MA102 Mathematical Proof and Analysis
and
MA103 Introduction to Abstract Mathematics

Exercises 3

The first five exercises count for your class grade and are due in **17:00 Monday, October 19$^{\text{th}}$**.

1 (a) Use the principle of induction to prove that, for all $n \in \mathbb{N}$, $n^3 + 5\,n$ is a multiple of 3.

(b) Use the principle of induction to prove that, for all $n \in \mathbb{N}$, $n^3 + 5\,n$ is a multiple of 6.

(*You may use the result that, for every* $k \in \mathbb{N}$, $k^2 + k$ *is an even number.*)

2 This question is the least-liked question this term. This is because it is hard to understand. Once you do understand it, you will discover that it is also one of the easiest questions this term.

(a) Use the Principle of Induction (with base case 1) to prove that the following is true for any predicate $P(n)$ defined for $n \in \mathbb{Z}$ and each integer $N \in \mathbb{Z}$:

$$\Big(P(N) \wedge \big(\forall n \in \{N, N+1, \dots\} : P(n) \implies P(n+1)\big)\Big) \implies \big(\forall n \in \{N, N+1, \dots\} : P(n)\big).$$

Note $\{N, N+1, \dots\}$ means the set $\{t \in \mathbb{Z} : t \geq N\}$. Which theorem in the lectures have you just proved?

(b) Let $P(n)$ be the statement $n^2 \geq 8n + 15$. Let $N = 9$. Prove that we have $P(N) \implies P(N+1)$. Is $P(N+1)$ true?

(c) Prove, using the Principle of Induction, that Strong Induction works, i.e. that the following statement is true for any predicate $P(n)$ defined for $n \in \mathbb{N}$.

If the statements

$$\begin{aligned} &P(1) \\ P(1) &\implies P(2) \\ \big(P(1) \wedge P(2)\big) &\implies P(3) \\ \big(P(1) \wedge P(2) \wedge P(3)\big) &\implies P(4) \\ &\dots \end{aligned}$$

are all True, then $P(n)$ is true for every $n \in \mathbb{N}$.

(d) Let $P(n)$ be any predicate defined for $n \in \mathbb{Z}$, and let N be an integer. Suppose that

$$\begin{aligned} &P(N) \\ P(N) &\implies P(N+1) \\ \big(P(N) \wedge P(N+1)\big) &\implies P(N+2) \\ \big(P(N) \wedge P(N+1) \wedge P(N+2)\big) &\implies P(N+3) \\ &\dots \end{aligned}$$

are all True. Prove that $P(n)$ is true for every integer $n \geq N$.

3 Write down explicit formulae for the expressions u_n, v_n and w_n defined as follows:

$$u_1 = 1; \quad u_n = u_{n-1} + 3 \quad (n \geq 2);$$
$$v_1 = 1; \quad v_n = n^2 v_{n-1} \quad (n \geq 2);$$
$$w_0 = 0; \quad w_n = w_{n-1} + 3n \quad (n \geq 1).$$

Start by working out the first few values in each case — without doing the arithmetic, so you might write, for instance, $u_2 = 1 + 3$, $u_3 = (1 + 3) + 3, \ldots$ — and try to spot the pattern emerging. It is a good idea to check that the formula you give is correct for the first one or two values of n.

4 The numbers $u_1, u_2, u_3, \ldots$ are defined by the following rule:

$$\begin{aligned} u_1 &= 2; \\ u_{n+1} &= 2u_n^2 - 2u_n + 1, \qquad \text{for } n = 1, 2, 3, \ldots. \end{aligned}$$

Is the following statement true or false?
'u_n is prime for each natural number n'
Justify your answer.

5 Let $a_1 = 1$, and define $a_n = 1 + 2017 \times \sum_{i=1}^{n-1} a_i$ for $n \geq 2$. Prove that $a_n = 2018^{n-1}$ for each $n \in \mathbb{N}$.

This starred question is best done with your peer study group. It doesn't count towards the class grade, and the deadline is **17:00, Monday 26$^{\text{th}}$ October**.

6∗ This question leads you through half of a Nobel Prize in Economics (Lloyd Shapley in 2012; this work is due to David Gale and Shapley, but Gale died in 2008) — if you Google it, you'll miss the fun. Bear in mind that, like all starred questions, this question is *not* course content (there will be nothing about this material on the exam). If you have time and energy to do some parts, I hope you'll enjoy it, but you **should not** spend so much time on this question that your other courses, or your social life, suffer.

In real life, this is used for things like assigning junior doctors to hospitals (not for arranging marriages), but I like this formulation.

Suppose that we have n men, call then $m_1, \ldots, m_n$, and n women $w_1, \ldots, w_n$. Each of the men wants to marry a woman and vice versa.

The men may have different preferences; maybe some like w_1 best, others like w_5 best, and so on. But each man has a (personal) preference ranking of all the women in some order.

The same is true for the women: each woman has a (personal) preference ranking of all the men in some order.

You need to pair off the men and women. But: if there is some man m and woman w who both prefer each other to their assigned partner, they will cheat and the fabric of society will be destroyed. We say a pairing (with n pairs) is *stable* if there is no such *cheating pair*.

The first two questions are supposed to give you an idea of how this works: try some examples.

(a) Suppose $n = 2$. There are two ways to pair the two men and the two women. Give an example of preference rankings where only one of these is stable. Give another example where both pairings are stable.

(b) Suppose $n = 3$. How many ways are there to pair the three men and three women? Give an example of preference rankings where only one of these is stable. Is it possible (for some other set of preference rankings) that all the pairings are stable? Is it possible that there is no stable pairing?

Now let's try to look at the general problem.

(c) Suppose that a set M of men all like w_1 best (i.e. she is top of all their preference rankings). Prove that, whatever stable pairing you pick, w_1 is either paired with a man not in M, or with her favourite man in M.

We call a pair (m, w) *plausible* if there is some stable pairing containing that pair. So (c) says that (m, w_1) is not plausible for any man $m \in M$ except w_1's favourite in M.

The *traditional marriage algorithm* is the following procedure. Each evening, each man goes to visit whichever woman is his favourite among those who haven't rejected him. Each woman looks at the collection of men visiting her, and rejects all but her most preferred among that collection. This repeats until the night when every woman is visited by exactly one man, at which point they get married.

Clarification: On the first evening, no man has been rejected before, so each man goes to visit the woman on the top of his preference list. It can happen that a woman is not visited by any man, in which case she won't reject anyone; similarly a woman can be visited by exactly one man, in which case she will not reject anyone. However if a woman is visited by two or more men, she rejects all but one; only that one will ever visit her again.

(d) Try running this algorithm by hand with a few of the preference lists for $n = 3$ you made earlier. Prove that the following is true for each woman w and each evening $i \geq 2$. If w was visited by a man m on evening $i - 1$, then w is visited by a man at least as good as m on evening i. Prove that it can never happen that a man is rejected by all the women (in which case the algorithm would fail).

(e) Prove that the traditional marriage algorithm always returns a stable pairing.

Note: this also proves that, for any n and whatever preference lists the n men and n women have, there *exists* a stable pairing!

(f) On the first evening, when any woman w rejects a man m, it is because (m, w) is not a plausible pair (as you proved in (c)). Prove that this is true on every evening, i.e. when any woman w rejects a man m, it is because (m, w) is not a plausible pair.

(g) Prove that when the traditional marriage algorithm is run, for each man m the following is true. Among all the women in

$$\{w : (m, w) \text{ is plausible }\},$$

the woman that m ends up paired with is his most favourite. And for each woman w the following is true. Among all the men in

$$\{m : (m, w) \text{ is plausible }\},$$

the man w ends up paired with is her least favourite. Think (at least a bit) about what this means in real life!

There are many things one could think about from this point. Here is one direction:

(h) Suppose that a man m should be paired with a woman w by the traditional marriage algorithm. But m (and only m) lies about his preferences, i.e. he visits women in some order not reflecting his true preferences. Is it possible that m ends up paired to someone other than w? Is it possible for m to benefit from lying, i.e. to end up with a woman he prefers to w? What about if instead a woman is lying: can she end up benefiting?

You might be interested in trying to take this further. Let's remember a real-world application is assigning junior doctors to hospitals; what if two junior doctors want to stay together? Can we still do something like this? What other real-world constraints might we want to introduce, and can we satisfy them? If not, can we at least get close? There is a lot known about this, both from a theory and a practical viewpoint, and it's still an active research field.

A closely related area (the second half of the 2012 Nobel Prize, awarded to Alvin Roth) is 'repugnant markets', where we want a solution but feel it is unethical for money to get involved (e.g. kidney transplants) — how can mathematics help us find solutions?