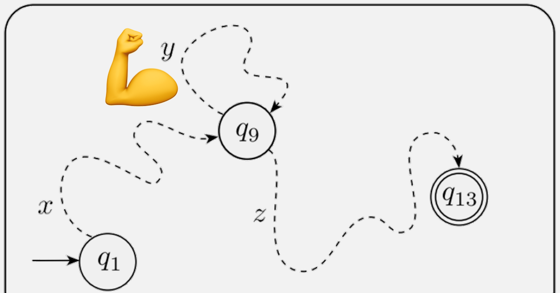
Examples with the Pumping Lemma



Logistics

- HW due Soon
- Questions?
- Exam accommodations for various reasons, lemme know/remind me

Last time: The Loop-State Pumping Lemma says:

For all strings in a regular language that are "long enough" (i.e., length p) ...

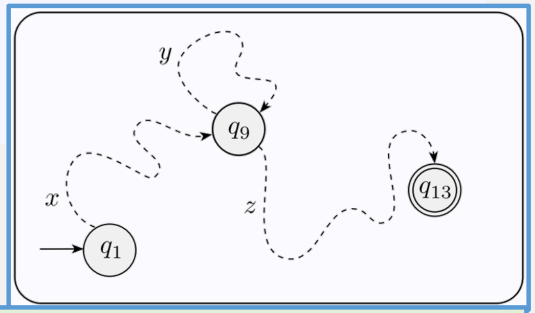
Pumping lemma If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

... these strings must be divisible into three pieces (call them x, y, and z) ...

... where repeating the middle piece y results in a "pumped" string is also in the language

Also, repeating part:
- can't be empty string
- must be in the first p characters



tl;dr:
Long enough strings means repeated states

IMPORTANT NOTE:
The pumping lemma **cannot** be used to show that a language is regular, only that is is non-regular

<u>Last time</u>: Equivalence of Contrapositive

- "If X then Y" is equivalent to … ?

X	Y	X → Y	¬X	¬Y	Y → X	¬X → ¬Y	¬Y → ¬X
T	T	**T**	F	F	T	T	**T**
T	F	**F**	F	T	T	T	**F**
F	T	**T**	T	F	F	F	**T**
F	F	**T**	T	T	T	T	**T**

- "If Y then X" (converse)
- "If not X then not Y" (inverse)

✓"If not Y then not X" (contrapositive)

Last time: Equivalence of Contrapositive

- "If X then Y" is equivalent to … ?
  - "If Y then X" (converse)
    - No!
  - "If not X then not Y" (inverse)
    - No!
  - ✓"If not Y then not X" (contrapositive)
    - **Yes!**
    - Proof by contradiction uses this equivalence

The Pumping Lemma is an If-Then Stmt

... then the language is **not** regular

Pumping lemma If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

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2. $|y| > 0$, and
3. $|xy| \leq p$.

Just need one class of counterexample!

Contrapositive: If (**any** of) these are **not** true ...

Ex: Rename. Rephrase. Restate. Reframe.

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Express your own meaningful reformulation.

The Pumping Lemma is an If-Then Stmt

Pumping lemma If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1.
2.
3.

IF A ∈ Reg
THEN ∃ p ∈ ℕ
    ST ∀ s ∈ A
        IF |s| ≥ p
        THEN 1, 2, & 3

IF $A \in \mathrm{Reg}$

THEN $\exists\ p \in \mathbb{N}$

    ST $\forall\ s \in A$

        IF $|s| \geq p$

        THEN 1, 2, & 3

IF $\neg$ ($\exists$ p $\in \mathbb{N}$
ST $\forall$ s $\in$ A
IF |s| $\geq$ p
THEN 1, 2, & 3)
THEN A $\notin$ Reg

"If

that whole big thing is false

then

A isn't regular"

IF $\forall p \in \mathbb{N}$

$\neg(\forall s \in A$

IF $|s| \geq p$

THEN 1, 2, & 3)

THEN $A \notin$ Reg

$$
\begin{aligned}
&\text{IF} \quad \forall\, p \in \mathbb{N} \\
&\qquad\qquad \exists\, s \in A \\
&\qquad\qquad \neg\,(\ \text{IF } |s| \geq p \\
&\qquad\qquad\qquad \text{THEN } 1, 2, \&\ 3) \\
&\text{THEN } A \notin \text{Reg}
\end{aligned}
$$

$$
\begin{aligned}
&\text{IF} \quad \forall\, p \in \mathbb{N} \\
&\qquad\qquad \exists\, s \in A \\
&\qquad\qquad \neg\,(\ \text{IF } |s| \geq p \\
&\qquad\qquad\qquad \text{THEN 1, 2, \& 3}) \\
&\text{THEN } A \notin \text{Reg}
\end{aligned}
$$

(Instead of confirming something about all strings, we're now asked to, for each number, disconfirm something about _some_ string)

IF $\forall\, p \in \mathbb{N}$

$\exists\, s \in A$

$|s| \geq p \land \neg(1,\ 2,\ \&\ 3)$

THEN $A \notin \mathrm{Reg}$

IF $\forall\, p \in \mathbb{N}$

$\exists\, s \in A$

$|s| \geq p \land (\neg 1 \lor \neg 2 \lor \neg 3)$

THEN $A \notin \mathrm{Reg}$

IF $\forall\ p \in \mathbb{N}$

$\exists\ s \in A$

$|s| \geq p \land (\neg 1 \lor \neg 2 \lor \neg 3)$

THEN $A \notin \text{Reg}$

*... But how can we come up with a string for *every* such number?*

Parameterize it!

Pumping Lemma: Non-Regularity Example

Pumping lemma If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
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Let B be the language $\{0^n1^n | n \geq 0\}$. We use the pumping lemma to prove that B is not regular. The proof is by contradiction.

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- Assumption: 0^n1^n is a regular language (must satisfy pumping lemma)

- The class 0^p1^p
  - Is this a class? Or a string?

Possible Split: y = all 0s

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- <u>Assumption</u>: 0^n1^n is a regular language (must satisfy pumpi
- <u>Counterexample</u> = 0^p1^p

But pumping lemma requires **only one** pumpable splitting

$$\overbrace{\underbrace{0}_{x}\underbrace{0\ \ldots\ 0}_{y}}^{p\text{ 0s}}\overbrace{\underbrace{11\ \ldots\ 1}_{z}}^{p\text{ 1s}}$$

- If xyz chosen so y contains
  - all 0s

So we must show that **every splitting** produces a contradiction

- Pumping y: produces a string with more 0s than 1s
  - This string is <u>not</u> in the language 0^n1^n
  - This means that 0^n1^n does <u>not</u> satisfy the pumping lemma
  - Which means that that 0^n1^n is a <u>not</u> regular lang
  - This is a **contradiction** of the assumption!

Pumping lemma If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

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Possible Split: y = all 1s

- Assumption: 0^n1^n is a regular language (must satisfy pumping lemma)
- Counterexample = 0^p1^p

$$\underbrace{\overbrace{00\ldots0}^{p\text{ 0s}}}_{x}\overbrace{\underbrace{11}_{y}\underbrace{\ldots1}_{z}}^{p\text{ 1s}}$$

- If xyz chosen so y contains
  - all 1s
- Is this string pumpable?
  - No!
  - By the same reasoning as in the previous slide

Possible Split: y = 0s and 1s

Pumping lemma If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
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- Assumption: 0^n1^n is a regular language (must satisfy pumping lemma)
- Counterexample = 0^p1^p

p 0s $\quad$ p 1s

00 ... 011 ... 1

$x \quad y \quad z$

- If xyz chosen so y contains
  - both 0s and 1s

Did we examine every possible splitting?

Yes. But maybe we don't have to.

- Is this string pumpable?
  - No!
  - Pumped string will have equal 0s and 1s
  - But they will be in the wrong order: so there is still a **contradiction**!

The Pumping Lemma says:

Pumping lemma If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

Also, repeating part y:
- can't be empty string
- must be in the first p characters

$$\overbrace{0\underbrace{0 \ldots 0}}^{p \text{ 0s}}11 \ldots 1$$

y must be in here!

Pumping Lemma: How to use Condition 3

Let $F = \{ww |\ w \in \{0,1\}^*\}$. We show that F is nonregular

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This is a creative process!

1. Is there a sub-language that expresses the non-regularity?
2. What's the "essence" of that non-regularity?
3. How do you use pumping to exploit the deficiency?

Pumping Lemma: Pumping Down

use the pumping lemma to show that $E = \{0^i 1^j \mid i > j\}$ is not regular.

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Check-in Quiz

On gradescope