Examples of modules:

(Question 6.5) Show that the set $M = \{\mathbf{p}\}$ with the addition $\mathbf{p}+\mathbf{p} = \mathbf{p}$, the scalar multiplication $\alpha \cdot \mathbf{p} = \mathbf{p}$ for any $\alpha \in R$ is a module over any ring R. What is the zero element? (*Hint:* in general, it's a pain to verify the properties above, but here what is the only possible outcome of evaluating something like $7(2\mathbf{p} + \mathbf{p}) - 5\mathbf{p}$?)

(Question 6.6) Show that R is a module over itself by identifying which properties of R immediately give each of the 8 requirements in the definition of a vector space. That is, suppose $M = R$ and we take the addition to be the addition in R, while to compute $\alpha \cdot \mathbf{x}$, we just use the multiplication from R and think of the result as a module element. Be careful when verifying properties (7) and (8) of a module. Verify that $\mathbf{0}_M = 0_R$.

(Question 6.7) Suppose that M is a module over a ring R (such as R itself), and let $S \subset R$ be a sub-ring – that is $S \subset R$ such that $1_R, 0_R \in S$ and adding and multiplying elements of S (using the operations from R) always produce elements in S (For example, $\mathbb{Q}$ is a sub-field of $\mathbb{R}$ and $\mathbb{R}$ is a sub-field of $\mathbb{C}$). Show that M is a module over S.

Example: $\mathbb{Z} \subset \mathbb{Q}, \mathbb{R}, \mathbb{C}$ and thus each of $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are modules over $\mathbb{Z}$. Similarly, $\mathbb{C}$ is a vector space over $\mathbb{R}$ (and $\mathbb{Q}$, but this isn't so useful.)

(Question 6.8) Show that $\mathbb{Z}/d\mathbb{Z}$ is a module over $\mathbb{Z}$, for the scalar multiplication $n \cdot \alpha = (n\alpha \mod d)$, $n \in \mathbb{Z}, \alpha \in \mathbb{Z}/d\mathbb{Z}$

Useful example: Let M be a module over R and let S be any *set*. Define $\mathcal{F}(S, M)$ to be the set of all functions $f : S \longrightarrow M$. We can define an addition, scalar multiplication, and zero in $\mathcal{F}(S, M)$ by using the properties of the module M:

(1) Given $f, g \in \mathcal{F}(S, M)$ we define $f + g \in \mathcal{F}(S, M)$ to be the function which takes $s \in S$ to $f(s) +_M g(s)$. Note that $f + g$ is the name of the function, so we will write its evaluation at s by $(f + g)(s)$. This equals $f(s) +_M g(s)$.
(2) Given $\alpha \in R$ and $f \in \mathcal{F}(S, M)$ we define $\alpha \cdot f$ to be the function which takes $s \in S$ to $\alpha \cdot_M f(s)$
(3) We define zero $\in \mathcal{F}(S, M)$ to be the function which takes $s \in S$ to 0_M for all s. That is, $\text{zero}(s) = 0_M$.

Then $\mathcal{F}(S, M)$, with this addition, scalar multiplication, and zero zero is a vector space over R. To see this,we need to explicitly verify each of the eight properties of a module in order (using the fact that they hold for M!). For example, to verify that additive inverses exist (property (4)) we start with an f in $\mathcal{F}(S, M)$ and find a function g with $f + g = \text{zero}$. We can do this using the properties of M: for each $s \in S$, $f(s) \in M$ and thus there is a unique element $y_s \in M$ such that $f(s) +_M y_s = 0_M$ (the subscript indicates that y_s could be different for different elements s). We define $g(s) = y_s$ and see that it works:

Proof of property (4): Let f be a function in $\mathcal{F}(S, M)$. Given $s \in S$, we know that $f(s) \in M$ has a unique additive inverse $y_s \in M$ since M is a module over R. For each $s \in S$, define $g(s) = y_s$. This is a well-defined function in $\mathcal{F}(S, M)$ since $y_s \in M$ is unique for s. Then $f + g = \text{zero}$ since $(f + g)(s) = f(s) +_M g(s) = f(s) +_M y_s = 0_M = \text{zero}(s)$ for all $s \in S$, and two functions in $\mathcal{F}(S, M)$ are equal if and only if they evaluate to the same element of M on each $s \in S$. Since we can construct such a function in $\mathcal{F}(S, M)$ for every $f \in \mathcal{F}(S, M)$, we have verified property (4) in the definition of a vector space for $\mathcal{F}(S, M)$.

(Question 6.9) Complete the argument showing that $\mathcal{F}(S, M)$, with this addition, scalar multiplication, and zero zero, is a vector space over R. Do this by explicitly verifying the remaining seven properties of a vector space. Each should follow from a property of M.

(Question 6.10) Let R be a ring. From the previous exercise, $\mathcal{F}(\{1, 2, 3, \ldots, n\}, R)$ is a module for any $n \in \mathbb{N}$. Given a vector f in this module, we write f_i for $f(i) \in R$, and present f by $(f_1, f_2, \ldots, f_n)$. Show that,

(1) The formula for $f + g$ in $\mathcal{F}(\{1, 2, 3, \ldots, n\}, R)$ in this presentation yields

$$(f_1, f_2, \ldots, f_n) + (g_1, \ldots, g_n) = (f_1 + g_1, f_2 + g_2, \ldots, f_n + g_n)$$

(2) Let $\lambda \in R$. Show that $\lambda \cdot f$ is computed as

$$\lambda \cdot (f_1, f_2, \ldots, f_n) = (\lambda f_1, \lambda f_2, \ldots, \lambda f_n)$$

(3) What is the zero element for this module?

Definition 26. *Let R be a commutative ring. The module $\mathcal{F}(\{1, 2, 3, \ldots, n\}, R)$ will be denoted R^n. Given $(x_1, \ldots, x_n)$ we call the element $x_i \in \mathbb{F}$ the i^{th} coordinate of $(x_1, \ldots, x_n)$, and the map $R^n \longrightarrow R$ given by $(x_1, \ldots, x_n) \longrightarrow x_i$ the i^{th} projection map.*

Convention: In R^n, we will name elements so that we don't have to write tuples: for example $(x_1, \ldots, x_n)$ will become $\mathbf{x}$. An element will be denoted by a lowercase, bold letter such as $\mathbf{v}$. The corresponding coordinates will generally be denoted by the same lowercase letter, in normal typeface, and with *subscripts* to indicate location, except for certain special vectors.

Note: In R^n there is a special set of vectors: the ordered set $[\mathbf{e}_1, \ldots, \mathbf{e}_n]$, where $\mathbf{e}_i$ be the vector $(0, \ldots, 1_R, \ldots, 0)$ where each entry is a 0 except for the i^{th} entry, which is a 1_R. Each vector in R^n is a unique linear combination of these:

$$(x_1, \ldots, x_n) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$$

(Question 6.11) Find all the vectors in $\left(\mathbb{Z}/2\mathbb{Z}\right)^4$. Give some examples of how to add and scale vectors in this vector space. How many vectors are there in $\left(\mathbb{Z}/d\mathbb{Z}\right)^n$?

(Question 6.12) Note that the set $M_{n\times m}(R)$ equals $\mathcal{F}(\{1, \ldots n\} \times \{1, \ldots, m\}, R)$ and is thus a module. Using the usual depiction of matrices as rectangular arrays of elements of $\mathbb{F}$, describe

(1) what happens to each entry when adding two matrices the vector addition from $\mathcal{F}(\{1, \ldots n\} \times \{1, \ldots, m\}, R)$.
(2) what happens to each entry when you multiply a matrix by an element $\lambda \in R$ using the scalar multiplication from $\mathcal{F}(\{1, \ldots n\} \times \{1, \ldots, m\}, R)$, and
(3) describe the zero vector in $\mathcal{F}(\{1, \ldots n\} \times \{1, \ldots, m\}, R)$.
(4) describe the additive inverse of a vector in $M_{n\times m}(\mathbb{F})$.

We will identify R^n and the set of column vectors $M_{n\times 1}(R)$ using the translation rule:

$$(x_1, x_2, \ldots, x_n) \leftrightarrow \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

(Question 6.13) Show that the translation is a bijection that preserves addition and scalar multiplication: that is, if we compute $\mathbf{x}+\mathbf{y}$ and then translate it into the column matrices, or translate $\mathbf{x}$ and $\mathbf{y}$ first and then add the resulting column vectors, gives the same result.

(Question 6.14) Show that $R[x]$ is a module over R.

Thus, $R[x]$ is a module over R which has an additional multiplication which makes it into a ring, such that $r \cdot (P(x) \cdot_{R[x]} Q(x)) = (r \cdot P(x)) \cdot_{R[x]} Q(x) = P(x) \cdot_{R[x]} (r \cdot Q(x))$. Such a structure is called an R-algebra. We will give a more precise definition later.

(Question 6.15) Show that the set of sequences $\mathcal{S}(\mathbb{F})$ used in example #2 above equals $\mathcal{F}(\mathbb{N}, \mathbb{F})$, and that the sum and scalar multiplication in the example is the same as that from the definition of $\mathcal{F}(\mathbb{N}, \mathbb{F})$. Conclude that $\mathcal{S}(\mathbb{F})$ is a vector space and find its zero vector. How does one find the additive inverse of a sequence?

Similarly, if $S = \mathbb{R}$ then the set of functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a vector space using the addition and scalar multiplication above. However, if $f(x) = \sin(x)$ and $g(x) = x^2$ then $(f+g)(x) = f(x) + g(x) = \sin(x) + x^2$, so the addition here recovers the usual sum of functions from calculus. This doesn't quite capture the space $C^k(\mathbb{R})$ used in Example #1 above, since the functions there were also differentiable or continuous, but we will see how to handle these additional complications soon.

There is an extension of the observation that $\mathcal{F}(S, M)$ is an R-module when M is. Suppose M_s is an R-module for all $s \in S$. Consider the set of all functions

$$f \colon S \longrightarrow \bigcup_{s \in S} M_s \qquad \text{such that} \qquad f(s) \in M_s \text{ for all } s \in S$$

If f and g are two such functions, then we can still add them using the formulas above: since $f(s), g(s)$ are in the *same* module M_s, we can set $(f+g)(s)$ to be $f(s) +_{M_s} g(s)$ to define a function in the set. We can similarly define a scalar multiplication, and then prove that the result is a vector space. However, as the proof is basically that for $\mathcal{F}(S, M)$ we will omit it here.

Definition 27. *Let S be a set, and let M_s be a module over R for each $s \in S$. The module* product *of the M_s is the set*

$$\prod_{s \in S} M_s = \left\{ f \colon S \longrightarrow \bigcup_{s \in S} M_s \,\middle|\, f(s) \in M_s \ \forall s \in S \right\}$$

when equipped with the addition and scalar multiplication defined by

(1) *For $f, g \in \prod_{s \in S} M_s$, let $f+g$ be the function such that*

$$(f+g)(s) = f(s) +_{M_S} g(s)$$

(2) *For $\lambda \in R$, and $f \in \prod_{s \in S} M_s$, let $\lambda \cdot f$ be the function such that*

$$(\lambda \cdot f)(s) = \lambda \cdot_{M_s} \big(f(s)\big)$$

Notes:

(1) When $S = \{1, \ldots, k\}$ is finite (and ordered), we will write $\prod_{s \in S} M_s$ as $M_1 \times \cdots \times M_k$.
(2) We will often write $(f_s)_{s \in S}$ instead of $f \in \prod_{s \in S} M_s$, where $f_s = f(s)$, as we do for the coordinates in $\mathbb{R}^n$.
(3) When $M_s = M$ for all $s \in S$, the product $\prod_{s \in S} M$ is just $\mathcal{F}(S, M)$, but the product notation is more customary. We will use the product notation from now on.