

Beam Loading

Shin-ichi YOSHIMOTO

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Contents

第 1 章	Static Beam Loading	5
1.1	Cavity の基礎	5
1.1.1	Particle Acceleration in an RF Field	5
1.1.2	Power dissipation	5
1.1.3	Shut impedance	5
1.1.4	Available Power	5
1.2	ビーム負荷付き空洞の RCL 等価回路	6
1.2.1	Cavity parameters	6
1.3	Cavity voltage	9
1.4	Steady-state conditions	9
1.5	Analysis of small perturbations	9
1.6	Equations for cavity voltage	9
第 2 章	Amplitude and phase modulation	11
2.1	Modulation Transfer Function	11
2.1.1	空洞共振器への適用	16
第 3 章	Robinson instability	17
付録 A	周波数伝達関数	19
A.1	部分分数展開を用いたラプラス逆変換の計算	19
付録 B	ラプラス変換	21
B.1	Pedersen Model	22
B.1.1	空洞を介した位相、振幅、チューニング伝送	22
付録 C	加速空洞の等価回路	27
C.1	RLC 並列共振回路の微分方程式	27

第 1 章 Static Beam Loading

1.1 Cavity の基礎

1.1.1 Particle Acceleration in an RF Field

$$E(z, t) = E(z) \cos(\omega t + \phi) \quad (1.1)$$

1.1.2 Power dissipation

$$P_{diss} = \frac{1}{2} \cdot \frac{V_{cav}^2}{R} = \frac{V_{cav}^2}{R_{sh}} \quad (1.2)$$

1.1.3 Shut impedance

$$R = \frac{1}{2} \cdot R_{sh} \quad (1.3)$$

空洞の入力カップラーの変圧比を $1 : n$ とすると (Fig. 1.1)、

$$V_{out} = N \cdot V_{in}, \quad I_{out} = \frac{1}{N} \cdot I_{in} \quad (1.4)$$

したがって、

$$Z_{out} = n^2 \cdot Z_{in} \quad (1.5)$$

1.1.4 Available Power

$$P_{diss} = \frac{1}{2} Y_c V_g^2 \quad (1.6)$$

$$V_g = \frac{I_g}{Y_L} = \frac{I_g}{Y_g + Y_c} \quad (1.7)$$

$$P_{diss} = \frac{1}{2} \frac{Y_c}{(Y_g + Y_c)^2} I_g^2 \quad (1.8)$$

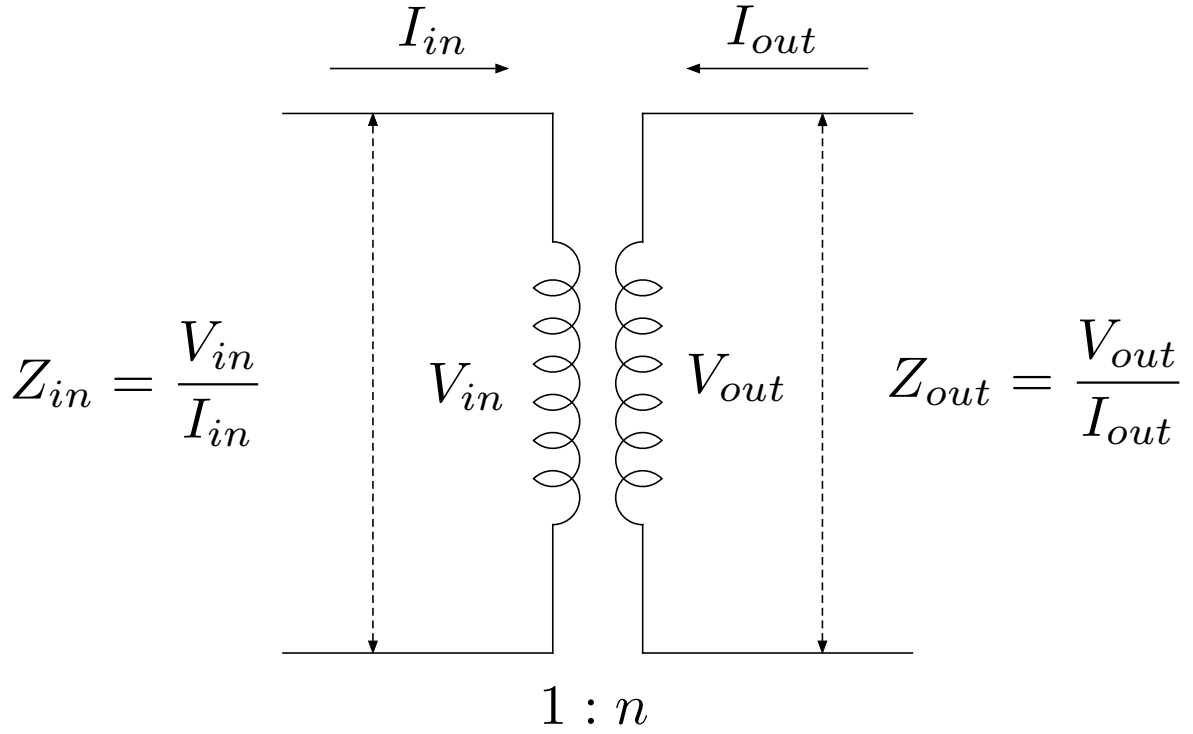


Fig. 1.1 理想的なトランスによる入力カップラー.

$$\begin{aligned} \frac{\partial P_{diss}}{\partial Y_c} &= \frac{1}{2} \frac{(Y_g + Y_c)^2 - 2Y_c(Y_g + Y_c)}{(Y_g + Y_c)^4} I_g^2 \\ &= \frac{1}{2} \frac{Y_g - Y_c}{(Y_g + Y_c)^3} I_g^2 \end{aligned} \quad (1.9)$$

$\partial P_{diss} / \partial Y_c = 0$ より、 $Y_g = Y_c$ の時、 P_{diss} は最大になり

$$P_{diss}^{max} = \frac{1}{8} Y_g I_g^2 = \frac{1}{8} \frac{R}{\beta} I_g^2 \equiv P_g \quad (1.10)$$

1.2 ビーム負荷付き空洞の RCL 等価回路

1.2.1 Cavity parameters

共振周波数

$$\omega_0 = \frac{1}{LC} \quad (1.11)$$

Quality factor

$$Q = 2\pi \frac{\text{stored energy in cavity}}{\text{dissipated energy per cycle}} = \frac{\omega_0 W}{P_{diss}} \quad (1.12)$$

Unloaded quality factor

$$Q_0 = \omega_0 \frac{(1/2)CV^2}{V^2/(2R)} = \omega_0 RC \quad (1.13)$$

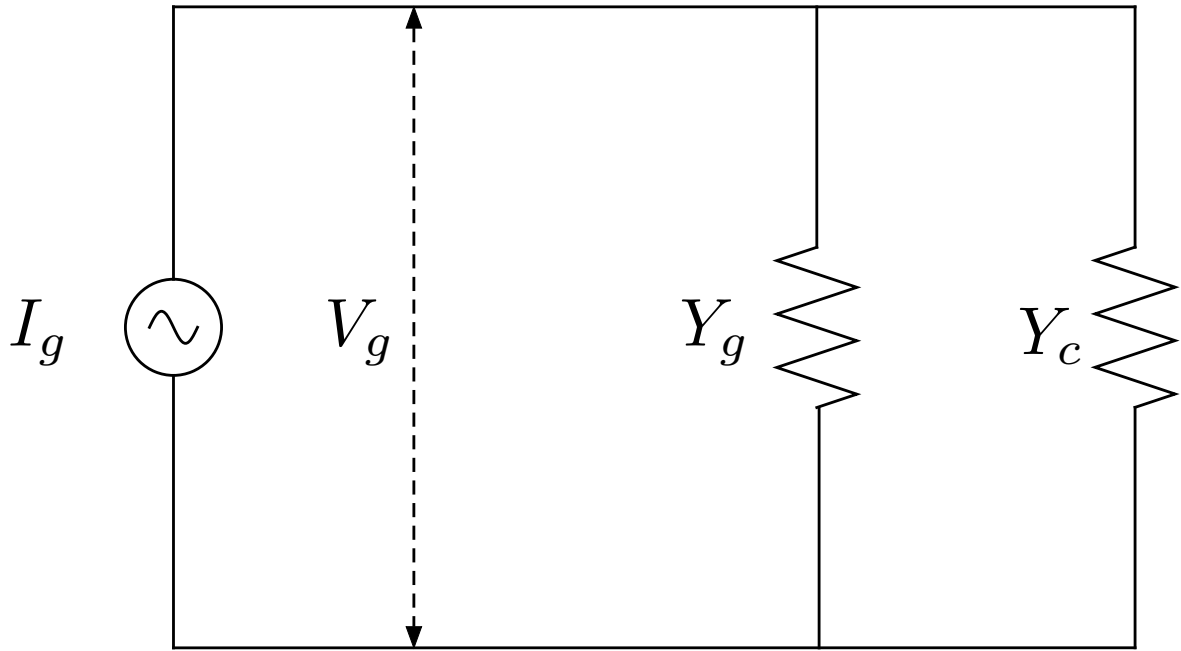


Fig. 1.2 Available Power.

External quality factor

$$Q_{ext} = 2\pi \frac{\text{stored energy in cavity}}{\text{dissipated energy in external devices per cycle}} = \frac{\omega_0 W}{P_{ext}} \quad (1.14)$$

Loaded quality factor

$$Q_L = 2\pi \frac{\text{stored energy in cavity}}{\text{total energy per cycle}} = \frac{\omega_0 W}{P_{tot}} \quad (1.15)$$

ここで、

$$P_{tot} = P_{diss} + P_{ext} \quad (1.16)$$

したがって、

$$\frac{1}{Q_L} = \frac{1}{Q_0} + \frac{1}{Q_{ext}} \quad (1.17)$$

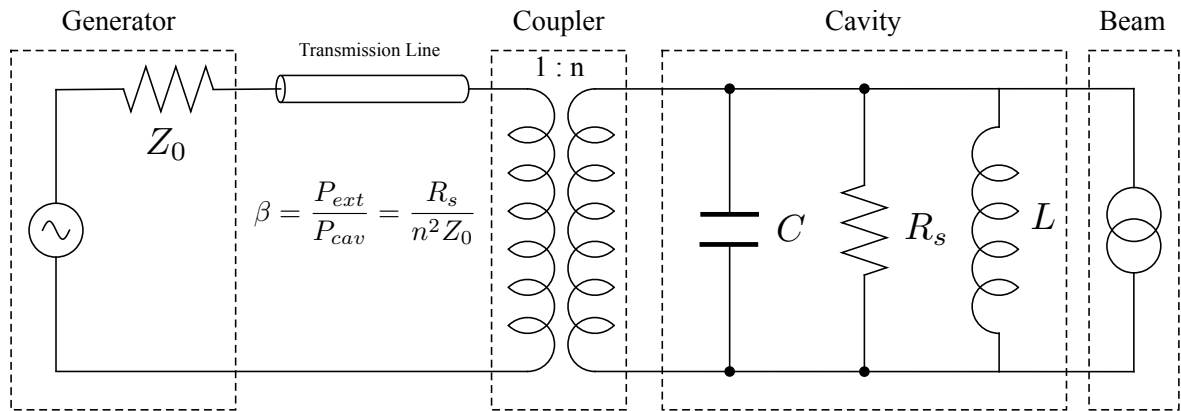


Fig. 1.3 Simplified model of cavity coupled to an RF generator by coupler and transmission lines.

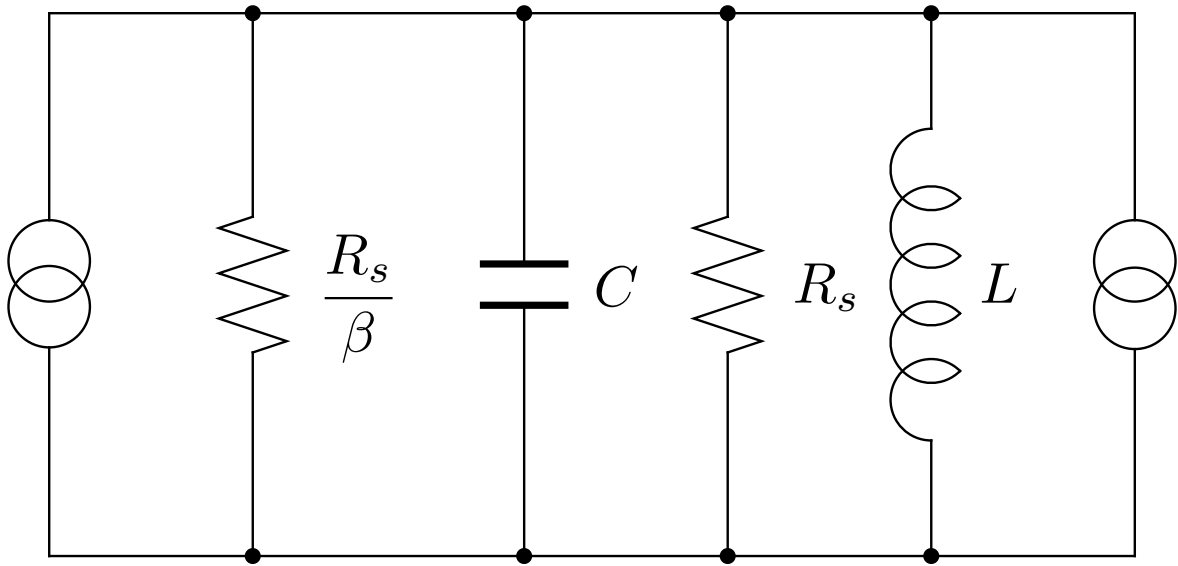


Fig. 1.4 等価回路

$$\frac{1}{R_L} = \frac{1}{R} + \frac{1}{Z_{ext}} \quad (1.18)$$

Coupling factor β

$$\beta = \frac{P_{ext}}{P_{cav}} = \frac{Q_0}{Q_{ext}} = \frac{R}{Z_{ext}} = \frac{R}{n^2 Z_0} \quad (1.19)$$

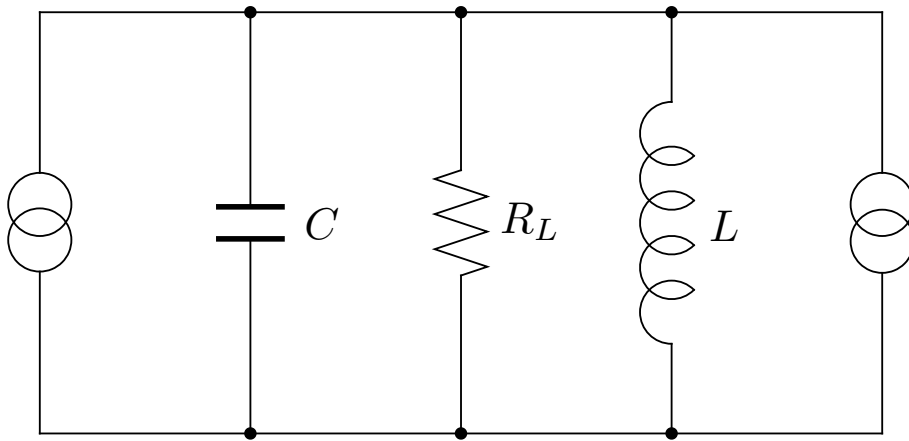


Fig. 1.5 等価回路 2

$$\ddot{V}(t) + \frac{1}{R_L C} \dot{V}(t) + \frac{1}{LC} V(t) = \frac{1}{C} \dot{I}(t) \quad (1.20)$$

Phasor

$$V(t) = \tilde{V} e^{j\omega_c t}, \quad I(t) = \tilde{I} e^{j\omega_c t} \quad (1.21)$$

1.3 Cavity voltage

$$\left(\frac{d^2}{dt^2} + \frac{\omega_c}{Q_L} \frac{d}{dt} + \omega_c^2 \right) V_c(t) = \frac{\omega_c R_L}{Q_L} \frac{dI_{tot}}{dt} \quad (1.22)$$

1.4 Steady-state conditions

1.5 Analysis of small perturbations

$$\delta V_c(t) = \frac{1}{2} \{ V_c e^{-j\omega_g t} U(t) + c.c \} \quad (1.23)$$

1.6 Equations for cavity voltage

$$\begin{aligned} \mathbf{V}_c(t) &= \tilde{\mathbf{V}}(t) e^{j\omega_c t}, & \tilde{\mathbf{V}}(t) &= \hat{\mathbf{V}}(1 + a_v(t)) e^{j(\phi_v(t) + \psi_v)} \\ \mathbf{I}_b(t) &= \tilde{\mathbf{I}}_b(t) e^{j\omega_c t}, & \tilde{\mathbf{I}}_b(t) &= \hat{\mathbf{I}}_b(1 + a_b(t)) e^{j(\phi_b(t) + \psi_b)} \\ \mathbf{I}_g(t) &= \tilde{\mathbf{I}}_g(t) e^{j\omega_c t}, & \tilde{\mathbf{I}}_g(t) &= \hat{\mathbf{I}}_g(1 + a_g(t)) e^{j(\phi_g(t) + \psi_g)} \\ \mathbf{I}_t(t) &= \mathbf{I}_g(t) + \mathbf{I}_b(t) = \tilde{\mathbf{I}}_t(t) e^{j\omega_c t} \end{aligned} \quad (1.24)$$

$$\dot{\mathbf{V}}_c(t) + 2\sigma \dot{\mathbf{V}}_c(t) + \omega_0^2 \mathbf{V}_c(t) = 2\sigma R \dot{\mathbf{I}}_t(t) \quad (1.25)$$

第 2 章 Amplitude and phase modulation

2.1 Modulation Transfer Function

振幅変調 (AM) や位相変調 (PM) された正弦波信号を伝達関数 $H(s)$ を持つシステムを介した場合、出力信号も一般に振幅変調と位相変調の両方を受けることになる。このようなシステムの完全な特性評価には、システムの伝達関数 $H(s)$ から導き出すことができる次の 4 つの異なる変調伝達関数 (Modulation Transfer Function) を求める必要がある。(Fig.2.1)

1. 入力の変調振幅が出力の振幅を変調する伝達関数 : $G_{aa}(s)$
2. 入力の変調位相が出力の位相を変調する伝達関数 : $G_{pp}(s)$
3. 入力の変調振幅が出力の位相を変調する伝達関数 : $G_{ap}(s)$
4. 入力の変調位相が出力の振幅を変調する伝達関数 : $G_{pa}(s)$

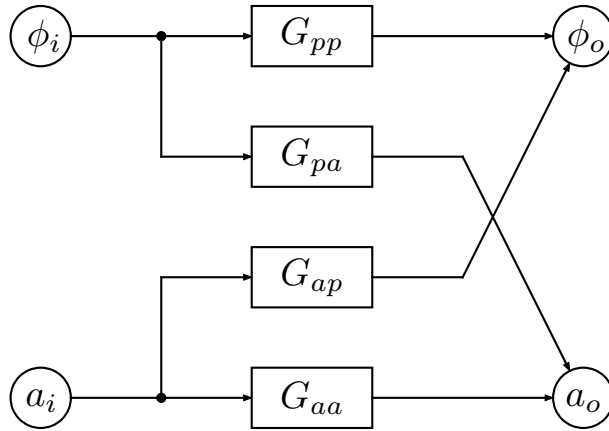


Fig. 2.1 Modulation Transfer Function.

変調の振幅が十分小さい場合、変調の伝送は線形であり、次のようにして求めることができる。

最初に、入力信号の振幅のみが $a_i(t)$ で変調された場合を考える。このとき、入力信号 $x(t)$ は次のようにあらわせる。

$$\begin{aligned} x(t) &= \text{Re} \left\{ \hat{X}(1 + a_i(t))e^{j\omega_c t} \right\} \\ &= \hat{X} \{ \cos \omega_c t + a_i(t) \cos \omega_c t \} \end{aligned} \quad (2.1)$$

出力信号 $y(t)$ は $a_i(t)$ によって振幅だけでなく位相も変調され、

$$y(t) = \text{Re} \left\{ \hat{Y} (1 + a_{o,a}(t)) e^{j(\omega_c t + \phi_0 + \phi_{o,a}(t))} \right\} \quad (2.2)$$

と表すことができる。この時、変調伝達関数 $G_{aa}(s)$, $G_{ap}(s)$ は、次のように定義される。

$$G_{aa}(s) = \frac{\hat{a}_{o,a}(s)}{\hat{a}_i(s)}, \quad G_{ap}(s) = \frac{\hat{\phi}_{o,a}(s)}{\hat{a}_i(s)} \quad (2.3)$$

ここで、

$$\begin{aligned} \mathcal{L}[\cos \omega_c t] &= \frac{1}{2} \mathcal{L}[(e^{j\omega_c t} + e^{-j\omega_c t})] \\ &= \frac{1}{2} \left(\int_0^\infty e^{j\omega_c t} e^{-st} dt + \int_0^\infty e^{-j\omega_c t} e^{-st} dt \right) \\ &= \frac{1}{2} \left(\int_0^\infty e^{-(s-j\omega_c)t} dt + \int_0^\infty e^{-(s+j\omega_c)t} dt \right) \\ &= \frac{1}{2} \left(\left[\frac{e^{-(s-j\omega_c)t}}{-(s-j\omega_c)} \right]_0^\infty + \left[\frac{e^{-(s+j\omega_c)t}}{-(s+j\omega_c)} \right]_0^\infty \right) \\ &= \frac{1}{2} \left(\frac{1}{s-j\omega_c} + \frac{1}{s+j\omega_c} \right) = \frac{s}{s^2 + \omega_c^2} \end{aligned} \quad (2.4)$$

また、 $\mathcal{L}\{a_i(t)\} = \hat{a}_i(s)$ とすると、

$$\begin{aligned} \mathcal{L}[a_i(t) \cos \omega_c t] &= \frac{1}{2} \mathcal{L}[(a_i(t) e^{j\omega_c t} + a_i(t) e^{-j\omega_c t})] \\ &= \frac{1}{2} \left(\int_0^\infty a_i(t) e^{j\omega_c t} e^{-st} dt + \int_0^\infty a_i(t) e^{-j\omega_c t} e^{-st} dt \right) \\ &= \frac{1}{2} \left(\int_0^\infty a_i(t) e^{-(s-j\omega_c)t} dt + \int_0^\infty a_i(t) e^{-(s+j\omega_c)t} dt \right) \\ &= \frac{1}{2} \{ \hat{a}_i(s-j\omega_c) + \hat{a}_i(s+j\omega_c) \} \end{aligned} \quad (2.5)$$

$X(s) = \mathcal{L}\{x(t)\}$ とし、(2.1) をラプラス変換すると、

$$X(s) = \frac{\hat{X}}{2} \left\{ \frac{2s}{s^2 + \omega_c^2} + a_i(s-j\omega_c) + a_i(s+j\omega_c) \right\} \quad (2.6)$$

一方、出力信号 $y(t)$ は $a_o \ll 1$, $\phi_o \ll 1$ の場合、以下のように近次できる。

$$\begin{aligned} y(t) &\simeq \text{Re} \left\{ \hat{Y} (1 + a_{o,a}(t)) (1 + j\phi_{o,a}(t)) e^{j(\omega_c t + \phi_0)} \right\} \\ &= \text{Re} \left\{ \hat{Y} (1 + a_{o,a}(t) + j\phi_{o,a}(t) + \underbrace{j a_{o,a}(t) \phi_{o,a}(t)}_0) e^{j(\omega_c t + \phi_0)} \right\} \\ &\simeq \text{Re} \left\{ \hat{Y} (1 + a_{o,a}(t) + j\phi_{o,a}(t)) (\cos(\omega_c t + \phi_0) + j \sin(\omega_c t + \phi_0)) \right\} \\ &= \hat{Y} \{ \cos(\omega_c t + \phi_0) + a_o(t) \cos(\omega_c t + \phi_0) - \phi_o(t) \sin(\omega_c t + \phi_0) \} \end{aligned} \quad (2.7)$$

ここで、

$$\begin{aligned}
\mathcal{L}[\cos(\omega_c t + \phi_o)] &= \frac{1}{2} \mathcal{L}[(e^{j(\omega_c t + \phi_o)} + e^{-j(\omega_c t + \phi_o)})] \\
&= \frac{1}{2} \left(\int_0^\infty e^{j(\omega_c t + \phi_o)} e^{-st} dt + \int_0^\infty e^{-j(\omega_c t + \phi_o)} e^{-st} dt \right) \\
&= \frac{e^{j\phi_o}}{2} \int_0^\infty e^{-(s-j\omega_c)t} dt + \frac{e^{-j\phi_o}}{2} \int_0^\infty e^{-(s+j\omega_c)t} dt \\
&= \frac{e^{j\phi_o}}{2} \left[\frac{e^{-(s-j\omega_c)t}}{-(s-j\omega_c)} \right]_0^\infty + \frac{e^{-j\phi_o}}{2} \left[\frac{e^{-(s+j\omega_c)t}}{-(s+j\omega_c)} \right]_0^\infty \\
&= \frac{1}{2} \left(\frac{e^{j\phi_o}}{s-j\omega_c} + \frac{e^{-j\phi_o}}{s+j\omega_c} \right) = \frac{\cos \phi_o s - \omega_c \sin \phi_o}{s^2 + \omega_c^2} \tag{2.8}
\end{aligned}$$

また、 $\mathcal{L}\{a_{o,a}(t)\} = \hat{a}_{o,a}(s)$, $\mathcal{L}\{\phi_{o,a}(t)\} = \hat{\phi}_{o,a}(s)$ とすると、

$$\begin{aligned}
\mathcal{L}[a_{o,a}(t) \cos(\omega_c t + \phi_o)] &= \frac{1}{2} \mathcal{L}[(a_{o,a}(t) e^{j(\omega_c t + \phi_o)} + a_{o,a}(t) e^{-j(\omega_c t + \phi_o)})] \\
&= \frac{1}{2} \left(\int_0^\infty a_{o,a}(t) e^{j(\omega_c t + \phi_o)} e^{-st} dt + \int_0^\infty a_{o,a}(t) e^{-j(\omega_c t + \phi_o)} e^{-st} dt \right) \\
&= \frac{e^{j\phi_o}}{2} \int_0^\infty a_{o,a}(t) e^{-(s-j\omega_c)t} dt + \frac{e^{-j\phi_o}}{2} \int_0^\infty a_{o,a}(t) e^{-(s+j\omega_c)t} dt \\
&= \frac{1}{2} \{ e^{j\phi_o} \hat{a}_{o,a}(s-j\omega_c) + e^{-j\phi_o} \hat{a}_{o,a}(s+j\omega_c) \} \tag{2.9}
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}[\phi_{o,a}(t) \sin(\omega_c t + \phi_o)] &= \frac{1}{2j} \mathcal{L}[(\phi_{o,a}(t) e^{j(\omega_c t + \phi_o)} - \phi_{o,a}(t) e^{-j(\omega_c t + \phi_o)})] \\
&= \frac{1}{2j} \left(\int_0^\infty \phi_{o,a}(t) e^{j(\omega_c t + \phi_o)} e^{-st} dt - \int_0^\infty \phi_{o,a}(t) e^{-j(\omega_c t + \phi_o)} e^{-st} dt \right) \\
&= \frac{e^{j\phi_o}}{2j} \int_0^\infty \phi_{o,a}(t) e^{-(s-j\omega_c)t} dt - \frac{e^{-j\phi_o}}{2j} \int_0^\infty \phi_{o,a}(t) e^{-(s+j\omega_c)t} dt \\
&= -\frac{j}{2} \{ e^{j\phi_o} \hat{\phi}_{o,a}(s-j\omega_c) - e^{-j\phi_o} \hat{\phi}_{o,a}(s+j\omega_c) \} \tag{2.10}
\end{aligned}$$

したがって、 $Y(s) = \mathcal{L}\{y(t)\}$ とし (2.7) をラプラス変換すると、

$$\begin{aligned}
Y(s) &= \frac{\hat{Y}}{2} \left[\frac{2(\cos \phi_o s - \omega_c \sin \phi_o)}{s^2 + \omega_c^2} + e^{j\phi_o} \hat{a}_{o,a}(s-j\omega_c) + e^{-j\phi_o} \hat{a}_{o,a}(s+j\omega_c) \right. \\
&\quad \left. + j \{ e^{j\phi_o} \hat{\phi}_{o,a}(s-j\omega_c) - e^{-j\phi_o} \hat{\phi}_{o,a}(s+j\omega_c) \} \right] \tag{2.11}
\end{aligned}$$

ここで、(2.3) より (2.11) は

$$\begin{aligned}
Y(s) &= \frac{\hat{Y}}{2} \left[\frac{2(\cos \phi_o s - \omega_c \sin \phi_o)}{s^2 + \omega_c^2} \right. \\
&\quad + e^{j\phi_o} G_{aa}(s-j\omega_c) \hat{a}_i(s-j\omega_c) + e^{-j\phi_o} G_{aa}(s+j\omega_c) \hat{a}_i(s+j\omega_c) \\
&\quad \left. + j \{ e^{j\phi_o} G_{ap}(s-j\omega_c) \hat{a}_i(s-j\omega_c) - e^{-j\phi_o} G_{ap}(s+j\omega_c) \hat{a}_i(s+j\omega_c) \} \right]
\end{aligned}$$

したがって、

$$Y(s) = \frac{\hat{Y}}{2} \left[\frac{2(\cos \phi_0 s - \omega_c \sin \phi_0)}{s^2 + \omega_c^2} + e^{j\phi_0} \{G_{aa}(s - j\omega_c) + jG_{ap}(s - j\omega_c)\} \hat{a}_i(s - j\omega_c) \right. \\ \left. + e^{-j\phi_0} \{G_{aa}(s + j\omega_c) - jG_{ap}(s + j\omega_c)\} \hat{a}_i(s + j\omega_c) \right] \quad (2.12)$$

ここで $Y(s) = H(s)X(s)$ より、(2.6) は、

$$Y(s) = \frac{\hat{X}H(s)}{2} \left[\frac{2s}{(s^2 + \omega_c^2)} + \hat{a}_i(s - j\omega_c) + \hat{a}_i(s + j\omega_c) \right] \quad (2.13)$$

(2.12) と (2.13) で係数を比較すると、

$$\hat{Y}(\cos \phi_0 s - \omega_c \sin \phi_0) = \hat{X}H(s)s \quad (2.14)$$

$$G_{aa}(s - j\omega_c) + jG_{ap}(s - j\omega_c) = \frac{\hat{X}}{\hat{Y}} e^{-j\phi_0} H(s) \\ G_{aa}(s + j\omega_c) - jG_{ap}(s + j\omega_c) = \frac{\hat{X}}{\hat{Y}} e^{j\phi_0} H(s) \quad (2.15)$$

(2.14) で、 $s = \pm j\omega_c$ の時を考えると、

$$H(\pm j\omega_c) = \frac{\hat{Y}}{\hat{X}} e^{\pm j\phi_0} \quad (2.16)$$

(2.15) と (2.16) より、

$$G_{aa}(s - j\omega_c) + jG_{ap}(s - j\omega_c) = \frac{H(s)}{H(j\omega_c)} \\ G_{aa}(s + j\omega_c) - jG_{ap}(s + j\omega_c) = \frac{H(s)}{H(-j\omega_c)} \quad (2.17)$$

以上より、入力の振幅変調に関する変調伝達関数 $G_{aa}(s)$, $G_{ap}(s)$ は、

$$G_{aa}(s) = \frac{1}{2} \left\{ \frac{H(s + j\omega_c)}{H(j\omega_c)} + \frac{H(s - j\omega_c)}{H(-j\omega_c)} \right\} = G_s(s) \\ G_{ap}(s) = \frac{-j}{2} \left\{ \frac{H(s + j\omega_c)}{H(j\omega_c)} - \frac{H(s - j\omega_c)}{H(-j\omega_c)} \right\} = G_c(s) \quad (2.18)$$

今度は、入力信号の位相のみが $\phi_i(t)$ で変調された信号を考えると、

$$x(t) = \text{Re}\{\hat{X}e^{j(\omega_c t + \phi_i(t))}\} \quad (2.19)$$

出力信号は先程と同様に考えると、

$$y(t) = \text{Re}\{\hat{Y}(1 + a_{o,p}(t))e^{j(\omega_c t + \phi_0 + \phi_{o,p}(t))}\} \quad (2.20)$$

入力の位相変調に関する変調伝達関数 $G_{pp}(s)$, $G_{pa}(s)$ は、

$$G_{pp}(s) = \frac{\hat{\phi}_{o,p}(s)}{\hat{\phi}_i(s)}, \quad G_{pa}(s) = \frac{\hat{a}_{o,p}(s)}{\hat{\phi}_i(s)} \quad (2.21)$$

(2.19) は、 $\phi_i(t) \ll 1$ の時、

$$\begin{aligned} x(t) &= \text{Re}\{\hat{X}e^{j(\omega_c t + \phi_i(t))}\} \\ &\simeq \text{Re}\{\hat{X}(1 + j\phi_i(t))e^{j\omega_c t}\} \\ &= \hat{X}(\cos \omega_c t - \phi_i(t) \sin \omega_c t) \end{aligned} \quad (2.22)$$

(2.22) をラプラス変換すると、

$$X(s) = \hat{X} \left[\frac{s}{s^2 + \omega_c^2} + j \left\{ \hat{\phi}_i(s - j\omega_c) - \hat{\phi}_i(s + j\omega_c) \right\} \right] \quad (2.23)$$

(2.11) と同様にして (2.20) もラプラス変換すると、

$$\begin{aligned} Y(s) = \frac{\hat{Y}}{2} \left[\frac{2(\cos \phi_0 s - \omega_c \sin \phi_0)}{s^2 + \omega_c^2} + e^{j\hat{\phi}_0} a_{o,p}(s - j\omega_c) + e^{-j\hat{\phi}_0} \hat{a}_{o,p}(s + j\omega_c) \right. \\ \left. + j \left\{ e^{j\hat{\phi}_0} \hat{\phi}_{o,p}(s - j\omega_c) - e^{-j\hat{\phi}_0} \hat{\phi}_{o,p}(s + j\omega_c) \right\} \right] \end{aligned} \quad (2.24)$$

同様に (2.24) は

$$\begin{aligned} Y(s) = \frac{\hat{Y}}{2} \left[\frac{2(\cos \phi_0 s - \omega_c \sin \phi_0)}{s^2 + \omega_c^2} + e^{j\hat{\phi}_0} \{G_{pa}(s - j\omega_c) + jG_{pp}(s - j\omega_c)\} \hat{\phi}_i(s - j\omega_c) \right. \\ \left. + e^{-j\hat{\phi}_0} \{G_{pa}(s + j\omega_c) - jG_{pp}(s + j\omega_c)\} \hat{\phi}_i(s + j\omega_c) \right] \end{aligned} \quad (2.25)$$

ここで $Y(s) = H(s)X(s)$ より、

$$Y(s) = \frac{\hat{X}H(s)}{2} \left[\frac{2s}{s^2 + \omega_c^2} + j \left\{ \hat{\phi}_i(s - j\omega_c) - \hat{\phi}_i(s + j\omega_c) \right\} \right] \quad (2.26)$$

(2.25) と (2.26) で係数を比較し、(2.16) を使うと、

$$\begin{aligned} G_{pa}(s - j\omega_c) + jG_{pp}(s - j\omega_c) &= j \frac{H(s)}{H(j\omega_c)} \\ G_{pa}(s + j\omega_c) - jG_{pp}(s + j\omega_c) &= -j \frac{H(s)}{H(-j\omega_c)} \end{aligned} \quad (2.27)$$

したがって、

$$\begin{aligned} G_{pa}(s) + jG_{pp}(s) &= j \frac{H(s + j\omega_c)}{H(j\omega_c)} \\ G_{pa}(s) - jG_{pp}(s) &= -j \frac{H(s - j\omega_c)}{H(-j\omega_c)} \end{aligned} \quad (2.28)$$

以上の結果から、変調伝達関数は以下の様になる。

$$\begin{aligned} G_{aa}(s) = G_{pp}(s) &= \frac{1}{2} \left\{ \frac{H(s + j\omega_c)}{H(j\omega_c)} + \frac{H(s - j\omega_c)}{H(-j\omega_c)} \right\} = G_s(s) \\ G_{ap}(s) = -G_{pa}(s) &= \frac{j}{2} \left\{ \frac{H(s + j\omega_c)}{H(j\omega_c)} - \frac{H(s - j\omega_c)}{H(-j\omega_c)} \right\} = G_c(s) \end{aligned} \quad (2.29)$$

2.1.1 空洞共振器への適用

$$Z(s) = \frac{2\sigma R_s}{s^2 + 2\sigma s + \omega_r^2} \quad (2.30)$$

$$G_{aa}(s) = G_{pp} = \frac{\sigma^2(1 + \tan^2 \phi_z) + \sigma s}{s^2 + 2\sigma s + \sigma^2(1 + \tan^2 \phi_z)} \quad (2.31)$$

$$G_{pa}(s) = -G_{ap} = \frac{\sigma \tan \phi_z s}{s^2 + 2\sigma s + \sigma^2(1 + \tan^2 \phi_z)} \quad (2.32)$$

第 3 章 Robinson instability

付録 A 周波数伝達関数

A.1 部分分数展開を用いたラプラス逆変換の計算

一般に、有理関数で表された伝達関数 $G(s)$ は、

$$G(s) = K \frac{(s + z_1)(s + z_2) \dots (s + z_m)}{(s + p_1)(s + p_2) \dots (s + p_n)} \quad (\text{A.1})$$

入力を、 $u(t) = e^{j\omega t}$ とすると、ラプラス変換は、次式で与えられる。

$$U(s) = \mathcal{L}[u(t)] = \mathcal{L}[e^{j\omega t}] = \frac{1}{s - j\omega} \quad (\text{A.2})$$

したがって、出力 $Y(s) = \mathcal{L}[y(t)]$ は、次のように部分分数展開できる。

$$\begin{aligned} Y(s) &= G(s)U(s) = \frac{k(s - z_1)(s - z_2) \dots (s - z_m)}{(s - p_1)(s - p_2) \dots (s - p_n)} \frac{1}{s - j\omega} \\ &= \frac{k_1}{s - p_1} + \frac{k_2}{s - p_2} + \dots + \frac{k_n}{s - p_n} + \frac{a}{s - j\omega} \end{aligned}$$

ただし、

$$\begin{aligned} k_i &= \lim_{s \rightarrow p_i} (s - p_i)G(s) \frac{s}{s - j\omega} \\ a &= \lim_{s \rightarrow j\omega} (s - j\omega)G(s) \frac{1}{s - j\omega} = \lim_{s \rightarrow j\omega} G(s) \cdot 1 = G(j\omega) \end{aligned} \quad (\text{A.3})$$

したがって、出力は $Y(s)$ を逆ラプラス変換して次式で求められる。

$$y(t) = k_1 e^{p_1 t} + k_2 e^{p_2 t} + \dots + k_n e^{p_n t} + a e^{j\omega t} \quad (\text{A.4})$$

極の実部は負であるから、定常状態では次式で与えられる。

$$y(t) = a e^{j\omega t} = G(j\omega) e^{j\omega t} \quad (\text{A.5})$$

付録 B ラプラス変換

$$\begin{aligned}
 \mathcal{L}[\cos \omega_c t] &= \frac{1}{2} \mathcal{L}[e^{j\omega_c t} + e^{-j\omega_c t}] \\
 &= \frac{1}{2} \left(\int_0^\infty e^{j\omega_c t} e^{-st} dt + \int_0^\infty e^{-j\omega_c t} e^{-st} dt \right) \\
 &= \frac{1}{2} \left(\int_0^\infty e^{-(s-j\omega_c)t} dt + \int_0^\infty e^{-(s+j\omega_c)t} dt \right) \\
 &= \frac{1}{2} \left(\left[\frac{e^{-(s-j\omega_c)t}}{-(s-j\omega_c)} \right]_0^\infty + \left[\frac{e^{-(s+j\omega_c)t}}{-(s+j\omega_c)} \right]_0^\infty \right) \\
 &= \frac{1}{2} \left(\frac{1}{s-j\omega_c} + \frac{1}{s+j\omega_c} \right) = \frac{s}{s^2 + \omega_c^2}
 \end{aligned} \tag{B.1}$$

$$\begin{aligned}
 \mathcal{L}[\cos(\omega_c t + \phi_o)] &= \frac{1}{2} \mathcal{L}[e^{j(\omega_c t + \phi_o)} + e^{-j(\omega_c t + \phi_o)}] \\
 &= \frac{1}{2} \left(\int_0^\infty e^{j(\omega_c t + \phi_o)} e^{-st} dt + \int_0^\infty e^{-j(\omega_c t + \phi_o)} e^{-st} dt \right) \\
 &= \frac{e^{j\phi_o}}{2} \int_0^\infty e^{-(s-j\omega_c)t} dt + \frac{e^{-j\phi_o}}{2} \int_0^\infty e^{-(s+j\omega_c)t} dt \\
 &= \frac{e^{j\phi_o}}{2} \left[\frac{e^{-(s-j\omega_c)t}}{-(s-j\omega_c)} \right]_0^\infty + \frac{e^{-j\phi_o}}{2} \left[\frac{e^{-(s+j\omega_c)t}}{-(s+j\omega_c)} \right]_0^\infty \\
 &= \frac{1}{2} \left(\frac{e^{j\phi_o}}{s-j\omega_c} + \frac{e^{-j\phi_o}}{s+j\omega_c} \right) = \frac{\cos \phi_o s - \omega_c \sin \phi_o}{s^2 + \omega_c^2}
 \end{aligned} \tag{B.2}$$

$\mathcal{L}\{a_i(t)\} = \hat{a}_i(s)$ とすると、

$$\begin{aligned}
 \mathcal{L}[a_i(t) \cos \omega_c t] &= \frac{1}{2} \mathcal{L}[(a_i(t)e^{j\omega_c t} + a_i(t)e^{-j\omega_c t})] \\
 &= \frac{1}{2} \left(\int_0^\infty a_i(t)e^{j\omega_c t} e^{-st} dt + \int_0^\infty a_i(t)e^{-j\omega_c t} e^{-st} dt \right) \\
 &= \frac{1}{2} \left(\int_0^\infty a_i(t)e^{-(s-j\omega_c)t} dt + \int_0^\infty a_i(t)e^{-(s+j\omega_c)t} dt \right) \\
 &= \frac{1}{2} \{ \hat{a}_i(s-j\omega_c) + \hat{a}_i(s+j\omega_c) \}
 \end{aligned} \tag{B.3}$$

$\mathcal{L}\{a_{o,a}(t)\} = \hat{a}_{o,a}(s)$, $\mathcal{L}\{\phi_{o,a}(t)\} = \hat{\phi}_{o,a}(s)$ とすると、

$$\begin{aligned}
\mathcal{L}[a_{o,a}(t) \cos(\omega_c t + \phi_o)] &= \frac{1}{2} \mathcal{L}[(a_{o,a}(t)e^{j(\omega_c t + \phi_o)} + a_{o,a}(t)e^{-j(\omega_c t + \phi_o)})] \\
&= \frac{1}{2} \left(\int_0^\infty a_{o,a}(t)e^{j(\omega_c t + \phi_o)}e^{-st}dt + \int_0^\infty a_{o,a}(t)e^{-j(\omega_c t + \phi_o)}e^{-st}dt \right) \\
&= \frac{e^{j\phi_o}}{2} \int_0^\infty a_{o,a}(t)e^{-(s-j\omega_c)t}dt + \frac{e^{-j\phi_o}}{2} \int_0^\infty a_{o,a}(t)e^{-(s+j\omega_c)t}dt \\
&= \frac{1}{2} \{ e^{j\phi_o} \hat{a}_{o,a}(s - j\omega_c) + e^{-j\phi_o} \hat{a}_{o,a}(s + j\omega_c) \} \tag{B.4}
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}[\phi_{o,a}(t) \sin(\omega_c t + \phi_o)] &= \frac{1}{2j} \mathcal{L}[(\phi_{o,a}(t)e^{j(\omega_c t + \phi_o)} - \phi_{o,a}(t)e^{-j(\omega_c t + \phi_o)})] \\
&= \frac{1}{2j} \left(\int_0^\infty \phi_{o,a}(t)e^{j(\omega_c t + \phi_o)}e^{-st}dt - \int_0^\infty \phi_{o,a}(t)e^{-j(\omega_c t + \phi_o)}e^{-st}dt \right) \\
&= \frac{e^{j\phi_o}}{2j} \int_0^\infty \phi_{o,a}(t)e^{-(s-j\omega_c)t}dt - \frac{e^{-j\phi_o}}{2j} \int_0^\infty \phi_{o,a}(t)e^{-(s+j\omega_c)t}dt \\
&= -\frac{j}{2} \{ e^{j\phi_o} \hat{\phi}_{o,a}(s - j\omega_c) - e^{-j\phi_o} \hat{\phi}_{o,a}(s + j\omega_c) \} \tag{B.5}
\end{aligned}$$

B.1 Pedersen Model

B.1.1 空洞を介した位相、振幅、チューニング伝送

位相および振幅が変調された正弦波信号は次のように表せる。

位相および振幅変調された正弦波信号を送信する場合：

$a_i(t)$ で振幅変調され、 $\phi(t)$ で位相変調された角振動数 ω_c の正弦波信号は、次のように表される。

$$x(t) = \text{Re} \left\{ A_i(1 + a_i(t))e^{j(\omega_c t + \phi_i(t))} \right\} \tag{B.6}$$

$a_i \ll 1$, $\phi_i \ll 1$ の場合、(B.6) は

$$\begin{aligned}
x(t) &\simeq \text{Re} \{ A_i(1 + a_i(t))(1 + j\phi_i(t))e^{j\omega_c t} \} \\
&= \text{Re} \left\{ A_i(1 + a_i(t) + j\phi_i(t) + \underbrace{ja_i(t)\phi_i(t)}_0)e^{j\omega_c t} \right\} \\
&\simeq \text{Re} \{ A_i(1 + a_i(t) + j\phi_i(t))e^{j\omega_c t} \} \\
&= \text{Re} \{ A_i(1 + a_i(t) + j\phi_i(t))(\cos \omega_c t + j \sin \omega_c t) \} \\
&= A_i(\cos \omega_c t + a_i(t) \cos \omega_c t - \phi(t) \sin \omega_c t) \tag{B.7}
\end{aligned}$$

ここで、

$$\begin{aligned}
\mathcal{L}[\cos \omega_c t] &= \frac{1}{2} \mathcal{L}[(e^{j\omega_c t} + e^{-j\omega_c t})] \\
&= \frac{1}{2} \left(\int_0^\infty e^{j\omega_c t} e^{-st} dt + \int_0^\infty e^{-j\omega_c t} e^{-st} dt \right) \\
&= \frac{1}{2} \left(\int_0^\infty e^{-(s-j\omega_c)t} dt + \int_0^\infty e^{-(s+j\omega_c)t} dt \right) \\
&= \frac{1}{2} \left(\left[\frac{e^{-(s-j\omega_c)t}}{-(s-j\omega_c)} \right]_0^\infty + \left[\frac{e^{-(s+j\omega_c)t}}{-(s+j\omega_c)} \right]_0^\infty \right) \\
&= \frac{1}{2} \left(\frac{1}{s-j\omega_c} + \frac{1}{s+j\omega_c} \right) = \frac{s}{s^2 + \omega_c^2}
\end{aligned} \tag{B.8}$$

また、 $\mathcal{L}\{a_i(t)\} = a_i(s)$, $\mathcal{L}\{\phi_i(t)\} = \phi_i(s)$ とすると、

$$\begin{aligned}
\mathcal{L}[a_i(t) \cos \omega_c t] &= \frac{1}{2} \mathcal{L}[(a_i(t)e^{j\omega_c t} + a_i(t)e^{-j\omega_c t})] \\
&= \frac{1}{2} \left(\int_0^\infty a_i(t)e^{j\omega_c t} e^{-st} dt + \int_0^\infty a_i(t)e^{-j\omega_c t} e^{-st} dt \right) \\
&= \frac{1}{2} \left(\int_0^\infty a_i(t)e^{-(s-j\omega_c)t} dt + \int_0^\infty a_i(t)e^{-(s+j\omega_c)t} dt \right) \\
&= \frac{a_i(s-j\omega_c) + a_i(s+j\omega_c)}{2}
\end{aligned} \tag{B.9}$$

$$\begin{aligned}
\mathcal{L}[\phi_i(t) \sin \omega_c t] &= \frac{1}{2j} \mathcal{L}[(\phi_i(t)e^{j\omega_c t} - \phi_i(t)e^{-j\omega_c t})] \\
&= \frac{1}{2j} \left(\int_0^\infty \phi_i(t)e^{j\omega_c t} e^{-st} dt - \int_0^\infty \phi_i(t)e^{-j\omega_c t} e^{-st} dt \right) \\
&= \frac{1}{2j} \left(\int_0^\infty \phi_i(t)e^{-(s-j\omega_c)t} dt - \int_0^\infty \phi_i(t)e^{-(s+j\omega_c)t} dt \right) \\
&= \frac{\phi_i(s-j\omega_c) - \phi_i(s+j\omega_c)}{2j}
\end{aligned} \tag{B.10}$$

以上より、 $X(s) = \mathcal{L}\{x(t)\}$ とし、(B.7) をラプラス変換すると、

$$X(s) = \frac{A_i}{2} \left\{ \frac{2s}{s^2 + \omega_c^2} + a_i(s-j\omega_c) + j\phi_i(s-j\omega_c) + a_i(s+j\omega_c) - j\phi_i(s+j\omega_c) \right\} \tag{B.11}$$

伝達関数 $H(s)$ を介して、出力される信号 $y(t)$ は一般的に振幅と位相の両方は変調され、次の様に表せる。

$$y(t) = \text{Re} \left\{ A_o(1 + a_o(t))e^{j(\omega_c t + \phi_z + \phi_o(t))} \right\} \tag{B.12}$$

$a_o \ll 1$, $\phi_o \ll 1$ の場合、(B.12) は

$$\begin{aligned}
y(t) &\simeq \text{Re} \left\{ A_o(1 + a_o(t))(1 + j\phi_o(t))e^{j(\omega_c t + \phi_z)} \right\} \\
&= \text{Re} \left\{ A_o(1 + a_o(t) + j\phi_o(t) + \underbrace{j a_o(t)\phi_o(t)}_0) e^{j(\omega_c t + \phi_z)} \right\} \\
&\simeq \text{Re} \{ A_o(1 + a_o(t) + j\phi_o(t))(\cos(\omega_c t + \phi_z) + j \sin(\omega_c t + \phi_z)) \} \\
&= A_o(\cos(\omega_c t + \phi_z) + a_o(t) \cos(\omega_c t + \phi_z) - \phi_o(t) \sin(\omega_c t + \phi_z))
\end{aligned} \tag{B.13}$$

ここで、

$$\begin{aligned}
\mathcal{L}[\cos(\omega_c t + \phi_z)] &= \frac{1}{2} \mathcal{L}[(e^{j(\omega_c t + \phi_z)} + e^{-j(\omega_c t + \phi_z)})] \\
&= \frac{1}{2} \left(\int_0^\infty e^{j(\omega_c t + \phi_z)} e^{-st} dt + \int_0^\infty e^{-j(\omega_c t + \phi_z)} e^{-st} dt \right) \\
&= \frac{e^{j\phi_z}}{2} \int_0^\infty e^{-(s-j\omega_c)t} dt + \frac{e^{-j\phi_z}}{2} \int_0^\infty e^{-(s+j\omega_c)t} dt \\
&= \frac{e^{j\phi_z}}{2} \left[\frac{e^{-(s-j\omega_c)t}}{-(s-j\omega_c)} \right]_0^\infty + \frac{e^{-j\phi_z}}{2} \left[\frac{e^{-(s+j\omega_c)t}}{-(s+j\omega_c)} \right]_0^\infty \\
&= \frac{1}{2} \left(\frac{e^{j\phi_z}}{s-j\omega_c} + \frac{e^{-j\phi_z}}{s+j\omega_c} \right) = \frac{\cos \phi_z s - \omega_c \sin \phi_z}{s^2 + \omega_c^2}
\end{aligned} \tag{B.14}$$

また、 $\mathcal{L}\{a_o(t)\} = a_o(s)$, $\mathcal{L}\{\phi_o(t)\} = \phi_o(s)$ とすると、

$$\begin{aligned}
\mathcal{L}[a_o(t) \cos(\omega_c t + \phi_z)] &= \frac{1}{2} \mathcal{L}[(a_o(t)e^{j(\omega_c t + \phi_z)} + a_o(t)e^{-j(\omega_c t + \phi_z)})] \\
&= \frac{1}{2} \left(\int_0^\infty a_o(t)e^{j(\omega_c t + \phi_z)} e^{-st} dt + \int_0^\infty a_o(t)e^{-j(\omega_c t + \phi_z)} e^{-st} dt \right) \\
&= \frac{e^{j\phi_z}}{2} \int_0^\infty a_o(t)e^{-(s-j\omega_c)t} dt + \frac{e^{-j\phi_z}}{2} \int_0^\infty a_o(t)e^{-(s+j\omega_c)t} dt \\
&= \frac{e^{j\phi_z}}{2} a_o(s-j\omega_c) + \frac{e^{-j\phi_z}}{2} a_o(s+j\omega_c)
\end{aligned} \tag{B.15}$$

$$\begin{aligned}
\mathcal{L}[\phi_o(t) \sin(\omega_c t + \phi_z)] &= \frac{1}{2j} \mathcal{L}[(\phi_o(t)e^{j(\omega_c t + \phi_z)} - \phi_o(t)e^{-j(\omega_c t + \phi_z)})] \\
&= \frac{1}{2j} \left(\int_0^\infty \phi_o(t)e^{j(\omega_c t + \phi_z)} e^{-st} dt - \int_0^\infty \phi_o(t)e^{-j(\omega_c t + \phi_z)} e^{-st} dt \right) \\
&= \frac{e^{j\phi_z}}{2j} \int_0^\infty \phi_o(t)e^{-(s-j\omega_c)t} dt - \frac{e^{-j\phi_z}}{2j} \int_0^\infty \phi_o(t)e^{-(s+j\omega_c)t} dt \\
&= \frac{e^{j\phi_z}}{2j} \phi_o(s-j\omega_c) - \frac{e^{-j\phi_z}}{2j} \phi_o(s+j\omega_c)
\end{aligned} \tag{B.16}$$

以上より、 $Y(s) = \mathcal{L}\{y(t)\}$ とし、(B.13) をラプラス変換すると、

$$\begin{aligned}
Y(s) &= A_o \left[\frac{\cos \phi_z s - \omega_c \sin \phi_z}{s^2 + \omega_c^2} + \frac{e^{j\phi_z}}{2} \{a_o(s-j\omega_c) + j\phi_o(s-j\omega_c)\} \right. \\
&\quad \left. + \frac{e^{-j\phi_z}}{2} \{a_o(s+j\omega_c) + j\phi_o(s+j\omega_c)\} \right]
\end{aligned} \tag{B.17}$$

ここで、

$$\begin{cases} a_o(s) = G_{aa}(s)a_i(s) + G_{pa}(s)\phi_i(s) \\ \phi_o(s) = G_{ap}(s)a_i(s) + G_{pp}(s)\phi_i(s) \end{cases} \quad (\text{B.18})$$

より、(B.18) を (B.17) に代入すると、

$$\begin{aligned} Y(s) = \frac{A_o}{2} \left[\frac{2(\cos \phi_z s - \omega_c \sin \phi_z)}{s^2 + \omega_c^2} \right. \\ + e^{j\phi_z} \{ (G_{aa}(s - j\omega_c) + jG_{ap}(s - j\omega_c))a_i(s - j\omega_c) \\ + (G_{pa}(s - j\omega_c) + jG_{pp}(s - j\omega_c))\} \phi_i(s - j\omega_c) \\ + e^{-j\phi_z} \{ (G_{aa}(s + j\omega_c) - jG_{ap}(s + j\omega_c))a_i(s + j\omega_c) \\ \left. + (G_{pa}(s + j\omega_c) - jG_{pp}(s + j\omega_c))\} \phi_i(s + j\omega_c) \right] \quad (\text{B.19}) \end{aligned}$$

ここで $Y(s) = H(s)X(s)$ より、

$$Y(s) = \frac{A_i H(s)}{2} \left\{ \frac{2s}{s^2 + \omega_c^2} + a_i(s - j\omega_c) + j\phi_i(s - j\omega_c) + a_i(s + j\omega_c) - j\phi_i(s + j\omega_c) \right\} \quad (\text{B.20})$$

(B.19) と (B.20) で係数を比較すると、

付録 C 加速空洞の等価回路

C.1 RLC 並列共振回路の微分方程式

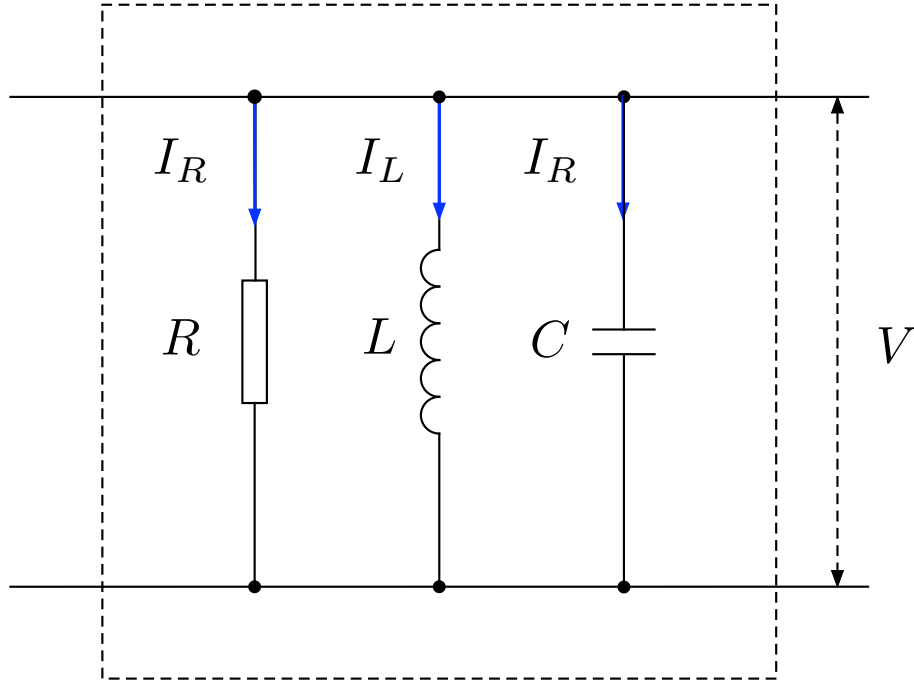


Fig. C.1 Cavity RLC Model

$$\begin{aligned}
 I(t) &= I_R(t) + I_L(t) + I_C(t) \\
 I_R(t) &= \frac{V(t)}{R}, \quad I_L(t) = \frac{1}{L} \int V(t) dt, \quad I_C(t) = C \frac{dV(t)}{dt}
 \end{aligned} \tag{C.1}$$

したがって、

$$\frac{V(t)}{R} + \frac{1}{L} \int V(t) dt + C \frac{dV(t)}{dt} = I(t) \tag{C.2}$$

両辺を時間 t で微分すると、

$$\frac{\dot{V}(t)}{R} + \frac{V(t)}{L} + CV\ddot{(t)} = \dot{I}(t) \tag{C.3}$$

$$\ddot{V}(t) + \frac{1}{CR} \dot{V}(t) + \frac{1}{LC} V(t) = \frac{1}{C} \dot{I}(t) \tag{C.4}$$

$$\omega_0^2 = \frac{1}{LC}, \quad \sigma = \frac{R}{2} \sqrt{\frac{C}{L}} = \frac{\omega_0}{2Q_L} \quad (\text{C.5})$$

$$\ddot{V}(t) + 2\sigma\dot{V}(t) + \omega_0^2 V(t) = 2\sigma R\dot{I}(t) \quad (\text{C.6})$$

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