Let us consider an optimisation problem

$$\begin{aligned}
\bar{\gamma} &:= \arg\min_{\gamma\in\Omega} L_{\Theta}(\gamma) \\
\gamma &:= [\gamma_1,\dots,\gamma_K] \in \mathbb{R}^{KT}, \\
\gamma_k &:= [\gamma_k(1),\dots,\gamma_k(T)] \in \mathbb{R}^{T} \\
L_{\Theta}^{\epsilon}(\gamma) &:= \tfrac{1}{T} b_{\Theta}^T \gamma + \tfrac{\epsilon^2}{T}\gamma^T H \gamma, \\
\Omega &:= \{\gamma \in \mathbb{R}^{KT} : \gamma \geq 0 \wedge \sum_{k=1}^{K} \gamma(t) = 1, \forall t = 1,\dots T\}
\end{aligned}$$

and $H \in \mathbb{R}^{KT,KT}$ is block-diagonal matrix, whose blocks $L_k \in \mathbb{R}^{T,T}$ are formed by Laplace matrix.
In what follows, we proved that this problem has always unique solution.

1 Properties

- H is SPS (since blocks are SPS) and

$$\begin{aligned}
\operatorname{Ker} H &= \operatorname{span}\{[c,\mathbf{0},\dots,\mathbf{0}]^T, [\mathbf{0},c,\mathbf{0}\dots,\mathbf{0}]^T,\dots,[\mathbf{0}\dots,\mathbf{0},c]^T\} \subset \mathbb{R}^{KT}, \\
c &:= [1,\dots,1] \in \mathbb{R}^T
\end{aligned} \tag{1}$$

- $L_{\Theta}^{\epsilon}(\gamma)$ is continuous (not strictly) convex function $\mathbb{R}^{KT} \to \mathbb{R}$,
- $\Omega \subset \mathbb{R}^{KT}$ is bounded closed convex set (has to be proved?),
- the matrix $B = [I,\dots I] \in \mathbb{R}^{T,KT}$ ($I \in \mathbb{R}^{T,T}$ denotes identity matrix) forms the equivalent definition of Ω given by

$$\Omega = \{\gamma \in \mathbb{R}^{KT} : \gamma \geq 0 \wedge B\gamma = c\}$$

- $\operatorname{Ker} H \cap \operatorname{Ker} B = \{0\}$, proof: let $d := [\alpha_1 c,\dots,\alpha_K c] \neq 0$ be a vector from $\operatorname{Ker} H$, then

$$Bd = \left[\sum_{k=1}^{K}\alpha_k,\dots,\sum_{k=1}^{K}\alpha_k\right]^T$$

and because d is nonzero (not all α_k is equal zero), then $Bd \neq 0$ and therefore $d \notin \operatorname{Ker} B$.

Lemma 1. *Let $A \in \mathbb{R}^{n\times n}$ be a SPS matrix, let $B \in \mathbb{R}^{m\times n}$, $\rho > 0$, and let $\operatorname{Ker} A \cap \operatorname{Ker} B = \{0\}$. Then matrix*

$$A_\rho = A + \rho B^T B$$

is SPD.

Proof. Let us follow proof by Dostál [?], Lemma 1.2.
If $x \in \mathbb{R}^n \setminus \{0\}$ and $\operatorname{Ker} A \cap \operatorname{Ker} B = \{0\}$, then either $Ax \neq 0$ or $Bx \neq 0$. Since $Ax \neq 0$ is equivalent to $A^{\frac{1}{2}}x \neq 0$, we get for $\rho > 0$

$$x^T A_\rho x = x^T(A + \rho B^T B)x = x^T Ax + \rho x^T B^T Bx = \|A^{\frac{1}{2}}x\|^2 + \rho\|Bx\|^2 > 0 \ .$$

Thus A_ρ is positive definite.

□

2 "The" proof

Let us define equivalent penalised optimisation problem

$$\bar{\gamma} = \arg\min_{\gamma \in \Omega} L_\Theta^\epsilon(\gamma) + \frac{\epsilon^2 \rho}{T}\|B\gamma - c\|^2,$$

where $\rho > 0$ is arbitrary number (*penalisation* parameter). Obviously, any solution of this problem is also the solution of original problem and vice versa (the penalisation term is equal to 0 for all feasible $\gamma \in \Omega$). However, the object function of the new problem can be written in form

$$L_\Theta^\epsilon(\gamma) + \frac{\epsilon^2\rho}{T}\|B\gamma - c\|^2 = \frac{\epsilon^2}{T}\gamma^T(H + \rho B^T B)\gamma + \frac{1}{T}\gamma^T(b_\Theta - 2\epsilon^2\rho B^T c) + \frac{\varepsilon^2\rho}{T}c^T c$$

and the Hessian matrix is given by $(H+\rho B^T B)$, which is SPD (see Lemma 1) and consequently, the object function is strictly convex (in this case, penalisation works as regularisation). Since strictly convex QP on closed convex set has always unique solution (see additional Lemma below), the solution of original problem is also unique.

Therefore, we can conclude that in the problems of this type, *equality constraints regularise original problem.*

3 QP solvability - strictly convex cost function on convex set

For completeness, let us review the proof of uniqueness of solution of QP with strictly convex cost function (SPD Hessian matrix) on closed convex set.

Let $\bar{x} \in \Omega$ be a solution of problem

$$\min_{x \in \Omega} f(x), \qquad f(x) := \frac{1}{2}x^T Ax - b^T x,$$

where $A \in \mathbb{R}^{n,n}$ is SPD matrix, $b \in \mathbb{R}^n$ and Ω is closed convex set.
Let us consider arbitrary $y \in \Omega \setminus \{\bar{x}\}$ (in what follows, we will show that

$f(y) > f(\bar{x})$) and let us denote $d := y - x \neq 0$. Then using the definition of f we can write

$$\begin{array}{rcl} f(y) = f(\bar{x}+d) & = & \frac{1}{2}(\bar{x}+d)^T A(\bar{x}+d) - b^T(\bar{x}+d) \\ & = & \frac{1}{2}\bar{x}^T A\bar{x} + \frac{1}{2}\bar{x}^T Ad + \frac{1}{2}d^T A\bar{x} + \frac{1}{2}d^T Ad - b^T\bar{x} - b^T d \\ \text{"}A = A^T\text{"} & = & f(\bar{x}) + d^T(A\bar{x} - b) + \frac{1}{2}d^T Ad \\ & > & f(\bar{x}) \end{array}$$

Last inequality holds since

- $d^T Ad > 0$ because $d \neq 0$ and A is SPD,
- $d^T(A\bar{x} - b) = d^T\nabla f(\bar{x}) \geq 0$ since $\bar{x}$ is minimiser of f and Ω is convex (every vector from $\bar{x}$ directed to fesible set is ascent (i.e. not descent) vector).

For curiosity: in proof we found that $f(\bar{x}+d) = f(\bar{x}) + d^T(A\bar{x} - b) + \frac{1}{2}d^T Ad$ which is in fact full Taylor expansion of f.