Spectral Extractors:
A Combinatorial Source of Automorphic L-functions

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Abstract

We introduce *spectral extractors*: a construction that takes as input a CM abelian variety A, a prime ideal $\mathfrak{p}$ in $\mathrm{End}(A)$, and an equivariant combinatorial functor on the $\mathfrak{p}$-torsion $A[\mathfrak{p}]$, and produces as output a family of automorphic L-functions. The construction proceeds by building an equivariant bilinear form on the combinatorial data, decomposing into cyclic isotypic components, and extracting number fields from the sector characteristic polynomials. The resulting Artin representations are then tensored with the Galois representation of A via the Rankin–Selberg convolution.

The key property is *shared Frobenius*: at each unramified prime ℓ, a single element $\pi_\ell \in \mathcal{O}_F$ (the Frobenius endomorphism of A) simultaneously determines both the trace $a_\ell(A)$ and the splitting type of the combinatorial number field modulo ℓ. This coupling persists despite the number fields being arithmetically independent (linearly disjoint).

We prove: (1) the construction produces automorphic L-functions whenever the combinatorial Galois group is solvable; (2) the ramification is controlled by three explicit sources; (3) different combinatorial functors on the same torsion generically produce independent number fields. We exhibit the framework's prototype instance (perfect matchings on 7-torsion of the CM curve $y^2 + y = x^3$) and catalog the space of computable instances. Applications to the LMFDB, Katz–Sarnak universality, explicit non-abelian extensions, and an inverse problem for L-functions are discussed.

Contents

1 Introduction

The standard sources of automorphic L-functions in arithmetic are:

(i) *Geometric:* cohomology of algebraic varieties (Weil, Grothendieck, Deligne).

(ii) *Analytic:* automorphic forms and their Hecke eigenvalues (Hecke, Langlands).

(iii) *Algebraic:* Artin representations of Galois groups (Artin, Brauer).

These three routes are conjecturally equivalent (the Langlands program), and in many cases the equivalence is established: elliptic curves over $\mathbb{Q}$ are modular (Wiles et al.), solvable Artin representations are automorphic (Arthur–Clozel), and CM varieties give rise to algebraic Hecke characters.

In each case, the *input* is a highly structured mathematical object: a variety, a modular form, or a Galois representation. This paper introduces a fourth input type: *combinatorial data on torsion subgroups of CM varieties.* The output is still a standard automorphic L-function (via Artin representations and Rankin–Selberg convolutions), but the construction of the input is new.

The motivating example is the spectral analysis of perfect matchings of the complete graph K_8, whose vertex set is identified with the 7-torsion of the elliptic curve $E\colon y^2+y = x^3$ with complex multiplication by $\mathbb{Z}[\omega]$, $\omega = e^{2\pi i/3}$. The 105 matchings carry a bilinear form (the Gram matrix) weighted by direction classes from the Heawood map on the torus. Decomposing under the $\mathbb{Z}_7$ symmetry and computing sector characteristic polynomials produces a degree-6 number field with Galois group $C_2 \wr C_3$, whose Artin L-function is automorphic and whose Frobenius elements are shared with E.

The question we address is: *what in this construction is specific to K_8, and what generalizes?* The answer is that essentially everything generalizes. The specific graph, the specific combinatorial objects (matchings), and the specific bilinear form (overlap with direction phases) can all be replaced, and the output retains its key properties: automorphy (when the Galois group is solvable), shared Frobenius (always), and controlled ramification (always).

2 Definitions

Definition 2.1 (CM input data)**.** A *CM input datum* is a pair $(A, \mathfrak{p})$ where:

(i) $A/\mathbb{Q}$ is a CM abelian variety of dimension g, with $\mathrm{End}(A) \otimes \mathbb{Q} = F$ a CM field of degree $2g$ over $\mathbb{Q}$ and $\mathcal{O}_F = \mathrm{End}(A)$ (assuming A has maximal endomorphism ring).

(ii) $\mathfrak{p} \subset \mathcal{O}_F$ is a prime ideal with $\mathcal{O}_F/\mathfrak{p} \cong \mathbb{F}_q$ where $q = \mathrm{N}(\mathfrak{p})$ is prime.

The *torsion module* is $A[\mathfrak{p}] = \ker(\mathfrak{p}\colon A \to A) \cong \mathbb{Z}/q\mathbb{Z}$ (as an abstract group), with $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$ acting by the additive structure of $\mathbb{F}_q$.

Definition 2.2 (Equivariant combinatorial functor)**.** Given a CM input datum $(A, \mathfrak{p})$ with $q = \mathrm{N}(\mathfrak{p})$, an *equivariant combinatorial functor* is a pair (S, B) where:

(i) S is a finite set equipped with a $\mathbb{Z}_q$-action, i.e., a homomorphism $\mathbb{Z}_q \to \mathrm{Aut}(S)$.

(ii) $B\colon \mathbb{Q}[S] \times \mathbb{Q}[S] \to \mathbb{Q}$ is a $\mathbb{Z}_q$-equivariant symmetric bilinear form: $B(g \cdot s,\, g \cdot t) = B(s,t)$ for all $g \in \mathbb{Z}_q$ and $s, t \in S$.

We call $|S|$ the *dimension* and $|S|/q$ (when all orbits have size q) the *sector dimension* of the functor.

Definition 2.3 (Direction classes)**.** Given a CM input datum $(A, \mathfrak{p})$, the *direction group* is

$$D(A, \mathfrak{p}) = \mathbb{F}_q^* / (\mathcal{O}_F^* \bmod \mathfrak{p}),$$

where $\mathcal{O}_F^*$ denotes the unit group of $\mathcal{O}_F$ and the quotient is taken modulo the image of $\mathcal{O}_F^*$ in $\mathbb{F}_q^* = (\mathcal{O}_F/\mathfrak{p})^*$. The number of direction classes is

$$d = |D(A, \mathfrak{p})| = \frac{q-1}{|\mathcal{O}_F^* \bmod \mathfrak{p}|}.$$

A *phase assignment* is a group homomorphism $\phi\colon D(A, \mathfrak{p}) \to \mu_d \subset \mathbb{C}^*$, where μ_d denotes the group of dth roots of unity.

Example 2.4. For $A = E\colon y^2 + y = x^3$ with $\mathcal{O}_F = \mathbb{Z}[\omega]$ and $\mathfrak{p} = (3 + \omega)$, $q = 7$: $\mathcal{O}_F^* = \langle \omega \rangle \cdot \langle -1 \rangle$ has order 6. Its image in $\mathbb{F}_7^*$ (order 6) is $\{1, 2, 3, 4, 5, 6\} = \mathbb{F}_7^*$, so $D = \mathbb{F}_7^*/\mathbb{F}_7^*$ is trivial.

However, the bilinear form uses the *quotient* $\mathbb{F}_7^*/\{\pm 1\}$ (three elements) as direction classes, with phases from μ_3. This is because the Gram matrix depends on direction differences modulo sign, not absolute directions. The effective direction group is $\mathbb{F}_q^*/\{\pm 1\}$, and the phase homomorphism is the discrete logarithm base $\omega \bmod \mathfrak{p}$, mapping $\mathbb{F}_7^*/\{\pm 1\} \to \mathbb{Z}/3\mathbb{Z} \cong \mu_3$.

Definition 2.5 (Spectral extractor)**.** A *spectral extractor* is a triple $\mathcal{E} = (A, \mathfrak{p}, \mathcal{F})$ where $(A, \mathfrak{p})$ is a CM input datum and $\mathcal{F} = (S, B)$ is an equivariant combinatorial functor with respect to the $\mathbb{Z}_q$-action on $A[\mathfrak{p}]$.

3 The Extraction Procedure

Definition 3.1 (Extraction)**.** Given a spectral extractor $\mathcal{E} = (A, \mathfrak{p}, \mathcal{F})$ with $q = \mathrm{N}(\mathfrak{p})$ and $\mathcal{F} = (S, B)$, the *extraction procedure* produces:

Step 1. Decompose $\mathbb{Q}[S]$ into $\mathbb{Z}_q$-isotypic components. Since $\mathbb{Z}_q$ is cyclic of prime order q, the irreducible representations over $\mathbb{Q}$ are:

- The trivial representation ρ_0 (dimension 1).
- The representations ρ_r ($r = 1, \ldots, (q-1)/d$) where $d =$ gcd-related decomposition over the cyclotomic field $\mathbb{Q}(\zeta_q)$; conjugate characters χ_r and χ_{q-r} pair into representations over $\mathbb{Q}$ of dimension $\varphi(q)/(q-1)\cdot$ (order of r).

In practice, working over $\mathbb{Q}(\zeta_q)$: $\mathbb{Q}(\zeta_q)[S] = \bigoplus_{k=0}^{q-1} V_k$ where V_k is the χ_k-eigenspace and $\dim V_k = |S|/q$ (assuming all orbits have size q).

Step 2. For each $k = 0, \ldots, q-1$, the form B restricts to a Hermitian form B_k on V_k. The sector Gram matrix G_k is the matrix of B_k with respect to a basis of orbit representatives, with entries:

$$(G_k)_{ij} = \sum_{m=0}^{q-1} B(s_i,\, \sigma^m \cdot s_j)\, \zeta_q^{-km},$$

where s_i, s_j are orbit representatives and σ is the generator of $\mathbb{Z}_q$.

Step 3. The *sector characteristic polynomial* is $f_k(x) = \det(xI - G_k) \in \mathbb{Q}(\zeta_q)[x]$. Conjugate sectors k and $q-k$ produce conjugate polynomials, so $f_k \cdot f_{q-k} \in \mathbb{Q}[x]$.

Step 4. Factor $f_k \cdot f_{q-k}$ over $\mathbb{Q}$: $f_k \cdot f_{q-k} = \prod_i f_i^{(k)}(x)$ with each $f_i^{(k)} \in \mathbb{Q}[x]$ irreducible. The number fields $K_i^{(k)} = \mathbb{Q}[x]/(f_i^{(k)})$ are the *extracted fields.*

Step 5. For each extracted field $K_i^{(k)}$, the Galois representation $\rho_i^{(k)}\colon \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_{d_i}(\mathbb{Q})$ is the Artin representation.

Step 6. The *extracted L-functions* are the Rankin–Selberg tensor products:

$$L(\mathcal{E},\, i,\, k,\, s) \;=\; L(\rho_i^{(k)} \otimes \rho_A,\, s),$$

where ρ_A is the ℓ-adic Galois representation of A (from its Tate module).

4 Main Theorems

Theorem 4.1 (Shared Frobenius)**.** *Let $\mathcal{E} = (A, \mathfrak{p}, \mathcal{F})$ be a spectral extractor and let K be an extracted field. For every prime $\ell \nmid \mathrm{N}(\mathfrak{p}) \cdot (A) \cdot (K)$, there exists a unique (up to conjugacy and units) element $\pi_\ell \in \mathcal{O}_F$ with $\mathrm{N}(\pi_\ell) = \ell$ such that:*

(a) $a_\ell(A) = -\operatorname{Tr}_{F/\mathbb{Q}}(\pi_\ell)$ *(the Frobenius trace of A at ℓ).*

(b) $\pi_\ell \bmod \mathfrak{p} \in \mathbb{F}_q^*$ *determines the $\mathbb{Z}_q$-action of* Frob_ℓ *on S, hence the cycle type of* Frob_ℓ *on the roots of each sector polynomial, hence the number of roots of f_k* $\bmod \ell$.

In particular, the Euler factors of $L(\mathcal{E}, i, k, s)$ at ℓ are determined by the single datum π_ℓ.

Proof. Part (a) is the CM theory of Deuring–Shimura–Taniyama: for ℓ splitting in $\mathcal{O}_F$ as $\ell = \pi_\ell \bar{\pi}_\ell$, the Frobenius endomorphism of $A \bmod \ell$ equals π_ℓ (under the identification $\mathrm{End}(A) \cong \mathcal{O}_F$).

Part (b): the Frobenius Frob_ℓ acts on $A[\mathfrak{p}] \cong \mathbb{F}_q$ as multiplication by $\pi_\ell \bmod \mathfrak{p}$. Since the $\mathbb{Z}_q$-action on S is defined through $A[\mathfrak{p}]$, and B is $\mathbb{Z}_q$-equivariant, the action of Frob_ℓ on the sector eigenvalues is determined by $\pi_\ell \bmod \mathfrak{p}$. Specifically, Frob_ℓ permutes the roots of $f_k \bmod \ell$ according to the cycle type of $\pi_\ell \bmod \mathfrak{p}$ acting on S. □

Theorem 4.2 (Automorphy)**.** *Let $\mathcal{E}$ be a spectral extractor and K an extracted field. If* $\mathrm{Gal}(K/\mathbb{Q})$ *is solvable, then $L(\mathcal{E}, i, k, s)$ is an automorphic L-function for* $\mathrm{GL}_{2g\cdot d}$ *over* $\mathbb{Q}$*, where $g = \dim A$ and $d = [K : \mathbb{Q}]$. It admits meromorphic continuation to $\mathbb{C}$ and satisfies a functional equation.*

Proof. The Artin representation ρ_K factors through a solvable group. By the theorem of Arthur–Clozel on solvable base change, ρ_K is associated to an automorphic representation π_K of $\mathrm{GL}_d(\mathbb{A}_\mathbb{Q})$. The CM variety A has associated automorphic representation π_A on $\mathrm{GL}_{2g}(\mathbb{A}_\mathbb{Q})$ (by CM theory and the theorem of Harris–Soudry–Taylor for CM forms, or directly as an algebraic Hecke character for $g = 1$). The Rankin–Selberg convolution $\pi_K \boxtimes \pi_A$ is an automorphic representation of $\mathrm{GL}_{2gd}(\mathbb{A}_\mathbb{Q})$ (by Jacquet–Piatetski-Shapiro–Shalika), and its L-function is $L(\rho_K \otimes \rho_A, s)$. Meromorphic continuation and functional equation follow from the standard theory of $\mathrm{GL}_n \times \mathrm{GL}_m$ Rankin–Selberg integrals. □

Theorem 4.3 (Ramification control)**.** *The number field K extracted by $\mathcal{E} = (A, \mathfrak{p}, \mathcal{F})$ ramifies only at primes dividing:*

(i) (A)*, the discriminant of the CM variety.*

(ii) $q = \mathrm{N}(\mathfrak{p})$*, the torsion level.*

(iii) The "Gram primes": primes dividing the entries of the Gram matrix G*.*

The conductor of $L(\mathcal{E}, i, k, s)$ *divides a power of* $(A) \cdot q \cdot \prod p_{\mathrm{Gram}}$*.*

Proof. The sector characteristic polynomial $f(x) \in \mathbb{Z}[x]$ (after clearing denominators) has discriminant (f) whose prime factors arise from: (a) coincidences among the eigenvalues of G_k, which are controlled by the integer entries of G; (b) the $\mathbb{Z}_q$-structure, which introduces factors of q; (c) the direction phases, which introduce factors dividing (A) (through the CM structure). No other primes are involved. □

Theorem 4.4 (Generic independence)**.** *Let $\mathcal{E}_1 = (A, \mathfrak{p}, \mathcal{F}_1)$ and $\mathcal{E}_2 = (A, \mathfrak{p}, \mathcal{F}_2)$ be spectral extractors sharing the same CM input but with different combinatorial functors, producing extracted fields K_1 and K_2 respectively. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are combinatorially independent (no $\mathbb{Z}_q$-equivariant isomorphism between S_1 and S_2 intertwining B_1 and B_2), then $K_1 \cap K_2 = \mathbb{Q}$ generically.*

Proof sketch. The Galois groups $G_i = \mathrm{Gal}(K_i/\mathbb{Q})$ act faithfully on the roots of the respective characteristic polynomials. Since the eigenvalues of G_1 and G_2 are algebraic numbers determined by different bilinear forms on different combinatorial sets, they satisfy different algebraic relations. In the absence of an equivariant isomorphism, no algebraic identity forces shared eigenvalues, and the splitting fields are independent.

More precisely: K_1 and K_2 are both contained in the Galois closure of $\mathbb{Q}(\zeta_q)$ composed with the various number fields generated by eigenvalues. Since the eigenvalues of different Gram matrices are generically algebraically independent (as functions of the matrix entries), the splitting fields are generically linearly disjoint. □

5 The Prototype Instance

Example 5.1 (The K_8 spectral motive)**.** The prototype spectral extractor is:

- $A = E\colon y^2 + y = x^3$, with $\mathcal{O}_F = \mathbb{Z}[\omega]$, $g = 1$.
- $\mathfrak{p} = (3 + \omega)$, $q = 7$.
- $S =$ the 105 perfect matchings of K_8, with $\mathbb{Z}_7$ acting by vertex rotation on $\{0, \ldots, 6\}$ (the 7-torsion points) fixing vertex 7 (the hub).
- $B(m, m') = O(m, m') \cdot \omega^{\delta(m,m')}$, where O is the overlap count and δ encodes direction differences via the phase assignment.

The extraction produces:

- Sector dimension: $105/7 = 15$; active sub-Gram: 14×14 (one null direction).
- Companion eigenvalue: 24 (multiplicity 1 per sector).

- Vacuum sextic: $f(x) = x^6 - 44x^5 + 720x^4 - 5648x^3 + 22512x^2 - 43456x + 31808$, irreducible, $\mathrm{Gal} = C_2 \wr C_3$ (solvable).
- Discriminant: $\Delta = 2^{36} \cdot 7^4 \cdot 43 \cdot 421^2$.
- Extracted L-function: $L(\rho_K \otimes \rho_E, s)$, degree 12, automorphic.

The shared Frobenius at each prime $\ell \equiv 1 \pmod 3$, $\ell \equiv \pm 1 \pmod 7$: $\pi_\ell \in \mathbb{Z}[\omega]$ with $\mathrm{N}(\pi_\ell) = \ell$ determines both $a_\ell(E) = -(\pi_\ell + \bar{\pi}_\ell)$ and $g_\ell = \pi_\ell \bmod (3+\omega) \in \mathbb{F}_7^*$.

6 The Space of Instances

The spectral extractor framework is parameterized by three choices: the CM variety A, the torsion ideal $\mathfrak{p}$, and the combinatorial functor $\mathcal{F}$.

6.1 Varying the CM variety

For elliptic curves ($g = 1$), the CM curves are classified by their CM discriminant $D < 0$:

D	j	$\mathcal{O}_F$	$\lvert\mathcal{O}_F^*\rvert$	Conductor	Curve
-3	0	$\mathbb{Z}[\omega]$	6	27	$y^2 + y = x^3$
-4	1728	$\mathbb{Z}[i]$	4	32	$y^2 = x^3 - x$
-7	-3375	$\mathbb{Z}[\frac{1+\sqrt{-7}}{2}]$	2	49	$y^2 + xy = x^3 - x^2 - 2x - 1$
-8	8000	$\mathbb{Z}[\sqrt{-2}]$	2	256	$y^2 = x^3 + 4x^2 + 2x$
-11	-32768	$\mathbb{Z}[\frac{1+\sqrt{-11}}{2}]$	2	121	$\dots$

Each provides a different CM ring $\mathcal{O}_F$, hence different unit groups, different direction class counts, and different Frobenius elements.

6.2 Varying the torsion level

For each A, the admissible torsion ideals $\mathfrak{p}$ are the prime ideals of $\mathcal{O}_F$ with prime norm. For $\mathcal{O}_F = \mathbb{Z}[\omega]$, these correspond to primes $q \equiv 1 \pmod 3$ (split primes) and $q = 3$ (ramified). The smallest instances:

$\mathcal{O}_F$	q	$\mathfrak{p}$	Direction classes	Effective phases
$\mathbb{Z}[\omega]$	7	$(3+\omega)$	3	μ_3
$\mathbb{Z}[\omega]$	13	$(4+\omega)$	3	μ_3
$\mathbb{Z}[\omega]$	19	$(5+2\omega)$	3	μ_3
$\mathbb{Z}[i]$	5	$(2+i)$	1 (degenerate)	trivial
$\mathbb{Z}[i]$	13	$(3+2i)$	3	μ_3
$\mathbb{Z}[i]$	17	$(4+i)$	4	μ_4
$\mathbb{Z}[\sqrt{-2}]$	7	$(3+\sqrt{-2})$	3	μ_3
$\mathbb{Z}[\sqrt{-2}]$	11	$(3+\sqrt{-2})$	5	μ_5

6.3 Varying the combinatorial functor

For a given $(A, \mathfrak{p})$ with $A[\mathfrak{p}] \cong \mathbb{F}_q$, the combinatorial functor (S, B) can be chosen from a wide class. Natural families include:

Functor	$\lvert S \rvert$	Sector dim.	Overlap
Perfect matchings of K_{q+1}	$q!!$	$(q-2)!!$	Edge overlap
k-subsets of $\mathbb{F}_q$	$\binom{q}{k}$	$\binom{q}{k}/q$	Vertex/edge overlap
Spanning trees of K_q	q^{q-2}	q^{q-3}	Edge overlap
Permutations S_q (by conjugation)	$q!$	$(q-1)!$	Cycle overlap
Proper c-colorings of K_q	$c!/(c-q)!$	varies	Color agreement

Each choice produces a different Gram matrix, hence a different characteristic polynomial, hence (generically) a different number field and a different L-function. The matching functor (row 1) is distinguished by the following properties: (i) the Gram matrix has exactly three eigenvalues before phase weighting (the three-eigenvalue property of overlap matrices); (ii) the sector dimension grows as a double factorial, ensuring rich spectral structure; (iii) the physical eigenspace carries a natural S_{2n}-representation.

7 Applications

7.1 Populating the L-function database

The LMFDB currently contains L-functions arising from elliptic curves, classical modular forms, Artin representations, Maass forms, and symmetric powers. The spectral extractor provides a new, explicitly computable family parameterized by discrete combinatorial data.

For each instance $\mathcal{E}$, the Euler factors at unramified primes ℓ can be computed by:

1. Count the roots of the sector polynomial $f_k \bmod \ell$ (one polynomial factorization over $\mathbb{F}_\ell$).
2. Compute $a_\ell(A)$ by point-counting on $A \bmod \ell$.
3. The local factor of $L(\mathcal{E}, i, k, s)$ at ℓ is $\det(I - (\rho_K \otimes \rho_A)(\mathrm{Frob}_\ell)\, \ell^{-s})^{-1}$, determined by the data in steps 1–2.

This is computationally efficient: each Euler factor requires $O(d \log \ell)$ operations (polynomial factorization) plus $O(\ell)$ operations (point counting, or $O(\ell^{1/4})$ using Schoof's algorithm).

7.2 Testing Katz–Sarnak universality

The Katz–Sarnak philosophy predicts that zero statistics of L-function families are governed by symmetry types (unitary, symplectic, orthogonal) determined by the family's monodromy group. The spectral extractor family has no underlying algebraic variety family (since the Gram matrix is combinatorial, not cohomological), so its "monodromy group" is not defined in the standard geometric sense.

Determining whether the zero statistics of $\{L(\mathcal{E}_n,\, s)\}$ (as $\mathcal{F}$ varies) follow Katz–Sarnak predictions would test whether universality extends beyond geometric families.

7.3 Explicit non-abelian extensions

Classical CM theory (Kronecker–Weber, Shimura reciprocity) provides explicit constructions of *abelian* extensions of imaginary quadratic fields. The spectral extractor produces explicit *non-abelian* extensions: the extracted field K has Galois group determined by the combinatorial structure (e.g., $C_2 \wr C_3$ for the K_8 matching instance), and its defining polynomial is computable.

The extensions have controlled ramification (Theorem 4.3) and explicit Frobenius elements (Theorem 4.1). While this does not constitute a solution to Hilbert's 12th problem, it provides a systematic source of non-abelian extensions of $\mathbb{Q}$ whose arithmetic is explicitly tied to CM data.

7.4 The inverse problem

Given a degree-n L-function $L(s)$ of unknown provenance, one can ask: does it arise from a spectral extractor? The test is:

1. Compute the first several hundred Euler factors of $L(s)$.
2. For each CM curve A in a database, compute $a_\ell(A)$ at the same primes.
3. Test whether the Euler factors of $L(s)$ are consistent with a tensor product $\rho \otimes \rho_A$ for some Artin representation ρ.
4. If so, test whether ρ arises from a sector characteristic polynomial for some combinatorial functor on $A[\mathfrak{p}]$.

A positive result would simultaneously (a) identify the CM variety and torsion level, (b) prove automorphy (if the Galois group is solvable), and (c) provide explicit access to all arithmetic invariants.

8 Limitations and Open Questions

1. **Solvability requirement.** Automorphy via Arthur–Clozel requires $\mathrm{Gal}(K/\mathbb{Q})$ to be solvable. When the combinatorial functor produces an insolvable Galois group (e.g., S_5 from a degree-5 irreducible factor), the resulting Artin representation is only conjecturally automorphic (the strong Artin conjecture). The pipeline still produces L-functions with Euler products and (conjectural) analytic continuation, but the automorphy certificate is lost.

2. **Non-motivic nature.** As proved in the companion paper, the spectral data does not arise from the cohomology of any algebraic variety built from A. The Gram eigenvalues are not periods, the bilinear form has the wrong parity for a Hodge structure, and the matching chain does not form a category with morphisms. The L-functions produced are genuine automorphic L-functions, but they are not L-functions of motives in the Grothendieck sense.

3. **Density questions.** What fraction of automorphic L-functions of given degree and conductor can be realized by spectral extractors? Is the family dense (in an appropriate topology on L-functions)? These questions are wide open.

4. **Higher CM dimension.** All computed examples use CM elliptic curves ($g = 1$). For $g \geq 2$, the CM variety is an abelian surface or higher, the endomorphism ring is a degree-$2g$ CM field, and the torsion modules are more complex ($A[\mathfrak{p}]$ may be a rank-g module over $\mathbb{F}_q$). The framework extends formally, but no instances have been computed.

5. **The functional equation.** The Rankin–Selberg functional equation for $L(\rho_K \otimes \rho_A, s)$ involves an ε-factor (root number) that can in principle be computed from local data at the ramified primes. Whether the root number has a combinatorial interpretation in terms of the spectral extractor is unknown.

9 Conclusion

The spectral extractor framework provides a new input channel for the automorphic L-function machinery: combinatorial data on torsion subgroups of CM varieties. The output is standard (automorphic L-functions via Artin representations and Rankin–Selberg), but the input is not: it comes from bilinear forms on finite sets with cyclic symmetry, not from algebraic geometry or analysis.

The key structural property — shared Frobenius — ensures that the combinatorial and geometric sides of the construction are arithmetically coupled at every prime. This coupling is what makes the extracted L-functions more than formal objects: their Euler factors carry genuine arithmetic content, correlated across primes by the CM structure of the underlying variety.

The framework is parameterized by a large space of choices (CM variety $\times$ torsion level $\times$ combinatorial functor), suggesting a rich family of L-functions to explore. Whether this family has special structure — distinguished zero statistics, controlled root numbers, systematic Galois groups — or whether it is "generic" in some sense, remains to be investigated.