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Symmetries in Classical and Quantum Field Theory

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Abstract: In this work, we detail the effect of continuous symmetries on classical gauge theories namely, Noether's theorems, before gauging the Dirac Lagrangian's axial symmetry to obtain a quantum electrodynamical analogue of the interaction between massless Dirac fermions and U(1) axial bosons; an 'axial theory'. Subsequently, we investigate the effect of a classical theory's symmetries on the quantum field theory (QFT) via Slavnov-Taylor and Ward-Takahashi identities, and we find that, for the class of symmetries with infinitesimal generators linear in the dynamical fields, a singular path integral Jacobian matrix characterises the presence of an anomaly in the quantised theory. Moreover, anomalies can be seen as topological invariants of a QFT, and gauging an anomalous 'linear' symmetry results in a violation of unitarity; an inconsistent quantum theory. In the case of the 'axial theory', we compute the anomaly explicitly.

Contents

1 Introduction **1**

2 Symmetries in Classical Gauge theory **2**
- 2.1 Classical field theory 2
- 2.2 Noether's first theorem 3
- 2.3 Noether's second theorem 4

3 The Dirac Lagrangian **5**
- 3.1 Gauging the vector symmetry 5
- 3.2 Gauging the axial symmetry 7

4 Quantum Field Theory **8**
- 4.1 Second Quantisation 8
- 4.2 The path integral 9

5 Symmetries in QFT **9**
- 5.1 Slavnov-Taylor identities 9
- 5.2 Ward-Takahashi identities 11

6 Anomalies in QFT **12**
- 6.1 The path integral measure & Fujikawa's method 12
- 6.2 Atiyah–Singer index theorem 14
- 6.3 Anomalies as obstructions to gauging linear symmetries 15

7 Conclusion **18**

A N-point function derivative **18**

1 Introduction

The study of continuous symmetries in dynamical systems has long been used by theorists to constrain possible Lagrangians. To this end, the seminal results in classical field theory are Noether's first and second theorems[1], which reveal the correspondence between quantities conserved on-shell and infinitesimal symmetries. Moreover, the analysis of local (gauge) symmetries has produced incredibly successful gauge theories such as quantum electrodynamics[2], which can be seen as resultant from 'gauged' symmetries. In this vein, we show that gauging the axial symmetry of the Dirac Lagrangian results in a well-formulated classical gauge theory; an 'axial theory' with a U(1) axial boson.

Thereafter, we investigate the effect of symmetries of the classical action on QFT via the quantum field-theoretic analogue of Noether's theorems; the Slavnov–Taylor and Ward-Takahashi identities [3–7]. Anomalous symmetries are symmetries of the classical theory that are violated by quantum processes in a QFT. In particular, the axial symmetry is famously violated by triangle feynman diagrams (the Adler-Bell-Jackiw anomaly [8, 9]) and can be viewed as a topological invariant of QFT via the Atiyah-Singer index theorem [10]. Using the Slavnov–Taylor identities, we find that it is necessary and sufficient for an anomalous symmetry with infinitesimal generators linear in the dynamical fields to have a singular path integral Jacobian, and gauging such symmetries produces a gauge-variant partition function; an inconsistent quantum theory.

Finally, we compute the gauge-variance of the 'axial theory' partition function using Fujikawa's method [11], although a topological approach is possible (cf. [12, 13]), and demonstrate an inconsistency in the quantum theory.

2 Symmetries in Classical Gauge theory

2.1 Classical field theory

In classical mechanics, we consider a countable set of particles each with finitely many degrees of freedom and generalised coordinates $q_i(t)$. These generalised coordinates $\{q_i(t)\}_i$ specify the system's configuration (position in configuration space), and together with the generalised conjugate momenta $\left\{p_i(t) := \frac{\partial L}{\partial \dot{q}_i}\right\}_i$, define the system's position in phase space [14]. In classical field theory, we generalise this notion of configuration space to a continuum with infinite degrees of freedom. The scalar field $\phi(t)$ can be seen as the generalised coordinates of a continuum $\left(q_i(t) \xrightarrow{i\to\vec{x}} \phi(\vec{x},t)\right)$, and for a system of continua the set of scalar fields, $\{\phi_k(\vec{x},t)\}_k$, specifies the system's configuration [15]. Subsequently, we generalise the classical Lagrangian $L(q_i(t), \dot{q}_i(t), t)$ via

$$L(t) = \int d^3\vec{x}\, \mathcal{L}(\phi_k(\vec{x},t), \partial_\mu\phi_k(\vec{x},t), \vec{x}, t), \tag{2.1}$$

where $\mathcal{L}$ is the Lagrangian density. With respect to a path in configuration space, $\vec{\Phi}(\vec{x},t)$, the classical action becomes

$$S\left[\vec{\Phi}(\vec{x},t)\right] = \int dt\, L = \int d^4x\, \mathcal{L}(\phi_k(\vec{x},t), \partial_\mu\phi_k(\vec{x},t), \vec{x}, t). \tag{2.2}$$

Using Hamilton's principle [16] ($\delta S[\vec{\Phi};\vec{\rho}] = 0, \forall \vec{\rho} \in C_0^\infty$),[1] we obtain the field-theoretic Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \phi_k} \approx \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)}\right], \tag{2.3}$$

where we use $\approx$ to denote an on-shell equality. In Lagrangian systems the Hamiltonian, $H(p_i(t), q_i(t), t)$, is characterised by the Legendre transform

$$H = \sum_i p_i(t)\dot{q}_i(t) - L, \tag{2.4}$$

[1] δS is a Gateaux derivative, and $\vec{\rho}$ vanishes at the endpoints of $\vec{\Phi}$.

to generalise this to field theory we introduce the momentum field conjugate to $\phi_k(\vec{x},t)$

$$\pi_k(\vec{x},t) \coloneqq \frac{\partial \mathcal{L}}{\partial \dot{\phi}_k}, \tag{2.5}$$

$$p_i(t) \coloneqq \frac{\partial L}{\partial \dot{q}_i} \xrightarrow{i\to\vec{x}} \frac{\partial}{\partial \dot{\phi}_k}\left[\int d^3\vec{x}\,\mathcal{L}\right] = \int d^3\vec{x}\,\pi_k(\vec{x},t). \tag{2.6}$$

Thus the classical Hamiltonian is generalised to

$$H(t) = \int d^3\vec{x}\,\mathcal{H}(\pi_k(\vec{x},t),\phi_k(\vec{x},t),\vec{x},t), \tag{2.7}$$

$$\mathcal{H} = \sum_k \pi_k(\vec{x},t)\dot{\phi}_k(\vec{x},t) - \mathcal{L}, \tag{2.8}$$

where $\mathcal{H}$ is the Hamiltonian density, and the field-theoretic analogue of Hamilton's equations [16] is the Hamiltonian field equations [15]

$$\dot{\phi}_k = \frac{\delta \mathcal{H}}{\delta \pi_k}, \quad \dot{\pi}_k = -\frac{\delta \mathcal{H}}{\delta \phi_k}, \tag{2.9}$$

where we use the functional derivative identify

$$\frac{\delta \mathcal{F}}{\delta g} = \frac{\partial \mathcal{F}}{\partial g} - \partial_\mu\left[\frac{\partial \mathcal{F}}{\partial(\partial_\mu g)}\right]. \tag{2.10}$$

To begin second quantisation we will also require the notion of a field-theoretic poisson bracket. Given two functionals, F and G,[2] of the dynamical fields given by

$$F = \int d^3\vec{x}\,\mathcal{F}(\pi_k,\phi_k,\vec{x},t), \quad G = \int d^3\vec{x}\,\mathcal{G}(\pi_k,\phi_k,\vec{x},t), \tag{2.11}$$

we define the poisson bracket in field theory as

$$\{F,G\}_f = \int d^3\vec{x}\sum_k\left[\frac{\delta\mathcal{F}}{\delta\phi_k}\frac{\delta\mathcal{G}}{\delta\pi_k} - \frac{\delta\mathcal{G}}{\delta\phi_k}\frac{\delta\mathcal{F}}{\delta\pi_k}\right]. \tag{2.12}$$

2.2 Noether's first theorem

In classical mechanics, Noether's first theorem shows the correspondence between global symmetries of the Lagrangian and conserved quantities (Noether charges). This generalises to global symmetries of the Lagrangian density corresponding to conserved currents in classical field theory. Consider an infinitesimal field transformation $\vartheta(\epsilon)$ such that

$$\phi_n \mapsto \phi_n + \delta_\epsilon\phi_n + \mathcal{O}(\epsilon^2); \quad \delta_\epsilon\phi_n = \epsilon\vartheta_n(\phi_k), \tag{2.13}$$

where the infinitesimal generators, ϑ_k, are independent of spacetime. A transformation is called a global symmetry if its effect on the Lagrangian density is

$$\mathcal{L} \mapsto \mathcal{L} + \epsilon\partial_\mu\Lambda^\mu(\phi_k,x) + \mathcal{O}(\epsilon^2). \tag{2.14}$$

[2] n.b. $K(x) = \int dy\,[K(y)\delta(x-y)]$.

Such a transformation changes the action by a surface term, leaving the Euler-Lagrange equations invariant. Using a Taylor expansion of the Lagrangian density under $\vartheta(\epsilon)$ we find that

$$\delta_\epsilon \mathcal{L} = \sum_k \delta_\epsilon \phi_k E_k + \partial_\mu \theta^\mu, \tag{2.15}$$

$$\theta^\mu = \sum_k \delta_\epsilon \phi_k \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)}, \tag{2.16}$$

where E_k are the Euler-Lagrange derivatives and θ^μ is the symplectic potential density [17]. By considering the shift in the action we arrive at Noether's first theorem [1]; an infinitesimal transformation $\vartheta(\epsilon)$ is a symmetry of the Lagrangian if, and only if,

$$\epsilon j^\mu := \theta^\mu - \epsilon \Lambda^\mu, \quad \epsilon \partial_\mu j^\mu = -\delta_\epsilon \phi_k E_k \approx 0, \tag{2.17}$$

where we use the summation convention for the field indices hereafter, and the on-shell divergence-free quantity j^μ is called the conserved Noether current.[3] This easily generalises to local symmetries: spacetime dependent generators $\vartheta_k(\phi, x)$. Moreover, a Lagrangian with no explicit spacetime dependence implies that the canonical energy-momentum tensor obeys divergence relations on-shell

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)} \partial_\nu \phi_k - \delta^\mu_\nu \mathcal{L}, \quad \partial_\mu T^\mu{}_\nu \approx 0. \tag{2.18}$$

This divergence property can be seen as the independent conservation relations of four Noether currents. By Noether's first theorem, the transformations corresponding to these Noether currents are the global symmetries

$$\phi_n(x^\mu) \mapsto \phi_n(x^\mu + \epsilon) = \phi_n(x^\mu) + \epsilon \partial_\mu \phi_n \big|_{x^\mu} + \mathcal{O}(\epsilon^2), \tag{2.19}$$

and it is clear that translation invariance of the equations of motion (for spacetime independent Lagrangians) is a direct corollary of Noether's first theorem.

2.3 Noether's second theorem

Noether's second theorem shows that a local symmetry introduces off-shell constraints on the equations of motion and leads to Noether current identities which are crucial for gauge theories. Under local infinitesimal symmetries (gauge symmetries), R_ξ, of the form

$$\phi_n \mapsto \phi_n + \delta_\xi \phi_n + \mathcal{O}(\xi^2); \quad \delta_\xi \phi_n = \sum_{m=0}^{N} R_n^{\mu_1 \cdots \mu_m}(\phi_k) \partial_{\mu_1} \cdots \partial_{\mu_m} \xi \lambda(x), \tag{2.20}$$

the action shifts by

$$\delta_\xi S = \int d^4x \, \delta_\xi \phi_n E_n + \partial_\mu \theta^\mu. \tag{2.21}$$

If $\lambda(x)$ is a compactly supported smooth function the boundary term vanishes, and we have

$$\int d^4x \, \delta_\xi \phi_n E_n = 0, \tag{2.22}$$

[3] n.b. j^μ is ϵ-independent.

and integration by parts leads to

$$\int d^4x\, \lambda(x)\Delta(E) = 0; \quad \Delta(E) \coloneqq \sum_{m=0}^{N} (-1)^m \partial_{\mu_1} \cdots \partial_{\mu_m} [R_n^{\mu_1 \cdots \mu_m}(\phi_k) E_n]. \tag{2.23}$$

By the fundamental lemma of the calculus of variations [18] we obtain the off-shell constraint on the Euler-Lagrange derivatives $\Delta(E) = 0$. Thus the equations of motion are not all independent, and the Euler-Lagrange equations ($E_n = 0$) are under-determined. However, observables must be uniquely determined by the theory, hence we conclude some degrees of freedom are gauge. Using the surface term S^μ from integration by parts we find an expression similar to (2.17)

$$\delta_\xi \phi_n E_n = \xi \partial_\mu S^\mu(E_k, \lambda), \tag{2.24}$$

with the important difference that S^μ vanishes on-shell. The quantity $j^\mu + S^\mu$ is divergence-free off-shell, thus by the Poincaré lemma we obtain Noether's second theorem [1, 19, 20]; the canonical Noether current j^μ is the divergence of a rank-2 tensor on-shell

$$j^\mu = -S^\mu(E_k, \lambda) + \partial_\nu \kappa^{\mu\nu} \approx \partial_\nu \kappa^{\mu\nu}; \quad \kappa^{\mu\nu} = -\kappa^{\nu\mu}. \tag{2.25}$$

The antisymmetric tensor $\kappa^{\mu\nu}$ is called a superpotential. Although the superpotential is arbitrary ($\kappa^{\mu\nu}$ is determined up to addition by a divergence-free quantity) it can be uniquely determined if one demands the superpotential obeys additional constraints (such as asymptotic conservation) [20]. In the case of electrodynamics, the superpotential is electromagnetic field tensor and (2.25) becomes Maxwell's equations

$$J^\mu \approx \partial_\nu F^{\nu\mu}, \quad F_{\mu\nu} \coloneqq \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.26}$$

3 The Dirac Lagrangian

3.1 Gauging the vector symmetry

In quantum mechanics, the Schrodinger equation is not Lorentz invariant and is thus incompatible with special relativity. A generalisation of the relativistic dispersion relation, using Hamiltonian and momentum operators, leads to the Klein-Gordon equation [21, 22].[4] However, the Klein-Gordon equation is a second-order partial differential equation and subsequently does not uniquely determine the time-evolution of the wavefunction. Alternatively, by finding the half-iterate of the d'Alembert operator,[5] the Klein-Gordon equation simplifies to a first-order partial differential equation; the Dirac equation [23]

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0. \tag{3.1}$$

We call the corresponding Lagrangian density the Dirac Lagrangian

$$\mathcal{L}_{\mathrm{D}} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi, \tag{3.2}$$

[4] $(\Box + m^2)\psi = 0$.
[5] $\Box \coloneqq \partial_t^2 - \nabla^2$.

where the adjoint bispinor is defined as $\bar{\psi} \coloneqq \psi^\dagger \gamma^0$. This Lagrangian admits a global U(1) symmetry called the vector symmetry

$$\psi \mapsto e^{i\vartheta}\psi, \quad \bar{\psi} \mapsto \bar{\psi} e^{-i\vartheta}. \tag{3.3}$$

This symmetry corresponds to a conserved current, called the vector current, via Noether's first theorem

$$j^\mu = \bar{\psi}\gamma^\mu\psi. \tag{3.4}$$

We now attempt to 'gauge' this symmetry by adding spacetime dependence to the infinitesimal generator

$$\psi \mapsto e^{i\vartheta(x)}\psi, \quad \bar{\psi} \mapsto \bar{\psi} e^{-i\vartheta(x)}. \tag{3.5}$$

However, the Lagrangian is not invariant under such a transformation, so we introduce a derivative operator which transforms covariantly

$$D_\mu \coloneqq \partial_\mu + ie\Pi_\mu, \tag{3.6}$$

$$D_\mu\psi \mapsto D'_\mu\left[e^{i\vartheta(x)}\psi\right] = e^{i\vartheta(x)} D_\mu\psi. \tag{3.7}$$

To satisfy this, the gauge field must transform as

$$\Pi_\mu \mapsto \Pi_\mu - \frac{1}{e}\partial_\mu\vartheta(x). \tag{3.8}$$

Thus we have a modified Lagrangian which admits a local U(1) gauge symmetry

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - e\Pi_\mu j^\mu, \tag{3.9}$$

where $\not{D} \coloneqq \gamma^\mu D_\mu$ is the Dirac operator. This is strikingly similar to the electromagnetic Lagrangian density which also has a U(1) gauge symmetry

$$\mathcal{L}_{\text{EM}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu J^\mu. \tag{3.10}$$

Thus we identify $\Pi_\mu = A_\mu$ and $ej^\mu = J^\mu$, and arrive at the Lagrangian for quantum electrodynamics

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - eA_\mu j^\mu + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi. \tag{3.11}$$

Schwinger arrived at the covariant formulation of quantum electrodynamics similarly, by constructing a Lagrangian invariant under Lorentz transformations, gauge transformations, and charge conjugation [2]. However, this is not the end of the story, the massless Dirac equation admits another global U(1) symmetry. In the next sections we construct a gauge-invariant Lagrangian and investigate the resultant gauge theory.

3.2 Gauging the axial symmetry

The massless Dirac Lagrangian admits another symmetry better seen in the Weyl basis

$$\psi = \begin{pmatrix} \psi_{\mathrm{L}} \\ \psi_{\mathrm{R}} \end{pmatrix}, \quad \gamma^\mu \partial_\mu = \begin{pmatrix} 0 & \sigma^\mu \partial_\mu \\ \bar{\sigma}^\mu \partial_\mu & 0 \end{pmatrix}. \tag{3.12}$$

In this basis the Dirac Lagrangian can be seen as the interaction between two Weyl fermions of opposite chirality [24]

$$\mathcal{L}_{\mathrm{D}} = \psi_{\mathrm{L}}^\dagger (i\sigma^\mu \partial_\mu)\psi_{\mathrm{L}} + \psi_{\mathrm{R}}^\dagger (i\bar{\sigma}^\mu \partial_\mu)\psi_{\mathrm{R}} - m(\psi_{\mathrm{L}}^\dagger \psi_{\mathrm{R}} + \psi_{\mathrm{R}}^\dagger \psi_{\mathrm{L}}). \tag{3.13}$$

In the massless case the Lagrangian gains a global U(1) symmetry known as the axial symmetry

$$\psi_{\mathrm{L}} \mapsto e^{i\vartheta}\psi_{\mathrm{L}}, \quad \psi_{\mathrm{R}} \mapsto e^{-i\vartheta}\psi_{\mathrm{R}}, \tag{3.14}$$

which is equivalent to

$$\psi \mapsto e^{i\vartheta\gamma^5}\psi, \tag{3.15}$$

and a Noether current

$$j_5^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi, \quad \partial_\mu j_5^\mu = 2im\bar{\psi}\gamma^5\psi. \tag{3.16}$$

We now proceed à la Schwinger, gauging the axial symmetry by promoting ∂_μ to a covariant derivative $\widetilde{D}_\mu$ as in (3.6) (with $q = e$), and determining the required gauge transformation[6]

$$\mathcal{L}_D|_{m=0} = \bar{\psi}(i\gamma^\mu\partial_\mu)\psi - q\bar{\psi}\gamma^\mu\Pi_\mu\psi, \tag{3.17}$$

$$\Pi_\mu \mapsto e^{i\vartheta\gamma^5}\Pi_\mu e^{-i\vartheta\gamma^5} - \frac{1}{q}\partial_\mu\vartheta\gamma^5. \tag{3.18}$$

This is the most general way to gauge the axial symmetry; theories with γ^5 couplings are a well-studied special case (cf. [25]) hence we investigate the implications of (3.18). Now we must add a kinetic term to the Lagrangian to determine the dynamics of the matrix-valued gauge field.[7] Such a term must be gauge-invariant to preserve the U(1) symmetry. We find a Lagrangian analogous to the Maxwell Lagrangian

$$\mathcal{L}_\Pi = -\frac{1}{4}\mathrm{Tr}(K_{\mu\nu}K^{\mu\nu}), \tag{3.19}$$

where the trace is taken over the internal indices and

$$K_{\mu\nu} = \partial_\mu\Pi_\nu - \partial_\nu\Pi_\mu + iq[\Pi_\mu, \Pi_\nu], \tag{3.20}$$

has the required gauge-invariance and is Lorentz invariant. Thus we have constructed a classical gauge theory of massless Dirac fermions

$$\mathcal{L}_{\mathrm{A}} = -\frac{1}{4}\mathrm{Tr}(K_{\mu\nu}K^{\mu\nu}) + \bar{\psi}i\widetilde{\not{D}}\psi, \tag{3.21}$$

[6] There are 4 components of Π_μ invariant under (3.18) which we set to 0, although this does not affect the following calculations.

[7] n.b. Π_μ^{ab} has two internal indices and one spacetime index.

$$\mathcal{L}_{\mathrm{A}} = -\frac{1}{4}\mathrm{Tr}(K_{\mu\nu}K^{\mu\nu}) - q\bar{\psi}\gamma^{\mu}\Pi_{\mu}\psi + \bar{\psi}(i\gamma^{\mu}\partial_{\mu})\psi. \tag{3.22}$$

By direct computation we find that the canonical energy-momentum tensor for this theory is

$$T^{\mu}{}_{\nu} = i\bar{\psi}\gamma^{\mu}\partial_{\nu}\psi + \mathrm{Tr}(K^{\kappa\mu}\partial_{\nu}\Pi_{\kappa}) - \delta^{\mu}_{\nu}\mathcal{L}_{\mathrm{A}}, \tag{3.23}$$

which is not gauge-invariant. However, using the procedure for interacting gauge theories in [26] we construct an improved energy-momentum tensor

$$\widetilde{T}^{\mu}{}_{\nu} = i\bar{\psi}\gamma^{\mu}\widetilde{D}_{\nu}\psi + \mathrm{Tr}(K_{\kappa\mu}K^{\nu\kappa}) - \delta^{\mu}_{\nu}\mathcal{L}_{\mathrm{A}}, \tag{3.24}$$

which is clearly gauge-invariant. Thus gauging the axial symmetry results in a well formulated classical gauge theory of axial bosons and massless Dirac fermions.

4 Quantum Field Theory

So far we have studied relativistic quantum mechanics, making use of classical fields to describe the dynamics of quantum states. However, such a quantum theory is inconsistent and predicts particles with negative energy which require peculiar interpretations to reconcile like Dirac's infamous hole theory [27]. To proceed to a quantum theory of particle interactions we must promote the dynamical fields to field operators; second quantisation. We shall see how this shift to a quantised field theory affects our calculation of expectation values and probabilities via the path integral.

4.1 Second Quantisation

In classical mechanics, the canonical transformations are exactly those which preserve the symplectic structure; invariance of poisson brackets of the dynamical variables (p_i, q_i) characterises transformations which leave Hamilton's equations unchanged [16]. A system's position in phase space specifies its classical state. However, in quantum mechanics all properties of a system are included in a quantum state, $|\psi\rangle$, inside of a Hilbert space upon which operators corresponding to observables act.

Dirac's famous canonical quantisation rule ($\{A, B\} \to \frac{1}{i\hbar}[\hat{A}, \hat{B}]$) [23] allows us quantise the canonical structure of classical mechanics (although Groenewold's theorem shows us that such a rule cannot hold for all functions of the dynamical variables [28]). To obtain a quantised field theory we consider the field-theoretic poisson brackets

$$\{\phi_n(z), \phi_m(w)\}_f = 0 = \{\pi_n(z), \pi_m(w)\}_f, \tag{4.1}$$

$$\{\phi_n(z), \pi_m(w)\}_f = \delta^n_m \delta^{(3)}(\vec{z} - \vec{w}). \tag{4.2}$$

Which are quantised to the canonical commutation relations for the dynamical field operators acting on a Fock space [29]

$$[\hat{\phi}_n(z), \hat{\phi}_m(w)] = 0 = [\hat{\pi}_n(z), \hat{\pi}_m(w)], \tag{4.3}$$

$$[\hat{\phi}_n(z), \hat{\pi}_m(w)] = i\hbar\delta^n_m \delta^{(3)}(\vec{z} - \vec{w}), \tag{4.4}$$

where the classical field $\phi(z)$ is an eigenvalue of the operator $\hat{\phi}(z)$. However, just like Dirac's rule this procedure does not always produce a consistent quantum theory. In the case of fermionic fields canonical quantisation produces states with negative probability and a Hamiltonian unbounded below [30]; we require anti-commutation relations (because of the spin-statistics theorem [31]) in natural units[8]

$$[\hat{\psi}_\alpha(z), \hat{\psi}^\dagger_\beta(w)]_+ = \delta^\alpha_\beta \delta^{(3)}(\vec{z} - \vec{w}), \tag{4.5}$$

where the unwritten anticommutators vanish. This implies the Dirac spinors are Grassmann (anticommuting) variables;[9] we require Berezin's notion of calculus on a Grassmann algebra [32] to define integrals over Dirac fields.

4.2 The path integral

In a quantised field theory the path integral can be used to express expectation values of operators and their compositions

$$\int \mathcal{D}\phi \, \{\varphi_1(x_1) \cdots \varphi_n(x_n)\} \, e^{iS[\phi]} \propto \langle \varphi_1(x_1) \cdots \varphi_n(x_n) \rangle = \langle 0 | \mathcal{T}\{\hat{\varphi}_1(x_1) \cdots \hat{\varphi}_n(x_n)\} | 0 \rangle \,, \tag{4.6}$$

where $\mathcal{T}$ is the time-ordering operator and we normalise probability amplitudes using the partition function

$$Z[J] = \int \mathcal{D}\phi \, \exp\left[iS[\phi] + i \int d^4x \, J(x)\phi(x) \right], \tag{4.7}$$

$$\frac{1}{Z[0]} \int \mathcal{D}\phi \, \varphi(x) e^{iS[\phi]} = \langle \varphi(x) \rangle \,, \tag{4.8}$$

and define the 'quantum action' via

$$W[J] \coloneqq -i \log Z[J]. \tag{4.9}$$

To perform calculations with the path integral we must perform a wick transformation so that the Lagrangian is integrated over Euclidean space, after which we transform the time axis back to $\mathbb{R}$ via analytic continuation. In Euclidean space we have

$$Z[J] = \int \mathcal{D}\phi \, \exp\left[-S_E[\phi] + \int d^4x_E \, J(x)\phi(x) \right], \quad Z[J] = e^{-W_E[J]}. \tag{4.10}$$

5 Symmetries in QFT

5.1 Slavnov-Taylor identities

We now extend our investigation of symmetries from classical fields to a quantised theory. Suppose the classical action admits a local symmetry such that the path integral measure is invariant

$$\phi_\mu \mapsto \phi_\mu + \delta_\epsilon \phi_\mu + \mathcal{O}(\epsilon^2); \quad \delta_\epsilon \phi_\mu = \epsilon \vartheta_\mu(\phi_k, x), \tag{5.1}$$

[8] n.b. the momentum conjugate to ψ_α is $i\psi^\dagger_\alpha$.

[9] n.b. ϕ^2 is an eigenvalue of $\hat{\phi}^2 = 0$.

$$\mathcal{D}\phi \mapsto \mathcal{D}\phi \ \det\left(\frac{\partial \phi'_\mu(x)}{\partial \phi_\nu(y)}\right) = \mathcal{D}\phi, \tag{5.2}$$

where $B_{\mu\nu} = \frac{\partial \phi'_\mu(x)}{\partial \phi_\nu(y)}$ is the transformation Jacobian. Thus the partition function $Z[J]$ is invariant, and for a source J^μ corresponding to ϕ_μ, we can write in Euclidean space

$$\begin{aligned} Z[J] &= \int \mathcal{D}\phi \ \exp\left[-S[\phi] + \int d^4x \, J^\mu(x)(\phi_\mu(x) + \delta_\epsilon \phi_\mu + \mathcal{O}(\epsilon^2))\right] \\ &= \int \mathcal{D}\phi \ \left\{1 + \int d^4x \, J^\mu(x)\delta_\epsilon\phi_\mu + \mathcal{O}(\epsilon^2)\right\} \exp\left[-S[\phi] + \int d^4 \, J^\mu(x)\phi_\mu(x)\right], \end{aligned} \tag{5.3}$$

and after interchanging the spacetime and path integrals we have

$$\int d^4x \, J^\mu(x)\langle\vartheta_\mu(\phi_k, x)\rangle_J = 0. \tag{5.4}$$

We define the effective quantum action as the Legendre transform

$$\Gamma[\varphi] = W[J] - \int d^4x \, J^\mu(x)\varphi_\mu(x); \qquad \varphi_\mu := \langle\phi_\mu\rangle_J \, . \tag{5.5}$$

Taking the functional derivative we obtain

$$J^\mu(x) = -\frac{\delta\Gamma[\varphi]}{\delta\varphi_\mu(x)}, \tag{5.6}$$

and after using (5.4) and the definition of a functional derivative we have

$$\delta\Gamma[\varphi_\mu; \langle\vartheta_\mu(\phi_k, x)\rangle_J] = 0. \tag{5.7}$$

Thus the effective quantum action is invariant under the transformations

$$\varphi_\mu \mapsto \varphi_\mu + \epsilon\langle\vartheta_\mu(\phi_k, x)\rangle_J \, , \tag{5.8}$$

such a relation is called a Slavnov-Taylor identity [3–5]. If the transformation is linear in the fields such that

$$\vartheta_\mu = \Theta_\mu[\phi, x] = \alpha_\mu(x) + \int d^4y \, \beta^\nu_\mu(x, y)\phi_\nu(y), \tag{5.9}$$

(e.g. linear $R^\mu_n(\phi_k)$ in (2.20)), the transformation Jacobian becomes field independent

$$B_{\mu\nu} = \frac{\delta}{\delta\phi_\nu(y)}(\phi_\mu + \epsilon\Theta_\mu[\phi, x]) = \delta_{\mu\nu}\delta^{(4)}(x - y) + \epsilon\beta^\nu_\mu(x, y), \tag{5.10}$$

and can be brought out of the path integral. Thus even if the path integral measure is not invariant the normalised correlators will be, granted the Jacobian is non-singular. Moreover, we have

$$\langle\Theta_\mu[\phi, x]\rangle_J = \alpha_\mu(x) + \int d^4y \, \beta^\nu_\mu(x, y) \, \langle\phi_\nu(y)\rangle_J = \Theta_\mu[\varphi, x], \tag{5.11}$$

which implies the effective quantum action is invariant under the same transformation as the classical action, namely,

$$\varphi_\mu \mapsto \varphi_\mu + \epsilon\Theta_\mu[\varphi, x], \tag{5.12}$$

is a symmetry of $\Gamma[\varphi]$. In this case, the quantum theory inherits the classical symmetry and it is impossible to violate the symmetry via quantum effects like loop diagrams because the corresponding correlators are invariant [5].

5.2 Ward-Takahashi identities

The classical conservation laws for Noether currents present in classical field theory have an analogue in the quantum theory, the Ward-Takahashi identities. Noether's first theorem in the classical theory induces conservation relations for correlators in the quantum field theory. Consider a bounded functional of the fields, $F[\phi]$, and its vacuum expectation value

$$\langle F[\phi]\rangle = \frac{1}{Z[0]}\int \mathcal{D}\phi'\, F[\phi']e^{iS[\phi']}, \tag{5.13}$$

$$\phi' = \phi + \delta_\epsilon \phi + \mathcal{O}(\epsilon^2); \quad \delta_\epsilon\phi = \epsilon\lambda(\phi, x), \tag{5.14}$$

which follows from relabeling the variable of integration. Suppose the field transformation preserves the path integral measure, then taking the functional taylor expansion leads to

$$\begin{aligned}\langle F[\phi]\rangle &= \frac{1}{Z[0]}\int \mathcal{D}\phi\, \left(F[\phi] + \frac{\delta F[\phi]}{\delta\phi}\epsilon\lambda + \mathcal{O}(\epsilon^2)\right)\exp\left[iS[\phi] + i\frac{\delta S[\phi]}{\delta\phi}\epsilon\lambda + \mathcal{O}(\epsilon^2)\right] \\ &= \frac{1}{Z[0]}\int \mathcal{D}\phi\, \left(F[\phi] + \frac{\delta F[\phi]}{\delta\phi}\epsilon\lambda\right)\left(1 + i\frac{\delta S[\phi]}{\delta\phi}\epsilon\lambda\right)e^{iS[\phi]} + \mathcal{O}(\epsilon^2),\end{aligned} \tag{5.15}$$

and to first-order in ϵ we have

$$\int \mathcal{D}\phi\, \lambda(\phi, x)\left(\frac{\delta F[\phi]}{\delta\phi} + iF[\phi]\frac{\delta S[\phi]}{\delta\phi}\right)e^{iS[\phi]} = 0. \tag{5.16}$$

Since $\lambda(\phi, x)$ is smooth and arbitrary this implies the Dyson-Schwinger equation [33, 34]

$$\left\langle \frac{\delta F[\phi]}{\delta\phi}\right\rangle = -i\left\langle F[\phi]\frac{\delta S[\phi]}{\delta\phi}\right\rangle. \tag{5.17}$$

Suppose the classical Noether current $j^\mu(x)$ has an analogue $\hat{\jmath}^\mu(x)$ in the operator formalism and the operators $\{\widehat{\mathcal{V}}_i(z_i)\}$ are functions of spacetime. The derivative of their correlator is

$$\begin{aligned}\partial^x_\mu\left\langle j^\mu(x)\prod_{i=1}^{n}\mathcal{V}_i(z_i)\right\rangle &= \left\langle \partial^x_\mu j^\mu(x)\prod_{i=1}^{n}\mathcal{V}_i(z_i)\right\rangle \\ &\quad + \sum_{k=1}^{n}\delta(x^0 - z^0_k)\left\langle \left[j^0(x), \mathcal{V}_k(z_k)\right]_{z_k}\prod_{i\neq k}^{n}\mathcal{V}_i(z_i)\right\rangle,\end{aligned} \tag{5.18}$$

where we define the commutator in the time-ordered product (time-ordered with respect to the subscript) as

$$\left[j(x), \mathcal{A}_m(y_m)\right]_{y_m} \coloneqq \Theta(y^0_\alpha - x^0)\hat{\jmath}(x)\widehat{\mathcal{A}}_m(y_m) - \Theta(x^0 - y^0_\beta)\widehat{\mathcal{A}}_m(y_m)\hat{\jmath}(x); \tag{5.19}$$

$$\mathcal{T}\prod_{i=1}^{n}\widehat{\mathcal{A}}_i = \cdots\widehat{\mathcal{A}}_\alpha(y_\alpha)\widehat{\mathcal{A}}_m(y_m)\widehat{\mathcal{A}}_\beta(y_\beta)\cdots, \tag{5.20}$$

using the Heaviside function Θ (see appendix for proof). The Noether current is only conserved on-shell so we cannot apply current conservation to the second term (the path integral is taken over all field configurations). Assuming that the symmetry corresponding to j^μ leaves the path integral measure invariant, and transforms $\mathcal{V}_i$ as

$$\mathcal{V}_i(\phi) \mapsto \mathcal{V}_i(\phi) + \delta_\epsilon \mathcal{V}_i(\phi) + \mathcal{O}(\epsilon^2); \quad \delta_\epsilon \mathcal{V}_i = \frac{\partial \mathcal{V}_i}{\partial \phi} \delta_\epsilon \phi, \tag{5.21}$$

after using the Dyson-Schwinger equation (with $F[\phi] = \prod_i \mathcal{V}_i$) and (2.15) we obtain, to first-order in ϵ, the relation

$$\left\langle \int d^4x \, \epsilon \partial_\mu^x j^\mu(x) \prod_{i=1}^n \mathcal{V}_i(z_i) \right\rangle = -i \sum_{k=1}^n \left\langle \delta_\epsilon \mathcal{V}_k(z_k) \prod_{i \neq k}^n \mathcal{V}_i(z_i) \right\rangle. \tag{5.22}$$

Demanding that the generator $\lambda(\phi, x)$ is smooth and compactly supported, and interchanging the spacetime and path integrals, leads to

$$\left\langle \partial_\mu^x j^\mu(x) \prod_{i=1}^n \mathcal{V}_i(z_i) \right\rangle = -i \sum_{k=1}^n \delta^{(4)}(x - z_k) \left\langle \lambda(\phi, x) \frac{\partial \mathcal{V}_k}{\partial \phi}(z_k) \prod_{i \neq k}^n \mathcal{V}_i(z_i) \right\rangle, \tag{5.23}$$

via the fundamental lemma of the calculus of variations.[10] The delta function terms are called Schwinger/contact terms, and in the case that none of the insertion times coincide we have the Ward-Takahashi identity [6, 7]

$$\partial_\mu^x \left\langle j^\mu(x) \prod_{i=1}^n \mathcal{V}_i(z_i) \right\rangle = 0; \quad x_0 \neq z_{i_0}. \tag{5.24}$$

These correlator derivatives can be computed perturbatively, and in the case of the axial current, j_5^μ, the three-point function corresponding to triangle Feynman diagrams violates the Ward-Takahashi identity, namely, the Adler–Bell–Jackiw anomaly [8, 9]. This suggests that the axial symmetry is not present in the quantum theory. Similarly, the Slavnov–Taylor identities imply it is a symmetry of the quantum theory if the path integral Jacobian is non-singular, because the axial transformation is linear in the dirac fields. Thus, for the class of symmetries linear in the dynamical fields in (5.9), a singular path integral Jacobian characterises the presence of an anomaly, and this can be viewed as the origin of the axial anomaly.

6 Anomalies in QFT

6.1 The path integral measure & Fujikawa's method

We shall now study transformations which are not necessarily symmetries of the path integral measure, to this end, we must introduce a more precise notion of the path integral over fermionic fields using Berezin's formal integration of Grassmann variables [32]. We decompose the Dirac spinor and its adjoint using an orthonormal basis, $\{\phi_n(x)\}$, of the Dirac operator's eigenfunctions and the independent Grassmann variables θ_n and $\bar{\xi}_m$

$$\psi = \sum_n \theta_n \phi_n(x) = \sum_n \theta_n \langle x|n\rangle, \tag{6.1}$$

[10] n.b. (2.17) $\implies \partial_\mu^x j^\mu = -\lambda E$.

$$\bar{\psi} = \sum_m \bar{\xi}_m \phi_m^\dagger(x) = \sum_m \bar{\xi}_m \langle m|x\rangle. \tag{6.2}$$

Hence the path integral 'measure' can be seen as a Berezin integral 'measure' under a coordinate transformation

$$\theta \mapsto \theta_n \langle x|n\rangle\,, \quad \bar{\xi} \mapsto \bar{\xi}_m \langle m|x\rangle\,, \tag{6.3}$$

$$\mathcal{D}\psi\mathcal{D}\bar{\psi} = [\det \langle x|n\rangle \det \langle m|x\rangle]^{-1} \lim_{\substack{N\to\infty \\ M\to\infty}} \left\{ \prod_n^N d\theta_n \prod_m^M d\bar{\xi}_m \right\} = \lim_{N\to\infty} \prod_n^N d\theta_n d\bar{\xi}_n. \tag{6.4}$$

Under the infinitesimal vector transformation the Dirac spinor becomes

$$\psi' = \sum_n \theta'_n \phi_n(x); \quad \theta'_n = e^{i\vartheta(x)}\theta_n, \tag{6.5}$$

and similarly for the adjoint spinor. Thus the path integral measure transforms as

$$\mathcal{D}\psi\mathcal{D}\bar{\psi} \mapsto \left[\det\left(e^{i\vartheta(x)\mathbb{I}}\right) \det\left(e^{-i\vartheta(x)\mathbb{I}}\right)\right]^{-1} \mathcal{D}\psi\mathcal{D}\bar{\psi}, \tag{6.6}$$

and we see the measure is invariant; the vector symmetry is retained by the quantum theory, and the correlators are gauge invariant by the Slavnov–Taylor identities. However, this is not the case for the axial transformation, the axial symmetry is anomalous. The following is an outline of Fujikawa's proof of the existence of the axial anomaly [11, 35]. Under the axial transformation, the Grassmann variables transform as

$$\theta'_n = \sum_m C_{nm}\theta_m, \quad \bar{\xi}'_n = \sum_m C_{nm}\bar{\xi}_m, \tag{6.7}$$

$$C_{nm} = \delta_{nm} + i\int d^4x\, \vartheta(x)\phi_n^\dagger(x)\gamma^5\phi_m(x). \tag{6.8}$$

Using the identities

$$\ln(\mathbb{I} + \epsilon A) = \epsilon A + \mathcal{O}(\epsilon^2), \tag{6.9}$$

$$\det\left(e^A\right) = e^{\mathrm{Tr}(A)}, \tag{6.10}$$

we find that the path integral measure changes by the Jacobian determinant

$$\begin{aligned} J[\vartheta] &= (\det C)^{-2} = \exp\left[-2\mathrm{Tr}(\ln C)\right] \\ &= \exp\left[-2\mathrm{Tr}\left(i\int d^4x\,\vartheta(x)\phi_n^\dagger(x)\gamma^5\phi_m(x)\right)\right] \\ &= \exp\left[-2i\int d^4x\,\vartheta(x)\sum_n \phi_n^\dagger(x)\gamma^5\phi_n(x)\right]. \end{aligned} \tag{6.11}$$

The sum appearing in the Jacobian is clearly divergent (by orthonormality) and requires the introduction of a regulator, or ultraviolet cut-off, to evaluate the integral. However, we will explore a more elegant topological approach that utilises topological and analytical indices in the next section, after we demonstrating the relationship between the axial anomaly and the axial current. Henceforth, we define the axial anomaly, $\mathcal{A}$, via the Jacobian of the path integral measure

$$J[\vartheta] = \exp\left[-\int d^4x\,\vartheta(x)\mathcal{A}\right]. \tag{6.12}$$

Under the axial transformation the partition function becomes

$$Z'[A] = \int \mathcal{D}\psi'\mathcal{D}\bar{\psi}'\,\exp\left[i\int d^4x\,-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}'(i\not{D} - m)\psi'\right], \tag{6.13}$$

but this is just a relabeling of the integration variables, thus the partition function is unchanged. Considering the change in the QED Lagrangian we have

$$Z[A] = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\,J[\vartheta]\exp\left[i\int d^4x\,\mathcal{L}_{\mathrm{QED}} + \vartheta(x)(\partial_\mu j_5^\mu - 2im\bar{\psi}\gamma^5\psi)\right]. \tag{6.14}$$

Thus, to first-order in $\vartheta(x)$ we have

$$Z[A] = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\,\left\{1 + i\int d^4x\,\vartheta(x)(\partial_\mu j_5^\mu - 2im\bar{\psi}\gamma^5\psi - \mathcal{A})\right\}\exp\left[i\int d^4x\,\mathcal{L}_{\mathrm{QED}}\right]. \tag{6.15}$$

Subsequently, by the fundamental lemma of the calculus of variations we have the anomalous divergence for the axial current

$$\partial_\mu j_5^\mu = 2im\bar{\psi}\gamma^5\psi + \mathcal{A}. \tag{6.16}$$

6.2 Atiyah–Singer index theorem

The Dirac operator $\not{D}$ is hermitian in Euclidean space. Given an eigenfunction of $\not{D}$, ϕ_n, with an eigenvalue $\lambda_n \neq 0$,$\gamma^5\phi_n$ is also an eigenfunction

$$\not{D}(\gamma^5\phi_n) = -\gamma^5\not{D}(\phi_n) = -\lambda_n\gamma^5\phi_n. \tag{6.17}$$

Moreover, hermiticity of $\not{D}$ implies that ϕ_n and $\gamma^5\phi_n$ are orthogonal (as they have different eigenvalues) by the spectral theorem [36]. When the eigenvalue is zero ϕ_n and $\gamma^5\phi_n$ are called zero modes of $\not{D}$. We also have that γ^5 is hermitian, hence the spectral theorem implies there exists an orthonormal basis that diagonalises γ^5 and spans $\mathrm{Ker}(\not{D})$. Ignoring the infinitesimal parameter for now, only the zero modes contribute to the integral in the Jacobian because of orthonormality, and we have

$$\int d^4x\,\sum_n \phi_n^\dagger\gamma^5\phi_n = \sum_n\int d^4x\,\phi_{n+}^{0\dagger}\phi_{n+}^0 - \sum_n\int d^4x\,\phi_{n-}^{0\dagger}\phi_{n-}^0 = n_+ - n_-, \tag{6.18}$$

$$\int d^4x\,\mathcal{A} = 2i(n_+ - n_-), \tag{6.19}$$

where $n_\pm$ are the number of positive and negative chirality zero modes, $\phi_{n\pm}^0$, respectively. This is precisely the analytical index of the Dirac operator projected onto the positive chirality subspace

$$\mathrm{index}(\not{D}_+) = n_+ - n_-, \tag{6.20}$$

$$\not{D}_{\pm} \coloneqq \not{D} P_{\pm} = \not{D}\big|_{\{\pm\}}, \tag{6.21}$$

where $P_{\pm} = \frac{1}{2}(1 \pm \gamma^5)$ is the chirality projection operator. By the Atiyah–Singer index theorem, this index coincides with the Dirac operator's topological index on a compact manifold [10]. This implies that the axial anomaly is a topological invariant of the quantum theory and no regularisation procedure can restore the axial symmetry. On a 4-dimensional compact manifold Ω, with no boundary and Riemannian curvature R, the topological index is [37, 38]

$$\text{index}(\not{D}_{+}) = \int_{\Omega} \hat{A}(\Omega)\text{ch}(F). \tag{6.22}$$

The integrand, called the index density, only includes 4-form terms and the curvature form is given by $F = e\,dA = \frac{e}{2} F_{\mu\nu} dx^{\mu} \wedge dx^{\nu}$. The characteristic classes $\text{ch}(F)$ and $\hat{A}(\Omega)$ are the Chern character of the curvature form, and Dirac genus of the manifold respectively

$$\text{ch}(F) = \text{Tr}\left(\exp\left[\frac{i}{2\pi}F\right]\right), \tag{6.23}$$

$$\hat{A}(\Omega) = \sqrt{\det\left(\frac{\frac{i}{4\pi}R}{\sinh\frac{i}{4\pi}R}\right)}. \tag{6.24}$$

Now we choose the manifold $S^4 = \Omega$ so the Riemannian curvature is constant and $\hat{A}(\Omega) = 1$. Thus the index becomes

$$\text{index}(\not{D}_{+}) = \int_{S^4} \text{ch}(F) = \frac{1}{2!}\left(\frac{i}{2\pi}\right)^2 \int_{S^4} \text{Tr}(F \wedge F). \tag{6.25}$$

Assuming the action due to the Maxwell Lagrangian converges

$$\int_{\mathbb{R}^4} d^4x\, F_{\mu\nu}F^{\mu\nu} < \infty \iff A_{\mu} \xrightarrow{|x|\to\infty} -\frac{1}{e}\partial_{\mu}\vartheta; \quad \vartheta(x) \in C^2(\mathbb{R}^4), \tag{6.26}$$

we can stereographically project our problem from S^4 to $\mathbb{R}^4$. Hence $\text{ch}(F)$ coincides with the Dirac operator's index density in Euclidean space if the gauge fields are asymptotically 'flat' (A_{μ} is a pure gauge at infinity) [39, 40]. Using antisymmetry of the wedge product, we identify the axial anomaly in Euclidean space as

$$\partial_{\mu} j_5^{\mu} - 2im\bar{\psi}\gamma^5\psi = \mathcal{A}[A_{\mu}] = \frac{-ie^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}. \tag{6.27}$$

6.3 Anomalies as obstructions to gauging linear symmetries

We have seen that a singular path integral Jacobian is a necessary and sufficient condition for anomalous linear symmetries. In the general case, gauging such a symmetry will produce a gauge-variant partition function, because constructing a gauge invariant classical action ensures the shift in the measure changes the partition function under a gauge transformation. This gauge symmetry breaking is tantamount to a violation of unitarity and is fatal for any quantum theory.

We now study the effect of gauging an anomalous symmetry in the case of the Dirac Lagrangian's axial symmetry. Using Fujikawa's method [11, 35], we shall calculate the path integral gauge-variance in the 'axial theory', although a topological approach is possible (cf. [12, 13]) using the Atiyah–Singer Families index theorem [41]. For the axial theory Lagrangian in (3.22), the path integral is

$$\int \mathcal{D}\psi \mathcal{D}\bar{\psi}\, \exp\left[i \int d^4x\, -\frac{1}{4}\mathrm{Tr}(K_{\mu\nu}K^{\mu\nu}) + \bar{\psi} i \widetilde{\not{D}} \psi\right]. \tag{6.28}$$

The classical action is gauge-invariant by construction, thus the only change in the path integral under a gauge transformation is from the Jacobian factor

$$\int \mathcal{D}\psi' \mathcal{D}\bar{\psi}'\, e^{iS[\psi',\bar{\psi}']} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi}\, J[\vartheta] e^{iS[\psi,\bar{\psi}]}, \tag{6.29}$$

where the Jacobian takes the same form as in (6.11), for an orthonormal basis of eigenfunctions $\{\phi_n(x)\}$ of the operator $\widetilde{\not{D}}$

$$J[\vartheta] = \exp\left[-2i \lim_{N\to\infty}\left\{\sum_{n=1}^{N} \int d^4x\, \vartheta(x) \phi_n^\dagger \gamma^5 \phi_n\right\}\right]. \tag{6.30}$$

We now proceed via Fujikawa's method of gauge-invariant mode cut-off to calculate the Jacobian; we introduce a decreasing regulator bump function $f(x)$, with $f(0) = 1$, to approximate the ultraviolet cut-off at N. Large eigenvalues of $\widetilde{\not{D}}$ correspond to ultraviolet modes,[11] hence $f(\lambda_n)$ is a good regulator that preserves gauge invariance of $J[\vartheta]$ and we write

$$\begin{aligned} J[\vartheta] &= \exp\left[-2i \lim_{M\to\infty}\left\{\sum_{n=1}^{\infty} \int d^4x\, \vartheta(x) \phi_n^\dagger \gamma^5 f\left(\frac{\lambda_n^2}{M^2}\right) \phi_n\right\}\right] \\ &= \exp\left[-2i \lim_{M\to\infty} \mathrm{Tr}\left\{\vartheta(x)\gamma^5 f\left(\frac{\widetilde{\not{D}}^2}{M^2}\right)\right\}\right], \end{aligned} \tag{6.31}$$

where the functional trace is taken over spacetime using the basis $\{\phi_n(x)\}$. Now we compute the series in momentum space

$$\begin{aligned} \sum_{n=1}^{\infty} \phi_n^\dagger \gamma^5 f\left(\frac{\widetilde{\not{D}}^2}{M^2}\right)\phi_n &= \int \frac{d^4k d^4p}{(2\pi)^4} \sum_{n=1}^{\infty} e^{-ip_\mu x^\mu} \tilde{\phi}_n^\dagger(p) \gamma^5 f\left(\frac{\widetilde{\not{D}}^2}{M^2}\right) \tilde{\phi}_n(k) e^{ik_\mu x^\mu} \\ &= \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left\{\gamma^5 f\left(\frac{(\widetilde{D}_\mu + ik_\mu)(\widetilde{D}^\mu + ik^\mu)}{M^2} + \frac{iq[\gamma^\mu,\gamma^\nu]K_{\mu\nu}}{4M^2}\right)\right\}; \end{aligned} \tag{6.32}$$

$$\phi_n(x) = \int \frac{d^4k}{(2\pi)^2}\, \tilde{\phi}_n(k) e^{ik_\mu x^\mu}, \tag{6.33}$$

where the trace is taken over the internal and γ indices, and we have use the identities

$$\sum_{n=1}^{\infty} \tilde{\phi}_n^\dagger(p) A \tilde{\phi}_n(k) = \delta(k-p)\mathrm{Tr}(A), \tag{6.34}$$

[11] n.b. ∂_μ is a momentum operator.

$$\widetilde{\not{D}}^2 = \widetilde{D}_\mu \widetilde{D}^\mu + \frac{iq}{4}[\gamma^\mu, \gamma^\nu] K_{\mu\nu}, \tag{6.35}$$

$$e^{-ik_\mu x^\mu} f(\partial_\mu) e^{ik_\mu x^\mu} = f(\partial_\mu + ik_\mu). \tag{6.36}$$

Now we rescale the momenta via $k_\mu \to M k_\mu$ and the series becomes

$$M^4 \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr} \left\{ \gamma^5 f \left(\frac{\widetilde{D}_\mu \widetilde{D}^\mu + 2iMk_\mu \widetilde{D}^\mu}{M^2} - k^2 + \frac{iq[\gamma^\mu, \gamma^\nu] K_{\mu\nu}}{4M^2} \right) \right\}. \tag{6.37}$$

Expanding the Taylor series of $f(x)$ about $-k^2 := -k_\mu k^\mu$, and using the trace properties of the gamma matrices, the only non-vanishing term in the limit $M \to \infty$ is the term quadratic in $K_{\mu\nu}$

$$\frac{M^4 q^2}{2! M^4} \epsilon^{\mu\nu\rho\sigma} \mathrm{Tr}(K_{\mu\nu} K_{\rho\sigma}) \int \frac{d^4k}{(2\pi)^4} f''(-k^2). \tag{6.38}$$

We calculate the regulator integral in hyperspherical coordinates

$$\begin{aligned} \int \frac{d^4k}{(2\pi)^4} f''(-k^2) &= \frac{1}{(2\pi)^4} \int dr\, d\varphi_1\, d\varphi_2\, d\varphi_3\, f''(r^2) r^3 \sin^2(\varphi_1) \sin(\varphi_2) \\ &= \frac{2\pi^2}{(2\pi)^4} \int_0^\infty dr\, r^3 f''(r^2) = \frac{1}{16\pi^2} \int_0^\infty du\, u f''(u) = \frac{1}{16\pi^2}, \end{aligned} \tag{6.39}$$

the last equality follows from integration by parts and compact support of $f(x)$ and its derivatives. Thus the Jacobian is

$$J[\vartheta] = \exp\left[\frac{-iq^2}{16\pi^2} \int d^4x\, \vartheta(x) \epsilon^{\mu\nu\rho\sigma} \mathrm{Tr}(K_{\mu\nu} K_{\rho\sigma}) \right]. \tag{6.40}$$

Using the cycle definition of the Levi-Civita symbol and invariance of the trace under cyclic permutations, we see that

$$\begin{aligned} \epsilon^{\mu\nu\rho\sigma} \mathrm{Tr}(K_{\mu\nu} K_{\rho\sigma}) &= 4\epsilon^{\mu\nu\rho\sigma} \mathrm{Tr}(\partial_\mu \Pi_\nu \partial_\rho \Pi_\sigma + 2iq \partial_\mu \Pi_\nu \Pi_\rho \Pi_\sigma) \\ &= 4\epsilon^{\mu\nu\rho\sigma} \partial_\mu \mathrm{Tr} \left(\Pi_\nu \partial_\rho \Pi_\sigma + \frac{2iq}{3} \Pi_\nu \Pi_\rho \Pi_\sigma \right). \end{aligned} \tag{6.41}$$

This is the divergence of a Chern-Simons current [42, 43]. Finally, we have

$$\begin{aligned} J[\vartheta] &= \exp\left[\frac{-iq^2}{4\pi^2} \int d^4x\, \vartheta(x) \epsilon^{\mu\nu\rho\sigma} \partial_\mu \mathrm{Tr} \left(\Pi_\nu \partial_\rho \Pi_\sigma + \frac{2iq}{3} \Pi_\nu \Pi_\rho \Pi_\sigma \right) \right] \\ &= \exp\left[\frac{-iq^2}{4\pi^2} \lim_{r\to\infty} \left\{ \int_{S_r^3} ds_\mu\, \vartheta(x) \epsilon^{\mu\nu\rho\sigma} \mathrm{Tr} \left(\Pi_\nu \partial_\rho \Pi_\sigma + \frac{2iq}{3} \Pi_\nu \Pi_\rho \Pi_\sigma \right) \right\} \right], \end{aligned} \tag{6.42}$$

where r is the radius of a 4-ball with boundary S_r^3. For constant $\vartheta(x)$ the integral is a Chern-Simons action, and hence a topological invariant independent of r. Thus, there is gauge-variance in the path integral. Moreover, the gauge-variance is a topological invariant of the quantum theory and no regularisation procedure can produce a consistent QFT.

7 Conclusion

In this work, we have shown the connection between symmetries in classical and quantum field theories, namely, Noether's theorem's in the classical theory induce the Ward-Takahashi identities in the quantum theory. Additionally, we have given a characterisation of anomalies in the case of linear symmetries as corresponding to singular path integral Jacobians and demonstrate the failure inherit to gauging an anomalous 'linear' symmetry. Moreover, we have shown that the axial symmetry admits a well-formulated classical gauge theory but the axial anomaly breaks gauge symmetry in the quantum theory. Due to the topological nature of the axial anomaly, such a violation of unitarity is irrecoverable and U(1) axial bosons coupled to Dirac fermions in this manner do not exist.

A N-point function derivative

We now give a proof of (5.18). Observe that

$$\partial^x_\mu \left\langle j^\mu(x) \prod_{i=1}^n \mathcal{V}_i(z_i) \right\rangle = \partial^x_\mu \left\langle 0 \middle| \mathcal{T}\left\{ \hat{\jmath}^\mu(x) \prod_{i=1}^n \widehat{\mathcal{V}}_i(z_i) \right\} \middle| 0 \right\rangle . \tag{A.1}$$

Let the vertex operators have time-ordering $\left\{\widehat{\mathcal{U}}_i(y_i)\right\} = \left\{\widehat{\mathcal{V}}_i(z_i)\right\}$ such that $y_i^0 > y_{i+1}^0$, then

$$\left\langle 0 \middle| \mathcal{T}\left\{ \hat{\jmath}^\mu(x) \prod_{i=1}^n \widehat{\mathcal{V}}_i(z_i) \right\} \middle| 0 \right\rangle = \sum_{i=0}^n \Theta(y_i^0 - x^0)\Theta(x^0 - y_{i+1}^0) \, \langle 0|\widehat{\mathcal{U}}_1 \cdots \widehat{\mathcal{U}}_i \, \hat{\jmath}^\mu(x) \widehat{\mathcal{U}}_{i+1} \cdots \widehat{\mathcal{U}}_n|0\rangle , \tag{A.2}$$

where we set $y_0^0 = -\infty$. Subsequently, by differentiation of the Heaviside functions we have

$$\begin{aligned} \partial^x_\mu \left\langle j^\mu(x) \prod_{i=1}^n \mathcal{V}_i(z_i) \right\rangle &= \left\langle 0 \middle| \mathcal{T}\left\{ \partial^x_\mu \hat{\jmath}^\mu(x) \prod_{i=1}^n \widehat{\mathcal{U}}_i(y_i) \right\} \middle| 0 \right\rangle \\ + \sum_{i=0}^n \left\{ \delta(x^0 - y_{i+1}^0)\Theta(y_i^0 - x^0) - \delta(x^0 - y_i^0)\Theta(x^0 - y_{i+1}^0) \right\} &\langle 0|\widehat{\mathcal{U}}_1 \cdots \widehat{\mathcal{U}}_i \, \hat{\jmath}^0(x) \widehat{\mathcal{U}}_{i+1} \cdots \widehat{\mathcal{U}}_n|0\rangle . \end{aligned} \tag{A.3}$$

After noting that $\delta(x_0 - y_0^0) = 0$, we see that the third term is

$$\sum_{i=1}^n \delta(x_0 - y_i^0) \, \langle 0|\widehat{\mathcal{U}}_1 \cdots \widehat{\mathcal{U}}_{i-1} [j^0(x), \mathcal{U}_i(y_i)]_{y_i} \, \widehat{\mathcal{U}}_{i+1} \cdots \widehat{\mathcal{U}}_n|0\rangle , \tag{A.4}$$

using the notation in (5.19). Each term in the sum is clearly a time-ordered product of the 'commutator' and the $\mathcal{U}_i$ operators (where we time-order the 'commutator' with respect to the subscript). Seeing that the time-ordered product with and $\mathcal{U}_i$ and $\mathcal{V}_i$ coincide, it is clear that (A.2) can be written as (5.18).

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