Available at WWW.MATHEMATICSWEB.ORG
POWERED BY SCIENCE DIRECT

Journal of Algebra 264 (2003) 186–198

JOURNAL OF Algebra

www.elsevier.com/locate/jalgebra

Differential operators on monomial curves

Eivind Eriksen

Institute of Mathematics, University of Warwick, Coventry CV4 7AL, UK

Received 12 February 2002

Communicated by Michel van den Bergh

Abstract

Let k be an algebraically closed field of characteristic 0, let $\Gamma \subseteq \mathbb{N}_0$ be a numerical semigroup, and let $A = k[\Gamma]$ be the corresponding semigroup algebra. We give an explicit description of the ring $D = \mathrm{D}(A)$ of k-linear differential operators on A, the associated graded ring $\mathrm{Gr}\,\mathrm{D}(A)$ and the module of derivations $\mathrm{Der}_k(A)$. We also classify all graded D-modules which are finitely generated torsion-free A-modules of rank 1, considered as modules over the sub-ring $A \subseteq \mathrm{D}(A)$.

Keywords: Rings of differential operators; Algebraic D-modules

Introduction

This is my version of joint work with Henrik Vosegaard. I would like to thank him for many interesting discussions in the process of preparing this paper.

Let k be a fixed algebraically closed field of characteristic 0, and consider an affine monomial curve X defined over k. Then $X = \mathrm{Spec}(A)$, where $A = k[\Gamma]$ is the semigroup algebra over k defined by a numerical semigroup $\Gamma \subseteq \mathbb{N}_0$. In this paper, we study the ring $D = \mathrm{D}(A) = \mathrm{D}(X)$ of k-linear differential operators on A in the sense of Grothendieck. We also study various other objects associated with differential operators on A: The associated graded ring $\mathrm{Gr}\,\mathrm{D}(A)$, the module of derivations $\mathrm{Der}_k(A)$, and the graded D-modules M which are finitely generated torsion-free A-modules of rank 1 considered as modules over the sub-ring $A \subseteq \mathrm{D}(A)$. In each case, we obtain a description in concrete terms of the objects in question. In the final section, we also show some explicit calculations for the case $\Gamma = \langle 3, 4, 5\rangle$.

E-mail address: eriksen@maths.warwick.ac.uk.

0021-8693/03/$ – see front matter
doi:10.1016/S0021-8693(03)00144-3

We are interested in the problem of describing differential operators, and in particular in classifying D-modules, when $D = \mathrm{D}(A)$ and $X = \mathrm{Spec}(A)$ is a singular variety. When X is an irreducible (possibly singular) curve, many strong results of a general nature are known about $\mathrm{D}(A)$, see Smith, Stafford [12]. We mention that $\mathrm{D}(A)$ is a Noetherian ring, and if the natural normalization map $\overline{X} \to X$ is injective, then $\mathrm{D}(A)$ is furthermore a simple Noetherian ring of global homological dimension 1, Morita-equivalent to $\mathrm{D}(\overline{X})$, and $\mathrm{Gr}\,\mathrm{D}(A)$ is a finitely generated k-algebra.

Observe that in the case of an affine monomial curve X, the normalization map is bijective and $\mathrm{D}(\overline{X}) = A_1(k)$, the first Weyl algebra. In this case, it was already shown in Musson [9] that $\mathrm{D}(A)$ is Morita equivalent to the Weyl algebra $A_1(k)$, and this result inspired the later work of Smith and Stafford mentioned above. Using these results, we conclude that $\mathrm{D}(A)$ is a simple Noetherian ring of global homological dimension 1, Morita equivalent to the first Weyl algebra $A_1(k)$, and that $\mathrm{Gr}\,\mathrm{D}(A)$ is a k-algebra of finite type.

The purpose of this paper is to show that it is possible to calculate with differential operators on monomial curves, and to show how such calculations can be carried out. As a result, we obtain a description of differential operators on monomial curves in concrete terms, using elementary methods. This will give additional information on, and in some cases re-prove, the structural results on rings of differential operators mentioned above, applied to monomial curves. We give a short summary of our results:

Let $\Gamma \subseteq \mathbb{N}_0$ be a numerical semigroup, and let $A = k[\Gamma]$ be the corresponding semigroup algebra. We use the following method to describe the differential operators on A: For each integer w, let $\Omega(w) = \{\gamma \in \Gamma\colon \gamma + w \notin \Gamma\}$, which is a finite set. Consider A as a subring of $k[t, t^{-1}]$ and let E denote the Euler derivation $E = t\partial$. We define

$$P_w = t^w \prod_{\gamma \in \Omega(w)} (E - \gamma)$$

for all integers w. Then P_w is a homogeneous differential operator in $\mathrm{D}(A)$ of weight w and order $|\Omega(w)|$, and we show that the set $\{P_w\colon w \in \mathbb{Z}\} \cup \{E\}$ generates $\mathrm{D}(A)$ as a k-algebra. Let $\sigma(w) = |\Omega(w)| = d(P_w)$ for all integers w. Based on a detailed study of the function $\sigma : \mathbb{Z} \to \mathbb{N}_0$, the results of which are described in Propositions 2 and 3, we obtain the following results:

Theorem A. *Let $\Gamma \subseteq \mathbb{N}_0$ be a numerical semigroup and let $A = k[\Gamma]$ be the corresponding semigroup algebra. Then the k-algebra $\mathrm{D}(A)$ is generated by the finite set*

$$\bigl\{P_w\colon |w| \in \{a_1, \dots, a_r\} \text{ or } |w| \in H\bigr\} \cup \{E\}$$

of cardinality $2r + 2h + 1$. Moreover, the ring $\mathrm{D}(A)$ is generated by the linear sub-space $\mathrm{D}^p(A)$ for $p = \max\{a_r, c\}$.

In general, the given set of generators is not minimal, and the ring $\mathrm{D}(A)$ might be generated by $\mathrm{D}^p(A)$ for $p < \max\{a_r, c\}$. In fact, this behaviour occurs already when $\Gamma = \langle 2, 3\rangle$. However, the following result shows that the given set of generators has good properties:

Theorem B. *Let $\Gamma \subseteq \mathbb{N}_0$ be a numerical semigroup, and let $A = k[\Gamma]$ be the corresponding semigroup algebra. Then the associated graded ring $\operatorname{Gr} \mathrm{D}(A)$ is a finitely generated $\mathbb{Z}^2$-graded k-algebra. If $\Gamma = \mathbb{N}_0$, it has a minimal homogeneous set of generators $\{t, u\}$, and otherwise it has a minimal homogeneous set of generators*

$$\left\{t^{\sigma(-w)}u^{\sigma(w)}\colon |w| \in \{a_1, \ldots, a_r\} \text{ or } |w| \in H\right\} \cup \{tu\}$$

of cardinality $2r + 2h + 1$. Moreover, $\operatorname{Gr} \mathrm{D}(A)$ is an affine semigroup algebra of Krull dimension 2.

We recall that A is a Cohen–Macaulay k-algebra, since A is an integral domain of Krull dimension 1. We denote by $t(A)$ the Cohen–Macaulay type of A, which is related to the module of derivations in the following way:

Theorem C. *Let $\Gamma \subseteq \mathbb{N}_0$ be a numerical semigroup with $\Gamma \neq \mathbb{N}_0$, and let $A = k[\Gamma]$ be the corresponding semigroup algebra. Then we have $\mu(\mathrm{Der}_k(A)) = t(A) + 1$. If $\mathrm{rk}(\Gamma) = 3$, then $2 \leqslant \mu(\mathrm{Der}_k(A)) \leqslant 3$, and $\mu = 2$ if and only if Γ is symmetric.*

We also show that the mapping $\Lambda \mapsto k[\Lambda]$ induces a bijective correspondence between the sets Λ such that $\Gamma \subseteq \Lambda \subseteq \mathbb{N}_0$ and $\Gamma + \Lambda \subseteq \Lambda$, and the equivalence classes of graded torsion-free A-modules of rank 1, where equivalence is given by graded isomorphisms. The graded D-module structures compatible with such an A-module is completely described by the following theorem:

Theorem D. *Let A be the coordinate ring of an affine monomial curve, and let M be a graded, torsion-free A-module of rank 1. Then there exists a graded D-module structure on M compatible with the A-module structure if and only if $M \cong A$. In this case, such a graded D-module structure is unique, up to graded isomorphism of D-modules, and it is given by the natural action of $D = \mathrm{D}(A)$ on A.*

Finally, we mention that a weaker form of Theorems A, B, and the first part of Theorem C appeared in the PhD thesis of Eriksson [5]. His results were obtained independently from ours, and the methods he used are very similar to ours. We also mention that the results in this paper appeared in the PhD thesis Eriksen [4] of the author.

1. Numerical semigroups

A *numerical semigroup* is a sub-semigroup Γ of $\mathbb{N}_0$ such that $\mathbb{N}_0 \backslash \Gamma$ is a finite set. We shall introduce some notation for numerical semigroups that will be useful later: We write $H = \mathbb{N}_0 \backslash \Gamma$ for the *set of holes* in Γ, by definition this is a finite set. We denote by h the cardinality of H, so this is the *number of holes* in Γ. There is also a largest integer $g = \max H$ in H, and this number is called the *Frobenius number* of Γ. The number $c = g + 1$ is the *conductor* of Γ, and it has the property that $c = \min\{i \in \mathbb{N}_0\colon i + \mathbb{N}_0 \subseteq \Gamma\}$. By convention, $g = -1$ and $c = 0$ when $\Gamma = \mathbb{N}_0$.

There is a unique minimal set of generators $S = \{a_1, \ldots, a_r\}$ of Γ, with the property that any other set of generators of Γ contains S. We may construct S explicitly in the following way: Let a_1 be the smallest non-zero number in Γ, and for all $i \geqslant 1$ such that the sub-semigroup $\langle a_1, \dots, a_i \rangle \neq \Gamma$, let a_{i+1} be the smallest integer in $\Gamma \setminus \langle a_1, \dots, a_i \rangle$. Since Γ is finitely generated, there exists a positive integer r such that $\langle a_1, \dots, a_r \rangle = \Gamma$. We shall always write $S = \{a_1, \dots, a_r\}$ with the given order $a_1 < a_2 < \cdots < a_r$.

We define the *rank* of Γ to be the number r of generators in the set S. This is clearly a uniquely defined positive integer, and we shall write $\mathrm{rk}(\Gamma) = r$ to denote it.

Let $\Gamma \subseteq \mathbb{N}_0$ be a numerical semigroup. We denote by $A = k[\Gamma]$ the corresponding semigroup algebra, and by $X = \mathrm{Spec}(A)$ the *affine monomial curve* corresponding to Γ. Notice that $A \subseteq k[t]$ is the graded sub-algebra of $k[t]$ generated by the monomials $\{t^{a_i}\colon 1 \leqslant i \leqslant r\}$, and that X is the affine irreducible curve parametrized by

$$X = \left\{\left(t^{a_1}, t^{a_2}, \dots, t^{a_r}\right) \in \mathbf{A}^r\colon\ t \in k\right\} \subseteq \mathbf{A}^r,$$

so the rank r equals the embedding dimension of $k[\Gamma]$. The rank of Γ is therefore often called the embedding dimension of Γ.

2. Differential operators

Let A be a commutative k-algebra and let $\mathrm{D}(A)$ be the ring of differential operators on A, see Coutinho [3, Chapter 3]. Then $\mathrm{D}(A)$ has a filtration $\{\mathrm{D}^p(A)\}$ given by the order of differential operators, and the associated graded ring

$$\mathrm{Gr}\,\mathrm{D}(A) = \bigoplus_p \mathrm{Gr}^p\,\mathrm{D}(A) = \bigoplus_p \mathrm{D}^p(A)/\mathrm{D}^{p-1}(A)$$

is commutative.

Assume that A is a $\mathbb{Z}$-graded k-algebra of finite type, and let A_γ denote the homogeneous component of A of degree γ for any $\gamma \in \mathbb{Z}$. Let $w \in \mathbb{Z}$. We say that a differential operator $P \in \mathrm{D}(A)$ is *homogeneous* of weight w if $P * A_\gamma \subseteq A_{\gamma+w}$ for all $\gamma \in \mathbb{Z}$, where $P*$ denotes the natural action of P on A. Let $\mathrm{D}(A)_w$ be the set of homogeneous differential operators on A of weight w. Then $\mathrm{D}(A)_w$ is a k-linear space, and since A is a finitely generated k-algebra, we have a direct sum decomposition

$$\mathrm{D}(A) = \bigoplus_w \mathrm{D}(A)_w$$

compatible with the ring structure of $\mathrm{D}(A)$. So the graded ring structure of A induces a graded ring structure on $\mathrm{D}(A)$ in a natural way.

Let $\Gamma \subseteq \mathbb{N}_0$ be a numerical semigroup, and let $A = k[\Gamma]$ be the corresponding semigroup algebra. We notice that A is a graded sub-algebra of T, where $T = k[\mathbb{Z}]$ is the affine coordinate ring of the 1-dimensional torus, and that A is of finite type over k. Moreover, the inclusion $A \subseteq T$ can be realized as a graded localization homomorphism. It follows that $\mathrm{D}(A)$ can be identified with the graded sub-algebra

$$\left\{P \in \mathrm{D}(T)\colon\ P * A \subseteq A\right\} \subseteq \mathrm{D}(T),$$

see Másson [8, Corollary 2.2.6]. The identification preserves the weight and order of differential operators, and we shall always consider $\mathrm{D}(A)$ as a sub-algebra of $\mathrm{D}(T)$ via this identification.

Let $T = k[t, t^{-1}]$ and consider the ring $\mathrm{D}(T)$ of differential operators on T. It is well known that any differential operator $P \in \mathrm{D}(T)$ can be written uniquely on the form

$$P = \sum_{\alpha,\beta} C_{\alpha,\beta} t^{\alpha} \partial^{\beta}, \tag{1}$$

where $\partial = \partial/\partial t$, $(\alpha, \beta) \in \mathbb{Z} \times \mathbb{N}_0$, and $c_{\alpha,\beta} \in k$ with $c_{\alpha,\beta} = 0$ for all but a finite number of indices (α, β). The order filtration and the graded structure of $\mathrm{D}(T)$ can be described in concrete terms: A monomial $t^{\alpha}\partial^{\beta}$ has order β and weight $\alpha - \beta$. We also notice that $\mathrm{D}(T)_0 = k[E]$, where $E = t\partial$ is the *Euler derivation*.

For all $w \in \mathbb{Z}$, let us define $\Omega(w) = \{\gamma \in \Gamma\colon \gamma + w \notin \Gamma\}$. Then $\Omega(w) \subseteq \mathbb{Z} \subseteq \mathbf{A}^1_k$, since k has characteristic 0. Let x be the coordinate of $\mathbf{A}^1_k$, and consider the ideal $I(\Omega(w)) = \{f \in k[x]\colon\ f(\Omega(w)) = 0\} \subseteq k[x]$. There is a natural isomorphism between $k[x]$ and $\mathrm{D}(T)_0 = k[E]$ such that $x \mapsto E$. This isomorphism identifies the ideal $I(\Omega(w)) \subseteq k[x]$ with an ideal in $k[E]$, and we shall denote this ideal $I(w)$.

Proposition 1. *Let $\Gamma \subseteq \mathbb{N}_0$ be a numerical semigroup, and let $A = k[\Gamma]$ be the corresponding semigroup algebra. Then $\mathrm{D}(A)_0 = k[E]$, and $\mathrm{D}(A)_w = t^w I(w)$ for all $w \in \mathbb{Z}$.*

Notice that $\Omega(w)$ is a finite set for all integers w. We shall denote the cardinality of $\Omega(w)$ by $\sigma(w)$, and this defines a function $\sigma : \mathbb{Z} \to \mathbb{N}_0$. The ideal $I(w)$ is clearly principal, with generator

$$\chi_w(E) = \prod_{\gamma \in \Omega(w)} (E - \gamma).$$

In particular, $\mathrm{D}(A)_w = P_w k[E]$ for all $w \in \mathbb{Z}$, where $P_w = t^w \chi_w(E)$. We see that P_w has order $\sigma(w)$, weight w and leading term $t^{w+\sigma(w)}\partial^{\sigma(w)}$. Furthermore, P_w has minimal order among the homogeneous differential operators in $\mathrm{D}(A)_w$, and it is uniquely defined by this property, up to a scalar in k^*.

We remark that the method used in this section can be used to describe the differential operators on affine semigroup algebras of higher dimension. In fact, a generalized version of Proposition 1 holds in this case, see Musson [10, Theorem 2.3]. We also mention that given the numerical semigroup Γ and the integer w, the standard form (1) of P_w can easily be found (for instance with the help of a Maple script). The coefficients of the standard form will be integers.

3. The numerical function σ

Let $\Gamma \subseteq \mathbb{N}_0$ be a numerical semigroup, and let $\sigma : \mathbb{Z} \to \mathbb{N}_0$ be the numerical function defined in the previous section. We have seen that $d(P_w) = \sigma(w)$ for all $w \in \mathbb{Z}$, so $\sigma(w)$

is the minimal order of (non-zero) homogeneous differential operators of weight w. This is the motivation for studying the function σ in more detail:

Proposition 2. *For all integers w, we have $\sigma(-w) = \sigma(w) + w$.*

Proof. See Perkins [11, Lemma 3.3]. □

Proposition 3. *For all integers w, w', we have $\sigma(w + w') \leqslant \sigma(w) + \sigma(w')$. Equality holds if and only if $ww' = 0$ or $w, w' \in \Gamma$ or $-w, -w' \in \Gamma$.*

Proof. Consider the map $f : \Omega(w + w')\backslash\Omega(w) \to \Omega(w')$ given by $f(l) = l + w$. This is a well-defined injection. But $\Omega(w + w')$ is the disjoint union of $\Omega(w + w') \cap \Omega(w)$ and $\Omega(w + w')\backslash\Omega(w)$. Since $\Omega(w + w') \cap \Omega(w) \subseteq \Omega(w)$, we obviously have that $\sigma(w + w') \leqslant \sigma(w) + \sigma(w')$. Furthermore, equality holds if and only if f is surjective and $\Omega(w) \subseteq \Omega(w + w')$. We show that this is the case if and only if $w = 0$ or $w' = 0$ or $w, w' \in \Gamma$ or $-w, -w' \in \Gamma$: One implication is obvious, so let us assume that $\sigma(w+w') = \sigma(w)+\sigma(w')$ and $w, w' \neq 0$. Then f is surjective and $\Omega(w) \subseteq \Omega(w+w')$ by the previous remark. If $w > 0$, then 0 is not in the image of f. But $im(f) = \Omega(w')$, and $0 \in \Omega(w')$ if and only if $w' \notin \Gamma$. Hence $w' \in \Gamma$. In particular $w' > 0$, so by interchanging the roles of w and w', we see that $w \in \Gamma$. If $w < 0$, we get from the symmetry formula above that

$$\sigma(-w) + \sigma(-w') = \sigma(w) + w + \sigma(w') + w' = \sigma(w + w') + w + w' = \sigma(-w - w').$$

By the argument in the case $w > 0$, we get $-w, -w' \in \Gamma$. □

If $\Gamma = \mathbb{N}_0$, we have $\Omega(w) = \{0, 1, \ldots, -(w + 1)\}$ if w is strictly negative, and $\Omega(w) = \emptyset$ otherwise. Hence, we obtain the formula $\sigma(w) = 1/2(|w| - w)$, and the above results are trivial in this special case.

4. The ring of differential operators

We recall that $\mathrm{D}(A)_w = P_w k[E]$ for all $w \in \mathbb{Z}$. It is therefore clear that the set $\{P_w\colon\ w \in \mathbb{Z}\} \cup \{E\}$ generates the k-algebra $\mathrm{D}(A)$. From Proposition 2, we also see that P_w has leading term $t^{\sigma(-w)}\partial^{\sigma(w)}$ for all $w \in \mathbb{Z}$. For all $w, w' \in \mathbb{Z}$, we therefore have that $P_w P_{w'} = P_{w+w'} E^n$ (modulo terms of lower order) for some $n \geqslant 0$, with $n = 0$ if and only if $\sigma(w + w') = \sigma(w) + \sigma(w')$. By Proposition 3, this condition is equivalent to $ww' = 0$, or $w, w' \in \Gamma$, or $-w, -w' \in \Gamma$.

Theorem 4. *Let $\Gamma \subseteq \mathbb{N}_0$ be a numerical semigroup and let $A = k[\Gamma]$ be the corresponding semigroup algebra. Then the k-algebra $\mathrm{D}(A)$ is generated by the finite set*

$$\left\{P_w\colon\ |w| \in \{a_1, \ldots, a_r\} \text{ or } |w| \in H\right\} \cup \{E\} \tag{2}$$

of cardinality $2r + 2h + 1$. Moreover, the ring $\mathrm{D}(A)$ is generated by the linear sub-space $\mathrm{D}^p(A)$ for $p = \max\{a_r, c\}$.

Proof. The first part follows from the remarks preceding the theorem. It is therefore enough to show that $d(P) \leqslant \max\{a_r, c\}$ for each differential operator P in the given generating set: We have $d(P_{a_i}) = 0$ and $d(P_{-a_i}) = a_i$ for $1 \leqslant i \leqslant r$, and $d(E) = 1$. If $\Gamma = \mathbb{N}_0$, there are no more generators, and we see that the conclusion holds. If $w \in H$, we have that $\Omega(w) \subseteq \{0, 1, \ldots, g - w\}$, so $\sigma(w) \leqslant g - w + 1 \leqslant g$. Similarly, if $-w \in H$, then we have that $\sigma(w) = \sigma(-w) - w \leqslant (g + w + 1) - w = g + 1 = c$. So if $|w| \in H$, then $\sigma(w) \leqslant c$, which proves that the conclusion holds if $\Gamma \neq \mathbb{N}_0$ as well. □

We remark that the generating set (2) is not minimal, in general. The easiest counter-example is found in the case $\Gamma = \mathbb{N}_0$: In this case, the set (2) is given by $\{t, \partial, E\}$, and the generator E is clearly superfluous.

Let us denote by $b(\Gamma)$ the least integer p such that the linear sub-space $\mathrm{D}^p(A)$ generates $\mathrm{D}(A)$. Then the last part of Theorem 4 shows that $b(\Gamma) \leqslant \max\{a_r, c\}$. We remark that this bound is not sharp, in general. The easiest counter-example is found in the case $\Gamma = \langle 2, 3\rangle$: In this case, the set $\{P_2, P_3, P_{-2}\}$ is found to be a set of generators of $\mathrm{D}(A)$, see Smith, Stafford [12] or Jones [7]. So $\mathrm{D}^2(A)$ generates $\mathrm{D}(A)$, while $a_r = 3$. However, it is true that there is a differential operator P in the generating set (2) such that $d(P) = \max\{a_r, c\}$, since $d(P_{-a_r}) = a_r$ and $d(P_{-g}) = c$.

5. The associated graded ring

The inclusion $\mathrm{D}(A) \subseteq \mathrm{D}(T)$ of $\mathbb{Z}$-graded rings gives rise to an injective homomorphism $\mathrm{Gr}\,\mathrm{D}(A) \to \mathrm{Gr}\,\mathrm{D}(T)$ of $\mathbb{Z}^2$-graded rings. Moreover, $\mathrm{Gr}\,\mathrm{D}(T) \cong k[t, t^{-1}, u]$, where u is the image in $\mathrm{Gr}\,\mathrm{D}(T)$ of the derivation $\partial \in \mathrm{D}(T)$. So there is a graded isomorphism between $\mathrm{Gr}\,\mathrm{D}(A)$ and a $\mathbb{Z}^2$-graded sub-algebra of $k[t, t^{-1}, u]$. We shall always identify $\mathrm{Gr}\,\mathrm{D}(A)$ with this sub-algebra. Notice that if $P \in \mathrm{D}(A)$ has leading term $t^\alpha \partial^\beta$, then the image of P in $\mathrm{Gr}\,\mathrm{D}(A)$ is $t^\alpha u^\beta$.

Let us consider the set $\Gamma' = \{(\alpha, \beta) \in \mathbb{Z}^2\colon\ t^\alpha u^\beta \in \mathrm{Gr}\,\mathrm{D}(A)\}$. Then $\Gamma' \subseteq \mathbb{Z} \times \mathbb{N}_0$ is obviously a semigroup. By construction, the corresponding semigroup algebra $k[\Gamma'] = \mathrm{Gr}\,\mathrm{D}(A)$. We know that the set

$$\bigl\{P_w\colon\ |w| \in \{a_1, \ldots, a_r\} \text{ or } |w| \in H\bigr\} \cup E$$

generates $\mathrm{D}(A)$ as a k-algebra. From the construction of this generating set, we see that the corresponding set of images in $\mathrm{Gr}\,\mathrm{D}(A)$ will generate $\mathrm{Gr}\,\mathrm{D}(A)$ as a k-algebra. But the leading term of P_w is $t^{\sigma(-w)}\partial^{\sigma(w)}$, and the leading term of E is $t\partial$, so

$$\bigl\{\bigl(\sigma(-w), \sigma(w)\bigr)\colon\ |w| \in \{a_1, \ldots, a_r\} \text{ or } |w| \in H\bigr\} \cup \bigl\{(1, 1)\bigr\} \tag{3}$$

generates Γ' as a semigroup. However, notice that there are sets of generators of the k-algebra $\mathrm{D}(A)$ such that the corresponding images in $\mathrm{Gr}\,\mathrm{D}(A)$ do not generate $\mathrm{Gr}\,\mathrm{D}(A)$.

It follows that Γ' is a finitely generated semigroup. Furthermore, $\Gamma' \subseteq \mathbb{N}_0^2$, the set $\mathbb{N}_0^2 \backslash \Gamma'$ is finite, and for all $(\alpha, \beta) \in \mathbb{N}_0^2$, we have $(\alpha, \beta) \in \Gamma'$ if and only if $(\beta, \alpha) \in \Gamma'$. We also see that $(\alpha, 0) \in \Gamma'$ if and only if $\alpha \in \Gamma$.

Lemma 5. *If $\Gamma \neq \mathbb{N}_0$, then the set* (3) *is a minimal set of generators for the semigroup Γ'.*

Proof. We have already seen that (3) is a set of generators for Γ'. To show that this is a minimal set of generators, it is enough to show that none of the generators are superfluous: First, notice that $(1,0), (0,1) \notin \Gamma'$, so $(1,1)$ is certainly not superfluous. Furthermore, $(\sigma(-w), \sigma(w))$ is given as $(a_i, 0)$ if $w = a_i$ and $(0, a_i)$ if $w = -a_i$. So by the minimality of the generating set $\{a_1, \ldots, a_r\}$ of Γ, none of these generators are superfluous. Let w be an integer such that $|w| \in H$, and assume that the corresponding generator $(\sigma(-w), \sigma(w))$ is superfluous. Then we have

$$\big(\sigma(-w), \sigma(w)\big) = \sum_{i \in I(w)} n_i \big(\sigma(-i), \sigma(i)\big) + n_0(1,1), \tag{4}$$

where $I(w) = \{i \in \mathbb{Z}\colon\ |i| \in \{a_1, \ldots, a_r\} \cup H,\ i \neq w\}$, and $n_i \in \mathbb{N}_0$ for $i \in I(w) \cup \{0\}$. By subtracting the second coordinate in (4) from the first, we get

$$w = \sum_{i \in I(w)} n_i i. \tag{5}$$

By Eq. (5) and repeated use of Proposition 3, we see that

$$\sigma(w) = \sigma\left(\sum_{i \in I(w)} n_i i\right) \leqslant \sum_{i \in I(w)} n_i \sigma(i) \leqslant \sum_{i \in I(w)} n_i \sigma(i) + n_0 = \sigma(w), \tag{6}$$

where the last equality follows from Eq. (4). So all terms in (6) are equal, and in particular $n_0 = 0$ and

$$\sum_{i \in I(w)} n_i \sigma(i) = \sigma\left(\sum_{i \in I(w)} n_i i\right).$$

Clearly $n_i > 0$ for some $i \in I(w)$, and we write $w = i + (w - i)$. Since i is a summand of w in expression (5), $\sigma(w) = \sigma(i) + \sigma(w - i)$ by (6) and another repeated use of Proposition 3. Again by Proposition 3, we see that $i, w - i \in \Gamma$, or $i, w - i \in -\Gamma$, or $i(w - i) = 0$. But $|w| \in H$ and $i \neq 0$, so $w = i$ and this contradicts $i \in I(w)$. So the generator $(\sigma(-w), \sigma(w))$ is not superfluous, and the generating set (3) is a minimal generating set. $\square$

Theorem 6. *Let $\Gamma \subseteq \mathbb{N}_0$ be a numerical semigroup, and let $A = k[\Gamma]$ be the corresponding semigroup algebra. Then the associated graded ring* $\operatorname{Gr} \mathrm{D}(A)$ *is a finitely generated $\mathbb{Z}^2$-graded k-algebra. If $\Gamma = \mathbb{N}_0$, it has a minimal homogeneous set of generators $\{t, u\}$, and otherwise it has a minimal homogeneous set of generators*

$$\big\{t^{\sigma(-w)} u^{\sigma(w)}\colon\ |w| \in \{a_1, \ldots, a_r\} \text{ or } |w| \in H\big\} \cup \{tu\}$$

of cardinality $2r + 2h + 1$. Moreover, $\operatorname{Gr} \mathrm{D}(A)$ *is an affine semigroup algebra of Krull dimension* 2.

Corollary 7 (Jones). *Let $\Gamma \subseteq \mathbb{N}_0$ be a numerical semigroup, and let $A = k[\Gamma]$ be the corresponding semigroup algebra. If $\Gamma \neq \mathbb{N}_0$, then*

$$\left|\mathbb{N}_0^2 \backslash \Gamma'\right| = 2\left(\sum_{w \in H} w\right) - h(h-1).$$

Proof. A proof was given by Jones in [7], but we include a new, simpler proof: The cardinality of $\{(\alpha, \beta) \in \mathbb{N}_0^2 \backslash \Gamma' \colon \alpha - \beta = w\}$ is $\sigma(w)$ for all $w \geqslant 0$, and $\sigma(w) = 0$ if $w \in \Gamma$. By symmetry, we therefore have that the cardinality of $\mathbb{N}_0^2 \backslash \Gamma'$ is given by $2\sum \sigma(w)$, where the sum is taken over the finite set H. But we have

$$\sum_{w \in H} \sigma(w) = \sum_{w' \in H} \#\{\gamma \in \Gamma \colon \gamma < w'\} = \sum_{w' \in H} \bigl(w' - \#\{w \in H \colon w < w'\}\bigr), \tag{7}$$

which gives the desired formula. $\square$

6. The module of derivations

Let w be an integer. There exists a homogeneous derivation $P \in \mathrm{Der}_k(A)$ of weight w if and only if $\sigma(w) \leqslant 1$. In this case, the space of such homogeneous derivations has dimension 1 and basis $t^w E$. But $\sigma(w) = 1$ implies $w + a_i \in \Gamma$ for $1 \leqslant i \leqslant r$. So if $\Gamma = \mathbb{N}_0$, then $\{\partial\}$ is a minimal homogeneous set of generators for $\mathrm{Der}_k(A)$ as a left A-module. Otherwise,

$$\left\{t^w E \colon w \in \Gamma''\right\} \cup \{E\}$$

is a minimal homogeneous set of generators for $\mathrm{Der}_k(A)$ considered as a left A-module, with $\Gamma'' = \{w \in \mathbb{Z} \colon \sigma(w) = 1\}$.

Let us recall some definitions: Clearly $A = k[\Gamma]$ is a positively graded k-algebra. So for any finitely generated graded A-module M, every minimal homogeneous set of generators has the same cardinality $\mu(M)$. It is also clear that A is Cohen–Macaulay (with respect to the unique graded maximal ideal $m \subseteq A$), and the *Cohen–Macaulay type* $t(A)$ of A is given by $t(A) = \dim_k \mathrm{Ext}_A^1(A/m, A)$.

Lemma 8. *Let Γ be a numerical semigroup. If $\Gamma \neq \mathbb{N}_0$, then $\Gamma'' \subseteq H$ has cardinality $t(A)$, and $\mu(\mathrm{Der}_k(A)) = t(A) + 1$.*

Proof. For the first part, assume that $\sigma(w) = 1$. If $w = -1$, then we have that $\sigma(w) = \sigma(-w) - w = \sigma(1) + 1 \geqslant 2$, so this is a contradiction. If $w \leqslant -2$, then $\sigma(w) = \sigma(-w) - w \geqslant 2$, so this is also a contradiction as well. So $w \geqslant 0$, and therefore $w \in H$. The last part follows from Fröberg [6]. $\square$

We say that a numerical semigroup Γ is *symmetric* if the following condition holds: For all $w \in \mathbb{Z}$ with $0 \leqslant w \leqslant g$, we have $w \in \Gamma$ if and only if $g - w \notin \Gamma$. Using the above lemma, we get a characterization of symmetric semigroups in terms of the set Γ'': Γ is symmetric if and only if $\Gamma'' = \{g\}$. If $\Gamma = \mathbb{N}_0$, then this is clear. Otherwise, notice that for an element $h \in H$ with $h \neq g$, we have $g - h \in \Gamma$ if and only if $g - h \in \Omega(h)$. The last condition means that $h \notin \Gamma''$, so Γ is symmetric if and only if $\Gamma'' = \{g\}$.

It is a theorem of Herzog and Kunz that Γ is symmetric if and only if $A = k[\Gamma]$ is Gorenstein, see Bruns, Herzog [1, Theorem 4.4.8]. But A is Gorenstein if and only if $t(A) = 1$, so the characterization above together with Lemma 8 gives another proof of this fact. We conclude that if Γ is symmetric, then $\mathrm{Der}_k(A)$ has a minimal homogeneous set of generators $\{\partial\}$ if $\Gamma = \mathbb{N}_0$, and $\{E, t^g E\}$ otherwise.

It is well known that all semigroups of rank $r < 3$ are symmetric. For the set of numerical semigroups with a fixed rank $r > 3$, there is no upper bound on the Cohen–Macaulay type $t(A)$ of the corresponding semigroup algebra, and therefore no bound on $\mu(\mathrm{Der}_k(A))$. A proof of this fact can be found in Cavaliere and Niesi [2]. However, the following interesting behaviour appears in the case $r = 3$:

Theorem 9. *Let $\Gamma \subseteq \mathbb{N}_0$ be a numerical semigroup, and let $A = k[\Gamma]$ be the corresponding semigroup algebra. If Γ has rank* 3*, then $t(A) \leqslant 2$. In particular, if Γ is a non-symmetric numerical semigroup of rank* 3*, then the module* $\mathrm{Der}_k(A)$ *of derivations has a minimal set of homogeneous generators*

$$\left\{E, t^f E, t^g E\right\},$$

where $f = \min\{w \in H\colon\ g - w \in H\}$.

Proof. It follows from Cavaliere and Niesi [2, Proposition 3.2] that $t(A) \leqslant 2$ when $\mathrm{rk}(\Gamma) = 3$. The rest is clear. □

7. Graded torsion-free A-modules and D-module structures

Let $\Gamma \subseteq \mathbb{N}_0$ be a numerical semigroup, and let $A = k[\Gamma]$ be the corresponding semigroup algebra. For each set Λ such that $\Gamma \subseteq \Lambda \subseteq \mathbb{N}_0$ and $\Gamma + \Lambda \subseteq \Lambda$, there corresponds an A-module $k[\Lambda]$ in a natural way: $k[\Lambda]$ is the graded A-submodule of $k[t]$ generated by the homogeneous elements $\{t^\lambda\colon\ \lambda \in \Lambda\}$. Since $\Lambda \setminus \Gamma$ is finite, $k[\Lambda]$ is clearly a finitely generated graded A-module. Moreover, it is easy to see that $k[\Lambda]$ is a torsion-free A-module of rank 1.

Let M, M' be $\mathbb{Z}$-graded A-modules. We say that M, M' are *gr-equivalent* if there exists a graded isomorphism $f : M \to M'$ (of any degree), and we denote by $\mathbf{TF}^i(A)$ the set of all equivalence classes of finitely generated $\mathbb{Z}$-graded A-modules which are torsion-free of rank i, up to gr-equivalence.

Proposition 10. *Let $\Gamma \subseteq \mathbb{N}_0$ be a numerical semigroup, and let $A = k[\Gamma]$ be the corresponding semigroup algebra. The assignment $\Lambda \mapsto k[\Lambda]$ induces a bijection between the set*

$$\{\Lambda\colon\ \Gamma \subseteq \Lambda \subseteq \mathbb{N}_0,\ \Gamma + \Lambda \subseteq \Lambda\}$$

and $\mathbf{TF}^1(A)$. In particular, the set $\mathbf{TF}^1(A)$ is finite.

Proof. We have seen that the given assignment induces a well-defined map. To show surjectivity, assume M represents an equivalence class in $\mathbf{TF}^1(A)$. Then M is gr-equivalent with a graded A-submodule of $k[t,t^{-1}]$ generated by monomials of the form t^λ with $\lambda \in \mathbb{Z}$. Define $\Lambda' = \{\lambda \in \mathbb{Z}\colon t^\lambda \in M\}$. Then $\Lambda' \subseteq \mathbb{Z}$ satisfies $\Gamma + \Lambda' \subseteq \Lambda'$. Since M is an A-module of finite type, $d = \min \Lambda'$ is well-defined. Let $\Lambda = \Lambda' - d$. Then $\Lambda \subseteq \mathbb{N}_0$, and since $0 \in \Lambda$, we have $\Gamma \subseteq \Lambda$. So Λ satisfies $\Gamma \subseteq \Lambda \subseteq \mathbb{N}_0$, $\Gamma + \Lambda \subseteq \Lambda$, and $k[\Lambda]$ is clearly gr-equivalent with M. To show injectivity, assume that $k[\Lambda]$, $k[\Lambda']$ are gr-equivalent with $\Gamma \subseteq \Lambda$, $\Lambda' \subseteq \mathbb{N}_0$. Then there is a graded isomorphism between $k[\Lambda]$ and $k[\Lambda']$ which is homogeneous of degree 0. So $\Lambda = \Lambda'$, and the assignment is bijective. The last part is clear, since $\mathbb{N}_0 \backslash \Gamma$ is finite. $\square$

Let $M = k[\Lambda]$ be a graded A-module, corresponding to a set Λ with $\Gamma \subseteq \Lambda \subseteq \mathbb{N}_0$ and $\Gamma + \Lambda \subseteq \Lambda$. For simplicity, we shall also write $D = \mathrm{D}(A)$. A *graded D-module structure on M compatible with the A-module structure* is a ring homomorphism $\rho : D \to \mathrm{End}_k(M)$ such that $\rho(D_w)(M_i) \subseteq M_{i+w}$ for all $w, i \in \mathbb{Z}$ and $\rho(a)(m) = am$ for all $a \in A$, $m \in M$.

Lemma 11. *Let $\rho : D \to \mathrm{End}_k(M)$ be a graded D-module structure on M compatible with the A-module structure. Then $\rho(E) = E + f$, where f denotes multiplication by a homogeneous element $f \in k[t]$ of degree* 0.

Proof. Let $f \in M \subseteq k[t]$ be given by $f = \rho(E)(1)$. Then $(\rho(E) - E)(1) = f$, and it is clearly enough to show that $\rho(E) - E$ is given by multiplication by f. Let $\lambda \in \Lambda$, and recall that $c, c + \lambda \in \Gamma$ when c is the conductor of Γ. Then we have

$$\begin{aligned}\big(\rho(E) - E\big)\big(t^{c+\lambda}\big) &= \rho\big(Et^{c+\lambda}\big)(1) - E * t^{c+\lambda}\\ &= \rho\big(t^{c+\lambda}E + (c+\lambda)t^{c+\lambda}\big)(1) - (c+\lambda)t^{c+\lambda} = t^{c+\lambda}\rho(E)(1),\end{aligned}$$

and

$$\begin{aligned}\big(\rho(E) - E\big)\big(t^{c+\lambda}\big) &= \rho\big(Et^{c}\big)\big(t^\lambda\big) - E * t^{c+\lambda} = \rho\big(t^cE + ct^c\big)\big(t^\lambda\big) - (c+\lambda)t^{c+\lambda}\\ &= t^c\big(\rho(E) - E\big)\big(t^\lambda\big).\end{aligned}$$

It follows that $t^c(\rho(E) - E)(t^\lambda) = t^c(t^\lambda f)$ for all $\lambda \in \Lambda$. But M is a torsion-free A-module, so $(\rho(E) - E)(t^\lambda) = ft^\lambda$, and therefore $\rho(E) = E + f$. Since $E \in \mathrm{D}(A)$ is homogeneous of degree 0 and ρ is a graded D-module structure, it follows that $f \in k[t]$ is homogeneous of degree 0. $\square$

Theorem 12. *Let A be the coordinate ring of an affine monomial curve, and let M be a graded, torsion-free A-module of rank* 1. *Then there exists a graded D-module structure on M compatible with the A-module structure if and only if $M \cong A$. In this case, such a graded D-module structure is unique, up to graded isomorphism of D-modules, and it is given by the natural action of $D = \mathrm{D}(A)$ on A.*

Proof. By assumption, $A = k[\Gamma]$ for a numerical semigroup $\Gamma \subseteq \mathbb{N}_0$, and $M = k[\Lambda]$ for a set Λ with $\Gamma \subseteq \Lambda \subseteq \mathbb{N}_0$ and $\Gamma + \Lambda \subseteq \Lambda$. The graded D-module structure $\rho : D \to \operatorname{End}_k(M)$ is of the form $\rho(E) = E + f$ with $f \in k$ by Lemma 11. For all $\lambda \in \Lambda$, $w \in \mathbb{Z}$, we have $\rho(P_w)(t^\lambda) = \chi_w(\lambda + f)t^{\lambda+w} \in M$. So $\rho(P_w)(t^\lambda)$ is homogeneous of degree $\lambda + w$. Consequently, $\lambda + w \in \Lambda$ or $\chi_w(\lambda + f) = 0$ for all $\lambda \in \Lambda$, $w \in \mathbb{Z}$. Let $\lambda = 0$ and $w < 0$. Then the above conditions gives that $f \in \Omega(w)$ for all $w < 0$, so $f = 0$. Assume that $\Lambda \neq \Gamma$, and choose $\lambda \in \Lambda \backslash \Gamma$. If $\chi_w(\lambda) = 0$, then $\lambda \in \Omega(w) \subseteq \Gamma$, which is impossible. So $\lambda + w \in \Lambda$ for all $w \in \mathbb{Z}$. But this is a contradiction, and we conclude that $\Lambda = \Gamma$. Since $f = 0$, the uniqueness is clear when $\Lambda = \Gamma$. $\square$

8. An example

Let us consider the monomial curve $X = \operatorname{Spec}(A)$, where $A = k[\Gamma]$ and the numerical semigroup Γ is given as $\Gamma = \langle 3, 4, 5\rangle$. In this case, $\Gamma = \mathbb{N}_0 \backslash \{1, 2\}$ is non-symmetric, and the corresponding space curve $X = \{(t^3, t^4, t^5) \in \mathbf{A}^3 \colon t \in k\}$ is therefore not Gorenstein.

In this example, we shall calculate the generating set of $\mathrm{D}(A)$ given in Theorem 4, the minimal generating set of $\operatorname{Gr}\mathrm{D}(A)$ given in Theorem 6, and the minimal generating set of $\operatorname{Der}_k(A)$. In order to do so, we need to look at $\Omega(w)$ when $|w| \in \{1, 2, 3, 4, 5\}$. But $\Omega(w) = \emptyset$ when $w = 3, 4, 5$, and in the remaining cases we have by definition

$$\begin{aligned}
\Omega(-5) &= \{0, 3, 4, 6, 7\},\\
\Omega(-4) &= \{0, 3, 5, 6\},\\
\Omega(-3) &= \{0, 4, 5\},\\
\Omega(-2) &= \{0, 3, 4\},\\
\Omega(-1) &= \{0, 3\},\\
\Omega(1) &= \{0\},\\
\Omega(2) &= \{0\}.
\end{aligned}$$

By Theorem 4, the ring $\mathrm{D}(A)$ is generated by the Euler derivation $E = t\partial$ and the following differential operators:

$$\begin{aligned}
P_{-5} &= t^{-5}E(E-3)(E-4)(E-6)(E-7)\\
&= \partial^5 - 10t^{-1}\partial^4 + 50t^{-2}\partial^3 - 140t^{-3}\partial^2 + 180t^{-4}\partial,\\
P_{-4} &= t^{-4}E(E-3)(E-5)(E-6) = \partial^4 - 8t^{-1}\partial^3 + 28t^{-2}\partial^2 - 40t^{-3}\partial,\\
P_{-3} &= t^{-3}E(E-4)(E-5) = \partial^3 - 6t^{-1}\partial^2 + 12t^{-2}\partial,\\
P_{-2} &= t^{-2}E(E-3)(E-4) = t\partial^3 - 4\partial^2 + 6t^{-1}\partial,\\
P_{-1} &= t^{-1}E(E-3) = t\partial^2 - 2\partial,\\
P_1 &= tE = t^2\partial,
\end{aligned}$$

$$\begin{aligned} P_2 &= t^2E = t^3\partial, \\ P_3 &= t^3, \\ P_4 &= t^4, \\ P_5 &= t^5. \end{aligned}$$

We remark that $P_{-5} = 1/3[P_{-3}, P_{-2}]$ and $P_{-4} = 1/3[P_{-3}, P_{-1}]$, so the generating set above is not minimal. In fact, $\mathrm{D}(A)$ is minimally generated by $\mathrm{D}^3(A)$. When we pass to the associated graded ring $\mathrm{Gr}\,\mathrm{D}(A)$, the corresponding set of generators

$$\left\{u^5, u^4, u^3, tu^3, tu^2, t^2u, t^3u, tu, t^3, t^4, t^5\right\}$$

is minimal. Finally, a minimal set of generators for $\mathrm{Der}_k(A)$ is given by $\{t\partial, t^2\partial, t^3\partial\}$.

Acknowledgments

This research has been supported by a Marie Curie Fellowship of the European Community program “Improving Human Research Potential and the Socio-economic Knowledge Base” under contract number HPMF-CT-2000-01099.

References

[1] W. Bruns, J. Herzog, Cohen–Macaulay Rings, in: Cambridge Stud. Adv. Math., Vol. 39, Cambridge University Press, 1993.
[2] M. Pia Cavaliere, G. Niesi, On monomial curves and Cohen–Macaulay type, Manuscripta Math. 42 (1983) 147–159.
[3] S.C. Coutinho, A Primer of Algebraic D-Modules, in: London Math. Soc. Student Texts, Vol. 33, Cambridge University Press, 1995.
[4] E. Eriksen, Graded D-modules on monomial curves, Ph.D. Thesis, University of Oslo, June 2000.
[5] A. Eriksson, Differential operators on some classes of rings, Ph.D. Thesis, Stockholm University, May 2000.
[6] R. Fröberg, The Frobenius number of some semigroups, Comm. Algebra 22 (14) (1994) 6021–6024.
[7] A. Graham Jones, Differential operators on semigroup algebras, Ph.D. Thesis, University of Sheffield, May 1991.
[8] G. Másson, Rings of differential operators and étale homomorphisms, Ph.D. Thesis, Massachusetts Institute of Technology, June 1991.
[9] I.M. Musson, Some rings of differential operators which are Morita-equivalent to the Weyl-algebra A_1, Proc. Amer. Math. Soc. 98 (1) (1986) 29–30.
[10] I.M. Musson, Rings of differential operators on invariant rings of tori, Trans. Amer. Math. Soc. 303 (2) (1987) 805–827.
[11] P.T. Perkins, Commutative subalgebras of the ring of differential operators on a curve, Pacific J. Math. 139 (2) (1989) 279–302.
[12] S.P. Smith, J.T. Stafford, Differential operators on an affine curve, Proc. London Math. Soc. (3) 56 (2) (1988) 229–259.