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String Theory

Contents

1 The Polyakov Action

1.1 The Relativistic point particle

We begin our discussion with the model of a point particle that is subject to relativistic effects. Such a particle traces out a line in spacetime known as the worldline. We know that the length of this world line is given by the equation

$$ds^2 = -dX^\mu g_{\mu\nu} dX^\nu \tag{1}$$

Where μ and $\nu = 0, 1, 2, ..., D$. D is the dimension of spacetime $g_{\mu\nu}$ is the spacetime metric tensor and 'ds' is the proper length. The simplest lorentz invarient action we can write for this particle would be proportional to the proper length

$$S = -mc \int ds \tag{2}$$

Where 'mc' is here for dimensional purposes. We can then write

$$\begin{aligned} S &= -mc \int \sqrt{-dX^\mu g_{\mu\nu} dX^\nu} \\ &= -mc \int \sqrt{-\frac{dX^\mu}{dt} dt g_{\mu\nu} \frac{dX^\nu}{dt} dt} \end{aligned} \tag{3}$$

If we assume that spacetime is flat then we write

$$\begin{aligned} S &= -mc \int dt \sqrt{-\frac{dX^\mu}{dt} \eta_{\mu\nu} \frac{dX^\nu}{dt}} \\ &= -mc \int dt \sqrt{-\frac{dX^\mu}{dt} \frac{dX_\mu}{dt}} \end{aligned} \tag{4}$$

Now if we contract the 0th term we can simply the action to the following quantity

$$\begin{aligned} &= -mc^2 \int dt \sqrt{1 - \frac{1}{c^2} \frac{dX^i}{dt} \frac{dX_i}{dt}} \\ &= -mc^2 \int dt \sqrt{1 - \frac{v^2}{c^2}} \end{aligned} \tag{5}$$

Where v^2 is the D-dimensional velocity squared. We can easily show that this reduces to the classical lagrangian in the non-relativitic limit.

$$\begin{aligned} L &= -mc^2 \int dt \sqrt{1 - \frac{v^2}{c^2}} \\ &\approx -mc^2 (1 - \frac{1}{2} \frac{v^2}{c^2}) = \frac{1}{2} m v^2 + \text{constant} \end{aligned} \tag{6}$$

So it reduces to the classical result in the non relativitic limit.

1.2 World Sheets and Nambu–Goto action

Now let us consider modeling a relativistic string. The simplest action we can construct would be proportional to the proper area traced out in spacetime by the string. This is called the Nambu-Goto action. We will begin by considering the proper area in General Relativity.

In General Relativity the proper area is given by the corresponding formula.

$$A = \int d\tau d\sigma \sqrt{-\det(h_{ab})} \tag{7}$$

Where h_{ab} is the induced metric of the surface. Here we have chosen Tau and sigma as parameterizations. Tau does not necessarily represent the proper time. Now there are a few caveats we need to take care of later. First lets motivate this result by consider breaking up the 'world sheet' into rectangles. Seeing how we are working with vectors, it makes sense to break it up into parallelograms by considering the identity in linear algebra.

$$dA = |dv_1 \times dv_2| = |dv_1||dv_2||\sin(\theta)| = dv_1||dv_2||\sqrt{1-\cos^2(\theta)|} \tag{8}$$

$$dA = \sqrt{|dv_1|^2|dv_2|^2 - |dv_1|^2|dv_2|^2\cos^2(\theta)|} \tag{9}$$

Now we know that dv1 and dv2 should be in different directions. We shall write this in terms of the tensor function $X^\mu(\tau,\sigma)$ that represents the parametrized world-sheet. We will assign the value dv1 to $\frac{dX^\mu}{d\tau}d\tau$ and dv1 to $\frac{dX^\mu}{d\sigma}d\sigma$ where sigma and tau are used to parametrize the world sheet. Now this prodcues the following result

$$\begin{aligned} dA &= \sqrt{(d\tau^2\frac{dX^\mu}{d\tau}g_{\mu\nu}\frac{dX^\nu}{d\tau})(d\sigma^2\frac{dX^\alpha}{d\sigma}g_{\alpha\beta}\frac{dX^\beta}{d\sigma}) - d\tau^2 d\sigma^2(\frac{dX^\mu}{d\tau}g_{\mu\nu}\frac{dX^\nu}{d\sigma})^2} \\ &= d\tau d\sigma\sqrt{(\frac{dX^\mu}{d\tau}g_{\mu\nu}\frac{dX^\nu}{d\tau})(\frac{dX^\alpha}{d\sigma}g_{\alpha\beta}\frac{dX^\beta}{d\sigma}) - (\frac{dX^\mu}{d\tau}g_{\mu\nu}\frac{dX^\nu}{d\sigma})^2} \end{aligned} \tag{10}$$

Now, if we believe equation (7) then we should be able to express equation (10) as the determinant of some metric. By inspection we can write an induced metric 'h' as the following.

$$\begin{aligned} h_{\alpha\beta} &= \begin{pmatrix} \frac{dX^\mu}{d\tau}g_{\mu\nu}\frac{dX^\nu}{d\tau} & \frac{dX^\mu}{d\tau}g_{\mu\nu}\frac{dX^\nu}{d\sigma} \\ \frac{dX^\mu}{d\tau}g_{\mu\nu}\frac{dX^\nu}{d\sigma} & \frac{dX^\alpha}{d\sigma}g_{\alpha\beta}\frac{dX^\beta}{d\sigma} \end{pmatrix} \\ &= \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu} \\ &= \partial_\alpha X^\mu \partial_\beta X_\mu \end{aligned} \tag{11}$$

And all together the Nambu-Goto action becomes.

$$\begin{aligned} S &= -T\int d\tau d\sigma\sqrt{-\det(h_{\alpha\beta})} \\ &\boxed{S = -T\int d\tau d\sigma\sqrt{-\det(\partial_\alpha X^\mu \partial_\beta X_\mu)}} \end{aligned} \tag{12}$$

Where the scaler '-T' is introduced for dimensional purposes. And the negative inside the square root comes as a consequence of the Pseudo-Riemannian space which ensures the determinant will be negative.

Equation (11) is the known as the induced metric. It shouldn't come as a surprise that we are now working with two metrics, since our strings are manifolds themselves. It also shouldn't be a surprise that the metric can be expressed as a 2x2 matrix since our world-sheet traces a two dimensional surface. The second equal sign in equation (11) can be easily checked to be true.

Now there is actually a problem with this action. The problems stems from renormalization, namely, the square root makes renomrlaization difficult. Correcting this problem requires us to rewrite the Nambu-Goto action using what is known as the auxillary world sheet metric. It is a metric that classically reduces the action to the Nambu-Goto action. This new action in terms of the auxiliary world sheet metric is known as the Polyakov action which is given by

$$\boxed{S = -\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma(-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu} \tag{13}$$

Where the $\gamma^{\alpha\beta}$ represents the auxiliary world sheet metric with both indices raised and the $\gamma = \det(\gamma_{\alpha\beta})$. This corrects the issue and makes the theory renormalizable. The scaler at the begining is there for dimensional purposes.

We now wish to show that this indeed reduces to equation (12) at the classical level. To show this we will find variation of the action with respect to the inverse world-sheet metric $\gamma^{\alpha\beta}$

$$\frac{\delta S}{\delta\gamma^{\alpha\beta}} = 0 \tag{14}$$

To accomplish this, we begin by finding the variation of equation (13)

$$\begin{aligned}\delta S &= -\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma\delta((-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu) \\ &= -\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma\delta(-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu - \frac{1}{4\pi\alpha'}\int_M d\tau d\sigma\ (-\gamma)^{\frac{1}{2}}\delta(\gamma^{\alpha\beta})\partial_\alpha X^\mu\partial_\beta X_\mu\end{aligned} \tag{15}$$

Using the following identity

$$\delta\gamma = -\gamma\gamma_{\alpha\beta}\delta\gamma^{\alpha\beta} \tag{16}$$

Which in our case implies that following relationship

$$\delta\sqrt{-\gamma} = -\frac{1}{2}\sqrt{-\gamma}\gamma_{\alpha\beta}\delta\gamma^{\alpha\beta} \tag{17}$$

Going back to equation (15) its easy to see that

$$\begin{aligned}\delta S &= \frac{1}{4\pi\alpha'}\int_M d\tau d\sigma\frac{1}{2}\sqrt{-\gamma}\gamma_{\alpha\beta}\delta\gamma^{\alpha\beta}\gamma^{\kappa\eta}\partial_\kappa X^\mu\partial_\eta X_\mu - \frac{1}{4\pi\alpha'}\int_M d\tau d\sigma\ (-\gamma)^{\frac{1}{2}}\delta(\gamma^{\alpha\beta})\partial_\alpha X^\mu\partial_\beta X_\mu \\ &= \frac{1}{4\pi\alpha'}\int_M d\tau d\sigma\delta\gamma^{\alpha\beta}\frac{1}{2}\sqrt{-\gamma}\gamma_{\alpha\beta}\gamma^{\kappa\eta}\partial_\kappa X^\mu\partial_\eta X_\mu - \frac{1}{4\pi\alpha'}\int_M d\tau d\sigma\delta\gamma^{\alpha\beta}(-\gamma)^{\frac{1}{2}}\partial_\alpha X^\mu\partial_\beta X_\mu \\ &= 0\end{aligned} \tag{18}$$

Dividing equation (18) by $\delta\gamma^{\alpha\beta}$ and canceling out the constants gives the equation of motion for the metric. In other words, it's easy to see that $\frac{\delta S}{\delta\gamma^{\alpha\beta}} = 0$ implies the following relationship.

$$\frac{1}{2}\gamma_{\alpha\beta}\gamma^{\kappa\eta}\partial_\kappa X^\mu \partial_\eta X_\mu = \partial_\alpha X^\mu \partial_\beta X_\mu \tag{19}$$

From here we can take the negative square root of the determinant from both sides wrt to the alpha and beta tensor,

$$\frac{1}{2}(-\gamma)^{\frac{1}{2}}\gamma^{\kappa\eta}\partial_\kappa X^\mu \partial_\eta X_\mu = \sqrt{-\det(\partial_\alpha X^\mu \partial_\beta X_\mu)} \tag{20}$$

But the right side is just the integrand of the Nambu-Goto action and the left side is the integrand of the Polyakov action. Therefore the equation of motion for the world sheet metric $\gamma_{\alpha\beta}$ implies the Nambu-Goto action. QED.

Now notice that we never explicitly defined a value for the world-sheet metric $\gamma_{\alpha\beta}$. This is to preserve the symmetries in the Polyakov action. Namely, these symmetries allows us to perform gauge fixing with the only requirement being that equation (19) is satisfied. We will now explore these symmetries in the next section to see which quantities we are allowed to gauge fix.

1.3 Symmetries in the Polyakov action

We would now like to talk about the symmetries in the Polyakov action

$$S = -\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma (-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu \tag{21}$$

...for which there are a lot. However, for now, lets restrict our analysis to that of Minkowski space. The Polyavok action is invarient under the following transformations

Poincare Transformation

$$X'^\mu = \Lambda^\mu_\nu X^\nu + a^\mu \tag{22}$$

$$\delta\gamma_{\alpha\beta} = 0 \tag{23}$$

Where Λ is a Lorentz transformation and a a^μ is a translation. Poincare invariance is a global symmetry in the action which implies it cannot be used for gauge fixing.

Diffeomorphism invariance

$$X'^\mu(\tau', \sigma') = X^\mu(\tau, \sigma) \tag{24}$$

$$\frac{\partial\sigma'^c}{\partial\sigma^a}\frac{\partial\sigma'^d}{\partial\sigma^b}\gamma'_{cd}(\tau', \sigma') = \gamma_{ab}(\tau, \sigma) \tag{25}$$

for some new choice of coordinates $\sigma'^a(\tau, \sigma)$. Equation (24) is also known as reparameterization invariance. This is an very important property of string theory so be sure to understand it. While equation (25) is nothing more than the tensor transformation law. Diffeomorphism invarice is a local symmetry which implies it allows for gauge fixing.

Weyl invariance

$$X'^{\mu}(\tau,\sigma) = X^{\mu}(\tau,\sigma) \tag{26}$$

$$\gamma'_{ab}(\tau,\sigma) = e^{2\omega(\tau,\sigma)}\gamma_{ab}(\tau,\sigma) \tag{27}$$

which holds for any $\omega(\tau,\sigma)$. This is also a local gauge symmetry, which implies that it can be used for gauge fixing.

This concludes the relevant symmetries in the Polyavok action. While in the future, we will talk more about the implication of these symmetries and treat them as more fundamental then the Polyakov action, for now they are simply symmetries in the Polyakov action.

1.4 Equations of motion and Boundary conditions

The goal of this section is find the equation of motion for X^{μ} in the Polyakov action.

$$S[X^{\mu},\gamma_{\alpha\beta}] = -\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma(-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\partial_{\alpha}X^{\mu}\partial_{\beta}X_{\mu} \tag{28}$$

We will now take the variation wrt to X^{μ}. Where the goal is to find equation of motion by demanding that

$$\frac{\delta S}{\delta X^{\mu}} = 0 \tag{29}$$

Taking the variation

$$\begin{aligned}\delta S &= -\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma\delta((-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\partial_{\alpha}X^{\mu}\partial_{\beta}X_{\mu}) \\ &= -\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma(-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\partial_{\alpha}\delta X^{\mu}\partial_{\beta}X_{\mu} - \frac{1}{4\pi\alpha'}\int_M d\tau d\sigma\ (-\gamma)^{\frac{1}{2}}(\gamma^{\alpha\beta})\partial_{\alpha}X^{\mu}\partial_{\beta}\delta X_{\mu}\end{aligned} \tag{30}$$

Where we used the easy to show identity $\delta\partial_a X^{\mu} = \partial_a\delta X^{\mu}$. By relabeling the indices we can combine them into a singal integral. (Specifically, we use the symmetry in the metric tensor)

$$\delta S = -\frac{1}{2\pi\alpha'}\int_M d\tau d\sigma(-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\partial_{\alpha}\delta X^{\mu}\partial_{\beta}X_{\mu} \tag{31}$$

Now from here we manipulate the integrand using the identity

$$(-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\partial_{\alpha}\delta X^{\mu}\partial_{\beta}X_{\mu} = \partial_{\alpha}((-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\delta X^{\mu}\partial_{\beta}X_{\mu}) - \delta X^{\mu}\partial_{\alpha}((-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\partial_{\beta}X_{\mu}) \tag{32}$$

Where this can be understood to be an integration by parts. Note that distributing the derivatives on the right side will equal the left side. Now, the term

$$\partial_{\alpha}((-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\delta X^{\mu}\partial_{\beta}X_{\mu}) = 0 \tag{33}$$

is usually understood to vanish due to boundary conditions we will impose in the next section. We will just assume it to be zero for now

From here, we are left with the action (after dividing by δX^{μ})

$$\frac{\delta S}{\delta X^{\mu}} = \frac{1}{2\pi\alpha'}\int_M d\tau d\sigma \partial_\alpha((-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\partial_\beta X_\mu) = 0 \tag{34}$$

The equation of motion is understood to mean the following equations

$$\partial_\alpha(-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\partial_\beta X_\mu = 0 \tag{35}$$

Along with

$$\begin{aligned}(-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\partial_\alpha\partial_\beta X_\mu &= (-\gamma)^{\frac{1}{2}}\partial_\alpha\partial^\alpha X_\mu = 0 \\ &= (-\gamma)^{\frac{1}{2}}\partial_\alpha\partial^\alpha X_\nu g^{\mu\nu} = g^{\mu\nu} * 0 \\ = (-\gamma)^{\frac{1}{2}}\partial_\alpha\partial^\alpha X^\mu &= \boxed{(-\gamma)^{\frac{1}{2}}\nabla^2 X^\mu = 0}\end{aligned} \tag{36}$$

This along with the previous section complete the equations of motion for the Polyakov action at this state. Its worth noting that there are a few terms that can be added to the Polyakov action that, although break Poincare invariance, are worth exploring. We will discuss these later.

Lets now talk about boundary conditions. There are two types of strings that are constructed from various boundary conditions. The two types are open strings and closed strings. Intuitively, we can think of closed strings as being topologically a circle and an open string as being topologically a line interval. Let us now discuss these boundary conditions.

For these conditions we will choose a parametrization for σ such that it lies inside the interval $0 \leq \sigma \leq \pi$. This is an arbitrary choice that is done for the sake of simplifying the analysis of the boundary conditions.

Closed String

$$X^\mu(\tau,\sigma) = X(\tau,\sigma+\pi) \tag{37}$$

This is a period condition that simply ensures the string is closed everywhere.

Open String with Neumann boundary conditions

$$\partial^\sigma X^\mu = 0 \text{ At } \sigma = 0, \pi \text{ (Generally)} \tag{38}$$

Now, under most gauge fixes we can write ∂_σ instead of ∂^σ but writing this way, its clearer that the boundary condition we assumed was zero *is indeed zero*. The general consequence of this boundary

condition is that the component of momentum normal to the worldsheet vanishes at the boundary.
Open String with Dirichlet Boundary condition

$$X^{\mu}|_{\sigma=0} = X_0^{\mu} \tag{39}$$

$$X^{\mu}|_{\sigma=\pi} = X_{\pi}^{\mu} \tag{40}$$

Where X_0^{μ} and X_{π}^{μ} are constants. This condition applies for $\mu = 1, 2, 3..D-p-1$ where d is the dimension of the theoy p is a p-dimensional subspace of the theory. We will talk more about Dp branes later.

1.5 Lightcone coordinates

The purpose of this section construct an easier set of coordintates to solve the equations of motion. These are known as light cone coordinates and they are merlely a new set of coordinates to makes solving our theory easier. In General relativity. we define the light cone component for any vector a^{μ} as the following

$$a^{+} \equiv \frac{1}{\sqrt{2}}(a^0 + a^1) \tag{41}$$

and

$$a^{-} \equiv \frac{1}{\sqrt{2}}(a^0 - a^1) \tag{42}$$

we let the rest of the indices run from i = 2,...,D

$$a^i \text{ runs from i = 2, . . . , D} \tag{43}$$

It's really just a change of basis in flat spacetime. We can imagine that, internally, the basis looks something like this

$$e_{\mu} = (a^{-}, a^{+}, a^{i}) \tag{44}$$

We can also define coordinates with 'lowered indices' as the following

$$a_{+} \equiv -a^{-} \tag{45}$$

and

$$a_{-} \equiv -a^{+} \tag{46}$$

These new coordinates are require us to re implement contractions which look like this in the new basis

$$\begin{aligned} a^{\mu}b_{\mu} &= -a^{+}b^{-} - a^{-}b^{+} + a^{i}b^{i} \\ &= a_{-}b^{-} + a_{+}b^{+} + a_{i}b^{i} \end{aligned} \tag{47}$$

Where we choose metric such that $a_i = a^i$

Back to string theory, we introduce world-sheet light cone coordinates defined by

$$\sigma^{\pm} = \tau \pm \sigma \text{ and } \partial_{\pm} = \frac{1}{2}(\partial_{\tau} \pm \partial_{\sigma}) \tag{48}$$

Along with the metric

$$\begin{pmatrix} \eta_{++} & \eta_{+-} \\ \eta_{-+} & \eta_{--} \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{49}$$

1.6 Lightcone gauge

The quickest way to get familiar with certain results in string theory is using the "lightcone gauge" as our fixing condition. As discussed earlier, the Polyakov action has a lot of symmetries that allow us to choose a convenient gauge.

Lets first look at the case for a point particle instead of the string world-sheet. The point particle action is given by

$$S = \frac{1}{2}\int d\tau(\eta^{-1}\dot{X}^{\mu}\dot{X}_{\mu} - \eta m^2) \tag{50}$$

Now the cannonical momentum is given by

$$\begin{aligned} p_{\mu} &= \frac{\partial L}{\partial \dot{X}^{\mu}} \\ &= \frac{\partial}{\partial \dot{X}^{\mu}}\frac{1}{2}(\eta^{-1}\dot{X}^{\nu}\dot{X}_{\nu} - \eta m^2) \\ &= \eta^{-1}\dot{X}_{\mu} \end{aligned} \tag{51}$$

Note that the dummy indicies that are being contracted were swapped in the second line to prevent the ambiguous notation (3 indices).

We now wish to construct Hamilonian which is usually done as follows

$$\begin{aligned} H &= p_{\mu}\dot{X}^{\mu} - L \\ &= \eta^{-1}\dot{X}_{\mu}\dot{X}^{\mu} - L \end{aligned} \tag{52}$$

Alright if we wish to carry out this process in lightcone coordinates, it can be down by first transforming (50) into lightcone coordinates. Using the identity. First lets gauge fix our parametrization. Specifically let

$$X^{+}(\tau) = \tau \text{ Where naturally } \dot{X}^{+}(\tau) = 1 \tag{53}$$

This lets us treat X^{+} as a sort of timelike variable. Now, we can use the contraction identity from the previous section

$$\dot{X}^{\mu}\dot{X}_{\mu} = -\dot{X}^{+}\dot{X}^{-} - \dot{X}^{-}\dot{X}^{+} + \dot{X}^{i}\dot{X}^{i} \tag{54}$$

To rewrite the action like this

$$
\begin{aligned}
S &= \frac{1}{2}\int d\tau(\eta^{-1}\dot{X}^{\mu}\dot{X}_{\mu} - \eta m^2) \\
&= \frac{1}{2}\int d\tau(\eta^{-1}(-\dot{X}^{+}\dot{X}^{-} - \dot{X}^{-}\dot{X}^{+} + \dot{X}^{i}\dot{X}^{i}) - \eta m^2) \\
&= \frac{1}{2}\int d\tau(-\eta^{-1}2\dot{X}^{-} + \eta^{-1}\dot{X}^{i}\dot{X}^{i} - \eta m^2)
\end{aligned}
\tag{55}
$$

Now we can find the canonical momentum

$$
\begin{aligned}
p_{-} &= \frac{\partial L}{\partial \dot{X}^{-}} = -\eta^{-1} \\
p_{+} &= \frac{\partial L}{\partial \dot{X}^{+}} = 0 \\
p_{i} &= \frac{\partial L}{\partial \dot{X}^{i}} = \eta^{-1}X_{i}
\end{aligned}
\tag{56}
$$

Which allows us to contruct the Hamiltonian like follows

$$
H = p_{\mu}\dot{X}^{\mu} - L = p_{-}\dot{X}^{-} + p^{i}\dot{X}^{i} - L \tag{57}
$$

Now it's easy to see that we can rewrite this as the following

$$
H = \frac{p^{i}p^{i} + m^2}{2p^{+}} \tag{58}
$$

We finish of this example by quantizing the theory. We accomplish this by imposing the following commutator relations

$$
\begin{aligned}
[p_{i}, X^{j}] &= -i\delta_{i}^{j} \\
[p_{-}, X^{-}] &= -i
\end{aligned}
\tag{59}
$$

Where the variables p_i, X^j, p_-, X^- are now promoted to operators. This completes the quantization procedure for the relativistic point particle.

We shall now turn to the Polyakov action and perform the same procedure.

$$
S[X^{\mu}, \gamma_{\alpha\beta}] = -\frac{1}{4\pi\alpha'}\int_{M} d\tau d\sigma(-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\partial_{\alpha}X^{\mu}\partial_{\beta}X_{\mu} \tag{60}
$$

Now we want to impose a similar gauge fixing condition. The specific conditions will be

$$
\boxed{
\begin{aligned}
X^{+} &= \tau \\
\partial_{\sigma}\gamma_{\sigma\sigma} &= 0 \\
\det(\gamma_{ab}) &= -1
\end{aligned}
}
\tag{61}
$$

We will also fix the parameters $-\infty < \tau < \infty$ and $0 \leq \sigma \leq \ell$ this suffices to gauge fix the symmetries in the Polyakov action. Now we can use the second condition in (103) to determine that $\gamma_{\sigma\sigma}$ is a function of

τ only $\gamma_{\sigma\sigma}(\tau)$. Now we can easily find the inverse of the metric γ_{ab} by invoking the result from general relativity. For any 2x2 metric $g_{\alpha\beta}$ the inverse is given by

$$g^{\alpha\beta} = \frac{1}{\det(g)} \begin{pmatrix} g_{11} & -g_{01} \\ -g_{01} & g_{00} \end{pmatrix} \tag{62}$$

Now back to string theory, we can use this formula to find the inverse of the worldsheet metric.

$$\begin{pmatrix} \gamma^{\tau\tau} & \gamma^{\tau\sigma} \\ \gamma^{\sigma\tau} & \gamma^{\sigma\sigma} \end{pmatrix} = \begin{pmatrix} -\gamma_{\sigma\sigma}(\tau) & \gamma_{\tau\sigma}(\tau,\sigma) \\ \gamma_{\tau\sigma}(\tau,\sigma) & -\gamma_{\tau\tau}(\tau,\sigma) \end{pmatrix} \tag{63}$$

Using the third gauge fixing condition in (103)

$$\begin{aligned} \det(\gamma_{ab}) &= -1 \\ \gamma_{\tau\tau}(\tau,\sigma)\gamma_{\sigma\sigma}(\tau) - \gamma^2_{\tau\sigma}(\tau,\sigma) &= -1 \\ \gamma_{\tau\tau}(\tau,\sigma) &= \frac{1}{\gamma_{\sigma\sigma}(\tau)}(-1+\gamma^2_{\tau\sigma}) \\ \gamma^{ab} &= \begin{pmatrix} -\gamma_{\sigma\sigma}(\tau) & \gamma_{\tau\sigma}(\tau,\sigma) \\ \gamma_{\tau\sigma}(\tau,\sigma) & \frac{1}{\gamma_{\sigma\sigma}(\tau)}(1-\gamma^2_{\tau\sigma}(\tau,\sigma)) \end{pmatrix} \end{aligned} \tag{64}$$

Now we wish to transform the Polyakov action into lightcone coordinates.

$$S[X^\mu, \gamma_{\alpha\beta}] = -\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma (-\gamma)^{\frac{1}{2}} \gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu \tag{65}$$

Start by defining

$$\begin{aligned} X' &= \frac{\partial X^\mu}{\partial\sigma} \\ \dot{X} &= \frac{\partial X^\mu}{\partial\tau} \end{aligned} \tag{66}$$

Now rewriting the lagrangian in light cone coordinates and using (47) to we can write.

$$\begin{aligned} L &= -\frac{1}{4\pi\alpha'}\int_0^\ell d\sigma(\gamma^{\tau\tau}\partial_\tau X^\mu\partial_\tau X_\mu + \gamma^{\tau\sigma}\partial_\tau X^\mu \partial_\sigma X_\mu + \gamma^{\sigma\tau}\partial_\sigma X^\mu\partial_\tau X_\mu + \gamma^{\sigma\sigma}\partial_\sigma X^\mu\partial_\sigma X_\mu) \\ &= -\frac{1}{4\pi\alpha'}\int_0^\ell d\sigma(-\gamma_{\sigma\sigma}(\tau)\partial_\tau X^\mu\partial_\tau X_\mu + 2\gamma_{\tau\sigma}(\tau,\sigma)\partial_\tau X^\mu\partial_\sigma X_\mu + \frac{1}{\gamma_{\sigma\sigma}(\tau)}(1-\gamma^2_{\tau\sigma}(\tau,\sigma))\partial_\sigma X^\mu\partial_\sigma X_\mu) \\ &= -\frac{1}{4\pi\alpha'}\int_0^\ell d\sigma(\gamma_{\sigma\sigma}(\tau)\partial_\tau X^\mu\partial_\tau X_\mu + 2\gamma_{\tau\sigma}(\tau,\sigma)\partial_\tau X^\mu\partial_\sigma X_\mu + \frac{1}{\gamma_{\sigma\sigma}(\tau)}(1-\gamma^2_{\tau\sigma}(\tau,\sigma))\partial_\sigma X^\mu\partial_\sigma X_\mu) \\ &= -\frac{1}{4\pi\alpha'}\int_0^\ell d\sigma[\gamma_{\sigma\sigma}(\tau)(2\dot{X}^- - \dot{X}^i\dot{X}^i) - 2\gamma_{\tau\sigma}(\tau,\sigma)(X' - \dot{X}^i X'^i) + \frac{1}{\gamma_{\sigma\sigma}(\tau)}(1-\gamma^2_{\tau\sigma}(\tau,\sigma))\partial_\sigma X^i\partial_\sigma X_i] \\ &= -\frac{1}{4\pi\alpha'}\int_0^\ell d\sigma[\gamma_{\sigma\sigma}(\tau)(2\partial_\tau X^- - \partial_\tau X^i\partial_\tau X^i) \\ &- 2\gamma_{\tau\sigma}(\tau,\sigma)(\partial_\sigma X^- - \partial_\tau X^i\partial_\sigma X^i) + \frac{1}{\gamma_{\sigma\sigma}(\tau)}(1-\gamma^2_{\tau\sigma}(\tau,\sigma))\partial_\sigma X^i\partial_\sigma X_i)] \end{aligned} \tag{67}$$

Where in the 4th line the identity $\frac{\partial X^+}{\partial \sigma} = 0$ along with $\frac{\partial X^+}{\partial \tau} = 1$. Now by convention we typically split the $\partial_\sigma X^-$ term as

$$x^- = \frac{1}{\ell}\int_0^\ell d\sigma X^-(\tau,\sigma) \tag{68}$$

$$Y^- = X^-(\tau,\sigma) - x^-(\tau) \tag{69}$$

Where it's clear that x^- is the mean value of X^- and the second term has an average value of zero for a fixed τ such that

$$\int_0^\ell d\sigma Y^-(\tau,\sigma) = 0 \tag{70}$$

This results in the following Lagrangian

$$\begin{aligned} L = -\frac{1}{4\pi\alpha'}\int_0^\ell d\sigma[\gamma_{\sigma\sigma}(\tau)(2\partial_\tau X^- - \partial_\tau X^i\partial_\tau X^i) \\ -2\gamma_{\tau\sigma}(\tau,\sigma)(\partial_\sigma Y^- - \partial_\tau X^i\partial_\sigma X^i) + \frac{1}{\gamma_{\sigma\sigma}(\tau)}(1-\gamma_{\tau\sigma}^2(\tau,\sigma))\partial_\sigma X^i\partial_\sigma X_i)] \end{aligned} \tag{71}$$

We also rewrite the X^- term using the identity $X^- = Y^- + x^-$. Looking at the first term

$$\begin{aligned} &\int_0^\ell d\sigma[\gamma_{\sigma\sigma}(\tau)2\partial_\tau X^-] \\ =&\int_0^\ell d\sigma[\gamma_{\sigma\sigma}(\tau)2\partial_\tau(Y^- + x^-)] \\ =&[\gamma_{\sigma\sigma}(\tau)2\partial_\tau(\int_0^\ell d\sigma Y^- + \int_0^\ell d\sigma x^-)] \end{aligned} \tag{72}$$

But the first integral is zero since by (70) which implies that the lagrangian can be written as follows.

$$\begin{aligned} L = -\frac{1}{4\pi\alpha'}\int_0^\ell d\sigma[\gamma_{\sigma\sigma}(\tau)(2\partial_\tau x^- - \partial_\tau X^i\partial_\tau X^i) \\ -2\gamma_{\tau\sigma}(\tau,\sigma)(\partial_\sigma Y^- - \partial_\tau X^i\partial_\sigma X^i) + \frac{1}{\gamma_{\sigma\sigma}(\tau)}(1-\gamma_{\tau\sigma}^2(\tau,\sigma))\partial_\sigma X^i\partial_\sigma X_i)] \end{aligned} \tag{73}$$

This concludes the groundwork required to solve the theory in these new coordinates. In the next sections we will apply boundary conditions and create a mode expansion along with quantizing the theory.

1.7 Open String in Lightcone gauge

We begin by applying the boundary conditions of an open string with the Neumann boundary condition (38) with the boundary at ℓ instead of at π just for convention.

$$\begin{aligned} \partial^\sigma X^\mu &= 0 \text{ (At } \sigma = 0, \ell) \\ = \gamma^{\sigma b}\partial_b X^\mu &= \gamma^{\sigma\sigma}\partial_\sigma X^\mu + \gamma^{\sigma\tau}\partial_\tau X^\mu \\ &= \gamma_{\tau\sigma}\partial_\tau X^\mu - \gamma_{\tau\tau}\partial_\sigma X^\mu = 0 \end{aligned} \tag{74}$$

Now since we are on lightcone gauge, we can let $\mu = +$ which implies

$$\gamma_{\tau\sigma}\partial_\tau X^+ - \gamma_{\tau\tau}\partial_\sigma X^+ = 0 \text{ (At } \sigma = 0, \ell) \tag{75}$$

But the second term is zero since taking the derivative τ wrt σ in this gauge choice. On the other hand, the first term will evaluate to simply $\gamma_{\tau\sigma}$ since the derivative will be 1.

$$\gamma_{\tau\sigma} = 0 \text{ (At } \sigma = 0, \ell) \tag{76}$$

Now taking the variation of the action wrt to the new field Y^- yields (ignoring the other terms which are trivially zero)

$$\begin{aligned}
\delta S &= \int d\tau d\sigma \delta(2\gamma_{\tau\sigma}(\tau,\sigma)\partial_\sigma Y^-) \\
&= 2\int d\tau d\sigma(\delta\partial_\sigma(\gamma_{\tau\sigma}Y^-) - \delta\partial_\sigma\gamma_{\tau\sigma}Y^-) \\
&= 2\int d\tau d\sigma(\partial_\sigma(\gamma_{\tau\sigma}\delta Y^-) - \partial_\sigma\gamma_{\tau\sigma}\delta Y^-) \\
&= 2\int d\tau d\sigma(-\partial_\sigma\gamma_{\tau\sigma}\delta Y^-) = 0
\end{aligned} \tag{77}$$

In the last line, we used the fact that the left side is zero because it is evaluated at the boundary which we just determined was zero. Now The most general solutions to the last equations is that such that $\partial_\sigma\gamma_{\tau\sigma}$ is just a function of τ. This allows the integral wrt σ to act on the δY^- which zeros the whole thing.

$$\partial_\sigma\gamma_{\tau\sigma} = F(\tau) \tag{78}$$

so

$$\partial_\sigma^2\gamma_{\tau\sigma} = 0 \tag{79}$$

Now this tells us that the derivative of $\gamma_{\tau\sigma}$ only has τ dependence and yet we must also meet the boundary conditions that $\gamma_{\tau\sigma}(\tau,0) = \gamma_{\tau\sigma}(\tau,\ell) = 0$ This constraints $\gamma_{\tau\sigma} = 0$ everywhere, otherwise we have no way of meeting both boundary conditions for any τ. Finally we can use the boundary condition again once again

$$\gamma_{\tau\sigma}\partial_\tau X^i - \gamma_{\tau\tau}\partial_\sigma X^i = 0 \text{ (At } \sigma = 0, \ell) \tag{80}$$

Where the first term is clearly zero and that leaves

$$\gamma_{\tau\tau}\partial_\sigma X^i = 0 \text{ (At } \sigma = 0, \ell) \tag{81}$$

Then the lagrangian reduces to

$$
\begin{aligned}
L = & -\frac{1}{4\pi\alpha'}\int_0^{\ell} d\sigma[\gamma_{\sigma\sigma}(\tau)(2\partial_\tau x^- - \partial_\tau X^i \partial_\tau X^i) \\
& - 2\gamma_{\tau\sigma}(\tau,\sigma)(\partial_\sigma Y^- - \partial_\tau X^i \partial_\sigma X^i) + \frac{1}{\gamma_{\sigma\sigma}(\tau)}(1-\gamma^2_{\tau\sigma}(\tau,\sigma)\partial_\sigma X^i \partial_\sigma X_i)] \\
= & \frac{1}{4\pi\alpha'}\int_0^{\ell} d\sigma[-\gamma_{\sigma\sigma}(\tau)(2\partial_\tau x^- + \partial_\tau X^i \partial_\tau X^i) - \frac{1}{\gamma_{\sigma\sigma}(\tau)}(\partial_\sigma X^i \partial_\sigma X_i)] \\
= & \boxed{-\frac{\ell}{2\pi\alpha'}\gamma_{\sigma\sigma}\partial_\tau x^- + \frac{1}{4\pi\alpha'}\int_0^{\ell} d\sigma[\gamma_{\sigma\sigma}\partial_\tau X^i \partial_\tau X^i - \frac{1}{\gamma_{\sigma\sigma}(\tau)}(\partial_\sigma X^i \partial_\sigma X_i)]}
\end{aligned}
\tag{82}
$$

Now we wish to find the conjugate momentum of this lagrangian along with the Hamiltonian, just like we did with the point particle case, in order to quantize the theory. We begin by finding the conjugate momentum of all the independent variables.

$$
p_- = -p^+ = \frac{\partial L}{\partial(\partial_\tau x^-)} = -\frac{\ell}{2\pi\alpha'}\gamma_{\sigma\sigma} \tag{83}
$$

On the other hand, the other independent variable X^i is also trivial to take.

$$
\Pi^i = \frac{\delta \mathcal{L}}{\delta(\partial_\tau X^i)} = \frac{1}{2\pi\alpha'}\gamma_{\sigma\sigma}\partial_\tau X^i \tag{84}
$$

Where Π the conjugate momentum density. The Hamiltonian is then

$$
H = p_- \partial_\tau x^- - L + \int d\sigma(\Pi^i \partial_\tau X^i) \tag{85}
$$

$$
\begin{aligned}
& = \int d\sigma[-\frac{1}{4\pi\alpha'}\gamma_{\sigma\sigma}\partial_\tau X^i \partial_\tau X^i + \frac{1}{4\pi\alpha'\gamma_{\sigma\ \sigma}}(\partial_\sigma X^i \partial_\sigma X_i) + \Pi^i \partial_\tau X^i] \\
& = \int d\sigma[-\frac{(2\pi\alpha')^2}{4\pi\alpha'\gamma_{\sigma\sigma}}\Pi^i\Pi^i + \frac{1}{4\pi\alpha'\gamma_{\sigma\ \sigma}}(\partial_\sigma X^i \partial_\sigma X_i) + \frac{2\pi\alpha'}{\gamma_{\sigma\sigma}}\Pi^i\Pi^i] \\
& = \int d\sigma[\frac{\pi\alpha'}{\gamma_{\sigma\sigma}}\Pi^i\Pi^i + \frac{1}{4\pi\alpha'\gamma_{\sigma\ \sigma}}(\partial_\sigma X^i \partial_\sigma X_i)] \\
& = \frac{1}{2\gamma_{\sigma\sigma}}\int d\sigma[2\pi\alpha'\Pi^i\Pi^i + \frac{1}{2\pi\alpha'}(\partial_\sigma X^i \partial_\sigma X_i)] \\
& = \frac{\ell}{4\pi\alpha' p^+}\int d\sigma[2\pi\alpha'\Pi^i\Pi^i + \frac{1}{2\pi\alpha'}(\partial_\sigma X^i \partial_\sigma X_i)]
\end{aligned}
\tag{86}
$$

At this point, we can use the Hamilton's equations to find the EOM. The first three are trivial. Let $c \equiv \frac{\ell}{2\pi\alpha' p^+}$

$$
\partial_\tau x^- = \frac{\partial H}{\partial p_-} = \frac{H}{p^+} \tag{87}
$$

$$
\partial_\tau X^- = \frac{\delta H}{\delta \Pi^i} = 2c\pi\alpha'\Pi^i \tag{88}
$$

$$
\partial_\tau p^+ = \frac{\partial H}{\partial p^+} = 0 \tag{89}
$$

While the last one requries an integration by parts.

$$\delta H = \frac{c}{2}\left(\frac{1}{2\pi\alpha'}\partial_\sigma \delta X^i \partial_\sigma X^i + \frac{1}{2\pi\alpha'}\partial_\sigma X^i \partial_\sigma \delta X^i\right)$$
$$= \frac{c}{2\pi\alpha'}\partial_\sigma \delta X^i \partial_\sigma X^i$$

Using

$$\partial_\sigma \delta X^i \partial_\sigma X^i = \partial_\sigma(\delta X^i \partial_\sigma X^i) - \delta X^i \partial_\sigma^2 X^i \tag{90}$$

Then

$$\delta H = \frac{c}{2\pi\alpha'}\int d\sigma[\partial_\sigma(\delta X^i \partial_\sigma X^i) - \delta X^i \partial_\sigma^2 X^i]$$
$$-\frac{\delta H}{\delta X^i} = \frac{c}{2\pi\alpha'}\partial_\sigma^2 X^i$$

Where in the last line we used the boundary condition to zero the first term.

Then the equation of motion is

$$\partial_\tau \Pi^i = -\frac{\delta \mathcal{H}}{\delta X^i} = \frac{c}{2\pi\alpha'}\partial_\sigma^2 X^i \tag{91}$$

If we combine the second and last equation we get the (simple) wave equation!

$$\partial_\tau^2 X^i = c^2 \partial_\sigma^2 X^i \tag{92}$$

Since we will rework all this in the path integral formalism later. Lets finish this section quickly by writing the mode expansion and quantizing the theory.

$$X^i = x^i + \frac{p^i}{p^+}\tau + i(2\alpha')^{\frac{1}{2}}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{1}{n}\alpha_n^i e^{-\frac{\pi i n c\tau}{\ell}}\cos\left(\frac{\pi n\sigma}{\ell}\right) \tag{93}$$

Where, classically, α_n are the fourier coefficients. The goal is not to quantize the theory. This is completed with the two commutators at a fixed time.

$$\begin{gathered}[x^-, p^+] = -i \\ [X^i(\sigma), \Pi^j(\sigma')] = i\delta^{ij}\delta(\sigma - \sigma')\end{gathered} \tag{94}$$

Now using the definitions

$$x^i = \frac{1}{\ell}\int_0^\ell d\sigma X^i(\tau,\sigma) \tag{95}$$

$$p^i = \int_0^\ell d\sigma \Pi^i(\tau,\sigma) \tag{96}$$

In other words the center of mass position and the average momentum

$$[x^i, p^j] = i\delta^{ij} \tag{97}$$

$$[\alpha_m^i, \alpha_n^j] = m\delta^{ij}\delta_{m-n} \tag{98}$$

These relationships essentially promote the variables $\alpha, \Pi, x^i, p+, x^-$ into operators which obey a similiar algebra to the creation and annihilation operators. To be clear, the α operators satisfy a similiar algebra

$$
\begin{gathered}
a^i_m \sim m^{\frac{1}{2}} a \qquad a^i_{-m} \sim m^{\frac{1}{2}} a^\dagger \\
\text{For } m > 0
\end{gathered}
\tag{99}
$$

Where a and $a^\dagger$ obey the harmonic osscilator creation and annhilation algebra. We then define a k to be the center of mass momentum and are eigenstates of p

$$
\begin{aligned}
p^+ |0; k\rangle &= k^+ |0; k\rangle \\
p^i |0; k\rangle &= k^i |0; k\rangle
\end{aligned}
\tag{100}
$$

Where it's clear the 0 represents the state of the oscillator and k is the momentum eigenvalue. Now just like any raising and lowering operator algebra we also impose that lowering the 0th state will result in zero.

$$
\alpha^i_m |0; k\rangle = 0 \tag{101}
$$

Finally we can write a general state

$$
|N; k\rangle = \prod_{i=2}^{D-1} \prod_{n=1}^{\infty} \frac{(\alpha^i_{-n})^{N_{in}}}{\sqrt{(n^{N_{in}} N_{in}!)}} |0; k\rangle \tag{102}
$$

Alright so don't let this scare you. The top part is just the creation operator and the bottom is just a normalization constant. The (i, n) subscript is just a tuple to index a specific direction and oscillator mode. This completes our discussion for the open string lightcone gauge (subject to change). Also (unlike QFT) we're not creating multiple strings using this. Rather we are changing the momentum of one string.

1.8 Closed String in Lightcone gauge

Lets quickly repeat the process for a closed string in the lightcone gauge. This section is still subject to modifications and rewrites in the future. Nevertheless, the analysis is largely a parallel except we add one more condition to the gauge fix.

$$
\begin{aligned}
X^+ &= \tau \\
\partial_\sigma \gamma_{\sigma\sigma} &= 0 \\
\det(\gamma_{ab}) &= -1 \\
\gamma_{\tau\sigma}(\tau, 0) &= 0
\end{aligned}
\tag{103}
$$

Now it turns out σ has symmetry under the following translations

$$
\sigma' = \sigma + \text{s modulus } \ell \tag{104}
$$

Where 'S' is a real scaler. This 'new freedom' comes from the periodic condition. We will ignore it for now. The mode expansion is then...

$$X^i = x^i + \frac{p^i}{p^+}\tau + i(\frac{\alpha'}{2})^{\frac{1}{2}} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (\frac{\alpha_n^i}{n} e^{-\frac{2\pi i n(\sigma + c\tau)}{\ell}} + \frac{\tilde{\alpha}_n^i}{n} e^{\frac{2\pi i n(\sigma - c\tau)}{\ell}}) \tag{105}$$

Now promote the following to operators that obey the following relationships.

$$[x^-, p^+] = -i \tag{106}$$

$$[x^i, p^j] = i\delta^{ij} \tag{107}$$

$$[\alpha_m^i, \alpha_n^j] = m\delta^{ij}\delta_{m-n} \tag{108}$$

$$[\tilde{\alpha}_m^i, \tilde{\alpha}_n^j] = m\delta^{ij}\delta_{m-n} \tag{109}$$

Finally a general state takes the following form

$$|N, \tilde{N}; k\rangle = \prod_{i=2}^{D-1} \prod_{n=1}^{\infty} \frac{(\alpha_{-n}^i)^{N_{in}} (\tilde{\alpha}_{-n}^i)^{\tilde{N}_{in}}}{\sqrt{(n^{N_{in}} N_{in}!)(n^{\tilde{N}_{in}} \tilde{N}_{in}!)}} |0, 0; k\rangle \tag{110}$$

This completes(subject to change) the discussion on the light cone gauge. We have built up a quantize string theory that

2 Conformal Field Theory