1 An Explicit Formula for an Elliptic Curve

By taking advantage of the Functional Equation of the Riemann Zeta Function and it's Euler Product, we can arrive at the Riemann - von Mangoldt Explicit Formula:

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{1}{2}\log(1 - x^{-2}) - \frac{\zeta'(0)}{\zeta(0)}$$

where $\psi(x) = \sum_{p^m < x} \log(p)$ is the Chebyshev ψ-function and ρ are the nontrivial zeros of $\zeta(s)$. Once it is known that there are no zeros on the boundary of the critical strip, this formula immediately gives the Prime Number Theorem as every term, except the first, is then $o(x)$.

If E is an elliptic curve over $\mathbb{Q}$, then Modularity gives the Functional Equation for $L(E, s)$:

$$N^{s/2}(2\pi)^{-s}\Gamma(s)L(E,s) = w_E N^{1-s/2}(2\pi)^{s-2}\Gamma(2-s)L(E, 2-s) \tag{1}$$

This allows the extension of $L(E, s)$ to an entire function on $\mathbb{C}$ as well as the dertermination of the trivial zeros of $L(E, s)$. We can then factor $L(E, s)$ into an Euler Product using the formula

$$L(E,s) = \frac{\zeta(s)\zeta(s-1)}{Z(E,s)}$$

and $Z(E, s)$ is the global Hasse-Weil Zeta Function for E. The combination of these should then give a kind of "Explicit Formula" for $L(E, s)$.

In particular we will show the following: If $\mathcal{N}_{p^m}(E) = \#E(\mathbb{F}_{p^m})$ and

$$a_{p^m}(E) = \begin{cases} 1 + p^m - \mathcal{N}_{p^m}(E) & \text{if } E \text{ has good reduction at } p \\ 1 & \text{if } E \text{ has split multiplicative reduction at } p \\ (-1)^m & \text{if } E \text{ has non-split multiplicative reduction at } p \\ 0 & \text{if } E \text{ has additive reduction at } p \end{cases}$$

and

$$\psi_E(x) = \sum_{p^m < x} -a_{p^m}(E)\log(p)$$

is a Chebyshev-type function for E then we have

$$\psi_E(x) = r_{\mathrm{an}} x + \sum_{\rho} \frac{x^{\rho}}{\rho} + \log\left(\frac{x-1}{x}\right) + B$$

where r_{an} is the Analytic Rank of E and ρ are the nontrivial zeros of $L(E, s)$ not equal to one and B is some constant. After establishing this formula, we will discuss some immediate results and where to proceed afterward.

1.1 The Hadamard Product of $L(E, s)$

Part of the Functional Equation says that the function $\Lambda(E, s) = N^{s/2}(2\pi)^{-s}\Gamma(s)L(E, s)$ only has zeros in the strip $1/2 \leq Re(s) \leq 3/2$. Since $s = -n$ is a pole of order 1 for $\Gamma(s)$ for $n \geq 0$ it follows that $L(E, s)$ has zeros of order 1 at $s = -n$.

Note that $L(E,s) = L(f,s)$ for some cusp form f, and the exponential decay of f at the cusps then shows that $L(E,s)$ is of order 1 in the sense of Hadamard and Weierstrass. Let $r = ord_{s=1}(L(E,s))$ then the theorem of Hadamard then gives the following factorization:

$$L(E,s) = e^{A+Bs}(1-s)^r e^{rs} \cdot s \cdot \prod_{n=1}^{\infty} \left(1+\frac{s}{n}\right) e^{-s/n} \prod_{\rho} \left(1-\frac{s}{\rho}\right) e^{s/\rho}$$

where the ρ are the zeros in the critical strip not equal to one. Note the logarithmic derivative formula

$$\frac{d}{ds} \log \left(\left(1-\frac{s}{\sigma}\right)^r e^{rs} \right) = r \cdot \frac{-s}{\sigma(s-\sigma)}$$

We then get

$$\frac{L'(E,s)}{L(E,s)} = B + r\frac{s}{s-1} + \frac{1}{s} - \sum_{n=1}^{\infty} \frac{s}{n(s+n)} + \sum_{\rho} \frac{s}{\rho(s-\rho)}$$

1.2 The Euler Factorization of $L(E,s)$

Consider the global Hasse-Weil Zeta Function for E:

$$\begin{aligned} Z(E,s) \quad &= \quad \prod_{p:\text{good}} \exp\left(\sum_{m=1}^{\infty} \frac{\mathcal{N}_{p^m}(E)}{m} p^{-sm}\right) \times \prod_{p:\text{split red}} \frac{1}{1-p^{-s+1}} \\ &\times \prod_{p:\text{nonsplit red}} \frac{1+p^{-s}}{(1-p^{-s})(1-p^{-s+1})} \times \prod_{p:\text{add red}} \frac{1}{(1-p^{-s})(1-p^{-s+1})} \end{aligned}$$

where the product runs over all primes of good reduction and $\mathcal{N}_{p^m}(E)$ is all $\mathbb{F}_{p^m}$ points of E. Now define $\mathcal{N}_q(E) := 1 + q - a_q(E)$ at the bad primes. The logarithmic derivative of $Z(E,s)$ is then

$$\frac{Z'(E,s)}{Z(E,s)} = -\sum_{p} \sum_{m=1}^{\infty} \mathcal{N}_{p^m}(E) \log(p) p^{-ms}$$

where the first product is over all primes.

Combining this with the Weil Conjecture for Elliptic Curves and the factorization for $L(E,s)$:

$$L(E,s) = \prod_{p} L_p^{-1}(E, p^{-s})$$

where

$$L_p(E,T) = \begin{cases} 1 - a_p(E)T + pT^2 & \text{if } E \text{ has good reduction at } p \\ 1 - a_p(E)T & \text{if } E \text{ has bad reduction at } p \end{cases}$$

gives

$$L(E,s) = \frac{\zeta(s)\zeta(s-1)}{Z(E,s)}$$

Using the logarithmic derivative for $\zeta(s)$ and $Z(E,s)$ then gives

$$\frac{L'(E,s)}{L(E,s)} = -\sum_{p} \sum_{m=1}^{\infty} a_{p^m} \log(p) p^{-ms}$$

1.3 The Explicit Formula

Equating the two expressions gives

$$-\sum_p \sum_{m=1}^{\infty} a_{p^m} \log(p) p^{-ms} = \frac{rs}{s-1} + \frac{1}{s} - \sum_{n=1}^{\infty} \frac{s}{n(s+n)} + \sum_\rho \frac{s}{\rho(s-\rho)} + B$$

Define the Chebyshev-type function

$$\psi_E(x) = \sum_{p^m < x} -a_{p^m}(E) \log(p)$$

We then have the following theorem

Theorem 1. If ρ denotes the "nontrivial" zeros of $L(E,s)$ then

$$\psi_E(x) = rx + \sum_\rho \frac{x^\rho}{\rho} + \log(x-1) + B \qquad (1)$$

Moreover

$$\lim_{x\to\infty} \frac{\psi_E(x)}{x} = r + \lim_{x\to\infty} \sum_\rho \frac{x^{\rho-1}}{\rho} \qquad (2)$$

Proof. Integrating both sides against the function x^s/s along a vertical line in the complex plane with a large enough real part gives the result. Note that the integral applied to $1/s$ is $\log(x)$. □

We can get some asymptotic bounds on this equation by taking advantage of Hasse's approximation $|a_q(E)| \leq 2\sqrt{q}$. Because of this we find that

$$\begin{aligned} |\psi_E(x)| &\leq \sum_{p^m < x} |a_{p^m}(E)| \log(p) \\ &\leq \sum_{p^m < x} 2\sqrt{x} \log(p) \\ &= 2\psi(x)\sqrt{x} \end{aligned}$$

where $\psi(x)$ is the ordinary Chebyshev Function. The Prime Number Theorem then gives

$$\lim_{x\to\infty} \frac{|\psi_E(x)|}{2x\sqrt{x}} \leq 1$$

From which we get

$$\lim_{x\to\infty} \left| \frac{r}{2\sqrt{x}} + \sum_\rho \frac{x^{\rho-\frac{3}{2}}}{2\rho} \right| = \lim_{x\to\infty} \left| \sum_\rho \frac{x^{\rho-\frac{3}{2}}}{2\rho} \right| \leq 1$$

We can do better, though, because there are no zeros on the boundary of the critical strip ($1/2 \leq Re(s) \leq 3/2$) which shows that this limit is zero. So

$$\lim_{x\to\infty} \frac{|\psi_E(x)|}{x\sqrt{x}} = 0$$

and $\psi_E(x) = o(x^{3/2})$. If the Generalized Riemann Hypothesis is true, then we'll get $\psi_E(x) = o(x^{1+\epsilon})$ for all $\epsilon > 0$. Is it then safe to conjecture that

$$\lim_{x\to\infty} \frac{|\psi_E(x)|}{x} = r,$$

or does $\sum_\rho \frac{x^{\rho-1}}{\rho}$ provide an error term? In general if we set $er_E = \lim_{x\to\infty} \sum_\rho \frac{x^{\rho-1}}{\rho}$, then

$$\lim_{x\to\infty} \frac{|\psi_E(x)|}{x} = r + er_E.$$