Options on Futures and Forwards: Black-76 and Alternatives

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Abstract

This project conducts a comparative analysis of various pricing models for options on futures and forwards, aiming to discern their respective strengths, weaknesses, and suitability under diverse market conditions. By examining models such as Black-76, Bachelier, Vasicek, and Ornstein-Uhlenbeck, this study provides insights crucial for effective decision-making in financial and energy markets.

1 Introduction

Options on futures and forwards serve as indispensable instruments in financial and energy markets, enabling investors to manage risk, speculate on future price movements and hedge against adverse market conditions. The accurate pricing of these options is paramount for informed decision-making and effective risk management strategies. However, the complexity and volatility of financial markets necessitate the utilization of sophisticated pricing models that can capture the intricacies of option pricing dynamics.

In this project, we delve into a comparative analysis of pricing models for options on futures and forwards. We evaluate the widely-used Black-76 model alongside alternative models such as Bachelier, Vasicek, and Ornstein-Uhlenbeck. By analyzing the underlying assumptions, methodologies, and practical implications of each model, we seek to provide practitioners and market participants with valuable insights into their applicability and performance across different market scenarios.

2 Definitions

2.1 Brownian Motion

Brownian motion is a stochastic process used to model random movements, such as particles in a fluid or gas. In finance, it serves to represent the random fluctuations in asset prices over time. Mathematically, Brownian motion is characterized by its continuous paths and constant variance, making it a useful tool for modelling the uncertainty inherent in financial markets. In the context of option pricing, Brownian motion is often used to model the stochastic behaviour of underlying asset prices, influencing the dynamics of option prices. The expected value of Brownian motion is 0, and its variance increases linearly with time.

Formally, a stochastic process $\{X(t), t \geq 0\}$ is said to be a Brownian motion process if it satisfies the following properties:

(i) $X(0) = 0$,

(ii) $\{X(t), t \geq 0\}$ has stationary and independent increments,

(iii) For every $t > 0$, $X(t)$ is normally distributed with mean 0 and variance $\sigma^2 t$.

[Ros23]

2.1.1 Geometric Brownian Motion

Geometric Brownian motion (GBM) is a type of Brownian motion commonly used in financial modelling to describe the stochastic behaviour of asset prices. It is characterized by a constant drift term and a volatility term, reflecting the continuous compounding of returns over time. Geometric Brownian motion is a key component of the Black-Scholes model and is instrumental in pricing derivative securities such as options.

A stochastic process $\{Y(t), t \geq 0\}$, is a Geometric Brownian motion process with drift coefficient μ and variance parameter σ^2, then the process $\{X(t), t \geq 0\}$ defined by $X(t) = e^{Y(t)}$.[Ros23]

2.2 Standard Normal Distribution

The standard normal distribution, also known as the Gaussian distribution, is a probability distribution that describes the variation of a random variable around its mean. It has a bell-shaped curve symmetric about the mean, with a mean of 0 and a standard deviation of 1. The cumulative distribution function of the standard normal distribution, denoted as $N(\cdot)$, plays a crucial role in option pricing models. It provides probabilities associated with specific values or ranges of values of a normally distributed random variable, facilitating the calculation of option prices and risk measures.

2.3 Risk-Neutral Pricing

Risk-neutral pricing is a fundamental concept in financial mathematics that forms the basis for derivative pricing models. It assumes that investors are indifferent to risk and value assets based on their expected future payoff discounted at the risk-free rate. In risk-neutral pricing, the expected return on an asset is adjusted using the risk-free interest rate to account for the time value of money and the uncertainty associated with future cash flows. This approach allows for the consistent valuation of derivative securities and enables the derivation of option pricing formulas that reflect market expectations under a risk-neutral framework.

3 Black-76

The Black-76 model, also known as the Black model, is an extension of the Black-Scholes model used for pricing options on futures contracts. It assumes constant interest rates and volatility and provides a formula for calculating the theoretical price of European options on futures. The formula takes into account the present value of the futures price and the risk-free interest rate. Despite its simplicity, the Black-76 model is widely used in financial markets for its ease of implementation and relatively accurate pricing predictions.

$$S(T) = S(0)e^{\sigma W_T - \frac{\sigma^2 T}{2}}$$

$$C(F, T_e) = e^{-r(T_e)}[F(T)N(d_1) - KN(d_2)] \tag{1}$$

where

$$d_1 = \frac{ln(F/K) + 0.5\sigma^2(T)}{\sigma\sqrt{T}}$$

$$d_2 = \frac{ln(F/K) - 0.5\sigma^2(T)}{\sigma\sqrt{T}}$$

[Swi20]

In this case, F is the price of the Futures or Forwards contract, K is the strike price, sigma is the volatility of the asset, T_e is the expiration of the call option, T is the expiration of the Future/Forward and $N(\cdot)$ is the cumulative distribution function of a Standard Normal Distribution.

3.1 Application: Pricing

Below we can see the implementation of the Black-76 formula in Python.

```python
def Black_76(F, K, T, r, sigma, option_type = 'call'):

  y_1 = (np.log(F/K)+T*((sigma**2/2)))/(sigma*np.sqrt(T))
  y_2 = (np.log(F/K)-T*((sigma**2/2)))/(sigma*np.sqrt(T))

  if option_type == 'call':
    option_price = np.exp(-r*T)*(F*stats.norm.cdf(y_1) - K*stats.norm.cdf(y_2))
  elif option_type == 'put':
    option_price = np.exp(-r*T)*(K*stats.norm.cdf(-y_2) - F*stats.norm.cdf(-y_1))
  else:
    return "Invalid Option Type"

  return round(option_price,2)
```

Figure 1: Python Code for Black-76 Pricing

Given a scenario where the Future price of an asset expiring in 6 months is \$50, the interest rate is 10%, volatility 20%, with a strike price of \$55. Then using the function created above, we can find the European Call and Put prices as $C^E(0) = \$1.05$ and $P^E(0) = \$5.81$, as seen below in Fig(2).

```python
F = 50     # Current Future price
K = 55    # Strike price
T = 0.5    # Time to expiration (in years)
r = 0.1     # Risk-free interest rate
sigma = 0.2  # Volatility

print("C^E(0) = ", Black_76(F, K, T, r, sigma, option_type='call'))
print("P^E(0) = ", Black_76(F, K, T, r, sigma, option_type='put'))
```

```
C^E(0) =  1.05
P^E(0) =  5.81
```

Figure 2: Black-76 Pricing Results

3.1.1 Negative Option Pricing

The Black-76 model, being an extension of the Black-Scholes model, assumes Geometric Brownian Motion for asset prices. This assumption inherently restricts its applicability to assets with positive prices only. In the case of oil futures contracts in April 2020, where the price of the contracts was negative, the Black-76 model could not be used to price options. Therefore, we will look at the Bachelier, Vasicek and Ornstein-Uhlenbeck(OU) models for negative prices.

4 Bachelier

The Bachelier model, named after Louis Bachelier, is another approach to pricing options on futures and forwards. Unlike the Black-76 model, which assumes Geometric Brownian Motion for asset prices, the Bachelier model assumes Arithmetic Brownian Motion. It is particularly suitable for markets where asset prices follow a random walk with normally distributed returns.

The Bachelier model has the stock price modelled as seen below.

$$S(t) = S_0 + \sigma W(t)$$

From the price of the stock, we can find the formula for pricing European Call options on the asset:

$$C(S_0, T_e) = E[S(T_e) - K]^+$$

$$= (S_0 - K)N\left(\frac{S_0 - K}{\sigma\sqrt{T_e}}\right) + \sigma\sqrt{T_e}N'\left(\frac{S_0 - K}{\sigma\sqrt{T_e}}\right)$$

Based on that formula, we can find the formula for pricing a European Call option on a Futures Contract for the asset:

$$C(F, T_e) = e^{-r(T_e - t)}[(F-K)N\left(\frac{F-K}{\sigma\sqrt{T_e - t}}\right) + \sigma\sqrt{T_e - t}N'\left(\frac{F-K}{\sigma\sqrt{T_e - t}}\right)] \tag{2}$$

[Swi20]

4.1 Application: Pricing

Below in Fig(3) we can see the Python implementation of the equation for pricing Options on Futures Contracts Eq(2) using the Bachelier Model.

```
def bachelier_option_price(F, K, T, sigma, r, option_type='call'):
    d1 = (F - K) / (sigma * np.sqrt(T))
    if option_type == 'call':
      option_price = np.exp(-r*T)*(((F - K) * stats.norm.cdf(d1)) + (sigma * np.sqrt(T) * stats.norm.pdf(d1)))
    elif option_type == 'put':
      option_price = np.exp(-r*T)*(((K - F) *stats.norm.cdf(-d1)) + (sigma * np.sqrt(T) * stats.norm.pdf(-d1)))

    return round(option_price,2)
```

Figure 3: Python Code for Bachelier Pricing

Given a scenario where the Future price of an asset expiring in 6 months is \$50, the interest rate is 10%, volatility 20%, with a strike price of \$55. Then using the function created above, we can find the European Call and Put prices as $C^E(0) = \$0.00$ and $P^E(0) = \$4.76$, as seen below in Fig(4).

```
# Example usage:
F = 50     # Current Future price
K = 55    # Strike price
T = 0.5    # Time to expiration (in years)
r = 0.1     # Risk-free interest rate
sigma = 0.2  # Volatility

print("C^E(0) = ", bachelier_option_price(F, K, T, sigma, r, option_type='call'))
print("P^E(0) = ", bachelier_option_price(F, K, T, sigma, r, option_type='put'))

C^E(0) =  0.0
P^E(0) =  4.76
```

Figure 4: Bachelier Pricing Results

4.2 Negative Option Pricing

The Bachelier model assumes Brownian Motion for asset prices, making it suitable for markets where asset prices follow a random walk with normally distributed returns. Since Brownian Motion can take both positive and negative values, the Bachelier model can accommodate pricing options on assets with negative prices as well as positive prices.

4.2.1 Application

Assuming the underlying asset price is negative, we will price a European call option based on this.

Given a scenario where the Future price of an asset expiring in 6 months is -\$10, the interest rate is 10%, volatility 20%, with a strike price of -\$1. Then using the function created above, we can find the European Call and Put prices as $C^E(0) = \$0.00$ and $P^E(0) = \$8.56$, as seen below in Fig(5).

```
F = -10    # Current Future price
K = -1    # Strike price
T = 0.5    # Time to expiration (in years)
r = 0.1     # Risk-free interest rate
sigma = 0.2  # Volatility

print("C^E(0) = ", bachelier_option_price(F, K, T,sigma, r, option_type='call'))
print("P^E(0) = ", bachelier_option_price(F, K, T, sigma, r, option_type='put'))

C^E(0) =  0.0
P^E(0) =  8.56
```

Figure 5: Bachelier Negative Pricing Results

5 Vasicek

The Vasicek model, developed by Oldrich Vasicek, is primarily utilized for interest rate modelling but can also be adapted for pricing options on futures and forwards. It describes interest rate movements as mean-reverting processes and considers the impact of mean reversion and volatility on option prices. The Vasicek model is valuable for its ability to capture the term structure of interest rates and provide insights into option pricing dynamics.

The Vasicek model describes the dynamics of the underlying asset $S(t)$ as follows:

$$dS(t) = a(b - S(t))dt + \sigma dW(t)$$

The formula for pricing a European call option on the asset is given by:

$$C(S_0, T_e) = E_Q(S(T_e) - K)^+ \tag{3}$$

$$= \left(e^{aT_e}(S_0 - b^*) - K\right) N\left(\frac{e^{-aT_e}(S_0 - b^*) - K}{\sigma\sqrt{\frac{1-e^{-2aT_e}}{2a}}}\right) + \sigma\sqrt{\frac{1-e^{-2aT_e}}{2a}} N'\left(\frac{e^{-aT_e}(S_0 - b^*) - K}{\sigma\sqrt{\frac{1-e^{-2aT_e}}{2a}}}\right)$$

The formula for pricing a European call option on a futures contract is given by:

$$C(F, T_e) = e^{-r(T_e - t)} E_Q(F(t,T) - K)^+ \tag{4}$$

$$= e^{-r(T_e-t)}\left[\left(e^{aT_e}(F - b^*) - K\right) N\left(\frac{e^{-aT_e}(F - b^*) - K}{\sigma\sqrt{\frac{1-e^{-2aT_e}}{2a}}}\right) + \sigma\sqrt{\frac{1-e^{-2aT_e}}{2a}} N'\left(\frac{e^{-aT_e}(F - b^*) - K}{\sigma\sqrt{\frac{1-e^{-2aT_e}}{2a}}}\right)\right]$$

Here, $b^* = b - \frac{\lambda\sigma}{a}$, and if $b^* = 0$, then we get the formula for the Ornstein-Uhlenbeck pricing [Swi20].

5.1 Application: Pricing

Below is the Python implementation of the Vasicek model for pricing options on futures contracts:

```python
def vasicek_option_price(F, K, T, sigma, r, b, a, lam, option_type='call'):
    B = b - (lam*sigma/a)
    d1 = ((np.exp(-a*T)*(F-B)) -K)/(sigma*np.sqrt((1-np.exp(-2*a*T))/(2*a)))
    if option_type == 'call':
        option_price = (np.exp(-r*T))*((np.exp(a*T)*(F-B) -K)*stats.norm.cdf(d1) + sigma*np.sqrt((1-np.exp(-2*a*T))/(2*a))*stats.norm.pdf(d1))
    elif option_type == 'put':
        option_price = ((np.exp(-r*T))*((np.exp(a*T)*(F-B) -K)*stats.norm.cdf(d1) + sigma*np.sqrt((1-np.exp(-2*a*T))/(2*a))*stats.norm.pdf(d1))) + K*np.exp(-r*T) - F*np.exp(-r*T)
    return round(option_price, 2)
```

Figure 6: Python Code for Vasicek Pricing

Given a scenario where the Future price of an asset expiring in 6 months is \$50, the interest rate is 10%, volatility 20%, with a strike price of \$55. The asset will also have a mean reversion level of 50, mean reversion speed of 2, and the market price of risk is 0.5. Then using the function created above, we can find the European Call and Put prices as $C^E(0) = \$0.00$ and $P^E(0) = \$4.76$, as seen below in Fig(7).

```
F = 50     # Current Future price
K = 55    # Strike price
T = 0.5    # Time to expiration (in years)
r = 0.1     # Risk-free interest rate
sigma = 0.2  # Volatility
b = 50  # Mean reversion level
a = 2  # Mean reversion speed
lam = 0.5  # Market price of risk

option_price_call = vasicek_option_price(F, K, T, sigma, r, b, a, lam, option_type='call')
print("C^E(0) = ", option_price_call)
option_price_put = vasicek_option_price(F, K, T, sigma, r, b, a, lam, option_type='put')
print("P^E(0) = ", option_price_put)

C^E(0) =  0.0
P^E(0) =  4.76
```

Figure 7: Vasicek Pricing Results

5.2 Negative Option Pricing

The Vasicek model is another model that is suitable for pricing options when the underlying asset has negative prices. This model does not assume positive asset prices and because of this, can be used as an alternative when pricing negative options.

5.2.1 Application

Given a scenario where the Future price of an asset expiring in 6 months is $-10, the interest rate is 10%, volatility 20%, with a strike price of -$1. The asset will also have a mean reversion level of 2, mean reversion speed of 2, and the market price of risk is 0.5. Then using the function created above, we can find the European Call and Put prices as $C^E(0) = -\$0.00$ and $P^E(0) = \$8.56$, as seen below in Fig(8).

```
F = -10     # Current Future price
K = -1    # Strike price
T = 0.5    # Time to expiration (in years)
r = 0.1     # Risk-free interest rate
sigma = 0.2  # Volatility
b = 2  # Mean reversion level
a = 2  # Mean reversion speed
lam = 0.5  # Market price of risk

option_price_call = vasicek_option_price(F, K, T, sigma, r, b, a, lam, option_type='call')
print("C^E(0) = ", option_price_call)
option_price_put = vasicek_option_price(F, K, T, sigma, r, b, a, lam, option_type='put')
print("P^E(0) = ", option_price_put)

C^E(0) =  -0.0
P^E(0) =  8.56
```

Figure 8: Vasicek Negative Pricing Results

6 Ornstein-Uhlenbeck (OU)

The Ornstein-Uhlenbeck model, also known as the mean-reverting process, is commonly employed in finance to model the behavior of interest rates and asset prices. It assumes that the underlying process returns to a long-term mean over time, making it suitable for modelling mean-reverting phenomena in financial markets. The Ornstein-Uhlenbeck model can be adapted to price options on futures and forwards by considering the mean reversion and volatility characteristics of the underlying asset.

The Ornstein-Uhlenbeck model describes the dynamics of the underlying asset $S(t)$ as follows:

$$dS(t) = -aS(t)dt + \sigma dW(t)$$

The formula for pricing a European call option on the asset is given by:

$$C(S_0, T_e) = E_Q(S(T_e) - K)^+$$

$$= \left(e^{a^* T_e} S_0 - K\right) N\left(\frac{e^{-a^* T_e} S_0 - K}{\sigma\sqrt{\frac{1-e^{-2a^* T_e}}{2a^*}}}\right) + \sigma\sqrt{\frac{1-e^{-2a^* T_e}}{2a^*}} N'\left(\frac{e^{-a^* T_e} S_0 - K}{\sigma\sqrt{\frac{1-e^{-2a^* T_e}}{2a^*}}}\right)$$

The formula for pricing a European call option on a futures contract is given by:

$$C(F, T_e) = e^{-r(T_e - t)} E_Q (F(t,T) - K)^+$$

$$= e^{-r(T_e - t)} \left[\left(e^{a^* T_e} F - K \right) N \left(\frac{e^{-a^* T_e} F - K}{\sigma \sqrt{\frac{1 - e^{-2a^* T_e}}{2a^*}}} \right) + \sigma \sqrt{\frac{1 - e^{-2a^* T_e}}{2a^*}} N' \left(\frac{e^{-a^* T_e} F - K}{\sigma \sqrt{\frac{1 - e^{-2a^* T_e}}{2a^*}}} \right) \right] \quad (5)$$

where

$$a* = a + \lambda \sigma$$

if $a^* = 0$ then this formula coincides with the Bachelier formula[Swi20]

6.1 Application: Pricing

Below is the Python implementation of the Vasicek model for pricing options on futures contracts:

```
def ou_option_price(F, K, T, sigma, r, a, lam, option_type='call'):
  A = a + lam*sigma
  d2 = (np.exp(-A*T)*F-K)
  d3 = (sigma * np.sqrt((1-np.exp(-2*A*T))/(2*A)))
  d1 =d2/d3
  if option_type == 'call':
        option_price = np.exp(-r*T)*(d2*stats.norm.cdf(d1) + d3*stats.norm.pdf(d1))
  elif option_type == 'put':
        option_price = (np.exp(-r*T)*(d2*stats.norm.cdf(d1) + d3*stats.norm.pdf(d1))) + K*np.exp(-r*T) - F*np.exp(-r*T)
  return round(option_price, 2)
```

Figure 9: Python Code for OU Pricing

Given a scenario where the Future price of an asset expiring in 6 months is \$50, the interest rate is 10%, volatility 20%, with a strike price of \$55. The asset will also have a mean reversion speed of 2, and the market price of risk is 0.5. Then using the function created above, we can find the European Call and Put prices as $C^E(0) = \$0.00$ and $P^E(0) = \$4.76$, as seen below in Fig(10).

```
F = 50     # Current Future price
K = 55    # Strike price
T = 0.5    # Time to expiration (in years)
r = 0.1     # Risk-free interest rate
sigma = 0.2  # Volatility
a = 0.2  # Mean reversion speed
lambda_ = 0.5  # Market price of risk

option_price_call = ou_option_price(F, K, T, sigma, r, a, lam, option_type='call')
print("C^E(0) = ", option_price_call)

# Calculate put option price
option_price_put = ou_option_price(F, K, T, sigma, r, a, lam, option_type='put')
print("P^E(0) = ", option_price_put)

C^E(0) =  0.0
P^E(0) =  4.76
```

Figure 10: OU Pricing Results

6.2 Negative Option Pricing

The Ornstein-Uhlenbeck model is another model that is suitable for pricing options when the underlying asset has negative prices. This model does not assume positive asset prices and because of this, can be used as an alternative when pricing negative options.

6.2.1 Application

Given a scenario where the Future price of an asset expiring in 6 months is \$-10, the interest rate is 10%, volatility 20%, with a strike price of -\$1. The asset will also have a mean reversion speed of 2, and the market price of risk is 0.5. Then using the function created above, we can find the European Call and Put prices as $C^E(0) = \$0.00$ and $P^E(0) = \$8.56$, as seen below in Fig(11).

```
F = -10     # Current Future price
K = -1    # Strike price
T = 0.5    # Time to expiration (in years)
r = 0.1     # Risk-free interest rate
sigma = 0.2  # Volatility
a = 2  # Mean reversion speed
lambda_ = 0.5  # Market price of risk

option_price_call = ou_option_price(F, K, T, sigma, r, a, lam, option_type='call')
print("C^E(0) = ", option_price_call)

# Calculate put option price
option_price_put = ou_option_price(F, K, T, sigma, r, a, lam, option_type='put')
print("P^E(0) = ", option_price_put)

C^E(0) =  0.0
P^E(0) =  8.56
```

Figure 11: OU Negative Pricing Results

7 When to use which model

Selecting the appropriate pricing model for options on futures and forwards depends on various factors, including market conditions, the nature of the underlying asset, and the assumptions of each model. Below, we outline the key considerations for choosing between the Black-76 model, Bachelier model, Vasicek model, and Ornstein-Uhlenbeck model:

Future Price	Mean-Reversion Level	Model
Positive	None	Black-76
Positive and Negative	None	Bachelier
Positive and Negative	b	Vasicek
Positive and Negative	0	Ornstein-Uhlenbeck

Table 1: Caption

The Black-76 model is suitable for pricing options on futures contracts when dealing with positive future prices and assuming no mean reversion. Its simplicity and ease of implementation make it a popular choice in financial and energy markets. However, practitioners should be aware of its limitations in accommodating mean-reverting processes and negative asset prices.

Unlike the Black-76 model, the Bachelier model can handle both positive and negative future prices, making it more versatile in volatile markets. It assumes arithmetic Brownian motion, which aligns well with assets exhibiting random walk behaviour with normally distributed returns. Market participants can leverage the Bachelier model for pricing options in scenarios where asset prices deviate significantly from their mean values.

The Vasicek model incorporates mean reversion and volatility characteristics, providing insights into option pricing dynamics for assets with both positive and negative prices. By considering the mean-reverting nature of asset prices, the Vasicek model offers a more nuanced approach to pricing options on futures and forwards. It is particularly useful in markets where asset prices exhibit mean-reverting behaviour over time.

Similar to the Vasicek model, the Ornstein-Uhlenbeck model accounts for mean reversion but assumes a mean reversion level of zero. It is applicable to assets demonstrating mean-reverting behaviour and can accommodate both positive and negative prices. Market participants can utilize the Ornstein-Uhlenbeck model in situations where assets exhibit mean-reverting tendencies with no bias towards a specific mean level.

8 Conclusion

In this project, we have examined various pricing models for options on futures and forwards, comparing the Black-76 model with alternatives like Bachelier, Vasicek, and Ornstein-Uhlenbeck. Each model offers unique strengths and weaknesses, catering to different market conditions and asset characteristics.

The Black-76 model is straightforward and suitable for positive future prices but lacks consideration for mean reversion. On the other hand, the Bachelier model accommodates both positive and negative future prices, assuming arithmetic Brownian motion, making it versatile in volatile markets.

For assets with mean-reverting behaviour, the Vasicek and Ornstein-Uhlenbeck models are more appropriate. While the Vasicek model incorporates mean reversion and volatility, allowing for both positive and negative prices, the Ornstein-Uhlenbeck model assumes mean reversion with a mean reversion level of zero, suitable for assets exhibiting mean-reverting tendencies without a specific mean level bias.

The choice of model depends on the specific characteristics of the underlying asset and prevailing market conditions. By understanding the assumptions and features of each model, market participants can make informed decisions when pricing options on futures and forwards, enhancing risk management and decision-making in financial and energy markets.

References

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