Glossary of real and complex analysis

This is a glossary of concepts and results in real analysis and complex analysis in mathematics.

See also: list of real analysis topics, list of complex analysis topics and glossary of functional analysis.

A

Abel

1. Abel sum
2. Abel integral

analytic capacity

analytic capacity.

analytic continuation

An analytic continuation of a holomorphic function is a unique holomorphic extension of the function (on a connected open subset of $\mathbb{C}$).

argument principle

argument principle

Ascoli

Ascoli's theorem says that an equicontinous bounded sequence of functions on a compact subset of $\mathbb{R}^n$ has a convergent subsequence with respect to the sup norm.

B

Borel

1. A Borel measure is a measure whose domain is the Borel σ-algebra.
2. The Borel σ-algebra on a topological space is the smallest σ-algebra containing all open sets.
3. Borel's lemma says that a given formal power series, there is a smooth function whose Taylor series coincides with the given series.

bounded

A subset A of a metric space (X, d) is bounded if there is some $C > 0$ such that $d(a, b) < C$ for all $a, b \in A$.

bump

A bump function is a nonzero compactly-supported smooth function, usually constructed using the exponential function.

C

Calderón

Calderón–Zygmund lemma

capacity

Capacity of a set is a notion in potential theory.

Carathéodory

Carathéodory's extension theorem

Cartan

Cartan's theorems A and B.

Cauchy

1. The Cauchy–Riemann equations are a system of differential equations such that a function satisfying it (in the distribution sense) is a holomorphic function.
2. Cauchy integral formula.
3. Cauchy residue theorem.
4. Cauchy's estimate.
5. The Cauchy principal value is, when possible, a number assigned to a function when the function is not integrable.
6. On a metric space, a sequence x_n is called a Cauchy sequence if $d(x_n, x_m) \to 0$; i.e., for each $\epsilon > 0$, there is an $N > 0$ such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$.

Cesàro

Cesàro summation is one way to compute a divergent series.

continuous

A function $f : X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is continuous if for any convergent sequence $x_n \to x$ in X, we have $f(x_n) \to f(x)$ in Y.

contour

The contour integral of a measurable function f over a piece-wise smooth curve $\gamma : [0, 1] \to \mathbb{C}$ is $\int_\gamma f\,dz := \int_0^1 \gamma^*(f\,dz)$.

converge

1. A sequence x_n in a topological space is said to converge to a point x if for each open neighborhood U of x, the set $\{n \mid x_n \notin U\}$ is finite.
2. A sequence x_n in a metric space is said to converge to a point x if for all $\epsilon > 0$, there exists an $N > 0$ such that for all $n > N$, we have $d(x_n, x) < \epsilon$.
3. A series $x_1 + x_2 + \cdots$ on a normed space (e.g., $\mathbb{R}^n$) is said to converge if the sequence of the partial sums $s_n := \sum_1^n x_j$ converges.

convolution

The convolution $f * g$ of two functions on a convex set is given by

$$(f * g)(x) = \int f(y - x) g(y)\,dy,$$

provided the integration converges.

Cousin

Cousin problems.

cutoff

cutoff function.

D

Dedekind
: A Dedekind cut is one way to construct real numbers.

derivative
: Given a map $f : E \to F$ between normed spaces, the derivative of f at a point *x* is a (unique) linear map $T : E \to F$ such that $\lim_{h \to 0} \|f(x+h) - f(x) - Th\|/\|h\| = 0$.

differentiable
: A map between normed space is differentiable at a point *x* if the derivative at *x* exists.

differentiation
: Lebesgue's differentiation theorem says: $f(x) = \lim_{r \to 0} \frac{1}{\mathrm{vol}(B(x,r))} \int_{B(x,r)} f \, d\mu$ for almost all *x*.

Dini
: Dini's theorem.

Dirac
: The Dirac delta function δ_0 on $\mathbb{R}^n$ is a distribution (so not exactly a function) given as $\langle \delta_0, \varphi \rangle = \varphi(0)$.

distribution
: A distribution is a type of a generalized function; precisely, it is a continuous linear functional on the space of test functions.

divergent
: A divergent series is a series whose partial sum does not converge. For example, $\sum_1^\infty \frac{1}{n}$ is divergent.

dominated
: Lebesgue's dominated convergence theorem says $\int f_n \, d\mu$ converges to $\int f \, d\mu$ if f_n is a sequence of measurable functions such that f_n converges to f pointwise and $|f_n| \le g$ for some integrable function g.

E

edge
: Edge-of-the-wedge theorem.

entire
: An entire function is a holomorphic function whose domain is the entire complex plane.

equicontinuous
: A set S of maps between fixed metric spaces is said to be equicontinuous if for each $\epsilon > 0$, there exists a $\delta > 0$ such that $\sup_{f \in S} d(f(x), f(y)) < \epsilon$ for all x, y with $d(x, y) < \delta$. A map f is uniformly continuous if and only if $\{f\}$ is equicontinuous.

F

Fatou

Fatou's lemma

Fourier

1. The Fourier transform of a function f on $\mathbb{R}^n$ is: (provided it makes sense)

$$\widehat{f}(\xi) = \int f(x)e^{-2\pi i x\cdot\xi}\,dx.$$

2. The Fourier transform $\widehat{f}$ of a distribution f is $\langle \widehat{f}, \varphi\rangle = \langle f, \widehat{\varphi}\rangle$. For example, $\widehat{\delta_0} = 1$ (Fourier's inversion formula).

G

Gauss

1. The Gauss–Green formula
2. Gaussian kernel

generalized

A generalized function is an element of some function space that contains the space of ordinary (e.g., locally integrable) functions. Examples are Schwartz's distributions and Sato's hyperfunctions.

H

Hardy-Littlewood maximal inequality

The Hardy-Littlewood maximal function of $f \in L^1(\mathbb{R}^n)$ is

$$Hf(x) := \sup_{r>0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f|.$$

The Hardy-Littlewood maximal inequality states that there is some constant C such that for all $f \in L^1(\mathbb{R}^n)$ and all $\alpha > 0$,

$$m(\{x : Hf(x) > \alpha\}) < \frac{C}{\alpha}\int_{\mathbb{R}^n} |f|.$$

Hardy space

Hardy space

Hartogs

1. Hartogs extension theorem
2. Hartogs's theorem on separate holomorphicity

harmonic

A function is harmonic if it satisfies the Laplace equation (in the distribution sense if the function is not twice differentiable).

Hausdorff

The Hausdorff–Young inequality says that the Fourier transformation $\widehat{\cdot} : L^p(\mathbb{R}^n) \to L^{p'}(\mathbb{R}^n)$ is a well-defined bounded operator when $1/p + 1/p' = 1$.

Heaviside

The Heaviside function is the function H on $\mathbb{R}$ such that $H(x) = 1,\ x \ge 0$ and $H(x) = 0,\ x < 0$.

Hilbert space

A Hilbert space is a real or complex inner product space that is a complete metric space with the metric induced by the inner product.

holomorphic function

A function defined on an open subset of $\mathbb{C}^n$ is holomorphic if it is complex differentiable. Equivalently, a function is holomorphic if it satisfies the Cauchy–Riemann equations (in the distribution sense if the function is not differentiable).

I

integrable

A measurable function f is said to be integrable if $\int |f| \, d\mu < \infty$.

integral

1. The integral of the indicator function on a measurable set is the measure (volume) of the set.
2. The integral of a measurable function is then defined by approximating the function by linear combinations of indicator functions.

isometry

An isometry between metric spaces (X, d_X) and (Y, d_Y) is a bijection $f : X \to Y$ that preserves the metric: $d_X(x, x') = d_Y(f(x), f(x'))$ for all $x, x' \in X$.

L

Lebesgue differentiation theorem

The Lebesgue differentiation theorem states that for locally integrable $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, the equalities

$$\lim_{r\to 0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y) - f(x)| \, dy = 0$$

and

$$\lim_{r\to 0} \frac{1}{m(B_r(x))} \int_{B_r(x)} f = f(x)$$

hold for almost every x. The set where they hold is called the Lebesgue set of f, and points in the Lebesgue set are called Lebesgue points.

Lebesgue integral

Lebesgue integral.

Lebesgue measure

Lebesgue measure.

Lelong

Lelong number.

Levi

Levi's problem asks to show a pseudoconvex set is a domain of holomorphy.

line integral

Line integral.

Liouville

Liouville's theorem says a bounded entire function is a constant function.

Lipschitz

1. A map f between metric spaces is said to be Lipschitz continuous if $\sup_{x \neq y} \frac{d(f(x), f(y))}{d(x, y)} < \infty$.
2. A map is locally Lipschitz continuous if it is Lipschitz continuous on each compact subset.

M

maximum

The maximum principle says that a maximum value of a harmonic function in a connected open set is attained on the boundary.

measurable function

A measurable function is a structure-preserving function between measurable spaces in the sense that the preimage of any measurable set is measurable.

measurable set

A measurable set is an element of a σ-algebra.

measurable space

A measurable space consists of a set and a σ-algebra on that set which specifies what sets are measurable.

measure

A measure is a function on a measurable space that assigns to each measurable set a number representing its measure or size. Specifically, if X is a set and Σ is a σ-algebra on X, then a set-function μ from Σ to the extended real number line is called a measure if the following conditions hold:

- **Non-negativity**: For all $E \in \Sigma, \;\; \mu(E) \geq 0.$
- $\mu(\varnothing) = 0.$
- **Countable additivity** (or σ-additivity): For all countable collections $\{E_k\}_{k=1}^{\infty}$ of pairwise disjoint sets in Σ,

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k).$$

measure space

A measure space consists of a measurable space and a measure on that measurable space.

meromorphic

A meromorphic function is an equivalence class of functions that are locally fractions of holomorphic functions.

method of stationary phase

The method of stationary phase.

metric space

A metric space is a set X equipped with a function $d : X \times X \to \mathbb{R}_{\geq 0}$, called a metric, such that (1) $d(x, y) = 0$ iff $x = y$, (2) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$, (3) $d(x, y) = d(y, x)$ for all $x, y \in X$.

microlocal

The notion microlocal refers to a consideration on the cotangent bundle to a space as opposed to that on the space itself. Explicitly, it amounts to considering functions on both points and momenta; not just functions on points.

Minkowski

Minkowski inequality

monotone

Monotone convergence theorem.

Morera

Morera's theorem says a function is holomorphic if the integrations of it over arbitrary closed loops are zero.

Morse

Morse function.

N

Nash

1. Nash function.
2. Nash–Moser theorem.

Nevanlinna theory

Nevanlinna theory concerns meromorphic functions.

net

A net is a generalization of a sequence.

normed vector space

A normed vector space, also called a normed space, is a real or complex vector space V on which a norm is defined. A norm is a map $\|\cdot\| : V \to \mathbb{R}$ satisfying four axioms:

1. Non-negativity: for every $x \in V$, $\|x\| \geq 0$.
2. Positive definiteness: for every $x \in V$, $\|x\| = 0$ if and only if x is the zero vector.
3. Absolute homogeneity: for every scalar λ and $x \in V$,

$$\|\lambda x\| = |\lambda| \, \|x\|$$

4. Triangle inequality: for every $x \in V$ and $y \in V$,

$$\|x + y\| \leq \|x\| + \|y\|.$$

O

Oka

Oka's coherence theorem says the sheaf $\mathcal{O}_{\mathbb{C}^n}$ of holomorphic functions is coherent.

open

The open mapping theorem (complex analysis)

oscillatory integral

An oscillatory integral can give a sense to a formal integral expression like $\delta_0(x) = \int e^{2\pi i x\cdot\xi}\, d\xi$.

P

Paley
: Paley–Wiener theorem

phase
: The phase space to a configuration space X (in classical mechanics) is the cotangent bundle T^*X to X.

plurisubharmonic
: A function f on an open subset $U \subset \mathbb{C}$ is said to be plurisubharmonic if $t \mapsto f(z+tw)$ is subharmonic for t in a neighborhood of zero in $\mathbb{C}$ and points z, w in U.

Poisson
: Poisson kernel

power series
: A power series is informally a polynomial of infinite degree; i.e., $\sum_{n=1}^{\infty} a_n x^n$.

pseudoconex
: A pseudoconvex set is a generalization of a convex set.

R

Radon measure
: Let X be a locally compact Hausdorff space and let I be a positive linear functional on the space of continuous functions with compact support $C_c(X)$. Positivity means that $I(f) \geq 0$ if $f \geq 0$. There exist Borel measures μ on X such that $I(f) = \int f\, d\mu$ for all $f \in C_c(X)$. A Radon measure on X is a Borel measure that is finite on all compact sets, outer regular on all Borel sets, and inner regular on all open sets. These conditions guarantee that there exists a *unique* Radon measure μ on X such that $I(f) = \int f\, d\mu$ for all $f \in C_c(X)$.

real-analytic
: A real-analytic function is a function given by a convergent power series.

Rellich
: Rellich's lemma tells when an inclusion of a Sobolev space to another Sobolev space is a compact operator.

Riemann
: 1. The Riemann integral of a function is either the upper Riemann sum or the lower Riemann sum when the two sums agree.
2. The Riemann zeta function is a (unique) analytic continuation of the function $z \mapsto \sum_1^{\infty} \frac{1}{n^z}$, $\mathrm{Re}(z) > 1$ (it's more traditional to write s for z).

3. The Riemann hypothesis, still a conjecture, says each nontrivial zero of the Riemann zeta function has real part equal to $\frac{1}{2}$.
4. Riemann's existence theorem.

Runge

1. Runge's approximation theorem.
2. Runge domain.

S

Sato
: Sato's hyperfunction, a type of a generalized function.

Schwarz
: A Schwarz function is a function that is both smooth and rapid-decay.

semianalytic
: The notion of semianalytic is an analog of semialgebraic.

semicontinuous
: A semicontinuous function.

sequence
: A sequence on a set X is a map $\mathbb{N} \to X$.

series
: A series is informally an infinite summation process $x_1 + x_2 + \cdots$. Thus, mathematically, specifying a series is the same as specifying the sequence of the terms in the series. The difference is that, when considering a series, one is often interested in whether the sequence of partial sums $s_n := x_1 + \cdots + x_n$ converges or not and if so, to what.

σ-algebra
: A σ-algebra on a set is a nonempty collection of subsets closed under complements, countable unions, and countable intersections.

Stieltjes
: Stieltjes–Vitali theorem

Stone–Weierstrass theorem
: The Stone–Weierstrass theorem is any one of a number of related generalizations of the Weierstrass approximation theorem, which states that any continuous real-valued function defined on a closed interval can be uniformly approximated by polynomials. Let X be a compact Hausdorff space and let $C(X, \mathbb{R})$ have the uniform metric. One version of the Stone–Weierstrass theorem states that if $\mathcal{A}$ is a closed subalgebra of $C(X, \mathbb{R})$ that separates points and contains a nonzero constant function, then in fact $\mathcal{A} = C(X, \mathbb{R})$. If a subalgebra is not closed, taking the closure and applying the previous version of the Stone–Weierstrass theorem reveals a different version of the theorem: if $\mathcal{A}$ is a subalgebra of $C(X, \mathbb{R})$ that separates points and contains a nonzero constant function, then $\mathcal{A}$ is dense in $C(X, \mathbb{R})$.

subanalytic
: subanalytic.

subharmonic
: A twice continuously differentiable function f is said to be subharmonic if $\Delta f \geq 0$ where Δ is the Laplacian. The subharmonicity for a more general function is defined by a limiting process.

subsequence

A subsequence of a sequence is another sequence contained in the sequence; more precisely, it is a composition $\mathbb{N} \xrightarrow{j} \mathbb{N} \xrightarrow{x} X$ where j is a strictly increasing injection and x is the given sequence.

support

1. The support of a function is the closure of the set of points where the function does not vanish.
2. The support of a distribution is the support of it in the sense in sheaf theory.

T

Tauberian

Tauberian theory is a set of results (called tauberian theorems) concerning a divergent series; they are sort of converses to abelian theorems but with some additional conditions.

Taylor

Taylor expansion

tempered

A tempered distribution is a distribution that extends to a continuous linear functional on the space of Schwarz functions.

test

A test function is a compactly-supported smooth function.

U

uniform

1. A sequence of maps $f_n : X \to E$ from a topological space to a normed space is said to converge uniformly to $f : X \to E$ if $\sup \|f_n - f\| \to 0$.
2. A map between metric spaces is said to be uniformly continuous if for each $\epsilon > 0$, there exist a $\delta > 0$ such that $d(f(x), f(y)) < \epsilon$ for all x, y with $d(x, y) < \delta$.

V

Vitali covering lemma

The Vitali covering lemma states that if $\mathcal{C}$ is a collection of open balls in $\mathbb{R}^n$ and

$$c < m\left(\bigcup_{B \in \mathcal{C}} B\right),$$

then there exists a finite number of balls $B_1, \dots, B_n \in \mathcal{C}$ such that

$$3^n \sum_{j=1}^{n} m(B_j) > c.$$

W

Weierstrass

1. Weierstrass preparation theorem.
2. Weierstrass M-test.

Weyl

1. Weyl calculus.
2. Weyl quantization.

Whitney

1. The Whitney extension theorem gives a necessary and sufficient condition for a function to be extended from a closed set to a smooth function on the ambient space.
2. Whitney stratification

References

- Grauert, Hans; Remmert, Reinhold (1984). *Coherent Analytic Sheaves* (https://link.springer.com/book/10.1007/978-3-642-69582-7). Grundlehren der mathematischen Wissenschaften. Vol. 265. Springer. doi:10.1007/978-3-642-69582-7 (https://doi.org/10.1007%2F978-3-642-69582-7). ISBN 978-3-642-69584-1.
- Halmos, Paul R. (1974) [1950], *Measure Theory* (https://archive.org/details/measuretheory00halm), Graduate Texts in Mathematics, vol. 18, New York, Heidelberg, Berlin: Springer-Verlag, ISBN 978-0-387-90088-9, MR 0033869 (https://mathscinet.ams.org/mathscinet-getitem?mr=0033869), Zbl 0283.28001 (https://zbmath.org/?format=complete&q=an:0283.28001)
- Hörmander, Lars (1983), *The analysis of linear partial differential operators I*, Grundl. Math. Wissenschaft., vol. 256, Springer, doi:10.1007/978-3-642-96750-4 (https://doi.org/10.1007%2F978-3-642-96750-4), ISBN 3-540-12104-8, MR 0717035 (https://mathscinet.ams.org/mathscinet-getitem?mr=0717035).
- Hörmander, Lars (1966). *An Introduction to Complex Analysis in Several Variables*. Van Nostrand.
- Rudin, Walter (1976). *Principles of Mathematical Analysis*. Walter Rudin Student Series in Advanced Mathematics (3rd ed.). McGraw-Hill. ISBN 9780070542358.
- Rudin, Walter (1986). *Real and Complex Analysis (International Series in Pure and Applied Mathematics)*. McGraw-Hill. ISBN 978-0-07-054234-1.
- Folland, Gerald B. (2007). *Real Analysis: Modern Techniques and Their Applications* (2nd ed.). Wiley.
- Jost, Jürgen (1998). *Postmodern Analysis*. Springer.

- Ahlfors, Lars V. (1978), *Complex analysis. An introduction to the theory of analytic functions of one complex variable*, International Series in Pure and Applied Mathematics (3rd ed.), McGraw-Hill

Further reading

- Semiclassical Microlocal Analysis（2020 Fall) (http://staff.ustc.edu.cn/~wangzuoq/Courses/20F-SMA/index.html) by 王作勤 (wangzuoq)