NEGATIVE THIRD-ORDER RECURSION

ETHAN LEVITT

1. Introduction

Everyone knows the Fibonacci sequence. The famous $1, 1, 2, 3, 5...$ that defines so many facets of nature and mathematics. However, the Fibonacci sequence provides no variation. So, over the next few pages, we will be exploring an alteration of the Fibonacci sequence that opens new doors of variation.

Definition 1.1. *Let the sequence a_n be defined as $a_n = a_{n-1} + a_{n-2} - a_{n-3}$ for $n \geqslant 3$.*

With this sequence, we can provide different seed values of a_0, a_1, and a_2, yielding many interesting patterns. For example, seeds like $3, 4, 5$ will present a simple sequence of $6, 7, 8, 9 \dots$ while a seed of $5, 12, 13$ will yield an output of $20, 21, 28, 29, 36, 37, \dots,$ providing a realm of possibilities for new sequences that can be generated.

2. Results

First, we ought to recognize some different variations on the original equation

$$a_n = a_{n-1} + a_{n-2} - a_{n-3}$$

that will make the algebra in our results simpler. The following lemmas are all simple manipulation of our first equation, and the proof will be left to the reader.

Lemma 2.1. *Variations on a_n.*

(1) $a_{n-1} = a_n - a_{n-2} + a_{n-3}$.
(2) $a_{n-2} = a_n - a_{n-1} + a_{n-3}$.
(3) $a_{n-3} = -a_n + a_{n-1} + a_{n-2}$.

In the scenario where one is solving for a high-valued a_n without an explicit formula (or a computer to run the recursion), the following theorem can be used to quickly find that specific a_n:

Theorem 2.2. $a_{n+1} = 2a_{n-1} - a_{n-3}$.

Proof. If $a_n = a_{n-1} + a_{n-2} - a_{n-3}$, then $a_{n+1} = a_n + a_{n-1} - a_{n-2}$. It follows that

$$a_{n+1} = a_n + a_{n-1} - a_{n-2} = (a_{n-1} + a_{n-2} - a_{n-3}) + a_{n-1} - a_{n-2} = 2a_{n-1} - a_{n-3}.$$

□

Another straightforward situation is if all the seeds are the same number:

Date: 2019.

Theorem 2.3. *If $a_0 = a_1 = a_2$, then $a_n = a_0$.*

Proof. Proof by induction.

Base case: Since a_0, a_1 and a_2 are accounted for, let $n = 3$. Then,

$$a_3 = a_2 + a_1 - a_0 = a_0 + a_0 - a_0 = a_0.$$

Inductive step: Assume $a_0 = a_1 = a_2$ and $a_n = a_0$ for all $n \leqslant k$. Then by induction hypothesis,

$$a_{k+1} = a_k + a_{k-1} - a_{k-2} = a_0 + a_0 - a_0 = a_0.$$

□

The two following theorems are closely related, and provide similar results from their base assumptions.

Theorem 2.4. *If $a_0 \neq a_1 = a_2$, then $a_3 = a_4$, $a_5 = a_6, \ldots$*

Aside: To effectively prove this theorem, I believe that a more technical expressions is necessary. These expressions have the same meaning, but the following will give us a more tangible argument to prove: If $a_0 \neq a_1 = a_2$, then $a_{2\ell+1} = a_{2\ell+2}$ for some $\ell \in \mathbb{N}$.

Proof. Proof by induction.

Base case: Let $\ell = 1$. Then,

$$a_{2\ell+2} = a_4 = a_3 + a_2 - a_1 = a_3 + a_2 - a_2 = a_3 = a_{2\ell+1}.$$

Inductive step: Assume $a_0 \neq a_1 = a_2$ and $a_{2\ell+1} = a_{2\ell+2}$ for $\ell = k$. Then,

$$a_{2(k+1)+2} = a_{2k+4} = a_{2k+3} + a_{2k+2} - a_{2k+1} = a_{2k+3} + a_{2k+2} - a_{2k+2} = a_{2k+3} = a_{2(k+1)+1}.$$

□

Theorem 2.5. *If $a_0 = a_1 \neq a_2$, then $a_2 = a_3$, $a_4 = a_5, \ldots$*

Aside: Once again, a more technical iteration of this theorem is required for a rigorous proof. This theorem can be restated as: If $a_0 = a_1 \neq a_2$, then $a_{2\ell} = a_{2\ell+1}$ for some $\ell \in \mathbb{N}$.

Proof. Proof by induction.

Base case: Let $\ell = 1$. Then,

$$a_{2\ell+1} = a_3 = a_2 + a_1 - a_0 = a_2 = a_{2\ell}$$

Inductive step: Assume $a_0 = a_1 \neq a_2$ and $a_{2\ell} = a_{2\ell+1}$ for some $\ell = k$. Then,

$$a_{2(k+1)+1} = a_{2k+3} = a_{2k+2} + a_{2k+1} - a_{2k} = a_{2k+2} + a_{2k+1} - a_{2k+1} = a_{2k+2} = a_{2(k+1)}.$$

□

While one may think that an assumption can now be made in the case of $a_0 = a_2 \neq a_1$, the following theorem could provide a bit of a surprise.

Theorem 2.6. *If $a_0 = a_2 \neq a_1$, then $a_n = a_1$ for all odd n, and $a_n = a_2$ for all even n.*

Proof. This theorem can be proved most effectively with two cases: one for even numbers and one for odd numbers.

Case 1: Even numbers. Proof by induction.

Base case: Since a_0 and a_2 are accounted for, let $n = 4$. Then, by theorem 2.2,

$$a_4 = 2a_2 - a_0 = 2a_2 - a_2 = a_2.$$

Inductive step:[1] Assume $a_0 = a_2$, $a_n = a_2$ for all even $n \leqslant k$ where k is an even number. Then by theorem 2.2,

$$a_{k+2} = 2a_k - a_{k-2}$$

and since $a_n = a_2$ for all even $n \leqslant k$,

$$a_{k+2} = 2a_k - a_{k-2} = 2a_2 - a_2 = a_2.$$

Case 2: Odd numbers. Proof by induction.

Base case: Since a_1 is accounted for, let $n = 3$, and recall that $a_0 = a_2$. Then,

$$a_3 = a_2 + a_1 - a_0 = a_2 + a_1 - a_2 = a_1.$$

Inductive step:[2] Assume $a_n = a_1$ for all odd $n \leqslant k$ where k is an odd number. Then, by theorem 2.2,

$$a_{k+2} = 2a_k - a_{k-2}$$

and since $a_n = a_1$ for all odd $n \leqslant k$,

$$a_{k+2} = 2a_k - a_{k-2} = 2a_1 - a_1 = a_1.$$

□

While a_n is a third order recursive sequence, this seeding can effectively make it a first order sequence.

Theorem 2.7. *For some number ℓ, If $a_0 + 2\ell = a_1 + \ell = a_2$, then $a_n = a_{n-1} + \ell$.*

Proof. Proof by induction.

Base case: Since a_0, a_1 and a_2 are accounted for, let $n = 3$. Then,

$$a_3 = a_2 + a_1 - a_0 \iff a_3 = (a_0 + 2\ell) + a_1 - a_0 \iff a_3 = 2\ell + a_1.$$

And since $a_1 + \ell = a_2$, it can be concluded that $a_1 = a_2 - \ell$, and therefore $a_3 = a_2 + \ell$.

Inductive step. Assume $a_0 + 2\ell = a_1 + \ell = a_2$ and $a_n = a_{n-1} + \ell$ for all $n = 2, 3, ..., k$. Then by induction hypothesis,

$$a_{k+1} = a_k + a_{k-1} - a_{k-2} \iff a_{k+1} = a_k + a_{k-2} + \ell - a_{k-2} \iff a_{k+1} = a_k + \ell.$$

□

[1] Since a_{k+1} will be an odd number, we need to set our inductive step to a_{k+2}.

[2] Again, a_{k+1} will be an even number. Therefore our inductive step needs to prove true for a_{k+2}.

3. Conclusion and Further Directions

While deciding upon a sequence to study, I found a similar sequence that yielded some promising patterns. For this, we shall define

$$s_n = b_n - f_n,$$

where

$$b_n = b_{n-1} + b_{n-2} + b_{n-3}$$

for $n \geqslant 3$, and b_0, b_1, and b_2 are defined as needed. Then,

$$f_n = f_{n-1} + f_{n-2}$$

for $n \geqslant 2$, and we define $f_0 = 0,\ f_1 = 1$. This is effectively the Fibonacci Sequence, and some interesting results come from certain seedings of b_n. None of these yield stable patterns, but rather interesting beginnings that eventually diverge from the targeted pattern.

Theorem 3.1. *Mimicking 2^{n-2} with s_n*

Let $b_0 = 1$, $b_1 = 1$, and $b_2 = 2$. Then, the first few terms of s_n are as follows:

$$s_2 = 1 \quad s_3 = 2 \quad s_4 = 4 \quad s_5 = 8 \quad s_6 = 16 \quad s_7 = 31 \quad s_8 = 60 \quad s_9 = 115$$

Notice how terms s_2 through s_6 are all equivalent to 2^{n-2}, and then $s_n < 2^{n-2}$, with the gap between the two values increasing more and more as n increases.

Theorem 3.2. *Mimicking prime numbers with s_n*

Let $b_0 = 0$, $b_1 = 1$, and $b_2 = 1$. Then, the first few terms of s_n are as follows:

$$s_3 = 0 \quad s_4 = 1 \quad s_5 = 2 \quad s_6 = 5 \quad s_7 = 11 \quad s_8 = 23 \quad s_9 = 47 \quad s_{10} = 94$$

Here, terms s_5 to s_9 are all prime numbers. After s_9, however, values of s_n are exclusively composite numbers until $s_{23} = 382087$ and then $s_{28} = 8328253$.[3] It seemed that this pattern would continue to yield interesting results but it does not, in fact, produce exclusively prime or exclusively composite numbers after a certain point. Another object of interest in this scenario is the difference between values of s_n. Notice that

$$s_6 - s_5 = 3 \quad s_7 - s_6 = 6 \quad s_8 - s_7 = 12 \quad s_9 - s_8 = 24 \quad s_{10} - s_9 = 47$$

The differences between the the prime numbers follows the form of $3 * (2^\ell)$, where ℓ starts as 0 and increases by 1 for each difference from $s_6 - s_5$ to $s_9 - s_8$. Then, this form is broken at $s_{10} - s_9$, providing an interesting correlation to the end of prime numbers with s_{10}. Let's call this new sequence p_n, as to not confuse it with our regular s_n.

Definition 3.3. $p_n = p_{n-1} + 3 * 2^{n-1}$ *where* $p_0 = 2$

With this equation, p_n would start with the folowing numbers:

$$p_0 = 2 \quad p_1 = 5 \quad p_2 = 11 \quad p_3 = 23 \quad p_4 = 47 \quad p_5 = 95 \quad p_6 = 191 \quad p_7 = 383$$

Disappointingly, even we continued to model p_n past where s_n fails to generate primes, we would not have a continued list of prime numbers. Note that p_0 through p_7 is a list of prime

[3] Up to the term $s_{30} = 28417385$. I have not found an efficient way to see if larger numbers come out prime.

numbers, other than $p_5 = 95$. Past this, we only get the prime number of $p_{11} = 6143$ and $p_{18} = 786431$ in our observable list of prime numbers.[4]

So, while there is little pattern that plays out in the long run with either of these theorems, the first few numbers provide interesting promises at synthesizing other numeric patterns. With some manipulation, perhaps the sequences can in fact follow the expected form.

[4] Up to $p_{30} = 3221225471$. Again, my prime number testing is not as advanced as I would like.