7. Submodules

The original definition provides a monolithic account of an R-module. We now start examining techniques to explore the structure of an R-module. In particular, there are subspaces which are themselves R-modules, for the addition and scalar multiplication from the larger module:

Definition 28. *Let M be a module over R. A subset $N \subset M$ is a submodule of M if it is closed under the addition and scalar multiplication on M. That is, if $\mathbf{n}_1, \mathbf{n}_2 \in N$ implies $\mathbf{n}_1 +_M \mathbf{n}_2 \in N$ and $\lambda \in R$ implies $\lambda \cdot_M \mathbf{n}_1 \in N$.*

When V is a vector space, W is called a (vector) sub-space.

(Question 7.1) Let $N \subset M$ be a submodule of M. Show that N is an R- module for the operations $+\colon W \times W \longrightarrow W$, $\cdot\colon R \times W \longrightarrow W$ defined by

$$\begin{aligned}(\mathbf{n}_1, \mathbf{n}_2) &\longrightarrow \mathbf{n}_1 +_M \mathbf{n}_2 \\ (\lambda, \mathbf{n}) &\longrightarrow \lambda \cdot_M \mathbf{n}\end{aligned}$$

(*Hint:* Justify this using just the closure properties and the fact that M is an R-module.)

Together, these imply the following criterion: to see that $S \subset M$ is a submodule it is enough to show that

(1) for all if $\mathbf{s}_1, \mathbf{s}_2 \in S$ we have $\mathbf{s}_1 +_M \mathbf{s}_2 \in S$
(2) for all $\mathbf{s} \in S$ and $\lambda \in \mathbb{F}$, we have $\lambda \cdot_M \mathbf{s} \in S$.

If either of these fails for any vectors or scalars, then the subset S is NOT a submodule.

We note for the future, that $N \subset M$ means there is an inclusion map $I_N : N \longrightarrow M$ where $\mathbf{n} \in N \longrightarrow \mathbf{n} \in M$. The requirements that N be a sub-module are then

$$\begin{aligned}I_N(\mathbf{n}_1 +_N \mathbf{n}_2) &= I_N(\mathbf{n}_1) +_M I_N(\mathbf{n}_2) \\ I_N(\lambda \cdot_N \mathbf{n}) &= \lambda \cdot_M I_N(\mathbf{n})\end{aligned}$$

(Question 7.2) Show that $\{\mathbf{0}_M\}$ is a submodule of any R-module M.

(Question 7.3) If $N \subset M$ is a submodule, and $P \subset N$ is a submodule (for the operations which make N a subspace of M), show that P is a submodule of M.

(Question 7.4) If $N \subset M$ and N is a submodule, then N must have a zero vector. Show that the zero vector in N is $\mathbf{0}_M$, the zero vector in V.

In particular, if $\mathbf{0}_M \notin N$ then N is not a submodule!

(Question 7.5) Show that $\{\, (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \,\}$ is not a subspace of $\mathbb{R}^2$.

(Question 7.6) Determine if any of the following are subspaces of $\mathbb{R}^3$:

(1) The set of (x, y, z) such that $2x - 3y + z = 1$,
(2) The set of (x, y, z) such that $2x - 3y + z = 0$,
(3) The set of (x, y, z) such that $xy \geq 0$

(Question 7.7) Show that both of the following are subspaces of $M_{n\times n}(\mathbb{F})$:

(1) $\mathrm{Sym}(n) = \left\{ \mathbf{A} \in M_{n\times n}(\mathbb{F}) \,\middle|\, \mathbf{A} = \mathbf{A}^T \right\}$
(2) $\mathrm{Asym}(n) = \left\{ \mathbf{A} \in M_{n\times n}(\mathbb{F}) \,\middle|\, \mathbf{A} = -\mathbf{A}^T \right\}$

(Question 7.8) Explain why the following are *NOT* a subspaces of $M_{n\times n}(\mathbb{F})$:

(1) $\left\{ \mathbf{A} \in M_{n\times n}(\mathbb{F}) \,\middle|\, \mathbf{A} + \mathbf{A}^T = \mathbf{I}_n \right\}$, where $\mathbf{I}_n$ is the $n \times n$ identity matrix.
(2) $\left\{ \mathbf{M} \in M_{2\times 2}(\mathbb{R}) \,\middle|\, \mathbf{M}^2 = \mathbf{0}_2 \right\}$ where $\mathbf{0}_2$ is the 2×2 zero matrix.

(Question 7.9) Show that the set $C^k(\mathbb{R})$, of functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ whose first k derivatives are continuous, is a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$. (*Hint:* Derivatives of sums can be found by ..., sums of continuous functions are ...)

Note: The set $C^\infty(\mathbb{R})$ consists of all the functions with continuous derivatives of every finite order. It is also a real vector space.

(Question 7.10) Show that the set of continuous functions $f \in C^0(\mathbb{R})$ such that $f(1) = 0$ is a subspace of $C^0(\mathbb{R})$. What about the set where $f(1) = 2$? The set of functions which vanish on $(0, 1) \subset \mathbb{R}$?

(Question 7.11) Let $\mathcal{D}(\mathbb{F}) \subset \mathcal{S}(\mathbb{F})$ consist of those sequences $(s_i)_{i=1}^\infty$ such that $s_i \neq 0$ for only finitely many i's. Show that $\mathcal{D}(\mathbb{F})$ is a *proper* sub-module of $\mathcal{S}(\mathbb{F})$.

This is a simple example of a more general construction:

Definition 29. *Let V_s be a module over R for each $s \in S$. The* direct sum $\bigoplus_{s\in S} V_s$ *is the submodule of $\prod_{s\in S} V_s$ given by*

$$\bigoplus_{s\in S} V_s = \left\{ f \in \prod_{s\in S} V_s \,\middle|\, f(s) \neq 0 \text{ for only finitely many } s \in S \right\}$$

Note 1: When S is finite, $\oplus_{s\in S} V_s = \prod_{s\in S} V_s$, and we will generally write $M_1 \oplus M_2 \oplus \cdots \oplus M_k$ instead of $M_1 \times M_2 \times \cdots \times M_k$. However, the product and the direct sum are different when S is infinite, as can seen when comparing $\mathcal{D}(\mathbb{F})$ and $\mathcal{S}(\mathbb{F})$ in the previous exercise.

Note 2: The direct sum will be more useful for us than the product. We'll see why in the next section.

Not every subset of a module M is a submodule, but is every subset contained in a submodule? M is a sub-module of itself, so yes. But if you look at $\mathbf{0}_M$, it is contained in a much smaller submodule: $\{\mathbf{0}_M\}$. So we adjust, our question: is every subset $S \subset M$ contained in a *smallest* submodule of M?

(Question 7.12) Let W_i, $i \in I$ be submodules of M. Show that $\bigcap_{i\in I} M_i$ is a submodule of M. (*Note:* there is no reason to assume I is finite, so don't use induction!)

Definition 30. *Let M be an R-module and $S \subset M$. The* span *of S is the intersection of all the subspaces of M which contain S*

$$\mathrm{Span}_R(S) = \bigcap \left\{ W \subset M \,\middle|\, W \text{ is a submodule of } M \text{ and } S \subset M \right\}$$

We have seen that S is contained in at least one subspace of M, so the intersection is not trivial. Furthermore, $\mathbf{0}_M \in W$ for all sub-modules, so the intersection is non-empty, and thus $\mathrm{Span}(S)$ is a non-empty subset of M. Since it is the intersection of sub-modules, it is a sub-module. It is the smallest sub-module in the following sense:

(Question 7.13) Suppose $S \subset W$ where W is a sub-module of M. Explain why $\mathrm{Span}(S) \subset W$.

(Question 7.14) Prove that

$$\mathrm{Span}_R(S) = \left\{ \alpha_1 \mathbf{p}_1 + \cdots \alpha_k \mathbf{p}_k \,\middle|\, k \in \mathbb{N}, \alpha_i \in R, \mathbf{p}_i \in S \right\}$$

Example: Let $\mathbf{v} \in M$, then $\mathrm{Span}_R(\{\mathbf{v}\}) = \left\{ r \cdot \mathbf{v} \,\middle|\, r \in R \right\}$ is an R-submodule of M. We will also denote this sub-module as $R\mathbf{v}$.

Note: If $r_1, \ldots, r_k \in R$ then $\mathrm{Span}(\{r_1, \ldots, r_k\})$ is a sub-module of R.

There is an important distinction between rings and fields: R is a module over itself, just as $\mathbb{F}$ is a vector space over itself. However,

Definition 31. *An ideal in a ring R is a proper subset $I \subset R$ which is a sub-module of R, thought of as a module over itself.*

Example: Let $v \in R$ and let $(v) = \mathrm{Span}(\{v\}) = \left\{ \lambda \cdot v \,\middle|\, \lambda \in R \right\}$. Then (v) is a called *the principal ideal* associated to v. For example,

$$(d) = d\mathbb{Z} = \{\ldots, -2d, -d, 0, d, 2d, \ldots\}$$

is a principal ideal in $\mathbb{Z}$.

Fields have no non-trivial ideals (that is, only $\{0_{\mathbb{F}}\}$ is an ideal).

(Question 7.15) Show that the only subspace of a field $\mathbb{F}$ is $\{\mathbf{0}\}$. That is $6\mathbb{Z} \subset \mathbb{R}$ is *not* a subspace of $\mathbb{R}$. (*Hint:* if $v \in W \subset \mathbb{F}$ is non-zero then show that $1 = w^{-1}w \in W$.)

In fact, rings whose only ideal is the zero sub-module have to be fields: if $v \neq 0$, then $(v) = R$ since if it is not the zero sub-module it must be everything.

(Question 7.16) Show that $(v) = R$ if and only if $v \in R^{\times}$

On the other hand, for Euclidean rings more can be said:

(Question 7.17) Suppose R is Euclidean with structure map ϕ and $I \subset R$ is an non-trivial ideal.

(1) Explain why the set of natural numbers $\{\phi(n) | n \in I\}$ has a minimal non-zero value γ.
(2) Explain why there is a $g \in I$ with $\phi(g) = \gamma$.
(3) Suppose $n \in I$. Since R is Euclidean there are $q, r \in R$ with $n = qg + r$ for $q, r \in R$ and $0 \leq \phi(r) < \phi(g)$. Explain why this implies n is a multiple of g.
(4) Conclude that the ideals in R are all principal ideals (g).

It is *VERY MUCH NOT* true that ideals in other types of rings need to be principal!