Applied Statistics

for Computer Science BSc, Exam

Probability Theory and Mathematical Statistics

for Computer Science Engineering BSc, Term grade

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Main topics

1. Probability theory
2. Statistics

Mathematical tools: combinatorics, calculus
Computer tool: Matlab
Book:
Yates, Goodman:
Probability and Stochastic Processes: A Friendly Introduction for Electrical and Computer Engineers

Lecture 1

Combinatorics

Permutations

Definition. An ordered sequence of n distinguishable objects is called an n-permutation.

Theorem. The number of n-permutations is n-factorial, that is $n! = n \cdot (n-1) \cdots 2 \cdot 1$.

Remark. $0! = 1$

Proofs.

1. Mathematical induction. $(n+1)! = (n+1)n!$
2. Use boxes!
3. Use a tree graph!

Permutations when there are identical elements.
Let k red and $n-k$ white balls in a box. The balls having the same color are indistinguishable! Denote by X the number of permutations of these n elements. Then

$$Xk!(n-k)! = n!$$

So

$$X = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

Let h red, k blue and m white elements $(h+k+m=n)$. Denote by X the number of permutations of these n elements. Then

$$Xh!k!m! = n!$$

So

$$X = \frac{n!}{h!k!m!}$$

Ordered selections

Ordered selections without replacement
Choose k out of n distinguishable objects and order them.
Theorem. The number of ordered selections without replacement is

$$n \cdot (n-1) \cdot (n-2) \cdots (n-k+1) = \frac{n!}{(n-k)!}$$

Proofs.

1. Use boxes!
2. Use a tree graph!

Ordered selections with replacement

Choose k out of n distinguishable objects so that after each choice we replace the object. The order of the elements is important.

Theorem. The number of ordered selections with replacement is

$$n \cdot n \cdots n \cdot n = n^k$$

Proofs.

1. Use boxes!
2. Use a tree graph!

Combinations

Here the order of the elements is not important.

Theorem. The number of ways to choose k objects out of n distinguishable objects is

$$\binom{n}{k}$$

Here n choose k or the binomial coefficient is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Proofs.

1. Let C denote the number of combinations, S the number of ordered selections, then

$$Ck! = S$$

and $S = \frac{n!}{(n-k)!}$.

2. The number of combinations is equal to the number of permutations of k ones and $n-k$ zeros.

Combinations with replacement

Here the order of the elements is not important but we replace the elements.

Theorem. The number of ways to choose k objects out of n distinguishable objects when we replace the chosen elements is

$$\binom{n+k-1}{k}$$

Binomial theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k =$$

$$\binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \cdots + \binom{n}{n} a^0 b^n .$$

Proofs.

1. Calculate the coefficient of $a^{n-k} b^k$ in the product

$$(a+b)^n = (a+b)(a+b)\cdots(a+b)$$

2. Use mathematical induction.

Examples for the binomial theorem

$$(a+b)^1 = \binom{1}{0} a^1 b^0 + \binom{1}{1} a^0 b^1 = 1a + 1b$$

$$(a+b)^2 = \binom{2}{0} a^2 b^0 + \binom{2}{1} a^1 b^1 + \binom{2}{2} a^0 b^2 = 1a^2 b^0 + 2a^1 b^1 + 1a^0 b^2 .$$

$$(a+b)^3 = \binom{3}{0} a^3 b^0 + \binom{3}{1} a^2 b^1 + \binom{3}{2} a^1 b^2 + \binom{3}{3} a^0 b^3 =$$

$$= 1a^3 b^0 + 3a^2 b^1 + 3a^1 b^2 + 1a^0 b^3 .$$

Pascal triangle

The binomial coefficients are contained in the Pascal triangle.
Remark. The rule of the Pascal triangle is

$$\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$$

Proofs.
1. Use direct calculations.
2. We have $n+1$ balls, among them there is 1 black and n white balls. Choose $k+1$ balls out of these $n+1$ balls. The number of these choices is $\binom{n+1}{k+1}$. We classify these choices in two parts:
1. only white balls were chosen $\binom{n}{k+1}$
2. the black ball and k white balls were chosen $\binom{n}{k}$

Applications of the binomial theorem

Exercise 1. Prove

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n.$$

Hint. Use binomial theorem for $(1+1)^n$.
Exercise 2. Prove

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots \pm \binom{n}{n} = 0.$$

Hint. Use binomial theorem for $(1-1)^n$.

Number of subsets

Theorem. The number of subsets of an n-element set is 2^n.

Proofs. Let $A = \{a_1, a_2, \ldots, a_n\}$

1. We write 1 if a_i belongs to the subset, and write 0 if not. The number of length n binary numbers is 2^n.

2. Apply ordered selections with replacement!

3. We can choose a k-element subset from A $\binom{n}{k}$ different ways. Then apply $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$.