Glossary of functional analysis

This is a glossary for the terminology in a mathematical field of functional analysis.

Throughout the article, unless stated otherwise, the base field of a vector space is the field of real numbers or that of complex numbers. Algebras are not assumed to be unital.

See also: List of Banach spaces, glossary of real and complex analysis.

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*-homomorphism between involutive Banach algebras is an algebra homomorphism preserving *.

A

abelian

Synonymous with "commutative"; e.g., an abelian Banach algebra means a commutative Banach algebra.

Anderson–Kadec

The Anderson–Kadec theorem says a separable infinite-dimensional Fréchet space is isomorphic to $\mathbb{R}^{\mathbb{N}}$.

Alaoglu

Alaoglu's theorem states that the closed unit ball in a normed space is compact in the weak-* topology.

adjoint

The adjoint of a bounded linear operator $T: H_1 \to H_2$ between Hilbert spaces is the bounded linear operator $T^*: H_2 \to H_1$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for each $x \in H_1, y \in H_2$.

approximate identity

In a not-necessarily-unital Banach algebra, an approximate identity is a sequence or a net $\{u_i\}$ of elements such that $u_i x \to x, x u_i \to x$ as $i \to \infty$ for each *x* in the algebra.

approximation property

A Banach space is said to have the approximation property if every compact operator is a limit of finite-rank operators.

B

Baire

The Baire category theorem states that a complete metric space is a Baire space; if U_i is a sequence of open dense subsets, then $\cap_1^\infty U_i$ is dense.

Banach

1. A Banach space is a normed vector space that is complete as a metric space.
2. A Banach algebra is a Banach space that has a structure of a possibly non-unital associative algebra such that

$$\|xy\| \leq \|x\|\|y\| \text{ for every } x, y \text{ in the algebra.}$$

3. A Banach disc is a continuous linear image of a unit ball in a Banach space.

balanced

A subset *S* of a vector space over real or complex numbers is balanced if $\lambda S \subset S$ for every scalar λ of length at most one.

barrel

1. A barrel in a topological vector space is a subset that is closed, convex, balanced and absorbing.
2. A topological vector space is barrelled if every barrell is a neighborhood of zero (that is, contains an open neighborhood of zero).

Bessel

Bessel's inequality states: given an orthonormal set *S* and a vector *x* in a Hilbert space,

$$\sum_{u \in S} |\langle x, u\rangle|^2 \leq \|x\|^2,$$ [1]

where the equality holds if and only if *S* is an orthonormal basis; i.e., maximal orthonormal set.

bipolar

bipolar theorem.

bounded

A bounded operator is a linear operator between Banach spaces for which the image of the unit ball is bounded.

bornological

A bornological space.

Birkhoff orthogonality

Two vectors *x* and *y* in a normed linear space are said to be **Birkhoff orthogonal** if $\|x + \lambda y\| \geq \|x\|$ for all scalars λ. If the normed linear space is a Hilbert space, then it is equivalent to the usual orthogonality.

Borel

Borel functional calculus

C

c

c space.

Calkin

The Calkin algebra on a Hilbert space is the quotient of the algebra of all bounded operators on the Hilbert space by the ideal generated by compact operators.

Cauchy–Schwarz inequality

The Cauchy–Schwarz inequality states: for each pair of vectors x, y in an inner-product space,

$$|\langle x, y\rangle| \leq \|x\|\|y\|.$$

closed

1. The closed graph theorem states that a linear operator between Banach spaces is continuous (bounded) if and only if it has closed graph.
2. A closed operator is a linear operator whose graph is closed.
3. The closed range theorem says that a densely defined closed operator has closed image (range) if and only if the transpose of it has closed image.

commutant

1. Another name for "centralizer"; i.e., the commutant of a subset *S* of an algebra is the algebra of the elements commuting with each element of *S* and is denoted by S'.
2. The von Neumann double commutant theorem states that a nondegenerate *-algebra $\mathfrak{M}$ of operators on a Hilbert space is a von Neumann algebra if and only if $\mathfrak{M}'' = \mathfrak{M}$.

compact

A compact operator is a linear operator between Banach spaces for which the image of the unit ball is precompact.

Connes

Connes fusion.

C*

A C* algebra is an involutive Banach algebra satisfying $\|x^*x\| = \|x^*\|\|x\|$.

convex

A locally convex space is a topological vector space whose topology is generated by convex subsets.

cyclic

Given a representation (π, V) of a Banach algebra A, a cyclic vector is a vector $v \in V$ such that $\pi(A)v$ is dense in V.

D

dilation

dilation (operator theory).

direct

Philosophically, a direct integral is a continuous analog of a direct sum.

Douglas

Douglas' lemma

Dunford

Dunford–Schwartz theorem

dual

1. The continuous dual of a topological vector space is the vector space of all the continuous linear functionals on the space.
2. The algebraic dual of a topological vector space is the dual vector space of the underlying vector space.

E

Eidelheit

A theorem of Eidelheit.

F

factor

A factor is a von Neumann algebra with trivial center.

faithful

A linear functional ω on an involutive algebra is faithful if $\omega(x^*x) \neq 0$ for each nonzero element x in the algebra.

Fréchet

A Fréchet space is a topological vector space whose topology is given by a countable family of seminorms (which makes it a metric space) and that is complete as a metric space.

Fredholm

A Fredholm operator is a bounded operator such that it has closed range and the kernels of the operator and the adjoint have finite-dimension.

G

Gelfand

1. The Gelfand–Mazur theorem states that a Banach algebra that is a division ring is the field of complex numbers.
2. The Gelfand representation of a commutative Banach algebra A with spectrum $\Omega(A)$ is the algebra homomorphism $F : A \to C_0(\Omega(A))$, where $C_0(X)$ denotes the algebra of continuous functions on X vanishing at infinity, that is given by $F(x)(\omega) = \omega(x)$. It is a *-preserving isometric isomorphism if A is a commutative C*-algebra.

Grothendieck

1. Grothendieck's inequality.
2. Grothendieck's factorization theorem.

H

Hahn–Banach

The Hahn–Banach theorem states: given a linear functional ℓ on a subspace of a complex vector space *V*, if the absolute value of ℓ is bounded above by a seminorm on *V*, then it extends to a linear functional on *V* still bounded by the seminorm. Geometrically, it is a generalization of the hyperplane separation theorem.

Heine

A topological vector space is said to have the Heine–Borel property if every closed and bounded subset is compact. Riesz's lemma says a Banach space with the Heine–Borel property must be finite-dimensional.

Hilbert

1. A Hilbert space is an inner product space that is complete as a metric space.
2. In the Tomita–Takesaki theory, a (left or right) Hilbert algebra is a certain algebra with an involution.

Hilbert–Schmidt

1. The Hilbert–Schmidt norm of a bounded operator T on a Hilbert space is $\sum_i \|Te_i\|^2$ where $\{e_i\}$ is an orthonormal basis of the Hilbert space.
2. A Hilbert–Schmidt operator is a bounded operator with finite Hilbert–Schmidt norm.

I

index

1. The index of a Fredholm operator $T : H_1 \to H_2$ is the integer $\dim(\ker(T^*)) - \dim(\ker(T))$.
2. The Atiyah–Singer index theorem.

index group

The index group of a unital Banach algebra is the quotient group $G(A)/G_0(A)$ where $G(A)$ is the unit group of *A* and $G_0(A)$ the identity component of the group.

inner product

1. An inner product on a real or complex vector space V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ such that for each $v, w \in V$, (1) $x \mapsto \langle x, v \rangle$ is linear and (2) $\langle v, w \rangle = \overline{\langle w, v \rangle}$ where the bar means complex conjugate.
2. An inner product space is a vector space equipped with an inner product.

involution

1. An involution of a Banach algebra *A* is an isometric endomorphism $A \to A,\ x \mapsto x^*$ that is conjugate-linear and such that $(xy)^* = (yx)^*$.
2. An involutive Banach algebra is a Banach algebra equipped with an involution.

isometry

A linear isometry between normed vector spaces is a linear map preserving norm.

K

Köthe

A Köthe sequence space. For now, see https://mathoverflow.net/questions/361048/on-k%C3%B6the-sequence-spaces

Krein–Milman

The Krein–Milman theorem states: a nonempty compact convex subset of a locally convex space has an extremal point.

Krein–Smulian

Krein–Smulian theorem

L

Linear

Linear Operators is a three-value book by Dunford and Schwartz.

Locally convex algebra

A locally convex algebra is an algebra whose underlying vector space is a locally convex space and whose multiplication is continuous with respect to the locally convex space topology.

M

Mazur
Mazur–Ulam theorem.

Montel
Montel space.

N

nondegenerate
A representation (π, V) of an algebra A is said to be nondegenerate if for each vector $v \in V$, there is an element $a \in A$ such that $\pi(a)v \neq 0$.

noncommutative
1. noncommutative integration
2. noncommutative torus

norm
1. A norm on a vector space *X* is a real-valued function $\|\cdot\| : X \to \mathbb{R}$ such that for each scalar a and vectors x, y in X, (1) $\|ax\| = |a|\|x\|$, (2) (triangular inequality) $\|x+y\| \leq \|x\| + \|y\|$ and (3) $\|x\| \geq 0$ where the equality holds only for $x = 0$.
2. A normed vector space is a real or complex vector space equipped with a norm $\|\cdot\|$. It is a metric space with the distance function $d(x, y) = \|x - y\|$.

normal
An operator is normal if it and its adjoint commute.

nuclear
See nuclear operator.

O

one
A one parameter group of a unital Banach algebra *A* is a continuous group homomorphism from $(\mathbb{R}, +)$ to the unit group of *A*.

open
The open mapping theorem says a surjective continuous linear operator between Banach spaces is an open mapping.

orthonormal
1. A subset *S* of a Hilbert space is orthonormal if, for each *u*, *v* in the set, $\langle u, v \rangle$ = 0 when $u \neq v$ and $= 1$ when $u = v$.
2. An orthonormal basis is a maximal orthonormal set (note: it is *not* necessarily a vector space basis.)

orthogonal
1. Given a Hilbert space *H* and a closed subspace *M*, the orthogonal complement of *M* is the closed subspace $M^{\perp} = \{x \in H | \langle x, y \rangle = 0, y \in M\}$.

2. In the notations above, the orthogonal projection P onto *M* is a (unique) bounded operator on *H* such that $P^2 = P, P^* = P, \mathrm{im}(P) = M, \ker(P) = M^\perp$.

P

Parseval

Parseval's identity states: given an orthonormal basis *S* in a Hilbert space, $\|x\|^2 = \sum_{u \in S} |\langle x, u\rangle|^2$.[1]

positive

A linear functional ω on an involutive Banach algebra is said to be positive if $\omega(x^* x) \geq 0$ for each element x in the algebra.

predual

predual.

projection

An operator *T* is called a projection if it is an idempotent; i.e., $T^2 = T$.

Q

quasitrace

Quasitrace.

R

Radon

See Radon measure.

Riesz decomposition

Riesz decomposition.

Riesz's lemma

Riesz's lemma.

reflexive

A reflexive space is a topological vector space such that the natural map from the vector space to the second (topological) dual is an isomorphism.

resolvent

The resolvent of an element *x* of a unital Banach algebra is the complement in $\mathbb{C}$ of the spectrum of *x*.

Ryll-Nardzewski

Ryll-Nardzewski fixed-point theorem.

S

Schauder

Schauder basis.

Schatten
: Schatten class

selection
: Michael selection theorem.

self-adjoint
: A self-adjoint operator is a bounded operator whose adjoint is itself.

separable
: A separable Hilbert space is a Hilbert space admitting a finite or countable orthonormal basis.

spectrum
: 1. The spectrum of an element *x* of a unital Banach algebra is the set of complex numbers λ such that $x - \lambda$ is not invertible.
2. The spectrum of a commutative Banach algebra is the set of all characters (a homomorphism to $\mathbb{C}$) on the algebra.

spectral
: 1. The spectral radius of an element *x* of a unital Banach algebra is $\sup_\lambda |\lambda|$ where the sup is over the spectrum of *x*.
2. The spectral mapping theorem states: if *x* is an element of a unital Banach algebra and *f* is a holomorphic function in a neighborhood of the spectrum $\sigma(x)$ of *x*, then $f(\sigma(x)) = \sigma(f(x))$, where $f(x)$ is an element of the Banach algebra defined via the Cauchy's integral formula.

state
: A state is a positive linear functional of norm one.

symmetric
: A linear operator *T* on a pre-Hilbert space is symmetric if $(Tx, y) = (x, Ty)$.

T

tensor product
: 1. See topological tensor product. Note it is still somewhat of an open problem to define or work out a correct tensor product of topological vector spaces, including Banach spaces.
2. A projective tensor product.

topological
: 1. A topological vector space is a vector space equipped with a topology such that (1) the topology is Hausdorff and (2) the addition $(x, y) \mapsto x + y$ as well as scalar multiplication $(\lambda, x) \mapsto \lambda x$ are continuous.
2. A linear map $f : E \to F$ is called a topological homomorphism if $f : E \to \mathrm{im}(f)$ is an open mapping.
3. A sequence $\cdots \to E_{n-1} \to E_n \to E_{n+1} \to \cdots$ is called topologically exact if it is an exact sequence on the underlying vector spaces and, moreover, each $E_n \to E_{n+1}$ is a topological homomorphism.

U

ultraweak
: ultraweak topology.

unbounded operator

An unbounded operator is a partially defined linear operator, usually defined on a dense subspace.

uniform boundedness principle

The uniform boundedness principle states: given a set of operators between Banach spaces, if $\sup_T |Tx| < \infty$, sup over the set, for each *x* in the Banach space, then $\sup_T \|T\| < \infty$.

unitary

1. A unitary operator between Hilbert spaces is an invertible bounded linear operator such that the inverse is the adjoint of the operator.
2. Two representations $(\pi_1, H_1), (\pi_2, H_2)$ of an involutive Banach algebra *A* on Hilbert spaces H_1, H_2 are said to be unitarily equivalent if there is a unitary operator $U : H_1 \to H_2$ such that $\pi_2(x)U = U\pi_1(x)$ for each *x* in *A*.

V

von Neumann

1. A von Neumann algebra.
2. von Neumann's theorem.
3. Von Neumann's inequality.

W

W*

A W*-algebra is a C*-algebra that admits a faithful representation on a Hilbert space such that the image of the representation is a von Neumann algebra.

References

1. Here, the part of the assertion is $\sum_{u \in S} \cdots$ is well-defined; i.e., when *S* is infinite, for countable totally ordered subsets $S' \subset S$, $\sum_{u \in S'} \cdots$ is independent of S' and $\sum_{u \in S} \cdots$ denotes the common value.

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Further reading

- Antony Wassermann's lecture notes at http://iml.univ-mrs.fr/~wasserm/
- Jacob Lurie's lecture notes on a von Neumann algebra at https://www.math.ias.edu/~lurie/261y.html
- https://mathoverflow.net/questions/408415/takesaki-theorem-2-6