Sound & Light (1)

1.1 Miscellaneous

$$\% \text{ error} = \frac{\text{observed} - \text{theoretical}}{\text{theoretical}} * 100\%$$

1.2 Kinematics

$$x = \frac{a}{2}(\Delta t)^2 + v_0 \Delta t + x_0 \qquad v = v_0 + a\Delta t$$

$$v^2 = v_0^2 + 2a\Delta x \qquad \Delta x = \frac{v_0 + v}{2} * \Delta t$$

1.3 Simple Harmonic Motion

$$x = A\cos(\omega t + \varphi) \quad v = -\omega A \cos(\omega t + \varphi) \quad a = -\omega^2 A \cos(\omega t + \varphi)$$

$$x_{\text{max}} = A \qquad v_{\text{max}} = \omega A \qquad a_{\text{max}} = \omega^2 A \qquad F_{\text{max}} = m\omega^2 A$$

1.3.1 Springs and Slinkies

x represents the distance from the equilibrium.

If you put a mass on top of the slinky, Δx_{eq} represents the difference between the original equilibrium and the new equilibrium.

$$F_s = kx = ma \qquad F_{s_{\text{max}}} = k\Delta x_{\text{eq}} = 9.8\Delta m$$

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad T = 2\pi\sqrt{\frac{m}{k}} \qquad \omega = 2\pi f = \sqrt{\frac{k}{m}}$$

$$SPE = \frac{1}{2}kx^2 \qquad KE = \frac{1}{2}mv^2$$

$$TME = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \frac{1}{2}kA^2 = \frac{1}{2}mv_{\text{max}}^2$$

1.3.2 Springs in parallel and series

Quantity	**In Series**	**In Parallel**
Equivalent spring constant	$\frac{1}{k_{\text{eq}}} = \frac{1}{k_1} + \frac{1}{k_2}$	$k_{\text{eq}} = k_1 + k_2$
Deflection (elongation)	$x_{\text{eq}} = x_1 + x_2$	$x_{\text{eq}} = x_1 = x_2$
Force	$F_{\text{eq}} = F_1 = F_2$	$F_{\text{eq}} = F_1 + F_2$
Stored energy	$E_{\text{eq}} = E_1 + E_2$	$E_{\text{eq}} = E_1 + E_2$

1.3.3 Pendulums

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{L}} \qquad T = 2\pi\sqrt{\frac{L}{g}}$$

1.4 Waves

$$T = \frac{1}{f} \qquad v = \lambda f \qquad v = \frac{\Delta x}{\Delta t}$$

1.4.1 Slinkies and strings with fixed ends

$$F_T = F_s = kx \qquad \mu = \frac{m}{L} \qquad v = \sqrt{\frac{F_T}{\mu}}$$

Given mass m_T hanging below a pulley, $F_T = m_T g$.

1.5 Standing waves

1.5.1 Open-open, closed-closed

n is the number of antinodes, or the n^{th} harmonic.

$$f_n = f_1 n = \frac{nv}{2L} \qquad f_1 = \frac{v}{2L} \qquad \lambda_n = \frac{2L}{n}$$

1.5.2 Open-closed

$$f_n = f_1 n = \frac{nv}{4L} \qquad f_1 = \frac{v}{4L} \qquad \lambda_n = \frac{2L}{n}$$

1.6 Sound

1.6.1 Speed of sound

$$v = 331\sqrt{\frac{T_{^\circ\text{C}} + 273}{273}} \qquad v \approx 331 + 0.59T$$

1.6.2 Sound intensity

$$I = \frac{\text{Power (W)}}{\text{Area}} = \frac{\text{Power (W)}}{4\pi r^2}$$

$$I_{\text{dB}} = 10\log_{10}(\frac{I}{10^{-12}}) \qquad I = 10^{\frac{I_{\text{dB}}}{10} - 12}$$

1.6.3 Doppler effect

$$\boxed{O}\rightarrow \ \boxed{S} \quad f_o = f_s\frac{v + v_o}{v} \qquad \leftarrow\boxed{O} \ \boxed{S} \quad f_o = f_s\frac{v - v_o}{v}$$

$$\boxed{O} \ \leftarrow\boxed{S} \quad f_o = f_s\frac{v}{v - v_s} \qquad \boxed{O} \ \boxed{S}\rightarrow \quad f_o = f_s\frac{v}{v + v_s}$$

$$\boxed{O}\rightarrow \ \leftarrow\boxed{S} \quad f_o = f_s\frac{v + v_o}{v - v_s} \qquad \boxed{O}\rightarrow \ \boxed{S}\rightarrow \quad f_o = f_s\frac{v + v_o}{v + v_s}$$

$$\leftarrow\boxed{O} \ \leftarrow\boxed{S} \quad f_o = f_s\frac{v - v_o}{v - v_s} \qquad \leftarrow\boxed{O} \ \boxed{S}\rightarrow \quad f_o = f_s\frac{v - v_o}{v + v_s}$$

1.6.4 Constructive and Destructive Interference (2 dimensions)

For a point on the m^{th} antinodal/nodal line playing the same frequency with the same phase:

$$PD = m\lambda$$

where PD is the path length difference.

1.6.5 Beats

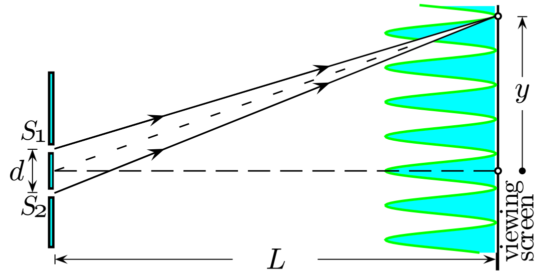
$$f_B = \Delta f$$

1.7 Light

1.7.1 Speed of light

$$c = 299\ 792\ 458 \frac{\text{m}}{\text{s}} \approx 3 * 10^8 \frac{\text{m}}{\text{s}}$$

1.7.2 Two-slit experiment



$$PD = \frac{dy}{L} = m\lambda$$

1.7.3 Mirror

$$r = 2f \qquad \frac{1}{f} = \frac{1}{p} + \frac{1}{q} \qquad M = \frac{h}{h_o} = \frac{-q}{p}$$

In a plane mirror, $p = -q$.

1.7.4 Lenses

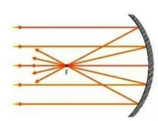
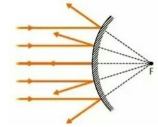
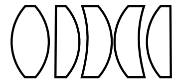
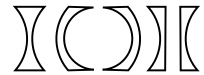
$$\frac{1}{f} = (n-1)(\frac{1}{r_1} - \frac{1}{r_2}) \qquad \frac{1}{f} = \frac{1}{p} + \frac{1}{q} \qquad M = \frac{h}{h_o} = \frac{-q}{p}$$

Multiple lenses

$$p_2 = \Delta x - q_1$$

1.7.5 Mirrors and Lenses

Real $\iff$ inverted, virtual $\iff$ upright.

	Converging	Diverging
Mirror		
Lens		
focal length	+	-
object distance	image	image
∞ to $2f$	Real smaller	Virtual smaller
$2f$ to f	Real larger	
f to 0	Virtual larger	
0 to $-f$	Virtual smaller	Virtual larger
$-f$ to $-2f$		Real larger
$-2f$ to $-\infty$		Real smaller

1.7.6 Refraction / Snell's Law

The **normal line** is the line perpendicular to the surface which touches the intersection of the surface and the light ray.

The **incident angle** is the angle between the ray of light and the normal line.

θ_1 and θ_2 are both measured from the normal line, not the surface.

Refraction occurs when the speed of light in two media are different and light hits the boundary of the two media. The frequency of the light will stay the same, but the speed, wavelength, and direction will change.

$$n = \frac{c}{v} \qquad n_1 \sin\theta_1 = n_2 \sin\theta_2$$

1.7.7 Ray diagrams

Mechanics (2)

2.1 Miscellaneous

$$\%\text{ error} = \frac{\text{observed} - \text{theoretical}}{\text{theoretical}} * 100\%$$

2.2 Kinematics

$$x(t) = \frac{a}{2}t^2 + v_0 t + x_0 \qquad v(t) = v_0 + at$$

$$v^2 = v_0^2 + 2a\Delta x \qquad \Delta x = \frac{v_0 + v}{2} * \Delta t$$

2.3 Forces

$$F_{\text{net}} = ma$$

F_T represents tension. It always points in the direction on which the rope pulls on the object.

F_N represents normal force. It takes the direction and magnitude necessary to prevent the object from passing through the surface that creates the normal force.

2.3.1 Friction

$$f_s \leq \mu_s F_N$$

The static friction takes the direction and magnitude necessary to prevent the object from moving in the component parallel to the surface, until the magnitude reaches $\mu_s F_N$. Upon reaching $\mu_s F_N$, the static friction is replaced by kinetic friction and the object starts moving:

$$k_s = \mu_k F_N$$

2.3.2 Centripetal force

The centripetal force always points towards the center of the circle representing the object's path, and therefore is perpendicular to the velocity. It is just another name for the net force in the centripetal direction. The centripetal acceleration is what causes the object to rotate.

$$F_c = ma_c = m\frac{v^2}{r} = mr\omega^2 \qquad a_c = \frac{v^2}{r}$$

2.4 Work and Energy

$$W = \int_a^b F(\vec{r}) \cdot d\vec{r}$$

In one dimension:

$$W = \int_a^b F(x)dx$$

Work-energy theorem. The work is the change in kinetic energy:

$$W = \Delta K$$

2.5 Simple Harmonic Motion

The object is at rest at the **equilibrium position**. $x = 0$ when the object is at equilibrium, and x represents the distance and direction from the equilibrium. When you pull it to one direction, the **restoring force** pulls the object back toward the equilibrium position. It oscillates back and forth, between $x = -A$ and $x = A$, where A is the **amplitude**. The **period** is the amount of time to complete one oscillation, and the **frequency** is the amount of oscillations that happen in one second (or some other time unit).

$$x(t) = A\cos(\omega t + \phi)$$
$$v(t) = -A\omega\sin(\omega t + \phi)$$
$$a(t) = -A\omega^2\cos(\omega t + \phi)$$

2.5.1 Spring force

$$F_s = -kx$$
$$U_s = -\tfrac{1}{2}kx^2$$

$$a = -\frac{k}{m}x$$
$$v = \sqrt{\frac{k}{m}(A^2 - x^2)}$$

2.6 Simple Harmonic Motion (old)

$$x = A\cos(\omega t+\varphi) \quad v = -\omega A\cos(\omega t+\varphi) \quad a = -\omega^2 A\cos(\omega t+\varphi)$$

$$x_{\max} = A \qquad v_{\max} = \omega A \qquad a_{\max} = \omega^2 A \qquad F_{\max} = m\omega^2 A$$

2.6.1 Springs and Slinkies

x represents the distance from the equilibrium.

If you put a mass on top of the slinky, Δx_{eq} represents the difference between the original equilibrium and the new equilibrium.

$$F_s = kx = ma \qquad F_{s_{\max}} = k\Delta x_{\text{eq}} = 9.8\Delta m$$

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad T = 2\pi\sqrt{\frac{m}{k}} \qquad \omega = 2\pi f = \sqrt{\frac{k}{m}}$$

$$SPE = \frac{1}{2}kx^2 \qquad KE = \frac{1}{2}mv^2$$

$$TME = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \frac{1}{2}kA^2 = \frac{1}{2}mv_{\max}^2$$

2.6.2 Springs in parallel and series

Quantity	In Series	In Parallel
Equivalent spring constant	$\frac{1}{k_{\text{eq}}} = \frac{1}{k_1} + \frac{1}{k_2}$	$k_{\text{eq}} = k_1 + k_2$
Deflection (elongation)	$x_{\text{eq}} = x_1 + x_2$	$x_{\text{eq}} = x_1 = x_2$
Force	$F_{\text{eq}} = F_1 = F_2$	$F_{\text{eq}} = F_1 + F_2$
Stored energy	$E_{\text{eq}} = E_1 + E_2$	$E_{\text{eq}} = E_1 + E_2$

2.7 Center of mass

The center of mass of an object which consists of masses at discrete points $\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots, \vec{r}_n$ is

$$\vec{r}_{\text{cm}} = \frac{\sum m_i \vec{r}_i}{\sum m_i}$$

The center of mass of a solid object with variable density is

$$\vec{r}_{\text{cm}} = \frac{\int \vec{r}dm}{\int dm}$$

To find the velocity of the center of mass, replace $\vec{r}_i$ with $\vec{v}_i$ in the above equations.

2.7.1 Density

$$\text{3D: density} \qquad \rho = \frac{M}{v} = \frac{dm}{dV} \qquad (\frac{\text{kg}}{\text{m}^3})$$

$$\text{2D: surface density} \qquad \sigma = \frac{M}{A} = \frac{dm}{dA} \qquad (\frac{\text{kg}}{\text{m}^2})$$

$$\text{1D: linear density} \qquad \lambda = \frac{M}{L} = \frac{dm}{dx} \qquad (\frac{\text{kg}}{\text{m}})$$

2.8 Momentum/Impulse

Definition of **Momentum**. **Momentum** ($\vec{p}$) is defined as:

$$\vec{p} = m\vec{v} \qquad (\text{Ns} = \text{kg}\frac{\text{m}}{\text{s}})$$

Definition of **Impulse**. **Impulse** is the integral of force over time. By the Impulse-Momentum Theorem, it is also the change in momentum.

$$\vec{J} = \int \vec{F}dt = \Delta p = \vec{F}_{\text{net,avg}}\Delta t$$

Additionally,

$$\vec{F}_{\text{net}} = \frac{dp}{dt}$$

Theorem (Conservation of momentum). Momentum is conserved when there are no external forces acting upon the object.

2.9 Rotational motion

2.9.1 Rotational kinematics

$\vec{\theta}$, angle, is the rotational equivalent of position.

$\Delta\theta$ is called angular displacement.

$$\text{Angular velocity} = \vec{\omega} = \frac{d\vec{\theta}}{dt} \quad \left(\frac{\text{rad}}{\text{s}}\right)$$
$$\text{Angular acceleration} = \vec{\alpha} = \frac{d\vec{\omega}}{dt} \quad \left(\frac{\text{rad}}{\text{s}^2}\right)$$
$$\text{Arc length} = s = R\theta$$
$$\text{Velocity} = v = R\omega$$
$$\text{Acceleration} = a = R\alpha$$

Constant acceleration

$$\omega_f = \omega_0 + \alpha t$$
$$\Delta\theta = \omega_0 t + \tfrac{1}{2}\alpha t^2$$
$$\omega_f^2 = \omega_0^2 + 2\alpha\delta\theta$$

2.9.2 Torque

$$\vec{\tau} = \vec{r} \times \vec{F} \qquad |\tau| = |r||F|\sin\theta$$

2.9.3 Rotational inertia

Where r represents the distance between the axis of rotation and the object, m represents mass, I represents rotational inertia, and K represents rotational kinetic energy:

$$I = mr^2 \qquad I_{\text{total}} := \sum m_i r_i^2 = \int r^2 dm \qquad K = \frac{1}{2}I\omega^2$$

Parallel Axis Theorem. If d is the distance between the axis through the center of mass and the desired axis of rotation parallel to the axis through the center of mass, then the rotational inertia through the desired axis of rotation is

$$I_{\text{II}} = I_{\text{cm}} + Md^2$$

2.9.4 Newton's Second Law

$$\sum \tau = I\alpha$$

2.9.5 Rotational kinetic energy

Definition of **Rotational kinetic energy**.

$$K_{\text{rot}} = \frac{1}{2}I\omega^2$$

Definition of **Rotational work**.

$$W = \int \tau\, d\theta = \Delta K$$

2.9.6 Rolling

During rolling, the velocity of the center of mass is

$$v_{\text{cm}} = R\omega$$

in the direction of rolling. Add $v_{\text{cm}} = R\omega$ to the tangential velocity, $R\omega$ in the direction perpendicular to the radius.

When rolling at the maximum speed possible without slipping, the friction is

$$f_s = \mu_s F_N$$

Otherwise,

$$f_s < \mu_s F_N$$

2.9.7 Angular momentum

$$\vec{L} = I\vec{\omega} \text{ for solid objects}$$

$$\vec{L} = \vec{r} \times \vec{p} \text{ for point masses}$$

$$L = rp\sin(\theta) = rmv\sin(\theta) \text{ for point masses}$$

Note: angular velocity's direction is specified by right-hand rule.

The derivative of angular momentum is the change in external net torque.

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{\text{net, ext}}$$

2.9.8 Equilibrium

If $\sum F = 0$ and $\sum \tau = 0$, then the object is in equilibrium and therefore is not accelerating.

If $\sum F = 0$ and $\sum \tau \neq 0$, then the object is accelerating in its rotation but has constant velocity.

If $\sum F \neq 0$ and $\sum \tau = 0$, then the object is linearly accelerating but is not accelerating rotationally.

If $\sum F \neq 0$ and $\sum \tau \neq 0$, then the object is in complex motion.

2.10 Gravitation

Using the equations which are similar to electrostatics:

Definition of **Gravitational constant**.

$$G \approx 6.67 * 10^{-11}\frac{\text{Nm}^2}{\text{kg}^2}$$

Definition of **Gravitational field**. The vector field g, which applies to all objects.

The electric field points towards masses.

g is also the acceleration caused by gravity.

Newton's law of universal gravitation. If stationary point or spherical masses m_1 and m_2 are near each other but not touching, then the magnitude of the gravitational force between the two objects is

$$|F_g| = \frac{Gm_1m_2}{r^2}$$

Law. The gravitational force on a stationary point or spherical mass g is $F_g = mg$, where g is the strength and direction of the gravitational field.

Law (by Newton's law of universal gravitation). The gravitational field at distance r due to a stationary point or spherical charge m is

$$g = \frac{GM}{r^2}.$$

The gravitational field at distance r due to a massive object whose total mass is $\int dm$ is

$$g = \int \frac{G\,dm}{r^2}$$

where r's tail is at the mass and points towards the point for which we want to calculate g.

2.10.1 Gravitational flux

Definition of **Gravitational flux (Φ)**. The flux through a surface with area vector $\vec{A}$ is

$$\Phi = \int \vec{g} \cdot d\vec{A}$$

where $\vec{g}$ is the gravitational field that passes through the surface, and the area vector's magnitude is the area and direction is normal/perpendicular to the surface.

Definition of **Net flux**. The flux through any surface that encloses a charge.

Gauss's Law. The net flux is proportional to the enclosed mass, and can be equated to the flux through the surface:

$$\Phi_{\text{net}} = -4\pi GM = \oint \vec{g} \cdot d\vec{A}$$

Electromagnetism (3)

3.1 Electrostatics

Definition of **Electric charge**. A scalar quantity that can describe an object. It can be positive or negative

Definition of **Elementary charge** (e). The smallest possible charge $e \approx 1.6 * 10^{-19}$ C.

3.1.1 Charging

Procedure (Charging by friction). Rub two nonconductive objects together. The one with the greater electron affinity (electronegativity) becomes negatively charged.

Procedure (Polarization). When two objects near each other, their charge distributions will change in order to ensure that similar charges in the different objects don't get too near each other.

3.1.2 Electric fields

Definition of **Permittivity of free space** (ε_0).

$$\varepsilon_0 \approx 8.85 * 10^{-12} \frac{\mathrm{C}^2}{\mathrm{Nm}^2}$$

Definition of **Coulomb's constant** (k).

$$k = \frac{1}{4\pi\varepsilon_0} \approx 8.99 * 10^9 \frac{\mathrm{Nm}^2}{\mathrm{C}^2}$$

Coulomb's Law. If stationary point or spherical charges q_1 and q_2 are near each other but not touching, then the magnitude of the electrostatic force between the two objects is

$$|F_e| = \frac{k|q_1||q_2|}{r^2} = \frac{|q_1||q_2|}{4\pi\varepsilon_0 r^2}$$

Definition of **Electric field**. The vector field E ($\frac{\mathrm{N}}{\mathrm{C}}$), which applies to all objects.

The electric field points away from positive charges and towards negative charges.

Law. The electrostatic force on a stationary point or spherical charge q is $F_e = Eq$, where E is the strength and direction of the electric field.

Law (by Coulomb's Law). The electric field at distance r due to a stationary point or spherical charge Q is

$$E = \frac{kQ}{r^2} = \frac{Q}{4\pi\varepsilon_0 r^2}$$

The electric field at distance r due to a charged object whose total charge is $\int dq$ is

$$E = \int \frac{k\,dq}{r^2} = \int \frac{dq}{4\pi\varepsilon_0 r^2}$$

where r's tail is at the charge and points towards the point for which we want to calculate E.

Definition of **Charge density**.

$$\begin{array}{rll} \text{3D: density} & \rho = \dfrac{Q}{v} = \dfrac{dq}{dV} & (\dfrac{\mathrm{C}}{\mathrm{m}^3}) \\ \text{2D: surface density} & \sigma = \dfrac{Q}{A} = \dfrac{dq}{dA} & (\dfrac{\mathrm{C}}{\mathrm{m}^2}) \\ \text{1D: linear density} & \lambda = \dfrac{Q}{L} = \dfrac{dq}{dx} & (\dfrac{\mathrm{C}}{\mathrm{m}}) \end{array}$$

Example (E-field of a sheet of charge). The magnitude of the E-field caused by a sheet of charge is

$$E = \frac{\sigma}{2\varepsilon_0}$$

where the E-field points away from the sheet if the charge is positive, and towards the sheet if the charge is negative.

Example (E-field of a cylindrical charge). The magnitude of the E-field caused by a cylindrical charge of radius R with negligible end effects, at radius r from its axis, is:

$$\frac{R\sigma}{r\varepsilon_0}$$

3.1.3 Electric flux

Definition of **Electric flux** (Φ). The flux through a surface with area vector $\vec{A}$ is

$$\Phi = \int \vec{E} \cdot d\vec{A} \qquad \left(\frac{\mathrm{Nm}^2}{\mathrm{C}}\right)$$

where $\vec{E}$ is the electric field that passes through the surface, and the area vector's magnitude is the area and direction is normal/perpendicular to the surface.

Definition of **Net flux**. The flux through any surface that encloses a charge.

Gauss's Law. The net flux is equivalent to the enclosed charge divided by the permittivity of free space, and can be equated to the flux through the surface:

$$\Phi_{\text{net}} = \frac{q}{\varepsilon_0} = 4\pi kq = \oint \vec{E} \cdot d\vec{A}$$

3.1.4 Electric potential

Definition of **Electric potential energy**. The work required to move a charge from a reference position to its current location in the electric field.

When the electrostatic force does work W_e on the object, its electric potential energy decreases and its kinetic energy increases:

$$\Delta U_e = -W_e$$

Definition of **Electric potential**. The work per unit charge required to move a charge from a reference position to its current location in the electric field.

Definition of **Electric potential difference / Voltage**. The difference in electric potential between two points.

$$\Delta V = \frac{\Delta U_e}{q} = -\frac{W_e}{q} = -\int \vec{E}(\vec{r}) \cdot d\vec{r}$$

Therefore,

$$\vec{E} = -\frac{dV}{dr}$$

Definition of **Equipotential line/surface**. A line/surface where for each point on the line/surface, the electric potential is the same.

Graphs of equipotential lines where each line's electric potential is an integer multiple of some number produce a topographic-like map of the electric field, where each of the equipotential lines can be viewed as contour lines.

Theorem (Potential caused by a point or spherical charge). The electric potential outside a point or spherical charge, at distance r from the center of the charge, is

$$\frac{kq}{r}$$

The electric potential inside a spherical charge of radius R is

$$\frac{kq}{R}$$

(This derives from Coulomb's Law.)

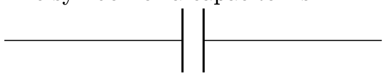
3.1.5 Capacitance

Definition of **Capacitor**. At least one charged conductor, usually two.

Common capacitor shapes:

- Parallel plate
- Cylindrical
- Spherical

The symbol for a capacitor is



Definition of **Capacitance**. Where Q is the magnitude of the charge on one of the plates, and V is the potential difference between two plates:

$$C = \frac{Q}{V} \qquad \left(\mathrm{F} = \frac{\mathrm{C}}{\mathrm{V}}\right)$$

The capacitance remains constant for a given capacitor regardless of any change in charge or voltage; it only depends on the shape of the capacitor.

The capacitance is a positive (unsigned) value.

Example (Parallel plate capacitor). The capacitance of a capacitor containing two parallel plates, where each plate has area A and the distance between the plates is d, is

$$C = \frac{\varepsilon_0 A}{d}$$

Example (Spherical shell capacitor). The capacitance of a capacitor containing two concentric spherical shells, where the smaller shell has radius a and the larger has radius b, is

$$C = \frac{ab}{bk - ak}$$

Example (Cylindrical capacitor). The capacitance of a capacitor containing two concentric cylindrical shells with negligible end effects, where the smaller shell has radius a and the larger has radius b and both have length L, is

$$C = \frac{2\pi\varepsilon_0 L}{\ln\left(\frac{b}{a}\right)}$$

Theorem (Energy stored in a capacitor).

$$U_e = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}VQ = \frac{1}{2}CV^2$$

Definition of **Dielectric**. An object that can be polarized.

Definition of **Permittivity**. The permittivity of space containing a dielectric is equal to

$$K\varepsilon_0$$

where K is the dielectric's **dielectric constant**.

3.1.6 Random circuit stuff

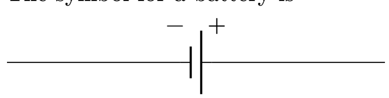
Definition of **Current** (i). The rate at which charges flow through a conductor.

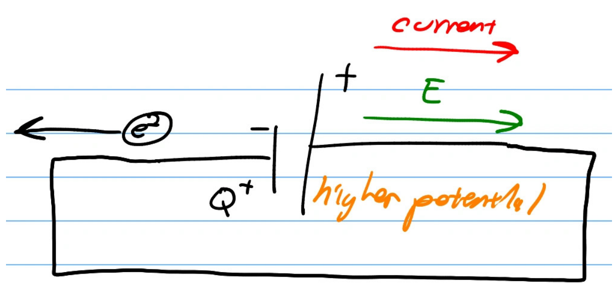
$$i = \frac{dq}{dt}$$

Definition of **Battery**. A battery has two terminals, a positive and a negative terminal. The positive terminal has higher electric potential than the negative terminal, and the battery creates an E-field between the positive and negative terminal. The potential difference between the positive and negative terminal is called the **voltage** of the battery, and is represented as $\mathcal{E}$.

The electric field causes positive current to flow from the positive to negative terminal. Equivalently, negative current flows from the negative to positive terminal, which is the way the actual electrons flow.

The symbol for a battery is





3.1.7 Loop Law

Loop Law. In a circuit, the sum of the signed potential differences along the entire circuit is 0.

$$\sum \Delta V = 0$$

In a circuit with a battery, this means that the voltage (unsigned) of the battery is equal to the sum of the other potential differences along the circuit.

Mathematics (4)

4.1 Logarithms

$$\log_b(MN) = \log_b(M) + \log_b(N)$$
$$\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)$$
$$\log_b(M^p) = p \cdot \log_b(M)$$
$$\log_b(a) = \frac{\log_x(a)}{\log_x(b)}$$
$$\log_b(b) = 1$$

4.2 Notation

deg $p(x)$ means the degree of polynomial p.

LC $p(x)$ means the leading coefficient of polynomial p.

4.3 Rational functions

For a rational function $f(x) = \frac{p(x)}{q(x)}$, cancel out any common factors, then:

- For all rational functions:
  - VA: roots of $q(x)$
  - Roots: roots of $p(x)$
- When deg $p(x) =$ deg $q(x)$:
  - HA: $y = \frac{\text{LC } p(x)}{\text{LC } q(x)}$
- When deg $p(x) <$ deg $q(x)$:
  - HA: $y = 0$
- When deg $p(x) >$ deg $q(x)$:
  - HA: none
  - slant asymptote: $\frac{p(x)}{q(x)}$ excluding remainder

4.4 Polynomials

4.4.1 Linear equations

$$\text{Slope-intercept form: } y = mx + b$$
$$\text{Point-slope form: } y - y_1 = m(x - x_1) \text{ for point } (x, y)$$
$$\text{Standard form: } ax + by = c$$

4.4.2 Quadratic equations

$$\text{Standard form: } y = ax^2 + bx + c$$
$$\text{Vertex form: } y = a(x-h)^2 + k \text{ for vertex } (h, k)$$
$$\text{Sum of roots: } \frac{-b}{a}$$
$$\text{Product of roots: } \frac{c}{a}$$

4.4.3 Higher-degree polynomials

In a polynomial

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

, with roots

$$r_1, r_2, r_3, \ldots, r_n$$

then:

$$r_1 + r_2 + r_3 + \cdots + r_n = \sum_{k=1} n r_k = -\frac{a_{n-1}}{a_n}$$

4.5 Sequences and Series

4.5.1 Explicit formulas

$$\text{Aritmetic sequence: } a_n = a_1 + r(n-1)$$
$$\text{Geometric sequence: } a_n = a_1 * r^{n-1}$$
$$\text{Harmonic sequence: } a_n = \frac{1}{a_1 + r(n-1)}$$

4.5.2 Arithmetic and Geometric Series

In the following equations, substituting $j = 1$ with $j = 0$, $j - 1$ with j, and a_1 with a_0 will produce the same result.

$$\sum_{j=1}^{n}(a_1 + r(j-1)) = \frac{n}{2}(2a_1 + (n-1)d)$$
$$\sum_{j=1}^{n}(a_1 * r^{j-1}) = \frac{a_1(1-r^n)}{1-r}$$
$$\sum_{j=1}^{\infty}(a_1 * r^{j-1}) = \frac{a_1}{1-r} \text{ for } r \in [-1, 1]$$

4.5.3 Special Sums

$$\sum_{j=1}^{n} c = nc \qquad \sum_{j=1}^{n} ca_j = c\sum_{j=1}^{n} a_j$$
$$\sum_{j=1}^{n}(a_j + b_j) = \sum_{j=1}^{n} a_j + \sum_{j=1}^{n} b_j \qquad \sum_{j=1}^{n} j = \frac{n(n+1)}{2}$$
$$\sum_{j=1}^{n} j^2 = \frac{n(n+\frac{1}{2})(n+1)}{3} = \frac{n(2n+1)(n+1)}{6} \qquad \sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4}$$

4.6 Trigonometry

°	rad	sin	cos	tan
0°	0	0	1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	$\frac{\pi}{2}$	1	0	undef

4.6.1 Law of Sines and Cosines

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c} \qquad c^2 = a^2 + b^2 - 2ab\cos(C)$$

4.6.2 Triangle area

$$K = \frac{1}{2}bh \qquad K = \frac{1}{2}bc\sin(A) \qquad K = \sqrt{s(s-a)(s-b)(s-c)}$$

4.6.3 More identities

$$(\sin A)^2 + (\cos A)^2 = 1 \qquad (\tan A)^2 + 1 = (\sec A)^2$$

$$\sin(\frac{\pi}{2} - x) = \cos(x) \qquad (\cot A)^2 + 1 = (\csc A)^2$$

$$\cos(-x) = \cos(x) \qquad \sin(-x) = \sin(x) \qquad \tan(-x) = \tan(x)$$

4.6.4 Slope

Where α is the angle between the line and the x-axis, and m is the slope of the line:

$$m = \tan\alpha$$

4.6.5 Sum and difference formulas

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$
$$\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B)$$
$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$
$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$
$$\tan(A+B) = \frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$$
$$\tan(A-B) = \frac{\tan(A)-\tan(B)}{1+\tan(A)\tan(B)}$$

$$\sin(2A) = 2\sin(A)\cos(A)$$
$$\cos(2A) = (\cos A)^2 - (\sin A)^2 = 2(\cos A)^2 - 1 = 1 - 2(\sin A)^2$$
$$\tan(2A) = \frac{2\tan(A)}{1-(\tan A)^2}$$

4.7 Vectors

$$\vec{v} + \vec{w} = \begin{bmatrix} v_x + w_x \\ v_y + w_y \\ v_z + w_z \end{bmatrix} \qquad c * \vec{v} = \begin{bmatrix} c * v_x \\ c * v_y \\ c * v_z \end{bmatrix}$$

$$\vec{v} \cdot \vec{w} = v_x w_x + v_y w_y + v_z w_z = |\vec{v}||\vec{w}|\cos(\theta)$$

$$|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}|\sin(\theta) = \text{area of parallelogram}$$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} \qquad \vec{v} \times \vec{w} \perp \vec{v} \qquad \vec{v} \times \vec{w} \perp \vec{w}$$

$$\vec{v} \perp \vec{w} \iff \vec{v} \times \vec{w} = \vec{0} \qquad \vec{v} \parallel \vec{w} \iff \vec{v} \cdot \vec{w} = 0$$

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|} \qquad \text{proj}_{\vec{b}}\vec{v} = \frac{\vec{v} \cdot \vec{b}}{\vec{b} \cdot \vec{b}} * \vec{b} = (|\vec{v}|\cos(\theta))$$

Right-hand rule

To determine the direction of $\vec{v} \times \vec{w}$, put the side of the right hand on $\vec{v}$ and curl the fingers toward $\vec{w}$. The direction the thumb is pointing is the direction of $\vec{v} \times \vec{w}$.

4.8 Polar

4.8.1 Polar and Cartesian sytems

With point $(x, y) = (r; \theta) = (r; \beta)$, where θ is CCW from the x-axis and β is a bearing, CW from the y-axis:

$$x = r\cos(\theta) = r\sin(\beta) \qquad y = r\sin(\theta) = r\cos(\beta)$$
$$r = \sqrt{x^2 + y^2} \qquad \theta \equiv \arctan(\frac{y}{x}) \qquad \beta \equiv \arctan(\frac{x}{y})$$

4.8.2 Converting functions

Try these substitutions in order:

$$x^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad x = r\cos\theta \qquad y = r\sin\theta$$

4.8.3 Limaçons and Petals

The function $y = A\cos(B(\theta + C)) + D$ is equivalent to $y = A\cos(B\theta) + D$ rotated C degrees/radians clockwise.

When C is 0 and B is 1, the x-intercepts are $A \pm D$ and the y-intercepts are $\pm D$, and it forms a limaçon.

When C is 0, but $B \neq 1$, then this sometimes still holds. The x-intercepts may also be $\pm A \pm D$.

There are B petals, with the axis of the first petal on the positive x-axis.

When B is even and $|D| < 1$, then the number of petals is $2B$.

Using sin instead of cos, limaçons have their axes on the positive y-axis, while for petals, the first petal starts from the positive x-axis and curves upwards.

4.9 Complex

$$\text{cis}(\theta) = e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

To find the n^{th} root of $x_r\,\text{cis}(x_\theta)$, solve the equation $z_r^n\,\text{cis}(nz_\theta) = x_r\,\text{cis}(x_\theta + 360^\circ k)$ for $k \in \mathbb{R}$.

4.10 Function domain

Function	Domain x	Range y
$\log(x)$	$(0, \infty)$	$\mathbb{R}$
$\sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$\arcsin(x)$	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$\arccos(x)$	$[-1, 1]$	$[0, \pi]$
$\arctan(x)$	$\mathbb{R}$	$(-\frac{\pi}{2}, \frac{\pi}{2})$

Calculus Theorems (5)

1 Completeness

Theorem (Completeness of the Real Numbers). Every nonempty subset S of $\mathbb{R}$ which is bounded above has a least upper bound $\sup S$.

Definition of **Supremum** ($\sup S$). A number such that

(1) $s \leq \sup S$ for every $s \in S$ (which just says that $\sup S$ is an upper bound for S)

(2) If u is any upper bound for S, then $\sup S \leq u$ (which says that $\sup S$ is the least upper bound for S).

Definition of **Infimum** ($\inf S$). A number such that

(1) $\inf S \leq s$ for every $s \in S$ (i.e. $\inf S$ is an lower bound for S)

(2) If l is any upper bound for S, then $l \leq \inf S$ (i.e. $\inf S$ is the greatest lower bound for S).

Theorem. Every nonempty subset S of $\mathbb{R}$ which is bounded below has a greatest lower bound.

Theorem. If $\min S$ exists, then $\min S = \inf S$.

Theorem. If $A \subset R$ and $c \geq 0$, and $cA := ca : a \in A$, $\sup cA = c \sup A$.

Theorem (Rationals between Reals). For every two real numbers a and b with $a < b$, there exists a rational number r satisfying $a < r < b$.

Nested Intervals Theorem.
If $I_n = [a_n, b_n] = \{x \in R : a_n \leq x \leq b_n\}$ s.t. $a_n \leq a_{n+1}$ and $b_{n+1} \leq b_n$ for $n \in \mathbb{N}$, so that $I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq ...$, then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

If $\inf\{b_n - a_n\} = 0$, then $\bigcap_{n=1}^{\infty} I_n = \{x\}$, where $x = \sup\{a_n\} = inf\{b_n\}$.

Capture Theorem. If A is a nonempty subset of $\mathbb{R}$, then:

(i) If A is bounded above, then any open interval containing $\sup A$ contains an element of A.

(ii) Similarly, if A is bounded below, then any open interval containing $\inf A$ contains an element of A.

Theorem (Binary Search (Bisection Method)). If we binary-search for x over $I_1 = [a_1, b_1]$ for $a_1, b_1 \in \mathbb{Q}$, we define I_n s.t. either $I_n := [a_{n-1}, \frac{a_{n-1}+b_{n-1}}{2}]$ or $I_n := [\frac{a_{n-1}+b_{n-1}}{2}, a_{n-1}]$, and we define $a_n := \inf I_n$ and $b_n := \sup I_n$. We define A to be the set of all a_n, and B to be the set of all b_n.

Then, the size of $I_n = \frac{b_1-a_1}{2^n} = b_n - a_n$, and $\bigcap_{n=1}^{\infty} I_n\{x\}$, where $x = \sup\{a_n\} = \inf\{b_n\}$.

2 Limits

Definition of **Limit**. If $\lim_{x\to a} f(x) = L$, then for any $\varepsilon > 0$, there exists $\delta > 0$ s.t. for any $x \in (a-\delta, a) \cup (a, a+\delta)$, $f(x) \in (L-\varepsilon, L+\varepsilon)$.

Alternatively,

Definition of **Limit**. If $\lim_{x\to a} f(x) = L$, then for any $\varepsilon > 0$, there exists $\delta > 0$ s.t. $|f(x) - L| < \varepsilon$ whenever $0 < |x-a| < \delta$.

Definition of **Right-sided limit**. If $\lim_{x\to a^+} f(x) = L$, then for any $\varepsilon > 0$, there exists $\delta > 0$ s.t. $|f(x) - L| < \varepsilon$ whenever $0 < x - a < \delta$.

Definition of **Left-sided limit**. If $\lim_{x\to a^-} f(x) = L$, then for any $\varepsilon > 0$, there exists $\delta > 0$ s.t. $|f(x) - L| < \varepsilon$ whenever $0 < a - x < \delta$.

Theorem (Limit Laws). Let $c \in R$ be a constant and suppose the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

- $\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
- $\lim_{x\to a}(cf(x)) = c \lim_{x\to a} f(x)$
- $\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$
- $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$, provided that $\lim_{x\to a} g(x) \neq 0$
- $\lim_{x\to a} x^n = (\lim_{x\to a} x)^n$
- $\lim_{x\to a} \frac{a(x)b(x)}{c(x)b(x)} = \lim_{x\to a} \frac{a(x)}{c(x)}$

These laws also apply to one-sided limits.

Theorem (L'Hopital's Rule). If f and g are differentiable and $g'(x) \neq 0$ on an open interval I that surrounds a, and $\lim_{x\to a} \frac{f(x)}{g(x)} \in \{\frac{0}{0}, \pm\frac{\infty}{\infty}\}$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$.

Theorem (Composition of Limits). If f is continuous at L and $\lim_{x\to a} g(x) = L$, then $\lim_{x\to a} f(g(x) = f(\lim_{x\to a} g(x))) = f(L)$

Theorem (Operations on infinity). For $x \in \mathbb{R}$,

$$\infty + x = \infty \qquad -\infty + x = -\infty \qquad \frac{x}{\pm\infty} = 0$$

$$x * \infty = \begin{cases} \infty & \text{if } x > 0 \\ -\infty & \text{if } x < 0 \end{cases}$$

$$x * -\infty = \begin{cases} -\infty & \text{if } x > 0 \\ \infty & \text{if } x < 0. \end{cases}$$

Definition of **Indeterminate forms**. The following forms are indeterminate and you cannot evaluate them.

$$\frac{0}{0}, \frac{\pm\infty}{\pm\infty}, 0 * \pm\infty, \infty - \infty$$

Squeeze Theorem. Let f , g, and h be defined for all $x \neq a$ over an open interval containing a. If

$$f(x) \leq g(x) \leq h(x)$$

for all $x \neq a$ in an open interval containing a and

$$\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$$

where $L \in \mathbb{R}$, then $\lim_{x\to a} g(x) = L$.

3 Continuity

Definition of **Continuity at a point**. Function f is continuous at point a if $\lim_{x\to a} f(x) = f(a)$.

Definition. f has a **removable discontinuity** if $\lim_{x\to a} f(x) = L \in \mathbb{R}$ (in this case either $f(a)$ is undefined, or $f(a)$ is defined by $L \neq f(a)$).

Definition. f has a **jump discontinuity** if $\lim_{x\to a^-} f(x) = L_1 \in \mathbb{R}$ and $\lim_{x\to a^+} f(x) = L_2 \in \mathbb{R}$ but $L1 \neq L2$.

Definition. f has an **infinite discontinuity** at a if $\lim_{x\to a^-} f(x) = \pm\infty$ or $\lim_{x\to a^+} f(x) = \pm\infty$

Intermediate Value Theorem. If f is continuous on $[a, b]$, then for any real number L between $f(a)$ and $f(b)$ there exists at least one $c \in [a, b]$ such that $f(c) = L$. In other words, if f is continuous on $[a, b]$, then the graph must cross the horizontal line $y = L$ at least once between the vertical lines $x = a$ and $x = b$.

Aura Theorem. If $f(x)$ is continuous and $f(a)$ is positive, then there exists an open interval containing a such that for all x in the interval, $f(x)$ is positive.

If $f(x)$ is continuous and $f(a)$ is negative, then there exists an open interval containing a such that for all x in the interval, $f(x)$ is negative.

Bolzano's Theorem. Let f be a continuous function defined on $[a, b]$. If 0 is between $f(a)$ and $f(b)$, then there exists $x \in [a, b]$ such that $f(x) = 0$.

4 Derivatives

The derivative is the instantaneous rate of change, and the slope of the tangent line to the point.

Definition of **Derivative** ($f'(a)$).

$$\frac{d}{da}f(a) = f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

Theorem (Tangent line to a point). The equation of the tangent line to the point $(a, f(a))$ is

$$y = f'(a)(x-a) + f(a)$$

Derivative Rules

Theorem (Difference Rule).

$$\frac{d}{dx}(f(x)-g(x)) = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

Theorem (Sum Rule).

$$\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

Theorem (Constant Multiple Rule).

$$\frac{d}{dx}(cf(x)) = c\frac{d}{dx}f(x)$$

Theorem (Product Rule).

$$\begin{aligned}\frac{d}{dx}(f(x)g(x)) &= f'(x)g(x) + f(x)g'(x)\\ \frac{d}{dx}(f(x)g(x)h(x)) &= f'(x)g(x)h(x) + f(x)g'(x)h(x)\\ &+ f(x)g(x)h'(x)\end{aligned}$$

and so on.

Theorem (Quotient Rule).

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$$

Theorem (Power Rule).

$$\frac{d}{dx}x^n = nx^{n-1}$$

for $n \in \mathbb{R}$

Theorem (Chain Rule).

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x) \qquad \frac{dy}{dx} = \frac{dy}{db}\frac{db}{dx}$$

Theorem (Derivative of inverse functions). Let $x \in \mathbb{R}$ and f be a differentiable, one-to-one function at x. Then if $f'(x) \neq 0$, then

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}$$

Theorem (Derivatives of exponentials and logs).

$$\begin{aligned}\frac{d}{dx}e^x &= e^x & \frac{d}{dx}\ln x &= \frac{1}{x}\\ \frac{d}{dx}a^x &= a^x\ln(a) & \frac{d}{dx}\log_a x &= \frac{1}{x\ln(a)}\end{aligned}$$

Theorem (Derivatives of trig functions).
Warning: x must be an angle in radians!

$$\begin{aligned}\sin'(x) &= \cos(x) & \cos'(x) &= -\sin(x)\\ \sec'(x) &= \sec(x)\tan(x) & \csc'(x) &= -\csc(x)\cot(x)\\ \tan'(x) &= \sec(x)^2 & \cot'(x) &= -\csc(x)^2\end{aligned}$$

$$\begin{aligned}\arcsin'(x) &= \frac{1}{\sqrt{1-x^2}} & \arccos'(x) &= -\frac{1}{\sqrt{1-x^2}}\\ \text{arcsec}'(x) &= \frac{1}{|x|\sqrt{x^2-1}} & \text{arccsc}'(x) &= -\frac{1}{|x|\sqrt{x^2-1}}\\ \arctan'(x) &= \frac{1}{1+x^2} & \text{arccot}'(x) &= -\frac{1}{1+x^2}\end{aligned}$$

5 Derivative Applications

5.7 Mean Value Theorem

Theorem (Mean Value Theorem). If the function f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ s.t.

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{\Delta f(x)}{\Delta x} \text{ on } [a, be]$$

Theorem (Some colloraries to the MVT). If $f(x)$ is differentiable on I, then:

- $f'(x) > 0$ for $x \in I \iff f(x)$ is strictly increasing for $x \in I$.
- $f'(x) \geq 0$ for $x \in I \iff f(x)$ is increasing or constant for $x \in I$.
- $f'(x) = 0$ for $x \in I \iff f(x)$ is constant for $x \in I$.
- $f'(x) \leq 0$ for $x \in I \iff f(x)$ is decreasing or constant for $x \in I$.
- $f'(x) < 0$ for $x \in I \iff f(x)$ is strictly decreasing for $x \in I$.

5.3, 5.10, 5.11, 5.16 Extrema

Definition of **Critical point of** f. A number c in the domain of f where $f'(c) = 0$ or $f'(c)$ does not exist.

Definition of **Stationary point of** f. A number c in the domain of f where $f'(c) = 0$

Fermat's Theorem. The local maxima and minima of f are critical points of f.

Exteme Value Theorem. If f is continuous on $[a, b]$, then it has an absolute max and an absolute min.

Theorem (Method to find absolute minima and maxima). Store the critical points of f in the array C. Then, the absolute maximum is $\max\{f(c) : c \in C\}$ and the absolute minimum is $\min\{f(c) : c \in C\}$.

Theorem (First Derivative Test). If f is continuous over I, and $c \in I$ is a critical point of f, and f is differentiable over $I \setminus c$, then:

- If $f'(x)$ is decreasing at c, then $f(c)$ is a local max.
- If $f'(x)$ is increasing at c, then $f(c)$ is a local min.
- If $f'(x)$ has the same sign before and after c, then $f(c)$ is neither a local max nor a local min.

Definition of **Concavity**. f is concave up on I if the tangent line to f at each point in I is lower than the graph of f.

f is concave down on I if the tangent line to f at each point in I is higher than the graph of f.

Theorem (Test for Concavity). If $f''(x) > 0$ for all $x \in I$, then f is concave up on I.

If $f''(x) < 0$ for all $x \in I$, then f is concave down on I.

Theorem (Second Derivative Test). If f'' is continuous on an interval containing c, where c is the x-value of a stationary point of f. Then,

- If $f''(c) > 0$, then $f(c)$ is a local max.
- If $f''(c) < 0$, then $f(c)$ is a local min.

Trimm's Single Extremum Theorem. If f is continuous on an interval I, and f has a single local extremum (max or min), then that extremum is a global max or min.

Miscellaneous

Definition of **Exponential function**.

$$\exp x = e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$$

6 Integrals

Antiderivative

Definition of **Antiderivative / Indefinite Integral**. The antiderivative F of a function f is the function such that $F'(x) = f(x)$.

$$F(x) = \int f(x)dx$$

Theorem (Antiderivative plus a constant). If F is the antiderivative of a function f, then $G(x) = F(x) + c$ where $c \in \mathbb{R}$ is also an antiderivative.

Definition of **Integral**.

$$(f_1[a,b]) \mapsto \int_a^b f(x)dx \in \mathbb{R}$$

such that the Properties of the Integral are true.

The definite integral takes in a function and a range $[a,b]$, and returns a number. The indefinite integral takes in a function and returns an infinitely large set of functions (the antiderivatives).

If $a > b$, then $\int_a^b f := -\int_b^a f$.

Let $\mathcal{R}([a,b])$ be the set of integrable functions, $\mathcal{C}([a,b])$ be the set of continuous functions, and $\mathcal{B}([a,b])$ be the set of bounded functions on $[a,b]$. Then

$$\mathcal{C}([a,b]) \subset \mathcal{R}([a,b]) \subset \mathcal{B}([a,b])$$

Theorem (Properties of the Integral). The integral is defined such that the following are true:

(I0) Every continuous function is integrable.

(I1) If $f(x) = c$, then $\int_a^b f(x)dx = c(b-a)$

(I2) If $f_1(x) \le f_2(x)$, then $\int_a^b f_1(x)dx \le \int_a^b f_2(x)dx$.

(I3) For any a, b, c, $\displaystyle\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.

Theorem (Fundamental Theorem of Calculus). Let $f \in \mathcal{R}([a,b])$ be some integrable function, where $a, b \in \mathbb{R}$. Let $\mathcal{F}(x) = \int_a^x f$ for $x \in [a,b]$. Then:

(a) $\mathcal{F}$ is continuous for every $c \in [a,b]$.

(b) If f is continuous at $c \in [a.b]$, then $\mathcal{F}$ is diffentiable at c, and $\mathcal{F}'(c) = f(c)$.

(c) If f is continuous on $[a,b]$, and F is an antiderivative of f, then $\int_a^b f = F(b) - F(a)$, or

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

Theorem (Substitution Rule). If g is a function that has a continuous derivative on an interval, another function f is continuous on the range of g, and F is an antiderivative of f on the range of g, then

$$\int f(g(x))g'(x)dx = F(g(x)) + C$$

For $a, b \in \mathbb{R}$,

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du = F(g(b)) - F(g(a))$$

Let $u := g(x)$. Then, if $du = E(g'(x)dx)$, where $E \in \mathbb{R}$, then

$$\int Ef(u)du = E\int f(u)du = F(u) + C$$

Theorem (Some Antiderivative Rules).

$$\int e^x dx = e^x + C \qquad \int x^a dx = \frac{x^{a+1}}{a+1} + C \text{ for } a \ne 1$$

$$\int a^x dx = \frac{a^x}{\ln(a)} + C \qquad \int x^{-1}dx = \ln|x| + C$$

$$\int f(x) + g(x)dx = \int f(x)dx + \int g(x)dx$$

Theorem (Integration By Parts). If f and g are integrable functions, then

$$\int f(x)g'(x)dx = f(x)g(x) + \int f(x)g(x)dx$$

Equivalently, if u and v are integrable functions of x, then

$$\int udv = uv - \int vdu$$

Additionally, if f' and g' are continuous, then

$$\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) + \int_a^b g(x)f'(x)dx$$

Theorem (Integrals of Trig Functions).

$$\int \sin(x)dx = -\cos(x) + C \qquad \int \cos(x)dx = \sin(x) + C$$

$$\int \tan(x)dx = -\ln|\cos(x)| + C = \ln|\sec(x)| + C$$

$$\int \cot(x)dx = \ln|\sin(x)| + C = -\ln|\csc(x)| + C$$

$$\int \sec(x)dx = \ln|\sec(x) + \tan(x)| + C$$

$$\int \csc(x)dx = -\ln|\cot(x) + \csc(x)| + C$$

Procedure (Integrals of Powers of Trig Functions). To solve $\int \sin^n(x)\,dx$ where n is even, substitute $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$.

To solve $\int \sin^n(x)\,dx$ where n is odd, substitute $\sin^2(x) = 1 - \cos^2(x)$ and perform u-sub with $u := \cos^2(x)$ and $du = -\sin(x)\,dx$.

To solve $\int \cos^n(x)\,dx$ where n is even, substitute $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$.

To solve $\int \cos^n(x)\,dx$ where n is odd, substitute $\cos^2(x) = 1 - \sin^2(x)$ and perform u-sub with $u := \sin^2(x)$ and $du = \cos(x)\,dx$.

To solve $\int \tan^n(x)\,dx$, substitute $\tan^2(x) = \sec^2(x) - 1$. If n is odd, perform u-sub with $u := \sec(x)$ and $du = \sec(x)\tan(x)\,dx$.

To solve $\int \cot^n(x)\,dx$, substitute $\cot^2(x) = \csc^2(x) - 1$. If n is odd, perform u-sub with $u := \csc(x)$ and $du = -\csc(x)\cot(x)\,dx$.

To solve $\int \sec^n(x)\,dx$ where n is even, substitute $\sec^2(x) = \tan^2(x) + 1$, but ensure that $\sec^2(x)\,dx$ remains. Then perform u-sub with $u := \tan(x)$ and $du = \sec^2(x)\,dx$.

To solve $\int \sec^n(x)\,dx$ where n is odd, substitute $\sec^n(x) = \frac{1}{\cos^n(x)}\frac{\cos(x)}{\cos(x)}$. Then substitute $\cos^2(x) = (1+u)(1-u)$ and u-sub with $u := \sin(x)$ and $du = \cos(x)dx$. Then perform partial fraction decomposition.

To solve $\int \csc^n(x)\,dx$ where n is even, substitute $\csc^2(x) = \csc^2(x) + 1$, but ensure that $\sec^2(x)\,dx$ remains. Then perform u-sub with $u := \cot(x)$ and $du = -\csc^2(x)\,dx$.

To solve $\int \csc^n(x)\,dx$ where n is odd, substitute $\csc^n(x) = \frac{1}{\sin^n(x)}\frac{\sin(x)}{\sin(x)}$. Then substitute $\sin^2(x) = (1+u)(1-u)$ and u-sub with $u := \cos(x)$ and $du = -\sin(x)dx$. Then perform partial fraction decomposition.

Remember that $\int \sec^2(x)\,dx = \tan(x)$ and $\int \csc^2(x)\,dx = -\cot(x)$.

Procedure (Trig Substitution).

Orig expression	Substitution	Pythagorean identity
$\sqrt{a^2 - x^2}$	$x := a\sin(\theta)$	$1 - \sin^2(\theta) = \cos^2(\theta)$
$\sqrt{a^2 + x^2}$	$x := a\tan(\theta)$	$1 + \tan^2(\theta) = \sec^2(\theta)$
$\sqrt{x^2 - a^2}$	$x := a\sec(\theta)$	$\sec^2(\theta) - 1 = \tan^2(\theta)$

Partial Fraction Decomposition

Theorem. Any polynomial $Q(x)$ with real coefficients can be factored over the reals as a product of two types of factors:

- linear factors (of the form $ax + b$)
- irreducible quadratic factors (of the form $ax^2 + bxx + c$, where $b^2 - 4ac < 0$)

Definition of **Proper rational function**. A rational function $\frac{P(x)}{Q(x)}$ where $\deg P < \deg Q$.

Theorem. Any rational function can be converted into a proper rational function plus a polynomial by continually long-dividing by the denominator.

Theorem. Let $R(x) = \frac{P(x)}{Q(x)}$ be a proper rational function, where the denominator $Q(x)$ has been factored into linear and irreducible quadratic factors. $R(x)$ can be written as a sum of partial fractions, where each factor in the denominator gives rise to terms in the partial fraction decomposition:

- For each factor of the form $(ax + b)^k$ in the denominator, add $\displaystyle\sum_{i=1}^{k} \frac{A_i}{(ax+b)^i}$ to the partial fraction decomposition.
- For each factor of the form $(ax^2 + bx + c)^k$ in the denominator, add $\displaystyle\sum_{i=1}^{k} \frac{A_i x + B_i}{(ax^2+bx+c)^i}$ to the partial fraction decomposition.

Riemann Sums

Definition of **Elementary function**. A function which is a polynomial, rational function, power function (x^a), exponential function (a^x), logarithmic functions, trigonometric and inverse trigonometric functions, or an addition, subtraction, multiplication, division, and composition of the above.

Definition of **Riemann sum**. Let $f : [a, b] \to \mathbb{R}$ be any bounded function and let P be a partition of $[a, b]$.

- A choice of a point $x_i^* \in [x_{i-1}, x_i]$ for all $i \in [1, n]$ is called a *tagging* of P, which we denote by $\tau = x_1^*, ..., x_n^*$.
- A pair (P, τ) is called a *tagged partition*.
- Given a bounded function $f : [a, b] \to \mathbb{R}$ and a tagged partition (P, τ) of $[a, b]$, the sum

$$R(f, P, \tau) = \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1})$$

is called the *Riemann sum* of f for (P, τ).

Theorem. If f is integrable on $[a, b]$, then for every $\varepsilon > 0$ there exists $\delta > 0$ s.t. if P is a partition of $[a, b]$ with $|P| := \max x_i - x_{i-1} < \delta$ and $\tau = \{x_i^*\}$, then

$$|R(f, P, \tau) - \int_a^b f(x)dx| < \varepsilon$$

Approximate Integration

Definition of **Left-endpoint approximation**. Take $x_i^* = x_{i-1} = a + \frac{(i-1)(b-a)}{n}$. Then the left-endpoint approximation of the function $f(x)$ is

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f\left(a + \frac{(i-1)(b-a)}{n}\right)\frac{b-a}{n}$$

The error bound is

$$E_n^L \leq \max\{|f'(x)|\}_{x\in[a,b]} \frac{(b-a)^2}{2n}$$

Definition of **Right-endpoint approximation**. Take $x_i^* = x_{i-1} = a + \frac{(i)(b-a)}{n}$. Then the right-endpoint approximation of the function $f(x)$ is

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f\left(a + \frac{(i)(b-a)}{n}\right)\frac{b-a}{n}$$

The error bound is

$$E_n^R \leq \max\{|f'(x)|\}_{x\in[a,b]} \frac{(b-a)^2}{2n}$$

Definition of **Midpoint approximation**. Take $x_i^* = x_{i-1} = a + \frac{(i-\frac{1}{2})(b-a)}{n}$. Then the midpoint approximation of the function $f(x)$ is

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f\left(a + \frac{(i-\frac{1}{2})(b-a)}{n}\right)\frac{b-a}{n}$$

The error bound is

$$E_n^M \leq \max\{|f''(x)|\}_{x\in[a,b]} \frac{(b-a)^3}{24n^2}$$

Definition of **Trapezoidal approximation**. Then the trapezoidal approximation of the function $f(x)$ is

$$\int_a^b f(x)dx \approx \frac{b-a}{2n}\left(f(a) + 2\sum_{i=1}^{n-1} f\left(a + \frac{(i)(b-a)}{n}\right) + f(b)\right)$$

The error bound is

$$E_n^T \leq \max\{|f''(x)|\}_{x\in[a,b]} \frac{(b-a)^3}{12n^2}$$

Definition of **Simpson's approximation**. For even n (greater values of n give more precise more accuracy):

$$\frac{b-a}{3n}\left(f(a) + \sum_{i=1}^{n-1}(3-(-1)^i) f\left(a + \frac{(i)(b-a)}{n}\right) + f(b)\right)$$

The error bound is

$$E_n^S \leq \max\{|f''''(x)|\}_{x\in[a,b]} \frac{(b-a)^5}{180n^4}$$

8 Integral Applications

Volumes

Generally, if $A(x)$ is the cross-section of a solid that intersects the x-axis at x, then the volume of the solid is

$$V = \int_a^b A(x)dx$$

Theorem (Disk Method). Let $f(x)$ be a continuous, nonnegative function defined on $[a, b]$, and R be the region bounded above by the graph of $f(x)$ and below by the x-axis. Then, the volume of the solid of revolution formed by revolving R around the x-axis is given by

$$V = \int_a^b \pi(f(x))^2 dx$$

Theorem (Washer Method). Let $f(x)$ be a continuous, nonnegative function defined on $[a, b]$, and R be the region bounded above by the graph of $f(x)$ and below by the graph of $g(x)$. Then, the volume of the solid of revolution formed by revolving R around the x-axis is given by

$$V = \int_a^b \pi((f(x))^2 - (g(x))^2)dx$$

Theorem (Cylindrical Shells Method). Let $f(x)$ be a continuous, nonnegative function defined on $[a, b]$, and R be the region bounded above by the graph of $f(x)$ and below by the x-axis. Then, the volume of the solid of revolution formed by revolving R around the y-axis is given by

$$V = \int_a^b 2\pi x f(x)dx$$

Other things

Theorem (Arc length of a curve). The arc length of the curve $f(x)$ on $[a,b]$, when $f'(x)$ exists and is continuous on $[a,b]$, is

$$\int_a^b \sqrt{1+(f'(x))^2}dx$$

Theorem (Surface area of a solid of revolution). Let $f(x)$ be a continuous, nonnegative function defined on $[a,b]$, and R be the region bounded above by the graph of $f(x)$ and below by the x-axis. Then, the surface area of the solid of revolution formed by revolving R around the x-axis is given by

$$S = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}dx$$

Theorem (Mass of a thin rod). Let there be a rod whose left end is located at $x=a$ and whose right end is located at $x=b$, and which is on the x-axis. Let $\rho(x)$ be the linear density at the point $x=a$. Then the mass of the rod is

$$M = \int_a^b \rho(x)\,dx$$

Theorem (Mass of a thin disk). Let there be a disk of radius R whose center is at the origin of the xy-plane. Let the mass be distributed in a rotationally-symmetric way. Let $\rho(r)$ be the radial density at the radius r. Then the mass of the disk is

$$M = \int_0^R 2\pi r\rho(r)dr$$

Theorem (Work). If an object moves along the x-axis from a to b, and $F(x)$ is the force applied to the object when the object is at the point x on the x-axis, then the work is

$$\int_a^b F(x)\,dx$$

Definition of **Average value of a function**. If f is continuous on $[a,b]$, then the average value of f on $[a,b]$ is

$$f_{\text{avg}} := \lim_{n\to\text{inf}} \frac{1}{b-a}\int_a^b f(x)\,dx$$

Theorem (Mean Value Theorem for Integrals). If f is continuous on $[a,b]$, then there exists $c \in [a,b]$ such that

$$f(c) = f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Equivalently,

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

If f is positive on $[a,b]$, then there is a number c such that the rectangle with base $[a,b]$ and height $f(c)$ has the same area as $\int_a^b f(x)\,dx$.

9 Improper Integrals

Definition of **Improper Integrals with Infinite Bounds**.

(a) If $\int_a^t f(x)$ exists for every $t \geq a$, then we define

$$\int_a^\infty f(x)dx := \lim_{t\to\infty}\int_a^t f(x)dx$$

provided that this limit exists and is finite.

(b) If $\int_t^b f(x)dx$ exists for every $t \leq b$, then we define

$$\int_{-\infty}^b f(x)dx := \lim_{t\to-\infty}\int_t^b f(x)dx$$

provided that this limit exists and is finite.

$\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are *convergent* if the corresponding limit exists and *divergent* if the limit doesn't exist.

(c) If both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are convergent, then we define

$$\begin{aligned}\int_{-\infty}^\infty f(x)dx &= \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx \\ &= \lim_{s\to-\infty}\int_s^a f(x)dx + \lim_{t\to\infty}\int_a^t f(x)dx\end{aligned}$$

Definition of **Improper Integrals with Discontinuous Integrand**.

(a) If f is continuous on $[a,b)$ and is discontinuous at b, then define

$$\int_a^b f(x)dx := \lim_{t\to b^-}\int_a^t f(x)dx$$

provided that this limit exists and is finite.

(b) If f is continuous on $(a,b]$ and is discontinuous at a, then define

$$\int_a^b f(x)dx := \lim_{t\to a^+}\int_t^b f(x)dx$$

provided that this limit exists and is finite.

$\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are *convergent* if the corresponding limit exists and *divergent* if the limit doesn't exist.

(c) If f has a discontinuity at c, where $a<c<b$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then we define

$$\begin{aligned}\int_a^b f(x)dx &= \int_a^c f(x)dx + \int_c^b f(x)dx \\ &= \lim_{s\to c^-}\int_a^s f(x)dx + \lim_{t\to c^+}\int_c^b f(x)dx\end{aligned}$$

Theorem (Comparison Test). Suppose f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

(a) If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.

(b) If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent.

Additionally,

(a) If $\int_{-\infty}^b f(x)dx$ is convergent, then $\int_{-\infty}^b g(x)dx$ is convergent.

(b) If $\int_{-\infty}^b g(x)dx$ is divergent, then $\int_{-\infty}^b f(x)dx$ is divergent.

Theorem (Limit Comparison Test). Suppose $f(x)$ and $g(x)$ are positive continuous functions defined on $[a,\infty)$ such that $\lim_{x\to\infty}\frac{f(x)}{g(x)} = c$ for some positive real number c. Then $\int_a^\infty f(x)dx$ converges iff $\int_a^\infty g(x)dx$.

10 Differential Equations

Definition of **Differential equation**. An equation involving an unknown function $y = f(x)$ and one or more of its derivatives.

Definition of **Solution to a differential equation**. A function $y=f(x)$ that satisfies the differential equation when f and its derivatives are substituted into the equation.

Procedure (Euler's Method). To numerically approximates the solution to the differential equation $y' = F(x,y)$ with $y(x_0) = y_0$,

$$y_n = y_{n-1} + F(x_{n-1}, y_{n-1})(x_n - x_{n-1})$$

Definition of **Separable Equation**. A separable equation is a differential equation where

$$\frac{dy}{dx} = g(x)f(y)$$

for some function $g(x)$ which depends only on x and $f(y)$ which depends only on y.

Theorem. For a separable equation,

$$\int \frac{1}{f(y)}dy = \int g(x)dx$$

Definition of **Logistic Differential Equation**. For a population P which increases exponentially ($\frac{dP}{dt} \approx kP$) when the population is small compared to the carrying capacity M but where the environment cannot sustain a population larger than M,

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

Then

$$P(t) = \frac{M}{1 + \left(\frac{M}{P_0} - 1\right)e^{-kt}}.$$

and

$$\frac{d^2P}{dt^2} = k^2P\left(1 - \frac{P}{M}\right)\left(1 - \frac{2P}{M}\right)$$

11 Sequences and Series

Definition of **Sequence**. A **sequence** is an ordered list of real numbers (**elements**) $a_1, a_2, a_3, \ldots$. It can also be considered a function $a : \mathbb{N} \to \mathbb{R}$.

Definition of **Convergence**. A sequance converges if $\lim_{n\to\infty} a_n$ exists.

Theorem. The Limit Laws and Squeeze Theorem also apply to limits of sequences.

Theorem. If $\{a_n\}$ is convergent, then it is bounded.

Theorem (Bounded Monotonic Sequence Theorem). Every bounded monotonic sequence converges. If $\{a_n\}$ is increasing, then $\lim\limits_{n\to\infty} a_n = \sup\{a_n : n \geq 1\}$. If $\{a_n\}$ is decreasing, then $\lim\limits_{n\to\infty} a_n = \inf\{a_n : n \geq 1\}$.

Definition of **Series**. The n**th partial sum** of $\{a_i\}_{i=1}^n$ is $s_n := \sum_{i=1}^n a_n$. Then the **series** is $\sum_{i=1}^\infty a_i := \lim_{n\to\infty} s_n$

Theorem (Convergence Test). If the series $\sum_{i=1}^n a_i$ is convergent, then $\lim_{i\to\infty} a_i = 0$.

Theorem (Divergence Test). If $\lim_{i\to\infty} a_i \neq 0$, then $\sum_{i=1}^n a_i$ diverges.

Theorem (Integral Test). If f is a continuous positive decreasing function for $x \geq 1$ and $a_n = f(n)$ for $n \in \mathbb{N}$, then

$$\sum_{n=1}^\infty a_n \qquad \text{and} \qquad \int_1^\infty f(x)\,dx$$

either both converge or both diverge.

Theorem (Remainder Estimate for Integral Test). If f is a continuous positive decreasing function for $x \geq 1$ and $a_n = f(n)$ for $n \in \mathbb{N}$, then

$$\int_{n+1}^\infty f(x)dx \leq R_n = s - s_n = \sum_{i=n+1}^\infty a_i \leq \int_n^\infty f(x)dx$$

and therefore

$$s_n + \int_{n+1}^\infty f(x)dx \leq \sum_{i=1}^\infty a_i \leq s_n + \int_n^\infty f(x)dx$$

Theorem (Comparison Test for Series). If $\sum a_n$ and $\sum b_n$ are series with positive terms:

- If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.
- If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also convergent.

Theorem (Limit Comparison Test for Series). If $\sum a_n$ and $\sum b_n$ are series with positive terms, then

- If $\lim_{n\to\infty} \frac{a_n}{b_n} \in \mathbb{R}^+$, then either both series converge or both series diverge.
- If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
- If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Theorem (Well-known series).
$\sum \frac{1}{n^p}$ (the p-**series**) converges if $p > 1$ and diverges if $p \leq 1$.

$\sum ar^{n-1}$ (the **geometric series**) converges if $|r| < 1$ and diverges if $|r| \geq 1$. If it converges, $\displaystyle\sum_{n=1}^\infty ar^{n-1} = \frac{a}{1-r}$

Definition of **Alternating series**. A series $\sum_{n=1}^\infty a_n = \sum_{n=1}^\infty (-1)^n b_n$ or $\sum_{n=1}^\infty (-1)^{n-1} b_n$ where $b_n > 0$.

Theorem (Alternating Series Test). If the alternating series $\sum_{n=1}^\infty (-1)^{n-1} b_n$ satisfies

(i) $b_{n+1} \leq b_n$ for all n, (i.e. $\{b_n\}$ is decreasing) and

(ii) $\lim\limits_{n\to\infty} b_n = 0$,

then the series is convergent.

Theorem (Alternating Series Divergence Test). If $\lim\limits_{n\to\infty} b_n \neq 0$, then the series is divergent.

Theorem (Alternating Series Estimation Theorem). If $s = \sum_{n=1}^\infty (-1)^{n-1} b_n$ is a series that satisfies the conditions of the Alternating Series Test, then

$$|R_n| := |s - s_n| \leq b_{n+1}$$

and therefore

$$s_n - b_{n+1} \leq s \leq s_n + b_{n+1}$$

Definition of **Absolute convergence**. A series $\sum_{n=1}^\infty a_n$ **converges absolutely** if the series of absolute values $\sum_{n=1}^\infty |a_n|$ converges.

Definition of **Conditional convergence**. A series **converges conditionally** if it converges but does not converge absolutely.

Theorem (Rearrangement of terms of absolutely convergent series). The terms of an absolutely convergent series can be rearranged without affecting the value/sum of the series.

Theorem (Absolute convergence implies convergence). If a series converges absolutely, then it converges.

Theorem (Ratio Test).

- If $\lim_{n\to\infty} \left|\frac{a_n+1}{a_n}\right| < 1$, then the series $\sum_{n=1}^\infty a_n$ converges absolutely and therefore converges.
- If $\lim_{n\to\infty} \left|\frac{a_n+1}{a_n}\right| > 1$ or $\lim_{n\to\infty} \frac{a_n+1}{a_n} = \infty$, then the series $\sum_{n=1}^\infty a_n$ diverges.
- If $\lim_{n\to\infty} \left|\frac{a_n+1}{a_n}\right| = 1$, the Ratio Test is inconclusive.

Theorem (Root Test).

- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$, then the series $\sum_{n=1}^\infty a_n$ converges absolutely and therefore converges.
- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^\infty a_n$ diverges.
- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Ratio Test is inconclusive.

12 Power Series

Definition of **Power series**. A power series centered at a is a series of the form

$$\sum_{k=0}^\infty c_k(x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

where x is a variable, a is a fixed real number called the *center* of the series, and the c_ks are constants called the *coefficients* of the series.

Theorem. For a power series $\sum_{k=0}^\infty c_k(x-a)^k$, there are only three possibilities:

(i) The series converges only when $x = a$.

(ii) The series converges for all x.

(iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$. R is called the **radius of convergence**.

Theorem. Suppose $\sum_{n=0}^\infty c_n(x-a)^n$ has radius of convergence $R > 0$. Then $f(x) = \sum_{n=0}^\infty c_n(x-a)^n$ is differentiable (and therefore continuous and integrable) on $(a-R, a+R)$, and

(i) $f'(x) = \displaystyle\sum_{n=0}^\infty \frac{d}{dx} c_n(x-a)^n = \sum_{n=0}^\infty nc_n(x-a)^{n-1}$

(ii) $\displaystyle\int f(x)dx = \sum_{n=0}^{\infty} \int c_n(x-a)^n = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$

Definition of **Taylor series**. The **Taylor series** of a function f centered at a is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Definition of **Maclaurin series**. The **Maclaurin series** of a function f is the Taylor series of f centered at a:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

Theorem. If f has a power series representation centered at a, then that power series is equivalent to the Taylor series of f centered at a.

Theorem (Basic Maclaurin Series).

$$\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} && \text{for } x \in \mathbb{R}\\
\sin(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} && \text{for } x \in \mathbb{R}\\
\cos(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} && \text{for } x \in \mathbb{R}\\
\ln(1+x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} && \text{for } |x| < 1\\
\arctan(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} && \text{for } |x| < 1\\
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n && \text{for } |x| < 1\\
\frac{1}{(1-x)^2} &= \sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty} (n+1)x^n && \text{for } |x| < 1
\end{aligned}$$

Definition of **Taylor polynomial**. The nth-degree Taylor polynomial of f at a is

$$\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Theorem (Taylor's Theorem). If f has $(n+1)$ continuous derivatives on an open interval I ccontaining a, then for all x in I,

$$f(x) = T_n(x) + R_n(x),$$

where T_n is the nth-order Taylor polynomial for f with center a and the remainder is

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some point c between x and a. This form of R_n is also called **Lagrange error bound** or the **Lagrange form of the remainder**.

Theorem. If R_n converges to 0, then $f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$.

13 Polar

Theorem. If r is a function of θ on the xy-plane, then

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \qquad \text{and} \qquad \frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\theta}}$$

Theorem (Arc length). If r is a function of θ on the xy-plane, then

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

and therefore

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

14 Vectors

$$\frac{d}{dt}\vec{r}(t) = \begin{matrix} x'(t) \\ y'(t) \\ z'(t) \end{matrix}$$

$$\int \vec{r}(t)dt = \begin{matrix} \int x(t)dt \\ \int y(t)dt \\ \int z(t)dt \end{matrix}$$

Multivariable Calculus (6)

Colley (2012) refers to *Vector Calculus, Fourth Edition* by Susan Jane Colley.

6.1 Vectors, lines, planes

6.1.1 $\mathbb{R}^n$

Definition of **Two-dimensional real-coordinate space** ($\mathbb{R}^2$).

$$\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$$

Definition of **Three-dimensional real-coordinate space** ($\mathbb{R}^2$).

$$\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$$

x, y, z should be presented such that the coordinate system is right-handed ($\hat{k} = \hat{\imath} \times \hat{\jmath}$ should have direction according to the right-hand rule).

Definition of **Standard basis vectors**. The standard basis vectors of a space are the unit vectors that go along the axes of the space. All vectors in that space can be expressed as sums of scalar multiples of the standard basis vectors of that space.

The standard basis vectors of $\mathbb{R}^2$ are $\hat{\imath}$ and $\hat{\jmath}$, also called $\mathbf{e_1}$ and $\mathbf{e_2}$.

The standard basis vectors of $\mathbb{R}^3$ are $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$, also called $\mathbf{e_1}$, $\mathbf{e_2}$, and $\mathbf{e_3}$.

6.1.2 Vectors

See the properties of fields in the LinAlg notes for definitions of addition and scalar multiplication.

Definition of **Displacement vector**. The vector from the end of $\vec{A}$ to the end of $\vec{B}$ when their starts are in the same location.

$$\overrightarrow{AB} = \vec{B} - \vec{A}$$

6.1.3 Dot and cross products

Definition of **Dot product**. Where $a, b \in \mathbb{R}^n$ and θ is the angle between a and b:

$$a \cdot b = \sum a_i b_i = |a||b| \cos\theta$$

Theorem (Properties of dot product). For $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^n$ and $k \in \mathbb{R}$:

- $\vec{a} \cdot \vec{a} = |\vec{a}|^2$
- $\vec{a} \cdot \vec{a} = 0$ iff $\vec{a} = 0$
- Commutativity: $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
- Distributivity: $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
- Distributivity: $(k\vec{a}) \cdot b = k(\vec{a} \cdot \vec{b}) = a \cdot (k\vec{b})$
- $\vec{a} \cdot \vec{b} = 0$ iff $a \perp b$, $a = 0$, or $b = 0$.

Definition of **Cross product**. For $\vec{a}, \vec{b} \in \mathbb{R}^3$, the unique vector $a \times b$ satisfying

- $|a \times b|$ is the area of the parallelogram spanned by a and b
- $a \times b = 0$ iff $a \parallel b$, $a = 0$, or $b = 0$.
- $a \times b$ is orthogonal to a and b.
- $(a, b, a \times b)$ is right-handed (if the coordinate system is right-handed)

Theorem (Properties of cross product). For $a, b, c \in \mathbb{R}^3$ and $k \in \mathbb{R}$:

- $a \times b = (-b) \times a$
- $a \times (b + c) = a \times b + a \times c$
- $(a + b) \times c = a \times c + b \times c$
- $k(a \times b) = (ka) \times b = a \times (kb)$

Theorem (Calculation of cross product). Where $a, b \in \mathbb{R}^n$ and θ is the angle between a and b:

$$a \times b = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$
$$= \hat{\imath} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} + \hat{\jmath} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$
$$|a \times b| = |a||b| \sin\theta$$

6.1.4 Lines

The most useful notation for a line is in parametric form:

Definition of **Parametric form of a line**. Where $r_0 \in \mathbb{R}^3$ is a point on the line, and $t \in \mathbb{R}^3$:

$$r(t) = r_0 + vt$$

t is called the **direction vector**.

Theorem. Two lines are parallel iff their direction vectors are scalar multiples of each other.

Definition of **Skew lines**. Two lines are skew iff they do not intersect but are not parallel, i.e. they lie in different parallel planes.

Procedure (Finding the intersection of two lines). Given lines with parametric equations

$$r_1(t) = a_1 + v_1 t \qquad r_2(t) = a_2 + v_2 t$$

solve the system of equations

$$a_1 + v_1 t_1 = a_2 + v_2 t_2$$

for t_1 or t_2, then plug it in to the appropriate equation.

(Break up the equation into its x, y, and z components, or whichever is appropriate for your coordinate space.)

6.1.5 Planes

Colley [1.5]

Definition of **Plane**. A plane Π is determined uniquely by a point P in the plane and a normal vector n.

A plane is the set of points A in space such that $\vec{AP}$ is perpendicular to n.

Theorem (Scalar equation for a plane in $\mathbb{R}^3$). If $n \in \mathbb{R}^3$ is the **normal vector** to the plane (vector perpendicular to the plane), and $P \in \mathbb{R}^3$ is a point on the plane:

$$n_x(x - P_x) + n_y(x - P_y) + n_z(z - P_z) = 0$$

or equivalently:

$$n_x x + n_y y + n_z z = n_x P_x + n_y P_y + n_z P_z$$

Procedure (Equation of plane containing three points). If $A, B, C \in \mathbb{R}^3$ are points on our plane, then we can find the normal vector by performing
$n = \overrightarrow{AB} \times \overrightarrow{AC} = (B - A) \times (C - A)$ (since $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are vectors on the plane).

Theorem (Parametric equation for a plane in $\mathbb{R}^3$). If $a, b \in \mathbb{R}^3$ are nonparallel nonzero vectors on the plane, and $P \in \mathbb{R}^3$ is a point on the plane, then the parametric equation for the plane is:

$$x(s, t) = P + sa + tb$$

6.1.6 Distance

Colley [1.5]

Procedure (Distance between point and line). Let P be the point, and $A + Lt$ be the line. Then the distance is

$$|\overrightarrow{AP} - \operatorname{proj}_L \overrightarrow{AP}| = \vec{P}$$

Procedure (Distance between parallel planes). Let Π_1 and Π_2 be the two planes.

If n is normal to both planes, and $P_1 \in \Pi_1$ and $P_2 \in \Pi_2$, then the answer is

$$|\operatorname{proj}_n \overrightarrow{P_1P_2}$$

6.1.7 Cylindrical and spherical coordinates

Trimm [5.6, 5.7] Brummet [08, MVCWUP:Feb3(29-33)]
Colley [1.7]

Definition of **Cylindrical coordinate**. An ordered pair (r, θ, z) where r is the distance between the point and the z-axis, θ is the angle counterclockwise from the positive x-axis along the xy-plane, and z is the position on the z-axis.

Definition of **Spherical coordinate**. An ordered pair (ρ, ϕ, θ), where ρ is the distance between the point and the origin, ϕ is the angle clockwise from the positive z-axis going downwards towards the xy-plane, and θ is the angle counterclockwise from the positive x-axis along the xy-plane.

Typically we use the following restrictions:

$$\rho > 0 \qquad 0 \leq \theta \leq 2\pi \qquad 0 \leq \phi \leq \pi$$

Theorem (Useful formulas).

$$r = \rho \sin\phi \qquad z = \rho\cos\phi$$

$$x = \rho\sin\phi\cos\theta \qquad y = \rho\sin\phi\sin\theta$$

$$r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta$$

6.2 Functions, limits, differentiation

6.2.1 Multivariable functions

Trimm [3.1, 3.2] Brummet [MVCWUP:Feb4(35-41)] Colley [2.1]

Definition of **Function**. All functions $f : X \Rightarrow Y$ are defined by:

- A domain set X
- A codomain set Y
- A rule of assignment that associates a unique element $y \in Y$ to each element $x \in X$

Definition of **Graph**. The graph of $f : X \subseteq \mathbb{R}^n \Rightarrow \mathbb{R}$ is the set

$$\{(x_1, \dots, x_n, f(x)) : x = (x_1, \dots, x_n)\}$$

Specifically, for $f : \mathbb{R}^2 \Rightarrow \mathbb{R}$ the graph is the set

$$\{(x, y, z) : (x, y) \in X \text{ and } z = f(x, y)\}$$

Definition of **Level set**. Let $f : X \subseteq \mathbb{R}^n \Rightarrow \mathbb{R}$. The **level set at height c of f** is the set in $\mathbb{R}^n$ defined by the equation $f(\vec{a}) = c$, where c is a constant. This is equivalent to the set

$$\{\vec{x} \in \mathbb{R}^n : f(\vec{x}) = c\}$$

In $\mathbb{R}^2$, this is also called a **level curve**.

Definition of **Contour set**. Let $f : X \subseteq \mathbb{R}^n \Rightarrow \mathbb{R}$. The **contour set at height c of f** is the set in $\mathbb{R}^{n+1}$ defined by the two equations $z = f(\vec{a})$ and $z = c$, where c is a constant. This is equivalent to the set

$$\{\vec{x} \in \mathbb{R}^{n+1} : z = f(\vec{x}) = c\}$$

If $f : X \subseteq \mathbb{R}^2 \Rightarrow \mathbb{R}$, this is also called a **contour curve**. It is equivalent to the level curve, except it is located in $\mathbb{R}^3$ rather than $\mathbb{R}^2$.

6.2.2 Limits

Trimm [3.3, DiffEq-1.0] Brummet [09, MVCWUP:Feb11(42-46)]
Colley [2.2]

Definition of **Limit**. $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = \vec{L}$ if for all $\varepsilon > 0$, there exists $\delta > 0$ s.t. if $0 < |\vec{x} - \vec{a}| < \delta$ then $|f(\vec{a}) - \vec{L}| < \varepsilon$.

Definition of **Continuity**. Let $f : X \subseteq \mathbb{R}^n \Rightarrow \mathbb{R}^m$ and let $\vec{a} \in X$. Then f is continuous at point $\vec{a}$ iff

$$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f(\vec{a})$$

If f is continuous at all $\vec{a} \in X$, then we say that f is continuous.

6.2.3 Differentiation

Trimm [3.4, DiffEq-1.0]
Brummet [11, 12.5, MVCWUP:Feb12/22(48-54)]
Colley [2.3, 2.4]

Definition of **Partial derivative with respect to x**. The partial derivative of $f(x, y)$ with respect to x is

$$\lim_{h\to 0} \frac{f(a+h, b) - f(a, b)}{h}$$

Let $z = f(x, y)$. Then the partial derivative is denoted by

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = D_x f$$

Definition of **Partial derivative**. The partial derivative of $f(\vec{x})$ with respect to the ith variable is

$$\frac{\partial f(\vec{x})}{\partial x_i} = \lim_{h\to 0} \frac{f\left(\begin{bmatrix} x_0 \\ \vdots \\ x_i + h \\ \vdots \\ x_n \end{bmatrix}\right) - f(\vec{x})}{h}$$

This is equivalent to letting $F(x_i) = f(\vec{x})$ and finding $F'(x_i)$.

Definition of **Higher-order partial**. The result of taking the partial derivative of a partial derivative, which may be higher-order.

A partial derivative that is not higher-order is called a **first-order partial**. A partial derivative of a first-order partial is a second-order partial, a partial derivative of a second-order partial is a third-order partial, etc.

A higher-order partial which is the result of taking the partial with respect to x_1, then with respect to x_2, then with respect to $x_3, \dots, x_n$, is denoted by

$$f_{x_1 \cdots x_n} = \frac{\partial}{\partial x_n} \cdots \frac{\partial}{\partial x_1} f$$

$x_1 \cdots x_n$ do not have to be distinct. If $x_1 \cdots x_n$ are not all the same then the higher-order partial is called a **mixed partial derivative**.

Theorem. Let $f : X \in \mathbb{R}^n \Rightarrow \mathbb{R}$ whose k-th order and lower partials exist and are continuous on X. Then its k-th order and lower partials may be evaluated in any order, i.e.

$$f_{x_1 \cdots x_n} = f_{x_n \cdots x_1} = f_{x_1 x_3 x_{27} \cdots x_4} = \cdots$$

Definition of **Gradient**.

$$\nabla f(\vec{x}) = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{bmatrix}$$

6.2.4 Implicit surfaces

Brummet [12]

Definition of **Implicit surface**. A surface in $\mathbb{R}^3$ defined by an equation which is not solved for x, y, nor z.

We often express it as

$$F(x, y, z) = 0,$$

in which case the surface is the set of points which satisfy $F(x, y, z) = 0$.

Theorem. The gradient $\nabla F(\vec{x})$ is the normal vector to the tangent plane to the implicit surface defined by $F(\vec{x}) = k$, where k is a constant.

Equivalently, if x_0 is a point on the level set $S = \{x \in X : F(x) = k\}$ where $F : X \subseteq \mathbb{R}^n \Rightarrow \mathbb{R}$, then the vector $\nabla F(x_0)$ is perpendicular to S.

6.2.5 Chain rule

Trimm [3.8, 6.5, DiffEq-1.0] Brummet [12, MVCWUP:Feb24(58-61)]

Colley [2.5]

Definition of **Jacobian**. If $f : X \subseteq \mathbb{R}^n \Rightarrow \mathbb{R}^m$ is a vector-valued function, then the Jacobian is

$$Df\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} \nabla f_1 \\ \nabla f_2 \\ \vdots \\ \nabla f_m \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{x_n} \end{bmatrix}$$

Theorem (Multivariable chain rule). Suppose $X \subseteq \mathbb{R}^m$ and $T \subseteq \mathbb{R}^n$ are open and $f : X \Rightarrow \mathbb{R}^p$ and $r : T \Rightarrow \mathbb{R}^m$ are defined so that $T \subseteq X$. If x is differentiable at $t_0 \in T$ and f is differentiable at $x_0 = r(t_0)$, then the composite $f \circ r$ is differentiable at t_0, and we have

$$(f \circ r)'(t) = \nabla f(x_0) \cdot r'(t_0)$$

Equivalently,

$$D(f \circ r)(t_0) = Df(x_0)Dr(t_0)$$

In $\mathbb{R}^3$,

$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt}$$

6.2.6 Paths

Brummet [MVCWUP:Feb24(55-56)]

Definition of **Path**. A path in $\mathbb{R}^n$ is a function $x : I \Rightarrow \mathbb{R}^n$, where I is a set of scalars. If $I = [a, b]$, then the endpoints of the path are $f(a)$ and $f(b)$.

Definition of **Tangent vector**. Given a path $r : \mathbb{R} \Rightarrow \mathbb{R}^3$, the tangent vector to said path at some point P is given by $r'(t)$, provided that $r'(t) \neq 0$. In $\mathbb{R}^3$,

$$r'(t) = \lim_{h \to 0} \frac{r(t+h) - r(t)}{h} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{bmatrix}$$

Definition of **Derivative of vector-valued function**. Let $f : T \subseteq \mathbb{R} \Rightarrow \mathbb{R}^m$. Then

$$f'(t) = \begin{bmatrix} f_1'(t) \\ f_2'(t) \\ \vdots \\ f_m'(t) \end{bmatrix}$$

6.2.7 Differentiability

Trimm [3.5] Brummet [13.5] Colley [2.3]

Definition of **Linear approximation** $(\mathbb{R}^n \Rightarrow \mathbb{R})$. The **linear approximation** or **tangent plane** $(\mathbb{R}^3)$ **or hyperplane** to the graph of a function f at the point $\vec{a}$ is expressed by

$$L(\vec{x}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a})$$

In $\mathbb{R}^3$, this is equivalent to the plane

$$z = L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Definition of **Linear approximation** $(\mathbb{R}^n \Rightarrow \mathbb{R}^m)$. The **linear approximation** to a vector-valued function f at the point $\vec{a}$ is expressed by

$$L(\vec{x}) = f(\vec{a}) + Df(\vec{a})(\vec{x} - \vec{a})$$

Definition of **Differentiability**. Let $f : X \subseteq \mathbb{R}^n \Rightarrow \mathbb{R}^m$, where X is an open subset of $\mathbb{R}^n$, and let $\vec{a} \in X$. f is differentiable at a iff all of its partial derivatives exist and

$$\lim_{\vec{x} \to \vec{a}} \frac{f(\vec{x}) - L(\vec{x})}{|\vec{x} - \vec{a}|} = 0$$

where $L(\vec{x})$ is the linear approximation to f at $\vec{a}$.

Theorem (Differentiability shortcut). Let $f : X \subseteq \mathbb{R}^n \Rightarrow \mathbb{R}^m$ be a vector-valued function. If all partial derivatives $\frac{\partial f_i}{\partial x_j}$ exist and are continuous in a neighborhood of $\vec{a}$ in X, then F is differentiable at $\vec{a}$.

6.2.8 Directional derivative

Trimm [3.7] Brummet [14] Colley [2.6]

Definition of **Directional derivative**. Let $f : X \subseteq \mathbb{R}^n \Rightarrow \mathbb{R}$, where X is an open subset of $\mathbb{R}^n$, and let $\vec{a} \in X$. If $\vec{v}$ is any unit vector in X, then the directional derivative of f at a in the direction of v is

$$D_{\vec{v}}f(\vec{a}) = \lim_{h \to 0} \frac{f(\vec{a} + h\vec{v}) - f(\vec{a})}{h}$$

Theorem. If f is differentiable at a, then

$$D_{\vec{v}}f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{v}$$

Theorem. The gradient is the path of steepest ascent, i.e.

$$D_{\widehat{\nabla f(\vec{a})}}f(\vec{a}) = \max\{D_{\vec{v}}f(\vec{a}) : \vec{v} \in \mathbb{R}^n\}$$

where $f : X \subseteq \mathbb{R}^n \Rightarrow \mathbb{R}$.

Theorem. Let $f : X \subseteq \mathbb{R}^2 \Rightarrow \mathbb{R}$, and let $(a, b, c) \in \mathbb{R}^3$. Then $\nabla f(a, b)$ is orthogonal to the level curve at height c.

6.3 unit 3

6.3.1 Extrema

Theorem. If f has a local maximum or minimum at (a, b) and the first order partial derivatives exist, then $f_x(a, b) = f_y(a, b) = 0$.

Extreme Value Theorem. Let X be a closed and bounded subset of $\mathbb{R}^n$ and suppose $f : X \Rightarrow \mathbb{R}^n$ is continuous. Then f attains an absolute maximum and an absolute minimum somewhere on X.

Definition of **Critical point of** f. A point $\vec{c}$ in the domain of f where all of the partial derivatives of f at $\vec{c}$ equal 0.

Definition of **Saddle point**. A critical point that is not a max or min.

Linear Algebra (7)

Let $F = \mathbb{R}$ or $F = \mathbb{C}$.

1 Linear equations

Definition of **Linear equation**. An equation that can be written in the form

$$\sum_k a_k x_k = y$$

where all $a_k \in F$ and $y \in F$.

Definition of **Solution to a linear equation**. The solution to a linear equation is a set $\{s_k\}$ such that $\sum_k a_k s_k = y$, i.e. substituting $x_k = s_k$ results in the equation being true.

Definition of **Linear system**. A set of linear equations.

Let m be the number of linear equations in the system. Let n be the number of variables in the system. Then the jth equation can be written as

$$\sum_{k=1}^{n} A_{jk} x_k = y_j$$

Let:

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad Y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

Then the system can be written as $AX = Y$.

Definition of **Consistent linear system**. A system that has at least one solution.

Definition of **Linear combination**. The linear combination of the equations of a linear system is a linear equation formed by multiplying each equation by c_j where $c_j \in F$.

This linear combination can be written as

$$\sum_{j=1}^{m} \sum_{k=1}^{n} c_j A_{jk} x_k = \sum_{j=1}^{m} y_j$$

Theorem. All solutions of a linear system are solutions to the linear combination of the equations of the system.

Definition of **Equivalent linear systems**. Two systems are equivalent if they have the same set of solutions.

Theorem. Two systems are equivalent if each equation in each system is a linear combination of the equations in the other system.

7.0.1 Matrices and rows

Definition of **Elementary row operations**. The elementary row operations are:

Definition of **Scaling**. $R_i \mapsto cR_i$ where c is a nonzero scalar.

Definition of **Replacement**. $R_i \mapsto R_i + cR_j$ where c is a scalar.

Definition of **Interchange**. Swap R_i and R_j.

Theorem (Elementary row operations are invertible). For any elementary row operation e, there exists an elementary row operation e^{-1} such that $e^{-1}(e(A)) = A$ for any matrix A.

Definition of **Row equivalence**. Two matrices are row-equivalent if each can be derived from the other using a finite number of elementary row operations.

Definition of **Row echelon form (REF)**. A matrix is in REF if it satisfies:

1. All nonzero rows are above all rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

Definition of **Reduced row echelon form (RREF)**. A matrix is in RREF if it is in REF and additionally satisfies:

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column

Definition of **Pivot position**. A location A_{ij} where $\mathrm{RREF}(A)_{ij}$ is a leading 1.

Definition of **Pivot column**. A column which contains a pivot position.

Definition of **Pivot**. A nonzero number at a pivot position.

Procedure (Gauss-Jordan elimination).

Procedure (Gaussian elimination). Iterate through the pivot columns of A from left to right. For each pivot column, use elementary row operations to ensure that the pivot position is nonzero and that all entries in the column below the pivot position are zero. This produces $\mathrm{REF}(A)$.

Procedure (Jordan elimination). Iterate through the pivot columns of $\mathrm{REF}(A)$ from right to left. For each pivot column, use elementary row operations to ensure that all other entries in the column other than the pivot are zero and that the pivot is equal to 1. This produces $\mathrm{RREF}(A)$.

Definition of **Leading variable, determined variable, basic variable**. A variable in a pivot column.

Definition of **Free variable**. A variable not in a pivot column.

7.0.2 Homogeneous linear systems

Definition of **Homogeneous linear system**. A system where $y_0 = y_1 = \cdots = y_m = 0$. It can be written as $AX = 0$.

Theorem (Trivial solution). For any homogeneous system, $x_0 = x_1 = \cdots = x_n = 0$ is a solution to the system. Therefore, all homogeneous systems are consistent.

Theorem. (a) If there are less equations than there are variables ($m < n$), then $AX = 0$ has an infinite number of solutions.

(b) If there are an equal number of equations and variables, then A is row-equivalent to the $n \times n$ identity iff $AX = 0$ has only the trivial solution.

(c) If there are more equations than there are variables ($m > n$), then

$$\mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

iff $AX = 0$ has only the trivial solution.

Procedure (Solution). To solve a homogeneous system, perform Gauss-Jordan elimination on A so that $R = \mathrm{RREF}(A)$. Then solve $RX = 0$. The variables which are not in pivot columns are free variables and may be set to any value, typically denoted $u_1, u_2, \ldots$.

7.0.3 Inhomogeneous linear systems

Procedure (Solution). To solve an homogeneous system, perform Gauss-Jordan elimination on $A' = [A|Y]$ so that $R' = [R|Z] = \mathrm{RREF}(A')$. Then solve $RX = Z$. Note that not all inhomogeneous systems are solvable (consistent).

2 Fields

7.0.4 Real and complex numbers

Definition of **Field properties**. The following properties:

Theorem (Properties of addition). For all $x, y, z \in F$:

(A1) Commutativity: $x + y = y + x$

(A2) Associativity: $(x + y) + z = x + (y + z)$

(A3) Identity: $\exists 0 \in \mathbb{R}$ s.t. $0 + x = x$

(A4) Additive inverse: For $x \in F$, $\exists -x \in F$ s.t. $x + (-x) = 0$

Theorem (Properties of multiplication). For all $x, y, z \in F$:

(M1) Commutativity: $xy = yx$

(M2) Associativity: $(xy)z = x(yz)$

(M3) Identity: $\exists 1 \in \mathbb{R}$ s.t. $1x = x$ and $1 \neq 0$

(M4) Additive inverse: For $x \in F \setminus \{0\}$, $\exists x^{-1} \in F$ s.t. $xx^{-1} = 1$

Theorem (Distributive property).

(D) $x(y + z) = xy + xz$ for all $x, y, z \in F$.

7.0.5 Fields

Definition of **Field**. A set F which defines the following two operations:

- Addition: an operation that maps $x, y \in F \Rightarrow c \in F$ and satisfies the properties of addition
- Multiplication: an operation that maps $x, y \in F \Rightarrow c \in F$ and satisfies the properties of multiplication

for which the distributive property also holds.

Theorem. $\mathbb{R}$ and $\mathbb{C}$ are fields.

Definition of **Complex number**. A number which can be defined by a pair of real numbers a, b where the value of the number is equal to $a + bi$.

Theorem (Useful things about complex numbers). Let $z = a + bi$ and $w = c + di$ be complex numbers. Then:

- $z + w = (a + c) + (b + d)i$
- $zw = (ac - bd) + (bc + ad)i$

Definition of F^n. For a field F, F^n is the set of all ordered n-tuples of elements of F:

$$F^n := \{(x_1, \dots, x_n) : x_1, \dots, x_n \in F\}$$

Definition of **Addition in** F^n. If $a, b \in F^n$:

$$a + b = (a_1 + b_1, \dots, a_n + b_n)$$

Addition follows the properties of additon (A1-A4).

Definition of **Product of element of F and element of F^n**. If $\alpha \in F$ and $x \in F^n$, then

$$\alpha x = (\alpha x_1, \dots, \alpha x_n)$$

2.3 Vector spaces

Definition of **Vector space**. A vector space over F is a set V with the following operations:

- **Vector addition**: $u \in V, v \in V \mapsto (u + v) \in V$, which satisfies the properties of addition
- **Scalar multiplication**: $\alpha \in F, v \in V \mapsto \alpha v \in V$, which satisfies the properties of scalar multiplication

Definition of **Properties of addition**. For all $u, v, w \in V$:

(A1) Commutativity: $u + v = v + u$

(A2) Associativity: $(u + v) + w = u + (v + w)$

(A3) Identity: $\exists 0 \in \mathbb{R}$ s.t. $0 + u = u$

(A4) Additive inverse: For $u \in F$, $\exists -u \in F$ s.t. $u + (-u) = 0$

Definition of **Properties of scalar multiplication**. For all $\alpha, \beta \in F$, $v, w \in V$:

(S1) Associativity: $(\alpha\beta)v = \alpha(\beta v)$

(S2) Distributivity over scalar addition: $(\alpha + \beta)v = \alpha v + \beta v$

(S3) Distributivity over vector additon: $\alpha(v + w) = \alpha v + \alpha w$

(S4) Multiplicative identity: $1v = v$

Theorem. F^n is a vector space.

Theorem. All inverses and identities are unique in a vector space.

Definition of F^ω. The set of all sequences of elements of F:

$$F^\omega := \{(x_1, x_2, \dots) : x_k \in F \text{ for } k \in \mathbb{N}\}$$

where addition and scalar multiplication are defined similarly to F^n:

$$a + b := (a_1 + b_1, \dots, a_n + b_n)$$
$$\alpha x := (\alpha x_1, \dots, \alpha x_n)$$

Definition of $F^{m,n}$. The set of all $m \times n$ matrices with entries in F, where addition and scalar multiplication are defined as:

$$(A + B)_{ij} := A_{ij} + B_{ij}$$
$$(\alpha A)_{ij} := \alpha A_{ij}$$

Theorem (Vector space of functions). Let V be a vector space, S be a set, and

$$V^S = \{f : S \to V\}$$

(the set of all functions that map members of S to members of V). Then V^S is a vector space, if we define

$$(f + g)(s) = f(s) + g(s) \qquad (\alpha f)(s) = \alpha(f(s))$$

7.0.6 Subspaces

Definition of **Subspace**. Let V be any vector space, and let W be a subset of V. Define vector addition and scalar multiplication on W by restricting the corresponding operations of V to W. If W is a vector space with respect to the restricted operations of V , then W is said to be a subspace of V.

Definition of **Closed**. An operation is closed under a set if applying the operation to elements of the set always results in an element of the set.

Definition of **Subspace**. A subspace of a vector space V is a subset W of V which contains the zero vector and is closed under addition and scalar multiplication.

Theorem. A subset W of a vector space V is a subspace iff

(i) W is nonempty

(ii) $\alpha \in F$ and $w_1, w_2 \in W$ implies $\alpha w_1 + w_2 \in W$

Typically, we prove (i) by proving that $0 \in W$.

7.0.7 Subspaces of F^n

Theorem (Subspaces of $\mathbb{R}^n$). $\mathbb{R}^n$ contains the following subspaces:

- $\{0\}$
- $\mathbb{R}^n$
- Any line through the origin
- Any plane through the origin
- etc

Definition of **Spanning**. For vectors to span a space is for it to be sufficient to be able to reach any point in the space using the vectors.

Definition of **Independence**. If a vector can be made out of other vectors, then the vector is independent

7.0.8 Intersections and unions of subspaces

Theorem. The intersection of any collection of subspaces of V is a subspace of V.

Theorem. The union of two subspaces of V is a subspace of V iff one of the subspaces is contained in the other.

Definition of **Sum of subspaces**. If U and W are subspaces of a vector space V, then

$$U + W := \{u + w : u \in U \text{ and } w \in W\}$$

Theorem. If U and W are subspaces of V, then $U + W$ is the smallest subspace of V containing both U and W.

Definition of **Direct sum**. If $V_1, \dots, V_n$ are subspaces of V such that each element of $+_{k=1}^{n} V_k = V_1 + \dots + V_n$ can be written uniquely as $\sum_{k=1}^{n} v_k = v_1 + \dots + v_n$ where $v_k \in V_k$, then $+_{k=1}^{n} V_k$ is a **direct sum** and can be written as $\bigoplus_{k=1}^{n} V_k = V_1 \oplus \dots \oplus V_n$.

Theorem. Let $V_1, \dots, V_n$ be subspaces of V. Then they are direct sums iff the only way to write $0 = v_1 + \dots + v_n$ is to take $v_1 = \dots = v_n = 0$.

Theorem. If U and W are subspaces of V, then $U + W$ is a direct sum iff $U \cap W = \{0\}$. This does not generalize to higher numbers of subspaces.

7.0.9 Spanning

Definition of **Linear combination**. A linear combination of a collection $v_1, \dots, v_n$ of vectors in vector space V is a vector of the form

$$\alpha_1 v_1 + \dots + \alpha_n v_n$$

where each $\alpha_k \in F$.

Definition of **Span**. Given $W \subseteq V$ where V is a vector field, the set of all linear combinations of vectors in W is called the span of W.

$$\operatorname{span}(W) := \left\{ \sum_{i=1}^{n} \alpha_i w_i : \alpha_i \in F, w_i \in W \right\}$$

Additionally, we define

$$\operatorname{span}(\emptyset) = \{0\}$$

Definition of **Subspace generated by a set**. Given $W \subseteq V$ where V is a vector field, the subpace generated by W is the smallest subspace of V containing W, or equivalently the intersection of all subspaces of V containing W.

Theorem. The span of W is the subspace generated by W, i.e. W is the smallest subspace of V containing W.

Definition of **Spanning, spanning set**. If $\operatorname{span}(W) = V$, then W spans V and W is a spanning set for V.

Definition of **Finite-dimensional vector space**. A vector space is finite-dimensional iff it has a finite spanning set.

Otherwise, it is infinite-dimensional.

7.0.10 Linear independence

Definition of **Linear independence**. Let V be a vector space. If $W \subseteq V$ is a finite set, it is linearly independent iff the only way to write 0 as a combination

$$\alpha_1 v_1 + \dots + \alpha_m v_m = 0$$

is by taking $\alpha_1 = \dots = \alpha_m = 0$. We also define $\emptyset$ to be linearly independent.

If $W \subseteq V$ is an infinite set, it is linearly independent if every finite subset of W is linearly independent.

Theorem. If $W \subseteq V$ is linearly independent, any subset $U \subseteq W$ is linearly independent.

If one vector in W is a linear combination of the other vectors (including if $0 \in W$), then W is linearly dependent.

7.0.11 Basis

Definition. Basis A basis of V is a subset of V which is linearly independent and spans V.

Definition of **Standard basis of F^n**.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$$

Definition of $P_m(F)$.

Definition of **Standard basis of $P_m(F)$**.

7.0.12 Dimension

Plus/minus lemma. Let $S \subseteq V$ (where V is a vector space).

- If S is linearly independent, and v is not in the span of S, then $S \cup \{v\}$ is linearly independent.
- If $v \in \operatorname{span}(S \setminus \{v\})$, then $\operatorname{span}(S) = \operatorname{span}(S \setminus \{v\})$.

Theorem. Let V be a finite-dimensional vector space and $S \subseteq V$. Then,

- If $\operatorname{span}(S) = V$, then S contains a subset B which is a basis of V
- If S is linearly independent, then S can be extended to a basis of V

Theorem. Let V be a finite-dimensional vector space spanned by a set of m vectors. Then any linearly independent set of vectors in V is finite and contains no more than m elements.

7.1 Unit 4

7.1.1 Linear map

Definition of **Linear map**. Let V and W be vector spaces over F. A **linear map** (also called **linear function** or **linear transformation**) from V to W is a function $T : V \Rightarrow W$ with the property that

$$T(\alpha v_1 + v_2) = \alpha T(v_1) + T(v_2)$$

Equivalently, it is a function with the two properties

$$T(v_1 + v_2) = Tv_1 + Tv_2 T(\alpha_v) \qquad = \alpha T v$$

Equivalently, it is a function with the property that

$$T(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 T v_1 + \alpha_2 T v_2$$

Definition of **Linear operator**. A function $T : V \Rightarrow V$ which is a linear map.

Differential Equations (8)

A *differential equation* is an equation involving a quantity and one or more of its derivatives.

1 Ordinary Differential Equations

An ODE involves the derivative of the dependent variable with respect to a single independent variable.

8.0.1 Solving by Integration

Definition of **Solution to a differential equation**. A function $y = f(x)$ that satisfies the differential equation when f and its derivatives are substituted into the equation.

Procedure (Euler's Method). To numerically approximates the solution to the differential equation $y' = F(x, y)$ with $y(x_0) = y_0$,

$$y_n = y_{n-1} + F(x_{n-1}, y_{n-1})(x_n - x_{n-1})$$

Definition of **Separable Equation**. A separable equation is a differential equation where

$$\frac{dy}{dx} = g(x)f(y)$$

for some function $g(x)$ which depends only on x and $f(y)$ which depends only on y.

Theorem. For a separable equation,

$$\int \frac{1}{f(y)} dy = \int g(x) dx$$

Definition of **Logistic Differential Equation**. For a population P which increases exponentially ($\frac{dP}{dt} \approx kP$) when the population is small compared to the carrying capacity M but where the environment cannot sustain a population larger than M,

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

Then

$$P(t) = \frac{M}{1 + \left(\frac{M}{P_0} - 1\right)e^{-kt}}.$$

and

$$\frac{d^2P}{dt^2} = k^2P\left(1 - \frac{P}{M}\right)\left(1 - \frac{2P}{M}\right)$$

8.0.2 Initial-Value Problems

Definition of **Initial-Value problem**. Assuming the function f is continuous, then the function y is a solution of the IVP given that

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0,$$

Where x_0 is called the *initial point* for the IVP and y_0 is the *initial value*.

8.0.3 Existence and Uniqueness of Solutions

Theorem (Existence and Uniqueness). If f is continuous, then the function f as previously defined has at least one solution on the interval of continuity. If at least one solution exists and $\frac{\partial f}{\partial y}$ is continuous on the same interval, the solution is unique.

8.0.4 Autonomous Equations

Definition of **Autonomous Equation**. An autonomous equation is an equation where the derivative of the dependent variable can be expressed as a function of the dependent variable alone, assuming continuity.

An example of an autonomous equation is

$$\frac{dy}{dx} = f(y) + g(x)$$

Whereas

$$\frac{dy}{dx} = f(y)g(x)$$

Is not autonomous.

Since $f(y)$ is independent of x, the resulting slope field has translational symmetry across the x-axis. A snapshot of the slope field through a vertical line $x = x_0$ is called a *phase line*. A line where the direction of the slopes is indicated with arrows either going down or up is called a *one-dimensional phase portrait* of the autonomous ODE.

At y values where $f(y) = 0$, when the slope is zero, those points are called *equilibrium points* or *stationary points*.

1. If surrounding solutions approach $y = y_0$ asymptomatically then the equilibrium point is *asymptomatically stable* or an *attractor*.
2. If surrounding solutions move away from $y = y_0$ then the equilibrium point is *unstable* or a *repeller*.
3. If surrounding solutions approach from one side and repel from another then the equilibrium point is *semi-stable*.

8.0.5 Bifurcations

A differential equation that depends on a parameter *bifurcates* if there is a qualitative change in the solutions as parameter changes, meaning that the phase line changes.

We can see this change in phase line by plotting the parameter and the solution of the differential equation. The resulting graph would look like an array of phase lines. Each point on the phase lines is called a *bifurcation point*.

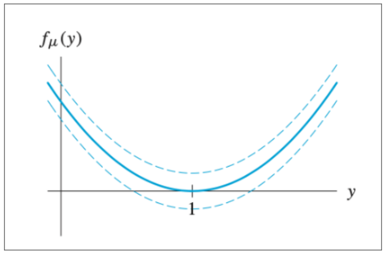


Figure: Graph of $f_\mu(y) = y^2 - 2y + \mu$ for $\mu < 1$, $\mu = 1$, and $\mu > 1$.

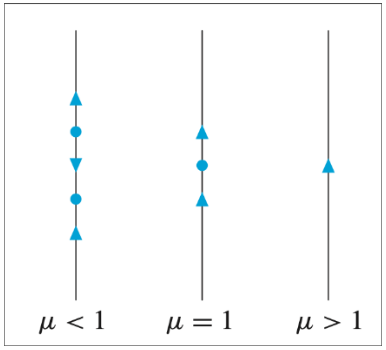


Figure: Corresponding phase portraits for $\frac{dy}{dx} = y^2 - 2y + \mu$.

The typical way to visualize bifurcations is through *bifurcation diagrams*. The parabola on the following figure is called a *bifurcation line*.

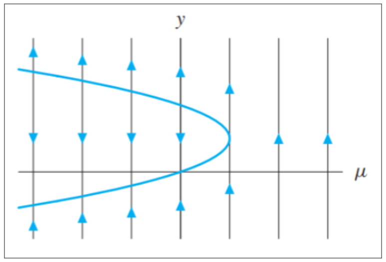


Figure: Bifurcation diagram for $\frac{dy}{dx} = y^2 - 2y + \mu$.

8.0.6 Separable Equations

Definition of **Separable first-order ODE**. An ODE is separable if it can be expressed as

$$\frac{dy}{dx} = g(x)h(y)$$

where g and h are continuous.

We can simplify the process of solving a separable equations by defining the following where $H(y)$ and $G(x)$ are antiderivatives of $\frac{1}{h(y)}$ and $g(x)$, respectively, and $c \in \mathbb{R}$ is an arbitrary constant.

$$H(y) = G(x) + c$$

which can then be expressed as

$$\int \frac{1}{h(y)} dy = \int g(x) dx$$

8.0.7 Implicitly-Defined Solutions

Sometimes a separable equation can't be solved for y explicitly. Let's define $F(x, y)$ as

$$F(x, y) := H(y) - G(x) - c$$

so that

$$F(x, y) = 0$$

The equation above implicitly defines y as a function of x only if it follows the *Implicit Function Theorem* from Multivariable Calculus.

Theorem (Implicit Function Theorem). If F is defined on a disc containing (x_0, y_0), where

1. $F(x, y) = 0$
2. $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ are continuous on the disc
3. $\frac{\partial F}{\partial x} \neq 0$

then the equation $F(x, y) = 0$ defines y as a function of x on some open set containing the point (x_0, y_0).

8.0.8 Singular Solutions

Definition of **Singular Solution**. A solution is singular if it cannot be obtained by any choice of c in the solution equation of the separable ODE.

When either $h(y)$ or $g(x)$ are equal to 0 inside a separable equation, then they would be valid solutions, but may not show up in the integration method for finding solutions as defined in the Separable Equations section. If that is the case, then those solutions are unique.

8.0.9 Orthogonal Trajectories

An *orthogonal trajectory* of a family of curves is a curve that intersects each curve of the family orthogonally.

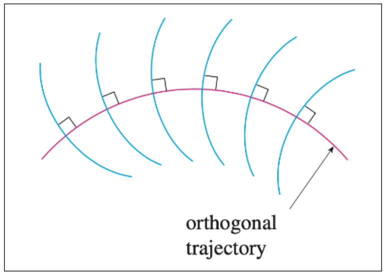


Figure: An example of an orthogonal trajectory.

For example,

$$x^2 + y^2 = r^2$$

and

$$y = mx$$

are orthogonal trajectories of each other.

To find orthogonal trajectories of an equation in terms of x, y, and another constant variable:

1. Take derivative of both sides in respect to x
2. Solve for $\frac{dy}{dx}$ from previous equation
3. Use the original equation to eliminate the constant and write an expression that only depends on x and y
4. The previous equation is a slope $m(x, y)$; find the orthogonal slope at point (x, y)
5. Solve the differential equation $\frac{dy}{dx} = m(x, y)$ to find the family of orthogonal trajectories

8.0.10 Linear Equations

Definition of **Linear Equation**. An equation is linear if it can be expressed in the from

$$\frac{dy}{dx} = p(x)y + q(x)$$

where $p, q : (a, b) \to \mathbb{R}, -\infty \leq a < b \geq \infty$ are continuous.

If $q(x) = 0$, then the linear equation is called *homogeneous*. If $p(x)$ is a constant, but not necessarily $q(x)$, it is called *constant-coefficient*.

8.0.11 Variation of Parameters

If $y_h(x)$ is a solution of a homogeneous linear equation, then $cy_h(x)$ is also a solution of the same homogeneous equation. A general solution of a homogeneous linear equation is

$$y_h(x) = ce^{\int p(x)dx}$$

A general solution of nonhomogeneous linear equation is

$$y(x) = ce^{\int p(x)dx} + e^{\int p(x)dx} \int e^{-\int p(x)dx} q(x) dx$$

8.0.12 Integrating Factors

This can be simplified by defining

$$\mu(x) := e^{-\int p(x)dx}$$

$$y(x) = \frac{1}{\mu(x)} \int \mu(x) q(x) dx$$

8.0.13 Singular Points

If we represent a linear equation in the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

and divide both sides by $a_1(x)$, then p and q will be discontinuous when $a_1(x) = 0$. These points are called *singular points*. These discontinuous points carry over to the final solution.

8.0.14 Bernoulli's Equation

An ODE in the from

$$\frac{dy}{dx} = p(x)y + f(x)y^n$$

where $n \in \mathbb{R}$, is called *Bernoulli's equation*.

This equation is linear if $n = 0, 1$. By substituting $u = y^{1-n}$, the equation can be rewritten in a linear form as

$$\frac{du}{dx} = (1 - n)(p(x)u + q(x))$$

where $n \neq 0, 1$.

8.0.15 Exact Equations

An exact equation is a differential equation in the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0$$

where M, N are continuously differentiable. They also represent partial derivatives of the potential function ψ

$$\psi_x+\psi_y\frac{dy}{dx}=0$$

where $\psi(x,y)$ is an existing *potential function*, as seen in Multivariable Calculus. The equation is exact only if,

- $\psi_{xy}=\psi_{yx}$
- Domain of functions above is open and topologically simply-connected (continuous)

So if the potential function exists, then the solution to the equation would be

$$\psi(x,y)=0$$

8.0.16 Homogeneous Equations

A real-valued function is *homogeneous of degree* α if it can be rewritten in the form

$$f(tx,ty)=t^\alpha f(x,y)$$

where $\alpha, t\in\mathbb{R}$.

A first-order differential equation of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0$$

is called *homogeneous* if both M and N are homogeneous of the same degree. To solve it, simply remember the substitution

$$y=ux$$

8.0.17 nth-Order Linear Equations

An *nth-order differential equation* is in the form

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

where the functions $a_i, g:(a,b)\to\mathbb{R}$ are continuous. An *nth-order initial value problem* (IVP) is to find the solution of the equation above where x_0 on interval I is subject to n initial conditions

$$y(x_0)=y_0,\qquad y'(x_0)=y_1,\qquad \ldots,\qquad y^{n-1}(x_0)=y_{n-1}$$

Theorem. If $a_n(x)\neq 0$ on I and $x_0\in I$, then a solution y of the nth-order IVP exists on I and is unique.

8.0.18 Boundary Value Problems

A boundary value problem (BVP) is when a linear differential equation of order two or greater has the dependent variable y or its derivatives specified at different points. E.g. $y(a)=y_0, y(b)=y_1$. Those values are called *boundary conditions*. A solution to this BVP would satisfy the differential equation on I, whose graph passes through (a,y_0) and (b,y_1). Those boundary conditions can be often written as

$$\alpha_1y(a)+\beta_1y'(a)=\gamma_1,$$
$$\alpha_2y(a)+\beta_2y'(a)=\gamma_2,$$

where $\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2$ are arbitrary constants. Conclusions of the earlier defined theorem regarding IVP for nth-order equations does not apply to BVP.

8.0.19 Homogeneous (nth-Order Linear) Equations

An nth-order linear differential equation is homogeneous if $g(x)$ in

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

is equal to zero. First the associated homogeneous equation has to be solved before solving the nonhomogeneous equation.

8.0.20 Differential Operators

The symbol D is the *differential operator*, where

$$D^ny=\frac{d^ny}{dx^n}.$$

The *n-th order differential operator* or *polynomial operator* is

$$L=a_n(x)D^n+a_{n-1}(x)D^{n-1}+\cdots+a_1(x)D+a_0(x)$$

Using L, nth-order linear equations can be written as

$$Ly=0\qquad\text{and}\qquad Ly=g(x)$$

Theorem (Superposition Principle for Homogeneous Linear Equations). Let $y_1,\ldots,y_k$ be solution of $Ly=0$ on interval I. Then the linear combination

$$y=c_1y_1+\cdots+c_ky_k$$

where c_i are constants, is also a solution of $Ly=0$.

By this theorem, every homogeneous nth-order linear equation has a solution of $y=0$.

8.0.21 Linear Dependence and Independence

Definition of **Linear Dependence**. A set of functions is *linearly dependent* on interval I if there exists constants c_i, not all zero, where

$$c_1f_1(c)+\cdots+c_nf_n(x)=0$$

for all x in I. If the only constants for which the equation above is satisfied are

$$c_1=c_2=\cdots=c_n=0$$

then the set is *linearly independent.*