Insights from Discrete Mathematics

from *Integer Partitions* to *Rooted Trees*

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Agenda

Introduction

Visualization of the Partitions

Euler's Generating Function

Hardy-Ramanujan's Asymptotic Expression

The Coins Change Problem

Rooted Trees

Introduction

Integer Partitions

Definition

A partition of a positive integer n is a way of expressing n as a sum of positive integers. Different orders of the same partitions do not count as separate partitions.

Example

Let $P(n)$ output the number of partitions for any number n.

$$P(0) = 1 \tag{1}$$

$$P(1) = 1 \tag{2}$$

$$P(4) = 5 \tag{3}$$

$P(4)$ as 4 can be written as $1{+}1{+}1{+}1$, $1{+}1{+}2$, $1{+}3$, $2{+}2$ and 4

Representations of the Partitions

Ferrers diagram

A partition can be represented graphically through a Ferrers diagram

$5 = 2+2+1$ can be illustrated as follows:

$$\begin{array}{ccc} 2 & 2 & 1 \\ * & * & * \\ * & * & \end{array}$$

If a partition's reverse is just itself, it's called self-conjugate. $10 = (4, 3, 2, 1)$ can be shown as:

$$\begin{array}{ccccc} & 4 & 3 & 2 & 1 \\ 4 & * & * & * & * \\ 3 & * & * & * & \\ 2 & * & * & & \\ 1 & * & & & \end{array}$$

Visualization of the Partitions

Young Diagram

Click here for animation demo using Python Manim

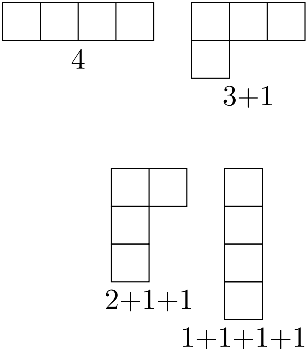


Figure: Young diagram for all partitions of 4

Euler's Generating Functions

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}, \quad \text{where } |x| < 1 \tag{4}$$

Example

To find the number of partitions of 7, we examine the coefficient of x^7:

$$(1+x^{1\cdot(1)}+x^{1\cdot(2)}+x^{1\cdot(3)}+\dots)(1+x^{2\cdot(1)}+x^{2\cdot(2)}+x^{2\cdot(3)}+\dots)\dots \tag{5}$$

Expanding above to get

$$1+x+2x^2+3x^3+5x^4+7x^5+11x^6+15x^7+22x^8+\dots \tag{6}$$

$$P(7) = 15 \tag{7}$$

Hardy-Ramanujan's Asymptotic Expression

Definition

Godfrey Hardy and Srinivasa Ramanujan proved an asymptotic formula for partitioning a number n using the circle method and modular functions

$$p(n) \approx \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}} \quad \text{as} \quad n \to \infty \tag{8}$$

Click here for animations demo using Python Manim

The Coins Change Problem

How many ways to change $\$1$?

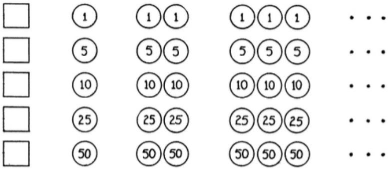


FIG. 1. A complete survey of alternatives.

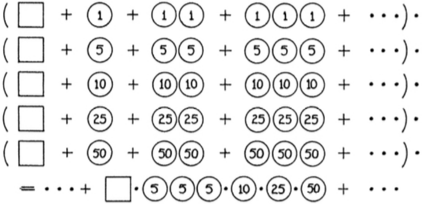


FIG. 2. Genesis of the figurate series.

Figure: Click here for animation demo using Python Manim

The Coins Change Problem

Using the Coins Change problem to derive the Generating Function

Definition

Interpret the coins change problem in mathematical form as follows:

Let P_n denote the number of ways of paying n cents with cents, nickels, dimes, quarters, and half dollars. Given $P_4 = 1$, $P_6 = 2$, and $P_{10} = 4$, what is P_{100} ?

Proof.

Please visit the section 4 of my research paper available on the GitHub (link) for complete derivation for the generating function based on the coins change problem. □

Generating Function

for Coin Change Problem

Statement of Result

If P_n is the number of ways to change n dollars with cents, nickels, dimes, quareters, and half dollars, then

$$1+P_1x+P_2x^2+\cdots = \frac{1}{1-x}\cdot\frac{1}{1-x^5}\cdot\frac{1}{1-x^{10}}\cdot\frac{1}{1-x^{25}}\cdot\frac{1}{1-x^{50}}$$

Rooted trees

Introduction

Definition

A *tree* is a graph consisting of vertices and edges but contains no closed path. A *rooted tree* is a tree with one distinguished vertex called the *root*. A non-root vertex is called a *knot*.

Question

How many rooted trees with n knots?

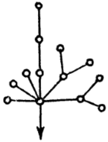
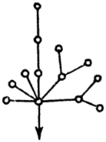


Figure: A rooted tree

Rooted Trees

From Trees to Cayley's Generating Function



Let there be T_n such trees, we look for a generating function with T_n as coefficient.

$$T_1x + T_2x^2 + T_3x^3 + \cdots \tag{9}$$

$$= x(1-x)^{-T_1}(1-x^2)^{-T_2}(1-x^3)^{-T_3}\cdots \tag{10}$$

Rooted Trees

A Visual Proof

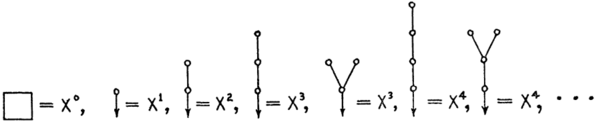


Figure: Convert trees into algebra

$$
\begin{aligned}
& T_1 x + T_2 x^2 + T_3 x^3 + \cdots \\
= & \underbrace{x}_{\text{this } x \text{ accounts for the root}} (1-x)^{-T_1}(1-x^2)^{-T_2}(1-x^3)^{-T_3}\cdots
\end{aligned}
$$

Proof.

Please visit the section 5 of my research paper available on the GitHub (link) for details. □