

Mathematics (1)

1.1 Notation

deg $p(x)$ means the degree of polynomial p .

LC $p(x)$ means the leading coefficient of polynomial p .

1.2 Rational functions

For a rational function $f(x) = \frac{p(x)}{q(x)}$, cancel out any common factors, then:

- When $\deg p(x) = \deg q(x)$:
 - HA: $y = \frac{\text{LC } p(x)}{\text{LC } q(x)}$
 - VA: roots of $q(x)$
- When $\deg p(x) < \deg q(x)$:
 - HA: $y = 0$
 - x-intercept: roots of $p(x)$
 - VA: roots of $q(x)$
- When $\deg p(x) > \deg q(x)$:
 - HA: none
 - slant asymptote: $\frac{p(x)}{q(x)}$ excluding remainder
 - VA: roots of $q(x)$

1.3 Polynomials

1.3.1 Linear equations

- Slope-intercept form: $y = mx + b$
- Point-slope form: $y - y_1 = m(x - x_1)$ for point (x, y)
- Standard form: $ax + by = c$

1.3.2 Quadratic equations

- Standard form: $y = ax^2 + bx + c$
- Vertex form: $y = a(x - h)^2 + k$ for vertex (h, k)
- Sum of roots: $\frac{-b}{a}$
- Product of roots: $\frac{c}{a}$

1.3.3 Higher-degree polynomials

In a polynomial

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

, with roots

$$r_1, r_2, r_3, \dots, r_n$$

then:

$$r_1 + r_2 + r_3 + \cdots + r_n = \sum_{k=1}^n nr_k = -\frac{a_{n-1}}{a_n}$$

1.4 Sequences and Series

1.4.1 Explicit formulas

- Aritmetic sequence: $a_n = a_1 + r(n - 1)$
- Geometric sequence: $a_n = a_1 * r^{n-1}$
- Harmonic sequence: $a_n = \frac{1}{a_1 + r(n - 1)}$

1.4.2 Arithmetic and Geometric Series

In the following equations, substituting $j = 1$ with $j = 0$, $j - 1$ with j , and a_1 with a_0 will produce the same result.

$$\sum_{j=1}^n (a_1 + r(j - 1)) = \frac{n}{2} (2a_1 + (n - 1)d)$$

$$\sum_{j=1}^n (a_1 * r^{j-1}) = \frac{a_1(1 - r^n)}{1 - r}$$

$$\sum_{j=1}^\infty (a_1 * r^{j-1}) = \frac{a_1}{1 - r} \text{ for } r \in [-1, 1]$$

1.4.3 Special Sums

$$\sum_{j=1}^n c = nc$$

$$\sum_{j=1}^n ca_j = c \sum_{j=1}^n a_j$$

$$\sum_{j=1}^n (a_j + b_j) = \sum_{j=1}^n a_j + \sum_{j=1}^n b_j$$

$$\sum_{j=1}^n j = \frac{n(n + 1)}{2}$$

$$\sum_{j=1}^n j^2 = \frac{n(n + \frac{1}{2})(n + 1)}{3}$$

$$\sum_{j=1}^n j^3 = \frac{n^2(n + 1)^2}{4}$$

1.5 Trigonometry

| ° | rad | sin | cos | tan |
|-----|-----------------|----------------------|----------------------|----------------------|
| 0° | 0 | 0 | 1 | 0 |
| 30° | $\frac{\pi}{6}$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{3}}$ |
| 45° | $\frac{\pi}{4}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | 1 |
| 60° | $\frac{\pi}{3}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\sqrt{3}$ |
| 90° | $\frac{\pi}{2}$ | 1 | 0 | undef |

1.5.1 Law of Sines and Cosines

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c} \qquad c^2 = a^2 + b^2 - 2ab \cos(C)$$

1.5.2 Triangle area

$$K = \frac{1}{2}bh \qquad K = \frac{1}{2}bc \sin(A) \qquad K = \sqrt{s(s - a)(s - b)(s - c)}$$

1.5.3 More identities

$$(\sin A)^2 + (\cos A)^2 = 1 \qquad (\tan A)^2 + 1 = (\sec A)^2$$

$$\sin(\frac{\pi}{2} - x) = \cos(x) \qquad (\cot A)^2 + 1 = (\csc A)^2$$

$$\cos(-x) = \cos(x) \qquad \sin(-x) = \sin(x) \qquad \tan(-x) = \tan(x)$$

1.5.4 Slope

Where α is the angle between the line and the x-axis, and m is the slope of the line:

$$m = \tan \alpha$$

1.5.5 Sum and difference formulas

$$\sin(A + B) = \sin(A) \cos(B) + \cos(A) \sin(B)$$

$$\sin(A - B) = \sin(A) \cos(B) - \cos(A) \sin(B)$$

$$\cos(A + B) = \cos(A) \cos(B) - \sin(A) \sin(B)$$

$$\cos(A - B) = \cos(A) \cos(B) + \sin(A) \sin(B)$$

$$\tan(A + B) = \frac{\tan(A) + \tan(B)}{1 - \tan(A) \tan(B)}$$

$$\tan(A - B) = \frac{\tan(A) - \tan(B)}{1 + \tan(A) \tan(B)}$$

$$\sin(2A) = 2 \sin(A) \cos(A)$$

$$\cos(2A) = (\cos A)^2 - (\sin A)^2 = 2(\cos A)^2 - 1 = 1 - 2(\sin A)^2$$

$$\tan(2A) = \frac{2 \tan(A)}{1 - (\tan A)^2}$$

1.6 Vectors

$$\vec{v} + \vec{w} = \begin{bmatrix} v_x + w_x \\ v_y + w_y \\ v_z + w_z \end{bmatrix} \qquad c * \vec{v} = \begin{bmatrix} c * v_x \\ c * v_y \\ c * v_z \end{bmatrix}$$

$$\vec{v} \cdot \vec{w} = v_x w_x + v_y w_y + v_z w_z = |\vec{v}| |\vec{w}| \cos(\theta)$$

$$|\vec{v} \times \vec{w}| = |\vec{v}| |\vec{w}| \sin(\theta) = \text{area of parallelogram}$$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} \qquad \vec{v} \times \vec{w} \perp \vec{v} \qquad \vec{v} \times \vec{w} \perp \vec{w}$$

$$\vec{v} \perp \vec{w} \iff \vec{v} \times \vec{w} = \vec{0} \qquad \vec{v} \parallel \vec{w} \iff \vec{v} \cdot \vec{w} = 0$$

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|} \qquad \text{proj}_{\vec{b}} \vec{v} = \frac{\vec{v} \cdot \vec{b}}{\vec{b} \cdot \vec{b}} * \vec{b} = (|\vec{v}| \cos(\theta))$$

Right-hand rule

To determine the direction of $\vec{v} \times \vec{w}$, put the side of the right hand on \vec{v} and curl the fingers toward \vec{w} . The direction the thumb is pointing is the direction of $\vec{v} \times \vec{w}$.

1.7 Polar

1.7.1 Polar and Cartesian sytems

With point $(x, y) = (r; \theta) = (r; \beta)$, where θ is CCW from the x -axis and β is a bearing, CW from the y -axis:

$x = r \cos(\theta) = r \sin(\beta) \quad y = r \sin(\theta) = r \cos(\beta)$

$r = \sqrt{x^2 + y^2} \qquad \theta \equiv \arctan(\frac{y}{x}) \quad \beta \equiv \arctan(\frac{x}{y})$

1.7.2 Converting functions

Try these substitutions in order:

$x^2 = x^2 + y^2 \qquad \tan \theta = \frac{y}{x} \qquad x = r \cos \theta \qquad y = r \sin \theta$

1.7.3 Limaçons and Petals

The function $y = A \cos(B(\theta + C)) + D$ is equivalent to $y = A \cos(B\theta) + D$ rotated C degrees/radians clockwise.

When C is 0 and B is 1, the x-intercepts are $A \pm D$ and the y-intercepts are $\pm D$, and it forms a limaçon.

When C is 0, but $B \neq 1$, then this sometimes still holds. The x-intercepts may also be $\pm A \pm D$.

There are B petals, with the axis of the first petal on the positive x-axis.

When B is even and $|D| < 1$, then the number of petals is $2B$.

Using sin instead of cos, limaçons have their axes on the positive y-axis, while for petals, the first petal starts from the positive x-axis and curves upwards.

1.8 Complex

$\text{cis}(\theta) = e^{i\theta} = \cos(\theta) + i \sin(\theta)$

To find the n^{th} root of $x_r \text{cis}(x_\theta)$, solve the equation $z_r^n \text{cis}(nz_\theta) = x_r \text{cis}(x_\theta + 360^\circ k)$ for $k \in \mathbb{R}$.

1.9 Function domain

| Function | Domain x | Range y |
|--------------|---------------|-----------------------------------|
| $\log(x)$ | $(0, \infty)$ | \mathbb{R} |
| \sqrt{x} | $[0, \infty)$ | $[0, \infty)$ |
| $\arcsin(x)$ | $[-1, 1]$ | $[-\frac{\pi}{2}, \frac{\pi}{2}]$ |
| $\arccos(x)$ | $[-1, 1]$ | $[0, \pi]$ |
| $\arctan(x)$ | \mathbb{R} | $(-\frac{\pi}{2}, \frac{\pi}{2})$ |

Calculus Theorems (2)

1 Completeness

1.3 Completeness

Theorem (Completeness of the Real Numbers). Every nonempty subset S of \mathbb{R} which is bounded above has a least upper bound $\sup S$.

Definition of Supremum ($\sup S$). A number such that

- (1) $s \leq \sup S$ for every $s \in S$ (which just says that $\sup S$ is an upper bound for S)
- (2) If u is any upper bound for S , then $\sup S \leq u$ (which says that $\sup S$ is the least upper bound for S).

Definition of Infimum ($\inf S$). A number such that

- (1) $\inf S \leq s$ for every $s \in S$ (i.e. $\inf S$ is an lower bound for S)
- (2) If l is any upper bound for S , then $l \leq \inf S$ (i.e. $\inf S$ is the greatest lower bound for S).

Theorem. Every nonempty subset S of \mathbb{R} which is bounded below has a greatest lower bound.

Theorem. If $\min S$ exists, then $\min S = \inf S$.

Theorem. If $A \subset \mathbb{R}$ and $c \geq 0$, and $cA := \{ca : a \in A\}$, $\sup cA = c \sup A$.

1.4 Consequences of Completeness

Theorem (Rationals between Reals). For every two real numbers a and b with $a < b$, there exists a rational number r satisfying $a < r < b$.

1.5 Nested Intervals Theorem

Nested Intervals Theorem.

If $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$ s.t. $a_n \leq a_{n+1}$ and $b_{n+1} \leq b_n$ for $n \in \mathbb{N}$, so that $I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \dots$, then $\bigcap_{n=1}^\infty I_n \neq \emptyset$.

If $\inf\{b_n - a_n\} = 0$, then $\bigcap_{n=1}^\infty I_n \{x\}$, where

$x = \sup\{a_n\} = \inf\{b_n\}$.

1.6 Capture Theorem

Capture Theorem. If A is a nonempty subset of \mathbb{R} , then:

- (i) If A is bounded above, then any open interval containing $\sup A$ contains an element of A .
- (ii) Similarly, if A is bounded below, then any open interval containing $\inf A$ contains an element of A .

1.7 Binary Search

If we binary-search for x over $I_1 = [a_1, b_1]$ for $a_1, b_1 \in \mathbb{Q}$, we define I_n s.t. either $I_n := [a_{n-1}, \frac{a_{n-1}+b_{n-1}}{2}]$ or $I_n := [\frac{a_{n-1}+b_{n-1}}{2}, a_{n+1}]$, and we define $a_n := \inf I_n$ and $b_n := \sup I_n$. We define A to be the set of all a_n , and B to be the set of all b_n .

Then, the size of $I_n = \frac{b_1-a_1}{2^n} = b_n - a_n$, and $\bigcap_{n=1}^\infty I_n \{x\}$, where $x = \sup\{a_n\} = \inf\{b_n\}$.

2 Limits

2.4 ε - δ definition of a Limit

Definition of Limit. If $\lim_{x \rightarrow a} f(x) = L$, then for any $\varepsilon > 0$, there exists $\delta > 0$ s.t. for any $x \in (a - \delta, a) \cup (a, a + \delta)$, $f(x) \in (L - \varepsilon, L + \varepsilon)$.

Alternatively,

Definition of Limit. If $\lim_{x \rightarrow a} f(x) = L$, then for any $\varepsilon > 0$, there exists $\delta > 0$ s.t. for any $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.

2.6 Limit Laws

Theorem (Limit Laws). Let $c \in \mathbb{R}$ be a constant and suppose the limits $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist. Then

- (i) $\lim_{x \rightarrow a} (f(x) \pm g(x)) = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$
- (ii) $\lim_{x \rightarrow a} (cf(x)) = c \lim_{x \rightarrow a} f(x)$
- (iii) $\lim_{x \rightarrow a} (f(x)g(x)) = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x)$
- (iv) $\lim_{x \rightarrow a} f(x)g(x) = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x)$, provided that $\lim_{x \rightarrow a} g(x) \neq 0$
- (v) See (i).
- (vi) $\lim_{x \rightarrow a} x^n = (\lim_{x \rightarrow a} x)^n$
- (vii) $\lim_{x \rightarrow a} \sqrt{f(x)} = \sqrt{\lim_{x \rightarrow a} f(x)}$
- (viii) $\lim_{x \rightarrow a} \frac{a(x)b(x)}{c(x)b(x)} = \lim_{x \rightarrow a} \frac{a(x)}{c(x)}$

Theorem (Operations on infinity). For $x \in \mathbb{R}$,

$$\begin{aligned} \infty + x &= \infty \\ -\infty + x &= -\infty \\ x * \infty &= \begin{cases} \infty & \text{if } x > 0 \\ -\infty & \text{if } x < 0 \end{cases} \\ x * -\infty &= \begin{cases} -\infty & \text{if } x > 0 \\ \infty & \text{if } x < 0. \end{cases} \end{aligned}$$

$$\frac{x}{\pm \infty} = 0$$

Definition of Indeterminate forms. The following forms are indeterminate and you cannot evaluate them.

$$\frac{0}{0}, \frac{\pm \infty}{\pm \infty}, 0 * \pm \infty, \infty - \infty$$

Other theorems

Composite Function Theorem. If f is continuous at L and $\lim_{x \rightarrow a} g(x) = L$, then $\lim_{x \rightarrow a} f(g(x)) = f(\lim_{x \rightarrow a} g(x)) = f(L)$

2.12 Squeeze Theorem

Squeeze Theorem. Let f , g , and h be defined for all $x \neq a$ over an open interval containing a . If

$$f(x) \leq g(x) \leq h(x)$$

for all $x \neq a$ in an open interval containing a and

$$\lim_{x \rightarrow a} f(x) = L = \lim_{x \rightarrow a} h(x)$$

where $L \in \mathbb{R}$, then $\lim_{x \rightarrow a} g(x) = L$.

3 Continuity

Definition of Continuity at a point. Function f is continuous at point a if $\lim_{x \rightarrow a} f(x) = f(a)$.

Definition. f has a **removable discontinuity** if $\lim_{x \rightarrow a} f(x) = L \in \mathbb{R}$ (in this case either $f(a)$ is undefined, or $f(a)$ is defined by $L \neq f(a)$).

Definition. f has a **jump discontinuity** if $\lim_{x \rightarrow a^-} f(x) = L_1 \in \mathbb{R}$ and $\lim_{x \rightarrow a^+} f(x) = L_2 \in \mathbb{R}$ but $L_1 \neq L_2$.

Definition. f has an **infinite discontinuity** at a if $\lim_{x \rightarrow a^-} f(x) = \pm \infty$ or $\lim_{x \rightarrow a^+} f(x) = \pm \infty$

Intermediate Value Theorem. If f is continuous on $[a, b]$, then for any real number L between $f(a)$ and $f(b)$ there exists at least one $c \in [a, b]$ such that $f(c) = L$. In other words, if f is continuous on $[a, b]$, then the graph must cross the horizontal line $y = L$ at least once between the vertical lines $x = a$ and $x = b$.

Aura Theorem. If $f(x)$ is continuous and $f(a)$ is positive, then there exists an open interval containing a such that for all x in the interval, $f(x)$ is positive.

If $f(x)$ is continuous and $f(a)$ is negative, then there exists an open interval containing a such that for all x in the interval, $f(x)$ is negative.

Bolzano's Theorem. Let f be a continuous function defined on $[a, b]$. If 0 is between $f(a)$ and $f(b)$, then there exists $x \in [a, b]$ such that $f(x) = 0$.

4 Derivatives

The derivative is the instantaneous rate of change, and the slope of the tangent line to the point.

Definition of Derivative ($f'(a)$).

$$\frac{d}{da}f(a) = f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

Theorem (Tangent line to a point). The equation of the tangent line to the point $(a, f(a))$ is

$$y = f'(a)(x - a) + f(a)$$

Derivative Rules

Theorem (Difference Rule).

$$\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

Theorem (Sum Rule).

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

Theorem (Constant Multiple Rule).

$$\frac{d}{dx}(cf(x)) = c\frac{d}{dx}f(x)$$

Theorem (Product Rule).

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

Theorem (Quotient Rule).

$$\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

Theorem (Power Rule).

$$\frac{d}{dx}x^n = nx^{n-1}$$

Theorem (Chain Rule).

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

Theorem (Other useful derivatives).

$$\sin'(x) = \cos(x) \qquad \cos'(x) = -\sin'(x)$$