Episodic Ologs : Category Theoretic Knowledge Representation Based on Episodic Logic with Coherence Based Knowledge Acquisition

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Abstract

[1] We propose a semantic representation/knowledge representation (SR/KR) based on Spivak and Kent's ologs and Schubert's Episodic Logic with attention to issues of bias in knowledge acquisition. Ologs are a category theoretic approach to SR/KR where concepts are objects and conceptual relations are morphisms, Episodic Logic is a SR/KR with a Montague style logical form, oriented towards deep natural language understanding (NLU) vis-à-vis attention to sematic expressivity and reasoning/inference. One of the main challenges in NLU is how to populate an NLU system with sufficient data to make natural derivations of meaning and inferences about the meaning in general or in some situation. Part of the challenge of acquiring knowledge for an NLU system is the discrepancy between reality and its descriptions in text (reporting bias). We will sketch how ologs could be extended to incorporate the semantic features of Episodic Logic and how the formalization of instance data present in ologs could be used to establish a notion of knowledge coherence which could mitigate issues of reporting bias.

1 Introduction

Schubert (2015) argues that a SR/KR oriented towards comprehensive NLU suitable for reasoning needs to capture the full spectrum of different kinds of semantic content and do so in a form suited to inference. In the same work the author argues that Episodic Logic is especially competitive in both of these regards compared to other SR/KRs. An additional issue discussed in the same paper is the knowledge acquisition bottleneck, where acquiring sufficient contextual and general knowledge to guide deep NLU and inference is difficult to accomplish. What's more, textual sources of knowledge aiming to be factual often differ from reality in ways that are difficult to discern without sufficient background knowledge, whether via benign issues of implication and emphasis of content or non-factuality and bias proper, a circular dependency in the problem barring exhaustive hand-authoring of verifiable knowledge. We will sketch how a category theoretic approach to SR/KR, in particular Spivak and Kent's ologs (Spivak and Kent, 2012), can provide a unified foundation for the the expressive and potentially the inferential features of Episodic Logic, an approach to issues of consistency and factuality in knowledge acquisition, as a means of grounding and coordinating multiple modular SR/KRs, and in general, an approach to SR/KR and knowledge acquisition which models some of the core mechanisms are used in semantic representation of data and knowledge acquisition at the human level.

[1] I am using ACL style guidelines and citation format.

2 Background

2.1 Overview of Category-Theoretic Terminology

Category theory, is a theory of mathematical structure and structural relationships. Roughly speaking, a *category* is a collection of *objects* of concern, and *morphisms* or *arrows*, describing relationships between objects in the category that satisfy certain basic properties, namely, every object has a morphism connecting it to itself, and morphisms between objects compose transitively, i.e. given for a morphism f from an object A to an object B, and another morphism g from objects B to C, written:

$$f : A \to B$$

$$g : B \to C$$

there is always another morphism:

$$g \circ f : A \to C$$

There are also *functors*, roughly, ways of relating a category $\mathcal{C}$ to another other category $\mathcal{D}$, and *natural transformations*, ways of relating two functors connecting the same categories, e.g. $F, G : \mathcal{C} \to \mathcal{D}$. There are many additional structures and concepts that category theory captures in addition to these (see appendix **A** for relevant definitions), all of which have in common category theory's relational emphasis and its resulting potential for abstraction. This emphasis is less apparent and requires work to recover in logic and type theory and their variants and extensions which ground most SR/KRs, although these category theoretic ideas *can* be be recovered in logic and type theory. As we will see, this relational focus of category theory has potential for developing a flexible SR/KR which can better reflect human organization and coordination of knowledge, in addition to providing a better setting for a SR/KR to fulfill Schubert's requirements.

2.2 Ologs

Spivak and Kent's ologs and Patterson's relational ologs both describe a category theoretic approach to SR/KR, where objects stand in for concepts and morphisms stand in for relations between concepts, and functors are used to model instances or realizations of concepts or relations between concepts. Relational ologs are oriented towards modeling Description Logic, which is decidable, unlike first order logic, at the expense of being less expressive than first order logic (Patterson, 2017), and have the further advantage of examination due to the beginnings of a computer implementation [2]. Ologs are also designed with the possibility of being easily converted into databases, which has a possible advantange over other SR/KRs in terms of efficient retrieval of information. On the other hand there seem to be no fully formed advancements in automated inference for ologs, although there published efforts towards automated inference in category theory (Kozen et al., 2006). Given the maturity and uniqueness of ologs as a category theoretic SR/KR, these will be our starting point for developing a category theoretic SR/KR with Schubert's requirements in mind, in particular, to recover as many of the expressive devices available in Episodic Logic in an olog based SR/KR.

[2]https://github.com/AlgebraicJulia/Catlab.jl

3 Episodic Logic

Episodic Logic is a Montague-style logical form based SR/KR with relative strength in semantic expressivity and inferetiability in comparison to other SR/KRs, making it better comparatively better suited for deep NLU (Schubert, 2015). Episodic Logic allows for generalized quantifiers, lambda abstraction, reification and modification of sentences and predicates, intensional predicates, unreliable generalizations, and explicit situational variables (Schubert and Hwang, 2000). Episodic Logic with its inference engine EPILOG provide a way to capture a relatively comprehensive range of semantic phenomena compared to other SR/KRs, in a way which affords inference about semantic data at a comparable efficiency with automated inference engines for first order logic (Schubert, 2015). It also has an associated knowledge base KNEXT which is capable of parsing sentences into factoids, generalizing them (through a process called quantificational sharpening), and making certain judgements about whether a generalized factoid is redudant or inconsistent with anything established in the knowledge base.

4 Category Theory and Modeling Conceptual Understanding

Human knowledge acquisition makes heavy use of comparison and analogy. Finding differences and commonalities between currently understood concepts, situations, or data in general, and a new instance of the same kind of data can help to understand the new data in a systematic way (Brown and Porter, 2006). This can be extended to cases where the known and new data might have some overlap, in which case the burden of labor for acquiring the new knowledge depends on where the new data differs from known data, e.g. if one already knows how to print a document from a device, then learning how to print the document double sided is mostly an issue of remembering how to select a relevant option during the rest of a process which is already known. Category theory and its focus on structure, relationships between structures, and abstraction over structure provide a convenient setting for formalizing certain formal notions of analogy and comparison (Brown and Porter, 2006), which adds further credence to the case that a category theoretic SR/KR can be used to better model human level knowledge acquisition. Navarrete and Dartnell (2017) further develops this idea in a way that situates category theoretic treatments of different kinds of analogy explicitly within human learning and cognition, developing concrete models of different kinds of analogy that could be used in an SR/KR. Phillips and Wilson (2010) even asserts that category theory as a means of modeling relationships between structures or representations, and relationships between such relationships etc., can be used to model core aspects of higher-cognition in general. All of these support the potential for such a category theoretic SR/KR's ability to better model human-level reasoning and knowledge acquisition.

5 Category Theory in Knowledge Acquisition

Gordon and Durme (2013) explain how reporting bias can effect development of a SR/KR and its knowledge base during knowledge acquisition. Reporting bias is a collection of phenomena in language data, namely, how what is described in text can fail to be factual depending on genre and other context, and how the lexical and world knowledge humans use to discern whether textual material can be possibly considered factual, is something that generally is not represented in text. This poses a problem for knowledge acquisition in that there is not just the issue of how to formally judge newly presented information against some set of ground truths, but acquiring these ground truths cannot be automatic by and large, e.g. there is no sufficiently comprehensive set of textual material from which an SR/KR could be populated in order to make sufficiently educated judgements. Healy and Caudell (2004) develop the idea of a category of concepts, i.e. the objects are concepts and the morphisms are conceptual relations, which can encode a hierarchy of concepts via considering the notions of sub-concept or super-concept relations as concept morphisms. They further develop considerations of different realizations (or instance data) among the same representation of diagram (structure of objects and morphisms) in the concept category. In the case of the Healy and Caudell (2004), neural networks are the representational context where concept structures are realized. The authors treat neural networks categorically like concepts, and incoporate functors from the concept category to the neural network category. From here, the authors develop a notion of *knowledge coherence* across the different realizations of a concept at a particular level using natural transformations (a structured relationship between two functors which map in the same direction between the same categories).

This formalization of conceptual hierarchy and conceptual relations and the way it can be used to develop an idea of knowledge coherence seems like it could be the beginning of a way to address reporting bias during knowledge acquisition. When some new information data is extracted, it can be evaluated at least based on coherency with previously seen information, a more general version of the comparison done during knowledge acquisition in KNEXT. A layered approach could also be taken along with conceptual hierarchy to model realizations of concepts at different levels of certainty. However, it is not clear in what senses or under what circumstances these ideas would be tractable to fully formalize, implement, or compute.

For the alterations relevant to knowledge acquisition, using analogy to enhance assimilation of new knowledge as discussed by Navarrete and Dartnell (2017) is another avenue of investigation, and in particular using the notion of knowledge coherence from Healy and Caudell (2004) to assess whether some new data is consistent with the existing data instances and/or the related concepts in general, as a means to address reporting bias encountered in knowledge acquisition. Another direction to explore is how to utilize the notion of instance data in ologs to create a multi-tiered understanding of some collection of concepts, which can be further used with the notion of knowledge coherence to model levels of certainty about how a concept can be realized.

6 Recovering the Expressive Devices of Episodic Logic In Ologs

Most of the semantic features of Episodic Logic can be recovered in in a set theoretic context, thus are amenable to being recovered in **cartesian closed** categories (CCC) (MacLane and Moerdijk, 1992). A CCC:

- Has a **terminal** object (every object in the category has a unique morphism to this object).
- Any two objects X, Y in a CCC have:
  - A **product** object $X \times Y$ (the categorical generalization of set-theoretic product).
  - An **exponential** object Y^X (roughly, the collection of morphisms from X to Y can be seen as an object in the category)

Generalized Quantifiers Classical quantifiers can be accounted for in in elementary topoi (a kind of CCC). The existential and universal quantifiers are formalized respectively as left and right **adjoints** of a functor between powersets (Goldblatt, 2006). As a result, quantifiers in general seem to be definable in any category theoretic context where one can define a notion of powersets and subsets for objects in the category in the same way that these notions are available in the category of sets. This is also the case for non-classical quantifiers (Hedges and Sadrzadeh, 2019), except further notions of set membership and cardinality need to be recovered for quantifiers like *some* or *most*, presumably through something like subobject classifiers. In all cases this would need to take place on the instance data/realization level of ologs given these set theoretic notions (especially the powerset functor) would not be able to be guaranteed at the concept level.

λ-abstraction Lambda abstractions can be modeled via exponential objects provided the category in question has an exponential object for every pair of objects, e.g. where $g : X$ and $\phi(g) : Y$, the term $\lambda g.\phi(g) : Y^X$. This seems possible to guarantee at the conceptual level, since "the collection of conceptual relations from concept A to concept B" itself can be considered a concept.

Sentence and Predicate Modification Adverbial modifiers would likely be represented by an object in the category describing the state of the subject of the modified predicate, connecting both the subject and the predicate (in category theoretic terms the morphism from the subject to the predicate *factors through* the morphism from the subject to the modifier). Non-adverbial modifiers like tense would ideally have to take on some kind of similar relational inpterpretation, whether in the same olog to some other diagram of objects, or to another olog ecoding some previous or future point in time. This has issues depending on the level of granularity requried, and while there are category theoretic approaches to describing time, this is an open area of research and it's unclear whether such approaches (Kato, 2017) are technically feasible or necessary to implement given their complexity.

Sentence and Predicate Reification Reification of sentences and predicates are broadly, cases of mention instead of use, e.g. *Carmen put in new lighting.* is a use of the sentence whereas *Carmine said Carmen put in new lighting.* involves the mention of the sentence, thus reification in EL involves reference of propositional content by other propositional content or object. The first impulse in this case is to first describe the reified content as some diagram and the entity referring to the content as a *limit* of the diagram, and find some way to have the morphisms of the limit (the morphisms modeling the referrence of the entity to the content) reflect the nature of referrence, which on one level seems as if it would be an issue of labeling morphisms as is typically done in ologs, but on the other hand there is a difference in interpretation of reified content which would need to also be reflected, since for instance, *Carmen put in new lighting.* and *Carmen didn't put in new lighting.* cannot both be true, but *Carmine said Carmen put in new lighting.* and *Charlotte said Carmen didn't put in new lighting.* can both be true as their truth values do not depend on the reified content. In this sense of relating content in the form of diagram or otherwise to some other object or diagram that requires a difference in interpretation, capturing this is similar to capturing intentional predicates or sentential modifiers.

Intentional Predicates Similar considerations for reified content and sentential modification would likely have to be made for intensional predicates (having to do with an agent believing, wanting) in terms of interpretation and how to encode the relationship between the agent and different kinds of themes, such as objects versus propositional content, e.g. *Claire believed the paper* vs *Claire believed that the paper's reasoning was correct if poorly presented.* Ultimately an agent in an olog would be represented by an object and the agent-specific predicate would be a labeled morphisms or a collection of labeled morphisms from the agent to some other object or diagram describing a situation (again the notion of a *limit* of a diagram). As with reification and sentential modification this could also potentially involve a functor into another olog to model the difference in interpreting the content versus interpreting the agent's relationship to the content.

Non-Logical Generalization Non-logical/defeasible generalizations would likely be heavily dependent on context and thus, where not hand-authored, would likely benefit from the idea of knowledge coherence used in knowledge acquisition. In the case of hand-authoring they would be encoded into domain specific ologs.

Situational Variables Casual and and situational relationships can be encoded in general or domain specific ologs at the conceptual level, and where necessary can be realized in instance data, where they would be susceptible in principle to the same reasoning tools at the conceptual level.

7 Experiments and Future Work

7.1 Implementation

EPILOG, the inference engine for Episodic Logic, and KNEXT a knowledge base utilizing Episodic Logic are both written in Lisp. In general functional programming seems to be a setting amenable to implementation of logical or semantic structures, in particular Haskell lends itself to this (van Eijck and Unger, 2010) along with implementation of category theoretic formalisms (Milewski, 2018). Tentatively we can say an implementation of Episodic ologs will use Haskell to implement these category theoretic formalisms which capture the expressive devices of Episodic Logic in a way informed by the way EPILOG and KNEXT handle these in the context of Lisp. A complete implementation of Episodic Ologs would require an implementation of ologs, implementation of the additional structures needed to capture the semantic features of EL within ologs, an implementation of some form of automated inference in order to make it competitive with other SR/KRs oriented towards deep NLU, and at least an attempt at implementing tests of knowledge coherence towards negotiating issues of reporting bias.

7.1.1 A Knowledge Base: Ologs and Databases

Any valid olog by design admits a database schema and any instance data or realization of of that olog can be seen as a database state fitting that schema (Spivak and Kent, 2012). This potentially affords the possibility of a knowledge base that can make use of more standard and efficient database tools. Haskell has libraries for interfacing with database engines which support SQL (HDBC, HSQL), but finding a way to do this in a way that is formally acceptable with the rest of the implementation (set theoretic and category theoretic reasoning about instance data) might take a significant amount of effort compared to the rest of the project.

7.2 Knowledge Acquisition Experiments

In terms of eventual experimentation that compares this approach with other SR/KRs, some implementation of this SR/KR in a given language and especially the methods to detect reporting bias (via knowledge coherence as natural transformations over functors of instance data) would be necessary. There would need to be an investigation into whether other SR/KRs attempt this classification, and then development of some kind of annotated corpus of data of varying degrees of factuality, and evaluating our system (along with the others if applicable) on whether (comparatively) this approach actually has the potential for efficacy in determining the reliability and admissibility of new data. The other issue which this experiment is that it depends on having knowledge base utilizing this SR/KR populated sufficient basic data to make these judgements, thus there would initially have to be significantly rich hand-authored ologs in order to have a basis for judgement of incoming textual data. From these experiments (or the failure to implement them) possibilities of where using probabalistic methods, such as involving machine learning, could aid in the overall process, likely using probabalistic methods and olog where efficiency and verifiability, respectively, are needed.

7.3 Further Formalization and Automated Inference

As discussed before, category theoretic techniques seem to hold the potential for abstracting over different SR/KRs or at least their onologies and functioning as a foundation to integrate them. If something like this can be done with the implementation, where at the base is something more general like a basic olog structure and additional expressive and descriptive features for different ontologies can be captured in additional modular layers. Additionally, one way in which this olog-based approach to SR/KR is lacking compared to Episodic Logic, there have been efforts to integrate category theory into automated theorem proving environments like Coq, for instance (Gross et al., 2014) and [3], even though we have found no efforts which have been specific to ologs besides some plans for future research discussed in Patterson (2017). One possible avenue for automated inference given an implementation of ologs in Haskell, is that there are successful tools for using Coq to verify Haskell pro-

[3] https://github.com/jwiegley/category-theory

grams (Breitner et al., 2018). However this would likely incur not insignificant organizational and computational complexity in the implementation.

7.4 Scope and Timeline

Given the structural complexity of the different tasks, minus issues of getting certain libaries to interact with each other as intended, it seems most feasible to first make a naïve implementation of ologs in Haskell which can coordinate with a particular database engine, followed by implementation of the different semantic features of Episodic Logic, followed by implementation of automated inference, followed by implementation of knowledge acquisition and notions of knowledge coherence.

7.5 Weaknesses

There are many weaknesses to this proposal. All of the challenges in creating an implementation described, assuming they can be overcome even in a straightforward way, would require a significant number of person-hours, anything that could be built in the span of a semester, even with multiple people, would in all likelihood be very brittle and difficult to meaningfully test. For comparison, Episodic Logic and its corresponding software suite have been in development since the late 90s, and while impresive, they still have noticable shortcomings. Finally, while the abbreviation "SR/KR" has been used throughout this paper to suggest that semantic representation and knowledge representation are interchangeable in a number of critical ways, ologs were designed for the task of knowledge representation, ecoding meaning in a fixed, general, and conceptual way. Tasks oriented towards NLU, require dealing with information which is not knowledge, information that is less certain and potentially more subtle and granular, and working with it in ways that involve dynamically making assumptions, and in general using defeasible reasoning. If a category-theoretic SR/KR is to be used for NLU, it must capture these different levels of certainty and ambiguity, neither ologs nor any of the other category theoretic proposals we have encountered seem really adequate to capture this without significant alteration.

8 Conclusion

We have developed a case for using a category theoretic approach to SR/KR along with integrating features suited to deep NLU, and sketched how this might take place. We gave some background description of basic category theoretic concepts and ologs, a sketch of how to recover each of the expressive features of Episodic Logic in a category theoretic context, an explanation of how category theory can be used to mitigate issues of reporting bias in knowledge acquisition, how category theoretic ideas have a possible advantage in modeling knowledge acquisition and conceptual understanding in a way more concomittant with NLU. We discussed a general approach to implementing these ologs with the expressive devices of Episodic Logic and additional functionality the implementation would need for practical use. Given the advantages we've seen of using category theoretic techniques to coordinate different expressive devices, along with the features which could potentially be used to mitigate issues of knowledge acquisition, enriching ologs with these capabilities and the expressive devices of Episodic Logic seems to be a worthwhile avenue of exploration. Even if the only purpose it serves is to further articulate the weaknesses described in the previous subsection, and thereby develope a more informed understanding of what would be needed in the design of category-theoretic SR/KRs for NLU, or of SR/KRs for NLU in general.

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A Appendices

A.1 Preliminaries

A class is a collection of members (or elements). Like the notion of a set, it tentatively resists formal definition. A proper class is a class that cannot be a member of another class. Sets are classes which are not proper classes.

A.2 Categories

A Category $\mathcal{C}$ is the following entities and governing axioms:

1. A class of **objects**, $Obj(\mathcal{C})$

2. A class of **morphisms**, $Hom(\mathcal{C})$, where every morphism f has a source object A, and a target object B, both of which are in $Obj(\mathcal{C})$. The morphism (also called an **arrow**) is written $f : A \to B$. We write $\mathcal{C}[A, B]$ to denote the **Hom-class** of morphisms from A to B. (Informally, these are called **Hom-sets**).

3. **Composition** For every A, B and C in $Obj(\mathcal{C})$, a binary operation

   $$\mathcal{C}[A, B] \times \mathcal{C}[B, C] \to \mathcal{C}[A, C]$$

   called the composition of morphisms. We write the composition of $f : A \to B$ and $g : B \to C$ as:

   $$g \circ f : A \to C$$

4. **Associativity**

   For $f : A \to B$, $g : B \to C$ and $h : C \to D$:

   $$h \circ (g \circ f) = (h \circ g) \circ f$$

5. **Identity**

   For every object X, there exists the **identity** morphism $1_X : X \to X$, such that for every $f : A \to X$ and $g : X \to B$, $1_X \circ f = f$ and $g \circ 1_X = g$.

A.3 Diagrams

Intuitively, a diagram is a graph, depicting a category or a particular part of a category about which we are curious [1]. For instance consider the following objects in **Set**: $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$. Also the following morphisms in **Set**, the inclusion maps: $i_1 : \mathbb{N} \to \mathbb{Z}$, $i_2 : \mathbb{N} \to \mathbb{R}$, $i_3 : \mathbb{Z} \to \mathbb{C}$, and $i_4 : \mathbb{R} \to \mathbb{C}$. Given these, a diagram in **Set** could be:

$$\begin{tikzcd}
\mathbb{N} \arrow[r, "i_2"] \arrow[d, "i_1"'] & \mathbb{R} \arrow[d, "i_4"] \\
\mathbb{Z} \arrow[r, "i_3"'] & \mathbb{C}
\end{tikzcd}$$

A diagram, however leaves all the *nonessential* details for the reader to fill in. Because given the earlier objects and morphisms, with the category's axioms of identity and composition, the complete version of the earlier diagram is:

$$\begin{tikzcd}
\mathbb{N} \arrow[loop above left, "1_{\mathbb{N}}"] \arrow[r, "i_2"] \arrow[d, "i_1"'] \arrow[dr, bend left, "i_4 \circ i_2"] \arrow[dr, bend right, "i_3 \circ i_1"'] & \mathbb{R} \arrow[loop above right, "1_{\mathbb{R}}"] \arrow[d, "i_4"] \\
\mathbb{Z} \arrow[loop below left, "1_{\mathbb{Z}}"] \arrow[r, "i_3"'] & \mathbb{C} \arrow[loop below right, "1_{\mathbb{C}}"]
\end{tikzcd}$$

We say a diagram *commutes* if all directed paths in the diagram lead to the same result by composition. For instance our earlier diagram commutes:

$$\begin{tikzcd}
\mathbb{N} \arrow[r, "i_2"] \arrow[d, "i_1"'] & \mathbb{R} \arrow[d, "i_4"] \\
\mathbb{Z} \arrow[r, "i_3"'] & \mathbb{C}
\end{tikzcd}$$

since:

$$i_3 \circ i_1 = i_4 \circ i_2$$

A.4 Products and Exponential Objects

Products For a category $\mathcal{C}$, and X_1, X_2 in $Obj(\mathcal{C})$, a **product** of X_1 and X_2 is an object of $\mathcal{C}$ usually written as $X_1 \times X_2$ with a pair of morphisms, called *canonical projections*:

$$\pi_1 : X_1 \times X_2 \to X_1$$

$$\pi_2 : X_1 \times X_2 \to X_2$$

such that, for every Y in $Obj(\mathcal{C})$, every f_1 in $Hom_{\mathcal{C}}(Y, X_1)$, and every f_2 in $Hom_{\mathcal{C}}(Y, X_2)$, there is a unique morphism $f : Y \to X_1 \times X_2$ such that:

$$\begin{tikzcd}
& Y \arrow[dl, "f_1"'] \arrow[d, dashed, "f"'] \arrow[dr, "f_2"] & \\
X_1 & X_1 \times X_2 \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & X_2
\end{tikzcd}$$

commutes.

Exponential Objects Where $\mathcal{C}$ is a category, Z and Y objects of $\mathcal{C}$, and $\mathcal{C}$ has all binary products with Y. An exponential object consists of an object

$$Z^Y$$

and a morphism

$$\mathbf{eval} : (Z^Y \times Y) \to Z$$

such that for any object X and morphism

$$g : X \times Y \to Z$$

there is a unique morphism $\lambda g : X \to Z^Y$ such that the following diagram commutes.

$$\begin{tikzcd}
X \arrow[d, dashed, "\lambda g"'] & X \times Y \arrow[d, dashed, "\lambda g \times 1_Y"'] \arrow[dr, "g"] & \\
Z^Y & Z^Y \times Y \arrow[r, "\mathbf{eval}"'] & Z
\end{tikzcd}$$

The notation Z^Y for the collection of morphisms from Y to Z might seem backwards at first, but there is an analogy here with integer exponents. For example, the number of total functions from a set of three elements to a set of two elements is eight, in notation, $2^3 = 8$.

1. An **initial** object **False** in a category $\mathcal{C}$, is an object, where, for each object X in $Obj(\mathcal{C})$ there is a unique morphism from **False** to X.

2. A **terminal** object **True** in a category $\mathcal{C}$, is an object, where, for each object X in $Obj(\mathcal{C})$ there is a unique morphism from X to **True**.

[1]The formal definition is more complicated but equivalent.

A.5 Functors and Natural Transformations

Functors Given categories $\mathcal{C}$ and $\mathcal{D}$ a **functor** $F : \mathcal{C} \to \mathcal{D}$ is a mapping between categories with the following characteristics:

1. For every X in $Obj(\mathcal{C})$, there is an associated object $F(X)$ in $Obj(\mathcal{D})$

2. For every $f : X \to Y$ in $Hom(\mathcal{C})$, there is an associated morphism $F(f) : F(X) \to F(Y)$ in $Hom(\mathcal{D})$ such that the following hold:

   (a) F preserves identity morphisms, i.e. for any X in $Obj(\mathcal{C})$:

   $$F(1_X) = 1_{F(X)}$$

   (b) F preserves composition of morphisms, i.e. for any $f : X \to Y$ and $g : Y \to Z$ in $Hom(\mathcal{C})$:

   $$F(g \circ f) = F(g) \circ F(f)$$

Natural Transformations Given functors

$$F : \mathcal{C} \to \mathcal{D}$$

$$G : \mathcal{C} \to \mathcal{D}$$

a **natural transformation**, $\eta : F \to G$ consists of, for every X in $Obj(\mathcal{C})$, a **component map**:

$$\eta_X : F(X) \to G(X)$$

such that for every f in $\mathcal{C}[X, Y]$ the following diagram commutes:

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(X) \arrow[r, "\eta_Y"'] & G(Y) \end{tikzcd}$$

B Supplemental Material

C Category Theory as a Meta-SR/KR

A SR/KR generally constitutes an ontology and a logic (Sowa, 2000), and just as knowledge can be domain specific, so can SR/KRs, in which case the ontology and logic respectively concern the objects of the domain and their interactions, relations, and operations. Human level knowledge and reasoning however involves not only knowledge in multiple dependent and independent domains, but the ability to coordinate knowledge and reasoning among different domains, often in real time. Institutions are category theoretic treatment of the notion of a "logical system" which can provide the ability to range over multiple theories and manage ontologies, (Kent, 2018), (Schorlemmer and Kalfoglou, 2008) which enables multiple kinds of problem solving and domains of reasoning. However this meta approach isn't limited to institutions

Integrating mutliple ontologies and their corresponding logics is also necessary for an SR/KR which hopes to capture multiple domains of knowledge since, while different objects, structures, or patterns of reasoning might be meaningful in their respective domains, their combination is not necessarily meaningful given a reasonably comprehensive ontology, and for this reason, an idea or treatment of meaning, within SR/KRs, depends fundamentally on some interpretation (Sarbo, 2013). Similarly, category theory can be used to merge ontologies (Hitzler et al., 2005), though how this approach handles the different reasoning methodologies accompanying different ontologies, and whether this approach is advantageous for our goals in SR/KR over the aforementioned approaches, is something to investigate. Patterson (2017), cites that further work in coordinating different ontologies in a categorical SR/KR could potentially be accoomplished using doctrines or sketches, both category theoretic models of the idea of a "theory" in logic (a formal language used to axiomatize some collection of one or more formal systems or models.)