega_c)\}
\end{aligned} \tag{1.27}$$

$X(s) = \mathcal{L}\{x(t)\}$ とし、(1.23) をラプラス変換すると、

$$X(s) = \frac{\hat{X}}{2}\left\{\frac{2s}{s^2+\omega_c^2} + a_i(s-j\omega_c) + a_i(s+j\omega_c)\right\} \tag{1.28}$$

一方、出力信号 $y(t)$ は $a_o \ll 1,\ \phi_o \ll 1$ の場合、以下のように近次できる。

$$\begin{aligned}
y(t) &\simeq \mathrm{Re}\left\{\hat{Y}(1+a_{o,a}(t))(1+j\phi_{o,a}(t))e^{j(\omega_c t+\phi_o)}\right\} \\
&= \mathrm{Re}\left\{\hat{Y}(1+a_{o,a}(t)+j\phi_{o,a}(t)+j\underbrace{a_{o,a}(t)\phi_{o,a}(t)}_{0})e^{j(\omega_c t+\phi_0)}\right\} \\
&\simeq \mathrm{Re}\left\{\hat{Y}(1+a_{o,a}(t)+j\phi_{o,a}(t))(\cos(\omega_c t+\phi_0)+j\sin(\omega_c+\phi_0))\right\} \\
&= \hat{Y}\{\cos(\omega_c t+\phi_o) + a_o(t)\cos(\omega_c t+\phi_o) - \phi_o(t)\sin(\omega_c t+\phi_o)\}
\end{aligned} \tag{1.29}$$

ここで、

$$
\begin{aligned}
\mathcal{L}[\cos(\omega_c t+\phi_o)] &= \frac{1}{2}\mathcal{L}[(e^{j(\omega_c t+\phi_o)}+e^{-j(\omega_c t+\phi_o)})] \\
&= \frac{1}{2}\left(\int_0^\infty e^{j(\omega_c t+\phi_o)}e^{-st}dt+\int_0^\infty e^{-j()\omega_c t+\phi_o)}e^{-st}dt\right) \\
&= \frac{e^{j\phi_o}}{2}\int_0^\infty e^{-(s-j\omega_c)t}dt+\frac{e^{-j\phi_o}}{2}\int_0^\infty e^{-(s+j\omega_c)t}dt \\
&= \frac{e^{j\phi_o}}{2}\left[\frac{e^{-(s-j\omega_c)t}}{-(s-j\omega_c)}\right]_0^\infty+\frac{e^{-j\phi_o}}{2}\left[\frac{e^{-(s+j\omega_c)t}}{-(s+j\omega_c)}\right]_0^\infty \\
&= \frac{1}{2}\left(\frac{e^{j\phi_o}}{s-j\omega_c}+\frac{e^{-j\phi_o}}{s+j\omega_c}\right)=\frac{\cos\phi_o s-\omega_c\sin\phi_o}{s^2+\omega_c^2}
\end{aligned}
\tag{1.30}
$$

また、$\mathcal{L}\{a_{o,a}(t)\}=\hat{a}_{o,a}(s)$, $\mathcal{L}\{\phi_{o,a}(t)\}=\hat{\phi}_{o,a}(s)$ とすると、

$$
\begin{aligned}
\mathcal{L}[a_{o,a}(t)\cos(\omega_c t+\phi_o)] &= \frac{1}{2}\mathcal{L}[(a_{o,a}(t)e^{j(\omega_c t+\phi_o)}+a_{o,a}(t)e^{-j(\omega_c t+\phi_o)})] \\
&= \frac{1}{2}\left(\int_0^\infty a_{o,a}(t)e^{j(\omega_c t+\phi_o)}e^{-st}dt+\int_0^\infty a_{o,a}(t)e^{-j(\omega_c t+\phi_o)}e^{-st}dt\right) \\
&= \frac{e^{j\phi_o}}{2}\int_0^\infty a_{o,a}(t)e^{-(s-j\omega_c)t}dt+\frac{e^{-j\phi_o}}{2}\int_0^\infty a_{o,a}(t)e^{-(s+j\omega_c)t}dt \\
&= \frac{1}{2}\left\{e^{j\phi_o}\hat{a}_{o,a}(s-j\omega_c)+e^{-j\phi_o}\hat{a}_{o,a}(s+j\omega_c)\right\}
\end{aligned}
\tag{1.31}
$$

$$
\begin{aligned}
\mathcal{L}[\phi_{o,a}(t)\sin(\omega_c t+\phi_o)] &= \frac{1}{2j}\mathcal{L}[(\phi_{o,a}(t)e^{j(\omega_c t+\phi_o)}-\phi_{o,a}(t)e^{-j(\omega_c t+\phi_o)})] \\
&= \frac{1}{2j}\left(\int_0^\infty \phi_{o,a}(t)e^{j(\omega_c t+\phi_o)}e^{-st}dt-\int_0^\infty \phi_{o,a}(t)e^{-j(\omega_c t+\phi_o)}e^{-st}dt\right) \\
&= \frac{e^{j\phi_o}}{2j}\int_0^\infty \phi_{o,a}(t)e^{-(s-j\omega_c)t}dt-\frac{e^{-j\phi_o}}{2j}\int_0^\infty \phi_{o,a}(t)e^{-(s+j\omega_c)t}dt \\
&= -\frac{j}{2}\left\{e^{j\phi_o}\hat{\phi}_{o,a}(s-j\omega_c)-e^{-j\phi_o}\hat{\phi}_{o,a}(s+j\omega_c)\right\}
\end{aligned}
\tag{1.32}
$$

したがって、$Y(s)=\mathcal{L}\{y(t)\}$ とし (1.29) をラプラス変換すると、

$$
\begin{aligned}
Y(s)=\frac{\hat{Y}}{2}\Bigg[&\frac{2(\cos\phi_o s-\omega_c\sin\phi_o)}{s^2+\omega_c^2}+e^{j\phi_0}\hat{a}_{o,a}(s-j\omega_c)+e^{-j\phi_0}\hat{a}_{o,a}(s+j\omega_c) \\
&+j\left\{e^{j\phi_0}\hat{\phi}_{o,a}(s-j\omega_c)-e^{-j\phi_0}\hat{\phi}_{o,a}(s+j\omega_c)\right\}\Bigg]
\end{aligned}
\tag{1.33}
$$

ここで、(1.25) より (1.33) は

$$
\begin{aligned}
Y(s)=\frac{\hat{Y}}{2}\Bigg[&\frac{2(\cos\phi_o s-\omega_c\sin\phi_o)}{s^2+\omega_c^2} \\
&+e^{j\phi_0}G_{aa}(s-j\omega_c)\hat{a}_i(s-j\omega_c)+e^{-j\phi_0}G_{aa}(s+j\omega_c)\hat{a}_i(s+j\omega_c) \\
&+j\left\{e^{j\phi_0}G_{ap}(s-j\omega_c)\hat{a}_i(s-j\omega_c)-e^{-j\phi_0}G_{ap}(s+j\omega_c)\hat{a}_i(s+j\omega_c)\right\}\Bigg]
\end{aligned}
$$

したがって、

$$Y(s)=\frac{\hat{Y}}{2}\left[\frac{2(\cos\phi_0 s-\omega_c\sin\phi_0)}{s^2+\omega_c^2}+e^{j\phi_0}\{G_{aa}(s-j\omega_c)+jG_{ap}(s-j\omega_c)\}\hat{a}_i(s-j\omega_c)\right.$$
$$\left.+e^{-j\phi_0}\{G_{aa}(s+j\omega_c)-jG_{ap}(s+j\omega_c)\}\hat{a}_i(s+j\omega_c)\right] \quad (1.34)$$

ここで $Y(s)=H(s)X(s)$ より、(1.28) は、

$$Y(s)=\frac{\hat{X}H(s)}{2}\left[\frac{2s}{(s^2+\omega_c^2}+\hat{a}_i(s-j\omega_c)+\hat{a}_i(s+j\omega_c)\right] \quad (1.35)$$

(1.34) と (1.35) で係数を比較すると、

$$\hat{Y}(\cos\phi_0 s-\omega_c\sin\phi_0)=\hat{X}H(s)s \quad (1.36)$$

$$\begin{aligned} G_{aa}(s-j\omega_c)+jG_{ap}(s-j\omega_c)&=\frac{\hat{X}}{\hat{Y}}e^{-j\phi_0}H(s)\\ G_{aa}(s+j\omega_c)-jG_{ap}(s+j\omega_c)&=\frac{\hat{X}}{\hat{Y}}e^{j\phi_0}H(s) \end{aligned} \quad (1.37)$$

(1.36) で、$s=\pm j\omega_c$ の時を考えると、

$$H(\pm j\omega_c)=\frac{\hat{Y}}{\hat{X}}e^{\pm j\phi_0} \quad (1.38)$$

(1.37) と (1.38) より、

$$\begin{aligned} G_{aa}(s-j\omega_c)+jG_{ap}(s-j\omega_c)&=\frac{H(s)}{H(j\omega_c)}\\ G_{aa}(s+j\omega_c)-jG_{ap}(s+j\omega_c)&=\frac{H(s)}{H(-j\omega_c)} \end{aligned} \quad (1.39)$$

以上より、入力の振幅変調に関する変調伝達関数 $G_{aa}(s),\ G_{ap}(s)$ は、

$$\begin{aligned} G_{aa}(s)&=\frac{1}{2}\left\{\frac{H(s+j\omega_c)}{H(j\omega_c)}+\frac{H(s-j\omega_c)}{H(-j\omega_c)}\right\}=G_s(s)\\ G_{ap}(s)&=\frac{-j}{2}\left\{\frac{H(s+j\omega_c)}{H(j\omega_c)}-\frac{H(s-j\omega_c)}{H(-j\omega_c)}\right\}=G_c(s) \end{aligned} \quad (1.40)$$

今度は、入力信号の位相のみが $\phi_i(t)$ で変調された信号を考えると、

$$x(t)=\mathrm{Re}\{\hat{X}e^{j(\omega_c t+\phi_i(t))}\} \quad (1.41)$$

出力信号は先程と同様に考えると、

$$y(t)=\mathrm{Re}\{\hat{Y}(1+a_{o,p}(t))e^{j(\omega_c t+\phi_0+\phi_{o,p}(t))}\} \quad (1.42)$$

入力の位相変調に関する変調伝達関数 $G_{pp}(s),\ G_{pa}(s)$ は、

$$G_{pp}(s)=\frac{\hat{\phi}_{o,p}(s)}{\hat{\phi}_i(s)},\quad G_{pa}(s)=\frac{\hat{a}_{o,p}(s)}{\hat{\phi}_i(s)} \quad (1.43)$$

(1.41) は、$\phi_i(t) \ll 1$ の時、

$$\begin{aligned} x(t) &= \mathrm{Re}\{\hat{X}e^{j(\omega_c t+\phi_i(t))}\} \\ &\simeq \mathrm{Re}\{\hat{X}(1+j\phi_i(t))e^{j\omega_c t}\} \\ &= \hat{X}(\cos\omega_c t - \phi_i(t)\sin\omega_c t) \end{aligned} \tag{1.44}$$

(1.44) をラプラス変換すると、

$$X(s) = \hat{X}\left[\frac{s}{s^2+\omega_c^2} + j\left\{\hat{\phi}_i(s-j\omega_c) - \hat{\phi}_i(s+j\omega_c)\right\}\right] \tag{1.45}$$

(1.33) と同様にして (1.42) もラプラス変換すると、

$$\begin{aligned} Y(s) = \frac{\hat{Y}}{2}\Bigg[&\frac{2(\cos\phi_0 s - \omega_c\sin\phi_0)}{s^2+\omega_c^2} + e^{j\hat{\phi}_0}a_{o,p}(s-j\omega_c) + e^{-j\phi_0}\hat{a}_{o,p}(s+j\omega_c) \\ &+ j\left\{e^{j\phi_0}\hat{\phi}_{o,p}(s-j\omega_c) - e^{-j\phi_0}\hat{\phi}_{o,p}(s+j\omega_c)\right\}\Bigg] \end{aligned} \tag{1.46}$$

同様に (1.46) は

$$\begin{aligned} Y(s) = \frac{\hat{Y}}{2}\Bigg[&\frac{2(\cos\phi_0 s - \omega_c\sin\phi_0)}{s^2+\omega_c^2} + e^{j\phi_0}\{G_{pa}(s-j\omega_c) + jG_{pp}(s-j\omega_c)\}\hat{\phi}_i(s-j\omega_c) \\ &+ e^{-j\phi_0}\{G_{pa}(s+j\omega_c) - jG_{pp}(s+j\omega_c)\}\hat{\phi}_i(s+j\omega_c)\Bigg] \end{aligned} \tag{1.47}$$

ここで $Y(s) = H(s)X(s)$ より、

$$Y(s) = \frac{\hat{X}H(s)}{2}\left[\frac{2s}{s^2+\omega_c^2} + j\left\{\hat{\phi}_i(s-j\omega_c) - \hat{\phi}_i(s+j\omega_c)\right\}\right] \tag{1.48}$$

(1.47) と (1.48) で係数を比較すし、(1.38) を使うと、

$$\begin{aligned} G_{pa}(s-j\omega_c) + jG_{pp}(s-j\omega_c) &= j\frac{H(s)}{H(j\omega_c)} \\ G_{pa}(s+j\omega_c) - jG_{pp}(s+j\omega_c) &= -j\frac{H(s)}{H(-j\omega_c)} \end{aligned} \tag{1.49}$$

したがって、

$$\begin{aligned} G_{pa}(s) + jG_{pp}(s) &= j\frac{H(s+j\omega_c)}{H(j\omega_c)} \\ G_{pa}(s) - jG_{pp}(s) &= -j\frac{H(s-j\omega_c)}{H(-j\omega_c)} \end{aligned} \tag{1.50}$$

以上の結果から、変調伝達関数は以下の様になる。

$$\begin{aligned} G_{aa}(s) = G_{pp}(s) \quad &= \frac{1}{2}\left\{\frac{H(s+j\omega_c)}{H(j\omega_c)} + \frac{H(s-j\omega_c)}{H(-j\omega_c)}\right\} = G_s(s) \\ G_{ap}(s) = -G_{pa}(s) \quad &= \frac{j}{2}\left\{\frac{H(s+j\omega_c)}{H(j\omega_c)} - \frac{H(s-j\omega_c)}{H(-j\omega_c)}\right\} = G_c(s) \end{aligned} \tag{1.51}$$

1.4.2　空洞共振器への適用

$$Z(s) = \frac{2\sigma R_s}{s^2 + 2\sigma s + \omega_r^2} \tag{1.52}$$

$$G_{aa}(s) = G_{pp} = \frac{\sigma^2(1 + \tan^2\phi_z) + \sigma s}{s^2 + 2\sigma s + \sigma^2(1 + \tan^2\phi_z)} \tag{1.53}$$

$$G_{pa}(s) = -G_{ap} = \frac{\sigma \tan\phi_z s}{s^2 + 2\sigma s + \sigma^2(1 + \tan^2\phi_z)} \tag{1.54}$$

第 2 章　Robinson instability

付録 A　ラプラス変換

$$
\begin{aligned}
\mathcal{L}[\cos\omega_c t] &= \frac{1}{2}\mathcal{L}[(e^{j\omega_c t}+e^{-j\omega_c t})] \\
&= \frac{1}{2}\left(\int_0^\infty e^{j\omega_c t}e^{-st}dt + \int_0^\infty e^{-j\omega_c t}e^{-st}dt\right) \\
&= \frac{1}{2}\left(\int_0^\infty e^{-(s-j\omega_c)t}dt + \int_0^\infty e^{-(s+j\omega_c)t}dt\right) \\
&= \frac{1}{2}\left(\left[\frac{e^{-(s-j\omega_c)t}}{-(s-j\omega_c)}\right]_0^\infty + \left[\frac{e^{-(s+j\omega_c)t}}{-(s+j\omega_c)}\right]_0^\infty\right) \\
&= \frac{1}{2}\left(\frac{1}{s-j\omega_c}+\frac{1}{s+j\omega_c}\right) = \frac{s}{s^2+\omega_c^2}
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
\mathcal{L}[\cos(\omega_c t+\phi_o)] &= \frac{1}{2}\mathcal{L}[(e^{j(\omega_c t+\phi_o)}+e^{-j(\omega_c t+\phi_o)})] \\
&= \frac{1}{2}\left(\int_0^\infty e^{j(\omega_c t+\phi_o)}e^{-st}dt + \int_0^\infty e^{-j()\omega_c t+\phi_o)}e^{-st}dt\right) \\
&= \frac{e^{j\phi_o}}{2}\int_0^\infty e^{-(s-j\omega_c)t}dt + \frac{e^{-j\phi_o}}{2}\int_0^\infty e^{-(s+j\omega_c)t}dt \\
&= \frac{e^{j\phi_o}}{2}\left[\frac{e^{-(s-j\omega_c)t}}{-(s-j\omega_c)}\right]_0^\infty + \frac{e^{-j\phi_o}}{2}\left[\frac{e^{-(s+j\omega_c)t}}{-(s+j\omega_c)}\right]_0^\infty \\
&= \frac{1}{2}\left(\frac{e^{j\phi_o}}{s-j\omega_c}+\frac{e^{-j\phi_o}}{s+j\omega_c}\right) = \frac{\cos\phi_o s-\omega_c\sin\phi_o}{s^2+\omega_c^2}
\end{aligned}
\tag{A.2}
$$

$\mathcal{L}\{a_i(t)\}=\hat{a}_i(s)$ とすると、

$$
\begin{aligned}
\mathcal{L}[a_i(t)\cos\omega_c t] &= \frac{1}{2}\mathcal{L}[(a_i(t)e^{j\omega_c t}+a_i(t)e^{-j\omega_c t})] \\
&= \frac{1}{2}\left(\int_0^\infty a_i(t)e^{j\omega_c t}e^{-st}dt + \int_0^\infty a_i(t)e^{-j\omega_c t}e^{-st}dt\right) \\
&= \frac{1}{2}\left(\int_0^\infty a_i(t)e^{-(s-j\omega_c)t}dt + \int_0^\infty a_i(t)e^{-(s+j\omega_c)t}dt\right) \\
&= \frac{1}{2}\{\hat{a}_i(s-j\omega_c)+\hat{a}_i(s+j\omega_c)\}
\end{aligned}
\tag{A.3}
$$

$\mathcal{L}\{a_{o,a}(t)\} = \hat{a}_{o,a}(s),\ \mathcal{L}\{\phi_{o,a}(t)\} = \hat{\phi}_{o,a}(s)$ とすると、

$$\begin{aligned}
\mathcal{L}[a_{o,a}(t)\cos(\omega_c t + \phi_o)] &= \frac{1}{2}\mathcal{L}[(a_{o,a}(t)e^{j(\omega_c t+\phi_o)} + a_{o,a}(t)e^{-j(\omega_c t+\phi_o)})] \\
&= \frac{1}{2}\left(\int_0^\infty a_{o,a}(t)e^{j(\omega_c t+\phi_o)}e^{-st}dt + \int_0^\infty a_{o,a}(t)e^{-j(\omega_c t+\phi_o)}e^{-st}dt\right) \\
&= \frac{e^{j\phi_o}}{2}\int_0^\infty a_{o,a}(t)e^{-(s-j\omega_c)t}dt + \frac{e^{-j\phi_o}}{2}\int_0^\infty a_{o,a}(t)e^{-(s+j\omega_c)t}dt \\
&= \frac{1}{2}\left\{e^{j\phi_o}\hat{a}_{o,a}(s-j\omega_c) + e^{-j\phi_o}\hat{a}_{o,a}(s+j\omega_c)\right\} \qquad \text{(A.4)}
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}[\phi_{o,a}(t)\sin(\omega_c t + \phi_o)] &= \frac{1}{2j}\mathcal{L}[(\phi_{o,a}(t)e^{j(\omega_c t+\phi_o)} - \phi_{o,a}(t)e^{-j(\omega_c t+\phi_o)})] \\
&= \frac{1}{2j}\left(\int_0^\infty \phi_{o,a}(t)e^{j(\omega_c t+\phi_o)}e^{-st}dt - \int_0^\infty \phi_{o,a}(t)e^{-j(\omega_c t+\phi_o)}e^{-st}dt\right) \\
&= \frac{e^{j\phi_o}}{2j}\int_0^\infty \phi_{o,a}(t)e^{-(s-j\omega_c)t}dt - \frac{e^{-j\phi_o}}{2j}\int_0^\infty \phi_{o,a}(t)e^{-(s+j\omega_c)t}dt \\
&= -\frac{j}{2}\left\{e^{j\phi_o}\hat{\phi}_{o,a}(s-j\omega_c) - e^{-j\phi_o}\hat{\phi}_{o,a}(s+j\omega_c)\right\} \qquad \text{(A.5)}
\end{aligned}$$

付録 B　変調

B.1　定義とベクトル表記

高周波の搬送波で情報を伝送するには変調が必要である。実際、信号が時間の純粋な正弦関数から外れるたびに変調が行われる。

振幅 (a(t)) と位相 (p(t)) で変調された ω_c における搬送波正弦波の時間依存性は次式で表される:

$$x(t) = \mathrm{Re}\Big\{\hat{X}(1 + a(t))e^{j(\omega_c t + \phi(t))}\Big\} \tag{B.1}$$

純粋な正弦波変調の場合:

$$x(t) = \mathrm{Re}\Big\{\hat{X}(1 + \hat{a}\cos\omega_{\mathrm{AM}} t)e^{j(\omega_c t + \hat{\phi}\sin\omega_{\mathrm{PM}} t)}\Big\} \tag{B.2}$$

B.2 Pedersen Model

B.2.1 空洞を介した位相、振幅、チューニング伝送

位相および振幅が変調された正弦波信号は次のように表せる。

位相および振幅変調された正弦波信号を送信する場合：

$a_i(t)$ で振幅変調され、$\phi(t)$ で位相変調された角振動数 ω_c の正弦波信号は、次のように表される。

$$x(t) = \mathrm{Re}\Big\{A_i(1 + a_i(t))e^{j(\omega_c t + \phi_i(t))}\Big\} \tag{B.3}$$

$a_i \ll 1,\ \phi_i \ll 1$ の場合、(B.3) は

$$\begin{aligned} x(t) &\simeq \mathrm{Re}\big\{A_i(1 + a_i(t))(1 + j\phi_i(t))e^{j\omega_c t}\big\} \\ &= \mathrm{Re}\left\{A_i(1 + a_i(t) + j\phi_i(t) + j\underbrace{a_i(t)\phi_i(t)}_{0})e^{j\omega_c t}\right\} \\ &\simeq \mathrm{Re}\big\{A_i(1 + a_i(t) + j\phi_i(t))e^{j\omega_c t}\big\} \\ &= \mathrm{Re}\{A_i(1 + a_i(t) + j\phi_i(t))(\cos\omega_c t + j\sin\omega_c t)\} \\ &= A_i(\cos\omega_c t + a_i(t)\cos\omega_c t - \phi(t)\sin\omega_c t) \end{aligned} \tag{B.4}$$

ここで、

$$\begin{aligned} \mathcal{L}[\cos\omega_c t] &= \frac{1}{2}\mathcal{L}[(e^{j\omega_c t} + e^{-j\omega_c t})] \\ &= \frac{1}{2}\left(\int_0^\infty e^{j\omega_c t}e^{-st}dt + \int_0^\infty e^{-j\omega_c t}e^{-st}dt\right) \\ &= \frac{1}{2}\left(\int_0^\infty e^{-(s-j\omega_c)t}dt + \int_0^\infty e^{-(s+j\omega_c)t}dt\right) \\ &= \frac{1}{2}\left(\left[\frac{e^{-(s-j\omega_c)t}}{-(s-j\omega_c)}\right]_0^\infty + \left[\frac{e^{-(s+j\omega_c)t}}{-(s+j\omega_c)}\right]_0^\infty\right) \\ &= \frac{1}{2}\left(\frac{1}{s-j\omega_c} + \frac{1}{s+j\omega_c}\right) = \frac{s}{s^2+\omega_c^2} \end{aligned} \tag{B.5}$$

また、$\mathcal{L}\{a_i(t)\} = a_i(s),\ \mathcal{L}\{\phi_i(t)\} = \phi_i(s)$ とすると、

$$\begin{aligned} \mathcal{L}[a_i(t)\cos\omega_c t] &= \frac{1}{2}\mathcal{L}[(a_i(t)e^{j\omega_c t} + a_i(t)e^{-j\omega_c t})] \\ &= \frac{1}{2}\left(\int_0^\infty a_i(t)e^{j\omega_c t}e^{-st}dt + \int_0^\infty a_i(t)e^{-j\omega_c t}e^{-st}dt\right) \\ &= \frac{1}{2}\left(\int_0^\infty a_i(t)e^{-(s-j\omega_c)t}dt + \int_0^\infty a_i(t)e^{-(s+j\omega_c)t}dt\right) \\ &= \frac{a_i(s-j\omega_c) + a_i(s+j\omega_c)}{2} \end{aligned} \tag{B.6}$$

$$
\begin{aligned}
\mathcal{L}[\phi_i(t)\sin\omega_c t] &= \frac{1}{2j}\mathcal{L}[(\phi_i(t)e^{j\omega_c t} - \phi_i(t)e^{-j\omega_c t})] \\
&= \frac{1}{2j}\left(\int_0^\infty \phi_i(t)e^{j\omega_c t}e^{-st}dt - \int_0^\infty \phi_i(t)e^{-j\omega_c t}e^{-st}dt\right) \\
&= \frac{1}{2j}\left(\int_0^\infty \phi_i(t)e^{-(s-j\omega_c)t}dt - \int_0^\infty \phi_i(t)e^{-(s+j\omega_c)t}dt\right) \\
&= \frac{\phi_i(s-j\omega_c) - \phi_i(s+j\omega_c)}{2j}
\end{aligned} \tag{B.7}
$$

以上より、$X(s) = \mathcal{L}\{x(t)\}$ とし、(B.4) をラプラス変換すると、

$$
X(s) = \frac{A_i}{2}\left\{\frac{2s}{s^2+\omega_c^2} + a_i(s-j\omega_c) + j\phi_i(s-j\omega_c) + a_i(s+j\omega_c) - j\phi_i(s+j\omega_c)\right\} \tag{B.8}
$$

伝達関数 $H(s)$ を介して、出力される信号 $y(t)$ は一般的に振幅と位相の両方は変調され、次の様に表せる。

$$
y(t) = \mathrm{Re}\Big\{A_o(1+a_o(t))e^{j(\omega_c t+\phi_z+\phi_o(t))}\Big\} \tag{B.9}
$$

$a_o \ll 1,\ \phi_o \ll 1$ の場合、(B.9) は

$$
\begin{aligned}
y(t) &\simeq \mathrm{Re}\Big\{A_o(1+a_o(t))(1+j\phi_o(t))e^{j(\omega_c t+\phi_z)}\Big\} \\
&= \mathrm{Re}\left\{A_i(1+a_o(t)+j\phi_o(t)+j\underbrace{a_o(t)\phi_o(t)}_{0})e^{j(\omega_c t+\phi_z)}\right\} \\
&\simeq \mathrm{Re}\{A_o(1+a_o(t)+j\phi_o(t))(\cos(\omega_c t+\phi_z)+j\sin(\omega_c+\phi_z))\} \\
&= A_o(\cos(\omega_c t+\phi_z) + a_o(t)\cos(\omega_c t+\phi_z) - \phi_o(t)\sin(\omega_c t+\phi_z))
\end{aligned} \tag{B.10}
$$

ここで、

$$
\begin{aligned}
\mathcal{L}[\cos(\omega_c t+\phi_z)] &= \frac{1}{2}\mathcal{L}[(e^{j(\omega_c t+\phi_z)} + e^{-j(\omega_c t+\phi_z)})] \\
&= \frac{1}{2}\left(\int_0^\infty e^{j(\omega_c t+\phi_z)}e^{-st}dt + \int_0^\infty e^{-j()\omega_c t+\phi_z)}e^{-st}dt\right) \\
&= \frac{e^{j\phi_z}}{2}\int_0^\infty e^{-(s-j\omega_c)t}dt + \frac{e^{-j\phi_z}}{2}\int_0^\infty e^{-(s+j\omega_c)t}dt \\
&= \frac{e^{j\phi_z}}{2}\left[\frac{e^{-(s-j\omega_c)t}}{-(s-j\omega_c)}\right]_0^\infty + \frac{e^{-j\phi_z}}{2}\left[\frac{e^{-(s+j\omega_c)t}}{-(s+j\omega_c)}\right]_0^\infty \\
&= \frac{1}{2}\left(\frac{e^{j\phi_z}}{s-j\omega_c} + \frac{e^{-j\phi_z}}{s+j\omega_c}\right) = \frac{\cos\phi_z s - \omega_c\sin\phi_z}{s^2+\omega_c^2}
\end{aligned} \tag{B.11}
$$

また、$\mathcal{L}\{a_o(t)\} = a_o(s)$, $\mathcal{L}\{\phi_o(t)\} = \phi_o(s)$ とすると、

$$\begin{aligned}
\mathcal{L}[a_o(t)\cos(\omega_c t + \phi_z)] &= \frac{1}{2}\mathcal{L}[(a_o(t)e^{j(\omega_c t+\phi_z)} + a_o(t)e^{-j(\omega_c t+\phi_z)})] \\
&= \frac{1}{2}\left(\int_0^\infty a_o(t)e^{j(\omega_c t+\phi_z)}e^{-st}dt + \int_0^\infty a_o(t)e^{-j(\omega_c t+\phi_z)}e^{-st}dt\right) \\
&= \frac{e^{j\phi_z}}{2}\int_0^\infty a_o(t)e^{-(s-j\omega_c)t}dt + \frac{e^{-j\phi_z}}{2}\int_0^\infty a_o(t)e^{-(s+j\omega_c)t}dt \\
&= \frac{e^{j\phi_z}}{2}a_o(s-j\omega_c) + \frac{e^{-j\phi_z}}{2}a_o(s+j\omega_c) \qquad \text{(B.12)}
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}[\phi_o(t)\sin(\omega_c t + \phi_z)] &= \frac{1}{2j}\mathcal{L}[(\phi_o(t)e^{j(\omega_c t+\phi_z)} - \phi_o(t)e^{-j(\omega_c t+\phi_z)})] \\
&= \frac{1}{2j}\left(\int_0^\infty \phi_o(t)e^{j(\omega_c t+\phi_z)}e^{-st}dt - \int_0^\infty \phi_o(t)e^{-j(\omega_c t+\phi_z)}e^{-st}dt\right) \\
&= \frac{e^{j\phi_z}}{2j}\int_0^\infty \phi_o(t)e^{-(s-j\omega_c)t}dt - \frac{e^{-j\phi_z}}{2j}\int_0^\infty \phi_o(t)e^{-(s+j\omega_c)t}dt \\
&= \frac{e^{j\phi_z}}{2j}\phi_o(s-j\omega_c) - \frac{e^{-j\phi_z}}{2j}\phi_o(s+j\omega_c) \qquad \text{(B.13)}
\end{aligned}$$

以上より、$Y(s) = \mathcal{L}\{y(t)\}$ とし、(B.10) をラプラス変換すると、

$$\begin{aligned}
Y(s) = A_o\Bigg[\frac{\cos\phi_z s - \omega_c \sin\phi_z}{s^2+\omega_c^2} &+ \frac{e^{j\phi_z}}{2}\{a_o(s-j\omega_c) + j\phi_o(s-j\omega_c)\} \\
&+ \frac{e^{-j\phi_z}}{2}\{a_o(s+j\omega_c) + j\phi_o(s+j\omega_c)\}\Bigg] \qquad \text{(B.14)}
\end{aligned}$$

ここで、

$$\begin{cases} a_o(s) = G_{aa}(s)a_i(s) + G_{pa}(s)\phi_i(s) \\ \phi_o(s) = G_{ap}(s)a_i(s) + G_{pp}(s)\phi_i(s) \end{cases} \qquad \text{(B.15)}$$

より、(B.15) を (B.14) に代入すると、

$$\begin{aligned}
Y(s) = \frac{A_o}{2}\Bigg[&\frac{2(\cos\phi_z s - \omega_c\sin\phi_z)}{s^2+\omega_c^2} \\
&+ e^{j\phi_z}\{(G_{aa}(s-j\omega_c) + jG_{ap}(s-j\omega_c))a_i(s-j\omega_c) \\
&+ (G_{pa}(s-j\omega_c) + jG_{pp}(s-j\omega_c))\}\phi_i(s-j\omega_c) \\
&+ e^{-j\phi_z}\{(G_{aa}(s+j\omega_c) - jG_{ap}(s+j\omega_c))a_i(s+j\omega_c) \\
&+ (G_{pa}(s+j\omega_c) - jG_{pp}(s+j\omega_c))\}\phi_i(s-+j\omega_c)\Bigg] \qquad \text{(B.16)}
\end{aligned}$$

ここで $Y(s) = H(s)X(s)$ より、

$$Y(s) = \frac{A_i H(s)}{2}\left\{\frac{2s}{s^2+\omega_c^2} + a_i(s-j\omega_c) + j\phi_i(s-j\omega_c) + a_i(s+j\omega_c) - j\phi_i(s+j\omega_c)\right\} \qquad \text{(B.17)}$$

(B.16) と (B.17) で係数を比較すると、

付録 C　加速空洞の等価回路

C.1　RLC 並列共振回路の微分方程式

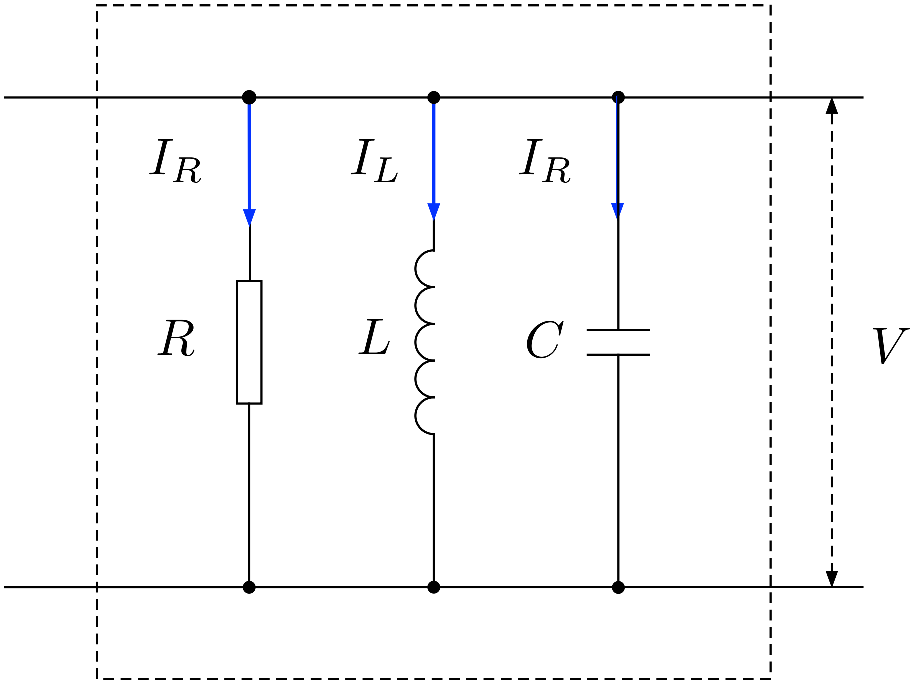

Fig. C.1　Cavity RLC Model

$$
\begin{aligned}
I(t) &= I_R(t) + I_L(t) + I_C(t) \\
I_R(t) &= \frac{V(t)}{R},\ I_L(t) = \frac{1}{L}\int V(t)dt,\ I_c(t) = C\frac{dV(t)}{dt}
\end{aligned}
\tag{C.1}
$$

したがって、

$$
\frac{V(t)}{R} + \frac{1}{L}\int V(t)dt + C\frac{dV(t)}{dt} = I(t) \tag{C.2}
$$

両辺を時間 t で微分すると、

$$
\frac{\dot{V}(t)}{R} + \frac{V(t)}{L} + C\ddot{V}(t) = I\dot{(}t) \tag{C.3}
$$

$$
\ddot{V}(t) + \frac{1}{CR}\dot{V}(t) + \frac{1}{LC}V(t) = \frac{1}{C}\dot{I}(t) \tag{C.4}
$$

$$\omega_0^2 = \frac{1}{LC}, \qquad \sigma = \frac{R}{2}\sqrt{\frac{C}{L}} = \frac{\omega_0}{2Q_L} \tag{C.5}$$

$$\ddot{V}(t) + 2\sigma\dot{V}(t) + \omega_0^2 V(t) = 2\sigma R\dot{I}(t) \tag{C.6}$$

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