Phases of String Stars in the Presence of a Spatial Circle

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ABSTRACT: In string theory, black holes are expected to transition into string stars as their Hawking temperature approaches the Hagedorn temperature. We study string stars and their phase transitions in the Euclidean spacetime $\mathbb{R}^d \times \mathbb{S}^1_{\tau} \times \mathbb{S}^1_{z}$. Using the Horowitz-Polchinski (HP) effective field theory, we discover novel solutions for d=2. The uniform string star exhibits a scaling symmetry that results in the absence of a critical point for its transition into the non-uniform solution. For d=4, we show that quartic corrections to the effective action resolve the mass degeneracy of uniform string stars. At d=5, we find that as non-uniformity increases, the quartic terms become significant (while higher-order terms remain negligible) and reverse the direction of temperature variation, leading to a swallowtail-type phase diagram in the canonical ensemble. Extending the quartic-corrected EFT to d=6, we find that string stars with small non-uniformity dominate the microcanonical ensemble but not the canonical ensemble, similar to the d=5 case. However, in the microcanonical ensemble, the uniform string star is anomalously (un)stable when the spatial circle is larger (smaller) than the critical size.

Contents

1	Introduction and summary	1
2	Horowitz-Polchinski EFT in $\mathbb{R}^d imes \mathbb{S}^1_z$	5
	2.1 Uniform solutions	7
	2.2 Non-uniform solutions	8
	2.3 $d=2$	10
	2.4 $d = 3$ and $d = 4$	13
3	Beyond HP EFT	15
	3.1 More on $d=3$ and $d=4$	15
	3.2 d = 5	17
	3.3 $d = 6$	20
4	Discussion	22

1 Introduction and summary

A Schwarzschild black hole is a nontrivial vacuum solution of general relativity (GR) that exhibits spherical symmetry. The black hole features an event horizon with a radius of r_H and a singularity at the center. Near the singularity, curvatures blow up, leading to the breakdown of GR. Due to this, the Schwarzschild black hole remains incompletely understood.

In the Euclidean signature, where the spacetime becomes $\mathbb{R}^d \times \mathbb{S}^1_{\tau}$ via a Wick rotation $\tau = it$, the black hole solution is under good control. In particular, the Euclidean time circle \mathbb{S}^1_{τ} is capped off at the horizon radius r_H , effectively removing the singularity. This can be seen from the metric,

$$ds^{2} = \left(1 - \left(\frac{r_{H}}{r}\right)^{d-2}\right)d\tau^{2} + \frac{dr^{2}}{1 - \left(\frac{r_{H}}{r}\right)^{d-2}} + r^{2}d\Omega_{d-1}^{2}.$$
 (1.1)

In (1.1), the circumference of the τ circle corresponds to the inverse Hawking temperature, which is related to the horizon radius by

$$r_H = \frac{(d-2)\beta}{4\pi} \ . \tag{1.2}$$

This relation also lets \mathbb{S}^1_{τ} smoothly capped off at r_H .

Embedded in string theory, the Schwarzschild metric (1.1) receives string corrections [1, 2]. When the length scale of the black hole is much larger than the string length, i.e., $\beta \gg \ell_s \equiv \sqrt{\alpha'}$, the string corrections are suppressed. However, as β approaches ℓ_s , GR breaks down, making the black hole solution less well understood.

When $\beta \sim \ell_s$, it is widely believed that the black hole transitions into a state of highly excited fundamental strings [3–5] (see also [6–11] for relevant works). Furthermore, near a string-scaled temperature known as the "Hagedorn temperature", denoted by β_H , there exists an effective field theory (EFT) proposed by Horowitz and Polchinski [12]. Similar to GR, the Horowitz-Polchinski (HP) EFT admits classical solutions, commonly referred to as "string stars". In particular, the spherically symmetric HP solution is of interest as an analog to the Schwarzschild black hole. For recent developments, see [13–29].

So far, we have assumed that the space is non-compact, i.e., \mathbb{R}^d . However, to explain certain physical phenomena, it is often useful to include compact spatial dimensions as well (e.g., the recent dark dimension scenario proposed within the Swampland program [30]). Therefore, understanding the solutions in the presence of compact dimensions becomes crucial. In this paper, we consider an additional spatial circle \mathbb{S}^1_z with periodicity $z \sim z + L$, focusing on the case where $L \gg \ell_s$.

The introduction of an extra spatial circle gives rise to new solutions, especially those that are non-uniform along \mathbb{S}^1_z . Consequently, there is a competition between uniform and non-uniform solutions. In the case of black holes, where the Schwarzschild metric (1.1) is uplifted to a black string uniformly wrapped around the z circle, the phase diagram and transition between uniform and non-uniform solutions have been extensively studied (see, for example, [31–42]). Notably, in Lorentzian spacetime, it was discovered by Gregory and Laflamme that the uniform black string becomes unstable to non-uniform perturbations when the spatial circle is large compared to the horizon radius [43, 44].

Unlike the black holes, the Horowitz-Polchinski (HP) solutions have no natural extensions in Lorentzian signature, making it ambiguous to discuss their Gregory-Laflamme instability. However, in Euclidean spacetime, we can analogously examine the instability of the free energy or entropy [16]. Specifically, by finding the time-independent non-uniform perturbation at the leading order, the critical point of the instability towards non-uniformity can be determined. Furthermore, the phase diagram can be constructed by comparing the free energy or the entropy of the uniform and non-uniform solutions.

In [26], we use the first-order non-uniform perturbation to determine the critical point of instability for the HP solution towards non-uniformity. By analyzing the second-order non-uniform perturbation, we further determine the order of phase transition at the critical point. It is shown that, in the canonical ensemble, at the critical temperature, the uniform HP solution undergoes a second-order phase transition and becomes non-uniform for $d < d_c$, where d_c is a critical dimension slightly

larger than 4. In the microcanonical ensemble, for d < 4, the uniform HP solution undergoes a first-order phase transition at the critical mass, with a discontinuous change in entropy. We also sketch the phase diagram for the microcanonical ensemble. Subsequently, [27] obtained the non-uniform HP solutions beyond perturbative analysis and presented the phase diagrams of both the canonical and microcanonical ensembles for d = 3, 4 and 5.

In this paper, we continue the study of string star solutions in the presence of the spatial circle \mathbb{S}^1_z . In section 2, we analyze the Horowitz-Polchinski EFT. For d=2, where the standard HP solution ceases to exist, we find novel solutions in which the Euclidean time circle does not asymptotically converge to a constant, namely β . Although β no longer has a physical interpretation, such as representing the inverse temperature, it serves as a parameter for the non-uniform solutions at a fixed length L of \mathbb{S}^1_z . In contrast, the uniform solutions are not determined by β due to a scaling transformation that relates different solutions. As a result, the uniform and non-uniform solutions are not connected with each other.

When we treat d as a continuous parameter and increase it slightly above 2, standard HP solutions emerge in which the τ circle converges to a length of β . In this case, we start with a localized higher-dimensional HP solution and gradually decrease β . We find that the non-uniform solution eventually transitions into a uniform solution at a critical temperature. Phase diagrams of both the canonical and microcanonical ensembles are presented for d between 2 and 3. In the canonical ensemble, the non-uniform phase connects continuously to the uniform phase at the critical point, indicating a second-order phase transition. This finding is consistent with the perturbative analysis in [26].

In the microcanonical ensemble, there are two branches of non-uniform HP solutions that meet at a mass distinct from the critical point. The lower non-uniform branch extends to the critical mass, where it intersects with the uniform branch. Together, the uniform and non-uniform solutions form a swallowtail-type phase diagram, indicating a first-order phase transition consistent with [26].

For d=3, we recover the phase diagrams presented in [27]. In the canonical ensemble, as β decreases, the non-uniform HP solution undergoes a second-order phase transition into a uniform HP solution, similar to the behavior observed for 2 < d < 3. However, in the microcanonical ensemble, the uniform solution consistently has a higher entropy than the non-uniform solution, which contrasts with the case of 2 < d < 3. Consequently, the uniform HP solution does not transition into a non-uniform HP solution as the mass varies.

Nevertheless, this does not rule out the possibility of the HP solution becoming non-uniform at the critical mass. Indeed, as proposed in [26], it is likely to undergo a first-order phase transition into a localized black hole at this point.

Similarly, for d = 4, the free energy of the non-uniform HP solution changes continuously to that of the uniform phase at the critical temperature. However, in

the microcanonical ensemble, the mass of the uniform HP solution remains fixed as β varies. To resolve the degeneracy of these solutions, we incorporate the quartic terms in the effective action¹, as shown in section 3. As a result, we find that the mass of the uniform string star increases with β .

In section 3, we also use the quartic-corrected effective action to study string stars in $\mathbb{R}^5 \times \mathbb{S}^1_z$. For the case of noncompact spatial dimensions, i.e., \mathbb{R}^d , it is known that the original HP EFT breaks down as d approaches 6 when treated as a continuous parameter. This can be demonstrated through a scaling analysis [18] (see also [26] for another derivation based on the free energy). However, the EFT can be restored by merely incorporating the quartic terms into the HP effective action. These suggest that in $\mathbb{R}^5 \times \mathbb{S}^1_z$, as the string star becomes increasingly non-uniform and approaches a localized higher-dimensional solution, the quartic terms become significant and must be taken into account.

By analyzing the quartic-corrected EFT, we find that for d=5, the transition between the uniform and non-uniform phases is first-order in the canonical ensemble, whereas it is second-order in the microcanonical ensemble, contrasting with the behavior observed for $d \leq 4$. These results are also consistent with the perturbative analysis in [26]. Notably, if the quartic terms are neglected, the free energy of the non-uniform phase is always higher than that of the uniform phase [27], leading to a puzzle regarding the end state of the phase transition from the uniform phase at the critical point. Our findings show that when the quartic terms are included, an additional branch of non-uniform solutions emerges, with a lower free energy than the uniform phase at the critical point, thereby resolving the puzzle.

Finally, in $\mathbb{R}^{d>6}$, the quartic-corrected effective action becomes invalid because the field amplitudes are not small at any temperature, and higher-order terms are no longer suppressed [18]. To gain further insights into higher-dimensional string stars, we study the quartic-corrected EFT in $\mathbb{R}^6 \times \mathbb{S}^1_z$ and examine the behavior of the solution as its non-uniformity increases. Specifically, we begin with the uniform solution and introduce non-uniform perturbations, allowing us to generate the entire non-uniform branch.

In the canonical ensemble, we find that the non-uniform branch emerges from a critical point on the uniform branch. Near this critical point, the non-uniform solutions correspond to inverse temperatures below the critical value and possess higher free energies than the uniform solutions. This means that below the critical inverse temperature, the uniform phase is stable against non-uniformity, and the phase transition at the critical temperature is first-order. Note that when the non-uniformity is small, the EFT is expected to remain valid.

As the non-uniformity increases, the free energy of the solution remains higher than that of the uniform solution. This suggests that the uniform string star does

¹I thank David Kutasov for pointing this out.

not transition into the non-uniform one. However, we also observe that the field amplitudes grow with increasing non-uniformity, signaling the breakdown of the EFT. Therefore, it remains possible that, if an exact worldsheet CFT is considered, the non-uniform string star could eventually have a lower free energy and dominate the canonical ensemble.

In the microcanonical ensemble, we unexpectedly find that the uniform solutions stable in the canonical ensemble become unstable, as they coexist with the non-uniform branch possessing higher entropy. This phenomenon does not occur in any of the other cases we study. Moreover, the uniform solutions that are unstable in the canonical ensemble are now stable, and undergo a second-order phase transition into the non-uniform string star at the critical mass.

2 Horowitz-Polchinski EFT in $\mathbb{R}^d \times \mathbb{S}^1_z$

First, consider a (d+1)-dimensional noncompact spacetime $\mathbb{R}^{d,1}$. To study the canonical ensemble, we perform a Wick rotation of the time coordinate, $t \to \tau = it$, resulting in the Euclidean spacetime $\mathbb{R}^d \times \mathbb{S}^1_{\tau}$. The circumference of the Euclidean time circle \mathbb{S}^1_{τ} is identified with the inverse temperature β of the canonical ensemble.

In string theory, when the inverse temperature β is near a critical value known as the Hagedorn temperature, β_H , the canonical ensemble can be described by the Horowitz-Polchinski (HP) effective field theory (EFT) [12]. The HP EFT includes massless fields, such as the graviton, as well as the field χ , which corresponds to the closed string tachyon with winding number ± 1 . This inclusion arises because the winding tachyon, with a mass of $(\beta^2 - \beta_H^2)/(2\pi\alpha')^2$, becomes light near β_H .

When all fields are small, the coupling between the field χ and the massless fields is dominated by its interaction with the radion φ , which captures the local variation of the Euclidean time circle. Specifically, at a given location x, the circumference of the τ circle is expressed as

$$\beta(x) = \beta e^{\varphi(x)} \ . \tag{2.1}$$

This leads to the effective mass of χ as follows

$$m_{\text{eff}}^2(x) = \frac{\beta^2 e^{2\varphi} - \beta_H^2}{(2\pi\alpha')^2} ,$$
 (2.2)

which establishes the coupling between the fields χ and φ .

Reducing on the τ circle, the EFT yields the following action in the flat space \mathbb{R}^d .

$$I_d = \frac{\beta}{16\pi G_N^{(d+1)}} \int d^d x \left[(\nabla \varphi)^2 + |\nabla \chi|^2 + \left(m_\infty^2 + \frac{\kappa}{\alpha'} \varphi \right) |\chi|^2 \right] . \tag{2.3}$$

Here, we truncate the action by neglecting higher-order terms and other fields that are either heavy or weakly coupled to χ . From (2.2), the mass term and coupling

coefficient are given by

$$m_{\infty}^2 = \frac{\beta^2 - \beta_H^2}{(2\pi\alpha')^2} , \quad \kappa = \frac{\beta_H^2}{2\pi^2\alpha'} .$$
 (2.4)

The precise values of β_H and κ depend on the type of string theory. For example, in bosonic string theory, $\kappa = 8$, while in type II string theory, $\kappa = 4$.

Next, we introduce an additional spatial circle \mathbb{S}^1_z with periodicity $z \sim z + L$. The higher-dimensional analog of the effective action (2.3) is given by

$$I_{d+1} = \frac{\beta}{16\pi G_N^{(d+2)}} \int_{-\frac{L}{2}}^{\frac{L}{2}} dz \int d^d x \left[(\partial_z \varphi)^2 + (\nabla_x \varphi)^2 + |\partial_z \chi|^2 + |\nabla_x \chi|^2 + \left(m_\infty^2 + \frac{\kappa}{\alpha'} \varphi \right) |\chi|^2 \right].$$

$$(2.5)$$

The (d+2)-dimensional Newton's constant, $G_N^{(d+2)}$, is related to its lower-dimensional counterpart by

$$G_N^{(d+2)} = G_N^{(d+1)} L . (2.6)$$

From the action (2.5), the e.o.m. for the fields χ and φ are given by

$$\nabla_x^2 \chi + \partial_z^2 \chi = \left(m_\infty^2 + \frac{\kappa}{\alpha'} \varphi \right) \chi ,$$

$$\nabla_x^2 \varphi + \partial_z^2 \varphi = \frac{\kappa}{2\alpha'} |\chi|^2 .$$
(2.7)

In this paper, we focus on the cases with spherical symmetry and reflection symmetry. In other words, the fields χ and φ depend only on the radial coordinate $r \equiv \sqrt{(x_1)^2 + \cdots + (x_d)^2}$ and are symmetric under $z \to -z$. Additionally, we fix the field χ to be real.

To simplify the differential equations further, we apply the following rescaling:

$$r = \hat{r}/m_{\infty} ,$$

$$z = \hat{z}/m_{\infty} ,$$

$$\chi(r) = \frac{\sqrt{2}\alpha'}{\kappa} m_{\infty}^2 \hat{\chi}(\hat{r}) ,$$

$$\varphi(r) = \frac{\alpha'}{\kappa} m_{\infty}^2 \hat{\varphi}(\hat{r}) .$$
(2.8)

Under this rescaling, the e.o.m. become

$$\partial_{\hat{r}}^{2}\hat{\chi} + \frac{d-1}{\hat{r}}\partial_{\hat{r}}\hat{\chi} + \partial_{\hat{z}}^{2}\hat{\chi} = (1+\hat{\varphi})\hat{\chi} ,$$

$$\partial_{\hat{r}}^{2}\hat{\varphi} + \frac{d-1}{\hat{r}}\partial_{\hat{r}}\hat{\varphi} + \partial_{\hat{z}}^{2}\hat{\varphi} = \hat{\chi}^{2} .$$
(2.9)

Note that in these rescaled variables (2.8), the periodicity of the z circle changes to $\hat{z} \sim \hat{z} + m_{\infty}L$.

2.1 Uniform solutions

For solutions that are uniform along the z circle, meaning the fields χ and φ are independent of z, the e.o.m. (2.9) reduce to the following ordinary differential equations.

$$\hat{\chi}''(\hat{r}) + \frac{d-1}{\hat{r}}\hat{\chi}'(\hat{r}) = (1+\hat{\varphi})\hat{\chi} ,$$

$$\hat{\varphi}''(\hat{r}) + \frac{d-1}{\hat{r}}\hat{\varphi}'(\hat{r}) = \hat{\chi}^2 .$$
(2.10)

At the action level, for uniform fields, the integration over z in (2.5) recovers the d-dimensional effective action (2.3).

The differential equations (2.10) can be solved numerically using the shooting method. This involves imposing the initial conditions $\hat{\chi}'(0) = \hat{\varphi}'(0) = 0$ and fine-tuning the values of $\hat{\chi}(0)$ and $\hat{\varphi}(0)$ to obtain solutions where both $\hat{\chi}$ and $\hat{\varphi}$ asymptotically approach zero (see, for example, [17]).

Substituting the HP solutions $\chi(r)$ and $\varphi(r)$ into the action yields the free energy of the canonical ensemble. Specifically, from (2.3), we have

$$F = \frac{I_d}{\beta} = \frac{\omega_{d-1}}{16\pi G_N^{(d+1)}} \int dr \ r^{d-1} \left[(\nabla \varphi)^2 + (\nabla \chi)^2 + \left(m_\infty^2 + \frac{\kappa}{\alpha'} \varphi \right) \chi^2 \right]$$

$$= \frac{\omega_{d-1}}{16\pi G_N^{(d+1)}} \int dr \ r^{d-1} \left[-\varphi \nabla^2 \varphi - \chi \nabla^2 \chi + \left(m_\infty^2 + \frac{\kappa}{\alpha'} \varphi \right) \chi^2 \right]$$

$$= \frac{\omega_{d-1}}{32\pi G_N^{(d+1)}} \frac{\kappa}{\alpha'} \int dr \ r^{d-1} \left(-\varphi \chi^2 \right) . \tag{2.11}$$

Note that in the second line we have integrated by parts and in the third line we have used the equations of motion. Using the rescaled variables and fields (2.8), the free energy can be explicitly expressed in terms of m_{∞} as

$$F = \frac{\omega_{d-1} m_{\infty}^{6-d}}{16\pi G_N^{(d+1)}} \left(\frac{\alpha'}{\kappa}\right)^2 \int d\hat{r} \ \hat{r}^{d-1} \left(-\hat{\varphi}\hat{\chi}^2\right) \ . \tag{2.12}$$

With the HP solution, we can also calculate the thermodynamic quantities of the microcanonical ensemble. First, the ADM mass can be determined from the asymptotic form of $\varphi(r)$, i.e.,

$$\varphi(r) \sim -\frac{C_{\varphi}}{r^{d-2}} \ .$$
(2.13)

By integrating the second equation in (2.7), the fall-off coefficient C_{φ} is found to be

$$C_{\varphi} = \frac{\kappa}{2(d-2)\alpha'} \int_0^{\infty} dr \ r^{d-1} \chi^2(r) \ .$$
 (2.14)

Thus, the mass of the HP solution can be expressed as

$$M = \frac{(d-2)\omega_{d-1}}{8\pi G_N^{(d+1)}} C_{\varphi} = \frac{\omega_{d-1}}{16\pi G_N^{(d+1)}} \frac{\kappa}{\alpha'} \int_0^{\infty} dr \ r^{d-1} \chi^2(r) \ . \tag{2.15}$$

Using the rescaling in (2.8), we can further extract the dependence of the mass on m_{∞} , which is given by

$$M = \frac{\omega_{d-1} m_{\infty}^{4-d}}{8\pi G_N^{(d+1)}} \frac{\alpha'}{\kappa} \int_0^\infty d\hat{r} \ \hat{r}^{d-1} \hat{\chi}^2(\hat{r}) \ . \tag{2.16}$$

This expression reveals that for d < 4, M increases with m_{∞} , while for d > 4, M decreases with m_{∞} . In the special case of d = 4, determining the dependence of M on m_{∞} requires including subleading terms in the effective action. We will come back to this issue in section 3.1.

For a given mass M, the entropy can be calculated using the formula

$$S = \beta M - \beta F . (2.17)$$

Expanding this expression to leading orders in $\beta - \beta_H$ yields

$$S = \beta_H M + (\beta - \beta_H) M - \beta_H F + O\left((\beta - \beta_H)^2\right)$$

= $\beta_H M + \frac{2(\pi \alpha')^2}{\beta_H} m_\infty^2 M - \beta_H F + O\left((\beta - \beta_H)^2\right)$, (2.18)

where m_{∞} is related to M as in (2.16).

2.2 Non-uniform solutions

More generally, for solutions that are non-uniform along the z circle, we need to solve the full partial differential equations (2.9). As pointed out by [27], the relaxation method is more practical for numerically solving these equations than the shooting method. Besides, it is useful to compactify the \hat{r} coordinate as

$$\hat{u} = \left(\frac{\hat{r}}{1+\hat{r}}\right)^2 \,, \tag{2.19}$$

which maps the domain $\hat{r} \in [0, \infty)$ to $\hat{u} \in [0, 1]$. This domain is then discretized using a Lobatto-Chebyshev grid (see, e.g., Appendix A of [45] for technical details). For the \hat{z} coordinate, reflection symmetry allows us to focus on the half-period $\hat{z} \in [0, m_{\infty}L/2]$, which is similarly discretized using a Lobatto-Chebyshev grid. Additionally, we impose the boundary conditions as

$$\partial_{\hat{z}}\hat{\chi}(\hat{u},\hat{z})|_{\hat{z}=0,\frac{m_{\infty}L}{2}} = \partial_{\hat{z}}\hat{\varphi}(\hat{u},\hat{z})|_{\hat{z}=0,\frac{m_{\infty}L}{2}} = \hat{\chi}(\hat{u},\hat{z})|_{\hat{u}=1} = \hat{\varphi}(\hat{u},\hat{z})|_{\hat{u}=1} = 0 \ . \tag{2.20}$$

To apply the relaxation method, an initial seed solution is needed. The rescaling in (2.8) implies that the size of the solution is characterized by $1/m_{\infty}$. Thus, when m_{∞} is large compared to 1/L, non-uniform solutions are localized in the z circle. These solutions can be approximated by the spherically symmetric solutions in the

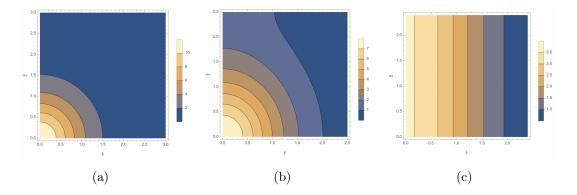


Figure 1: Contour plots of $\hat{\chi}(\hat{r},\hat{z})$ for d=4 and (a) $m_{\infty}L=6$, (b) $m_{\infty}L=5$, (c) $m_{\infty}L=4.8$.

noncompact space \mathbb{R}^{d+1} , i.e., the solutions of (2.10) with d replaced by d+1. Such solutions, obtained via the shooting method, can serve as seeds for the relaxation method by replacing \hat{r} with $\sqrt{\hat{r}^2 + \hat{z}^2}$. Once a solution of (2.9) is found, it can be iteratively used as a new seed to generate more solutions for progressively smaller values of m_{∞} .

In figure 1, we present the numerical solutions of $\hat{\chi}(\hat{r},\hat{z})$, for d=4 as an example. The plots show that when $m_{\infty}L$ is sufficiently large, the HP solution is indeed localized in \mathbb{S}^1_z . Furthermore, as m_{∞} decreases, the solution becomes increasingly uniform. At a critical value between 4.8/L and 5/L, the non-uniform solution transitions into a uniform solution.

Also note that the amplitudes of the solutions are of order 1 (e.g., figure 1). Consequently, from (2.8), the amplitudes of the unhatted fields χ and φ scale as $\alpha' m_{\infty}^2$. The EFT is valid only when the fields are small, which requires $\alpha' m_{\infty} \ll 1$. Furthermore, the critical point corresponds to $m_{\infty} = O(L^{-1})$, where the size of the solution is comparable to the size of the spatial circle. Therefore, for the solutions to remain valid near the critical point, the condition $L \gg \ell_s$ must be satisfied—precisely the regime of interest in this paper.

With the solutions for $\hat{\chi}(\hat{r}, \hat{z})$ and $\hat{\varphi}(\hat{r}, \hat{z})$, we can compute thermodynamic quantities such as the free energy, mass, and entropy. First, similar to (2.12), the free energy of the canonical ensemble is given by

$$F = \frac{\omega_{d-1}L^{d-5}}{8\pi G_N^{(d+2)}} \left(\frac{\alpha'}{\kappa}\right)^2 f , \qquad (2.21)$$

where

$$f \equiv (m_{\infty}L)^{5-d} \int_0^{\frac{m_{\infty}L}{2}} d\hat{z} \int_0^{\infty} d\hat{r} \, \hat{r}^{d-1} \left(-\hat{\varphi}\hat{\chi}^2\right) \,. \tag{2.22}$$

For the microcanonical ensemble, the mass of the HP solution can be generalized

from (2.16) to

$$M = \frac{\omega_{d-1}}{4\pi G_N^{(d+2)}} \frac{\alpha'}{\kappa} L^{d-3} m , \qquad (2.23)$$

where m is defined as

$$m \equiv (m_{\infty}L)^{3-d} \int_0^{\frac{m_{\infty}L}{2}} d\hat{z} \int_0^{\infty} d\hat{r} \ \hat{r}^{d-1} \hat{\chi}^2(\hat{r}, \hat{z}) \ . \tag{2.24}$$

With the mass M (2.23) and the free energy F (2.21), we can further calculate the entropy. More specifically, using (2.18), we obtain

$$S = \beta_H M + \frac{\omega_{d-1} \pi^3 \alpha'^4}{\beta_H^3 G_N^{(d+2)}} L^{d-5} s , \qquad (2.25)$$

where

$$s \equiv (m_{\infty}L)^2 m - \frac{f}{2} \ . \tag{2.26}$$

Here, the first term in (2.25) reflects the Hagedorn behavior, while the second term represents the leading-order correction to it.

2.3 d = 2

We are motivated to investigate the HP solutions for d=2 due to the interesting behavior of black holes in this context. In particular, it has been found that as the size increases and the horizon becomes uniform, a black hole would expand infinitely, swallowing the whole universe [46, 47]. Thus, it is intriguing to study what happens when we start with a localized HP solution at a sufficiently large m_{∞} and gradually decrease m_{∞} , thereby increasing the size of the solution.²

However, a subtlety arises in $\mathbb{R}^2 \times \mathbb{S}^1_z$. For any solution of $\hat{\chi}$ that vanishes asymptotically, the second line of the e.o.m. (2.9) implies that $\hat{\varphi}$ diverges logarithmically at large \hat{r} . In other words, the circumference of the Euclidean time circle cannot converge to a constant, meaning that the canonical and microcanonical ensembles may not be well-defined. Note that this situation also occurs in the black hole case.

Nevertheless, normalizable solutions for $\hat{\chi}$ can still be found. Specifically, as described in the last subsection, we begin with a localized HP solution as a seed for the relaxation method and obtain non-uniform solutions of (2.9) by progressively decreasing $m_{\infty}L$. Since the Euclidean time circle does not asymptotically approach a constant, m_{∞} no longer carries a direct physical interpretation, such as representing the asymptotic mass of the field χ . However, it remains a convenient parameter for characterizing the solutions.

²I am grateful to Roberto Emparan, Mikel Sanchez-Garitaonandia and Marija Tomašević for bringing this to my attention and for providing valuable insights on the topic.

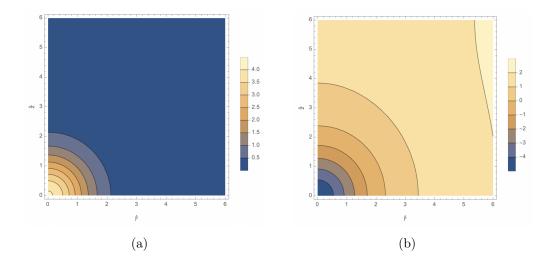


Figure 2: Non-uniform solutions of (a) $\hat{\chi}$ and (b) $\hat{\varphi}$ for d=2, with $m_{\infty}L=12$.

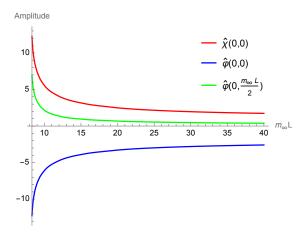


Figure 3: $\hat{\chi}(\hat{r}=0,\hat{z}=0)$, $\hat{\varphi}(\hat{r}=0,\hat{z}=0)$ and $\hat{\varphi}(\hat{r}=0,\hat{z}=\frac{m_{\infty}L}{2})$ as functions of $m_{\infty}L$.

Moreover, the divergence of $\hat{\varphi}$ at large \hat{r} prevents the field from satisfying the boundary conditions given in (2.20), complicating the numerical analysis. To address this, we redefine the field as

$$\hat{\varphi} = (1 + \hat{r}^2)\hat{\phi} , \qquad (2.27)$$

and accordingly modify the boundary conditions to

$$\partial_{\hat{z}}\hat{\chi}(\hat{u},\hat{z})|_{\hat{z}=0,\frac{m_{\infty}L}{2}} = \partial_{\hat{z}}\hat{\phi}(\hat{u},\hat{z})|_{\hat{z}=0,\frac{m_{\infty}L}{2}} = \hat{\chi}(\hat{u},\hat{z})|_{\hat{u}=1} = \hat{\phi}(\hat{u},\hat{z})|_{\hat{u}=1} = 0 \ . \tag{2.28}$$

We plot the solutions in figure 2, using $m_{\infty}L = 12$ as an example. It can be seen that at large \hat{r} , $\hat{\varphi}$ diverges to positive infinity, meaning that the Euclidean time circle opens up. Besides, it turns out that as m_{∞} decreases, the non-uniform solution does not transition into a uniform solution, similar to the behavior observed in the black

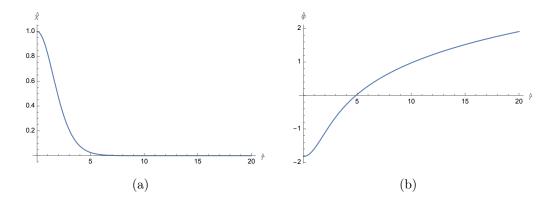


Figure 4: Uniform HP solutions $\hat{\chi}$ and $\hat{\varphi}$ for d=2 with $\hat{\chi}(0)=1$.

hole case. In fact, as shown in figure 3, the field amplitudes diverge as m_{∞} approaches a constant. Especially, $\hat{\varphi}(0,0)$ diverges to negative infinity, while $\hat{\varphi}(0,m_{\infty}L/2)$ diverges to positive infinity, which implies that the non-uniform solution never becomes uniform.

However, this does not imply that uniform solutions are absent. Indeed, by numerically solving the e.o.m. (2.10), we find solutions where $\hat{\chi}$ vanishes asymptotically. For instance, figure 4 shows a solution with $\hat{\chi}(0) = 1$.

Unlike the HP solutions for d > 2, the uniform solution is not uniquely determined at a given m_{∞} . This is because the boundary condition $\hat{\varphi} = 0$ at infinity is no longer applicable for d = 2. As a result, any scaling transformation of the solution shown in figure 4, with

$$\hat{\chi} \to \lambda^2 \hat{\chi} , \quad \hat{\varphi} \to \lambda^2 (\hat{\varphi} + 1) - 1 , \quad \hat{r} \to \frac{\hat{r}}{\lambda} ,$$
 (2.29)

yields another solution of (2.10).

This scaling symmetry also explains why the non-uniform solution does not transition into the uniform solution as m_{∞} varies. Suppose such a transition does occur at some critical value $m_{\infty,c}$. Then, the transformation (2.29), combined with $\hat{z} \to \hat{z}/\lambda$, under which the e.o.m. (2.9) is invariant, would map the critical value to $m_{\infty,c}/\lambda$. Thus, the critical point becomes ambiguous, making the transition scenario problematic.

Finally, note that the case of d=2 differs qualitatively from cases where d lies between 2 and 3. In particular, for 2 < d < 3, $\hat{\varphi}$ vanishes asymptotically, allowing the canonical and microcanonical ensembles to be well-defined. This also enables us to impose the boundary conditions (2.20) rather than (2.28).

Taking d=2.7, 2.8 and 2.9 as examples, we begin with localized solutions and progressively decrease $m_{\infty}L$ to generate additional solutions with increasingly uniformity. The free energy f as a function of $m_{\infty}L$ is plotted in figure 5. Unlike the case of d=2, as m_{∞} decreases, the non-uniform HP solution transitions into a

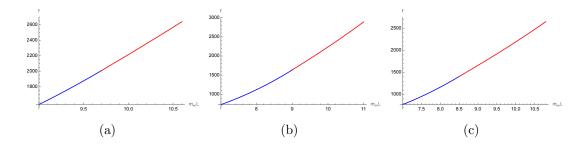


Figure 5: Phase diagrams of the canonical ensemble for (a) d = 2.7, (b) d = 2.8 and (c) d = 2.9. The red curves represent non-uniform solutions, and the blue curves represent uniform solutions.

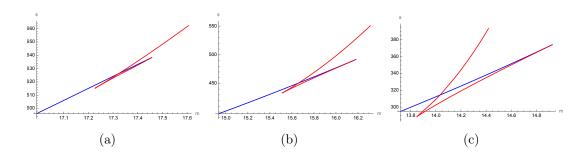


Figure 6: Phase diagrams of the microcanonical ensemble for (a) d = 2.7, (b) d = 2.8 and (c) d = 2.9.

uniform solution at a critical point. Since the free energy change is continuous at the critical point, the phase transition can be classified as second-order.

For the microcanonical ensemble, the phase diagrams are shown in figure 6. For each d, there are two branches of non-uniform solutions. Along with the uniform branch, they form a swallowtail-type phase diagram. This indicates that the phase transition between the uniform and non-uniform solutions is first-order, consistent with the perturbative analysis in [26].

2.4 d = 3 and d = 4

The phase diagrams of the canonical ensemble for d=3 and d=4 are presented in figure 7, which recover the results of [27]. Similar to the case of 2 < d < 3, as illustrated in figure 5, the non-uniform solution undergoes a second-order phase transition into the uniform solution, denoted by the solid blue curve, at the critical point $(m_{\infty} \approx 7.9/L \text{ for } d=3 \text{ and } m_{\infty} \approx 4.8/L \text{ for } d=4)$. Notably, the uniform solution lies at m_{∞} smaller than the critical value, where its size, characterized by m_{∞}^{-1} , is large compared to L.

In figure 7, we also include the uniform solutions above the critical value of m_{∞} , represented by the dashed blue curve. It can be seen that the free energies of these

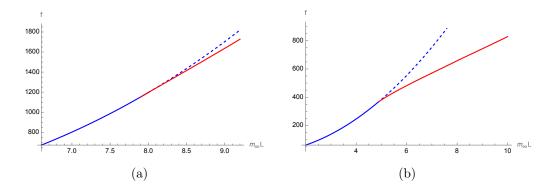


Figure 7: Phase diagrams of the HP solutions in the canonical ensemble for (a) d = 3 and (b) d = 4. The red curves denote the non-uniform solutions, and the blue curves represent the uniform solutions.

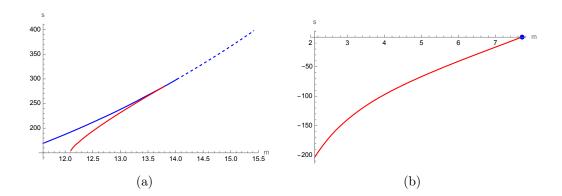


Figure 8: Phase diagrams of the HP solutions in the microcanonical ensemble for (a) d = 3 and (b) d = 4. The red curves correspond to non-uniform solutions, while the blue curves and dot represent uniform solutions.

uniform solutions are higher than those of the non-uniform solutions, indicating that they are unstable towards non-uniformity.

In the microcanonical ensemble, the phase diagrams for d=3 and d=4 differ qualitatively from those for 2 < d < 3 (e.g., figure 6). For d=3, as shown in figure 8(a), the non-uniform solution always has a lower entropy than the uniform solution [27]. Thus, the uniform solution cannot transition into a non-uniform HP solution.

However, this raises the question of what state the uniform HP solution transitions into at the critical mass. Notably, as m_{∞} increases, the non-uniform HP solution decreases in mass and becomes increasingly localized. When m_{∞} approaches $O(\ell_s^{-1})$, the HP EFT breaks down, and this is supposed to be the regime where the localized HP solution transitions into a localized black hole. It is then reasonable that the black hole branch complements figure 8(a) to form a swallowtail-type phase diagram, similar to those in figure 6. In other words, the uniform HP solution un-

dergoes a first-order phase transition into a localized black hole at the critical mass, as proposed in [26].

For d=4, the mass of the uniform HP solution is fixed, as discussed before. Thus, the uniform solution is denoted by a blue dot in figure 8(b). However, as we will see in the next section, taking the quartic terms in the effective action into account causes the mass of the uniform solution to increase slightly with m_{∞} . Therefore, when the uniform solution has a mass below the critical value, its size—determined by m_{∞}^{-1} —is larger than that at the critical point, imply that the solution is stable. We will show that this is indeed the case below.

3 Beyond HP EFT

As discussed in [15], the HP solution in \mathbb{R}^d ceases to exist for $d \geq 6$. Furthermore, [18, 26] show that when d is close to 6 as a continuous parameter, the effective action (2.3) breaks down because the quartic terms become significant. However, the higher-order terms, such as $|\chi|^6$, are still suppressed as long as $\chi, \varphi \ll 1$. In this section, we extend the original HP analysis by incorporating these quartic terms, which are derived from string scattering amplitudes in [48] (for bosonic and type II strings), leading to the following corrected action.

$$I_{d+1} = \frac{\beta}{16\pi G_N^{(d+2)}} \int_{-\frac{L}{2}}^{\frac{L}{2}} dz \int d^d x \left[(\partial_z \varphi)^2 + (\nabla_x \varphi)^2 + (\partial_z \chi)^2 + |\nabla_x \chi|^2 + \left(m_\infty^2 + \frac{\kappa}{\alpha'} \varphi + \frac{\kappa}{\alpha'} \varphi^2 \right) |\chi|^2 + \frac{\kappa}{4\alpha'} |\chi|^4 \right].$$

$$(3.1)$$

We will use the quartic-corrected action to study string stars for d = 5 and d = 6. Before proceeding, however, let us first explore how the quartic terms determine the dependence of M on m_{∞} for the uniform solution at d = 4.

3.1 More on d = 3 and d = 4

For the uniform solution, the e.o.m. derived from the quartic-corrected action (3.1) are given by

$$\chi''(r) + \frac{d-1}{r}\chi'(r) = \left(m_{\infty}^2 + \frac{\kappa}{\alpha'}\varphi(r) + \frac{\kappa}{\alpha'}\varphi^2(r) + \frac{\kappa}{2\alpha'}\chi^2(r)\right)\chi(r) ,$$

$$\varphi''(r) + \frac{d-1}{r}\varphi'(r) = \frac{\kappa}{2\alpha'}\left(1 + 2\varphi(r)\right)\chi^2(r) .$$
 (3.2)

To analyze corrections to the HP solution, we expand the fields perturbatively as

$$\chi = \frac{\sqrt{2}\alpha'}{\kappa} m_{\infty}^2 \left(\hat{\chi} + \frac{\alpha'}{\kappa} m_{\infty}^2 \hat{\hat{\chi}} + O((\alpha' m_{\infty}^2)^2) \right) ,$$

$$\varphi = \frac{\alpha'}{\kappa} m_{\infty}^2 \left(\hat{\varphi} + \frac{\alpha'}{\kappa} m_{\infty}^2 \hat{\hat{\varphi}} + O((\alpha' m_{\infty}^2)^2) \right) .$$
(3.3)

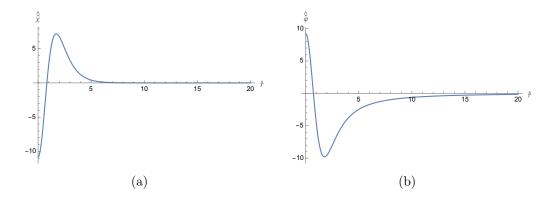


Figure 9: Profiles of the leading-order perturbations $\hat{\chi}$ and $-\hat{\varphi}$ to the Horowitz-Polchinski solution at d=4.

Substituting this expansion into (3.2) and using the rescaled variable $\hat{r} = m_{\infty}r$, we derive the leading-order and subleading-order differential equations. Specifically, the $O(m_{\infty}^2)$ equations govern $\hat{\chi}$ and $\hat{\varphi}$, and are identical to those in (2.10). At $O(m_{\infty}^4)$, the equations for $\hat{\chi}$ and $\hat{\varphi}$ are given by

$$\hat{\hat{\chi}}''(\hat{r}) + \frac{d-1}{\hat{r}}\hat{\hat{\chi}}'(\hat{r}) = (1+\hat{\varphi}(\hat{r})\hat{\hat{\chi}}(\hat{r}) + \hat{\chi}(\hat{r})\hat{\hat{\varphi}}(\hat{r}) + \hat{\chi}^{3}(\hat{r}) + \hat{\varphi}^{2}(\hat{r})\hat{\chi}(\hat{r}) ,$$

$$\hat{\varphi}''(\hat{r}) + \frac{d-1}{\hat{r}}\hat{\varphi}'(\hat{r}) = 2\hat{\chi}(\hat{r})\hat{\hat{\chi}}(\hat{r}) + 2\hat{\chi}^{2}(\hat{r})\hat{\varphi}(\hat{r}) .$$
(3.4)

As before, we can first use the shooting method to numerically solve (2.10). Once the solutions of $\hat{\chi}$ and $\hat{\varphi}$ are obtained, they can be plugged in (3.4), and the shooting method can be applied again to solve for the fields $\hat{\chi}$ and $\hat{\varphi}$. For d=4, the solutions are plotted in figure 9. Notably, while $\hat{\varphi}$ is positive at small \hat{r} , it eventually becomes negative as \hat{r} increases. In other words, the asymptotic behavior $\hat{\varphi}$ is given by

$$\hat{\hat{\varphi}} \sim -\frac{\hat{C}_{\varphi}}{\hat{r}d-2} , \qquad (3.5)$$

with a positive fall-off coefficient $\hat{\hat{C}}_{\varphi}$.

We similarly define the fall-off coefficient of $\hat{\varphi}$ as \hat{C}_{φ} , such that $\hat{\varphi} \sim -\hat{C}_{\varphi}/\hat{r}^{d-2}$. Then, combining (3.3), (3.5) and $r = \hat{r}/m_{\infty}$, the fall-off coefficient of φ in (2.13) can be written as

$$C_{\varphi} = \frac{\alpha' m_{\infty}^{4-d}}{\kappa} \left(\hat{C}_{\varphi} + \frac{\alpha'}{\kappa} m_{\infty}^2 \hat{C}_{\varphi} + O((\alpha' m_{\infty}^2)^2) \right). \tag{3.6}$$

Since $\hat{C}_{\varphi} > 0$ for d = 4, it follows that the mass of the uniform solution increases with m_{∞} .

With this new result, we now provide additional comments on the phase diagrams of the microcanonical ensembles for d = 3 and d = 4. In figure 8(a) (combined with

figure 7(a)), we see that for d=3, as m_{∞} increases, the mass of the non-uniform HP solution decreases. For sufficiently large m_{∞} , the non-uniform solution becomes localized in \mathbb{S}^1_z and can be approximated by the spherically symmetric HP solution in four-dimensional noncompact space. However, as discovered above, incorporating the quartic terms in the effective action reveals that the mass of the four-dimensional HP solution increases with m_{∞} . This suggests that as m_{∞} continues to increase, the mass of the non-uniform solution will eventually stop decreasing and begin to increase. This scenario is consistent with the conjecture in section 2.4 that the HP solutions, along with the black hole branch, form a swallowtail-type phase diagram.

For d=4, since the mass varies slightly with m_{∞} as shown in (3.6), the blue dot in figure 8(b) can be replaced by a short curve. Besides, the stability of the uniform solution below the critical mass can be verified by comparing its entropy with that of the non-uniform solution. For the latter, the subleading term in the entropy (2.25) is negative and scales as $1/G_N^{(d+2)}L \sim m_{\infty}/G_N^{(d+2)}$ near the critical point. In contrast, this term vanishes for original uniform HP solution [15]. Furthermore, when the quartic terms in the effective action are included, it follows from (3.3) that the subleading term in the entropy (2.18) becomes proportional to $m_{\infty}^2/G_N^{(d+2)}L \sim m_{\infty}^3/G_N^{(d+2)}$, which is much smaller than the (absolute value of) corresponding term for the non-uniform solution, given that $\alpha'm_{\infty}^2 \ll 1$. Therefore, the uniform solution has a higher entropy than the non-uniform solution, indicating its stability against non-uniformity.

At the critical mass, rather than transitioning into a non-uniform HP solution, the uniform solution would transition into a localized black hole, which has a much higher entropy [26, 27]. As discussed in section 2.4, the black hole branch is expected to be connected with the non-uniform HP branch at $m_{\infty} \sim \ell_s^{-1}$. Consequently, in the microcanonical ensemble, the black hole, uniform string star and non-uniform HP branches together form a swallowtail-type phase diagram³, similar to the one in the d=3 case.

3.2 d = 5

In $\mathbb{R}^5 \times \mathbb{S}^1_z$, the localized string star approaches the spherically symmetric solution in \mathbb{R}^6 . As discussed before, the original HP EFT breaks down, and the quartic terms have to be taken into account, (3.1), with the higher-order terms still negligible. Note that these quartic terms break the rescaling property associated with (2.8). Thus, we will no longer use the hatted variables and fields.

³Note that the uniform branch spans only a small range of M around the critical value, as discussed above. However, when $\alpha' m_{\infty}^2$ is not much larger than $G_N^{(d+1)}$, quantum effects become significant, and the string star becomes a state of free strings, which extends to lower mass values [15].

From the action (3.1), we derive the e.o.m. as

$$\nabla_x^2 \chi + \partial_z^2 \chi = \left(m_\infty^2 + \frac{\kappa}{\alpha'} \varphi + \frac{\kappa}{\alpha'} \varphi^2 + \frac{\kappa}{2\alpha'} \chi^2 \right) \chi ,$$

$$\nabla_x^2 \varphi + \partial_z^2 \varphi = \frac{\kappa}{2\alpha'} (1 + 2\varphi) \chi^2 .$$
(3.7)

As in the previous section, we can use a localized solution as a seed for the relaxation method to generate non-uniform solutions. Specifically, the localized solution can be approximated by [18]

$$\chi(r,z) = -\sqrt{2}\varphi(r,z) = \sqrt{\frac{140\alpha'}{3\kappa}} \frac{m_{\infty}}{\left(1 + \frac{1}{24}\sqrt{\frac{70\kappa}{3\alpha'}}m_{\infty}(r^2 + z^2)\right)^2} . \tag{3.8}$$

This analytic expression suggests that, in the numerical computation, it is convenient to rescale

$$r \to \frac{r}{\sqrt{m_{\infty}}} , \quad z \to \frac{z}{\sqrt{m_{\infty}}} ,$$

$$\chi \to m_{\infty} \chi , \quad \varphi \to m_{\infty} \varphi . \tag{3.9}$$

Besides, one can see that χ and φ are still small in the regime $\alpha' m_{\infty}^2 \ll 1$, keeping the higher-order terms beyond the quartic-corrected action (3.1) suppressed.

For a solution of (3.7), substituting the e.o.m. into the on-shell action yields the following expression for the free energy.

$$F = \frac{\omega_{d-1}\kappa}{32\pi\alpha' G_N^{(d+2)}}\tilde{F} , \qquad (3.10)$$

where

$$\tilde{F} \equiv \int_0^{\frac{L}{2}} dz \int_0^{\infty} dr \ r^{d-1} \left(-2\varphi - 4\varphi^2 - \chi^2 \right) \chi^2 \ . \tag{3.11}$$

Besides, using the second line of the e.o.m. (3.7), the mass can be computed similarly to (2.15) and is given by

$$M = \frac{\omega_{d-1}}{8\pi G_N^{(d+2)}} \frac{\kappa}{\alpha'} \tilde{M} , \qquad (3.12)$$

with

$$\tilde{M} \equiv \int_0^{\frac{L}{2}} dz \int_0^{\infty} dr \ r^{d-1} \left(1 + 2\varphi(r, z) \right) \chi^2(r, z) \ . \tag{3.13}$$

From the expressions for F and M, the entropy can be calculated using (2.17) as follows,

$$S = \beta_H M + \frac{\omega_{d-1} \kappa \beta_H}{32\pi \alpha' G_N^{(d+2)}} \tilde{S} . \tag{3.14}$$

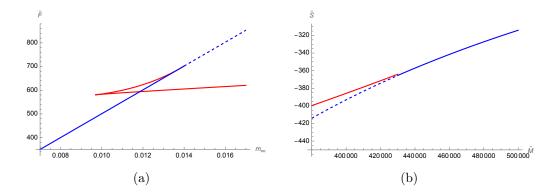


Figure 10: Phase diagrams for the string stars at d=5 in (a) the canonical ensemble and (b) the microcanonical ensemble, for L=200 with $\frac{\kappa}{\alpha'}=1$. The red curves represent the non-uniform solutions. The blue curves denote the uniform solutions.

Here, we have defined

$$\tilde{S} \equiv \frac{4\alpha' m_{\infty}^2}{\kappa} \tilde{M} - \tilde{F} \ . \tag{3.15}$$

In figure 10, we plot the phase diagrams for L=200, with $\kappa/\alpha'=1$. In the canonical ensemble, as shown in figure 10(a), the non-uniform solution exhibits a turning point as m_{∞} varies. This behavior contrasts with the cases of d=3 and d=4. Specifically, as m_{∞} decreases, we find non-uniform solutions forming the lower red branch in figure 10(a). However, our numerical code fails to produce any solution at $m_{\infty}=0.0096$. This implies that solutions with greater uniformity exist at larger values of m_{∞} .

To find these solutions, we create a seed for the relaxation method by linearly combining lower-branch solutions at $m_{\infty} = 0.0097$ and $m_{\infty} = 0.0098$,

$$2(\chi,\varphi)|_{m_{\infty}=0.0097, \text{ lower branch}} - (\chi,\varphi)|_{m_{\infty}=0.0098, \text{ lower branch}}$$
 (3.16)

Using this seed, we can generate the upper-branch solution at $m_{\infty} = 0.0098$. Gradually increasing m_{∞} , we then construct the upper red branch, which extends up to $m_{\infty} = 0.014$, where the solution becomes uniform.

As shown in figure 10(a), the uniform and non-uniform solutions form a swallowtail-type phase diagram. In particular, as m_{∞} increases, the uniform HP solution undergoes a first-order phase transition at the critical point $m_{\infty} \approx 0.014$ (for L = 200 with $\kappa/\alpha' = 1$) and becomes non-uniform. Conversely, if we begin with a non-uniform solution at sufficiently large m_{∞} (but still much smaller than ℓ_s^{-1}) and decrease m_{∞} , the solution transitions into the uniform phase at the turning point $m_{\infty} \approx 0.0097$.

Note that perturbed non-uniform solutions are found at m_{∞} values smaller than the critical one, consistent with analyses that neglect the quartic terms in the action [26, 27]. However, as the non-uniformity grows, the quartic corrections gradually

become significant, causing the non-uniform branch to turn around and extend towards larger values of m_{∞} .

Finally, in the microcanonical ensemble, starting with a uniform solution at large mass, figure 10(b) shows that as M decreases, a second-order phase transition occurs at the critical point, resulting in a non-uniform solution. This finding aligns with [26, 27].

3.3 d = 6

We now consider the case of d=6, where both the HP effective action (2.5) and the quartic-corrected effective action (3.1) become invalid for localized solutions. In this situation, we begin with the uniform solution at a small value of m_{∞} , denoted by $\chi_*(r)$ and $\varphi_*(r)$, for which the effective action (3.1) is still applicable. By introducing non-uniform perturbations to the uniform solution, we aim to shed light on higher-dimensional solutions, which are not yet well understood.⁴

First, we perturb the uniform solutions as follows,

$$\chi(r,z) = \chi_*(r) + \lambda \cos\left(\frac{2\pi}{L}z\right) \chi_1(r) , \quad \varphi(r,z) = \varphi_*(r) + \lambda \cos\left(\frac{2\pi}{L}z\right) \varphi_1(r) , \quad (3.17)$$

where λ is the perturbative parameter. Plugging the perturbed fields in the e.o.m. (3.7), we derive the differential equations governing χ_1 and φ_1 . Specifically,

$$\nabla^{2}\chi_{1} - \left(\frac{2\pi}{L}\right)^{2}\chi_{1} = m_{\infty}^{2}\chi_{1} + \frac{\kappa}{\alpha'}\left((\varphi_{*} + \varphi_{*}^{2} + \frac{3}{2}\chi_{*}^{2})\chi_{1} + (\chi_{*} + 2\chi_{*}\varphi_{*})\varphi_{1}\right) ,$$

$$\nabla^{2}\varphi_{1} - \left(\frac{2\pi}{L}\right)^{2}\varphi_{1} = \frac{\kappa}{\alpha'}\left((\chi_{*} + 2\chi_{*}\varphi_{*})\chi_{1} + \chi_{*}^{2}\varphi_{1}\right) .$$
(3.18)

As before, we can use the shooting method to solve (3.18) for χ_1 and φ_1 . Since these differential equations are linear, we can, for instance, set $\chi_1(r) = 1$. Note that solving these equations amounts to solving an eigenvalue problem, as only a specific value of L yields normalizable solutions. Accordingly, the shooting method requires iteratively adjusting L until normalizable solutions are obtained.

Once the perturbed solutions (3.17) are obtained, they can be used as a seed for the relaxation method to generate non-uniform solutions.⁵ For numerical computations, we again employ the rescaled variables and fields given in (3.9) for convenience (Note that it is different from the rescaling in (2.8)).

⁴See a recent paper [29] for a non-perturbative prescription for the existence of higherdimensional string star solutions. Note that this paper also leverages the non-uniformity of the HP solutions for $d \leq 5$ in the presence of $\mathbb{T}^{n \geq 2}$ to explore higher-dimensional string star solutions. However, it is unclear to me whether the uniform solution transitions at the critical point directly into a string star localized in all directions of $\mathbb{T}^{n \geq 2}$, or only in one direction while remaining uniform along the others. This question motivated my investigation of non-uniform string stars for d = 6 in this subsection.

⁵This method of generating non-uniform solutions is also utilized in [27].

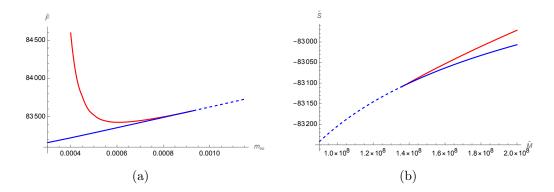


Figure 11: Phase diagrams for d=6 in the (a) canonical ensemble and (b) microcanonical ensemble, with $\frac{\kappa}{\alpha'}=1$ and L=180, near the critical point. The red curves represent the non-uniform solutions, while the blue curves denote the uniform solutions.

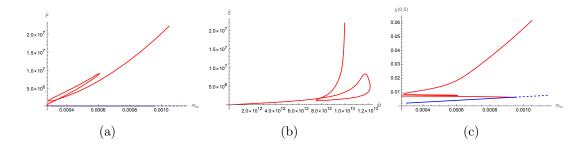


Figure 12: Complete phase diagrams for d=6: (a) canonical ensemble, (b) microcanonical ensemble, and (c) $\chi(0,0)$ as a function of m_{∞} , with $\frac{\kappa}{\alpha'}=1$ and L=180.

In figure 11, we plot the free energy and entropy for L=180 and $\kappa/\alpha'=1$ near the critical point $m_{\infty}\approx 0.00093$. As shown in figure 11(a), the perturbed non-uniform solutions are located at values of m_{∞} smaller than the critical one, where the uniform solutions (represented by the blue solid curve) are stable. This indicates that the uniform solution undergoes a first-order phase transition at the critical point.

For the microcanonical ensemble (figure 11(b)), the behavior is quite surprising. In particular, the uniform solutions that are stable in the canonical ensemble (represented by the solid blue curve) become unstable, as they exhibit lower entropy than the non-uniform solutions. On the other hand, the uniform solutions with m_{∞} greater than the critical value (represented by the dashed blue curve) become stable in the microcanonical ensemble. Thus, the uniform solution undergoes a second-order phase transition into a non-uniform solution at the critical mass.

The complete phase diagrams are presented in figure 12, which also includes the amplitude of the solution χ . As can be seen in figure 12(a), the free energy of the non-uniform solution is always higher than that of the uniform solution. This suggests that the uniform solution may never transition into a non-uniform solution in the canonical ensemble. In contrast, in the microcanonical ensemble (figure 12(b)), the

non-uniform solution always dominates over the uniform solution above the critical mass

However, as shown in figure 12(c), the fields grow as the non-uniform solution becomes increasingly localized, signaling the breakdown of the effective action (3.1). Therefore, if all α' corrections are taken into account, it is plausible that the free energy of the non-uniform solution, as it becomes highly localized, stops increasing, begins to decrease, and eventually drops below that of the uniform solution. This scenario would enable the uniform solution to transition into the non-uniform solution.

4 Discussion

The aim of this paper was to investigate the phase transitions between uniform and non-uniform string stars in the space $\mathbb{R}^d \times \mathbb{S}^1_z$. By analyzing the Horowitz-Polchinski EFT, we successfully reproduced the phase diagrams for d=3 and d=4 presented in [27]. In particular, the phase transition at the critical point is second-order in the canonical ensemble and first-order in the microcanonical ensemble. For d=2, where no standard HP solution exists, we found novel solutions in which the Euclidean time circle opens up at infinity. Interestingly, unlike the case of d>2, the uniform and non-uniform solutions are disconnected. As discussed in section 2.3, this absence of a critical length for L can be attributed to the scaling symmetry of the uniform solution, (2.29).

Besides, it is often necessary to go beyond the HP EFT by including the quartic terms in the effective action. First, this approach revealed that the mass of the uniform string star for d=4 increases with m_{∞} . Using the quartic-corrected effective action, we also obtained the phase diagrams for d=5 and d=6. Specifically, at the critical point towards non-uniformity, the uniform string star undergoes a first-order phase transition in the canonical ensemble and a second-order transition in the microcanonical ensemble. These phenomena contrast with those for d=3 and d=4.

In the case of d=6, we also noticed that as the non-uniform solution becomes localized, the fields grow large, signaling the breakdown of the EFT. To study higher-dimensional string stars further, we need to analyze the full world-sheet theory. To this end, note that at the Hagedorn temperature β_H , or $m_{\infty}=0$, the e.o.m. (3.7)

⁶One might question whether a string star solution even exists when all α' corrections are taken into account. However, as identified in [29], such a solution does exist and is independent of the specific details of the corrections.

⁷Note that this is also consistent with the fact that the localized black hole has a lower free energy than the uniform black string, combined with the assumption that the black objects' free energies are comparable to those of the string stars at $m_{\infty} \sim \ell_s^{-1}$. I thank Alek Bedroya and David Wu for highlighting this important point.

are consistent with the relation

$$\chi = -\sqrt{2}\varphi \ . \tag{4.1}$$

From the worldsheet analysis, this relation reflects an enhanced SU(2) symmetry, which is expected to hold even when the EFT is invalid [17].

Based on the enhanced SU(2) symmetry, at the Hagedorn temperature, the worldsheet theory can be given by a non-abelian Thirring model with a coupling that depends on the radial coordinate, as introduced in [17]. Specifically, the deformation to the sigma model is expressed as

$$\chi(r)J^a\bar{J}^a \,, \tag{4.2}$$

where J^a and \bar{J}^a denote the $SU(2)_L$ and $SU(2)_R$ currents. By imposing conformal symmetry on this worldsheet theory, one can determine $\chi(r)$ and, in turn, obtain string star solutions at higher dimensions, as discussed in [18]. We leave this calculation to future work.

Another issue is about the Lorentzian interpretation of the solutions studied in this paper. For black holes, the instability of the uniform solutions towards non-uniformity in Lorentzian spacetime is known as the Gregory-Laflamme instability, as mentioned before. Naturally, one might ask whether a similar Gregory-Laflamme instability exists for string stars. However, the winding tachyon χ has no counterpart in Lorentzian signature because Lorentzian time is not periodic. This produces an obstruction to continuing the solutions to Lorentzian spacetime. Thus, understanding the nature of string stars in Lorentzian spacetime remains an important open question (see, e.g., [9, 49, 50] for related explorations).

Relatedly, recall that for d=6, figure 11(b) indicates that at a given mass, when the spatial circle L is smaller than the critical size, the uniform solution is entropically disfavored and unstable to non-uniform perturbations. This behavior is anomalous because, in Lorentzian signature, the Gregory-Laflamme instability does not occur when L is smaller than the critical size. Therefore, our findings suggest that either something beyond classical dynamics is required for a Lorentzian interpretation of the string star, if such an interpretation exists, or the classical dynamics cannot be captured by the Euclidean EFT studied in this paper. This issue is also left to future work.

Acknowledgement

I would like to express my gratitude to Alek Bedroya, Roberto Emparan, David Kutasov, Mikel Sanchez-Garitaonandia, Marija Tomašević and David Wu for lots of interesting discussions and useful comments on the draft.

⁸I thank Roberto Emparan for pointing this out to me.

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