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List of integrals of hyperbolic functions

The following is a list of integrals (anti-derivative functions) of hyperbolic functions. For a complete list of integral functions, see list of integrals.

In all formulas the constant a is assumed to be nonzero, and C denotes the constant of integration.

Integrals involving only hyperbolic sine functions

$$\int \sinh ax\,dx = \frac{1}{a}\cosh ax + C$$

$$\int \sinh^2 ax\,dx = \frac{1}{4a}\sinh 2ax - \frac{x}{2} + C$$

$$\int \sinh^n ax\,dx = \begin{cases} \frac{1}{an}(\sinh^{n-1} ax)(\cosh ax) - \frac{n-1}{n}\int \sinh^{n-2} ax\,dx, & n>0 \\ \frac{1}{a(n+1)}(\sinh^{n+1} ax)(\cosh ax) - \frac{n+2}{n+1}\int \sinh^{n+2} ax\,dx, & n<0, n\neq -1 \end{cases}$$

$$\begin{aligned}\int \frac{dx}{\sinh ax} &= \frac{1}{a}\ln\left|\tanh\frac{ax}{2}\right| + C \\ &= \frac{1}{a}\ln\left|\frac{\cosh ax + 1}{\sinh ax}\right| + C \\ &= \frac{1}{a}\ln\left|\frac{\sinh ax}{\cosh ax + 1}\right| + C \\ &= \frac{1}{2a}\ln\left|\frac{\cosh ax - 1}{\cosh ax + 1}\right| + C\end{aligned}$$

$$\int \frac{dx}{\sinh^n ax} = -\frac{\cosh ax}{a(n-1)\sinh^{n-1} ax} - \frac{n-2}{n-1}\int \frac{dx}{\sinh^{n-2} ax} \qquad (\text{for } n\neq 1)$$

$$\int x\sinh ax\,dx = \frac{1}{a}x\cosh ax - \frac{1}{a^2}\sinh ax + C$$

$$\int (\sinh ax)(\sinh bx)\,dx = \frac{1}{a^2-b^2}\big(a(\sinh bx)(\cosh ax) - b(\cosh bx)(\sinh ax)\big) + C \qquad (\text{for } a^2\neq b^2)$$

Integrals involving only hyperbolic cosine functions

$$\int \cosh ax\,dx = \frac{1}{a}\sinh ax + C$$

$$\int \cosh^2 ax\,dx = \frac{1}{4a}\sinh 2ax + \frac{x}{2} + C$$

$$\int \cosh^n ax\,dx = \begin{cases} \frac{1}{an}(\sinh ax)(\cosh^{n-1} ax) + \frac{n-1}{n}\int \cosh^{n-2} ax\,dx, & n>0 \\ -\frac{1}{a(n+1)}(\sinh ax)(\cosh^{n+1} ax) + \frac{n+2}{n+1}\int \cosh^{n+2} ax\,dx, & n<0, n\neq -1 \end{cases}$$

$$\begin{aligned}\int \frac{dx}{\cosh ax} &= \frac{2}{a}\arctan e^{ax} + C \\ &= \frac{1}{a}\arctan(\sinh ax) + C\end{aligned}$$

$$\int \frac{dx}{\cosh^n ax} = \frac{\sinh ax}{a(n-1)\cosh^{n-1} ax} + \frac{n-2}{n-1}\int \frac{dx}{\cosh^{n-2} ax} \qquad (\text{for } n\neq 1)$$

$$\int x\cosh ax\,dx = \frac{1}{a}x\sinh ax - \frac{1}{a^2}\cosh ax + C$$

$$\int x^2\cosh ax\,dx = -\frac{2x\cosh ax}{a^2} + \left(\frac{x^2}{a} + \frac{2}{a^3}\right)\sinh ax + C$$

$$\int (\cosh ax)(\cosh bx)\,dx = \frac{1}{a^2-b^2}\big(a(\sinh ax)(\cosh bx) - b(\sinh bx)(\cosh ax)\big) + C \qquad (\text{for } a^2 \neq b^2)$$

$$\int \frac{dx}{1+\cosh(ax)} = \frac{2}{a}\frac{1}{1+e^{-ax}} + C \qquad \text{or } \frac{2}{a} \text{ times The Logistic Function}$$

Other integrals

Integrals of hyperbolic tangent, cotangent, secant, cosecant functions

$$\int \tanh x\,dx = \ln\cosh x + C$$

$$\int \tanh^2 ax\,dx = x - \frac{\tanh ax}{a} + C$$

$$\int \tanh^n ax\,dx = -\frac{1}{a(n-1)}\tanh^{n-1} ax + \int \tanh^{n-2} ax\,dx \qquad (\text{for } n \neq 1)$$

$$\int \coth x\,dx = \ln|\sinh x| + C,\ \text{for } x \neq 0$$

$$\int \coth^n ax\,dx = -\frac{1}{a(n-1)}\coth^{n-1} ax + \int \coth^{n-2} ax\,dx \qquad (\text{for } n \neq 1)$$

$$\int \operatorname{sech}\, x\,dx = \arctan(\sinh x) + C$$

$$\int \operatorname{csch}\, x\,dx = \ln\left|\tanh\frac{x}{2}\right| + C = \ln|\coth x - \operatorname{csch} x| + C,\ \text{for } x \neq 0$$

Integrals involving hyperbolic sine and cosine functions

$$\int (\cosh ax)(\sinh bx)\,dx = \frac{1}{a^2-b^2}\big(a(\sinh ax)(\sinh bx) - b(\cosh ax)(\cosh bx)\big) + C \qquad (\text{for } a^2 \neq b^2)$$

$$\begin{aligned}\int \frac{\cosh^n ax}{\sinh^m ax}\,dx &= \frac{\cosh^{n-1} ax}{a(n-m)\sinh^{m-1} ax} + \frac{n-1}{n-m}\int \frac{\cosh^{n-2} ax}{\sinh^m ax}\,dx \qquad (\text{for } m \neq n)\\ &= -\frac{\cosh^{n+1} ax}{a(m-1)\sinh^{m-1} ax} + \frac{n-m+2}{m-1}\int \frac{\cosh^n ax}{\sinh^{m-2} ax}\,dx \qquad (\text{for } m \neq 1)\\ &= -\frac{\cosh^{n-1} ax}{a(m-1)\sinh^{m-1} ax} + \frac{n-1}{m-1}\int \frac{\cosh^{n-2} ax}{\sinh^{m-2} ax}\,dx \qquad (\text{for } m \neq 1)\end{aligned}$$

$$\begin{aligned}\int \frac{\sinh^m ax}{\cosh^n ax}\,dx &= \frac{\sinh^{m-1} ax}{a(m-n)\cosh^{n-1} ax} + \frac{m-1}{n-m}\int \frac{\sinh^{m-2} ax}{\cosh^n ax}\,dx \qquad (\text{for } m \neq n)\\ &= \frac{\sinh^{m+1} ax}{a(n-1)\cosh^{n-1} ax} + \frac{m-n+2}{n-1}\int \frac{\sinh^m ax}{\cosh^{n-2} ax}\,dx \qquad (\text{for } n \neq 1)\\ &= -\frac{\sinh^{m-1} ax}{a(n-1)\cosh^{n-1} ax} + \frac{m-1}{n-1}\int \frac{\sinh^{m-2} ax}{\cosh^{n-2} ax}\,dx \qquad (\text{for } n \neq 1)\end{aligned}$$

Integrals involving hyperbolic and trigonometric functions

$$\int \sinh(ax+b)\sin(cx+d)\,dx = \frac{a}{a^2+c^2}\cosh(ax+b)\sin(cx+d) - \frac{c}{a^2+c^2}\sinh(ax+b)\cos(cx+d) + C$$

$$\int \sinh(ax+b)\cos(cx+d)\,dx = \frac{a}{a^2+c^2}\cosh(ax+b)\cos(cx+d) + \frac{c}{a^2+c^2}\sinh(ax+b)\sin(cx+d) + C$$

$$\int \cosh(ax+b)\sin(cx+d)\,dx = \frac{a}{a^2+c^2}\sinh(ax+b)\sin(cx+d) - \frac{c}{a^2+c^2}\cosh(ax+b)\cos(cx+d) + C$$

$$\int \cosh(ax+b)\cos(cx+d)\,dx = \frac{a}{a^2+c^2}\sinh(ax+b)\cos(cx+d) + \frac{c}{a^2+c^2}\cosh(ax+b)\sin(cx+d) + C$$
