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List of common coordinate transformations

This is a list of some of the most commonly used coordinate transformations.

2-dimensional

Let (x, y) be the standard Cartesian coordinates, and (r, θ) the standard polar coordinates.

To Cartesian coordinates

From polar coordinates

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ \frac{\partial(x,y)}{\partial(r,\theta)} &= \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \\ \text{Jacobian} = \det\frac{\partial(x,y)}{\partial(r,\theta)} &= r \end{aligned}$$

From log-polar coordinates

$$\begin{aligned} x &= e^\rho \cos\theta, \\ y &= e^\rho \sin\theta. \end{aligned}$$

By using complex numbers $(x, y) = x + iy'$, the transformation can be written as

$$x + iy = e^{\rho + i\theta}$$

That is, it is given by the complex exponential function.

From bipolar coordinates

$$\begin{aligned} x &= a\frac{\sinh\tau}{\cosh\tau - \cos\sigma} \\ y &= a\frac{\sin\sigma}{\cosh\tau - \cos\sigma} \end{aligned}$$

From 2-center bipolar coordinates

$$x = \frac{1}{4c}\left(r_1^2 - r_2^2\right)$$
$$y = \pm\frac{1}{4c}\sqrt{16c^2 r_1^2 - \left(r_1^2 - r_2^2 + 4c^2\right)^2}$$

From Cesàro equation

$$x = \int \cos\left[\int \kappa(s)\, ds\right] ds$$
$$y = \int \sin\left[\int \kappa(s)\, ds\right] ds$$

To polar coordinates

From Cartesian coordinates

$$r = \sqrt{x^2 + y^2}$$
$$\theta' = \arctan\left|\frac{y}{x}\right|$$

Note: solving for θ' returns the resultant angle in the first quadrant ($0 < \theta < \frac{\pi}{2}$). To find θ, one must refer to the original Cartesian coordinate, determine the quadrant in which θ lies (for example, (3,−3) [Cartesian] lies in QIV), then use the following to solve for θ :

$$\theta = \begin{cases} \theta' & \text{for } \theta' \text{ in QI:} & 0 < \theta' < \frac{\pi}{2} \\ \pi - \theta' & \text{for } \theta' \text{ in QII:} & \frac{\pi}{2} < \theta' < \pi \\ \pi + \theta' & \text{for } \theta' \text{ in QIII:} & \pi < \theta' < \frac{3\pi}{2} \\ 2\pi - \theta' & \text{for } \theta' \text{ in QIV:} & \frac{3\pi}{2} < \theta' < 2\pi \end{cases}$$

The value for θ must be solved for in this manner because for all values of θ, $\tan\theta$ is only defined for $-\frac{\pi}{2} < \theta < +\frac{\pi}{2}$, and is periodic (with period π). This means that the inverse function will only give values in the domain of the function, but restricted to a single period. Hence, the range of the inverse function is only half a full circle.

Note that one can also use

$$r = \sqrt{x^2 + y^2}$$
$$\theta' = 2\arctan\frac{y}{x + r}$$

From 2-center bipolar coordinates

$$r = \sqrt{\frac{r_1^2 + r_2^2 - 2c^2}{2}}$$

$$\theta = \arctan\left[\sqrt{\frac{8c^2(r_1^2 + r_2^2 - 2c^2)}{r_1^2 - r_2^2}} - 1\right]$$

Where $2c$ is the distance between the poles.

To log-polar coordinates from Cartesian coordinates

$$\rho = \log\sqrt{x^2 + y^2},$$

$$\theta = \arctan\frac{y}{x}.$$

Arc-length and curvature

In Cartesian coordinates

$$\kappa = \frac{x'y'' - y'x''}{\left(x'^2 + y'^2\right)^{\frac{3}{2}}}$$

$$s = \int_a^t \sqrt{x'^2 + y'^2}\, dt$$

In polar coordinates

$$\kappa = \frac{r^2 + 2r'^2 - rr''}{\left(r^2 + r'^2\right)^{\frac{3}{2}}}$$

$$s = \int_a^\varphi \sqrt{r^2 + r'^2}\, d\varphi$$

3-dimensional

Let (x, y, z) be the standard Cartesian coordinates, and (ρ, θ, φ) the spherical coordinates, with θ the angle measured away from the +Z axis (as [1] (https://commons.wikimedia.org/wiki/File:3D_Spherical.svg), see conventions in spherical coordinates). As φ has a range of 360° the same considerations as in polar (2 dimensional) coordinates apply whenever an arctangent of it is taken. θ has a range of 180°, running from 0° to 180°, and does not pose any problem when calculated from an arccosine, but beware for an arctangent.

If, in the alternative definition, θ is chosen to run from −90° to +90°, in opposite direction of the earlier definition, it can be found uniquely from an arcsine, but beware of an arccotangent. In this case in all formulas below all arguments in θ should have sine and cosine exchanged, and as derivative also a plus and minus exchanged.

All divisions by zero result in special cases of being directions along one of the main axes and are in practice most easily solved by observation.

To Cartesian coordinates

From spherical coordinates

$$\begin{aligned}
x &= \rho \, \sin\theta \cos\varphi \\
y &= \rho \, \sin\theta \sin\varphi \\
z &= \rho \, \cos\theta \\
\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} &= \begin{pmatrix} \sin\theta\cos\varphi & \rho\cos\theta\cos\varphi & -\rho\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & \rho\cos\theta\sin\varphi & \rho\sin\theta\cos\varphi \\ \cos\theta & -\rho\sin\theta & 0 \end{pmatrix}
\end{aligned}$$

So for the volume element:

$$dx\,dy\,dz = \det\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)}\,d\rho\,d\theta\,d\varphi = \rho^2\,\sin\theta\,d\rho\,d\theta\,d\varphi$$

From cylindrical coordinates

$$\begin{aligned}
x &= r \, \cos\theta \\
y &= r \, \sin\theta \\
z &= z \\
\frac{\partial(x,y,z)}{\partial(r,\theta,z)} &= \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

So for the volume element:

$$dV = dx\,dy\,dz = \det\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\,dr\,d\theta\,dz = r\,dr\,d\theta\,dz$$

To spherical coordinates

From Cartesian coordinates

$$\begin{aligned}\rho &= \sqrt{x^2 + y^2 + z^2} \\ \theta &= \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right) = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) \\ \varphi &= \arctan\left(\frac{y}{x}\right) = \arccos\left(\frac{x}{\sqrt{x^2 + y^2}}\right) = \arcsin\left(\frac{y}{\sqrt{x^2 + y^2}}\right) \\ \frac{\partial\left(\rho, \theta, \varphi\right)}{\partial\left(x, y, z\right)} &= \begin{pmatrix} \frac{x}{\rho} & \frac{y}{\rho} & \frac{z}{\rho} \\ \frac{xz}{\rho^2\sqrt{x^2+y^2}} & \frac{yz}{\rho^2\sqrt{x^2+y^2}} & -\frac{\sqrt{x^2+y^2}}{\rho^2} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{pmatrix}\end{aligned}$$

See also the article on atan2 for how to elegantly handle some edge cases.

So for the element:

$$d\rho\,d\theta\,d\varphi = \det\frac{\partial(\rho, \theta, \varphi)}{\partial(x, y, z)}\,dx\,dy\,dz = \frac{1}{\sqrt{x^2 + y^2}\sqrt{x^2 + y^2 + z^2}}\,dx\,dy\,dz$$

From cylindrical coordinates

$$\begin{aligned}\rho &= \sqrt{r^2 + h^2} \\ \theta &= \arctan\frac{r}{h} \\ \varphi &= \varphi \\ \frac{\partial(\rho, \theta, \varphi)}{\partial(r, h, \varphi)} &= \begin{pmatrix} \frac{r}{\sqrt{r^2+h^2}} & \frac{h}{\sqrt{r^2+h^2}} & 0 \\ \frac{h}{r^2+h^2} & \frac{-r}{r^2+h^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \det\frac{\partial(\rho, \theta, \varphi)}{\partial(r, h, \varphi)} &= \frac{1}{\sqrt{r^2 + h^2}}\end{aligned}$$

To cylindrical coordinates

From Cartesian coordinates

$$\begin{aligned}r &= \sqrt{x^2 + y^2} \\ \theta &= \arctan\left(\frac{y}{x}\right) \\ z &= z\end{aligned}$$

$$\frac{\partial(r,\theta,h)}{\partial(x,y,z)} = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} & 0 \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

From spherical coordinates

$$\begin{aligned} r &= \rho \sin\varphi \\ h &= \rho \cos\varphi \\ \theta &= \theta \\ \frac{\partial(r,h,\theta)}{\partial(\rho,\varphi,\theta)} &= \begin{pmatrix} \sin\varphi & \rho\cos\varphi & 0 \\ \cos\varphi & -\rho\sin\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \det \frac{\partial(r,h,\theta)}{\partial(\rho,\varphi,\theta)} &= -\rho \end{aligned}$$

Arc-length, curvature and torsion from Cartesian coordinates

$$\begin{aligned} s &= \int_0^t \sqrt{x'^2 + y'^2 + z'^2}\, dt \\ \kappa &= \frac{\sqrt{(z''y' - y''z')^2 + (x''z' - z''x')^2 + (y''x' - x''y')^2}}{\left(x'^2 + y'^2 + z'^2\right)^{\frac{3}{2}}} \\ \tau &= \frac{x'''\,(y'z'' - y''z') + y'''\,(x''z' - x'z'') + z'''\,(x'y'' - x''y')}{(x'y'' - x''y')^2 + (x''z' - x'z'')^2 + (y'z'' - y''z')^2} \end{aligned}$$

See also

- Geographic coordinate conversion
- Transformation matrix

References

- Arfken, George (2013). *Mathematical Methods for Physicists* (https://books.google.com/books?id=qLFo_Z-PoGIC). Academic Press. ISBN 978-0123846549.