Sound & Light (1)

1.1 Miscellaneous

$$\%\text{ error} = \frac{\text{observed} - \text{theoretical}}{\text{theoretical}} * 100\%$$

1.2 Kinematics

$$x = \frac{a}{2}(\Delta t)^2 + v_0 \Delta t + x_0 \qquad v = v_0 + a\Delta t$$

$$v^2 = v_0^2 + 2a\Delta x \qquad \Delta x = \frac{v_0 + v}{2} * \Delta t$$

1.3 Simple Harmonic Motion

$$x = A\cos(\omega t + \varphi) \quad v = -\omega A\cos(\omega t + \varphi) \quad a = -\omega^2 A\cos(\omega t + \varphi)$$

$$x_{\max} = A \qquad v_{\max} = \omega A \qquad a_{\max} = \omega^2 A \qquad F_{\max} = m\omega^2 A$$

1.3.1 Springs and Slinkies

x represents the distance from the equilibrium.

If you put a mass on top of the slinky, Δx_{eq} represents the difference between the original equilibrium and the new equilibrium.

$$F_s = kx = ma \qquad F_{s_{\max}} = k\Delta x_{\text{eq}} = 9.8\Delta m$$

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad T = 2\pi\sqrt{\frac{m}{k}} \qquad \omega = 2\pi f = \sqrt{\frac{m}{k}}$$

$$SPE = \frac{1}{2}kx^2 \qquad KE = \frac{1}{2}mv^2$$

$$TME = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \frac{1}{2}kA^2 = \frac{1}{2}mv_{\max}^2$$

1.3.2 Springs in parallel and series

Quantity	In Series	In Parallel
Equivalent spring constant	$\frac{1}{k_{\text{eq}}} = \frac{1}{k_1} + \frac{1}{k_2}$	$k_{\text{eq}} = k_1 + k_2$
Deflection (elongation)	$x_{\text{eq}} = x_1 + x_2$	$x_{\text{eq}} = x_1 = x_2$
Force	$F_{\text{eq}} = F_1 = F_2$	$F_{\text{eq}} = F_1 + F_2$
Stored energy	$E_{\text{eq}} = E_1 + E_2$	$E_{\text{eq}} = E_1 + E_2$

1.3.3 Pendulums

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{L}} \qquad T = 2\pi\sqrt{\frac{L}{g}}$$

1.4 Waves

$$T = \frac{1}{f} \qquad v = \lambda f \qquad v = \frac{\Delta x}{\Delta t}$$

1.4.1 Slinkies and strings with fixed ends

$$F_T = F_s = kx \qquad \mu = \frac{m}{L} \qquad v = \sqrt{\frac{F_T}{\mu}}$$

Given mass m_T hanging below a pulley, $F_T = m_T g$.

1.5 Standing waves

1.5.1 Open-open, closed-closed

n is the number of antinodes, or the n^{th} harmonic.

$$f_n = f_1 n = \frac{nv}{2L} \qquad f_1 = \frac{v}{2L} \qquad \lambda_n = \frac{2L}{n}$$

1.5.2 Open-closed

$$f_n = f_1 n = \frac{nv}{4L} \qquad f_1 = \frac{v}{4L} \qquad \lambda_n = \frac{2L}{n}$$

1.6 Sound

1.6.1 Speed of sound

$$v = 331\sqrt{\frac{T_{^\circ\text{C}} + 273}{273}} \qquad v \approx 331 + 0.59T$$

1.6.2 Sound intensity

$$I = \frac{\text{Power (W)}}{\text{Area}} = \frac{\text{Power (W)}}{4\pi r^2}$$

$$I_{\text{dB}} = 10\log_{10}\left(\frac{I}{10^{-12}}\right) \qquad I = 10^{\frac{I_{\text{dB}}}{10} - 12}$$

1.6.3 Doppler effect

$$\boxed{O}\rightarrow \quad \boxed{S} \quad f_o = f_s\frac{v + v_o}{v} \qquad \leftarrow\boxed{O} \quad \boxed{S} \quad f_o = f_s\frac{v - v_o}{v}$$

$$\boxed{O} \quad \leftarrow\boxed{S} \quad f_o = f_s\frac{v}{v - v_s} \qquad \boxed{O} \quad \boxed{S}\rightarrow \quad f_o = f_s\frac{v}{v + v_s}$$

$$\boxed{O}\rightarrow \quad \leftarrow\boxed{S} \quad f_o = f_s\frac{v + v_o}{v - v_s} \qquad \boxed{O}\rightarrow \quad \boxed{S}\rightarrow \quad f_o = f_s\frac{v + v_o}{v + v_s}$$

$$\leftarrow\boxed{O} \quad \leftarrow\boxed{S} \quad f_o = f_s\frac{v - v_o}{v - v_s} \qquad \leftarrow\boxed{O} \quad \boxed{S}\rightarrow \quad f_o = f_s\frac{v - v_o}{v + v_s}$$

1.6.4 Constructive and Destructive Interference (2 dimensions)

For a point on the m^{th} antinodal/nodal line playing the same frequency with the same phase:

$$PD = m\lambda$$

where PD is the path length difference.

1.6.5 Beats

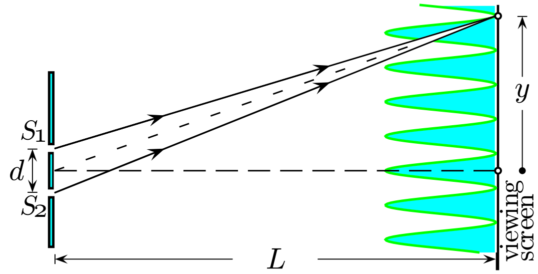
$$f_B = \Delta f$$

1.7 Light

1.7.1 Speed of light

$$c = 299\ 792\ 458 \frac{\text{m}}{\text{s}} \approx 3 * 10^8 \frac{\text{m}}{\text{s}}$$

1.7.2 Two-slit experiment



$$PD = \frac{dy}{L} = m\lambda$$

1.7.3 Mirror

$$r = 2f \qquad \frac{1}{f} = \frac{1}{p} + \frac{1}{q} \qquad M = \frac{h}{h_o} = \frac{-q}{p}$$

In a plane mirror, $p = -q$.

1.7.4 Lenses

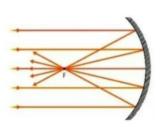
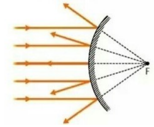
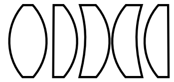
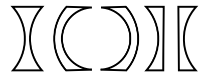
$$\frac{1}{f} = (n-1)(\frac{1}{r_1} - \frac{1}{r_2}) \qquad \frac{1}{f} = \frac{1}{p} + \frac{1}{q} \qquad M = \frac{h}{h_o} = \frac{-q}{p}$$

Multiple lenses

$$p_2 = \Delta x - q_1$$

1.7.5 Mirrors and Lenses

Real $\iff$ inverted, virtual $\iff$ upright.

	Converging	Diverging
Mirror		
Lens		
focal length	+	-
object distance	image	image
∞ to $2f$	Real smaller	Virtual smaller
$2f$ to f	Real larger	
f to 0	Virtual larger	
0 to $-f$	Virtual smaller	Virtual larger
$-f$ to $-2f$		Real larger
$-2f$ to $-\infty$		Real smaller

1.7.6 Refraction / Snell's Law

The **normal line** is the line perpendicular to the surface which touches the intersection of the surface and the light ray.

The **incident angle** is the angle between the ray of light and the normal line.

θ_1 and θ_2 are both measured from the normal line, not the surface.

Refraction occurs when the speed of light in two media are different and light hits the boundary of the two media. The frequency of the light will stay the same, but the speed, wavelength, and direction will change.

$$n = \frac{c}{v} \qquad n_1 \sin\theta_1 = n_2 \sin\theta_2$$

1.7.7 Ray diagrams

Mechanics (2)

2.1 Miscellaneous

$$\% \text{ error} = \frac{\text{observed} - \text{theoretical}}{\text{theoretical}} * 100\%$$

2.2 Kinematics

$$x(t) = \frac{a}{2}t^2 + v_0 t + x_0 \qquad v(t) = v_0 + at$$

$$v^2 = v_0^2 + 2a\Delta x \qquad \Delta x = \frac{v_0 + v}{2} * \Delta t$$

2.3 Forces

$$F_{\text{net}} = ma$$

F_T represents tension. It always points in the direction on which the rope pulls on the object.

F_N represents normal force. It takes the direction and magnitude necessary to prevent the object from passing through the surface that creates the normal force.

2.3.1 Friction

$$f_s \leq \mu_s F_N$$

The static friction takes the direction and magnitude necessary to prevent the object from moving in the component parallel to the surface, until the magnitude reaches $\mu_s F_N$. Upon reaching $\mu_s F_N$, the static friction is replaced by kinetic friction and the object starts moving:

$$k_s = \mu_k F_N$$

2.3.2 Centripetal force

The centripetal force always points towards the center of the circle representing the object's path, and therefore is perpendicular to the velocity. It is just another name for the net force in the centripetal direction. The centripetal acceleration is what causes the object to rotate.

$$F_c = ma_c = m\frac{v^2}{r} \qquad a_c = \frac{v^2}{r}$$

2.4 Work and Energy

$$W = \int_a^b F(\vec{r}) \cdot d\vec{r}$$

In one dimension:

$$W = \int_a^b F(x)dx$$

2.4.1 Spring force

2.5 Simple Harmonic Motion

The object is at rest at the **equilibrium position**. $x = 0$ when the object is at equilibrium, and x represents the distance and direction from the equilibrium. When you pull it to one direction, the **restoring force** pulls the object back toward the equilibrium position. It oscillates back and forth, between $x = -A$ and $x = A$, where A is the **amplitude**. The **period** is the amount of time to complete one oscillation, and the **frequency** is the amount of oscillations that happen in one second (or some other time unit).

2.5.1 Spring force

$$a(t) = -\frac{k}{m}x(t)$$

$$x(t) = A\cos(\omega t + \phi)$$
$$v(t) = -A\omega\sin(\omega t + \phi)$$
$$a(t) = -A\omega^2\cos(\omega t)$$

2.6 Simple Harmonic Motion (old)

$$x = A\cos(\omega t + \varphi) \qquad v = -\omega A\cos(\omega t + \varphi) \qquad a = -\omega^2 A\cos(\omega t + \varphi)$$

$$x_{\max} = A \qquad v_{\max} = \omega A \qquad a_{\max} = \omega^2 A \qquad F_{\max} = m\omega^2 A$$

2.6.1 Springs and Slinkies

x represents the distance from the equilibrium.

If you put a mass on top of the slinky, Δx_{eq} represents the difference between the original equilibrium and the new equilibrium.

$$F_s = kx = ma \qquad F_{s_{\max}} = k\Delta x_{\text{eq}} = 9.8\Delta m$$

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad T = 2\pi\sqrt{\frac{m}{k}} \qquad \omega = 2\pi f = \sqrt{\frac{m}{k}}$$

$$SPE = \frac{1}{2}kx^2 \qquad KE = \frac{1}{2}mv^2$$

$$TME = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \frac{1}{2}kA^2 = \frac{1}{2}mv_{\max}^2$$

2.6.2 Springs in parallel and series

Quantity	In Series	In Parallel
Equivalent spring constant	$\frac{1}{k_{\text{eq}}} = \frac{1}{k_1} + \frac{1}{k_2}$	$k_{\text{eq}} = k_1 + k_2$
Deflection (elongation)	$x_{\text{eq}} = x_1 + x_2$	$x_{\text{eq}} = x_1 = x_2$
Force	$F_{\text{eq}} = F_1 = F_2$	$F_{\text{eq}} = F_1 + F_2$
Stored energy	$E_{\text{eq}} = E_1 + E_2$	$E_{\text{eq}} = E_1 + E_2$

Mathematics (3)

3.1 Logarithms

$$\log_b(MN) = \log_b(M) + \log_b(N)$$
$$\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)$$
$$\log_b(M^p) = p \cdot \log_b(M)$$
$$\log_b(a) = \frac{\log_x(a)}{\log_x(b)}$$
$$\log_b(b) = 1$$

3.2 Notation

deg $p(x)$ means the degree of polynomial p.

LC $p(x)$ means the leading coefficient of polynomial p.

3.3 Rational functions

For a rational function $f(x) = \frac{p(x)}{q(x)}$, cancel out any common factors, then:

- For all rational functions:
  - VA: roots of $q(x)$
  - Roots: roots of $p(x)$
- When deg $p(x) =$ deg $q(x)$:
  - HA: $y = \frac{\text{LC } p(x)}{\text{LC } q(x)}$
- When deg $p(x) <$ deg $q(x)$:
  - HA: $y = 0$
- When deg $p(x) >$ deg $q(x)$:
  - HA: none
  - slant asymptote: $\frac{p(x)}{q(x)}$ excluding remainder

3.4 Polynomials

3.4.1 Linear equations

Slope-intercept form: $y = mx + b$

Point-slope form: $y - y_1 = m(x - x_1)$ for point (x, y)

Standard form: $ax + by = c$

3.4.2 Quadratic equations

Standard form: $y = ax^2 + bx + c$

Vertex form: $y = a(x - h)^2 + k$ for vertex (h, k)

Sum of roots: $\frac{-b}{a}$

Product of roots: $\frac{c}{a}$

3.4.3 Higher-degree polynomials

In a polynomial

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

, with roots

$$r_1, r_2, r_3, \ldots, r_n$$

then:

$$r_1 + r_2 + r_3 + \cdots + r_n = \sum_{k=1} n r_k = -\frac{a_{n-1}}{a_n}$$

3.5 Sequences and Series

3.5.1 Explicit formulas

Aritmetic sequence: $a_n = a_1 + r(n-1)$

Geometric sequence: $a_n = a_1 * r^{n-1}$

Harmonic sequence: $a_n = \frac{1}{a_1 + r(n-1)}$

3.5.2 Arithmetic and Geometric Series

In the following equations, substituting $j = 1$ with $j = 0$, $j - 1$ with j, and a_1 with a_0 will produce the same result.

$$\sum_{j=1}^{n}(a_1 + r(j-1)) = \frac{n}{2}(2a_1 + (n-1)d)$$
$$\sum_{j=1}^{n}(a_1 * r^{j-1}) = \frac{a_1(1 - r^n)}{1 - r}$$
$$\sum_{j=1}^{\infty}(a_1 * r^{j-1}) = \frac{a_1}{1 - r} \text{ for } r \in [-1, 1]$$

3.5.3 Special Sums

$$\sum_{j=1}^{n} c = nc \qquad \sum_{j=1}^{n} ca_j = c\sum_{j=1}^{n} a_j$$
$$\sum_{j=1}^{n}(a_j + b_j) = \sum_{j=1}^{n} a_j + \sum_{j=1}^{n} b_j \qquad \sum_{j=1}^{n} j = \frac{n(n+1)}{2}$$
$$\sum_{j=1}^{n} j^2 = \frac{n(n+\frac{1}{2})(n+1)}{3} = \frac{n(2n+1)(n+1)}{6} \qquad \sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4}$$

3.6 Trigonometry

$^\circ$	rad	sin	cos	tan
0°	0	0	1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	$\frac{\pi}{2}$	1	0	undef

3.6.1 Law of Sines and Cosines

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c} \qquad c^2 = a^2 + b^2 - 2ab\cos(C)$$

3.6.2 Triangle area

$$K = \frac{1}{2}bh \qquad K = \frac{1}{2}bc\sin(A) \qquad K = \sqrt{s(s-a)(s-b)(s-c)}$$

3.6.3 More identities

$$(\sin A)^2 + (\cos A)^2 = 1 \qquad (\tan A)^2 + 1 = (\sec A)^2$$
$$\sin(\frac{\pi}{2} - x) = \cos(x) \qquad (\cot A)^2 + 1 = (\csc A)^2$$
$$\cos(-x) = \cos(x) \qquad \sin(-x) = \sin(x) \qquad \tan(-x) = \tan(x)$$

3.6.4 Slope

Where α is the angle between the line and the x-axis, and m is the slope of the line:

$$m = \tan\alpha$$

3.6.5 Sum and difference formulas

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$
$$\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B)$$
$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$
$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$
$$\tan(A+B) = \frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$$
$$\tan(A-B) = \frac{\tan(A)-\tan(B)}{1+\tan(A)\tan(B)}$$

$$\sin(2A) = 2\sin(A)\cos(A)$$
$$\cos(2A) = (\cos A)^2 - (\sin A)^2 = 2(\cos A)^2 - 1 = 1 - 2(\sin A)^2$$
$$\tan(2A) = \frac{2\tan(A)}{1-(\tan A)^2}$$

3.7 Vectors

$$\vec{v} + \vec{w} = \begin{bmatrix} v_x + w_x \\ v_y + w_y \\ v_z + w_z \end{bmatrix} \qquad c * \vec{v} = \begin{bmatrix} c * v_x \\ c * v_y \\ c * v_z \end{bmatrix}$$
$$\vec{v} \cdot \vec{w} = v_x w_x + v_y w_y + v_z w_z = |\vec{v}||\vec{w}|\cos(\theta)$$
$$|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}|\sin(\theta) = \text{area of parallelogram}$$
$$\vec{v} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} \qquad \vec{v} \times \vec{w} \perp \vec{v} \qquad \vec{v} \times \vec{w} \perp \vec{w}$$
$$\vec{v} \perp \vec{w} \iff \vec{v} \times \vec{w} = \vec{0} \qquad \vec{v} \parallel \vec{w} \iff \vec{v} \cdot \vec{w} = 0$$
$$\hat{v} = \frac{\vec{v}}{|\vec{v}|} \qquad \text{proj}_{\vec{b}}\vec{v} = \frac{\vec{v} \cdot \vec{b}}{\vec{b} \cdot \vec{b}} * \vec{b} = (|\vec{v}|\cos(\theta))$$

Right-hand rule

To determine the direction of $\vec{v} \times \vec{w}$, put the side of the right hand on $\vec{v}$ and curl the fingers toward $\vec{w}$. The direction the thumb is pointing is the direction of $\vec{v} \times \vec{w}$.

3.8 Polar

3.8.1 Polar and Cartesian sytems

With point $(x, y) = (r; \theta) = (r; \beta)$, where θ is CCW from the x-axis and β is a bearing, CW from the y-axis:

$$x = r\cos(\theta) = r\sin(\beta) \qquad y = r\sin(\theta) = r\cos(\beta)$$
$$r = \sqrt{x^2 + y^2} \qquad \theta \equiv \arctan(\frac{y}{x}) \quad \beta \equiv \arctan(\frac{x}{y})$$

3.8.2 Converting functions

Try these substitutions in order:

$$x^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad x = r\cos\theta \qquad y = r\sin\theta$$

3.8.3 Limaçons and Petals

The function $y = A\cos(B(\theta + C)) + D$ is equivalent to $y = A\cos(B\theta) + D$ rotated C degrees/radians clockwise.

When C is 0 and B is 1, the x-intercepts are $A \pm D$ and the y-intercepts are $\pm D$, and it forms a limaçon.

When C is 0, but $B \neq 1$, then this sometimes still holds. The x-intercepts may also be $\pm A \pm D$.

There are B petals, with the axis of the first petal on the positive x-axis.

When B is even and $|D| < 1$, then the number of petals is $2B$.

Using sin instead of cos, limaçons have their axes on the positive y-axis, while for petals, the first petal starts from the positive x-axis and curves upwards.

3.9 Complex

$$\text{cis}(\theta) = e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

To find the n^{th} root of $x_r\,\text{cis}(x_\theta)$, solve the equation $z_r^n\,\text{cis}(nz_\theta) = x_r\,\text{cis}(x_\theta + 360°k)$ for $k \in \mathbb{R}$.

3.10 Function domain

Function	Domain x	Range y
$\log(x)$	$(0, \infty)$	$\mathbb{R}$
$\sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$\arcsin(x)$	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$\arccos(x)$	$[-1, 1]$	$[0, \pi]$
$\arctan(x)$	$\mathbb{R}$	$(-\frac{\pi}{2}, \frac{\pi}{2})$

Calculus Theorems (4)

1 Completeness

Theorem (Completeness of the Real Numbers). Every nonempty subset S of $\mathbb{R}$ which is bounded above has a least upper bound $\sup S$.

Definition of **Supremum** ($\sup S$). A number such that

(1) $s \leq \sup S$ for every $s \in S$ (which just says that $\sup S$ is an upper bound for S)

(2) If u is any upper bound for S, then $\sup S \leq u$ (which says that $\sup S$ is the least upper bound for S).

Definition of **Infimum** ($\inf S$). A number such that

(1) $\inf S \leq s$ for every $s \in S$ (i.e. $\inf S$ is an lower bound for S)

(2) If l is any upper bound for S, then $l \leq \inf S$ (i.e. $\inf S$ is the greatest lower bound for S).

Theorem. Every nonempty subset S of $\mathbb{R}$ which is bounded below has a greatest lower bound.

Theorem. If $\min S$ exists, then $\min S = \inf S$.

Theorem. If $A \subset R$ and $c \geq 0$, and $cA := ca : a \in A$, $\sup cA = c \sup A$.

Theorem (Rationals between Reals). For every two real numbers a and b with $a < b$, there exists a rational number r satisfying $a < r < b$.

Nested Intervals Theorem.
If $I_n = [a_n, b_n] = \{x \in R : a_n \leq x \leq b_n\}$ s.t. $a_n \leq a_{n+1}$ and $b_{n+1} \leq b_n$ for $n \in \mathbb{N}$, so that $I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \ldots$, then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

If $\inf\{b_n - a_n\} = 0$, then $\bigcap_{n=1}^{\infty} I_n = \{x\}$, where $x = \sup\{a_n\} = inf\{b_n\}$.

Capture Theorem. If A is a nonempty subset of $\mathbb{R}$, then:

(i) If A is bounded above, then any open interval containing $\sup A$ contains an element of A.

(ii) Similarly, if A is bounded below, then any open interval containing $\inf A$ contains an element of A.

Theorem (Binary Search (Bisection Method)). If we binary-search for x over $I_1 = [a_1, b_1]$ for $a_1, b_1 \in \mathbb{Q}$, we define I_n s.t. either $I_n := [a_{n-1}, \frac{a_{n-1}+b_{n-1}}{2}]$ or $I_n := [\frac{a_{n-1}+b_{n-1}}{2}, a_{n-1}]$, and we define $a_n := \inf I_n$ and $b_n := \sup I_n$. We define A to be the set of all a_n, and B to be the set of all b_n.

Then, the size of $I_n = \frac{b_1-a_1}{2^n} = b_n - a_n$, and $\bigcap_{n=1}^{\infty} I_n\{x\}$, where $x = \sup\{a_n\} = \inf\{b_n\}$.

2 Limits

Definition of **Limit**. If $\lim_{x\to a} f(x) = L$, then for any $\varepsilon > 0$, there exists $\delta > 0$ s.t. for any $x \in (a-\delta, a) \cup (a, a+\delta)$, $f(x) \in (L-\varepsilon, L+\varepsilon)$.

Alternatively,

Definition of **Limit**. If $\lim_{x\to a} f(x) = L$, then for any $\varepsilon > 0$, there exists $\delta > 0$ s.t. $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.

Definition of **Right-sided limit**. If $\lim_{x\to a^+} f(x) = L$, then for any $\varepsilon > 0$, there exists $\delta > 0$ s.t. $|f(x) - L| < \varepsilon$ whenever $0 < x - a < \delta$.

Definition of **Left-sided limit**. If $\lim_{x\to a^-} f(x) = L$, then for any $\varepsilon > 0$, there exists $\delta > 0$ s.t. $|f(x) - L| < \varepsilon$ whenever $0 < a - x < \delta$.

Theorem (Limit Laws). Let $c \in R$ be a constant and suppose the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

- $\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
- $\lim_{x\to a}(cf(x)) = c \lim_{x\to a} f(x)$
- $\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$
- $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$, provided that $\lim_{x\to a} g(x) \neq 0$
- $\lim_{x\to a} x^n = (\lim_{x\to a} x)^n$
- $\lim_{x\to a} \frac{a(x)b(x)}{c(x)b(x)} = \lim_{x\to a} \frac{a(x)}{c(x)}$

These laws also apply to one-sided limits.

Theorem (L'Hopital's Rule). If f and g are differentiable and $g'(x) \neq 0$ on an open interval I that surrounds a, and $\lim_{x\to a} \frac{f(x)}{g(x)} \in \{\frac{0}{0}, \pm\frac{\infty}{\infty}\}$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$.

Theorem (Composition of Limits). If f is continuous at L and $\lim_{x\to a} g(x) = L$, then $\lim_{x\to a} f(g(x) = f(\lim_{x\to a} g(x))) = f(L)$

Theorem (Operations on infinity). For $x \in \mathbb{R}$,

$$\infty + x = \infty \qquad -\infty + x = -\infty \qquad \frac{x}{\pm\infty} = 0$$

$$x * \infty = \begin{cases} \infty & \text{if } x > 0 \\ -\infty & \text{if } x < 0 \end{cases}$$

$$x * -\infty = \begin{cases} -\infty & \text{if } x > 0 \\ \infty & \text{if } x < 0. \end{cases}$$

Definition of **Indeterminate forms**. The following forms are indeterminate and you cannot evaluate them.

$$\frac{0}{0}, \frac{\pm\infty}{\pm\infty}, 0 * \pm\infty, \infty - \infty$$

Squeeze Theorem. Let f , g, and h be defined for all $x \neq a$ over an open interval containing a. If

$$f(x) \leq g(x) \leq h(x)$$

for all $x \neq a$ in an open interval containing a and

$$\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$$

where $L \in \mathbb{R}$, then $\lim_{x\to a} g(x) = L$.

3 Continuity

Definition of **Continuity at a point**. Function f is continuous at point a if $\lim_{x\to a} f(x) = f(a)$.

Definition. f has a **removable discontinuity** if $\lim_{x\to a} f(x) = L \in \mathbb{R}$ (in this case either $f(a)$ is undefined, or $f(a)$ is defined by $L \neq f(a)$).

Definition. f has a **jump discontinuity** if $\lim_{x\to a^-} f(x) = L_1 \in \mathbb{R}$ and $\lim_{x\to a^+} f(x) = L_2 \in \mathbb{R}$ but $L1 \neq L2$.

Definition. f has an **infinite discontinuity** at a if $\lim_{x\to a^-} f(x) = \pm\infty$ or $\lim_{x\to a^+} f(x) = \pm\infty$

Intermediate Value Theorem. If f is continuous on $[a, b]$, then for any real number L between $f(a)$ and $f(b)$ there exists at least one $c \in [a, b]$ such that $f(c) = L$. In other words, if f is continuous on $[a, b]$, then the graph must cross the horizontal line $y = L$ at least once between the vertical lines $x = a$ and $x = b$.

Aura Theorem. If $f(x)$ is continuous and $f(a)$ is positive, then there exists an open interval containing a such that for all x in the interval, $f(x)$ is positive.

If $f(x)$ is continuous and $f(a)$ is negative, then there exists an open interval containing a such that for all x in the interval, $f(x)$ is negative.

Bolzano's Theorem. Let f be a continuous function defined on $[a, b]$. If 0 is between $f(a)$ and $f(b)$, then there exists $x \in [a, b]$ such that $f(x) = 0$.

4 Derivatives

The derivative is the instantaneous rate of change, and the slope of the tangent line to the point.

Definition of **Derivative** ($f'(a)$).

$$\frac{d}{da}f(a) = f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

Theorem (Tangent line to a point). The equation of the tangent line to the point $(a, f(a))$ is

$$y = f'(a)(x-a) + f(a)$$

Derivative Rules

Theorem (Difference Rule).

$$\frac{d}{dx}(f(x)-g(x)) = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

Theorem (Sum Rule).

$$\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

Theorem (Constant Multiple Rule).

$$\frac{d}{dx}(cf(x)) = c\frac{d}{dx}f(x)$$

Theorem (Product Rule).

$$\begin{aligned}\frac{d}{dx}(f(x)g(x)) &= f'(x)g(x) + f(x)g'(x)\\ \frac{d}{dx}(f(x)g(x)h(x)) &= f'(x)g(x)h(x) + f(x)g'(x)h(x)\\ &+ f(x)g(x)h'(x)\end{aligned}$$

and so on.

Theorem (Quotient Rule).

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$$

Theorem (Power Rule).

$$\frac{d}{dx}x^n = nx^{n-1}$$

for $n \in \mathbb{R}$

Theorem (Chain Rule).

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x) \qquad \frac{dy}{dx} = \frac{dy}{db}\frac{db}{dx}$$

Theorem (Derivative of inverse functions). Let $x \in \mathbb{R}$ and f be a differentiable, one-to-one function at x. Then if $f'(x) \neq 0$, then

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}$$

Theorem (Derivatives of exponentials and logs).

$$\begin{aligned}\frac{d}{dx}e^x &= e^x & \frac{d}{dx}\ln x &= \frac{1}{x}\\ \frac{d}{dx}a^x &= a^x\ln(a) & \frac{d}{dx}\log_a x &= \frac{1}{x\ln(a)}\end{aligned}$$

Theorem (Derivatives of trig functions).
Warning: x must be an angle in radians!

$$\begin{aligned}\sin'(x) &= \cos(x) & \cos'(x) &= -\sin(x)\\ \sec'(x) &= \sec(x)\tan(x) & \csc'(x) &= -\csc(x)\cot(x)\\ \tan'(x) &= \sec(x)^2 & \cot'(x) &= -\csc(x)^2\end{aligned}$$

$$\begin{aligned}\operatorname{arcsin}'(x) &= \frac{1}{\sqrt{1-x^2}} & \operatorname{arccos}'(x) &= -\frac{1}{\sqrt{1-x^2}}\\ \operatorname{arcsec}'(x) &= \frac{1}{|x|\sqrt{x^2-1}} & \operatorname{arccsc}'(x) &= -\frac{1}{|x|\sqrt{x^2-1}}\\ \operatorname{arctan}'(x) &= \frac{1}{1+x^2} & \operatorname{arccot}'(x) &= -\frac{1}{1+x^2}\end{aligned}$$

5 Derivative Applications

5.7 Mean Value Theorem

Theorem (Mean Value Theorem). If the function f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ s.t.

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{\Delta f(x)}{\Delta x} \text{ on } [a, be]$$

Theorem (Some collloraries to the MVT). If $f(x)$ is differentiable on I, then:

- $f'(x) > 0$ for $x \in I \iff f(x)$ is strictly increasing for $x \in I$.
- $f'(x) \geq 0$ for $x \in I \iff f(x)$ is increasing or constant for $x \in I$.
- $f'(x) = 0$ for $x \in I \iff f(x)$ is constant for $x \in I$.
- $f'(x) \leq 0$ for $x \in I \iff f(x)$ is decreasing or constant for $x \in I$.
- $f'(x) < 0$ for $x \in I \iff f(x)$ is strictly decreasing for $x \in I$.

5.3, 5.10, 5.11, 5.16 Extrema

Definition of **Critical point of** f. A number c in the domain of f where $f'(c) = 0$ or $f'(c)$ does not exist.

Definition of **Stationary point of** f. A number c in the domain of f where $f'(c) = 0$

Fermat's Theorem. The local maxima and minima of f are critical points of f.

Exteme Value Theorem. If f is continuous on $[a, b]$, then it has an absolute max and an absolute min.

Theorem (Method to find absolute minima and maxima). Store the critical points of f in the array C. Then, the absolute maximum is $\max\{f(c) : c \in C\}$ and the absolute minimum is $\min\{f(c) : c \in C\}$.

Theorem (First Derivative Test). If f is continuous over I, and $c \in I$ is a critical point of f, and f is differentiable over $I \setminus c$, then:

- If $f'(x)$ is decreasing at c, then $f(c)$ is a local max.
- If $f'(x)$ is increasing at c, then $f(c)$ is a local min.
- If $f'(x)$ has the same sign before and after c, then $f(c)$ is neither a local max nor a local min.

Definition of **Concavity**. f is concave up on I if the tangent line to f at each point in I is lower than the graph of f.

f is concave down on I if the tangent line to f at each point in I is higher than the graph of f.

Theorem (Test for Concavity). If $f''(x) > 0$ for all $x \in I$, then f is concave up on I.

If $f''(x) < 0$ for all $x \in I$, then f is concave down on I.

Theorem (Second Derivative Test). If f'' is continuous on an interval containing c, where c is the x-value of a stationary point of f. Then,

- If $f''(c) > 0$, then $f(c)$ is a local max.
- If $f''(c) < 0$, then $f(c)$ is a local min.

Trimm's Single Extremum Theorem. If f is continuous on an interval I, and f has a single local extremum (max or min), then that extremum is a global max or min.

5 Integrals

Antiderivative

Definition of **Antiderivative / Indefinite Integral**. The antiderivative F of a function f is the function such that $F'(x) = f(x)$.

$$F(x) = \int f(x)dx$$

Theorem (Antiderivative plus a constant). If F is the antiderivative of a function f, then $G(x) = F(x) + c$ where $c \in \mathbb{R}$ is also an antiderivative.

Definition of **Integral**.

$$(f_1[a,b]) \mapsto \int_a^b f(x)dx \in \mathbb{R}$$

such that the Properties of the Integral are true.

The definite integral takes in a function and a range $[a,b]$, and returns a number. The indefinite integral takes in a function and returns an infinitely large set of functions (the antiderivatives).

If $a > b$, then $\int_a^b f := -\int_b^a f$.

Let $\mathcal{R}([a,b])$ be the set of integrable functions, $\mathcal{C}([a,b])$ be the set of continuous functions, and $\mathcal{B}([a,b])$ be the set of bounded functions on $[a,b]$. Then

$$\mathcal{C}([a,b]) \subset \mathcal{R}([a,b]) \subset \mathcal{B}([a,b])$$

Theorem (Properties of the Integral). The integral is defined such that the following are true:

(I0) Every continuous function is integrable.

(I1) If $f(x) = c$, then $\int_a^b f(x)dx = c(b-a)$

(I2) If $f_1(x) \le f_2(x)$, then $\int_a^b f_1(x)dx \le \int_a^b f_2(x)dx$.

(I3) For any a, b, c, $\displaystyle\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.

Theorem (Fundamental Theorem of Calculus). Let $f \in \mathcal{R}([a,b])$ be some integrable function, where $a, b \in \mathbb{R}$. Let $\mathcal{F}(x) = \int_a^x f$ for $x \in [a,b]$. Then:

(a) $\mathcal{F}$ is continuous for every $c \in [a,b]$.

(b) If f is continuous at $c \in [a.b]$, then $\mathcal{F}$ is diffentiable at c, and $\mathcal{F}'(c) = f(c)$.

(c) If f is continuous on $[a,b]$, and F is an antiderivative of f, then $\int_a^b f = F(b) - F(a)$, or

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

Theorem (Substitution Rule). If g is a function that has a continuous derivative on an interval, another function f is continuous on the range of g, and F is an antiderivative of f on the range of g, then

$$\int f(g(x))g'(x)dx = F(g(x)) + C$$

For $a, b \in \mathbb{R}$,

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du = F(g(b)) - F(g(a))$$

Let $u := g(x)$. Then, if $du = E(g'(x)dx)$, where $E \in \mathbb{R}$, then

$$\int Ef(u)du = E\int f(u)du = F(u) + C$$

Theorem (Some Antiderivative Rules).

$$\int e^x dx = e^x + C \qquad \int x^a dx = \frac{x^{a+1}}{a+1} + C \text{ for } a \neq 1$$

$$\int a^x dx = \frac{a^x}{\ln(a)} + C \qquad \int x^{-1} dx = \ln|x| + C$$

$$\int f(x) + g(x)dx = \int f(x)dx + \int g(x)dx$$

Theorem (Integration By Parts). If f and g are integrable functions, then

$$\int f(x)g'(x)dx = f(x)g(x) + \int f(x)g(x)dx$$

Equivalently, if u and v are integrable functions of x, then

$$\int udv = uv - \int vdu$$

Additionally, if f' and g' are continuous, then

$$\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) + \int_a^b g(x)f'(x)dx$$

Theorem (Integrals of Trig Functions).

$$\int \sin(x)dx = -\cos(x) + C \qquad \int \cos(x)dx = \sin(x) + C$$

$$\int \tan(x)dx = -\ln|\cos(x)| + C = \ln|\sec(x)| + C$$

$$\int \cot(x)dx = \ln|\sin(x)| + C = -\ln|\csc(x)| + C$$

$$\int \sec(x)dx = \ln|\sec(x) + \tan(x)| + C$$

$$\int \csc(x)dx = -\ln|\cot(x) + \csc(x)| + C$$

Procedure (Integrals of Powers of Trig Functions). To solve $\int \sin^n(x)\,dx$ where n is even, substitute $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$.

To solve $\int \sin^n(x)\,dx$ where n is odd, substitute $\sin^2(x) = 1 - \cos^2(x)$ and perform u-sub with $u := \cos^2(x)$ and $du = -\sin(x)\,dx$.

To solve $\int \cos^n(x)\,dx$ where n is even, substitute $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$.

To solve $\int \cos^n(x)\,dx$ where n is odd, substitute $\cos^2(x) = 1 - \sin^2(x)$ and perform u-sub with $u := \sin^2(x)$ and $du = \cos(x)\,dx$.

To solve $\int \tan^n(x)\,dx$, substitute $\tan^2(x) = \sec^2(x) - 1$. If n is odd, perform u-sub with $u := \sec(x)$ and $du = \sec(x)\tan(x)\,dx$.

To solve $\int \cot^n(x)\,dx$, substitute $\cot^2(x) = \csc^2(x) - 1$. If n is odd, perform u-sub with $u := \csc(x)$ and $du = -\csc(x)\cot(x)\,dx$.

To solve $\int \sec^n(x)\,dx$ where n is even, substitute $\sec^2(x) = \tan^2(x) + 1$, but ensure that $\sec^2(x)\,dx$ remains. Then perform u-sub with $u := \tan(x)$ and $du = \sec^2(x)\,dx$.

To solve $\int \sec^n(x)\,dx$ where n is odd, substitute $\sec^n(x) = \frac{1}{\cos^n(x)}\frac{\cos(x)}{\cos(x)}$. Then substitute $\cos^2(x) = (1+u)(1-u)$ and u-sub with $u := \sin(x)$ and $du = \cos(x)dx$. Then perform partial fraction decomposition.

To solve $\int \csc^n(x)\,dx$ where n is even, substitute $\csc^2(x) = \csc^2(x) + 1$, but ensure that $\sec^2(x)\,dx$ remains. Then perform u-sub with $u := \cot(x)$ and $du = -\csc^2(x)\,dx$.

To solve $\int \csc^n(x)\,dx$ where n is odd, substitute $\csc^n(x) = \frac{1}{\sin^n(x)}\frac{\sin(x)}{\sin(x)}$. Then substitute $\sin^2(x) = (1+u)(1-u)$ and u-sub with $u := \cos(x)$ and $du = -\sin(x)dx$. Then perform partial fraction decomposition.

Remember that $\int \sec^2(x)\,dx = \tan(x)$ and $\int \csc^2(x)\,dx = -\cot(x)$.

Procedure (Trig Substitution).

Orig expression	Substitution	Pythagorean identity
$\sqrt{a^2 - x^2}$	$x := a\sin(\theta)$	$1 - \sin^2(\theta) = \cos^2(\theta)$
$\sqrt{a^2 + x^2}$	$x := a\tan(\theta)$	$1 + \tan^2(\theta) = \sec^2(\theta)$
$\sqrt{x^2 - a^2}$	$x := a\sec(\theta)$	$\sec^2(\theta) - 1 = \tan^2(\theta)$

Partial Fraction Decomposition

Theorem. Any polynomial $Q(x)$ with real coefficients can be factored over the reals as a product of two types of factors:

- linear factors (of the form $ax+b$)
- irreducible quadratic factors (of the form $ax^2+bxx+c$, where $b^2-4ac<0$)

Definition of **Proper rational function**. A rational function $\frac{P(x)}{Q(x)}$ where $\deg P < \deg Q$.

Theorem. Any rational function can be converted into a proper rational function plus a polynomial by continually long-dividing by the denominator.

Theorem. Let $R(x) = \frac{P(x)}{Q(x)}$ be a proper rational function, where the denominator $Q(x)$ has been factored into linear and irreducible quadratic factors. $R(x)$ can be written as a sum of partial fractions, where each factor in the denominator gives rise to terms in the partial fraction decomposition:

- For each factor of the form $(ax+b)^k$ in the denominator, add $\displaystyle\sum_{i=1}^{k}\frac{A_i}{(ax+b)^i}$ to the partial fraction decomposition.
- For each factor of the form $(ax^2+bx+c)^k$ in the denominator, add $\displaystyle\sum_{i=1}^{k}\frac{A_ix+B_i}{(ax^2+bx+c)^i}$ to the partial fraction decomposition.

Riemann Sums

Definition of **Elementary function**. A function which is a polynomial, rational function, power function (x^a), exponential function (a^x), logarithmic functions, trigonometric and inverse trigonometric functions, or an addition, subtraction, multiplication, division, and composition of the above.

Definition of **Riemann sum**. Let $f : [a,b] arrow \mathbb{R}$ be any bounded function and let P be a partition of $[a,b]$.

- A choice of a point $x_i^* \in [x_{i-1}, x_i]$ for all $i \in [1,n]$ is called a *tagging* of P, which we denote by $\tau = x_1^*, ..., x_n^*$.
- A pair (P,τ) is called a *tagged partition.*
- Given a bounded function $f : [a,b] arrow \mathbb{R}$ and a tagged partition (P,τ) of $[a,b]$, the sum

$$R(f,P,\tau) = \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1})$$

is called the *Riemann sum* of f for (P,τ).

Theorem. If f is integrable on $[a,b]$, then for every $\varepsilon > 0$ there exists $\delta > 0$ s.t. if P is a partition of $[a,b]$ with $|P| := \max x_i - x_{i-1} < \delta$ and $\tau = \{x_i^*\}$, then

$$|R(f,P,\tau) - \int_a^b f(x)dx| < \varepsilon$$

Approximate Integration

Definition of **Left-endpoint approximation**. Take $x_i^* = x_{i-1} = a + \frac{(i-1)(b-a)}{n}$. Then the left-endpoint approximation of the function $f(x)$ is

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f\left(a + \frac{(i-1)(b-a)}{n}\right)\frac{b-a}{n}$$

The error bound is

$$E_n^L \le \max\{|f'(x)|\}_{x\in[a,b]}\frac{(b-a)^2}{2n}$$

Definition of **Right-endpoint approximation**. Take $x_i^* = x_{i-1} = a + \frac{(i)(b-a)}{n}$. Then the right-endpoint approximation of the function $f(x)$ is

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f\left(a + \frac{(i)(b-a)}{n}\right)\frac{b-a}{n}$$

The error bound is

$$E_n^R \le \max\{|f'(x)|\}_{x\in[a,b]}\frac{(b-a)^2}{2n}$$

Definition of **Midpoint approximation**. Take $x_i^* = x_{i-1} = a + \frac{(i-\frac{1}{2})(b-a)}{n}$. Then the midpoint approximation of the function $f(x)$ is

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f\left(a + \frac{(i-\frac{1}{2})(b-a)}{n}\right)\frac{b-a}{n}$$

The error bound is

$$E_n^M \le \max\{|f''(x)|\}_{x\in[a,b]}\frac{(b-a)^3}{24n^2}$$

Definition of **Trapezoidal approximation**. Then the trapezoidal approximation of the function $f(x)$ is

$$\int_a^b f(x)dx \approx \frac{b-a}{2n}\left(f(a) + 2\sum_{i=1}^{n-1} f\left(a + \frac{(i)(b-a)}{n}\right) + f(b)\right)$$

The error bound is

$$E_n^T \le \max\{|f''(x)|\}_{x\in[a,b]}\frac{(b-a)^3}{12n^2}$$

Definition of **Simpson's approximation**. For even n (greater values of n give more precise more accuracy):

$$\frac{b-a}{3n}\left(f(a) + \sum_{i=1}^{n-1}(3-(-1)^i)f\left(a + \frac{(i)(b-a)}{n}\right) + f(b)\right)$$

The error bound is

$$E_n^S \le \max\{|f''''(x)|\}_{x\in[a,b]}\frac{(b-a)^5}{180n^4}$$

8 Integral Applications

Volumes

Generally, if $A(x)$ is the cross-section of a solid that intersects the x-axis at x, then the volume of the solid is

$$V = \int_a^b A(x)dx$$

Theorem (Disk Method). Let $f(x)$ be a continuous, nonnegative function defined on $[a,b]$, and R be the region bounded above by the graph of $f(x)$ and below by the x-axis. Then, the volume of the solid of revolution formed by revolving R around the x-axis is given by

$$V = \int_a^b \pi(f(x))^2 dx$$

Theorem (Washer Method). Let $f(x)$ be a continuous, nonnegative function defined on $[a,b]$, and R be the region bounded above by the graph of $f(x)$ and below by the graph of $g(x)$. Then, the volume of the solid of revolution formed by revolving R around the x-axis is given by

$$V = \int_a^b \pi((f(x))^2 - (g(x))^2)dx$$

Theorem (Cylindrical Shells Method). Let $f(x)$ be a continuous, nonnegative function defined on $[a,b]$, and R be the region bounded above by the graph of $f(x)$ and below by the x-axis. Then, the volume of the solid of revolution formed by revolving R around the y-axis is given by

$$V = \int_a^b 2\pi x f(x)dx$$

Other things

Theorem (Arc length of a curve). The arc length of the curve $f(x)$ on $[a, b]$, when $f'(x)$ exists and is continuous on $[a, b]$, is

$$\int_a^b \sqrt{1 + (f'(x))^2} dx$$

Theorem (Surface area of a solid of revolution). Let $f(x)$ be a continuous, nonnegative function defined on $[a, b]$, and R be the region bounded above by the graph of $f(x)$ and below by the x-axis. Then, the surface area of the solid of revolution formed by revolving R around the x-axis is given by

$$S = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} dx$$

Theorem (Mass of a thin rod). Let there be a rod whose left end is located at $x = a$ and whose right end is located at $x = b$, and which is on the x-axis. Let $\rho(x)$ be the linear density at the point $x = a$. Then the mass of the rod is

$$M = \int_a^b \rho(x)\, dx$$

Theorem (Mass of a thin disk). Let there be a disk of radius R whose center is at the origin of the xy-plane. Let the mass be distributed in a rotationally-symmetric way. Let $\rho(r)$ be the radial density at the radius r. Then the mass of the disk is

$$M = \int_0^R 2\pi r \rho(r) dr$$

Theorem (Work). If an object moves along the x-axis from a to b, and $F(x)$ is the force applied to the object when the object is at the point x on the x-axis, then the work is

$$\int_a^b F(x)\, dx$$

Definition of **Average value of a function**. If f is continuous on $[a, b]$, then the average value of f on $[a, b]$ is

$$f_{\text{avg}} := \lim_{n\to\inf} \frac{1}{b-a} \int_a^b f(x)\, dx$$

Theorem (Mean Value Theorem for Integrals). If f is continuous on $[a, b]$, then there exists $c \in [a, b]$ such that

$$f(c) = f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x)\, dx$$

Equivalently,

$$\int_a^b f(x)\, dx = f(c)(b-a)$$

If f is positive on $[a, b]$, then there is a number c such that the rectangle with base $[a, b]$ and height $f(c)$ has the same area as $\int_a^b f(x)\, dx$.

4.1 Improper Integrals

Definition of **Improper Integrals with Infinite Bounds**.

(a) If $\int_a^t f(x)$ exists for every $t \geq a$, then we define

$$\int_a^\infty f(x)dx := \lim_{t\to\infty} \int_a^t f(x)dx$$

provided that this limit exists and is finite.

(b) If $\int_t^b f(x)dx$ exists for every $t \leq b$, then we define

$$\int_{-\infty}^b f(x)dx := \lim_{t\to-\infty} \int_t^b f(x)dx$$

provided that this limit exists and is finite.

$\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are *convergent* if the corresponding limit exists and *divergent* if the limit doesn't exist.

(c) If both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are convergent, then we define

$$\begin{aligned}\int_{-\infty}^\infty f(x)dx &= \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx \\ &= \lim_{s\to-\infty} \int_s^a f(x)dx + \lim_{t\to\infty} \int_a^t f(x)dx\end{aligned}$$

Definition of **Improper Integrals with Discontinuous Integrand**.

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then define

$$\int_a^b f(x)dx := \lim_{t\to b^-} \int_a^t f(x)dx$$

provided that this limit exists and is finite.

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then define

$$\int_a^b f(x)dx := \lim_{t\to a^+} \int_t^b f(x)dx$$

provided that this limit exists and is finite.

$\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are *convergent* if the corresponding limit exists and *divergent* if the limit doesn't exist.

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then we define

$$\begin{aligned}\int_a^b f(x)dx &= \int_a^c f(x)dx + \int_c^b f(x)dx \\ &= \lim_{s\to c^-} \int_a^s f(x)dx + \lim_{t\to c^+} \int_c^b f(x)dx\end{aligned}$$

Theorem (Comparison Test). Suppose f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

(a) If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.

(b) If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent.

Additionally,

(a) If $\int_{-\infty}^b f(x)dx$ is convergent, then $\int_{-\infty}^b g(x)dx$ is convergent.

(b) If $\int_{-\infty}^b g(x)dx$ is divergent, then $\int_{-\infty}^b f(x)dx$ is divergent.

Theorem (Limit Comparison Test). Suppose $f(x)$ and $g(x)$ are positive continuous functions defined on $[a, \infty)$ such that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = c$ for some positive real number c. Then $\int_a^\infty f(x)dx$ converges iff $\int_a^\infty g(x)dx$.

4.2 Differential Equations

Definition of **Differential equation**. An equation involving an unknown function $y = f(x)$ and one or more of its derivatives.

Definition of **Solution to a differential equation**. A function $y = f(x)$ that satisfies the differential equation when f and its derivatives are substituted into the equation.

Procedure (Euler's Method). To numerically approximates the solution to the differential equation $y' = F(x, y)$ with $y(x_0) = y_0$,

$$y_n = y_{n-1} + F(x_{n-1}, y_{n-1})(x_n - x_{n-1})$$

Definition of **Separable Equation**. A separable equation is a differential equation where

$$\frac{dy}{dx} = g(x)f(y)$$

for some function $g(x)$ which depends only on x and $f(y)$ which depends only on y.

Theorem. For a separable equation,

$$\int \frac{1}{f(y)} dy = \int g(x)dx$$