Calculus Theorems (1)

1 Completeness

1.3 Completeness

Theorem (Completeness of the Real Numbers). Every nonempty subset S of $\mathbb{R}$ which is bounded above has a least upper bound $\sup S$.

Definition of **Supremum ($\sup S$)**. A number.

2 Limits

2.4 ε-δ definition of a Limit

Definition of **Limit**. If $\lim_{x \to a} f(x) = L$, then for any $\varepsilon > 0$, there exists $\delta > 0$ s.t. for any $x \in (a - \delta, a) \cup (a, a + \delta)$, $f(x) \in (L - \varepsilon, L + \varepsilon)$.

Alternatively,

Definition of **Limit**. If $\lim_{x \to a} f(x) = L$, then for any $\varepsilon > 0$, there exists $\delta > 0$ s.t. for any $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$

2.12 Squeeze Theorem

Theorem (Squeeze Theorem). Let f , g, and h be defined for all $x \neq a$ over an open interval containing a. If

$$f(x) \leq g(x) \leq h(x)$$

for all $x \neq a$ in an open interval containing a and

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$$

where $L \in R$, then $\lim_{x \to a} g(x) = L$.

3 Continuity

Definition of **Continuity at a point**. Function f is continuous at point a if $\lim_{x \to a} f(x) = f(a)$.

Definition. f has a **removable discontinuity** if $\lim_{x \to a} f(x) = L \in \mathbb{R}$ (in this case either $f(a)$ is undefined, or $f(a)$ is defined by $L \neq f(a)$).

Definition. f has a **jump discontinuity** if $\lim_{x \to a^-} f(x) = L_1 \in \mathbb{R}$ and $\lim_{x \to a^+} f(x) = L_2 \in \mathbb{R}$ but $L1 \neq L2$.

Definition. f has an **infinite discontinuity** at a if $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$

Theorem (Intermediate Value Theorem). If f is continuous on $[a, b]$, then for any real number L between $f(a)$ and $f(b)$ there exists at least one $c \in [a, b]$ such that $f(c) = L$. In other words, if f is continuous on $[a, b]$, then the graph must cross the horizontal line $y = L$ at least once between the vertical lines $x = a$ and $x = b$.

4 Derivatives