Carl Somera

String Theory

Contents

1 The Polyakov Action

1.1 The Relativistic point particle

We begin our discussion with the model of a point particle that is subject to relativistic effects. Such a particle traces out a line in spacetime known as the worldline. We know that the length of this world line is given by the equation

$$ds^2 = -dX^\mu g_{\mu\nu} dX^\nu \tag{1}$$

Where μ and $\nu = 0, 1, 2, ..., D$. D is the dimension of spacetime $g_{\mu\nu}$ is the spacetime metric tensor and 'ds' is the proper length. The simplest lorentz invarient action we can write for this particle would be proportional to the proper length

$$S = -mc \int ds \tag{2}$$

Where 'mc' is here for dimensional purposes. We can then write

$$\begin{aligned} S &= -mc \int \sqrt{-dX^\mu g_{\mu\nu} dX^\nu} \\ &= -mc \int \sqrt{-\frac{dX^\mu}{dt} dt g_{\mu\nu} \frac{dX^\nu}{dt} dt} \end{aligned} \tag{3}$$

If we assume that spacetime is flat then we write

$$\begin{aligned} S &= -mc \int dt \sqrt{-\frac{dX^\mu}{dt} \eta_{\mu\nu} \frac{dX^\nu}{dt}} \\ &= -mc \int dt \sqrt{-\frac{dX^\mu}{dt} \frac{dX_\mu}{dt}} \end{aligned} \tag{4}$$

Now if we contract the 0th term we can simply the action to the following quantity

$$\begin{aligned} &= -mc^2 \int dt \sqrt{1 - \frac{1}{c^2} \frac{dX^i}{dt} \frac{dX_i}{dt}} \\ &= -mc^2 \int dt \sqrt{1 - \frac{v^2}{c^2}} \end{aligned} \tag{5}$$

Where v^2 is the D-dimensional velocity squared. We can easily show that this reduces to the classical lagrangian in the non-relativitic limit.

$$\begin{aligned} L &= -mc^2 \sqrt{1 - \frac{v^2}{c^2}} \\ &\approx -mc^2 (1 - \frac{1}{2} \frac{v^2}{c^2}) = \frac{1}{2} m v^2 + \text{constant} \end{aligned} \tag{6}$$

So it reduces to the classical result in the non relativitic limit.

1.2 World Sheets and Nambu–Goto action

Now let us consider modeling a relativistic string. The simplest action we can construct would be proportional to the proper area traced out in spacetime by the string. This is called the Nambu-Goto action. We will begin by considering the proper area in General Relativity.

In General Relativity the proper area is given by the corresponding formula.

$$A = \int d\tau d\sigma \sqrt{-\det(h_{ab})} \tag{7}$$

Where h_{ab} is the induced metric of the surface. Here we have chosen Tau and sigma as parameterizations. Tau does not necessarily represent the proper time. Now there are a few caveats we need to take care of later. First lets motivate this result by consider breaking up the 'world sheet' into rectangles. Seeing how we are working with vectors, it makes sense to break it up into parallelograms by considering the identity in linear algebra.

$$dA = |dv_1 \times dv_2| = |dv_1||dv_2||\sin(\theta)| = dv_1||dv_2||\sqrt{1-\cos^2(\theta)|} \tag{8}$$

$$dA = \sqrt{|dv_1|^2|dv_2|^2 - |dv_1|^2|dv_2|^2\cos^2(\theta)|} \tag{9}$$

Now we know that dv1 and dv2 should be in different directions. We shall write this in terms of the tensor function $X^\mu(\tau,\sigma)$ that represents the parametrized world-sheet. We will assign the value dv1 to $\frac{dX^\mu}{d\tau}d\tau$ and dv2 to $\frac{dX^\mu}{d\sigma}d\sigma$ where sigma and tau are used to parametrize the world sheet. Now this prodcues the following result

$$\begin{aligned} dA &= \sqrt{(d\tau^2\frac{dX^\mu}{d\tau}g_{\mu\nu}\frac{dX^\nu}{d\tau})(d\sigma^2\frac{dX^\alpha}{d\sigma}g_{\alpha\beta}\frac{dX^\beta}{d\sigma}) - d\tau^2 d\sigma^2(\frac{dX^\mu}{d\tau}g_{\mu\nu}\frac{dX^\nu}{d\sigma})^2} \\ &= d\tau d\sigma\sqrt{(\frac{dX^\mu}{d\tau}g_{\mu\nu}\frac{dX^\nu}{d\tau})(\frac{dX^\alpha}{d\sigma}g_{\alpha\beta}\frac{dX^\beta}{d\sigma}) - (\frac{dX^\mu}{d\tau}g_{\mu\nu}\frac{dX^\nu}{d\sigma})^2} \end{aligned} \tag{10}$$

Now, if we believe equation (7) then we should be able to express equation (10) as the determinant of some metric. By inspection we can write an induced metric 'h' as the following.

$$\begin{aligned} h_{\alpha\beta} &= \begin{pmatrix} \frac{dX^\mu}{d\tau}g_{\mu\nu}\frac{dX^\nu}{d\tau} & \frac{dX^\mu}{d\tau}g_{\mu\nu}\frac{dX^\nu}{d\sigma} \\ \frac{dX^\mu}{d\tau}g_{\mu\nu}\frac{dX^\nu}{d\sigma} & \frac{dX^\alpha}{d\sigma}g_{\alpha\beta}\frac{dX^\beta}{d\sigma} \end{pmatrix} \\ &= \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu} \\ &= \partial_\alpha X^\mu \partial_\beta X_\mu \end{aligned} \tag{11}$$

And all together the Nambu-Goto action becomes.

$$\begin{gathered} S = -T\int d\tau d\sigma\sqrt{-\det(h_{\alpha\beta})} \\ \boxed{S = -T\int d\tau d\sigma\sqrt{-\det(\partial_\alpha X^\mu \partial_\beta X_\mu)}} \end{gathered} \tag{12}$$

Where the scaler '-T' is introduced for dimensional purposes. And the negative inside the square root comes as a consequence of the Pseudo-Riemannian space which ensures the determinant will be negative.

Equation (11) is the known as the induced metric. It shouldn't come as a surprise that we are now working with two metrics, since our strings are manifolds themselves. It also shouldn't be a surprise that the metric can be expressed as a 2x2 matrix since our world-sheet traces a two dimensional surface. The second equal sign in equation (11) can be easily checked to be true.

Now there is actually a problem with this action. The problems stems from renormalization, namely, the square root makes renomrlaization difficult. Correcting this problem requires us to rewrite the Nambu-Goto action using what is known as the auxillary world sheet metric. It is a metric that classically reduces the action to the Nambu-Goto action. This new action in terms of the auxiliary world sheet metric is known as the Polyakov action which is given by

$$\boxed{S = -\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma(-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu} \tag{13}$$

Where the $\gamma^{\alpha\beta}$ represents the auxiliary world sheet metric with both indices raised and the $\gamma = \det(\gamma_{\alpha\beta})$. This corrects the issue and makes the theory renormalizable. The scaler at the begining is there for dimensional purposes.

We now wish to show that this indeed reduces to equation (12) at the classical level. To show this we will find variation of the action with respect to the inverse world-sheet metric $\gamma^{\alpha\beta}$

$$\frac{\delta S}{\delta\gamma^{\alpha\beta}} = 0 \tag{14}$$

To accomplish this, we begin by finding the variation of equation (13)

$$\begin{aligned}\delta S &= -\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma\delta((-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu) \\ &= -\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma\delta(-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu - \frac{1}{4\pi\alpha'}\int_M d\tau d\sigma\ (-\gamma)^{\frac{1}{2}}\delta(\gamma^{\alpha\beta})\partial_\alpha X^\mu\partial_\beta X_\mu\end{aligned} \tag{15}$$

Using the following identity

$$\delta\gamma = -\gamma\gamma_{\alpha\beta}\delta\gamma^{\alpha\beta} \tag{16}$$

Which in our case implies that following relationship

$$\delta\sqrt{-\gamma} = -\frac{1}{2}\sqrt{-\gamma}\gamma_{\alpha\beta}\delta\gamma^{\alpha\beta} \tag{17}$$

Going back to equation (15) its easy to see that

$$\begin{aligned}\delta S &= \frac{1}{4\pi\alpha'}\int_M d\tau d\sigma\frac{1}{2}\sqrt{-\gamma}\gamma_{\alpha\beta}\delta\gamma^{\alpha\beta}\gamma^{\kappa\eta}\partial_\kappa X^\mu\partial_\eta X_\mu - \frac{1}{4\pi\alpha'}\int_M d\tau d\sigma\ (-\gamma)^{\frac{1}{2}}\delta(\gamma^{\alpha\beta})\partial_\alpha X^\mu\partial_\beta X_\mu \\ &= \frac{1}{4\pi\alpha'}\int_M d\tau d\sigma\delta\gamma^{\alpha\beta}\frac{1}{2}\sqrt{-\gamma}\gamma_{\alpha\beta}\gamma^{\kappa\eta}\partial_\kappa X^\mu\partial_\eta X_\mu - \frac{1}{4\pi\alpha'}\int_M d\tau d\sigma\delta\gamma^{\alpha\beta}(-\gamma)^{\frac{1}{2}}\partial_\alpha X^\mu\partial_\beta X_\mu \\ &= 0\end{aligned} \tag{18}$$

Dividing equation (18) by $\delta\gamma^{\alpha\beta}$ and canceling out the constants gives the equation of motion for the metric. In other words, it's easy to see that $\frac{\delta S}{\delta\gamma^{\alpha\beta}} = 0$ implies the following relationship.

$$\frac{1}{2}\gamma_{\alpha\beta}\gamma^{\kappa\eta}\partial_\kappa X^\mu \partial_\eta X_\mu = \partial_\alpha X^\mu \partial_\beta X_\mu \tag{19}$$

From here we can take the negative square root of the determinant from both sides wrt to the alpha and beta tensor,

$$\frac{1}{2}(-\gamma)^{\frac{1}{2}}\gamma^{\kappa\eta}\partial_\kappa X^\mu \partial_\eta X_\mu = \sqrt{-\det(\partial_\alpha X^\mu \partial_\beta X_\mu)} \tag{20}$$

But the right side is just the integrand of the Nambu-Goto action and the left side is the integrand of the Polyakov action. Therefore the equation of motion for the world sheet metric $\gamma_{\alpha\beta}$ implies the Nambu-Goto action. QED.

Now notice that we never explicitly defined a value for the world-sheet metric $\gamma_{\alpha\beta}$. This is to preserve the symmetries in the Polyakov action. Namely, these symmetries allows us to perform gauge fixing with the only requirement being that equation (19) is satisfied. We will now explore these symmetries in the next section to see which quantities we are allowed to gauge fix.

1.3 Symmetries in the Polyakov action

We would now like to talk about the symmetries in the Polyakov action

$$S = -\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma (-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu \tag{21}$$

...for which there are a lot. However, for now, lets restrict our analysis to that of Minkowski space. The Polyakov action is invarient under the following transformations

Poincare Transformation

$$X'^\mu = \Lambda^\mu_\nu X^\nu + a^\mu \tag{22}$$

$$\delta\gamma_{\alpha\beta} = 0 \tag{23}$$

Where Λ is a Lorentz transformation and a a^μ is a translation. Poincare invariance is a global symmetry in the action which implies it cannot be used for gauge fixing.

Diffeomorphism invariance

$$X'^\mu(\tau', \sigma') = X^\mu(\tau, \sigma) \tag{24}$$

$$\frac{\partial\sigma'^c}{\partial\sigma^a}\frac{\partial\sigma'^d}{\partial\sigma^b}\gamma'_{cd}(\tau', \sigma') = \gamma_{ab}(\tau, \sigma) \tag{25}$$

for some new choice of coordinates $\sigma'^a(\tau, \sigma)$. Equation (24) is also known as reparameterization invariance. This is an very important property of string theory so be sure to understand it. While equation (25) is nothing more than the tensor transformation law. Diffeomorphism invarice is a local symmetry which implies it allows for gauge fixing.

Weyl invariance

$$X'^{\mu}(\tau,\sigma) = X^{\mu}(\tau,\sigma) \tag{26}$$

$$\gamma'_{ab}(\tau,\sigma) = e^{2\omega(\tau,\sigma)}\gamma_{ab}(\tau,\sigma) \tag{27}$$

which holds for any $\omega(\tau,\sigma)$. This is also a local gauge symmetry, which implies that it can be used for gauge fixing.

This concludes the relevant symmetries in the Polyakov action. While in the future, we will talk more about the implication of these symmetries and treat them as more fundamental then the Polyakov action, for now they are simply symmetries in the Polyakov action.

1.4 Equations of motion and Boundary conditions

The goal of this section is find the equation of motion for X^{μ} in the Polyakov action.

$$S[X^{\mu},\gamma_{\alpha\beta}] = -\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma(-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\partial_{\alpha}X^{\mu}\partial_{\beta}X_{\mu} \tag{28}$$

We will now take the variation wrt to X^{μ}. Where the goal is to find equation of motion by demanding that

$$\frac{\delta S}{\delta X^{\mu}} = 0 \tag{29}$$

Taking the variation

$$\begin{aligned}\delta S &= -\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma\delta((-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\partial_{\alpha}X^{\mu}\partial_{\beta}X_{\mu}) \\ &= -\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma(-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\partial_{\alpha}\delta X^{\mu}\partial_{\beta}X_{\mu} - \frac{1}{4\pi\alpha'}\int_M d\tau d\sigma\ (-\gamma)^{\frac{1}{2}}(\gamma^{\alpha\beta})\partial_{\alpha}X^{\mu}\partial_{\beta}\delta X_{\mu}\end{aligned} \tag{30}$$

Where we used the easy to show identity $\delta\partial_a X^{\mu} = \partial_a\delta X^{\mu}$. By relabeling the indices we can combine them into a singal integral. (Specifically, we use the symmetry in the metric tensor)

$$\delta S = -\frac{1}{2\pi\alpha'}\int_M d\tau d\sigma(-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\partial_{\alpha}\delta X^{\mu}\partial_{\beta}X_{\mu} \tag{31}$$

Now from here we manipulate the integrand using the identity

$$(-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\partial_{\alpha}\delta X^{\mu}\partial_{\beta}X_{\mu} = \partial_{\alpha}((-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\delta X^{\mu}\partial_{\beta}X_{\mu}) - \delta X^{\mu}\partial_{\alpha}((-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\partial_{\beta}X_{\mu}) \tag{32}$$

Where this can be understood to be an integration by parts. Note that distributing the derivatives on the right side will equal the left side. Now, the term

$$\partial_{\alpha}((-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\delta X^{\mu}\partial_{\beta}X_{\mu}) = 0 \tag{33}$$

is usually understood to vanish due to boundary conditions we will impose in the next section. We will just assume it to be zero for now

From here, we are left with the action (after dividing by δX^μ)

$$\frac{\delta S}{\delta X^\mu} = \frac{1}{2\pi\alpha'}\int_M d\tau d\sigma \partial_\alpha((-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\partial_\beta X_\mu) = 0 \tag{34}$$

The equation of motion is understood to mean the following equations

$$\partial_\alpha(-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\partial_\beta X_\mu = 0 \tag{35}$$

Along with

$$\begin{aligned}(-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\partial_\alpha\partial_\beta X_\mu &= (-\gamma)^{\frac{1}{2}}\partial_\alpha\partial^\alpha X_\mu = 0 \\ &= (-\gamma)^{\frac{1}{2}}\partial_\alpha\partial^\alpha X_\nu g^{\mu\nu} = g^{\mu\nu} * 0 \\ = (-\gamma)^{\frac{1}{2}}\partial_\alpha\partial^\alpha X^\mu &= \boxed{(-\gamma)^{\frac{1}{2}}\nabla^2 X^\mu = 0}\end{aligned} \tag{36}$$

This along with the previous section complete the equations of motion for the Polyakov action at this state. Its worth noting that there are a few terms that can be added to the Polyakov action that, although break Poincare invariance, are worth exploring. We will discuss these later.

Lets now talk about boundary conditions. There are two types of strings that are constructed from various boundary conditions. The two types are open strings and closed strings. Intuitively, we can think of closed strings as being topologically a circle and an open string as being topologically a line interval. Let us now discuss these boundary conditions.

For these conditions we will choose a parametrization for σ such that it lies inside the interval $0 \leq \sigma \leq \pi$. This is an arbitrary choice that is done for the sake of simplifying the analysis of the boundary conditions.

Closed String

$$X^\mu(\tau,\sigma) = X(\tau,\sigma+\pi) \tag{37}$$

This is a period condition that simply ensures the string is closed everywhere.

Open String with Neumann boundary conditions

$$\partial^\sigma X^\mu = 0 \text{ At } \sigma = 0, \pi \text{ (Generally)} \tag{38}$$

Now, under most gauge fixes we can write ∂_σ instead of ∂^σ but writing this way, its clearer that the boundary condition we assumed was zero *is indeed zero*. The general consequence of this boundary

condition is that the component of momentum normal to the worldsheet vanishes at the boundary.
Open String with Dirichlet Boundary condition

$$X^\mu|_{\sigma=0} = X_0^\mu \tag{39}$$

$$X^\mu|_{\sigma=\pi} = X_\pi^\mu \tag{40}$$

Where X_0^μ and X_π^μ are constants. This condition applies for $\mu = 1, 2, 3..D - p - 1$ where d is the dimension of the theoy p is a p-dimensional subspace of the theory. We will talk more about Dp branes later.

1.5 Lightcone coordinates

The purpose of this section construct an easier set of coordintates to solve the equations of motion. These are known as light cone coordinates and they are merlely a new set of coordinates to makes solving our theory easier. In General relativity. we define the light cone component for any vector a^μ as the following

$$a^+ \equiv \frac{1}{\sqrt{2}}(a^0 + a^1) \tag{41}$$

and

$$a^- \equiv \frac{1}{\sqrt{2}}(a^0 - a^1) \tag{42}$$

we let the rest of the indices run from i = 2,...,D

$$a^i \text{ runs from i = 2, . . . , D} \tag{43}$$

It's really just a change of basis in flat spacetime. We can imagine that, internally, the basis looks something like this

$$e_\mu = (a^-, a^+, a^i) \tag{44}$$

We can also define coordinates with 'lowered indices' as the following

$$a_+ \equiv -a^- \tag{45}$$

and

$$a_- \equiv -a^+ \tag{46}$$

These new coordinates are require us to re implement contractions which look like this in the new basis

$$\begin{aligned} a^\mu b_\mu &= -a^+ b^- - a^- b^+ + a^i b^i \\ &= a_- b^- + a_+ b^+ + a_i b^i \end{aligned} \tag{47}$$

Where we choose metric such that $a_i = a^i$

Back to string theory, we introduce world-sheet light cone coordinates defined by

$$\sigma^{\pm} = \tau \pm \sigma \text{ and } \partial_{\pm} = \frac{1}{2}(\partial_\tau \pm \partial_\sigma) \tag{48}$$

Along with the metric

$$\begin{pmatrix} \eta_{++} & \eta_{+-} \\ \eta_{-+} & \eta_{--} \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{49}$$

1.6 Lightcone gauge

The quickest way to get familiar with certain results in string theory is using the "lightcone gauge" as our fixing condition. As discussed earlier, the Polyakov action has a lot of symmetries that allow us to choose a convenient gauge.

Lets first look at the case for a point particle instead of the string world-sheet. The point particle action is given by

$$S = \frac{1}{2}\int d\tau (\eta^{-1}\dot{X}^\mu \dot{X}_\mu - \eta m^2) \tag{50}$$

Now the cannonical momentum is given by

$$\begin{aligned} p_\mu &= \frac{\partial L}{\partial \dot{X}^\mu} \\ &= \frac{\partial}{\partial \dot{X}^\mu}\frac{1}{2}(\eta^{-1}\dot{X}^\nu \dot{X}_\nu - \eta m^2) \\ &= \eta^{-1}\dot{X}_\mu \end{aligned} \tag{51}$$

Note that the dummy indicies that are being contracted were swapped in the second line to prevent the ambiguous notation (3 indices).

We now wish to construct Hamilonian which is usually done as follows

$$\begin{aligned} H &= p_\mu \dot{X}^\mu - L \\ &= \eta^{-1}\dot{X}_\mu \dot{X}^\mu - L \end{aligned} \tag{52}$$

Alright if we wish to carry out this process in lightcone coordinates, it can be done by first transforming (50) into lightcone coordinates. Using the identity. First lets gauge fix our parametrization. Specifically let

$$X^+(\tau) = \tau \text{ Where naturally } \dot{X}^+(\tau) = 1 \tag{53}$$

This lets us treat X^+ as a sort of timelike variable. Now, we can use the contraction identity from the previous section

$$\dot{X}^\mu \dot{X}_\mu = -\dot{X}^+\dot{X}^- - \dot{X}^-\dot{X}^+ + \dot{X}^i\dot{X}^i \tag{54}$$

To rewrite the action like this

$$
\begin{aligned}
S &= \frac{1}{2}\int d\tau(\eta^{-1}\dot{X}^{\mu}\dot{X}_{\mu} - \eta m^2) \\
&= \frac{1}{2}\int d\tau(\eta^{-1}(-\dot{X}^{+}\dot{X}^{-} - \dot{X}^{-}\dot{X}^{+} + \dot{X}^{i}\dot{X}^{i}) - \eta m^2) \\
&= \frac{1}{2}\int d\tau(-\eta^{-1}2\dot{X}^{-} + \eta^{-1}\dot{X}^{i}\dot{X}^{i} - \eta m^2)
\end{aligned}
\tag{55}
$$

Now we can find the canonical momentum

$$
\begin{aligned}
p_{-} &= \frac{\partial L}{\partial \dot{X}^{-}} = -\eta^{-1} \\
p_{+} &= \frac{\partial L}{\partial \dot{X}^{+}} = 0 \\
p_{i} &= \frac{\partial L}{\partial \dot{X}^{i}} = \eta^{-1}X_i
\end{aligned}
\tag{56}
$$

Which allows us to contruct the Hamiltonian like follows

$$H = p_{\mu}\dot{X}^{\mu} - L = p_{-}\dot{X}^{-} + p^{i}\dot{X}^{i} - L \tag{57}$$

Now it's easy to see that we can rewrite this as the following

$$H = \frac{p^i p^i + m^2}{2p^{+}} \tag{58}$$

We finish of this example by quantizing the theory. We accomplish this by imposing the following commutator relations

$$
\begin{aligned}
[p_i, X^j] &= -i\delta_i^j \\
[p_{-}, X^{-}] &= -i
\end{aligned}
\tag{59}
$$

Where the variables p_i, X^j, p_{-}, X^{-} are now promoted to operators. This completes the quantization procedure for the relativistic point particle.

We shall now turn to the Polyakov action and perform the same procedure.

$$S[X^{\mu}, \gamma_{\alpha\beta}] = -\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma(-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\partial_{\alpha}X^{\mu}\partial_{\beta}X_{\mu} \tag{60}$$

Now we want to impose a similar gauge fixing condition. The specific conditions will be

$$
\boxed{
\begin{aligned}
X^{+} &= \tau \\
\partial_{\sigma}\gamma_{\sigma\sigma} &= 0 \\
\det(\gamma_{ab}) &= -1
\end{aligned}
}
\tag{61}
$$

We will also fix the parameters $-\infty < \tau < \infty$ and $0 \leq \sigma \leq \ell$ this suffices to gauge fix the symmetries in the Polyakov action. Now we can use the second condition in (103) to determine that $\gamma_{\sigma\sigma}$ is a function of

τ only $\gamma_{\sigma\sigma}(\tau)$. Now we can easily find the inverse of the metric γ_{ab} by invoking the result from general relativity. For any 2x2 metric $g_{\alpha\beta}$ the inverse is given by

$$g^{\alpha\beta} = \frac{1}{\det(g)} \begin{pmatrix} g_{11} & -g_{01} \\ -g_{01} & g_{00} \end{pmatrix} \tag{62}$$

Now back to string theory, we can use this formula to find the inverse of the worldsheet metric.

$$\begin{pmatrix} \gamma^{\tau\tau} & \gamma^{\tau\sigma} \\ \gamma^{\sigma\tau} & \gamma^{\sigma\sigma} \end{pmatrix} = \begin{pmatrix} -\gamma_{\sigma\sigma}(\tau) & \gamma_{\tau\sigma}(\tau,\sigma) \\ \gamma_{\tau\sigma}(\tau,\sigma) & -\gamma_{\tau\tau}(\tau,\sigma) \end{pmatrix} \tag{63}$$

Using the third gauge fixing condition in (103)

$$\begin{aligned} \det(\gamma_{ab}) &= -1 \\ \gamma_{\tau\tau}(\tau,\sigma)\gamma_{\sigma\sigma}(\tau) - \gamma_{\tau\sigma}^2(\tau,\sigma) &= -1 \\ \gamma_{\tau\tau}(\tau,\sigma) &= \frac{1}{\gamma_{\sigma\sigma}(\tau)}(-1+\gamma_{\tau\sigma}^2) \\ \gamma^{ab} &= \begin{pmatrix} -\gamma_{\sigma\sigma}(\tau) & \gamma_{\tau\sigma}(\tau,\sigma) \\ \gamma_{\tau\sigma}(\tau,\sigma) & \frac{1}{\gamma_{\sigma\sigma}(\tau)}(1-\gamma_{\tau\sigma}^2(\tau,\sigma)) \end{pmatrix} \end{aligned} \tag{64}$$

Now we wish to transform the Polyakov action into lightcone coordinates.

$$S[X^\mu,\gamma_{\alpha\beta}] = -\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma (-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu \tag{65}$$

Start by defining

$$\begin{aligned} X' &= \frac{\partial X^\mu}{\partial\sigma} \\ \dot{X} &= \frac{\partial X^\mu}{\partial\tau} \end{aligned} \tag{66}$$

Now rewriting the lagrangian in light cone coordinates and using (47) to we can write.

$$\begin{aligned} L &= -\frac{1}{4\pi\alpha'}\int_0^\ell d\sigma(\gamma^{\tau\tau}\partial_\tau X^\mu\partial_\tau X_\mu + \gamma^{\tau\sigma}\partial_\tau X^\mu \partial_\sigma X_\mu + \gamma^{\sigma\tau}\partial_\sigma X^\mu\partial_\tau X_\mu + \gamma^{\sigma\sigma}\partial_\sigma X^\mu\partial_\sigma X_\mu) \\ &= -\frac{1}{4\pi\alpha'}\int_0^\ell d\sigma(-\gamma_{\sigma\sigma}(\tau)\partial_\tau X^\mu\partial_\tau X_\mu + 2\gamma_{\tau\sigma}(\tau,\sigma)\partial_\tau X^\mu\partial_\sigma X_\mu + \frac{1}{\gamma_{\sigma\sigma}(\tau)}(1-\gamma_{\tau\sigma}^2(\tau,\sigma))\partial_\sigma X^\mu\partial_\sigma X_\mu) \\ &= -\frac{1}{4\pi\alpha'}\int_0^\ell d\sigma(\gamma_{\sigma\sigma}(\tau)\partial_\tau X^\mu\partial_\tau X_\mu + 2\gamma_{\tau\sigma}(\tau,\sigma)\partial_\tau X^\mu\partial_\sigma X_\mu + \frac{1}{\gamma_{\sigma\sigma}(\tau)}(1-\gamma_{\tau\sigma}^2(\tau,\sigma))\partial_\sigma X^\mu\partial_\sigma X_\mu) \\ &= -\frac{1}{4\pi\alpha'}\int_0^\ell d\sigma[\gamma_{\sigma\sigma}(\tau)(2\dot{X}^- - \dot{X}^i\dot{X}^i) - 2\gamma_{\tau\sigma}(\tau,\sigma)(X' - \dot{X}^i X'^i) + \frac{1}{\gamma_{\sigma\sigma}(\tau)}(1-\gamma_{\tau\sigma}^2(\tau,\sigma))\partial_\sigma X^i\partial_\sigma X_i] \\ &= -\frac{1}{4\pi\alpha'}\int_0^\ell d\sigma[\gamma_{\sigma\sigma}(\tau)(2\partial_\tau X^- - \partial_\tau X^i\partial_\tau X^i) \\ &- 2\gamma_{\tau\sigma}(\tau,\sigma)(\partial_\sigma X^- - \partial_\tau X^i\partial_\sigma X^i) + \frac{1}{\gamma_{\sigma\sigma}(\tau)}(1-\gamma_{\tau\sigma}^2(\tau,\sigma))\partial_\sigma X^i\partial_\sigma X_i)] \end{aligned} \tag{67}$$

Where in the 4th line the identity $\frac{\partial X^+}{\partial \sigma} = 0$ along with $\frac{\partial X^+}{\partial \tau} = 1$. Now by convention we typically split the $\partial_\sigma X^-$ term as

$$x^- = \frac{1}{\ell}\int_0^\ell d\sigma X^-(\tau,\sigma) \tag{68}$$

$$Y^- = X^-(\tau,\sigma) - x^-(\tau) \tag{69}$$

Where it's clear that x^- is the mean value of X^- and the second term has an average value of zero for a fixed τ such that

$$\int_0^\ell d\sigma Y^-(\tau,\sigma) = 0 \tag{70}$$

This results in the following Lagrangian

$$\begin{aligned} L = -\frac{1}{4\pi\alpha'}\int_0^\ell d\sigma[\gamma_{\sigma\sigma}(\tau)(2\partial_\tau X^- - \partial_\tau X^i\partial_\tau X^i) \\ -2\gamma_{\tau\sigma}(\tau,\sigma)(\partial_\sigma Y^- - \partial_\tau X^i\partial_\sigma X^i) + \frac{1}{\gamma_{\sigma\sigma}(\tau)}(1-\gamma_{\tau\sigma}^2(\tau,\sigma))\partial_\sigma X^i\partial_\sigma X_i)] \end{aligned} \tag{71}$$

We also rewrite the X^- term using the identity $X^- = Y^- + x^-$. Looking at the first term

$$\begin{aligned} &\int_0^\ell d\sigma[\gamma_{\sigma\sigma}(\tau)2\partial_\tau X^-] \\ =&\int_0^\ell d\sigma[\gamma_{\sigma\sigma}(\tau)2\partial_\tau(Y^- + x^-)] \\ =&[\gamma_{\sigma\sigma}(\tau)2\partial_\tau(\int_0^\ell d\sigma Y^- + \int_0^\ell d\sigma x^-)] \end{aligned} \tag{72}$$

But the first integral is zero since by (70) which implies that the lagrangian can be written as follows.

$$\begin{aligned} L = -\frac{1}{4\pi\alpha'}\int_0^\ell d\sigma[\gamma_{\sigma\sigma}(\tau)(2\partial_\tau x^- - \partial_\tau X^i\partial_\tau X^i) \\ -2\gamma_{\tau\sigma}(\tau,\sigma)(\partial_\sigma Y^- - \partial_\tau X^i\partial_\sigma X^i) + \frac{1}{\gamma_{\sigma\sigma}(\tau)}(1-\gamma_{\tau\sigma}^2(\tau,\sigma))\partial_\sigma X^i\partial_\sigma X_i)] \end{aligned} \tag{73}$$

This concludes the groundwork required to solve the theory in these new coordinates. In the next sections we will apply boundary conditions and create a mode expansion along with quantizing the theory.

1.7 Open String in Lightcone gauge

We begin by applying the boundary conditions of an open string with the Neumann boundary condition (38) with the boundary at ℓ instead of at π just for convention.

$$\begin{aligned} &\partial^\sigma X^\mu = 0 \text{ (At } \sigma = 0,\ell) \\ =&\gamma^{\sigma b}\partial_b X^\mu = \gamma^{\sigma\sigma}\partial_\sigma X^\mu + \gamma^{\sigma\tau}\partial_\tau X^\mu \\ &= \gamma_{\tau\sigma}\partial_\tau X^\mu - \gamma_{\tau\tau}\partial_\sigma X^\mu = 0 \end{aligned} \tag{74}$$

Now since we are on lightcone gauge, we can let $\mu = +$ which implies

$$\gamma_{\tau\sigma}\partial_\tau X^+ - \gamma_{\tau\tau}\partial_\sigma X^+ = 0 \text{ (At } \sigma = 0, \ell) \tag{75}$$

But the second term is zero since taking the derivative τ wrt σ in this gauge choice. On the other hand, the first term will evaluate to simply $\gamma_{\tau\sigma}$ since the derivative will be 1.

$$\gamma_{\tau\sigma} = 0 \text{ (At } \sigma = 0, \ell) \tag{76}$$

Now taking the variation of the action wrt to the new field Y^- yields (ignoring the other terms which are trivially zero)

$$\begin{aligned}
\delta S &= \int d\tau d\sigma \delta(2\gamma_{\tau\sigma}(\tau,\sigma)\partial_\sigma Y^-) \\
&= 2\int d\tau d\sigma(\delta\partial_\sigma(\gamma_{\tau\sigma}Y^-) - \delta\partial_\sigma\gamma_{\tau\sigma}Y^-) \\
&= 2\int d\tau d\sigma(\partial_\sigma(\gamma_{\tau\sigma}\delta Y^-) - \partial_\sigma\gamma_{\tau\sigma}\delta Y^-) \\
&= 2\int d\tau d\sigma(-\partial_\sigma\gamma_{\tau\sigma}\delta Y^-) = 0
\end{aligned} \tag{77}$$

In the last line, we used the fact that the left side is zero because it is evaluated at the boundary which we just determined was zero. Now The most general solutions to the last equations is that such that $\partial_\sigma\gamma_{\tau\sigma}$ is just a function of τ. This allows the integral wrt σ to act on the δY^- which zeros the whole thing.

$$\partial_\sigma\gamma_{\tau\sigma} = F(\tau) \tag{78}$$

so

$$\partial_\sigma^2\gamma_{\tau\sigma} = 0 \tag{79}$$

Now this tells us that the derivative of $\gamma_{\tau\sigma}$ only has τ dependence and yet we must also meet the boundary conditions that $\gamma_{\tau\sigma}(\tau,0) = \gamma_{\tau\sigma}(\tau,\ell) = 0$ This constraints $\gamma_{\tau\sigma} = 0$ everywhere, otherwise we have no way of meeting both boundary conditions for any τ. Finally we can use the boundary condition again once again

$$\gamma_{\tau\sigma}\partial_\tau X^i - \gamma_{\tau\tau}\partial_\sigma X^i = 0 \text{ (At } \sigma = 0, \ell) \tag{80}$$

Where the first term is clearly zero and that leaves

$$\gamma_{\tau\tau}\partial_\sigma X^i = 0 \text{ (At } \sigma = 0, \ell) \tag{81}$$

Then the lagrangian reduces to

$$
\begin{aligned}
L = &-\frac{1}{4\pi\alpha'}\int_0^\ell d\sigma[\gamma_{\sigma\sigma}(\tau)(2\partial_\tau x^- - \partial_\tau X^i\partial_\tau X^i) \\
&- 2\gamma_{\tau\sigma}(\tau,\sigma)(\partial_\sigma Y^- - \partial_\tau X^i\partial_\sigma X^i) + \frac{1}{\gamma_{\sigma\sigma}(\tau)}(1-\gamma_{\tau\sigma}^2(\tau,\sigma)\partial_\sigma X^i\partial_\sigma X_i)] \\
= &\frac{1}{4\pi\alpha'}\int_0^\ell d\sigma[-\gamma_{\sigma\sigma}(\tau)(2\partial_\tau x^- + \partial_\tau X^i\partial_\tau X^i) - \frac{1}{\gamma_{\sigma\sigma}(\tau)}(\partial_\sigma X^i\partial_\sigma X_i)] \\
= &\boxed{-\frac{\ell}{2\pi\alpha'}\gamma_{\sigma\sigma}\partial_\tau x^- + \frac{1}{4\pi\alpha'}\int_0^\ell d\sigma[\gamma_{\sigma\sigma}\partial_\tau X^i\partial_\tau X^i - \frac{1}{\gamma_{\sigma\sigma}(\tau)}(\partial_\sigma X^i\partial_\sigma X_i)]}
\end{aligned}
\tag{82}
$$

Now we wish to find the conjugate momentum of this lagrangian along with the Hamiltonian, just like we did with the point particle case, in order to quantize the theory. We begin by finding the conjugate momentum of all the independent variables.

$$
p_- = -p^+ = \frac{\partial L}{\partial(\partial_\tau x^-)} = -\frac{\ell}{2\pi\alpha'}\gamma_{\sigma\sigma}
\tag{83}
$$

On the other hand, the other independent variable X^i is also trivial to take.

$$
\Pi^i = \frac{\delta\mathcal{L}}{\delta(\partial_\tau X^i)} = \frac{1}{2\pi\alpha'}\gamma_{\sigma\sigma}\partial_\tau X^i
\tag{84}
$$

Where Π the conjugate momentum density. The Hamiltonian is then

$$
H = p_-\partial_\tau x^- - L + \int d\sigma(\Pi^i\partial_\tau X^i)
\tag{85}
$$

$$
\begin{aligned}
&= \int d\sigma[-\frac{1}{4\pi\alpha'}\gamma_{\sigma\sigma}\partial_\tau X^i\partial_\tau X^i + \frac{1}{4\pi\alpha'\gamma_{\sigma\ \sigma}}(\partial_\sigma X^i\partial_\sigma X_i) + \Pi^i\partial_\tau X^i] \\
&= \int d\sigma[-\frac{(2\pi\alpha')^2}{4\pi\alpha'\gamma_{\sigma\sigma}}\Pi^i\Pi^i + \frac{1}{4\pi\alpha'\gamma_{\sigma\ \sigma}}(\partial_\sigma X^i\partial_\sigma X_i) + \frac{2\pi\alpha'}{\gamma_{\sigma\sigma}}\Pi^i\Pi^i] \\
&= \int d\sigma[\frac{\pi\alpha'}{\gamma_{\sigma\sigma}}\Pi^i\Pi^i + \frac{1}{4\pi\alpha'\gamma_{\sigma\ \sigma}}(\partial_\sigma X^i\partial_\sigma X_i)] \\
&= \frac{1}{2\gamma_{\sigma\sigma}}\int d\sigma[2\pi\alpha'\Pi^i\Pi^i + \frac{1}{2\pi\alpha'}(\partial_\sigma X^i\partial_\sigma X_i)] \\
&= \frac{\ell}{4\pi\alpha' p^+}\int d\sigma[2\pi\alpha'\Pi^i\Pi^i + \frac{1}{2\pi\alpha'}(\partial_\sigma X^i\partial_\sigma X_i)]
\end{aligned}
\tag{86}
$$

At this point, we can use the Hamilton's equations to find the EOM. The first three are trivial. Let $c \equiv \frac{\ell}{2\pi\alpha' p^+}$

$$
\partial_\tau x^- = \frac{\partial H}{\partial p_-} = \frac{H}{p^+}
\tag{87}
$$

$$
\partial_\tau X^- = \frac{\delta H}{\delta \Pi^i} = 2c\pi\alpha'\Pi^i
\tag{88}
$$

$$
\partial_\tau p^+ = \frac{\partial H}{\partial p^+} = 0
\tag{89}
$$

While the last one requries an integration by parts.

$$\delta H = \frac{c}{2}(\frac{1}{2\pi\alpha'}\partial_\sigma \delta X^i \partial_\sigma X^i + \frac{1}{2\pi\alpha'}\partial_\sigma X^i \partial_\sigma \delta X^i) \\ = \frac{c}{2\pi\alpha'}\partial_\sigma \delta X^i \partial_\sigma X^i$$

Using

$$\partial_\sigma \delta X^i \partial_\sigma X^i = \partial_\sigma(\delta X^i \partial_\sigma X^i) - \delta X^i \partial_\sigma^2 X^i \tag{90}$$

Then

$$\delta H = \frac{c}{2\pi\alpha'}\int d\sigma[\partial_\sigma(\delta X^i \partial_\sigma X^i) - \delta X^i \partial_\sigma^2 X^i] \\ -\frac{\delta H}{\delta X^i} = \frac{c}{2\pi\alpha'}\partial_\sigma^2 X^i$$

Where in the last line we used the boundary condition to zero the first term.

Then the equation of motion is

$$\partial_\tau \Pi^i = -\frac{\delta \mathcal{H}}{\delta X^i} = \frac{c}{2\pi\alpha'}\partial_\sigma^2 X^i \tag{91}$$

If we combine the second and last equation we get the (simple) wave equation!

$$\partial_\tau^2 X^i = c^2 \partial_\sigma^2 X^i \tag{92}$$

Since we will rework all this in the path integral formalism later. Lets finish this section quickly by writing the mode expansion and quantizing the theory.

$$X^i = x^i + \frac{p^i}{p^+}\tau + i(2\alpha')^{\frac{1}{2}} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n}\alpha_n^i e^{-\frac{\pi i n c \tau}{\ell}} \cos(\frac{\pi n \sigma}{\ell}) \tag{93}$$

Where, classically, α_n are the fourier coefficients. The goal is not to quantize the theory. This is completed with the two commutators at a fixed time.

$$[x^-, p^+] = -i \\ [X^i(\sigma), \Pi^j(\sigma')] = i\delta^{ij}\delta(\sigma - \sigma') \tag{94}$$

Now using the definitions

$$x^i = \frac{1}{\ell}\int_0^\ell d\sigma X^i(\tau, \sigma) \tag{95}$$

$$p^i = \int_0^\ell d\sigma \Pi^i(\tau, \sigma) \tag{96}$$

In other words the center of mass position and the average momentum

$$[x^i, p^j] = i\delta^{ij} \tag{97}$$

$$[\alpha_m^i, \alpha_n^j] = m\delta^{ij}\delta_{m-n} \tag{98}$$

These relationships essentially promote the variables $\alpha, \Pi, x^i, p+, x^-$ into operators which obey a similiar algebra to the creation and annihilation operators. To be clear, the α operators satisfy a similiar algebra

$$\begin{aligned} a^i_m \sim m^{\frac{1}{2}} a \qquad a^i_{-m} \sim m^{\frac{1}{2}} a^\dagger \\ \text{For } m > 0 \end{aligned} \tag{99}$$

Where a and $a^\dagger$ obey the harmonic osscilator creation and annhilation algebra. We then define a k to be the center of mass momentum and are eigenstates of p

$$\begin{aligned} p^+ |0;k\rangle &= k^+ |0;k\rangle \\ p^i |0;k\rangle &= k^i |0;k\rangle \end{aligned} \tag{100}$$

Where it's clear the 0 represents the state of the oscillator and k is the momentum eigenvalue. Now just like any raising and lowering operator algebra we also impose that lowering the 0th state will result in zero.

$$\alpha^i_m |0;k\rangle = 0 \tag{101}$$

Finally we can write a general state

$$|N;k\rangle = \prod_{i=2}^{D-1} \prod_{n=1}^{\infty} \frac{(\alpha^i_{-n})^{N_{in}}}{\sqrt{(n^{N_{in}} N_{in}!)}} |0;k\rangle \tag{102}$$

Alright so don't let this scare you. The top part is just the creation operator and the bottom is just a normalization constant. The (i, n) subscript is just a tuple to index a specific direction and oscillator mode. This completes our discussion for the open string lightcone gauge (subject to change). Also (unlike QFT) we're not creating multiple strings using this. Rather we are changing the momentum of one string.

1.8 Closed String in Lightcone gauge

Lets quickly repeat the process for a closed string in the lightcone gauge. This section is still subject to modifications and rewrites in the future. Nevertheless, the analysis is largely a parallel except we add one more condition to the gauge fix.

$$\begin{aligned} X^+ &= \tau \\ \partial_\sigma \gamma_{\sigma\sigma} &= 0 \\ \det(\gamma_{ab}) &= -1 \\ \gamma_{\tau\sigma}(\tau, 0) &= 0 \end{aligned} \tag{103}$$

Now it turns out σ has symmetry under the following translations

$$\sigma' = \sigma + \text{s modulus } \ell \tag{104}$$

Where 'S' is a real scaler. This 'new freedom' comes from the periodic condition. We will ignore it for now. The mode expansion is then...

$$X^i = x^i + \frac{p^i}{p^+}\tau + i(\frac{\alpha'}{2})^{\frac{1}{2}} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (\frac{\alpha_n^i}{n} e^{-\frac{2\pi i n(\sigma + c\tau)}{\ell}} + \frac{\tilde{\alpha}_n^i}{n} e^{\frac{2\pi i n(\sigma - c\tau)}{\ell}}) \tag{105}$$

Now promote the following to operators that obey the following relationships.

$$[x^-, p^+] = -i \tag{106}$$

$$[x^i, p^j] = i\delta^{ij} \tag{107}$$

$$[\alpha_m^i, \alpha_n^j] = m\delta^{ij}\delta_{m,-n} \tag{108}$$

$$[\tilde{\alpha}_m^i, \tilde{\alpha}_n^j] = m\delta^{ij}\delta_{m,-n} \tag{109}$$

Finally a general state takes the following form

$$|N, \tilde{N}; k\rangle = \prod_{i=2}^{D-1} \prod_{n=1}^{\infty} \frac{(\alpha_{-n}^i)^{N_{in}} (\tilde{\alpha}_{-n}^i)^{\tilde{N}_{in}}}{\sqrt{(n^{N_{in}} N_{in}!)(n^{\tilde{N}_{in}} \tilde{N}_{in}!)}} |0, 0; k\rangle \tag{110}$$

This completes(subject to change) the discussion on the light cone gauge. We have built up a quantize string theory that

2 The Feynman Path Integral

2.1 The Path Integral formalism of Quantum Mechanics

We begin this next section by building the mathematical foundation needed to actually solve (compute) string theory using modern methods. Indeed the ultimate goal of these lecture notes is to write down the Polyakov path integral and compute scattering amplitudes with it. To understand this, we must first understand the Feynman Path integral. We will begin by considering non relativistic Quantum Mechanics. Lets attempt to motivate and build the path integral formalism.

The one dimensional Hamiltonian of a simple system is given by

$$H = \frac{P^2}{2m} + V(Q) \tag{111}$$

Which satisfy the cannonical commutator relationship

$$[Q, P] = i\hbar \tag{112}$$

Lets set $\hbar = 1$ like we do for most of time for convenience. Now consider the Heisenberg picture where operators are defined to be time dependent. Specifically, we can consider an instantaneous eigenstate. Lets first use the construction from the Heisenberg picture

$$|q,t\rangle = e^{iHt}|q\rangle \tag{113}$$

Now operators are defined as follows.

$$Q(t) = e^{iHt}Q(0)e^{-iHt} \tag{114}$$

Which implies that instanous eigenstates are found like follows

$$Q(t)|q,t\rangle = q|q,t\rangle \tag{115}$$

Where the intermediate exponential cancel out. Now, we can write the transition amplitude as follows.

$$\langle q'',t''|q',t'\rangle = \langle q''|e^{-iHt''}e^{iHt'}|q'\rangle \tag{116}$$

Now, from the Campbell-baker-Hausdorf formula

$$e^{A+B} = e^{A}e^{B}e^{-\frac{1}{2}[A,B]+..} \tag{117}$$

All the terms of the right most exponential will cancel out of (117) if we can just show that

$$[A,B] = 0 \tag{118}$$

This turns out to be case for us

$$[-iHt'', iHt'] = 0 \tag{119}$$

The global time parameter and constants don't matter. The Hamiltonian will naturally commute with itself which implies that we can treat this like follows

$$e^{-iH(t''-t')} = e^{-iHt''}e^{iHt'} \tag{120}$$

Cool, so we can break up the exponential without worrying about the fact that they are operators instead of scalers. Now, we will split this up into discrete time intervals. Let $\delta t = \frac{t''-t'}{N+1}$ where N+1 is the total time intervals.

$$e^{-iH(\delta t+\delta t+\delta t+\delta t+..)} = e^{-iH\delta t}e^{-iHt\delta}e^{-iH\delta t}e^{-iH\delta t}e^{-iH\delta t}\,.\,. \tag{121}$$

Now at this point, we can intesert a complete set of states between every operator. Recall the identity from Quantum Mechanics

$$\int_{-\infty}^{\infty} |q\rangle \langle q| \, dq = 1 \tag{122}$$

Where we can think of this as a sum of all projection operators $|q_1\rangle \langle q_1|$ which just returns the original quantity. Let us apply this to every operator.

$$\begin{aligned}\langle q'', t''|q', t'\rangle &= \int dq_1 dq_2 dq_3 dq_4 \cdot \cdot dq_{N-1} dq_N \langle q''|e^{-iH\delta t}|q_N\rangle \langle q_N|e^{-iH\delta t}|q_{N-1}\rangle \cdot \cdot \langle q_2|e^{-iH\delta t}|q_1\rangle \langle q_1|e^{-iH\delta t}|q'\rangle \\ &= \int \prod_{j=1}^{N} dq_j \langle q''|e^{-iH\delta t}|q_N\rangle \langle q_N|e^{-iH\delta t}|q_{N-1}\rangle \cdot \cdot \langle q_2|e^{-iH\delta t}|q_1\rangle \langle q_1|e^{-iH\delta t}|q'\rangle\end{aligned} \tag{123}$$

Lets look at an individual inner product

$$\langle q_2|e^{-iH\delta t}|q_1\rangle = \langle q_2|e^{-i\delta t(\frac{P^2}{2m}+V(Q)}|q_1\rangle \tag{124}$$

Unfortunately P^2 does not commute with $V(Q)$ but we can instead use an approximation. Since the goal is to eventually take the limit as $\delta t \to 0$ then this should be acceptable.

$$e^{-i\delta t(\frac{P^2}{2m}+V(Q))} = e^{-i\delta t \frac{P^2}{2m}} e^{-i\delta t V(Q)} e^{O(\delta t^2)} \tag{125}$$

Where we usually neglect the last term because δt will approach zero. Now, insert the identity

$$\langle q_2|e^{-iH\delta t}|q_1\rangle = \int dp_1 \langle q_2|e^{-i\delta t \frac{P^2}{2m}}|p_1\rangle \langle p_1|e^{-i\delta t V(Q)}|q_1\rangle \tag{126}$$

Using lowercase p and q now... (The eigenvalue equation is used)

$$\langle q_2|e^{-iH\delta t}|q_1\rangle = \int dp_1 e^{-i\delta t \frac{p_1^2}{2m}} e^{-i\delta t V(q_1)} \langle q_2|p_1\rangle \langle p_1|q_1\rangle \tag{127}$$

But we know what the value of of $\langle q_2|p_1\rangle \langle p_1|q_1\rangle$ which is just the eigenstates of p projected into position space and its complex conjugate. Specifically

$$\langle p_1|q_1\rangle = \frac{1}{\sqrt{2\pi}} e^{ip_1 q_1} \tag{128}$$

So we write (127) as the following.

$$\begin{aligned}\langle q_2|e^{-iH\delta t}|q_1\rangle &= \int \frac{dp_1}{2\pi} e^{-i\delta t \frac{p_1^2}{2m}} e^{-i\delta t V(q_1)} e^{ip_1(q_2-q_1)} \\ &= \int \frac{dp_1}{2\pi} e^{-i\delta t H(q_1,p_1)} e^{ip_1(q_2-q_1)}\end{aligned} \tag{129}$$

Where we now have the semi classical Hamiltonian (well classical in the classical limit. In other words the Hamiltonian is a function of scalers) instead. Now here is something that is usually brushed under the rug, it actually turns out that we've been a little messy with the derivation. If we want to consider a theory that is weyl ordered (which we do) using a more general hamiltonian, we need to make a small replacement. It turns out, the Hamiltonian in (129) should be really be a function of $(\bar{q}, p)$ instead of (q, p). So we need to make the replacement $H(q, p) \rightarrow H(\bar{q}, p)$ where $\bar{q} = \frac{q_1+q_2}{2}$. Now repeating this process with all of the inner products, we end up with

$$\langle q'', t''|q', t'\rangle = \int \prod_{k=1}^{N} dq_k \prod_{j=0}^{N} \frac{dp_j}{2\pi} e^{ip_j(q_{j+1}-q_j)} e^{-i\delta t H(\bar{q}_j, p_j)} \tag{130}$$

Where $\bar{q_0} = \frac{1}{2}(q' + q_1)$ and $\bar{q_N} = \frac{1}{2}(q_N + q'')$. At this point, we can define the functional differential as the follwing

$$\mathcal{D}x = C \prod_{j}^{N} dx_j \tag{131}$$

Where C is an arbitrary constant. Now going back to the transition amplitude we can turn the product of exponentials into a sum.

$$\begin{aligned}
\langle q'', t''|q', t'\rangle &= \int \mathcal{D}q\mathcal{D}p \exp{(i \sum_{j=0}^{N} [p_j(q_{j+1} - q_j) - \delta t H(\bar{q}_j, p_j)])} \\
&= \int \mathcal{D}q\mathcal{D}p \exp{(i \sum_{j=0}^{N} [p_j(q_{j+1} - q_j)\frac{\delta t}{\delta t} - \delta t H(\bar{q}_j, p_j)])} \\
&= \int \mathcal{D}q\mathcal{D}p \exp{(i \sum_{j=0}^{N} [p_j \frac{(q_{j+1} - q_j)}{\delta t} - H(\bar{q}_j, p_j)]\delta t)}
\end{aligned} \tag{132}$$

At this point we can formally take the continuum limit letting $\delta t \rightarrow 0$. This produces

$$= \int \mathcal{D}q\mathcal{D}p \exp{(i \int_{t'}^{t''} [p(t)\dot{q}(t) - H(q, p)]dt)} \tag{133}$$

Where in the continuum limit the small perturbation of q disappears, in other words $\bar{q} \rightarrow q(t)$. Alright now we wish to simplfy this integral by evaluating one of then functional integrals $\mathcal{D}p$. To do this, we wiill assume H only has P^2 dependence and re-express the function as a discrete product and replacing $H = \frac{p^2}{2m} + v(q)$

$$\begin{aligned}
&= \int \mathcal{D}q \prod_{j=0}^{N} \frac{dp_j}{2\pi} \exp{(i[p_j \frac{(q_{j+1} - q_j)}{\delta t} - H(q_j, p_j)]\delta t)} \\
&= \int \mathcal{D}q \prod_{j=0}^{N} \frac{dp_j}{2\pi} \exp{(i[p_j \frac{(q_{j+1} - q_j)}{\delta t} - \frac{p_j^2}{2m} - v(q_j)]\delta t)}
\end{aligned} \tag{134}$$

Lets examine the stuff inside the exponential of one individual integral(out of the infinite ones).

$$\begin{aligned}
&\exp{(i[p_j \frac{(q_{j+1} - q_j)}{\delta t} - \frac{p_j^2}{2m} - v(q_j)]\delta t)} \\
&= \exp(i[p_j \delta t \frac{(q_{j+1} - q_j)}{\delta t} - \frac{p_j^2}{2m}\delta t]) \exp(-i\delta t v(q_j))
\end{aligned} \tag{135}$$

We can factor the part 'p' inside the exponential by using complete the square. Specifically

$$\begin{aligned}
&i[p_j\delta t\frac{(q_{j+1}-q_j)}{\delta t}-\frac{p_j^2}{2m}\delta t]\\
&=-\frac{i\delta t}{2m}(p_j^2-\frac{p_j(q_{j+1}-q_j)2m}{\delta t})\\
&=-\frac{i\delta t}{2m}(p_j-\frac{(q_{j+1}-q_j)m}{\delta t})^2+\frac{i(q_{j+1}-q_j)^2m}{2\delta t}
\end{aligned}\tag{136}$$

So we can split up the exponential into several factors and put it back into the path integrar

$$=\int \mathcal{D}q\prod_{j=0}^{N}\frac{dp_j}{2\pi}\exp\,(-\frac{i\delta t}{2m}(p_j-\frac{(q_{j+1}-q_j)m}{\delta t})^2)\exp(\frac{i(q_{j+1}-q_j)^2m}{2\delta t})\exp(-i\delta tv(q_j))\tag{137}$$

We now have a gaussian integral that can be evaluated for each 'j' step. Actually it is a fresnel integral that follows the identity.

$$\int_{-\infty}^{\infty}e^{-iax^2}=\sqrt{\frac{\pi}{a}}e^{-i\frac{\pi}{4}}\tag{138}$$

We can use this identity to evaluate the path integral exactly. Specifically $a=\frac{\delta t}{2m}$, After the substitution we have...

$$\begin{aligned}
&=\int \mathcal{D}q\prod_{j=0}^{N}\sqrt{\frac{m}{2\pi\delta t}}e^{-i\frac{\pi}{4}}\exp(\frac{i(q_{j+1}-q_j)^2m}{2\delta t})\exp(-i\delta tv(q_j))\\
&=\int \mathcal{D}q\left(\sqrt{\frac{m}{2\pi\delta t}}e^{-i\frac{\pi}{4}}\right)^{N+1}\prod_{j=0}^{N}\exp(\frac{i(q_{j+1}-q_j)^2m}{2\delta t})\exp(-i\delta tv(q_j))\\
&=\int \mathcal{D}q\prod_{j=0}^{N}\exp(\frac{i(q_{j+1}-q_j)^2m}{2\delta t})\exp(-i\delta tv(q_j))
\end{aligned}$$

Now, lets focus on just the product for now (139)

$$\begin{aligned}
&\to\prod_{j=0}^{N}\exp(\frac{i(q_{j+1}-q_j)^2m}{2\delta t})\exp(-i\delta tv(q_j))\\
&=\exp\left(\sum_{j=0}^{N}\frac{i(q_{j+1}-q_j)^2m}{2\delta t}-i\delta tv(q_j)\right)\\
&=\exp\left(i\sum_{j=0}^{N}\left(\frac{(q_{j+1}-q_j)^2m}{2\delta t^2}-v(q_j)\right)\delta t\right)
\end{aligned}$$

Where in the 3rd line, by by the definition of $\mathcal{D}q$, we can absorb any constant into it. In the 5th line we turned the product into a sum which we can do for exponentials.

Now taking the continuum limit, in other words letting $\delta t\to 0$

$$\begin{aligned}
&=\lim_{\delta t\to 0}\exp\left(i\sum_{j=0}^{N}\left(\frac{(q_{j+1}-q_j)^2m}{2\delta t^2}-v(q_j)\right)\delta t\right)\\
&=\exp\left(i\int_{t'}^{t''}\left(\frac{1}{2}m\dot{q}^2-v(q)\right)dt\right)\\
&=\exp\left(i\int_{t'}^{t''}L(q,\dot{q})dt\right)
\end{aligned}\tag{140}$$

Now putting it all together we arrive at the final result.

$$\boxed{\int_{-\infty}^{\infty} \mathcal{D}q \exp\left(i \int_{t'}^{t''} L(q,\dot{q}) dt \right)} \tag{141}$$

This is the Feynman path integral. It is a functional integral, where we can think of $\mathcal{D}q$ as integrating over all possible positions $dq_1 dq_2$....

Now in standard units this is given by

$$\int_{-\infty}^{\infty} \mathcal{D}q \exp\left(\frac{i}{\hbar} \int_{t'}^{t''} L(q,\dot{q}) dt \right) \tag{142}$$

There is another way to justify the final derivation we did to absorb the two functional integrals into one. Because most QFT books take this approach I will also include it here. Going back that specific step

$$= \int \mathcal{D}q\mathcal{D}p \exp\left(i \int_{t'}^{t''} [p(t)\dot{q}(t) - H(q,p)]dt\right) \tag{143}$$

We can instead consider an approximation where the $[p]$ functional integral is dominated by the stationary phase. We achieve this by demanding that

$$\begin{aligned} &\frac{\partial}{\partial p}(p(t)\dot{q}(t) - H(q,p)) = 0 \\ &\dot{q} = \frac{\partial H}{\partial p} \end{aligned} \tag{144}$$

This is called the stationary phase approximation in mathematics. Intuitively we can think of it assuming that the path integral will be dominated by paths with stationary phases. Now we regonize the second part of equation (144) to be the Legendre transformation for switching into the lagrangian formalism. Therefore,

$$\int_{-\infty}^{\infty} \mathcal{D}q \exp\left(i \int_{t'}^{t''} L(q,\dot{q}) dt \right) \tag{145}$$

We arrive at the same result now obtained by the stationary phase approximation. This result is exact if momentum is of square (p^2) power.

Free Particle Examle

We shall now solve our first system in this new formalism. It is the simpliest system we can conceive here. Namely we will consider the case of a free particle such that

$$L = \frac{m\dot{q}^2}{2} \tag{146}$$

Now the feynman path integral becomes.

$$\int_{-\infty}^{\infty} \mathcal{D}q \exp\left(\int \frac{m\dot{q}^2}{2} dt \right) \tag{147}$$

Converting this to a discrete sum we get

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \mathcal{D}q \prod \exp\left(i\frac{m(q_{n+1}-q_n)^2}{2\delta t^2}\delta t\right) \\
&= \int_{-\infty}^{\infty} \mathcal{D}q \exp\left(i\sum_{j=0}^{N}\left(\frac{m(q_{j+1}-q_j)^2}{2\delta t^2}\right)\delta t\right) \\
&= \int_{-\infty}^{\infty} \prod_{n=1}^{N}(q_n) \exp\left(i\sum_{j=0}^{N}\left(\frac{m(q_{j+1}-q_j)^2}{2\delta t^2}\right)\delta t\right)
\end{aligned}
\tag{148}
$$

Now we get an infinite number of integrals. We wish to look at these individually. Lets start with the dq_1 ignoring all the terms that would be held constant in the integration process. We then wish to compute...

$$
\int_{-\infty}^{\infty} dq_1 \exp\left(\frac{im}{2\delta t}\left((q_1-q')^2+(q_2-q_1)^2\right)\right) \tag{149}
$$

Now using expanding out the terms inside...

$$
\begin{aligned}
&\left(\frac{im}{2\delta t}\left((q_1-q')^2+(q_2-q_1)^2\right)\right) \\
&\left(\frac{im}{2\delta t}\left(q'^2+q_2^2+2q_1^2-2q_1(q_2+q')\right)\right)
\end{aligned}
\tag{150}
$$

Now putting this back into (149) we get

$$
\exp\left(\frac{im}{2\delta t}\left(q'^2+q_2^2\right)\right)\int_{-\infty}^{\infty} dq_1 \exp\left(\frac{im}{\delta t}\left(q_1^2-q_1(q_2+q')\right)\right) \tag{151}
$$

Now using the identity

$$
\int_{-\infty}^{\infty} dx e^{ia(x^2+bx)} = \sqrt{\frac{\pi i}{a}}e^{-ia\frac{b^2}{4}} \tag{152}
$$

We end up with the following expression if we replace $a = m/\delta t$ and $b = -(q_2+q')$. This results in (151) becoming..

$$
\begin{aligned}
&\exp\left(\frac{im}{2\delta t}\left(q'^2+q_2^2\right)\right)\sqrt{\frac{i\pi\delta t}{m}}\exp\left(-i\frac{m}{\delta t}\frac{(q_2+q')^2}{4}\right) \\
&= \sqrt{\frac{i\pi\delta t}{m}}\exp\left(\frac{im}{4\delta t}\left(q_2-q'\right)^2\right)
\end{aligned}
\tag{153}
$$

we can then plug this result back into (148) and calculate the dq_2 integral

$$
\begin{aligned}
&\sqrt{\frac{i\pi\delta t}{m}}\int_{-\infty}^{\infty}\prod_{n=2}^{N}(q_n)\exp\left(\frac{im}{4\delta t}\left(q_2-q'\right)^2\right)\exp\left(i\sum_{j=2}^{N}\left(\frac{m(q_{j+1}-q_j)^2}{2\delta t^2}\right)\delta t\right) \\
&= \sqrt{\frac{i\pi\delta t}{m}}\int_{-\infty}^{\infty}\prod_{n=2}^{N}(q_n)\exp\left(\frac{im}{4\delta t}\left(q_2-q'\right)^2\right)\exp\left(\frac{im}{2\delta t}\left(q_3-q_2\right)^2\right)\exp\left(i\sum_{j=2}^{N}\left(\frac{m(q_{j+1}-q_j)^2}{2\delta t^2}\right)\delta t\right)
\end{aligned}
\tag{154}
$$

2.2 Harmonic oscillator example

2.3 Fermionic Fields

2.4 Interacting theory

3 Conformal Field Theory

3.1 Complex Analysis

We begin with a short overview of complex analysis. Indeed several machinary is needed from here to understand conformal field theory. This section is meant to provide a brief overview with an emphasis on computation instead of justifications. The defining property of complex numbers is a number that satisfies the expression

$$i^2 = -1 \tag{155}$$

where this is defined in this matter to prevent the ambiguous definition of $\sqrt{-1} = i$. Now there are several ways to represent complex numbers but the most useful are the cartesian and exponential form

$$\begin{aligned} z &= a + bi \\ z &= re^{i\theta} \end{aligned} \tag{156}$$

Where a, b, r and θ are real numbers. Every single complex number can be represented in either of these forms. Now most stuff from standard analysis carries over to complex analysis but there are a few things we can define specific to the complex plane. Consider a point at $z = 3 + 6i$. What if we wanted to define an operation that gives us the euclidean distance from the origin (treating the complex plane as a pseudo 2D space). This is accomplished by

$$|z| = \sqrt{z\bar{z}} = \sqrt{(3+6i)(3-6i)} = \sqrt{45} \tag{157}$$

This is often called the modulus of z and, intuitively, we can think of it as giving us the distance from the origin to a specific point on the complex plane.

We now move onto discussion of analytic functions. In a sense, a function is said to be analytic if it is equal to its own Taylor series. That is,

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} a_k (z - z_0)^k \\ a_k &= \frac{f^{(k)}(z_0)}{k!} \end{aligned} \tag{158}$$

Where z_0 is the center of the power series and a The implications of this is that the function is infinitely differentiable. This is also known as c^∞ in terms of smoothness.

Fundamental Theorem of Algebra

Now it turns out that the inclusion of complex numbers into our set allows us to factor out every polynomial. That is, a polynomial of order m will have m complex factors with multiplicies included. In practice this means that given a polynomial such as $x^2 + 1$. We can find two roots

$$(x - i)(x + i) = x^2 + 1 \tag{159}$$

namely $\pm i$ for this case. If we have a polynomial of order m, we can instead find m roots. Another example is $x^4 - 4x^3 + 13x^2 - 36x + 36$. In this case, m = 4 we can factor out like following

$$\begin{aligned} &= x^4 - 4x^3 + 13x^2 - 36x + 36 \\ &= (x - 2)^2(x^2 + 9) \\ &= (x - 2)^2(x + 3i)(x - 3i) \end{aligned} \tag{160}$$

So we end up with a repeated root of 2 along with $\pm 3i$ for a total of 4 roots.

Cauchy-Riemann equations

One of the most useful theorems in complex analysis is the Cauchy-Riemann theorem. Indeed, it allows check if a function is analytic. The theorem goes as follows. Consider a general complex function

$$z = u(x, y) + iv(x, y) \tag{161}$$

Where 'u' and 'v' are real functions and we have assumed the cartesian basis. Now it turns out that if Z is continuously differentiable then z is analytic if it satsifes the following PDEs

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \text{ and} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned} \tag{162}$$

And that's it. Those two PDE's essentially describe every single analytic function provided it is continuous and differentiable. Lets do an example. Consider the complex function

$$\begin{aligned} z &= re^{i\theta} \\ z &= r\sin(\theta) + ir\cos(\theta) \end{aligned} \tag{163}$$

Where it's clear that in polar form u(r, θ) v(r, θ). Now in order to check the cauchy rieman equation we must switch our basis. We do this by expanding into the differential form

$$\begin{aligned} du &= \frac{\partial u}{\partial r}dr + \frac{\partial u}{\partial \theta}d\theta \\ dv &= \frac{\partial v}{\partial r}dr + \frac{\partial v}{\partial \theta}d\theta \end{aligned} \tag{164}$$

Now using the polar coordinates transformation

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2} \\
\theta &= \tan^{-1}(\frac{y}{x})
\end{aligned} \tag{165}
$$

we can compute all the derivatives

$$
\begin{aligned}
&\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} = \cos^2(\theta) + \sin^2(\theta) = 1 \\
&\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial y} = \cos^2(\theta) + \sin^2(\theta) = 1 \\
&\text{and} \\
&\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} = \cos(\theta)\sin(\theta) - \cos(\theta)\sin(\theta) = 0 \\
&\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x} = \cos(\theta)\sin(\theta) - \cos(\theta)\sin(\theta) = 0
\end{aligned} \tag{166}
$$

Where it's clear the theorem is satisfied since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ which implies that the function $z = re^{i\theta}$ is analytic.

Contour integration

We want to define a way to integrate in the complex plane. Indeed we quickly run into an issue when defining integration. In the real number line, the bounds are not ambigious. Namely, if we integrate from say 0 to infinity, we don't need to worry about the particular path that we choose. Indeed, there is only one available path. However, once we introduce the complex plane, we have an infinite amount of paths that can be taken to reach the bounds.Therefore, instead of considering integration we contsruct what is known as a contour integral. The idea is analogous to that of a line integral in mulitvariable calculus. Namely, we define a parametrized path $\gamma(t)$ that we choose in order to specify a path.

$$
\int_{\gamma} f(z)dz \tag{167}
$$

And that it! Thats how we define integration in the complex plane. Well when we actually compute this, we typically express it in a parametrized form that makes it clear that we are specifying a particular path. Using the definition of a differential

$$
\begin{aligned}
z &= \gamma(t) \\
dz &= \frac{d\gamma}{dt}dt
\end{aligned} \tag{168}
$$

So..

$$
\int_0^{t_0} f(\gamma(t))\frac{d\gamma}{dt}dt \tag{169}
$$

Finally if we have a Contour integral that forms a simple closed loop then we call this a closed contour integral and write it like follows. The contour is always implies to take a counter clockwise path unless specified otherwise.

$$\oint_\gamma f(z)dz \tag{170}$$

Now, there is a very important theorem when it comes to closed contour integrals that roughly the next section will also cover. For now lets consider the following construction.

$$z = u(x,y) + iv(x,y) \tag{171}$$

Now consider the cartesian vector

$$w = \begin{pmatrix} u(x,y) \\ -v(x,y) \end{pmatrix} \tag{172}$$

It turns out that

$$\begin{aligned} \nabla \cdot w &= \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \\ (\nabla \times w)^z &= -\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \end{aligned} \tag{173}$$

But these are just the Cauchy-Riemann equations. Which implies that this vector fields will have zero divergence and zero curl as long as the function f is analytic. This result is extremely important because we can split up closed a contour integral into the two components.

$$\oint_\gamma f(z)dz = f^{\text{Work}} + if^{\text{Flux}} \tag{174}$$

Now the divergence theorem and Stokes' theorem say

$$\begin{aligned} \oint_\gamma F \cdot d\ell &= \iint_R (\nabla \times F) dA \\ \oint_\gamma F \cdot \hat{n} dl &= \iint_R (\nabla \cdot F) dA \end{aligned} \tag{175}$$

Where the first one is the f^{Work} and the second integral is f^{Flux} however we just established that these two quantities are zero if the function is analytic since the right hand side is zero. Which implies that for any analytic function the closed contour integral will be zero provided the right hand side of the expression remains zero everywhere. In other words, provided the area enclosed by the contour is analytic, then

$$\oint_\gamma f(z)dz = 0 \tag{176}$$

3.2 Residue Theorem

We now want to explore what happens when the function is mostly analytic but has what is known as a pole. To understand, consider a function $f = \frac{c_{-1}}{z}$. Where c_{-1} is a constant. We know that the function has a singularity at the point $z = 0$ which implies it isn't analytic there, Lets try performing a contour integral around it. Choosing the parametrization $z(\theta) = re^{i\theta}$ then we can write

$$\oint \frac{c_{-1}}{z}dz = \oint \frac{c_{-1}}{re^{i\theta}}dz$$
$$dz = ire^{i\theta}d\theta \tag{177}$$
$$\int_0^{2\pi} \frac{c_{-1}}{re^{i\theta}}ire^{i\theta}d\theta = 2\pi i c_{-1}$$

Interestingly, we end up with a term that is proportional to 'i'. This implies that contour integral has flux by no work. Also, the result is independent of the radius of the contour. It only depends on the constant coefficient c_0. Indeed it turns out the singularity causes there to be a positive flux. It turns out that whenever we have a 'pole' which we can think of as a singularities that are analytic in the neighborhood around it except at the location of the pole. This is known as the function being Meromorphic. Which again, just means the function is analytic an infinitesimal distance around a point except at the point

Now if you think the math here resembles that of gauss's law then you would be correct. Indeed we can think of the poles as charges with a charge of c_1 and naturally the flux will give us the sum of the total amount of charge times some constant which is $2\pi i$ in this case.

It turns out that any closed contour integral of the form

$$\oint_\gamma \frac{1}{(z-a)^n}dz = 0 \tag{178}$$
For n $\neq 1$

Where n can be positive or negative. The implications of this result are massive but first we need to understand one thing.

For any meromorphic function, a laruent series lets us expand any reasonable f(z) function as the following.

$$f(z) = \sum_{n=\infty}^{\infty} c_n(z-a)^n$$

Where 'a' is the location of the pole. c_n are coefficients Now...

$$\oint f(z)dz = \oint \sum_{n=\infty}^{\infty} c_n(z-a)^n dz \tag{179}$$
$$= \oint dz[... + \frac{c_{-2}}{(z-a)^2} + \frac{c_{-1}}{(z-a)^1} + \frac{c_0}{(z-a)^0} + c_1(z-a)^1 + c_2(z-a)^2 + ...]$$

But we just established that only the c_{-1} will be non zero therefore

$$\oint f(z)dz = 2\pi i c_{-1}$$

We typically call c_{-1} the residue of a specific pole. This is analogues to the charge is gauss's law. The coefficients in the Laurent series are defined to be

$$c_n = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{(z-a)^{n+1}}dz \tag{180}$$

Where 'a' is the location of a pole. Now we finally reach the residue the theorem. Which states that any closed contour integral can be expressed as a sum of the residues times $2\pi i$

$$\boxed{\oint_{\gamma} f(z)dz = 2\pi i \sum \text{Res}(f, a_k)} \tag{181}$$

In other words, the closed contour integral is proportional to the sum of all the residues enclosed. This is by far the most important theorem in complex analysis. If this sounds like Gauss's law, it should. Now we won't provide a formal proof for the purpose of not derailing the subject.

The most straight forward way to compute the residues is by applying the formula

$$\text{Res(f,a)} = \frac{1}{p-1} \lim_{z \to a} \frac{d^{p-1}}{dz^{p-1}} ((z-a)^p f(z)) \tag{182}$$

Where p is the order of the pole. This quantity is found by demanding that the quantity

$$(z-a)^n f(z) \tag{183}$$

is analytic instead of meromorphic for the smallest 'n' possible. And that same value of 'n' turns out to be the order of the pole.

Other modes of finding the c_{-1} coefficient including doing a mode expansion. For example consider $e^{\frac{1}{z}}$. We can find the residue by doing a taylor expansion of e^z. Consider the taylor expansion centered at zero

$$e^z = 1 + z + \frac{z^2}{2!} + ... \tag{184}$$

Now replace $z \to \frac{1}{z}$

$$e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{z^2 2!} + ... \tag{185}$$

Where now it's clear the residue is encoded in the second term and is simply 1.

We finish this section by including another power theorem in complex analysis. Given an analytic function we can get the value at a point 'a' using the formula

$$f(a) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z-a} dz \tag{186}$$

This shows us just how powerful analytic functions are. Indeed we can get the value of any point inside a contour by just knowing the values around it. Differentiating the above equation n times provides this convinent formula

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z-a)^{n+1}} dz \tag{187}$$

This completes our discussion of most of the machinery required from complex analysis. There is just one more thing we need to take care of before we resume our discussion on Conformal Field Theory.

3.3 Analytic Continuation

We now wish to explore our first conformal transformation. Indeed, an analytic function is said to be conformal in all places where the derivative doesn't vanish. But what exactly do we mean by a conformal transformation? The definition is quite simple actually.

A conformal transformation is a transformation that preservers angles after the transformation. A few examples you may be familiar with are rotations and translations. Naturally, angles are preserved under these transformations. Analytic functions are similar but they persevere angles $\mathcal{locally}$ instead of globally at places where it's first derivative isn't zero. So when we say conformal transformation, we usually mean angle preserving transformation.

The idea of analytic continuations stems from expending a function beyond it's original domain. The extension is chosen suchs that the transofrmation reamins locally conformal including in regions that extend beyond the domain of the function. Well the rigorous definition is that the functions is extended, such that, the resulting function is analytic but intuitively we can think of analytic continuation as an extensions that preserves angles in regions where it's first derivative isn't zero.

The best way to understand is from an example. Consider the function

$$f(z) = \sum_{k=0}^{\infty} (-1)^k (z-1)^k \tag{188}$$

The series converges in the open set defined by $U = \{|z-1| < 1\}$ which just means the set of points around '1' with a modulus of less than one. In other words a circle in the complex plane of radius one. Now lets try to create a power series in the complex plane center somewhere close to the edge of the boundary at a point 'a'.

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} a_k (z-a)^k \\ a_k &= \frac{f^{(k)}(a)}{k!} \\ a_k &= \frac{1}{2\pi i} \oint_\gamma \frac{f(t)}{(t-a)^{n+1}} dt \end{aligned} \tag{189}$$

Expanding f(t) using the definition of f(z)

$$a_k = \frac{1}{2\pi i} \oint_\gamma \frac{\sum_{n=0}^{\infty} (-1)^n (t-1)^n}{(t-a)^{k+1}} dt \tag{190}$$

Now we choose a parameterization such that the contour stays inside the analytic portion of the function.

$$a_k = \frac{1}{2\pi i} \int_0^{2\pi} \frac{\sum_{n=0}^{\infty} (-1)^n (a + re^{i\theta} - 1)^n}{(a + re^{i\theta} - a)^{k+1}} ire^{i\theta} d\theta \tag{191}$$

Where we make sure to choose a parametrization $a+re^{i\theta}$ such that the contour stays inside the analytically defined portion. Now using the binomial expansion

$$\begin{aligned} a_k &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{\sum_{n=0}^{\infty}(-1)^n(a+re^{i\theta}-1)^n}{(re^{i\theta})^{k+1}} ire^{i\theta}d\theta \\ &= \frac{1}{2\pi}\sum_{n=0}^{\infty}(-1)^n\int_0^{2\pi} \frac{\sum_{m=0}^{n}\begin{pmatrix} n \\ m \end{pmatrix}(a-1)^{n-m}(re^{i\theta})^m}{(re^{i\theta})^k}d\theta \\ &= \frac{1}{2\pi}\sum_{n=0}^{\infty}(-1)^n\int_0^{2\pi}\sum_{m=0}^{n}\begin{pmatrix} n \\ m \end{pmatrix}(a-1)^{n-m}r^{m-k}e^{i(m-k)\theta}d\theta \end{aligned} \tag{192}$$

Now the integral is kronecker delta which implies its zero unless $m = k$. We an contract the second sum to force this.

$$\begin{aligned} a_k &= \frac{1}{2\pi}\int_0^{2\pi}\sum_{m=k}^{n}(-1)^n\begin{pmatrix} n \\ k \end{pmatrix}(a-1)^{n-k}d\theta \\ &= (-1)^k a^{-k-1} \end{aligned} \tag{193}$$

If we use these coefficients to the original for the taylor expansion we can get.

$$\frac{1}{a}\sum_{k=0}^{\infty}(1-\frac{z}{a})^k \tag{194}$$

Using the geomertric series identity we can see.

$$f(z) \rightarrow \frac{1}{(z-a)+a} \tag{195}$$

Which has a radius of convergence of |a| which we can pick such that |a| > 1 but where 'a' is still in U. This allows the series to converge for areas beyond it's original domain outside of U which only had a radius of |1|. Indeed the analytic continuation of the original function is $1/z$. This extends the domain of the original function in all except the point at $z = 0$ where the function has a pole. There are many functions that are defined via analytic continuation. For instance the gamma function

$$\Gamma(z) \tag{196}$$

is defined for complex values via analytic continuation.

Most useful for string theory, the infinite series

$$\sum_{n=1}^{\infty}\frac{1}{n^z} \tag{197}$$

converges for values where $\text{Re}(z) > 1$. We then define the analytically continued function $\zeta(z)$ to be defined for all values except for a pole at z = 1.

Perhaps most useful to string theory, we can let z = -1 which allows us to formally make the replacement

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + ... \rightarrow \zeta(-1) = -\frac{1}{12} \tag{198}$$

This concludes the discussion on complex analysis. We will continue in the next section exploring conformal field theory.

3.4 Complex Coordinates in CFT

We will now define a new set of "Complex coordinates" to simply analysis of conformal field theory. We begin by defining the following variables

$$\begin{aligned} z &= \sigma^1 + i\sigma^2 \\ \bar{z} &= \sigma^1 - i\sigma^2 \\ \partial_z &= \frac{1}{2}(\partial_1 - i\partial_2) \\ \partial_{\bar{z}} &= \frac{1}{2}(\partial_1 + i\partial_2) \end{aligned} \tag{199}$$

These satisfy the following useful properties.

$$\partial_z z = 1 \ \partial_z \bar{z} = 0 \ \partial_{\bar{z}} z = 0 \ \partial_{\bar{z}} \bar{z} = 1 \tag{200}$$