Insights from Discrete Mathematics from *Integer Partitions* to *Rooted Trees*

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1 Introduction

1.1 Definition of Partitioning

A partition of a positive integer n is written as a sum of positive integers. Different orders of the same partitions do not count as separate partitions.

1.2 Examples

Let $P(n)$ output the number of partitions for any number n.

By default, $P(0) = 1$

1 can be written as 1, hence $P(1) = 1$

2 can be written as $1 + 1$ and 2, hence $P(2) = 2$

3 can be written as $1 + 1 + 1$, $1 + 2$ and 3. $1 + 2$ is the same thing as $2 + 1$, it is only counted once, making $P(3) = 3$

4 can be written as $1 + 1 + 1 + 1$, $1 + 1 + 2$, $1 + 3$, $2 + 2$ and 4, which results in $P(4) = 5$

1.3 Different Representations of the Partitions

1.3.1 Ferrers diagram

A partition can be represented graphically through a Ferrers diagram, named after British mathematician Norman Macleod Ferrers. For example, $5 = 2+2+1$ can be illustrated as follows:

```
2 2 1
* * *
* *
```

A conjugate for this diagram can be created by swapping the columns with rows. For example, $5 = 3 + 2$ is represented as:

```
3 2
* *
* *
*
```

Therefore, the conjugate of (2, 2, 1) is (3, 2).

If the conjugate of a partition happens to be the same as the partition itself, it is called a self-conjugate. For instance, $10 = (4, 3, 2, 1)$ can be shown as:

```
  4 3 2 1
4 * * * *
3 * * *
2 * *
1 *
```

1.3.2 Young Diagram

Alfred Young created the Young Diagram, which is similar to the Ferres diagram but uses squares instead of dots. Although Young diagrams are very similar to Ferres diagrams, they are particularly useful for studying group representation theory and symmetric functions.

Animations demo for Young Diagram using Python Manim.[1]

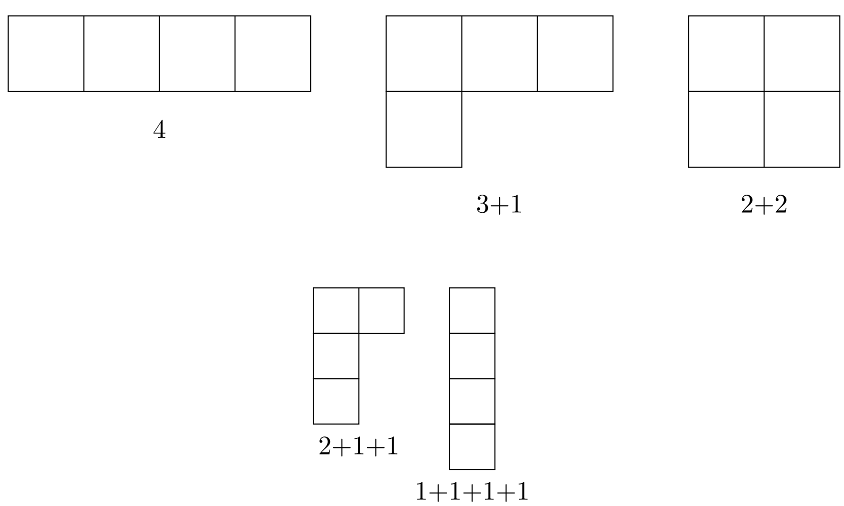


Figure 1: Young diagram for all partitions of 4.

2 Euler's Generating Function

2.1 Intro to Euler's Generating Function

Euler discovered several theorems related to integer partitioning, but his generating function is particularly notable. It is defined as:

$$\sum_{n=0}^{\infty} p(n)x^n = \frac{1}{\prod_{p=1}^{\infty}(1-x^p)}, \quad \text{where } |x| < 1$$

For instance, to find the number of partitions of 7, we examine the coefficient of x^7.

$(1+x^{1\cdot(1)}+x^{1\cdot(2)}+x^{1\cdot(3)}+\ldots)(1+x^{2\cdot(1)}+x^{2\cdot(2)}+x^{2\cdot(3)}+\ldots)(1+x^{3\cdot(1)}+x^{3\cdot(2)}+x^{3\cdot(3)}+\ldots)\ldots$

Expanding above to get $1+x+2x^2+3x^3+5x^4+7x^5+11x^6+15x^7+22x^8+\ldots$

Hence, $P(7) = 15$

2.2 Euler's Odd-Partitions & Distinct-Partitions Theorem

Another interesting theorem by Euler states that the number of odd-partitions is equal to the number of distinct parts:

(1) Odd parts: A partition of any number n which contains only odd numbers.
(2) Distinct parts: A partition that contains no repeated numbers.

Here is an example to demonstrate this theorem, 5 can be written:

Partition	Type
(5)	Odd part, Distinct part
(4, 1)	Distinct part
(3, 2)	Distinct part
(3, 1, 1)	Odd part
(2, 2, 1)	N/A
(2, 1, 1, 1)	N/A
(1, 1, 1, 1, 1)	Odd part

Table 1: Euler - Types of partitions for Integer 5

Hence, we have 3 of each and the number of odd parts is equal to the number of distinct parts.

3 Hardy-Ramanujan's Asymptotic Expression

In 1918, Godfrey Hardy and Srinivasa Ramanujan developed an asymptotic formula for the number of partitions of a number n, using the circle method along with modular functions to create an asymptotic solution:

$$p(n) \approx \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}} \quad \text{as} \quad n \to \infty$$

This formula shows the rapid increase in the number of partitions as n becomes larger.

For example, when plugging in the numbers 20, 50 and 100, we have the following outcomes:

Animations demo for Hardy-Ramanujan's Asymptotic Expression using Python Manim.[1]

n	$p(n)$	$p(n)$ using formula	$p(n)$ using formula/$p(n)$
20	627	692.3846405	1.104281723
50	204226	217590.4992	1.065439754
100	190569292	199280893.3	1.045713563
$\vdots$	$\vdots$	$\vdots$	$\vdots$
∞	∞	∞	1

Table 2: The outcomes of the Hardy-Ramanujan's Asymptotic Expression

4 The Coins Change Problem

Animations demo for Coin Change Combination Problem using Python Manim.[1]

4.1 The Coins Change Problem

Question: "*In how many ways can you change one dollar?*" [2]

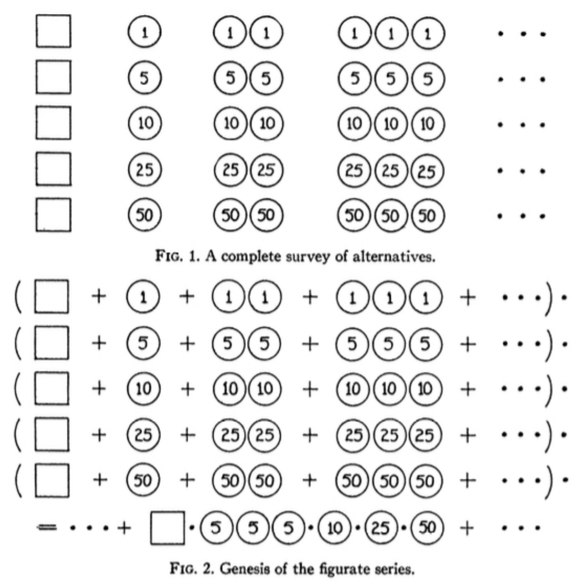


Figure 2: How many ways can you change one dollar?

Interpret the coins change problem in mathematical form as follows:

Let P_n denote the number of ways of paying n cents with cents, nickels, dimes, quarters, and half dollars. Given $P_4 = 1$, $P_6 = 2$, and $P_{10} = 4$, what is P_{100} ?

Now let's explore the various possibilities here. For instance, we could use denominations of 1 cent, 2 cents, 5 cents, and so on, as demonstrated in the diagram below:

In the first diagram, if we interpret each line as the sum of pictures in it, and then consider the product of these five infinite sums, we get the second diagram. Once this is expanded, it gives us the number of different ways of making payments.

For instance, if we take the term with 2 coins in line two, it represents paying two nickels.

In order to make this work, we substitute each pictorial variable as a power of a new variable x. The exponent of x would be the joint value of the coins represented by the picture. Then we get:

$$P_0 + P_1 x + P_2 x^2 + \cdots + P_n x^n + \dots$$

In above series, the coefficient of x^n counts the number of different ways of paying the amounts of n cents. This is called the enumerating series. We can substitute the second figure into a **geometric series** as follows:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

This changes each of the first five lines into a geometric series and equation in the second figure. So we have:

$$(1-x)^{-1}(1-x^5)^{-1}(1-x^{10})^{-1}(1-x^{25})^{-1}(1-x^{50})^{-1} = P_0 + P_1 x + P_2 x^2 + \cdots + P_n x^n$$

This sum is usually termed as the generating function. This function expanded in powers of x generates the numbers $P_0, P_1, \dots, P_n$, our starting point.

To get our generating function, we put every single power of x instead of just the coin values. This gives us all the partitions of an integer.

Now, we have reduced this combinatorial problem to a different kind of problem! We must expand it in powers of x. Let's assume we have already obtained the expansion of the product of the first two factors:

$$(1-x)^{-1}(1-x^5)^{-1} = a_0 + a_1 x + a_2 x^2 + \dots$$

We can use this to find the expansion for 3 factors

$$(1-x)^{-1}(1-x^5)^{-1}(1-x^{10})^{-1} = b_0 + b_1x + b_2x^2 + \dots$$

This means:

$$a_0 + a_1x + a_2x^2 + \cdots = (b_0 + b_1x + b_2x^2 + \dots)(1 - x^{10})$$

If we compare the coefficients of x^n, we find that

$$b_n = a_n + b_{n-10}$$

With this, we can conveniently compute the coefficients of b_n if the coefficients of a_n are already known.

Let's add a table that computes P_{50}. The table below represents the coefficients for the series expansions of $(1-x)^{-1}$, $(1-x^5)^{-1}$, $(1-x^{10})^{-1}$, $(1-x^{25})^{-1}$, and $(1-x^{50})^{-1}$:

$n =$	0	5	10	15	20	25	30	35	40	45	50
$(1-x)^{-1}$	1	1	1	1	1	1	1	1	1	1	1
$(1-x^5)^{-1}$	1	2	3	4	5	6	7	8	9	10	11
$(1-x^{10})^{-1}$	1	2	4	6	9	12	16	20	25	30	36
$(1-x^{25})^{-1}$	1	2	4	6	9	13					49
$(1-x^{50})^{-1}$	1										50

This table above yields $P_{50} = 50$, which means one can pay 50 cents in 50 different ways. Although this table only computes up to P_{50}, we can continue this computation and verify that $P_{100} = 292$.

5 Rooted Trees

5.1 Introduction

Definition 5.1. A *tree* is a graph consisting of vertices and edges but contains no closed path. A *rooted tree* is a tree with one distinguished vertex called the *root*. A non-root vertex is called a *knot*.

Key Question 5.1. *How many rooted trees with n knots?*

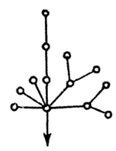
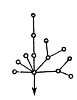


Figure 3: A rooted tree

5.2 From Trees to Cayley's Generating Function

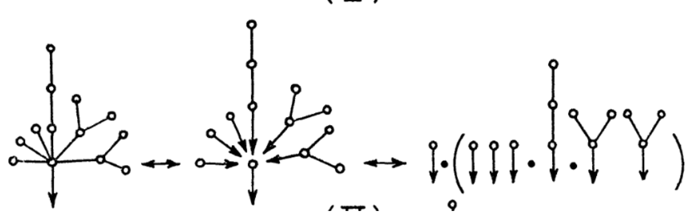


Let there be T_n such trees, we look for a generating function with T_n as coefficient.

$$T_1 x + T_2 x^2 + T_3 x^3 + \cdots = x(1-x)^{-T_1}(1-x^2)^{-T_2}(1-x^3)^{-T_3}\cdots$$

5.3 A Visual Proof

(I)



Combinatorial insight: a rooted tree consists of the root +

1. some number of 1-trees
2. some number of 2-trees ...

where the total number of knots add up to n.

5.4 Important Observation

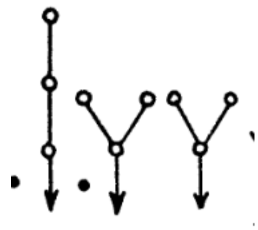


Figure 4: A 3-tree can be chosen T_3 different ways!

5.5 Making choices by multiplication

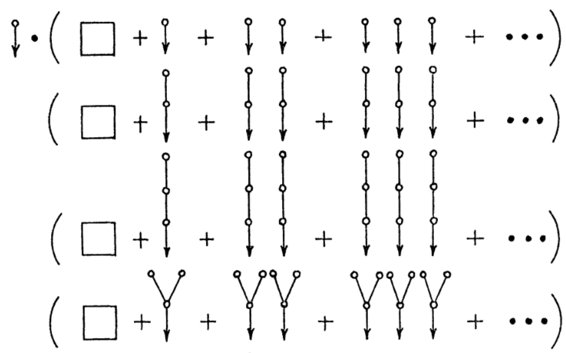


Figure 5: Infinitely many rows: T_k rows for k-trees

Insight: when you expand this product you are actually making choices

5.6 Visual generating function

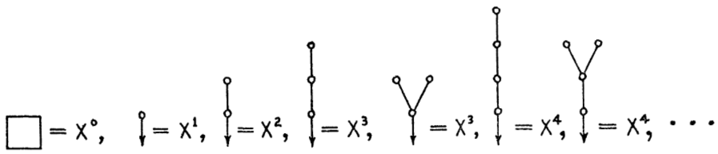


Figure 6: convert pictures into algebra

$$
\begin{aligned}
& T_1 x + T_2 x^2 + T_3 x^3 + \cdots \\
= & \underbrace{x}_{\text{this } x \text{ accounts for the root}} (1-x)^{-T_1}(1-x^2)^{-T_2}(1-x^3)^{-T_3}\cdots
\end{aligned}
$$

6 Sources Cited

References

[1] Ethan Li. *Using Python Manim Library for Mathematical Animations.* 2024. URL: `https://github.com/ethan201not404/Math-Mentorship-2024/blob/main` (visited on 05/05/2024).

[2] G. Polya. "On Picture-Writing". In: *The American Mathematical Monthly, Dec., 1956, Vol. 63, No. 10, pp. 689-697* (Dec. 1956).