

Let us consider an optimisation problem

$$\begin{aligned}
\bar{\gamma} &:= \arg \min_{\gamma \in \Omega} L_{\Theta}(\gamma) \\
\gamma &:= [\gamma_1, \dots, \gamma_K] \in \mathbb{R}^{KT}, \\
\gamma_k &:= [\gamma_k(1), \dots, \gamma_k(T)] \in \mathbb{R}^T \\
L_{\Theta}^{\epsilon}(\gamma) &:= \frac{1}{T} b_{\Theta}^T \gamma + \frac{\epsilon^2}{T} \gamma^T H \gamma, \\
\Omega &:= \{\gamma \in \mathbb{R}^{KT} : \gamma \geq 0 \wedge \sum_{k=1}^K \gamma(t) = 1, \forall t = 1, \dots, T\}
\end{aligned}$$

and  $H \in \mathbb{R}^{KT, KT}$  is block-diagonal matrix, whose blocks  $L_k \in \mathbb{R}^{T, T}$  are formed by Laplace matrix.

In what follows, we proved that this problem has always unique solution.

## 1 Properties

- $H$  is SPS (since blocks are SPS) and

$$\begin{aligned}
\text{Ker } H &= \text{span}\{[c, \mathbf{0}, \dots, \mathbf{0}]^T, [\mathbf{0}, c, \mathbf{0}, \dots, \mathbf{0}]^T, \dots, [\mathbf{0}, \dots, \mathbf{0}, c]^T\} \subset \mathbb{R}^{KT}, \\
c &:= [1, \dots, 1] \in \mathbb{R}^T
\end{aligned} \tag{1}$$

- $L_{\Theta}^{\epsilon}(\gamma)$  is continuous (not strictly) convex function  $\mathbb{R}^{KT} \rightarrow \mathbb{R}$ ,
- gradient is given by  $\nabla_{\gamma} L_{\Theta}^{\epsilon}(\gamma) = \frac{1}{T} b_{\Theta} + \frac{2\epsilon^2}{T} H \gamma$  ("well known" - can be proved using Taylor expansion),
- $\Omega \subset \mathbb{R}^{KT}$  is bounded closed convex set (has to be proved?),
- the matrix  $B = [I, \dots, I] \in \mathbb{R}^{T, KT}$  ( $I \in \mathbb{R}^{T, T}$  denotes identity matrix) forms the equivalent definition of  $\Omega$  given by

$$\Omega = \{\gamma \in \mathbb{R}^{KT} : \gamma \geq 0 \wedge B\gamma = c\}$$

- $\text{Ker } H \cap \text{Ker } B = \{0\}$ , proof: let  $d := [\alpha_1 c, \dots, \alpha_K c] \neq 0$  be a vector from  $\text{Ker } H$ , then

$$Bd = \left[ \sum_{k=1}^K \alpha_k, \dots, \sum_{k=1}^K \alpha_k \right]^T$$

and because  $d$  is nonzero (not all  $\alpha_k$  is equal zero), then  $Bd \neq 0$  and therefore  $d \notin \text{Ker } B$ .

**Lemma 1.** Let  $A \in \mathbb{R}^{n \times n}$  be a SPS matrix, let  $B \in \mathbb{R}^{m \times n}$ ,  $\rho > 0$ , and let  $\text{Ker } A \cap \text{Ker } B = \{0\}$ . Then matrix

$$A_{\rho} = A + \rho B^T B$$

is SPD.

*Proof.* Let us follow Dostál [?], Lemma 1.2.

If  $x \in \mathbb{R}^n \setminus \{0\}$  and  $\text{Ker } A \cap \text{Ker } B = \{0\}$ , then either  $Ax \neq 0$  or  $Bx \neq 0$ . Since  $Ax \neq 0$  is equivalent to  $A^{\frac{1}{2}}x \neq 0$ , we get for  $\rho > 0$

$$x^T A_\rho x = x^T (A + \rho B^T B)x = x^T Ax + \rho x^T B^T Bx = \|A^{\frac{1}{2}}x\|^2 + \rho \|Bx\|^2 > 0 .$$

Thus  $A_\rho$  is positive definite. □

## 2 ”The” proof

Let us define equivalent penalised optimisation problem

$$\bar{\gamma} = \arg \min_{\gamma \in \Omega} L_\Theta^\epsilon(\gamma) + \frac{\epsilon^2 \rho}{T} \|B\gamma - c\|^2 .$$

Obviously, any solution of this problem is also the solution of original problem and vice versa (the penalisation term is equal to 0 for all feasible  $\gamma \in \Omega$ ). However, the object function of the new problem can be written in form

$$L_\Theta^\epsilon(\gamma) + \frac{\epsilon^2 \rho}{T} \|B\gamma - c\|^2 = \frac{\epsilon^2}{T} \gamma^T (H + \rho B^T B) \gamma + \frac{1}{T} \gamma^T (b_\Theta - 2\epsilon^2 \rho B^T c) + \frac{\epsilon^2 \rho}{T} c^T c$$

and the Hessian matrix is given by  $(H + \rho B^T B)$ , which is SPD (see Lemma 1) and consequently, the object function is strictly convex (in this case, penalisation works as regularisation). Since strictly convex QP on closed convex set has always unique solution, the solution of original problem is also unique.