Sound & Light (1)

1.1 Miscellaneous

$$\%\text{ error} = \frac{\text{observed} - \text{theoretical}}{\text{theoretical}} * 100\%$$

1.2 Kinematics

$$x = \frac{a}{2}(\Delta t)^2 + v_0\Delta t + x_0 \qquad v = v_0 + a\Delta t$$

$$v^2 = v_0^2 + 2a\Delta x \qquad \Delta x = \frac{v_0 + v}{2} * \Delta t$$

1.3 Simple Harmonic Motion

$$x = A\cos(\omega t + \varphi) \quad v = -\omega A\cos(\omega t + \varphi) \quad a = -\omega^2 A\cos(\omega t + \varphi)$$

$$x_{\text{max}} = A \qquad v_{\text{max}} = \omega A \qquad a_{\text{max}} = \omega^2 A \qquad F_{\text{max}} = m\omega^2 A$$

1.3.1 Springs and Slinkies

x represents the distance from the equilibrium.

If you put a mass on top of the slinky, Δx_{eq} represents the difference between the original equilibrium and the new equilibrium.

$$F_s = kx = ma \qquad F_{s_{\text{max}}} = k\Delta x_{\text{eq}} = 9.8\Delta m$$

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad T = 2\pi\sqrt{\frac{m}{k}} \qquad \omega = 2\pi f = \sqrt{\frac{k}{m}}$$

$$SPE = \frac{1}{2}kx^2 \qquad KE = \frac{1}{2}mv^2$$

$$TME = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \frac{1}{2}kA^2 = \frac{1}{2}mv_{\text{max}}^2$$

1.3.2 Springs in parallel and series

Quantity	In Series	In Parallel
Equivalent spring constant	$\frac{1}{k_{\text{eq}}} = \frac{1}{k_1} + \frac{1}{k_2}$	$k_{\text{eq}} = k_1 + k_2$
Deflection (elongation)	$x_{\text{eq}} = x_1 + x_2$	$x_{\text{eq}} = x_1 = x_2$
Force	$F_{\text{eq}} = F_1 = F_2$	$F_{\text{eq}} = F_1 + F_2$
Stored energy	$E_{\text{eq}} = E_1 + E_2$	$E_{\text{eq}} = E_1 + E_2$

1.3.3 Pendulums

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{L}} \qquad T = 2\pi\sqrt{\frac{L}{g}}$$

1.4 Waves

$$T = \frac{1}{f} \qquad v = \lambda f \qquad v = \frac{\Delta x}{\Delta t}$$

1.4.1 Slinkies and strings with fixed ends

$$F_T = F_s = kx \qquad \mu = \frac{m}{L} \qquad v = \sqrt{\frac{F_T}{\mu}}$$

Given mass m_T hanging below a pulley, $F_T = m_T g$.

1.5 Standing waves

1.5.1 Open-open, closed-closed

n is the number of antinodes, or the n^{th} harmonic.

$$f_n = f_1 n = \frac{nv}{2L} \qquad f_1 = \frac{v}{2L} \qquad \lambda_n = \frac{2L}{n}$$

1.5.2 Open-closed

$$f_n = f_1 n = \frac{nv}{4L} \qquad f_1 = \frac{v}{4L} \qquad \lambda_n = \frac{2L}{n}$$

1.6 Sound

1.6.1 Speed of sound

$$v = 331\sqrt{\frac{T_{^\circ\text{C}} + 273}{273}} \qquad v \approx 331 + 0.59T$$

1.6.2 Sound intensity

$$I = \frac{\text{Power (W)}}{\text{Area}} = \frac{\text{Power (W)}}{4\pi r^2}$$

$$I_{\text{dB}} = 10\log_{10}\left(\frac{I}{10^{-12}}\right) \qquad I = 10^{\frac{I_{\text{dB}}}{10} - 12}$$

1.6.3 Doppler effect

$$\boxed{O}\rightarrow \ \boxed{S} \quad f_o = f_s\frac{v + v_o}{v} \qquad \leftarrow\boxed{O} \ \boxed{S} \quad f_o = f_s\frac{v - v_o}{v}$$

$$\boxed{O} \ \leftarrow\boxed{S} \quad f_o = f_s\frac{v}{v - v_s} \qquad \boxed{O} \ \boxed{S}\rightarrow \quad f_o = f_s\frac{v}{v + v_s}$$

$$\boxed{O}\rightarrow \ \leftarrow\boxed{S} \quad f_o = f_s\frac{v + v_o}{v - v_s} \qquad \boxed{O}\rightarrow \ \boxed{S}\rightarrow \quad f_o = f_s\frac{v + v_o}{v + v_s}$$

$$\leftarrow\boxed{O} \ \leftarrow\boxed{S} \quad f_o = f_s\frac{v - v_o}{v - v_s} \qquad \leftarrow\boxed{O} \ \boxed{S}\rightarrow \quad f_o = f_s\frac{v - v_o}{v + v_s}$$

1.6.4 Constructive and Destructive Interference (2 dimensions)

For a point on the m^{th} antinodal/nodal line playing the same frequency with the same phase:

$$PD = m\lambda$$

where PD is the path length difference.

1.6.5 Beats

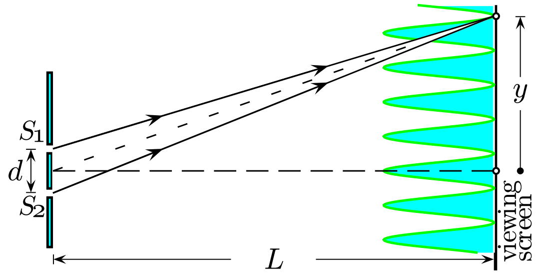
$$f_B = \Delta f$$

1.7 Light

1.7.1 Speed of light

$$c = 299\ 792\ 458 \frac{\mathrm{m}}{\mathrm{s}} \approx 3 * 10^8 \frac{\mathrm{m}}{\mathrm{s}}$$

1.7.2 Two-slit experiment



$$PD = \frac{dy}{L} = m\lambda$$

1.7.3 Mirror

$$r = 2f \qquad \frac{1}{f} = \frac{1}{p} + \frac{1}{q} \qquad M = \frac{h}{h_o} = \frac{-q}{p}$$

In a plane mirror, $p = -q$.

1.7.4 Lenses

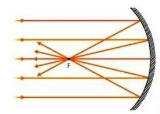
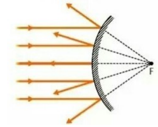
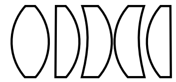
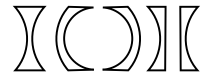
$$\frac{1}{f} = (n-1)(\frac{1}{r_1} - \frac{1}{r_2}) \qquad \frac{1}{f} = \frac{1}{p} + \frac{1}{q} \qquad M = \frac{h}{h_o} = \frac{-q}{p}$$

Multiple lenses

$$p_2 = \Delta x - q_1$$

1.7.5 Mirrors and Lenses

Real $\iff$ inverted, virtual $\iff$ upright.

	Converging	Diverging
Mirror		
Lens		
focal length	+	-
object distance	image	image
∞ to $2f$	Real smaller	Virtual smaller
$2f$ to f	Real larger	
f to 0	Virtual larger	
0 to $-f$	Virtual smaller	Virtual larger
$-f$ to $-2f$		Real larger
$-2f$ to $-\infty$		Real smaller

1.7.6 Refraction / Snell's Law

The **normal line** is the line perpendicular to the surface which touches the intersection of the surface and the light ray.

The **incident angle** is the angle between the ray of light and the normal line.

θ_1 and θ_2 are both measured from the normal line, not the surface.

Refraction occurs when the speed of light in two media are different and light hits the boundary of the two media. The frequency of the light will stay the same, but the speed, wavelength, and direction will change.

$$n = \frac{c}{v} \qquad n_1 \sin\theta_1 = n_2 \sin\theta_2$$

1.7.7 Ray diagrams

Mechanics (2)

2.1 Miscellaneous

$$\% \text{ error} = \frac{\text{observed} - \text{theoretical}}{\text{theoretical}} * 100\%$$

2.2 Kinematics

$$x(t) = \frac{a}{2}t^2 + v_0 t + x_0 \qquad v(t) = v_0 + at$$

$$v^2 = v_0^2 + 2a\Delta x \qquad \Delta x = \frac{v_0 + v}{2} * \Delta t$$

2.3 Forces

$$F_{\text{net}} = ma$$

F_T represents tension. It always points in the direction on which the rope pulls on the object.

F_N represents normal force. It takes the direction and magnitude necessary to prevent the object from passing through the surface that creates the normal force.

2.3.1 Friction

$$f_s \leq \mu_s F_N$$

The static friction takes the direction and magnitude necessary to prevent the object from moving in the component parallel to the surface, until the magnitude reaches $\mu_s F_N$. Upon reaching $\mu_s F_N$, the static friction is replaced by kinetic friction and the object starts moving:

$$k_s = \mu_k F_N$$

2.3.2 Centripetal force

The centripetal force always points towards the center of the circle representing the object's path, and therefore is perpendicular to the velocity. It is just another name for the net force in the centripetal direction. The centripetal acceleration is what causes the object to rotate.

$$F_c = ma_c = m\frac{v^2}{r} = mr\omega^2 \qquad a_c = \frac{v^2}{r}$$

2.4 Work and Energy

$$W = \int_a^b F(\vec{r}) \cdot d\vec{r}$$

In one dimension:

$$W = \int_a^b F(x)dx$$

Work-energy theorem. The work is the change in kinetic energy:

$$W = \Delta K$$

2.5 Simple Harmonic Motion

The object is at rest at the **equilibrium position**. $x = 0$ when the object is at equilibrium, and x represents the distance and direction from the equilibrium. When you pull it to one direction, the **restoring force** pulls the object back toward the equilibrium position. It oscillates back and forth, between $x = -A$ and $x = A$, where A is the **amplitude**. The **period** is the amount of time to complete one oscillation, and the **frequency** is the amount of oscillations that happen in one second (or some other time unit).

$$x(t) = A\cos(\omega t + \phi)$$
$$v(t) = -A\omega \sin(\omega t + \phi)$$
$$a(t) = -A\omega^2 \cos(\omega t + \phi)$$

2.5.1 Spring force

$$F_s = -kx$$
$$U_s = -\tfrac{1}{2}kx^2$$

$$a = -\frac{k}{m}x$$
$$v = \sqrt{\frac{k}{m}(A^2 - x^2)}$$

2.6 Simple Harmonic Motion (old)

$$x = A\cos(\omega t + \varphi) \quad v = -\omega A\cos(\omega t + \varphi) \quad a = -\omega^2 A\cos(\omega t + \varphi)$$

$$x_{\max} = A \qquad v_{\max} = \omega A \qquad a_{\max} = \omega^2 A \qquad F_{\max} = m\omega^2 A$$

2.6.1 Springs and Slinkies

x represents the distance from the equilibrium.

If you put a mass on top of the slinky, Δx_{eq} represents the difference between the original equilibrium and the new equilibrium.

$$F_s = kx = ma \qquad F_{s_{\max}} = k\Delta x_{\text{eq}} = 9.8\Delta m$$

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad T = 2\pi\sqrt{\frac{m}{k}} \qquad \omega = 2\pi f = \sqrt{\frac{k}{m}}$$

$$SPE = \frac{1}{2}kx^2 \qquad KE = \frac{1}{2}mv^2$$

$$TME = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \frac{1}{2}kA^2 = \frac{1}{2}mv_{\max}^2$$

2.6.2 Springs in parallel and series

Quantity	In Series	In Parallel
Equivalent spring constant	$\frac{1}{k_{\text{eq}}} = \frac{1}{k_1} + \frac{1}{k_2}$	$k_{\text{eq}} = k_1 + k_2$
Deflection (elongation)	$x_{\text{eq}} = x_1 + x_2$	$x_{\text{eq}} = x_1 = x_2$
Force	$F_{\text{eq}} = F_1 = F_2$	$F_{\text{eq}} = F_1 + F_2$
Stored energy	$E_{\text{eq}} = E_1 + E_2$	$E_{\text{eq}} = E_1 + E_2$

2.7 Center of mass

The center of mass of an object which consists of masses at discrete points $\vec{r}_1, \vec{r}_2, \vec{r}_3, ..., \vec{r}_n$ is

$$\vec{r}_{\text{cm}} = \frac{\sum m_i \vec{r}_i}{\sum m_i}$$

The center of mass of a solid object with variable density is

$$\vec{r}_{\text{cm}} = \frac{\int \vec{r}dm}{\int dm}$$

To find the velocity of the center of mass, replace $\vec{r}_i$ with $\vec{v}_i$ in the above equations.

2.7.1 Density

$$\text{3D: density} \quad \rho = \frac{M}{v} = \frac{dm}{dV} \quad (\frac{\text{kg}}{\text{m}^3})$$
$$\text{2D: surface density} \quad \sigma = \frac{M}{A} = \frac{dm}{dA} \quad (\frac{\text{kg}}{\text{m}^2})$$
$$\text{1D: linear density} \quad \lambda = \frac{M}{L} = \frac{dm}{dx} \quad (\frac{\text{kg}}{\text{m}})$$

2.8 Momentum/Impulse

Definition of **Momentum**. **Momentum** ($\vec{p}$) is defined as:

$$\vec{p} = m\vec{v} \qquad (\text{Ns} = \text{kg}\frac{\text{m}}{\text{s}})$$

Definition of **Impulse**. **Impulse** is the integral of force over time. By the Impulse-Momentum Theorem, it is also the change in momentum.

$$\vec{J} = \int \vec{F}dt = \Delta p = \vec{F}_{\text{net,avg}}\Delta t$$

Additionally,

$$\vec{F}_{\text{net}} = \frac{dp}{dt}$$

Theorem (Conservation of momentum). Momentum is conserved when there are no external forces acting upon the object.

2.9 Rotational motion

2.9.1 Rotational kinematics

$\vec{\theta}$, angle, is the rotational equivalent of position.

$\Delta\theta$ is called angular displacement.

$$\text{Angular velocity} = \vec{\omega} = \frac{d\vec{\theta}}{dt} \quad \left(\frac{\text{rad}}{\text{s}}\right)$$
$$\text{Angular acceleration} = \vec{\alpha} = \frac{d\vec{\omega}}{dt} \quad \left(\frac{\text{rad}}{\text{s}^2}\right)$$
$$\text{Arc length} = s = R\theta$$
$$\text{Velocity} = v = R\omega$$
$$\text{Acceleration} = a = R\alpha$$

Constant acceleration

$$\omega_f = \omega_0 + \alpha t$$
$$\Delta\theta = \omega_0 t + \tfrac{1}{2}\alpha t^2$$
$$\omega_f^2 = \omega_0^2 + 2\alpha\delta\theta$$

2.9.2 Torque

$$\vec{\tau} = \vec{r} \times \vec{F} \qquad |\tau| = |r||F|\sin\theta$$

2.9.3 Rotational inertia

Where r represents the distance between the axis of rotation and the object, m represents mass, I represents rotational inertia, and K represents rotational kinetic energy:

$$I = mr^2 \qquad I_{\text{total}} := \sum m_i r_i^2 = \int r^2 dm \qquad K = \frac{1}{2}I\omega^2$$

Parallel Axis Theorem. If d is the distance between the axis through the center of mass and the desired axis of rotation parallel to the axis through the center of mass, then the rotational inertia through the desired axis of rotation is

$$I_{\text{II}} = I_{\text{cm}} + Md^2$$

2.9.4 Newton's Second Law

$$\sum \tau = I\alpha$$

2.9.5 Rotational kinetic energy

Definition of **Rotational kinetic energy**.

$$K_{\text{rot}} = \frac{1}{2}I\omega^2$$

Definition of **Rotational work**.

$$W = \int \tau \, d\theta = \Delta K$$

2.9.6 Rolling

During rolling, the velocity of the center of mass is

$$v_{\text{cm}} = R\omega$$

in the direction of rolling. Add $v_{\text{cm}} = R\omega$ to the tangential velocity, $R\omega$ in the direction perpendicular to the radius.

When rolling at the maximum speed possible without slipping, the friction is

$$f_s = \mu_s F_N$$

Otherwise,

$$f_s < \mu_s F_N$$

2.9.7 Angular momentum

$$\vec{L} = I\vec{\omega} \text{ for solid objects}$$

$$\vec{L} = \vec{r} \times \vec{p} \text{ for point masses}$$

$$L = rp\sin(\theta) = rmv\sin(\theta) \text{ for point masses}$$

Note: angular velocity's direction is specified by right-hand rule.

The derivative of angular momentum is the change in external net torque.

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{\text{net, ext}}$$

2.9.8 Equilibrium

If $\sum F = 0$ and $\sum \tau = 0$, then the object is in equilibrium and therefore is not accelerating.

If $\sum F = 0$ and $\sum \tau \neq 0$, then the object is accelerating in its rotation but has constant velocity.

If $\sum F \neq 0$ and $\sum \tau = 0$, then the object is linearly accelerating but is not accelerating rotationally.

If $\sum F \neq 0$ and $\sum \tau \neq 0$, then the object is in complex motion.

2.10 Gravitation

Using the equations which are similar to electrostatics:

Definition of **Gravitational constant**.

$$G \approx 6.67 * 10^{-11} \frac{\text{Nm}^2}{\text{kg}^2}$$

Definition of **Gravitational field**. The vector field g, which applies to all objects.

The electric field points towards masses.

g is also the acceleration caused by gravity.

Newton's law of universal gravitation. If stationary point or spherical masses m_1 and m_2 are near each other but not touching, then the magnitude of the gravitational force between the two objects is

$$|F_g| = \frac{Gm_1m_2}{r^2}$$

Law. The gravitational force on a stationary point or spherical mass g is $F_g = mg$, where g is the strength and direction of the gravitational field.

Law (by Newton's law of universal gravitation). The gravitational field at distance r due to a stationary point or spherical charge m is

$$g = \frac{GM}{r^2}.$$

The gravitational field at distance r due to a massive object whose total mass is $\int dm$ is

$$g = \int \frac{G\,dm}{r^2}$$

where r's tail is at the mass and points towards the point for which we want to calculate g.

2.10.1 Gravitational flux

Definition of **Gravitational flux (Φ)**. The flux through a surface with area vector $\vec{A}$ is

$$\Phi = \int \vec{g} \cdot d\vec{A}$$

where $\vec{g}$ is the gravitational field that passes through the surface, and the area vector's magnitude is the area and direction is normal/perpendicular to the surface.

Definition of **Net flux**. The flux through any surface that encloses a charge.

Gauss's Law. The net flux is proportional to the enclosed mass, and can be equated to the flux through the surface:

$$\Phi_{\text{net}} = -4\pi GM = \oint \vec{g} \cdot d\vec{A}$$

Electromagnetism (3)

3.1 Electrostatics

Definition of **Electric charge**. A scalar quantity that can describe an object. It can be positive or negative

Definition of **Elementary charge** (e). The smallest possible charge $e \approx 1.6 * 10^{-19}$ C.

3.1.1 Charging

Procedure (Charging by friction). Rub two nonconductive objects together. The one with the greater electron affinity (electronegativity) becomes negatively charged.

Procedure (Polarization). When two objects near each other, their charge distributions will change in order to ensure that similar charges in the different objects don't get too near each other.

3.1.2 Electric fields

Definition of **Permittivity of free space** (ε_0).

$$\varepsilon_0 \approx 8.85 * 10^{-12} \frac{\mathrm{C}^2}{\mathrm{Nm}^2}$$

Definition of **Coulomb's constant** (k).

$$k = \frac{1}{4\pi\varepsilon_0} \approx 8.99 * 10^9 \frac{\mathrm{Nm}^2}{\mathrm{C}^2}$$

Coulomb's Law. If stationary point or spherical charges q_1 and q_2 are near each other but not touching, then the magnitude of the electrostatic force between the two objects is

$$|F_e| = \frac{k|q_1||q_2|}{r^2} = \frac{|q_1||q_2|}{4\pi\varepsilon_0 r^2}$$

Definition of **Electric field**. The vector field E ($\frac{\mathrm{N}}{\mathrm{C}}$), which applies to all objects.

The electric field points away from positive charges and towards negative charges.

Law. The electrostatic force on a stationary point or spherical charge q is

$$F_e = Eq$$

where E is the strength and direction of the electric field.

Law (by Coulomb's Law). The electric field at distance r due to a stationary point or spherical charge Q is

$$E = \frac{kQ}{r^2} = \frac{Q}{4\pi\varepsilon_0 r^2}$$

The electric field at distance r due to a charged object whose total charge is $\int dq$ is

$$E = \int \frac{k\,dq}{r^2} = \int \frac{dq}{4\pi\varepsilon_0 r^2}$$

where r's tail is at the charge and points towards the point for which we want to calculate E.

Definition of **Charge density**.

$$\begin{array}{lll} \text{3D: density} & \rho = \frac{Q}{v} = \frac{dq}{dV} & (\frac{\mathrm{C}}{\mathrm{m}^3}) \\ \text{2D: surface density} & \sigma = \frac{Q}{A} = \frac{dq}{dA} & (\frac{\mathrm{C}}{\mathrm{m}^2}) \\ \text{1D: linear density} & \lambda = \frac{Q}{L} = \frac{dq}{dx} & (\frac{\mathrm{C}}{\mathrm{m}}) \end{array}$$

Example (E-field of a sheet of charge). The magnitude of the E-field caused by a sheet of charge is

$$E = \frac{\sigma}{2\varepsilon_0}$$

where the E-field points away from the sheet if the charge is positive, and towards the sheet if the charge is negative.

Example (E-field of a cylindrical charge). The magnitude of the E-field caused by a cylindrical charge of radius R with negligible end effects, at radius r from its axis, is:

$$\frac{R\sigma}{r\varepsilon_0}$$

3.1.3 Electric flux

Definition of **Electric flux** (Φ). The flux through a surface with area vector $\vec{A}$ is

$$\Phi_E = \int \vec{E} \cdot d\vec{A} \qquad \left(\frac{\mathrm{Nm}^2}{\mathrm{C}}\right)$$

where $\vec{E}$ is the electric field that passes through the surface, and the area vector's magnitude is the area and direction is normal/perpendicular to the surface.

Definition of **Net flux**. The flux through any surface that encloses a charge.

Gauss's Law. The net flux is equivalent to the enclosed charge divided by the permittivity of free space, and can be equated to the flux through the surface:

$$\Phi_{\text{net}} = \frac{q}{\varepsilon_0} = 4\pi kq = \oint \vec{E} \cdot d\vec{A}$$

3.1.4 Electric potential

Definition of **Electric potential energy**. The work required to move a charge from a reference position to its current location in the electric field.

When the electrostatic force does work W_e on the object, its electric potential energy decreases and its kinetic energy increases:

$$\Delta U_e = -W_e$$

Definition of **Electric potential**. The work per unit charge required to move a charge from a reference position to its current location in the electric field.

Definition of **Electric potential difference / Voltage**. The difference in electric potential between two points.

$$\Delta V = \frac{\Delta U_e}{q} = -\frac{W_e}{q} = -\int \vec{E}(\vec{r}) \cdot d\vec{r}$$

Therefore,

$$\vec{E} = -\frac{dV}{dr}$$

Lemma. If an electric potential V between two points separated by distance L is caused by a uniform electric field of strength E,

$$V = -EL$$

Definition of **Equipotential line/surface**. A line/surface where for each point on the line/surface, the electric potential is the same.

Graphs of equipotential lines where each line's electric potential is an integer multiple of some number produce a topographic-like map of the electric field, where each of the equipotential lines can be viewed as contour lines.

Theorem (Potential caused by a point or spherical charge). The electric potential outside a point or spherical charge, at distance r from the center of the charge, is

$$\frac{kq}{r}$$

The electric potential inside a spherical charge of radius R is

$$\frac{kq}{R}$$

(This derives from Coulomb's Law.)

3.2 Circuits

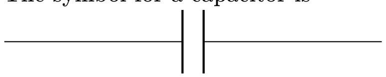
3.2.1 Capacitance

Definition of **Capacitor**. At least one charged conductor, usually two.

Common capacitor shapes:

- Parallel plate
- Cylindrical
- Spherical

The symbol for a capacitor is



Definition of **Capacitance**. Where Q is the magnitude of the charge on one of the plates, and V is the potential difference between two plates:

$$C = \frac{Q}{V} \qquad \left(\mathrm{F} = \frac{\mathrm{C}}{\mathrm{V}}\right)$$

The capacitance remains constant for a given capacitor regardless of any change in charge or voltage; it only depends on the shape of the capacitor.

The capacitance is a positive (unsigned) value.

Example (Parallel plate capacitor). The capacitance of a capacitor containing two parallel plates, where each plate has area A and the distance between the plates is d, is

$$C = \frac{\varepsilon_0 A}{d}$$

Example (Spherical shell capacitor). The capacitance of a capacitor containing two concentric spherical shells, where the smaller shell has radius a and the larger has radius b, is

$$C = \frac{ab}{bk - ak}$$

Example (Cylindrical capacitor). The capacitance of a capacitor containing two concentric cylindrical shells with negligible end effects, where the smaller shell has radius a and the larger has radius b and both have length L, is

$$C = \frac{2\pi\varepsilon_0 L}{\ln\left(\frac{b}{a}\right)}$$

Theorem (Energy stored in a capacitor).

$$U_e = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}VQ = \frac{1}{2}CV^2$$

Definition of **Dielectric**. An object that can be polarized.

Definition of **Permittivity**. The permittivity of space containing a dielectric is equal to

$$K\varepsilon_0$$

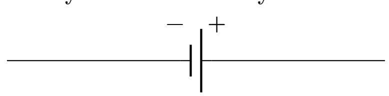
where K is the dielectric's **dielectric constant**.

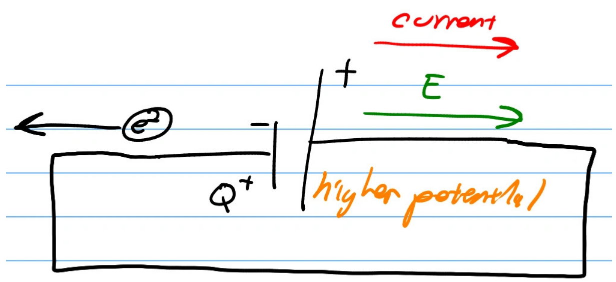
3.2.2 Miscellaneous circuit stuff

Definition of **Battery**. A battery has two terminals, a positive and a negative terminal. The positive terminal has higher electric potential than the negative terminal, and the battery creates an E-field between the positive and negative terminal. The potential difference between the positive and negative terminal is called the **voltage** of the battery, and is represented as $\mathcal{E}$.

The electric field causes positive current to flow from the positive to negative terminal. Equivalently, negative current flows from the negative to positive terminal, which is the way the actual electrons flow.

The symbol for a battery is





3.2.3 Loop Law

Loop Law. In a circuit, the sum of the signed potential differences along the entire circuit is 0.

$$\sum \Delta V = 0$$

In a circuit with a battery, this means that the voltage (unsigned) of the battery is equal to the sum of the other potential differences along the circuit.

If a set of capacitors were replaced by a single capacitor without changing any of the observable properties of the circuit, then C_{eq} (the **equivalent capacitance**) is equal to the capacitance of that single capcitor and Q_{eq} (the **equivalent charge**) is equal to the capacitance of one plate of that single capacitor.

Example (Capacitors in parallel).

$$C_{\text{eq}} = \sum C_i$$
$$\mathcal{E} = V_1 = \cdots = V_n$$
$$Q_{\text{eq}} = \sum Q_i$$

Example (Capacitors in series).

$$\frac{1}{C_{\text{eq}}} = \sum \frac{1}{C_i}$$
$$\mathcal{E} = \sum V_i$$
$$Q_{\text{eq}} = Q_1 = \cdots = Q_n$$

3.2.4 Current and resistance

Definition of **Current / amperage (i or I)**. The rate at which charges flow through a conductor.

$$i = I = \frac{dq}{dt} \qquad \left(\mathrm{C} = \frac{\mathrm{C}}{\mathrm{s}}\right)$$

Definition of **Junction**. A place in a circuit at which the current can change.

Law (Junction Rule). The sum of the current into a junction equals the sum of the current out of the junction.

$$\sum I_{\text{into}} = \sum I_{\text{out}}$$

Definition of **Current density**.

$$J = \frac{i}{A}$$

Definition of **Resistance** (R).

$$R = \frac{V}{i} \qquad \left(\mathrm{C} = \frac{\mathrm{V}}{\mathrm{C}}\right)$$

The resistance is a property of the conducting object.

Ohm's Law.

$$V = iR$$

Definition of **Resistivity**.

$$\rho = \frac{E}{J} \qquad (\mathrm{Cm})$$

The resistivity is a property of the conducting material.

Lemma. The resistance of a uniform conductor of length L, cross-sectional area A, and resistivity ρ is

$$R = \frac{\rho L}{A}$$

Law.

$$Q = Nq$$

where N is the number of charge carriers and q is the charge of each charge carrier.

If the charge carrier is an electron (as usual), then

$$Q = Ne$$

Definition of **Drift speed**.

$$v_{\text{drift}} = \frac{J}{nq}$$

where $n = \frac{N}{\text{Volume}}$ (the charge carrier density), and q is the charge of each charge carrier ($q = e$ if the charge carrier is an electron).

Theorem (Power consumed by a resistor). Most of this energy is converted to heat.

$$P = IV = I^2R = \frac{V^2}{R}$$

Lemma (Resistors in parallel). By the Loop Law,

$$V_{eq} = V_1 = \cdots = V_n$$

By the Junction Rule,

$$\frac{\mathcal{E}}{R_{eq}} = i_{\text{total}} = \sum i_i$$

Therefore by Ohm's Law,

$$\frac{1}{R_{eq}} = \sum \frac{1}{R_i}$$

Lemma (Resistors in series). By the Loop Law,

$$V_{eq} = \sum V_i$$

Therefore by Ohm's Law,

$$R_{eq} = \sum R_i$$

3.2.5 RC circuits

Definition of **RC circuit**. A circuit with a resistor and a capacitor.

Definition of **Time constant**. For an RC circuit,

$$\tau := R_{eq}C_{eq} \qquad \text{(s)}$$

Law (Charging a capacitor). A capacitor can be charged by applying an $\mathcal{EMF}$ such as a battery on both sides of the capacitor, possibly including a resistor or resistors with equivalent resistance R.

Initial and final conditions:

$$\begin{aligned} t &= 0 & t &\to \infty \\ q_0 &= 0 & q_{\max} &= C\mathcal{E} \\ i_0 &= \frac{\mathcal{E}}{R} & i_f &= 0 \end{aligned}$$

Applying the loop law:

$$\mathcal{E} - R\frac{dq}{dt} - \frac{q}{c} = 0$$

Solving the equation, we get:

$$\begin{aligned} q(t) &= q_{\max}\left(1 - e^{-\frac{t}{\tau}}\right) \\ i(t) &= i_0 e^{-\frac{t}{\tau}} \end{aligned}$$

Law (Discharging a capacitor). A capacitor can be discharged by shorting both plates, possibly including a resistor or resistors with equivalent resistance R in between.

Initial and final conditions:

$$\begin{aligned} t &= 0 & t &\to \infty \\ q_{\max} &= C\mathcal{E} & q_f &= 0 \\ i_0 &= -\frac{\mathcal{E}}{R} & i_f &= 0 \end{aligned}$$

Applying the loop law:

$$-R\frac{dq}{dt} - \frac{q}{c} = 0$$

Solving the equation, we get:

$$\begin{aligned} q(t) &= q_{\max} e^{-\frac{t}{\tau}} \\ i(t) &= i_0 e^{-\frac{t}{\tau}} \end{aligned}$$

(ensure $i_0 < 0$)

3.2.6 Magnetism

Definition of **Magnetic field** ($\vec{B}$). The field lines exit through the north pole of the magnet and enter through the south pole of the magnet. They always form a closed loop.

A magnetic field is caused by a moving charged particle, and is perpendicular to the velocity of a particle, as described by the right hand rule - if the velocity $\vec{v}$ is represented by the right-hand thumb, then curling the rest of the fingers will give the direction of the $\vec{B}$-field.

Definition of **Permanent magnet**. A magnet (north and south pole) which stays magnetic without a current or other external force being applied.

It is caused when the electron spins within the magnet are all aligned.

Definition of **Solenoid**. A coiled conducting wire.

If a current is run through the solenoid, it becomes an **electromagnet**.

Definition of **Magnetic force**.

$$\vec{F}_B = q\vec{v} \times \vec{B}$$

therefore

$$|F_B| = q|v||B|\sin\theta$$

Lemma (Magnetic force on a straight wire). Given a wire of length l (define $\vec{l}$ to be the vector whose magnitude is the length of the wire and whose direction is the direction of the current), with current i passing through it, in a uniform external magnetic field $\vec{B}_{\text{ext}}$,

$$\vec{F}_B = i\vec{l} \times \vec{B}_{\text{ext}}$$

Definition of **Permeability of free space**.

$$\mu_0 = 4\pi * 10^{-7} \qquad \frac{\text{Tm}^2}{\text{C}}$$

Biot-Savart Law. The B-field at point P caused by a point current part of a longer current-carrying object is

$$d\vec{B} = \frac{\mu_0 i}{4\pi}\frac{d\vec{s} \times \hat{r}}{r^2}$$

where $d\vec{s}$ is the length of the point current and $\hat{r}$ is the unit vector of the vector from the point current to P.

Ampère's Law. The B-field along an Amperian loop (a closed loop) is related to the enclosed current by

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{\text{enc}}$$

where $d\vec{s}$ represents the length of a infinitesimally small segment of the Amperian loop and i_{enc} represents the current enclosed by the Amperian loop. ($\vec{B}$ is measured at each point along the Amperian loop.)

Example (B-field due to a straight wire). The B-field due to a straight wire carrying current i, measured at a distance r from the center of the wire, is

$$B = \frac{\mu_0 i}{2\pi r}$$

Within the straight wire with a radius of R, at a distance r from the center of the wire,

$$B = \frac{\mu_0 i r}{2\pi R^2}$$

Example (B-field inside a solenoid). The B-field along the central axis inside a solenoid with n loops of wire per unit length carrying current i, is

$$B_0 = \mu_0 n i$$

3.2.7 Magnetic flux

Definition of **Magnetic flux**. The flux through a surface with area vector $\vec{A}$ is

$$\Phi_B = \int \vec{B} \cdot d\vec{A} \qquad (\mathrm{W} = \mathrm{Tm}^2)$$

where $\vec{B}$ is the B-field that passes through the surface, and the area vector's magnitude is the area and direction is normal/perpendicular to the surface.

Faraday's Law. The emf induced by a changing magnetic flux through the surface through a closed conductive loop is

$$\mathcal{E} = -N\frac{d\Phi_B}{dt}$$

Lenz's Law. The direction of the induced emf (and thus the induced current) is such that the induced magnetic field it produces opposes the change in the magnetic flux that caused it.

Theorem (by Lenz's Law). An increasing magnetic flux induces a B-field in the opposite direction of the changing flux.

An decreasing magnetic flux induces a B-field in the same direction as the changing flux.

The induced current's direction is determined by the induced B-field using the right-hand rule.

3.2.8 Inductors

Definition. A device which stores energy in a magnetic field, based on the current flowing through it.

Definition of **Inductance**. The inductance of an inductor with N loops, with current I flowing through it, and where Φ_B is the flux through the surface bounded by one loop of the inductor:

$$L = \frac{N\Phi_B}{I} \qquad \left(\mathrm{H}\frac{\mathrm{Tm}^2}{\mathrm{C}}\right)$$

Theorem (Inductance of a solenoidal inductor). The inductance of a solenoid with n loops of wire per unit length, of length l, where A is the area of the surface bounded by one loop of the solenoid, is

$$L = \mu_0 n^2 A l$$

Theorem (Constitutive equation). The EMF induced by an inductor is given by

$$\mathcal{E} = -L\frac{di}{dt}$$

The direction is specified by Lenz's Law.

Theorem (Energy stored in an inductor). The energy stored in an inductor is

$$U_L = \frac{1}{2}LI^2$$

3.2.9 RL circuits

Definition of **RL circuit**. A circuit with a resistor and an inductor.

Definition of **Inductive time constant**. For an RL circuit,

$$\tau_L := \frac{L}{R} \qquad (\mathrm{s})$$

Theorem (Behavior of an RL circuit with a battery). By the Loop Law,

$$\mathcal{E} - IR - L\frac{di}{dt} = 0$$

Solving the diffeq, we get

$$I(t) = \frac{\mathcal{E}}{R}\left(1 - e^{-\frac{t}{\tau_L}}\right)$$

The circuit starts off with no current, and as time goes on the current increases until it reaches its final value $I = \frac{\mathcal{E}}{R}$ at which point the inductor stores its max energy.

Theorem (Behavior of an RL circuit after the battery is disconnected). By the Loop Law,

$$-IR - L\frac{di}{dt} = 0$$

Solving the diffeq, we get

$$I(t) = \frac{\mathcal{E}}{R}e^{-\frac{t}{\tau_L}}$$

The circuit starts off with current of $I = \frac{\mathcal{E}}{R}$ and ends with no current, at which point the inductor stores no energy.

3.2.10 LC circuits

Definition of **LC circuit**. A circuit with an inductor and a capacitor.

This circuit stores energy. At some times the capacitor stores energy, but the capacitor discharges into the inductor until the inductor stores all the energy. Then the inductor releases energy towards the capacitor, causing the capacitor to store all the energy but with the positive and negative side swapped.

Theorem.

$$\omega = \frac{1}{\sqrt{LC}} \qquad f = \frac{1}{2\pi\sqrt{LC}}$$

The charge and current on one plate of the capacitor is given by

$$q(t) = Q\cos(\omega t) \qquad i(t) = q'(t) = -Q\omega\sin(\omega t)$$

where Q is the maximum charge on one plate of the capacitor.

The energy in the LC circuit is:

$$U_{\text{total}} = \frac{1}{2}\frac{Q^2}{C} = U_e + U_L = \frac{1}{2}\frac{q^2}{C} + \frac{1}{2}Li^2$$

Mathematics (4)

4.1 Logarithms

$$\log_b(MN) = \log_b(M) + \log_b(N)$$
$$\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)$$
$$\log_b(M^p) = p \cdot \log_b(M)$$
$$\log_b(a) = \frac{\log_x(a)}{\log_x(b)}$$
$$\log_b(b) = 1$$

4.2 Notation

deg $p(x)$ means the degree of polynomial p.

LC $p(x)$ means the leading coefficient of polynomial p.

4.3 Rational functions

For a rational function $f(x) = \frac{p(x)}{q(x)}$, cancel out any common factors, then:

- For all rational functions:
  - VA: roots of $q(x)$
  - Roots: roots of $p(x)$
- When deg $p(x) =$ deg $q(x)$:
  - HA: $y = \frac{\text{LC } p(x)}{\text{LC } q(x)}$
- When deg $p(x) <$ deg $q(x)$:
  - HA: $y = 0$
- When deg $p(x) >$ deg $q(x)$:
  - HA: none
  - slant asymptote: $\frac{p(x)}{q(x)}$ excluding remainder

4.4 Polynomials

4.4.1 Linear equations

Slope-intercept form: $y = mx + b$

Point-slope form: $y - y_1 = m(x - x_1)$ for point (x, y)

Standard form: $ax + by = c$

4.4.2 Quadratic equations

Standard form: $y = ax^2 + bx + c$

Vertex form: $y = a(x-h)^2 + k$ for vertex (h, k)

Sum of roots: $\frac{-b}{a}$

Product of roots: $\frac{c}{a}$

4.4.3 Higher-degree polynomials

In a polynomial

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

, with roots

$$r_1, r_2, r_3, \ldots, r_n$$

then:

$$r_1 + r_2 + r_3 + \cdots + r_n = \sum_{k=1} n r_k = -\frac{a_{n-1}}{a_n}$$

4.5 Sequences and Series

4.5.1 Explicit formulas

Aritmetic sequence: $a_n = a_1 + r(n-1)$

Geometric sequence: $a_n = a_1 * r^{n-1}$

Harmonic sequence: $a_n = \frac{1}{a_1 + r(n-1)}$

4.5.2 Arithmetic and Geometric Series

In the following equations, substituting $j = 1$ with $j = 0$, $j - 1$ with j, and a_1 with a_0 will produce the same result.

$$\sum_{j=1}^{n}(a_1 + r(j-1)) = \frac{n}{2}(2a_1 + (n-1)d)$$
$$\sum_{j=1}^{n}(a_1 * r^{j-1}) = \frac{a_1(1-r^n)}{1-r}$$
$$\sum_{j=1}^{\infty}(a_1 * r^{j-1}) = \frac{a_1}{1-r} \text{ for } r \in [-1, 1]$$

4.5.3 Special Sums

$$\sum_{j=1}^{n} c = nc \qquad \sum_{j=1}^{n} ca_j = c\sum_{j=1}^{n} a_j$$
$$\sum_{j=1}^{n}(a_j + b_j) = \sum_{j=1}^{n} a_j + \sum_{j=1}^{n} b_j \qquad \sum_{j=1}^{n} j = \frac{n(n+1)}{2}$$
$$\sum_{j=1}^{n} j^2 = \frac{n(n+\frac{1}{2})(n+1)}{3} \qquad \sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4}$$
$$= \frac{n(2n+1)(n+1)}{6}$$

4.6 Trigonometry

$\circ$	rad	sin	cos	tan
0°	0	0	1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	$\frac{\pi}{2}$	1	0	undef

4.6.1 Law of Sines and Cosines

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c} \qquad c^2 = a^2 + b^2 - 2ab\cos(C)$$

4.6.2 Triangle area

$$K = \frac{1}{2}bh \qquad K = \frac{1}{2}bc\sin(A) \qquad K = \sqrt{s(s-a)(s-b)(s-c)}$$

4.6.3 More identities

$$(\sin A)^2 + (\cos A)^2 = 1 \qquad (\tan A)^2 + 1 = (\sec A)^2$$
$$\sin(\frac{\pi}{2} - x) = \cos(x) \qquad (\cot A)^2 + 1 = (\csc A)^2$$
$$\cos(-x) = \cos(x) \qquad \sin(-x) = \sin(x) \qquad \tan(-x) = \tan(x)$$

4.6.4 Slope

Where α is the angle between the line and the x-axis, and m is the slope of the line:

$$m = \tan\alpha$$

4.6.5 Sum and difference formulas

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$
$$\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B)$$
$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$
$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$
$$\tan(A+B) = \frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$$
$$\tan(A-B) = \frac{\tan(A)-\tan(B)}{1+\tan(A)\tan(B)}$$

$$\sin(2A) = 2\sin(A)\cos(A)$$
$$\cos(2A) = (\cos A)^2 - (\sin A)^2 = 2(\cos A)^2 - 1 = 1 - 2(\sin A)^2$$
$$\tan(2A) = \frac{2\tan(A)}{1-(\tan A)^2}$$

4.7 Vectors

$$\vec{v} + \vec{w} = \begin{bmatrix} v_x + w_x \\ v_y + w_y \\ v_z + w_z \end{bmatrix} \qquad c * \vec{v} = \begin{bmatrix} c * v_x \\ c * v_y \\ c * v_z \end{bmatrix}$$
$$\vec{v} \cdot \vec{w} = v_x w_x + v_y w_y + v_z w_z = |\vec{v}||\vec{w}|\cos(\theta)$$
$$|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}|\sin(\theta) = \text{area of parallelogram}$$
$$\vec{v} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} \qquad \vec{v} \times \vec{w} \perp \vec{v} \qquad \vec{v} \times \vec{w} \perp \vec{w}$$
$$\vec{v} \perp \vec{w} \iff \vec{v} \times \vec{w} = \vec{0} \qquad \vec{v} \parallel \vec{w} \iff \vec{v} \cdot \vec{w} = 0$$
$$\hat{v} = \frac{\vec{v}}{|\vec{v}|} \qquad \text{proj}_{\vec{b}}\vec{v} = \frac{\vec{v} \cdot \vec{b}}{\vec{b} \cdot \vec{b}} * \vec{b} = (|\vec{v}|\cos(\theta))$$

Right-hand rule

To determine the direction of $\vec{v} \times \vec{w}$, put the side of the right hand on $\vec{v}$ and curl the fingers toward $\vec{w}$. The direction the thumb is pointing is the direction of $\vec{v} \times \vec{w}$.

4.8 Polar

4.8.1 Polar and Cartesian sytems

With point $(x, y) = (r; \theta) = (r; \beta)$, where θ is CCW from the x-axis and β is a bearing, CW from the y-axis:

$$x = r\cos(\theta) = r\sin(\beta) \qquad y = r\sin(\theta) = r\cos(\beta)$$
$$r = \sqrt{x^2 + y^2} \qquad \theta \equiv \arctan(\frac{y}{x}) \qquad \beta \equiv \arctan(\frac{x}{y})$$

4.8.2 Converting functions

Try these substitutions in order:

$$x^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad x = r\cos\theta \qquad y = r\sin\theta$$

4.8.3 Limaçons and Petals

The function $y = A\cos(B(\theta + C)) + D$ is equivalent to $y = A\cos(B\theta) + D$ rotated C degrees/radians clockwise.

When C is 0 and B is 1, the x-intercepts are $A \pm D$ and the y-intercepts are $\pm D$, and it forms a limaçon.

When C is 0, but $B \neq 1$, then this sometimes still holds. The x-intercepts may also be $\pm A \pm D$.

There are B petals, with the axis of the first petal on the positive x-axis.

When B is even and $|D| < 1$, then the number of petals is $2B$.

Using sin instead of cos, limaçons have their axes on the positive y-axis, while for petals, the first petal starts from the positive x-axis and curves upwards.

4.9 Complex

$$\text{cis}(\theta) = e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

To find the n^{th} root of $x_r \,\text{cis}(x_\theta)$, solve the equation $z_r^n \,\text{cis}(nz_\theta) = x_r \,\text{cis}(x_\theta + 360^\circ k)$ for $k \in \mathbb{R}$.

4.10 Function domain

Function	Domain x	Range y
$\log(x)$	$(0, \infty)$	$\mathbb{R}$
$\sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$\arcsin(x)$	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$\arccos(x)$	$[-1, 1]$	$[0, \pi]$
$\arctan(x)$	$\mathbb{R}$	$(-\frac{\pi}{2}, \frac{\pi}{2})$

Calculus Theorems (5)

1 Completeness

Theorem (Completeness of the Real Numbers). Every nonempty subset S of $\mathbb{R}$ which is bounded above has a least upper bound $\sup S$.

Definition of **Supremum** ($\sup S$). A number such that

(1) $s \le \sup S$ for every $s \in S$ (which just says that $\sup S$ is an upper bound for S)

(2) If u is any upper bound for S, then $\sup S \le u$ (which says that $\sup S$ is the least upper bound for S).

Definition of **Infimum** ($\inf S$). A number such that

(1) $\inf S \le s$ for every $s \in S$ (i.e. $\inf S$ is an lower bound for S)

(2) If l is any upper bound for S, then $l \le \inf S$ (i.e. $\inf S$ is the greatest lower bound for S).

Theorem. Every nonempty subset S of $\mathbb{R}$ which is bounded below has a greatest lower bound.

Theorem. If $\min S$ exists, then $\min S = \inf S$.

Theorem. If $A \subset R$ and $c \ge 0$, and $cA := ca : a \in A$, $\sup cA = c \sup A$.

Theorem (Rationals between Reals). For every two real numbers a and b with $a < b$, there exists a rational number r satisfying $a < r < b$.

Nested Intervals Theorem.
If $I_n = [a_n, b_n] = \{x \in R : a_n \le x \le b_n\}$ s.t. $a_n \le a_{n+1}$ and $b_{n+1} \le b_n$ for $n \in \mathbb{N}$, so that $I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq ...$, then $\bigcap_{n=1}^{\infty} I_n \ne \emptyset$.

If $\inf\{b_n - a_n\} = 0$, then $\bigcap_{n=1}^{\infty} I_n = \{x\}$, where $x = \sup\{a_n\} = inf\{b_n\}$.

Capture Theorem. If A is a nonempty subset of $\mathbb{R}$, then:

(i) If A is bounded above, then any open interval containing $\sup A$ contains an element of A.

(ii) Similarly, if A is bounded below, then any open interval containing $\inf A$ contains an element of A.

Theorem (Binary Search (Bisection Method)). If we binary-search for x over $I_1 = [a_1, b_1]$ for $a_1, b_1 \in \mathbb{Q}$, we define I_n s.t. either $I_n := [a_{n-1}, \frac{a_{n-1}+b_{n-1}}{2}]$ or $I_n := [\frac{a_{n-1}+b_{n-1}}{2}, a_{n-1}]$, and we define $a_n := \inf I_n$ and $b_n := \sup I_n$. We define A to be the set of all a_n, and B to be the set of all b_n.

Then, the size of $I_n = \frac{b_1 - a_1}{2^n} = b_n - a_n$, and $\bigcap_{n=1}^{\infty} I_n\{x\}$, where $x = \sup\{a_n\} = \inf\{b_n\}$.

2 Limits

Definition of **Limit**. If $\lim_{x\to a} f(x) = L$, then for any $\varepsilon > 0$, there exists $\delta > 0$ s.t. for any $x \in (a-\delta, a) \cup (a, a+\delta)$, $f(x) \in (L - \varepsilon, L + \varepsilon)$.

Alternatively,

Definition of **Limit**. If $\lim_{x\to a} f(x) = L$, then for any $\varepsilon > 0$, there exists $\delta > 0$ s.t. $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.

Definition of **Right-sided limit**. If $\lim_{x\to a^+} f(x) = L$, then for any $\varepsilon > 0$, there exists $\delta > 0$ s.t. $|f(x) - L| < \varepsilon$ whenever $0 < x - a < \delta$.

Definition of **Left-sided limit**. If $\lim_{x\to a^-} f(x) = L$, then for any $\varepsilon > 0$, there exists $\delta > 0$ s.t. $|f(x) - L| < \varepsilon$ whenever $0 < a - x < \delta$.

Theorem (Limit Laws). Let $c \in R$ be a constant and suppose the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

- $\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
- $\lim_{x\to a}(cf(x)) = c \lim_{x\to a} f(x)$
- $\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$
- $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$, provided that $\lim_{x\to a} g(x) \ne 0$
- $\lim_{x\to a} x^n = (\lim_{x\to a} x)^n$
- $\lim_{x\to a} \frac{a(x)b(x)}{c(x)b(x)} = \lim_{x\to a} \frac{a(x)}{c(x)}$

These laws also apply to one-sided limits.

Theorem (L'Hopital's Rule). If f and g are differentiable and $g'(x) \ne 0$ on an open interval I that surrounds a, and $\lim_{x\to a} \frac{f(x)}{g(x)} \in \{\frac{0}{0}, \pm\frac{\infty}{\infty}\}$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$.

Theorem (Composition of Limits). If f is continuous at L and $\lim_{x\to a} g(x) = L$, then $\lim_{x\to a} f(g(x) = f(\lim_{x\to a} g(x))) = f(L)$

Theorem (Operations on infinity). For $x \in \mathbb{R}$,

$$\infty + x = \infty \qquad -\infty + x = -\infty \qquad \frac{x}{\pm\infty} = 0$$

$$x * \infty = \begin{cases} \infty & \text{if } x > 0 \\ -\infty & \text{if } x < 0 \end{cases}$$

$$x * -\infty = \begin{cases} -\infty & \text{if } x > 0 \\ \infty & \text{if } x < 0. \end{cases}$$

Definition of **Indeterminate forms**. The following forms are indeterminate and you cannot evaluate them.

$$\frac{0}{0}, \frac{\pm\infty}{\pm\infty}, 0 * \pm\infty, \infty - \infty$$

Squeeze Theorem. Let f , g, and h be defined for all $x \ne a$ over an open interval containing a. If

$$f(x) \le g(x) \le h(x)$$

for all $x \ne a$ in an open interval containing a and

$$\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$$

where $L \in \mathbb{R}$, then $\lim_{x\to a} g(x) = L$.

3 Continuity

Definition of **Continuity at a point**. Function f is continuous at point a if $\lim_{x\to a} f(x) = f(a)$.

Definition. f has a **removable discontinuity** if $\lim_{x\to a} f(x) = L \in \mathbb{R}$ (in this case either $f(a)$ is undefined, or $f(a)$ is defined by $L \ne f(a)$).

Definition. f has a **jump discontinuity** if $\lim_{x\to a^-} f(x) = L_1 \in \mathbb{R}$ and $\lim_{x\to a^+} f(x) = L_2 \in \mathbb{R}$ but $L1 \ne L2$.

Definition. f has an **infinite discontinuity** at a if $\lim_{x\to a^-} f(x) = \pm\infty$ or $\lim_{x\to a^+} f(x) = \pm\infty$

Intermediate Value Theorem. If f is continuous on $[a, b]$, then for any real number L between $f(a)$ and $f(b)$ there exists at least one $c \in [a, b]$ such that $f(c) = L$. In other words, if f is continuous on $[a, b]$, then the graph must cross the horizontal line $y = L$ at least once between the vertical lines $x = a$ and $x = b$.

Aura Theorem. If $f(x)$ is continuous and $f(a)$ is positive, then there exists an open interval containing a such that for all x in the interval, $f(x)$ is positive.

If $f(x)$ is continuous and $f(a)$ is negative, then there exists an open interval containing a such that for all x in the interval, $f(x)$ is negative.

Bolzano's Theorem. Let f be a continuous function defined on $[a, b]$. If 0 is between $f(a)$ and $f(b)$, then there exists $x \in [a, b]$ such that $f(x) = 0$.

4 Derivatives

The derivative is the instantaneous rate of change, and the slope of the tangent line to the point.

Definition of **Derivative** ($f'(a)$).

$$\frac{d}{da}f(a) = f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

Theorem (Tangent line to a point). The equation of the tangent line to the point $(a, f(a))$ is

$$y = f'(a)(x-a) + f(a)$$

Derivative Rules

Theorem (Difference Rule).

$$\frac{d}{dx}(f(x)-g(x)) = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

Theorem (Sum Rule).

$$\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

Theorem (Constant Multiple Rule).

$$\frac{d}{dx}(cf(x)) = c\frac{d}{dx}f(x)$$

Theorem (Product Rule).

$$\begin{aligned}\frac{d}{dx}(f(x)g(x)) &= f'(x)g(x) + f(x)g'(x)\\ \frac{d}{dx}(f(x)g(x)h(x)) &= f'(x)g(x)h(x) + f(x)g'(x)h(x)\\ &+ f(x)g(x)h'(x)\end{aligned}$$

and so on.

Theorem (Quotient Rule).

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$$

Theorem (Power Rule).

$$\frac{d}{dx}x^n = nx^{n-1}$$

for $n \in \mathbb{R}$

Theorem (Chain Rule).

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x) \qquad \frac{dy}{dx} = \frac{dy}{db}\frac{db}{dx}$$

Theorem (Derivative of inverse functions). Let $x \in \mathbb{R}$ and f be a differentiable, one-to-one function at x. Then if $f'(x) \neq 0$, then

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}$$

Theorem (Derivatives of exponentials and logs).

$$\begin{aligned}\frac{d}{dx}e^x &= e^x & \frac{d}{dx}\ln x &= \frac{1}{x}\\ \frac{d}{dx}a^x &= a^x\ln(a) & \frac{d}{dx}\log_a x &= \frac{1}{x\ln(a)}\end{aligned}$$

Theorem (Derivatives of trig functions).
Warning: x must be an angle in radians!

$$\begin{aligned}\sin'(x) &= \cos(x) & \cos'(x) &= -\sin(x)\\ \sec'(x) &= \sec(x)\tan(x) & \csc'(x) &= -\csc(x)\cot(x)\\ \tan'(x) &= \sec(x)^2 & \cot'(x) &= -\csc(x)^2\end{aligned}$$

$$\begin{aligned}\arcsin'(x) &= \frac{1}{\sqrt{1-x^2}} & \arccos'(x) &= -\frac{1}{\sqrt{1-x^2}}\\ \text{arcsec}'(x) &= \frac{1}{|x|\sqrt{x^2-1}} & \text{arccsc}'(x) &= -\frac{1}{|x|\sqrt{x^2-1}}\\ \arctan'(x) &= \frac{1}{1+x^2} & \text{arccot}'(x) &= -\frac{1}{1+x^2}\end{aligned}$$

5 Derivative Applications

5.7 Mean Value Theorem

Theorem (Mean Value Theorem). If the function f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ s.t.

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{\Delta f(x)}{\Delta x} \text{ on } [a, be]$$

Theorem (Some collorаries to the MVT). If $f(x)$ is differentiable on I, then:

- $f'(x) > 0$ for $x \in I \iff f(x)$ is strictly increasing for $x \in I$.
- $f'(x) \geq 0$ for $x \in I \iff f(x)$ is increasing or constant for $x \in I$.
- $f'(x) = 0$ for $x \in I \iff f(x)$ is constant for $x \in I$.
- $f'(x) \leq 0$ for $x \in I \iff f(x)$ is decreasing or constant for $x \in I$.
- $f'(x) < 0$ for $x \in I \iff f(x)$ is strictly decreasing for $x \in I$.

5.3, 5.10, 5.11, 5.16 Extrema

Definition of **Critical point of** f. A number c in the domain of f where $f'(c) = 0$ or $f'(c)$ does not exist.

Definition of **Stationary point of** f. A number c in the domain of f where $f'(c) = 0$

Fermat's Theorem. The local maxima and minima of f are critical points of f.

Exteme Value Theorem. If f is continuous on $[a, b]$, then it has an absolute max and an absolute min.

Theorem (Method to find absolute minima and maxima). Store the critical points of f in the array C. Then, the absolute maximum is $\max\{f(c) : c \in C\}$ and the absolute minimum is $\min\{f(c) : c \in C\}$.

Theorem (First Derivative Test). If f is continuous over I, and $c \in I$ is a critical point of f, and f is differentiable over $I \setminus c$, then:

- If $f'(x)$ is decreasing at c, then $f(c)$ is a local max.
- If $f'(x)$ is increasing at c, then $f(c)$ is a local min.
- If $f'(x)$ has the same sign before and after c, then $f(c)$ is neither a local max nor a local min.

Definition of **Concavity**. f is concave up on I if the tangent line to f at each point in I is lower than the graph of f.

f is concave down on I if the tangent line to f at each point in I is higher than the graph of f.

Theorem (Test for Concavity). If $f''(x) > 0$ for all $x \in I$, then f is concave up on I.

If $f''(x) < 0$ for all $x \in I$, then f is concave down on I.

Theorem (Second Derivative Test). If f'' is continuous on an interval containing c, where c is the x-value of a stationary point of f. Then,

- If $f''(c) > 0$, then $f(c)$ is a local max.
- If $f''(c) < 0$, then $f(c)$ is a local min.

Theorem (Single local extremum is global extremum of continuous function). If f is continuous on an interval I, and f has a single local extremum (max or min), then that extremum is a global max or min.

Miscellaneous

Definition of **Exponential function**.

$$\exp x = e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

6 Integrals

Antiderivative

Definition of **Antiderivative / Indefinite Integral**. The antiderivative F of a function f is the function such that $F'(x) = f(x)$.

$$F(x) = \int f(x)dx$$

Theorem (Antiderivative plus a constant). If F is the antiderivative of a function f, then $G(x) = F(x) + c$ where $c \in \mathbb{R}$ is also an antiderivative.

Definition of **Integral**.

$$(f_1[a,b]) \mapsto \int_a^b f(x)dx \in \mathbb{R}$$

such that the Properties of the Integral are true.

The definite integral takes in a function and a range $[a,b]$, and returns a number. The indefinite integral takes in a function and returns an infinitely large set of functions (the antiderivatives).

If $a > b$, then $\int_a^b f := -\int_b^a f$.

Let $\mathcal{R}([a,b])$ be the set of integrable functions, $\mathcal{C}([a,b])$ be the set of continuous functions, and $\mathcal{B}([a,b])$ be the set of bounded functions on $[a,b]$. Then

$$\mathcal{C}([a,b]) \subset \mathcal{R}([a,b]) \subset \mathcal{B}([a,b])$$

Theorem (Properties of the Integral). The integral is defined such that the following are true:

(I0) Every continuous function is integrable.

(I1) If $f(x) = c$, then $\int_a^b f(x)dx = c(b-a)$

(I2) If $f_1(x) \leq f_2(x)$, then $\int_a^b f_1(x)dx \leq \int_a^b f_2(x)dx$.

(I3) For any a, b, c, $\displaystyle\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.

Theorem (Fundamental Theorem of Calculus). Let $f \in \mathcal{R}([a,b])$ be some integrable function, where $a, b \in \mathbb{R}$. Let $\mathcal{F}(x) = \int_a^x f$ for $x \in [a,b]$. Then:

(a) $\mathcal{F}$ is continuous for every $c \in [a,b]$.

(b) If f is continuous at $c \in [a.b]$, then $\mathcal{F}$ is diffentiable at c, and $\mathcal{F}'(c) = f(c)$.

(c) If f is continuous on $[a,b]$, and F is an antiderivative of f, then $\int_a^b f = F(b) - F(a)$, or

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

Theorem (Substitution Rule). If g is a function that has a continuous derivative on an interval, another function f is continuous on the range of g, and F is an antiderivative of f on the range of g, then

$$\int f(g(x))g'(x)dx = F(g(x)) + C$$

For $a, b \in \mathbb{R}$,

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du = F(g(b)) - F(g(a))$$

Let $u := g(x)$. Then, if $du = E(g'(x)dx)$, where $E \in \mathbb{R}$, then

$$\int Ef(u)du = E\int f(u)du = F(u) + C$$

Theorem (Some Antiderivative Rules).

$$\int e^x dx = e^x + C \qquad \int x^a dx = \frac{x^{a+1}}{a+1} + C \text{ for } a \neq 1$$

$$\int a^x dx = \frac{a^x}{\ln(a)} + C \qquad \int x^{-1} dx = \ln|x| + C$$

$$\int f(x) + g(x)dx = \int f(x)dx + \int g(x)dx$$

Theorem (Integration By Parts). If f and g are integrable functions, then

$$\int f(x)g'(x)dx = f(x)g(x) + \int f(x)g(x)dx$$

Equivalently, if u and v are integrable functions of x, then

$$\int udv = uv - \int vdu$$

Additionally, if f' and g' are continuous, then

$$\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) + \int_a^b g(x)f'(x)dx$$

Theorem (Integrals of Trig Functions).

$$\int \sin(x)dx = -\cos(x) + C \qquad \int \cos(x)dx = \sin(x) + C$$

$$\int \tan(x)dx = -\ln|\cos(x)| + C = \ln|\sec(x)| + C$$

$$\int \cot(x)dx = \ln|\sin(x)| + C = -\ln|\csc(x)| + C$$

$$\int \sec(x)dx = \ln|\sec(x) + \tan(x)| + C$$

$$\int \csc(x)dx = -\ln|\cot(x) + \csc(x)| + C$$

Procedure (Integrals of Powers of Trig Functions). To solve $\int \sin^n(x)\,dx$ where n is even, substitute $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$.

To solve $\int \sin^n(x)\,dx$ where n is odd, substitute $\sin^2(x) = 1 - \cos^2(x)$ and perform u-sub with $u := \cos^2(x)$ and $du = -\sin(x)\,dx$.

To solve $\int \cos^n(x)\,dx$ where n is even, substitute $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$.

To solve $\int \cos^n(x)\,dx$ where n is odd, substitute $\cos^2(x) = 1 - \sin^2(x)$ and perform u-sub with $u := \sin^2(x)$ and $du = \cos(x)\,dx$.

To solve $\int \tan^n(x)\,dx$, substitute $\tan^2(x) = \sec^2(x) - 1$. If n is odd, perform u-sub with $u := \sec(x)$ and $du = \sec(x)\tan(x)\,dx$.

To solve $\int \cot^n(x)\,dx$, substitute $\cot^2(x) = \csc^2(x) - 1$. If n is odd, perform u-sub with $u := \csc(x)$ and $du = -\csc(x)\cot(x)\,dx$.

To solve $\int \sec^n(x)\,dx$ where n is even, substitute $\sec^2(x) = \tan^2(x) + 1$, but ensure that $\sec^2(x)\,dx$ remains. Then perform u-sub with $u := \tan(x)$ and $du = \sec^2(x)\,dx$.

To solve $\int \sec^n(x)\,dx$ where n is odd, substitute $\sec^n(x) = \frac{1}{\cos^n(x)}\frac{\cos(x)}{\cos(x)}$. Then substitute $\cos^2(x) = (1+u)(1-u)$ and u-sub with $u := \sin(x)$ and $du = \cos(x)dx$. Then perform partial fraction decomposition.

To solve $\int \csc^n(x)\,dx$ where n is even, substitute $\csc^2(x) = \csc^2(x) + 1$, but ensure that $\sec^2(x)\,dx$ remains. Then perform u-sub with $u := \cot(x)$ and $du = -\csc^2(x)\,dx$.

To solve $\int \csc^n(x)\,dx$ where n is odd, substitute $\csc^n(x) = \frac{1}{\sin^n(x)}\frac{\sin(x)}{\sin(x)}$. Then substitute $\sin^2(x) = (1+u)(1-u)$ and u-sub with $u := \cos(x)$ and $du = -\sin(x)dx$. Then perform partial fraction decomposition.

Remember that $\int \sec^2(x)\,dx = \tan(x)$ and $\int \csc^2(x)\,dx = -\cot(x)$.

Procedure (Trig Substitution).

Orig expression	Substitution	Pythagorean identity
$\sqrt{a^2 - x^2}$	$x := a\sin(\theta)$	$1 - \sin^2(\theta) = \cos^2(\theta)$
$\sqrt{a^2 + x^2}$	$x := a\tan(\theta)$	$1 + \tan^2(\theta) = \sec^2(\theta)$
$\sqrt{x^2 - a^2}$	$x := a\sec(\theta)$	$\sec^2(\theta) - 1 = \tan^2(\theta)$

Partial Fraction Decomposition

Theorem. Any polynomial $Q(x)$ with real coefficients can be factored over the reals as a product of two types of factors:

- linear factors (of the form $ax + b$)
- irreducible quadratic factors (of the form $ax^2 + bxx + c$, where $b^2 - 4ac < 0$)

Definition of **Proper rational function**. A rational function $\frac{P(x)}{Q(x)}$ where $\deg P < \deg Q$.

Theorem. Any rational function can be converted into a proper rational function plus a polynomial by continually long-dividing by the denominator.

Theorem. Let $R(x) = \frac{P(x)}{Q(x)}$ be a proper rational function, where the denominator $Q(x)$ has been factored into linear and irreducible quadratic factors. $R(x)$ can be written as a sum of partial fractions, where each factor in the denominator gives rise to terms in the partial fraction decomposition:

- For each factor of the form $(ax + b)^k$ in the denominator, add $\displaystyle\sum_{i=1}^{k} \frac{A_i}{(ax+b)^i}$ to the partial fraction decomposition.
- For each factor of the form $(ax^2 + bx + c)^k$ in the denominator, add $\displaystyle\sum_{i=1}^{k} \frac{A_i x + B_i}{(ax^2+bx+c)^i}$ to the partial fraction decomposition.

Riemann Sums

Definition of **Elementary function**. A function which is a polynomial, rational function, power function (x^a), exponential function (a^x), logarithmic functions, trigonometric and inverse trigonometric functions, or an addition, subtraction, multiplication, division, and composition of the above.

Definition of **Riemann sum**. Let $f : [a, b] \to \mathbb{R}$ be any bounded function and let P be a partition of $[a, b]$.

- A choice of a point $x_i^* \in [x_{i-1}, x_i]$ for all $i \in [1, n]$ is called a *tagging* of P, which we denote by $\tau = x_1^*, ..., x_n^*$.
- A pair (P, τ) is called a *tagged partition*.
- Given a bounded function $f : [a, b] \to \mathbb{R}$ and a tagged partition (P, τ) of $[a, b]$, the sum

$$R(f, P, \tau) = \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1})$$

is called the *Riemann sum* of f for (P, τ).

Theorem. If f is integrable on $[a, b]$, then for every $\varepsilon > 0$ there exists $\delta > 0$ s.t. if P is a partition of $[a, b]$ with $|P| := \max x_i - x_{i-1} < \delta$ and $\tau = \{x_i^*\}$, then

$$|R(f, P, \tau) - \int_a^b f(x)dx| < \varepsilon$$

Approximate Integration

Definition of **Left-endpoint approximation**. Take $x_i^* = x_{i-1} = a + \frac{(i-1)(b-a)}{n}$. Then the left-endpoint approximation of the function $f(x)$ is

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f\left(a + \frac{(i-1)(b-a)}{n}\right)\frac{b-a}{n}$$

The error bound is

$$E_n^L \leq \max\{|f'(x)|\}_{x\in[a,b]} \frac{(b-a)^2}{2n}$$

Definition of **Right-endpoint approximation**. Take $x_i^* = x_{i-1} = a + \frac{(i)(b-a)}{n}$. Then the right-endpoint approximation of the function $f(x)$ is

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f\left(a + \frac{(i)(b-a)}{n}\right)\frac{b-a}{n}$$

The error bound is

$$E_n^R \leq \max\{|f'(x)|\}_{x\in[a,b]} \frac{(b-a)^2}{2n}$$

Definition of **Midpoint approximation**. Take $x_i^* = x_{i-1} = a + \frac{(i-\frac{1}{2})(b-a)}{n}$. Then the midpoint approximation of the function $f(x)$ is

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f\left(a + \frac{(i-\frac{1}{2})(b-a)}{n}\right)\frac{b-a}{n}$$

The error bound is

$$E_n^M \leq \max\{|f''(x)|\}_{x\in[a,b]} \frac{(b-a)^3}{24n^2}$$

Definition of **Trapezoidal approximation**. Then the trapezoidal approximation of the function $f(x)$ is

$$\int_a^b f(x)dx \approx \frac{b-a}{2n}\left(f(a) + 2\sum_{i=1}^{n-1} f\left(a + \frac{(i)(b-a)}{n}\right) + f(b)\right)$$

The error bound is

$$E_n^T \leq \max\{|f''(x)|\}_{x\in[a,b]} \frac{(b-a)^3}{12n^2}$$

Definition of **Simpson's approximation**. For even n (greater values of n give more precise more accuracy):

$$\frac{b-a}{3n}\left(f(a) + \sum_{i=1}^{n-1}(3-(-1)^i) f\left(a + \frac{(i)(b-a)}{n}\right) + f(b)\right)$$

The error bound is

$$E_n^S \leq \max\{|f''''(x)|\}_{x\in[a,b]} \frac{(b-a)^5}{180n^4}$$

8 Integral Applications

Volumes

Generally, if $A(x)$ is the cross-section of a solid that intersects the x-axis at x, then the volume of the solid is

$$V = \int_a^b A(x)dx$$

Theorem (Disk Method). Let $f(x)$ be a continuous, nonnegative function defined on $[a, b]$, and R be the region bounded above by the graph of $f(x)$ and below by the x-axis. Then, the volume of the solid of revolution formed by revolving R around the x-axis is given by

$$V = \int_a^b \pi(f(x))^2 dx$$

Theorem (Washer Method). Let $f(x)$ be a continuous, nonnegative function defined on $[a, b]$, and R be the region bounded above by the graph of $f(x)$ and below by the graph of $g(x)$. Then, the volume of the solid of revolution formed by revolving R around the x-axis is given by

$$V = \int_a^b \pi((f(x))^2 - (g(x))^2)dx$$

Theorem (Cylindrical Shells Method). Let $f(x)$ be a continuous, nonnegative function defined on $[a, b]$, and R be the region bounded above by the graph of $f(x)$ and below by the x-axis. Then, the volume of the solid of revolution formed by revolving R around the y-axis is given by

$$V = \int_a^b 2\pi x f(x)dx$$

Other things

Theorem (Arc length of a curve). The arc length of the curve $f(x)$ on $[a, b]$, when $f'(x)$ exists and is continuous on $[a, b]$, is

$$\int_a^b \sqrt{1 + (f'(x))^2}\,dx$$

Theorem (Surface area of a solid of revolution). Let $f(x)$ be a continuous, nonnegative function defined on $[a, b]$, and R be the region bounded above by the graph of $f(x)$ and below by the x-axis. Then, the surface area of the solid of revolution formed by revolving R around the x-axis is given by

$$S = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,dx$$

Theorem (Mass of a thin rod). Let there be a rod whose left end is located at $x = a$ and whose right end is located at $x = b$, and which is on the x-axis. Let $\rho(x)$ be the linear density at the point $x = a$. Then the mass of the rod is

$$M = \int_a^b \rho(x)\,dx$$

Theorem (Mass of a thin disk). Let there be a disk of radius R whose center is at the origin of the xy-plane. Let the mass be distributed in a rotationally-symmetric way. Let $\rho(r)$ be the radial density at the radius r. Then the mass of the disk is

$$M = \int_0^R 2\pi r\rho(r)dr$$

Theorem (Work). If an object moves along the x-axis from a to b, and $F(x)$ is the force applied to the object when the object is at the point x on the x-axis, then the work is

$$\int_a^b F(x)\,dx$$

Definition of **Average value of a function**. If f is continuous on $[a, b]$, then the average value of f on $[a, b]$ is

$$f_{\text{avg}} := \lim_{n\to\inf} \frac{1}{b-a}\int_a^b f(x)\,dx$$

Theorem (Mean Value Theorem for Integrals). If f is continuous on $[a, b]$, then there exists $c \in [a, b]$ such that

$$f(c) = f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Equivalently,

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

If f is positive on $[a, b]$, then there is a number c such that the rectangle with base $[a, b]$ and height $f(c)$ has the same area as $\int_a^b f(x)\,dx$.

9 Improper Integrals

Definition of **Improper Integrals with Infinite Bounds**.

(a) If $\int_a^t f(x)$ exists for every $t \geq a$, then we define

$$\int_a^\infty f(x)dx := \lim_{t\to\infty}\int_a^t f(x)dx$$

provided that this limit exists and is finite.

(b) If $\int_t^b f(x)dx$ exists for every $t \leq b$, then we define

$$\int_{-\infty}^b f(x)dx := \lim_{t\to-\infty}\int_t^b f(x)dx$$

provided that this limit exists and is finite.

$\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are *convergent* if the corresponding limit exists and *divergent* if the limit doesn't exist.

(c) If both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are convergent, then we define

$$\begin{aligned}\int_{-\infty}^\infty f(x)dx &= \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx \\ &= \lim_{s\to-\infty}\int_s^a f(x)dx + \lim_{t\to\infty}\int_a^t f(x)dx\end{aligned}$$

Definition of **Improper Integrals with Discontinuous Integrand**.

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then define

$$\int_a^b f(x)dx := \lim_{t\to b^-}\int_a^t f(x)dx$$

provided that this limit exists and is finite.

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then define

$$\int_a^b f(x)dx := \lim_{t\to a^+}\int_t^b f(x)dx$$

provided that this limit exists and is finite.

$\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are *convergent* if the corresponding limit exists and *divergent* if the limit doesn't exist.

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then we define

$$\begin{aligned}\int_a^b f(x)dx &= \int_a^c f(x)dx + \int_c^b f(x)dx \\ &= \lim_{s\to c^-}\int_a^s f(x)dx + \lim_{t\to c^+}\int_c^b f(x)dx\end{aligned}$$

Theorem (Comparison Test). Suppose f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

(a) If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.

(b) If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent.

Additionally,

(a) If $\int_{-\infty}^b f(x)dx$ is convergent, then $\int_{-\infty}^b g(x)dx$ is convergent.

(b) If $\int_{-\infty}^b g(x)dx$ is divergent, then $\int_{-\infty}^b f(x)dx$ is divergent.

Theorem (Limit Comparison Test). Suppose $f(x)$ and $g(x)$ are positive continuous functions defined on $[a, \infty)$ such that $\lim_{x\to\infty}\frac{f(x)}{g(x)} = c$ for some positive real number c. Then $\int_a^\infty f(x)dx$ converges iff $\int_a^\infty g(x)dx$.

10 Differential Equations

Definition of **Differential equation**. An equation involving an unknown function $y = f(x)$ and one or more of its derivatives.

Definition of **Solution to a differential equation**. A function $y = f(x)$ that satisfies the differential equation when f and its derivatives are substituted into the equation.

Procedure (Euler's Method). To numerically approximates the solution to the differential equation $y' = F(x, y)$ with $y(x_0) = y_0$,

$$y_n = y_{n-1} + F(x_{n-1}, y_{n-1})(x_n - x_{n-1})$$

Definition of **Separable Equation**. A separable equation is a differential equation where

$$N(y)\frac{dy}{dx} = M(x)$$

for some function $M(x)$ which depends only on x and $N(y)$ which depends only on y.

Theorem. For a separable equation,

$$\int N(y)\frac{dy}{dx}dx = \int M(x)\,dx$$

and therefore

$$\int N(y)\,dy = \int M(x)\,dx$$

Definition of **Logistic Differential Equation**. For a population P which increases exponentially ($\frac{dP}{dt} \approx kP$) when the population is small compared to the carrying capacity M but where the environment cannot sustain a population larger than M,

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

Then

$$P(t) = \frac{M}{1 + \left(\frac{M}{P_0} - 1\right)e^{-kt}}.$$

and

$$\frac{d^2P}{dt^2} = k^2P\left(1 - \frac{P}{M}\right)\left(1 - \frac{2P}{M}\right)$$

11 Sequences and Series

Definition of **Sequence**. A **sequence** is an ordered list of real numbers (**elements**) $a_1, a_2, a_3, \ldots$. It can also be considered a function $a : \mathbb{N} \to \mathbb{R}$.

Definition of **Convergence**. A sequance converges if $\lim_{n\to\infty} a_n$ exists.

Theorem. The Limit Laws and Squeeze Theorem also apply to limits of sequences.

Theorem. If $\{a_n\}$ is convergent, then it is bounded.

Theorem (Bounded Monotonic Sequence Theorem). Every bounded monotonic sequence converges. If $\{a_n\}$ is increasing, then $\lim\limits_{n\to\infty} a_n = \sup\{a_n : n \geq 1\}$. If $\{a_n\}$ is decreasing, then $\lim\limits_{n\to\infty} a_n = \inf\{a_n : n \geq 1\}$.

Definition of **Series**. The n**th partial sum** of $\{a_i\}_{i=1}^n$ is $s_n := \sum_{i=1}^n a_n$. Then the **series** is $\sum_{i=1}^\infty a_i := \lim_{n\to\infty} s_n$

Theorem (Convergence Test). If the series $\sum_{i=1}^n a_i$ is convergent, then $\lim_{i\to\infty} a_i = 0$.

Theorem (Divergence Test). If $\lim_{i\to\infty} a_i \neq 0$, then $\sum_{i=1}^n a_i$ diverges.

Theorem (Integral Test). If f is a continuous positive decreasing function for $x \geq 1$ and $a_n = f(n)$ for $n \in \mathbb{N}$, then

$$\sum_{n=1}^\infty a_n \qquad \text{and} \qquad \int_1^\infty f(x)\,dx$$

either both converge or both diverge.

Theorem (Remainder Estimate for Integral Test). If f is a continuous positive decreasing function for $x \geq 1$ and $a_n = f(n)$ for $n \in \mathbb{N}$, then

$$\int_{n+1}^\infty f(x)dx \leq R_n = s - s_n = \sum_{i=n+1}^\infty a_i \leq \int_n^\infty f(x)dx$$

and therefore

$$s_n + \int_{n+1}^\infty f(x)dx \leq \sum_{i=1}^\infty a_i \leq s_n + \int_n^\infty f(x)dx$$

Theorem (Comparison Test for Series). If $\sum a_n$ and $\sum b_n$ are series with positive terms:

- If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.
- If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also convergent.

Theorem (Limit Comparison Test for Series). If $\sum a_n$ and $\sum b_n$ are series with positive terms, then

- If $\lim_{n\to\infty} \frac{a_n}{b_n} \in \mathbb{R}^+$, then either both series converge or both series diverge.
- If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
- If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Theorem (Well-known series).
$\sum \frac{1}{n^p}$ (the p-**series**) converges if $p > 1$ and diverges if $p \leq 1$.

$\sum ar^{n-1}$ (the **geometric series**) converges if $|r| < 1$ and diverges if $|r| \geq 1$. If it converges, $\displaystyle\sum_{n=1}^\infty ar^{n-1} = \frac{a}{1-r}$

Definition of **Alternating series**. A series $\sum_{n=1}^\infty a_n = \sum_{n=1}^\infty (-1)^n b_n$ or $\sum_{n=1}^\infty (-1)^{n-1} b_n$ where $b_n > 0$.

Theorem (Alternating Series Test). If the alternating series $\sum_{n=1}^\infty (-1)^{n-1} b_n$ satisfies

(i) $b_{n+1} \leq b_n$ for all n, (i.e. $\{b_n\}$ is decreasing) and

(ii) $\lim\limits_{n\to\infty} b_n = 0$,

then the series is convergent.

Theorem (Alternating Series Divergence Test). If $\lim\limits_{n\to\infty} b_n \neq 0$, then the series is divergent.

Theorem (Alternating Series Estimation Theorem). If $s = \sum_{n=1}^\infty (-1)^{n-1} b_n$ is a series that satisfies the conditions of the Alternating Series Test, then

$$|R_n| := |s - s_n| \leq b_{n+1}$$

and therefore

$$s_n - b_{n+1} \leq s \leq s_n + b_{n+1}$$

Definition of **Absolute convergence**. A series $\sum_{n=1}^\infty a_n$ **converges absolutely** if the series of absolute values $\sum_{n=1}^\infty |a_n|$ converges.

Definition of **Conditional convergence**. A series **converges conditionally** if it converges but does not converge absolutely.

Theorem (Rearrangement of terms of absolutely convergent series). The terms of an absolutely convergent series can be rearranged without affecting the value/sum of the series.

Theorem (Absolute convergence implies convergence). If a series converges absolutely, then it converges.

Theorem (Ratio Test).

- If $\lim_{n\to\infty} \left|\frac{a_n+1}{a_n}\right| < 1$, then the series $\sum_{n=1}^\infty a_n$ converges absolutely and therefore converges.
- If $\lim_{n\to\infty} \left|\frac{a_n+1}{a_n}\right| > 1$ or $\lim_{n\to\infty} \frac{a_n+1}{a_n} = \infty$, then the series $\sum_{n=1}^\infty a_n$ diverges.
- If $\lim_{n\to\infty} \left|\frac{a_n+1}{a_n}\right| = 1$, the Ratio Test is inconclusive.

Theorem (Root Test).

- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$, then the series $\sum_{n=1}^\infty a_n$ converges absolutely and therefore converges.
- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^\infty a_n$ diverges.
- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Ratio Test is inconclusive.

12 Power Series

Definition of **Power series**. A power series centered at a is a series of the form

$$\sum_{k=0}^\infty c_k(x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

where x is a variable, a is a fixed real number called the *center* of the series, and the c_ks are constants called the *coefficients* of the series.

Theorem. For a power series $\sum_{k=0}^\infty c_k(x-a)^k$, there are only three possibilities:

(i) The series converges only when $x = a$.

(ii) The series converges for all x.

(iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$. R is called the **radius of convergence**.

Theorem. Suppose $\sum_{n=0}^\infty c_n(x-a)^n$ has radius of convergence $R > 0$. Then $f(x) = \sum_{n=0}^\infty c_n(x-a)^n$ is differentiable (and therefore continuous and integrable) on $(a-R, a+R)$, and

(i) $f'(x) = \displaystyle\sum_{n=0}^\infty \frac{d}{dx} c_n(x-a)^n = \sum_{n=0}^\infty nc_n(x-a)^{n-1}$

(ii) $\int f(x)dx = \sum_{n=0}^{\infty} \int c_n(x-a)^n = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$

Definition of **Taylor series**. The **Taylor series** of a function f centered at a is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Definition of **Maclaurin series**. The **Maclaurin series** of a function f is the Taylor series of f centered at a:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

Theorem. If f has a power series representation centered at a, then that power series is equivalent to the Taylor series of f centered at a.

Theorem (Basic Maclaurin Series).

$$\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} && \text{for } x \in \mathbb{R} \\
\sin(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} && \text{for } x \in \mathbb{R} \\
\cos(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} && \text{for } x \in \mathbb{R} \\
\ln(1+x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} && \text{for } |x| < 1 \\
\arctan(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} && \text{for } |x| < 1 \\
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n && \text{for } |x| < 1 \\
\frac{1}{(1-x)^2} &= \sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty} (n+1)x^n && \text{for } |x| < 1
\end{aligned}$$

Definition of **Taylor polynomial**. The nth-degree Taylor polynomial of f at a is

$$\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Theorem (Taylor's Theorem). If f has $(n+1)$ continuous derivatives on an open interval I ccontaining a, then for all x in I,

$$f(x) = T_n(x) + R_n(x),$$

where T_n is the nth-order Taylor polynomial for f with center a and the remainder is

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some point c between x and a. This form of R_n is also called **Lagrange error bound** or the **Lagrange form of the remainder**.

Theorem. If R_n converges to 0, then $f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$.

13 Polar

Theorem. If r is a function of θ on the xy-plane, then

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \qquad \text{and} \qquad \frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\theta}}$$

Theorem (Arc length). If r is a function of θ on the xy-plane, then

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

and therefore

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Theorem. The area of a region bounded by a function $r = R(\theta)$ between $\theta = \alpha$ and $\theta = \beta$ is

$$\frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta$$

This can be derived using MVC by evaluating

$$\int_{\alpha}^{\beta}\int_0^r r\, dr\, d\theta$$

14 Vectors

$$\frac{d}{dt}\vec{r}(t) = \begin{matrix} x'(t) \\ y'(t) \\ z'(t) \end{matrix}$$

$$\int \vec{r}(t)dt = \begin{matrix} \int x(t)dt \\ \int y(t)dt \\ \int z(t)dt \end{matrix}$$

Multivariable Calculus (6)

Colley (2012) refers to *Vector Calculus, Fourth Edition* by Susan Jane Colley.

6.1 Vectors, lines, planes

6.1.1 $\mathbb{R}^n$

Definition of **Two-dimensional real-coordinate space** ($\mathbb{R}^2$).

$$\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$$

Definition of **Three-dimensional real-coordinate space** ($\mathbb{R}^2$).

$$\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$$

x, y, z should be presented such that the coordinate system is right-handed ($\hat{k} = \hat{\imath} \times \hat{\jmath}$ should have direction according to the right-hand rule).

Definition of **Standard basis vectors**. The standard basis vectors of a space are the unit vectors that go along the axes of the space. All vectors in that space can be expressed as sums of scalar multiples of the standard basis vectors of that space.

The standard basis vectors of $\mathbb{R}^2$ are $\hat{\imath}$ and $\hat{\jmath}$, also called $\mathbf{e_1}$ and $\mathbf{e_2}$.

The standard basis vectors of $\mathbb{R}^3$ are $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$, also called $\mathbf{e_1}$, $\mathbf{e_2}$, and $\mathbf{e_3}$.

6.1.2 Vectors

See the properties of fields in the LinAlg notes for definitions of addition and scalar multiplication.

Definition of **Displacement vector**. The vector from the end of $\vec{A}$ to the end of $\vec{B}$ when their starts are in the same location.

$$\overrightarrow{AB} = \vec{B} - \vec{A}$$

6.1.3 Dot and cross products

Definition of **Dot product**. Where $a, b \in \mathbb{R}^n$ and θ is the angle between a and b:

$$a \cdot b = \sum a_i b_i = |a||b| \cos\theta$$

Theorem (Properties of dot product). For $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^n$ and $k \in \mathbb{R}$:

- $\vec{a} \cdot \vec{a} = |\vec{a}|^2$
- $\vec{a} \cdot \vec{a} = 0$ iff $\vec{a} = 0$
- Commutativity: $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
- Distributivity: $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
- Distributivity: $(k\vec{a}) \cdot b = k(\vec{a} \cdot \vec{b}) = a \cdot (k\vec{b})$
- $\vec{a} \cdot \vec{b} = 0$ iff $a \perp b$, $a = 0$, or $b = 0$.

Definition of **Cross product**. For $\vec{a}, \vec{b} \in \mathbb{R}^3$, the unique vector $a \times b$ satisfying

- $|a \times b|$ is the area of the parallelogram spanned by a and b
- $a \times b = 0$ iff $a \parallel b$, $a = 0$, or $b = 0$.
- $a \times b$ is orthogonal to a and b.
- $(a, b, a \times b)$ is right-handed (if the coordinate system is right-handed)

Theorem (Properties of cross product). For $a, b, c \in \mathbb{R}^3$ and $k \in \mathbb{R}$:

- $a \times b = (-b) \times a$
- $a \times (b + c) = a \times b + a \times c$
- $(a + b) \times c = a \times c + b \times c$
- $k(a \times b) = (ka) \times b = a \times (kb)$

Theorem (Calculation of cross product). Where $a, b \in \mathbb{R}^n$ and θ is the angle between a and b:

$$a \times b = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$
$$= \hat{\imath} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} + \hat{\jmath} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$
$$|a \times b| = |a||b| \sin\theta$$

6.1.4 Lines

The most useful notation for a line is in parametric form:

Definition of **Parametric form of a line**. Where $r_0 \in \mathbb{R}^3$ is a point on the line, and $t \in \mathbb{R}^3$:

$$r(t) = r_0 + vt$$

t is called the **direction vector**.

Theorem. Two lines are parallel iff their direction vectors are scalar multiples of each other.

Definition of **Skew lines**. Two lines are skew iff they do not intersect but are not parallel, i.e. they lie in different parallel planes.

Procedure (Finding the intersection of two lines). Given lines with parametric equations

$$r_1(t) = a_1 + v_1 t \qquad r_2(t) = a_2 + v_2 t$$

solve the system of equations

$$a_1 + v_1 t_1 = a_2 + v_2 t_2$$

for t_1 or t_2, then plug it in to the appropriate equation.

(Break up the equation into its x, y, and z components, or whichever is appropriate for your coordinate space.)

6.1.5 Planes

Colley [1.5]

Definition of **Plane**. A plane Π is determined uniquely by a point P in the plane and a normal vector n.

A plane is the set of points A in space such that $\vec{AP}$ is perpendicular to n.

Theorem (Scalar equation for a plane in $\mathbb{R}^3$). If $n \in \mathbb{R}^3$ is the **normal vector** to the plane (vector perpendicular to the plane), and $P \in \mathbb{R}^3$ is a point on the plane:

$$n_x(x - P_x) + n_y(x - P_y) + n_z(z - P_z) = 0$$

or equivalently:

$$n_x x + n_y y + n_z z = n_x P_x + n_y P_y + n_z P_z$$

Procedure (Equation of plane containing three points). If $A, B, C \in \mathbb{R}^3$ are points on our plane, then we can find the normal vector by performing $n = \overrightarrow{AB} \times \overrightarrow{AC} = (B - A) \times (C - A)$ (since $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are vectors on the plane).

Theorem (Parametric equation for a plane in $\mathbb{R}^3$). If $a, b \in \mathbb{R}^3$ are nonparallel nonzero vectors on the plane, and $P \in \mathbb{R}^3$ is a point on the plane, then the parametric equation for the plane is:

$$x(s, t) = P + sa + tb$$

6.1.6 Distance

Colley [1.5]

Procedure (Distance between point and line). Let P be the point, and $A + Lt$ be the line. Then the distance is

$$|\overrightarrow{AP} - \operatorname{proj}_L \overrightarrow{AP}| = \vec{P}$$

Procedure (Distance between parallel planes). Let Π_1 and Π_2 be the two planes.

If n is normal to both planes, and $P_1 \in \Pi_1$ and $P_2 \in \Pi_2$, then the answer is

$$|\operatorname{proj}_n \overrightarrow{P_1 P_2}$$

6.1.7 Cylindrical and spherical coordinates

Colley [1.7] Trimm [5.6, 5.7]
Brummet [08, MVCWUP:Feb3(29-33)]

Definition of **Cylindrical coordinate**. An ordered pair (r, θ, z) where r is the distance between the point and the z-axis, θ is the angle counterclockwise from the positive x-axis along the xy-plane, and z is the position on the z-axis.

Definition of **Spherical coordinate**. An ordered pair (ρ, ϕ, θ), where ρ is the distance between the point and the origin, ϕ is the angle clockwise from the positive z-axis going downwards towards the xy-plane, and θ is the angle counterclockwise from the positive x-axis along the xy-plane.

Typically we use the following restrictions:

$$\rho > 0 \qquad 0 \le \theta \le 2\pi \qquad 0 \le \phi \le \pi$$

Theorem (Useful formulas).

$$r = \rho \sin\phi \qquad z = \rho\cos\phi$$

$$x = \rho\sin\phi\cos\theta \qquad y = \rho\sin\phi\sin\theta$$

$$r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta$$

6.2 Functions, limits, differentiation

6.2.1 Multivariable functions

Colley [2.1] Trimm [3.1, 3.2] Brummet [MVCWUP:Feb4(35-41)]

Definition of **Function**. All functions $f : X \to Y$ are defined by:

- A domain set X
- A codomain set Y
- A rule of assignment that associates a unique element $y \in Y$ to each element $x \in X$

Definition of **Graph**. The graph of $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is the set

$$\{(x_1, \dots, x_n, f(x)) : x = (x_1, \dots, x_n)\}$$

Specifically, for $f : \mathbb{R}^2 \to \mathbb{R}$ the graph is the set

$$\{(x, y, z) : (x, y) \in X \text{ and } z = f(x, y)\}$$

Definition of **Level set**. Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$. The **level set at height c of f** is the set in $\mathbb{R}^n$ defined by the equation $f(\vec{a}) = c$, where c is a constant. This is equivalent to the set

$$\{\vec{x} \in \mathbb{R}^n : f(\vec{x}) = c\}$$

In $\mathbb{R}^2$, this is also called a **level curve**.

Definition of **Contour set**. Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$. The **contour set at height c of f** is the set in $\mathbb{R}^{n+1}$ defined by the two equations $z = f(\vec{a})$ and $z = c$, where c is a constant. This is equivalent to the set

$$\{\vec{x} \in \mathbb{R}^{n+1} : z = f(\vec{x}) = c\}$$

If $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$, this is also called a **contour curve**. It is equivalent to the level curve, except it is located in $\mathbb{R}^3$ rather than $\mathbb{R}^2$.

6.2.2 Limits

Trimm [3.3, DiffEq-1.0] Brummet [09, MVCWUP:Feb11(42-46)]
Colley [2.2]

Definition of **Limit**. $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = \vec{L}$ if for all $\varepsilon > 0$, there exists $\delta > 0$ s.t. if $0 < |\vec{x} - \vec{a}| < \delta$ then $|f(\vec{a}) - \vec{L}| < \varepsilon$.

Definition of **Continuity**. Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and let $\vec{a} \in X$. Then f is continuous at point $\vec{a}$ iff

$$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f(\vec{a})$$

If f is continuous at all $\vec{a} \in X$, then we say that f is continuous.

6.2.3 Differentiation

Colley [2.3, 2.4]
Trimm [3.4, DiffEq-1.0]
Brummet [11, 12.5, MVCWUP:Feb12/22(48-54)]

Definition of **Partial derivative with respect to x**. The partial derivative of $f(x, y)$ with respect to x is

$$\lim_{h\to 0} \frac{f(a+h, b) - f(a, b)}{h}$$

Let $z = f(x, y)$. Then the partial derivative is denoted by

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = D_x f$$

Definition of **Partial derivative**. The partial derivative of $f(\vec{x})$ with respect to the ith variable is

$$\frac{\partial f(\vec{x})}{\partial x_i} = \lim_{h\to 0} \frac{f\left(\begin{bmatrix} x_0 \\ \vdots \\ x_i + h \\ \vdots \\ x_n \end{bmatrix}\right) - f(\vec{x})}{h}$$

This is equivalent to letting $F(x_i) = f(\vec{x})$ and finding $F'(x_i)$.

Definition of **Higher-order partial**. The result of taking the partial derivative of a partial derivative, which may be higher-order.

A partial derivative that is not higher-order is called a **first-order partial**. A partial derivative of a first-order partial is a second-order partial, a partial derivative of a second-order partial is a third-order partial, etc.

A higher-order partial which is the result of taking the partial with respect to x_1, then with respect to x_2, then with respect to $x_3, \dots, x_n$, is denoted by

$$f_{x_1 \cdots x_n} = \frac{\partial}{\partial x_n} \cdots \frac{\partial}{\partial x_1} f$$

$x_1 \cdots x_n$ do not have to be distinct. If $x_1 \cdots x_n$ are not all the same then the higher-order partial is called a **mixed partial derivative**.

Definition of **C^k function**. Where k is a nonnegative integer, a function $f : X \in \mathbb{R}^n \to \mathbb{R}$ is of order C^k at point $\vec{x} \in X$ iff its k-th order and lower partials exist and are continuous at $\vec{x}$.

It is of order C^∞ at point $\vec{x}$ iff it is of order C^k at $\vec{x}$ for all $k \in \mathbb{N}$.

It is of order C^k iff it is of order C^k at all $x \in \vec{x}$.

Theorem. Let $f : X \in \mathbb{R}^n \to \mathbb{R}$ whose k-th order and lower partials exist and are continuous on X. Then its k-th order and lower partials may be evaluated in any order, i.e.

$$f_{x_1 \cdots x_n} = f_{x_n \cdots x_1} = f_{x_1 x_3 x_{27} \cdots x_4} = \cdots$$

Definition of **Gradient**.

$$\nabla f(\vec{x}) = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{bmatrix}$$

6.2.4 Implicit surfaces

Brummet [12]

Definition of **Implicit surface**. A surface in $\mathbb{R}^3$ defined by an equation which is not solved for x, y, nor z.

We often express it as

$$F(x, y, z) = 0,$$

in which case the surface is the set of points which satisfy $F(x, y, z) = 0$.

Theorem. The gradient $\nabla F(\vec{x})$ is the normal vector to the tangent plane to the implicit surface defined by $F(\vec{x}) = k$, where k is a constant.

Equivalently, if x_0 is a point on the level set $S = \{x \in X : F(x) = k\}$ where $F : X \subseteq \mathbb{R}^n \to \mathbb{R}$, then the vector $\nabla F(x_0)$ is perpendicular to S.

6.2.5 Chain rule

Colley [2.5] Trimm [3.8, 6.5, DiffEq-1.0]
Brummet [12, MVCWUP:Feb24(58-61)]

Definition of **Jacobian**. If $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is a vector-valued function, then the Jacobian is

$$Df\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} \nabla f_1 \\ \nabla f_2 \\ \vdots \\ \nabla f_m \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{x_n} \end{bmatrix}$$

Theorem (Multivariable chain rule). Suppose $X \subseteq \mathbb{R}^m$ and $T \subseteq \mathbb{R}^n$ are open and $f : X \to \mathbb{R}^p$ and $r : T \to \mathbb{R}^m$ are defined so that $T \subseteq X$. If x is differentiable at $t_0 \in T$ and f is differentiable at $x_0 = r(t_0)$, then the composite $f \circ r$ is differentiable at t_0, and we have

$$(f \circ r)'(t) = \nabla f(x_0) \cdot r'(t_0)$$

Equivalently,

$$D(f \circ r)(t_0) = Df(x_0)Dr(t_0)$$

In $\mathbb{R}^3$,

$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt}$$

6.2.6 Paths

Brummet [MVCWUP:Feb24(55-56)]

Definition of **Path**. A path in $\mathbb{R}^n$ is a function $x : I \to \mathbb{R}^n$, where I is a set of scalars. If $I = [a, b]$, then the endpoints of the path are $f(a)$ and $f(b)$.

Definition of **Tangent vector**. Given a path $r : \mathbb{R} \to \mathbb{R}^3$, the tangent vector to said path at some point P is given by $r'(t)$, provided that $r'(t) \neq 0$. In $\mathbb{R}^3$,

$$r'(t) = \lim_{h\to 0} \frac{r(t+h) - r(t)}{h} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{bmatrix}$$

Definition of **Derivative of vector-valued function**. Let $f : T \subseteq \mathbb{R} \to \mathbb{R}^m$. Then

$$f'(t) = \begin{bmatrix} f_1'(t) \\ f_2'(t) \\ \vdots \\ f_m'(t) \end{bmatrix}$$

6.2.7 Differentiability

Colley [2.3] Trimm [3.5] Brummet [13.5]

Definition of **Linear approximation** ($\mathbb{R}^n \to \mathbb{R}$). The **linear approximation** or **tangent plane** ($\mathbb{R}^3$) **or hyperplane** to the graph of a function f at the point $\vec{a}$ is expressed by

$$L(\vec{x}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a})$$

In $\mathbb{R}^3$, this is equivalent to the plane

$$z = L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Definition of **Linear approximation** ($\mathbb{R}^n \to \mathbb{R}^m$). The **linear approximation** to a vector-valued function f at the point $\vec{a}$ is expressed by

$$L(\vec{x}) = f(\vec{a}) + Df(\vec{a})(\vec{x} - \vec{a})$$

Definition of **Differentiability**. Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$, where X is an open subset of $\mathbb{R}^n$, and let $\vec{a} \in X$. f is differentiable at a iff all of its partial derivatives exist and

$$\lim_{\vec{x}\to\vec{a}} \frac{f(\vec{x}) - L(\vec{x})}{|\vec{x} - \vec{a}|} = 0$$

where $L(\vec{x})$ is the linear approximation to f at $\vec{a}$.

Theorem (Differentiability shortcut). Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a vector-valued function. If all partial derivatives $\frac{\partial f_i}{\partial x_j}$ exist and are continuous in a neighborhood of $\vec{a}$ in X, then F is differentiable at $\vec{a}$.

6.2.8 Directional derivative

Colley [2.6] Trimm [3.7] Brummet [14]

Definition of **Directional derivative**. Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$, where X is an open subset of $\mathbb{R}^n$, and let $\vec{a} \in X$. If $\vec{v}$ is any unit vector in X, then the directional derivative of f at a in the direction of v is

$$D_{\vec{v}}f(\vec{a}) = \lim_{h\to 0} \frac{f(\vec{a} + h\vec{v}) - f(\vec{a})}{h}$$

Theorem. If f is differentiable at a, then

$$D_{\vec{v}}f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{v}$$

Theorem. The gradient is the path of steepest ascent, i.e.

$$D_{\widehat{\nabla f(\vec{a})}}f(\vec{a}) = \max\{D_{\vec{v}}f(\vec{a}) : \vec{v} \in \mathbb{R}^n\}$$

where $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$.

Theorem. Let $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$, and let $(a, b, c) \in \mathbb{R}^3$. Then $\nabla f(a, b)$ is orthogonal to the level curve at height c.

6.3 Extrema

6.3.1 Absolute extrema

Colley [4.1] Brummet [15, MVCWUP:69-74(Mar 4-6)]

Theorem (Quasi-First Derivative Test). If $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ has a local maximum or minimum at $\vec{a}$ and the first order partial derivatives exist, then $\nabla f \cdot \vec{a} = 0$, or equivalently all the partials are equal to 0.

Extreme Value Theorem. Let X be a closed and bounded subset of $\mathbb{R}^n$ and suppose $f : X \to \mathbb{R}^n$ is continuous. Then f attains an absolute maximum and an absolute minimum somewhere on X.

Definition of **Critical point of** f. A point $\vec{c}$ in the domain of f where all of the partial derivatives of f at $\vec{c}$ equal 0.

Definition of **Saddle point**. A critical point that is not a max or min.

Theorem (Method to find absolute minima and maxima). Let C be the set of all critical points of f. Then, the absolute maximum is $\max\{f(\vec{c}) : \vec{c} \in C\}$ and the absolute minimum is $\min\{f(\vec{c}) : \vec{c} \in C\}$.

Theorem (Method to find absolute minima and maxima with a constraint). Let C be the set of all critical points of f. Let S be the union of C and the boundary of the constraint (the constraint constrains the domain on which we are finding absolute minima and maxima). Then, the absolute maximum is $\max\{f(\vec{c}) : \vec{c} \in C\}$ and the absolute minimum is $\min\{f(\vec{c}) : \vec{c} \in C\}$.

6.3.2 Some linalg stuff

Definition of **Matrix multiplication**. The matrix multiplication of the $m \times n$ matrix A and the $n \times p$ matrix B is made by dot-producting the rows of the first by the columns of the second:

$$[AB_{ij}] = [A_{i*} \cdot B_{*j}] = \left[\sum_{k=1}^{n} A_{ik}B_{kj}\right]$$

Definition of **Positive definite**. Let A be a matrix. Then A is positive definite iff for all $v \in \mathbb{R}^n \setminus \{0\}$, $v^T Av > 0$.

Definition of **Negative definite**. Let A be a matrix. Then A is negative definite iff for all $v \in \mathbb{R}^n \setminus \{0\}$, $v^T Av < 0$.

positive semidefinite and **negative semidefinite** are the same except that the determinant/eigenvalue/pivot/$v^T Av$ could also be 0.

Definition of **Principal minor**. The determinant of a submatrix of a matrix.

Definition of **Leading principal minor**. The kth-order leading principal minor is the determinant of the top left submatrix of a matrix, where the 1st-order leading principal minor is the determinant of the 1x1 matrix at its top left corner, the 2nd-order is the determinant of the 2x2 matrix at its top left corner, etc.

Theorem (Equivalent conditions for positive definiteness). Let A be a matrix. Then A is positive definite iff

- All leading principal minors of A are positive
- All eigenvalues of A are positive
- All pivots of A are positive

Theorem (Equivalent conditions for negative definiteness). Let A be a matrix. Then A is negative definite iff

- The kth-order leading principal minor is negative if k is odd and positive if k is even
- All eigenvalues of A are negative
- All pivots of A are negative
- $-A$ is positive definite

6.3.3 Local extrema, Second Derivative Test and Taylor series

Colley [4.1] Brummet [16, 17, MVCWUP:75-80(Mar 13-14)]

Definition of **Hessian matrix**. The Hessian matrix Hf of a function $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is the matrix of second-order partials

$$[Hf_{ij}] = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]$$

If $X \subseteq \mathbb{R}^2$, then

$$Hf = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

The **first-order Taylor polynomial** is just the linear approximation

$$T_1(\vec{x}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a})$$

Definition of **Second-order Taylor polynomial**. The second degree Taylor polynomial for a function $f \in \mathbb{R}^n \to \mathbb{R}$ at point $\vec{a}$ evaluated at point $\vec{x}$, where $\vec{h} := \vec{x} - \vec{a}$, is:

$$\begin{aligned} T_2(\vec{x}) &= f(\vec{a}) + \sum_{i=1}^{n} f_{x_i}(\vec{a})h_i + \frac{1}{2}\sum_{i,j=1}^{n} f_{x_i x_j}(\vec{a})h_i h_j \\ &= f(\vec{a}) + \nabla f(\vec{a}) \cdot \vec{h} + \frac{1}{2}\vec{h}^T Hf(\vec{a})\vec{h} \end{aligned}$$

Higher-order Taylor polynomials are not very useful.

Theorem (Second Derivative Test). Let X be an open subset of $\mathbb{R}^n$ and $f : X \to \mathbb{R}$ whose 2nd-order and lower partials exist and are continuous on X (f is of class C^2). Let $\vec{a} \in X$ be a critical point of f. Then

- If the Hessian $Hf(\vec{a})$ is positive definite, then f has a local minimum at $\vec{a}$.
- If the Hessian $Hf(\vec{a})$ is negative definite, then f has a local maximum at $\vec{a}$.
- If $\det Hf(\vec{a}) \neq 0$ but $Hf(\vec{a})$ is neither positive nor negative definite, then f has a saddle point at $\vec{a}$.

Equivalently if $X \subseteq \mathbb{R}^2$, let

$$D := f_{xx}(\vec{a})f_{yy}(\vec{a}) - \left(f_{xy}(\vec{a})\right)^2 = \det Hf(\vec{a})$$

Then

- If $D > 0$ and $f_{xx}(\vec{a}) > 0$, then f has a local minimum at $\vec{a}$.
- If $D > 0$ and $f_{xx}(\vec{a}) < 0$, then f has a local maximum at $\vec{a}$.
- If $D < 0$, then f has a saddle point at $\vec{a}$.
- If $D = 0$ the test is inconclusive.

(Note that if $f_{xx}(\vec{a}) = 0$ then $D \leq 0$.)

6.3.4 Lagrange multiplier

Theorem. If $f(\vec{x}_0) = c$ is an extreme value (absolute max or min) of f on g (the constraint is $\{\vec{x} : g(\vec{x}) = k\}$) and $\nabla g(\vec{x}_0) \neq 0$, then at $\vec{x}_0$, the level set $\{\vec{x} : f(\vec{x}) = c\}$ is tangent to $g(\vec{x}) = k$.

Equivalently, if $f(\vec{x}_0) = c$ is an extreme value (absolute max or min) of f on g and $\nabla g(\vec{x}_0) \neq 0$, then $\nabla f(\vec{x}_0) = \lambda \nabla g(\vec{x}_0)$, where $\lambda \in \mathbb{R}$ is called the **Lagrange multiplier**.

6.4 Integration

6.4.1 Double integrals

Colley [5.1, 5.2] Paul's Notes [15.1, 15.2, 15.3]

Brummet [08, MVCWUP:Feb3(29-33)]

Definition of **Double integral**. The double integral of $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$ over the rectangle R is

$$\iint_R f(x,y)\,dA = \lim_{n,\,m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} f\left(x_i^*, y_j^*\right)\Delta A$$

Generally, the double integral of f over the region R is

$$\iint_R f(x,y)\,dA = \lim_{n\to\infty} \sum_{(x_i^*, y_j^*)\in R} f\left(x_i^*, y_j^*\right)\Delta A$$

(choose n points (x_i^*, y_j^*) in R, then sum each $f\left(x_i^*, y_j^*\right)\Delta A$)

Theorem (Fubini's Theorem). If $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$ is continuous on $[a,b] \times [c,d]$, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Theorem. If $f(x,y) = g(x)h(y)$ and $R = [a,b] \times [c,d]$, then

$$\iint_R f(x,y)\,dA = \iint_R g(x)\,h(y)\,dA = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right)$$

Theorem (Type I integrals). If $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$ is defined on $D = \{(x,y) : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$, then

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

Theorem (Type II integrals). If $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$ is defined on $D = \{(x,y) : h_1(y) \leq x \leq h_2(x), c \leq y \leq d\}$, then

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Theorem (Reversing the order of integration). Sketch the bounds of the region of integration, then just redo the bounds from scratch.

Theorem (Converting double integrals to polar).

$$\int_a^b \int_c^d f\,dx\,dy = \int_\alpha^\beta \int_\gamma^\delta f\,r\,dr\,d\theta$$

$a, b, c, d, \alpha, \beta, \gamma, \delta$ may be constants or may depend on the variables. To find the new bounds sketch the bounds and redo the bounds.

Definition of **Triple integral**. The triple integral of f over the region R is

$$\iiint_R f\,dA = \lim_{n\to\infty} \sum_{(x_i^*, y_j^*, z_k^*)\in R} f\left(x_i^*, y_j^*, z_k^*\right)\Delta A$$

(choose n points (x_i^*, y_j^*, z_k^*) in R, then sum each $f\left(x_i^*, y_j^*, z_k^*\right)\Delta A$)

6.4.2 General change of variables

Theorem (General change of variables). Given a function $f : \mathbb{R}^n \to \mathbb{R}^m$ which takes in an argument in the coordinate system X, and $T_{XU} : \mathbb{R}^n \to \mathbb{R}^n$ is a transformation from coordinates in X to coordinates in U,

$$\int \cdots \int_R f(\vec{x})\, dx_1 \cdots dx_n = \int \cdots \int_R f(T_{XU}(\vec{u})) \left|\det DT_{XU}^{-1}\right| du_1 \cdots du_n$$

where $\left|\det DT_{XU}^{-1}\right|$ is the absolute determinant of the Jacobian of T_{XU}^{-1} (the transformation from coordinates in U to coordinates in X), which is also denoted by

$$\left|\frac{\partial(x_1, \dots, x_n)}{\partial(u_1, \dots, u_n)}\right|$$

It can also be computed more easily by taking the inverse of DT_{XU}:

$$\left|\det DT_{XU}^{-1}\right| = \left|\frac{1}{\det DT_{XU}}\right|$$

Example (Cartesian to spherical). Converting from Cartesian coordinates to spherical coordinates in $\mathbb{R}^3$:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = T_{XP}^{-1}\left(\begin{bmatrix} \rho \\ \phi \\ \theta \end{bmatrix}\right) = \begin{bmatrix} \rho \sin\phi \cos\theta \\ \rho \sin\phi \sin\theta \\ \rho \cos\phi \end{bmatrix}$$

so then the absolute Jacobian determinant is

$$\left|\det T_{XP}^{-1}\right| = \rho^2 \sin\phi$$

6.5 Vector fields

6.5.1 Vector fields

Definition of **Vector field**. A vector field in $\mathbb{R}^n$ is a mapping $F : X \subseteq \mathbb{R}^n \to \mathbb{R}^n$.

Definition of **Flow line**. A flow line of a vector field $F : X \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a differentiable path $\vec{x} : I \to \mathbb{R}^n$ (where I is an interval on $\mathbb{R}$) such that

$$\vec{x}'(t) = F(\vec{x}(t))$$

That is, the velocity vector of $\vec{x}$ at time t is given by the value of the vector field F at the point on x at time t.

Procedure (Approximating flow line). Begin at a vector along the flow line. Then the next vector on the vector field in the direction pointed to by this vector is approximately along the flow line. So you can draw a curve through the vectors following the arrows.

6.5.2 Conservative vector field

Definition of **Conservative vector field**. A vector field $\vec{F}$ is conservative iff there exists $f : \mathbb{R}^n \to \mathbb{R}$ such that $\vec{F} = \nabla f$ at all points in $\mathbb{R}^n$. Then f is called the **potential function** for F.

Note: Sometimes in physics, the potential function is defined such that $\vec{F} = -\nabla f$ - for example, $\vec{E} = \nabla V$ in E+M.

Lemma. A conservative vector field is irrotational.

If a vector field is irrotational and is defined on a domain with no holes, then it is conservative.

6.5.3 Divergence and curl

Definition of **Del operator**. In $\mathbb{R}^3$, del is defined by

$$\nabla := \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$$

In $\mathbb{R}^n$, del is defined by

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{bmatrix}$$

Del is an operator; it takes in a function and outputs a function.

Definition of **Divergence**. Let $\vec{F} : X \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable vector field. Then the divergence of $\vec{F}$ is the scalar field

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x_1} + \cdots + \frac{\partial F_n}{\partial x_n}$$

Lemma. If $\vec{F}$ represents the flow rate of a fluid, then $\operatorname{div} \vec{F}$ represents the net mass flow through each point in the domain of $\vec{F}$:

- If $\operatorname{div} \vec{F} > 0$, then more fluid is flowing out than in.
- If $\operatorname{div} \vec{F} < 0$, then more fluid is flowing in than out.
- If $\operatorname{div} \vec{F} = 0$, then the same amount of fluid flows in as flows out. In this case, $\vec{F}$ is considered **incompressible** and **solenoidal**.

Definition of **Curl** ($\mathbb{R}^3$). Let $\vec{F} : X \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ be a differentiable vector field on $\mathbb{R}^3$. Then the curl of $\vec{F}$ is the vector field

$$\operatorname{curl} \vec{F} = \nabla \times \vec{F}$$

Definition of **Curl** ($\mathbb{R}^2$). Let $\vec{F} : X \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable vector field on $\mathbb{R}^2$. Then the curl of $\vec{F}$ is the scalar field

$$\operatorname{curl} \vec{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

This is equivalent to the magnitude of $\nabla \times \vec{F}$, where counterclockwise is positive and clockwise is negative (by right-hand rule).

Definition of **Irrotational**. If $\nabla \times \vec{F} = 0$ everywhere on the vector field $\vec{F} : X \subseteq \mathbb{R}^n \to \mathbb{R}^n$, then $\vec{F}$ is considered **irrotational**.

Lemma. Let there exist an infinitesimally small sphere at the point $\vec{x} \in X$. Let $\vec{F} : X \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field that represents the velocity of a fluid at each point in X. Then $\operatorname{curl} \vec{F}$ is the unique vector such that

- The direction of $\operatorname{curl} \vec{F}$ is along the axis of rotation of the sphere, following the right-hand rule
- The magnitude of $\operatorname{curl} \vec{F}$ is the speed of the rotation of the sphere

6.5.4 Line integrals

Definition of **Scalar line integral**. If C is a smooth plane curve defined by $x = x(t), y = y(t), a \le t \le b$, then

$$\int_C f(x, y)\, ds := \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

This generalizes to higher dimensions - if C is a smooth curve defined by $\vec{x} = \vec{x}(t), a \le t \le b$, where $x \in \mathbb{R}^n$, then

$$\int_C f(\vec{x}) ds = \int_a^b f(\vec{x}(t)) \sqrt{\left(\frac{dx_1}{dt}\right)^2 + \cdots + \left(\frac{dx_n}{dt}\right)^2}\, dt$$

Definition of **Line integral of a vector field along a smooth curve**. If F is any continuous vector field defined on a smooth curve C defined by $\vec{r}(t), a \le t \le b$, then

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \hat{T} ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$$

It represents the work done by moving a particle along the curve C, if F is a force field.

6.5.5 Green's Theorem

Green's Theorem. Let ∂D be a positively oriented, piecewise smooth, simple closed curve in the xy-plane, and D be the region bounded by ∂D. If P and Q have continuous partial derivatives on an open region containing D, then

$$\oint_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA$$

Equivalently, if $\vec{F} = x, y \mapsto \begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix}$, then

$$\oint_{\partial D} \vec{F} \cdot d\vec{r} = \iint_D \operatorname{curl} \vec{F}\,dA$$

In other words, the circulation of a vector field along a curve is the same as the sum of the curls within the region bounded by the curve.

Definition of **Circulation**. The circulation of the vector field $\vec{F}$ around the curve C is

$$\oint_C \vec{F} \cdot d\vec{r}$$

It measures how much F aligns with the curve C

6.6 Surfaces

6.6.1 Parametric surfaces

Definition of **Parametric/parameterized surface**. Let $\vec{X} : D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ be a one-to-one function (except possibly at the boundary of D). Then the image of X is called a parameterized surface.

Definition of **Normal vector to a parameterized surface**. Let $\vec{X} = \begin{bmatrix} u \\ v \end{bmatrix} \mapsto \begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{bmatrix}$ be a parameterization of a surface, and let $\vec{u}$ be a vector in the domain of $\vec{X}$. Then the tangent vector along the u-axis is $\vec{X}_u(\vec{u})$, where

$$\vec{X}_u = \begin{bmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial u} \end{bmatrix}$$

and similarly the tangent vector along the v-axis is $\vec{X}_v(\vec{u})$.

Then the normal vector to the parameterized surface at the point $\vec{u}$ is

$$\vec{N} = \vec{X}_u(\vec{u}) \times \vec{X}_v(\vec{v})$$

Definition of **Smooth**. A paramaterization $\vec{X}$ of a surface is smooth at a point if its normal vector is not equal to 0 at that point.

A surface is smooth at a point if there exists a paramaterization for that surface which is smooth at that point.

Note that a smooth surface can have non-smooth paramaterizations.

6.6.2 Surface integrals

Definition of **Scalar surface integral**. The surface integral over the surface S which is paramaterized by $\vec{X} : D \subseteq \mathbb{R}^2 \to \mathbb{R}^3 = (u,v) \to (x,y,z)$ and where D is the domain of $\vec{X}$ is

$$\iint_S f\,dS = \iint_D f(\vec{X}(u,v))\,|\vec{X}_u \times \vec{X}_v|\,dA$$

(dS is a part of the surface area, dA is a part of the domain)

Linear Algebra (7)

Let $F = \mathbb{R}$ or $F = \mathbb{C}$.

7.1 Linear equations

Definition of **Linear equation**. An equation that can be written in the form

$$\sum_k a_k x_k = y$$

where all $a_k \in F$ and $y \in F$.

Definition of **Solution to a linear equation**. The solution to a linear equation is a set $\{s_k\}$ such that $\sum_k a_k s_k = y$, i.e. substituting $x_k = s_k$ results in the equation being true.

Definition of **Linear system**. A set of linear equations.

Let m be the number of linear equations in the system. Let n be the number of variables in the system. Then the jth equation can be written as

$$\sum_{k=1}^{n} A_{jk} x_k = y_j$$

Let:

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad Y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

Then the system can be written as $AX = Y$.

Definition of **Consistent linear system**. A system that has at least one solution.

Definition of **Linear combination**. The linear combination of the equations of a linear system is a linear equation formed by multiplying each equation by c_j where $c_j \in F$.

This linear combination can be written as

$$\sum_{j=1}^{m} \sum_{k=1}^{n} c_j A_{jk} x_k = \sum_{j=1}^{m} y_j$$

Theorem. All solutions of a linear system are solutions to the linear combination of the equations of the system.

Definition of **Equivalent linear systems**. Two systems are equivalent if they have the same set of solutions.

Theorem. Two systems are equivalent if each equation in each system is a linear combination of the equations in the other system.

7.1.1 Matrices and rows

Definition of **Elementary row operations**. The elementary row operations are:

Definition of **Scaling**. $R_i \mapsto cR_i$ where c is a nonzero scalar.

Definition of **Replacement**. $R_i \mapsto R_i + cR_j$ where c is a scalar.

Definition of **Interchange**. Swap R_i and R_j.

Theorem (Elementary row operations are invertible). For any elementary row operation e, there exists an elementary row operation e^{-1} such that $e^{-1}(e(A)) = A$ for any matrix A.

Definition of **Row equivalence**. Two matrices are row-equivalent if each can be derived from the other using a finite number of elementary row operations.

Definition of **Row echelon form (REF)**. A matrix is in REF if it satisfies:

1. All nonzero rows are above all rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

Definition of **Reduced row echelon form (RREF)**. A matrix is in RREF if it is in REF and additionally satisfies:

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column

Definition of **Pivot position**. A location A_{ij} where $\mathrm{RREF}(A)_{ij}$ is a leading 1.

Definition of **Pivot column**. A column which contains a pivot position.

Definition of **Pivot**. A nonzero number at a pivot position.

Procedure (Gauss-Jordan elimination).

Procedure (Gaussian elimination). Iterate through the pivot columns of A from left to right. For each pivot column, use elementary row operations to ensure that the pivot position is nonzero and that all entries in the column below the pivot position are zero. This produces $\mathrm{REF}(A)$.

Procedure (Jordan elimination). Iterate through the pivot columns of $\mathrm{REF}(A)$ from right to left. For each pivot column, use elementary row operations to ensure that all other entries in the column other than the pivot are zero and that the pivot is equal to 1. This produces $\mathrm{RREF}(A)$.

Definition of **Leading variable, determined variable, basic variable**. A variable in a pivot column.

Definition of **Free variable**. A variable not in a pivot column.

7.1.2 Homogeneous linear systems

Definition of **Homogeneous linear system**. A system where $y_0 = y_1 = \cdots = y_m = 0$. It can be written as $AX = 0$.

Theorem (Trivial solution). For any homogeneous system, $x_0 = x_1 = \cdots = x_n = 0$ is a solution to the system. Therefore, all homogeneous systems are consistent.

Theorem. (a) If there are less equations than there are variables ($m < n$), then $AX = 0$ has an infinite number of solutions.

(b) If there are an equal number of equations and variables, then A is row-equivalent to the $n \times n$ identity iff $AX = 0$ has only the trivial solution.

(c) If there are more equations than there are variables ($m > n$), then

$$\mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

iff $AX = 0$ has only the trivial solution.

Procedure (Solution). To solve a homogeneous system, perform Gauss-Jordan elimination on A so that $R = \mathrm{RREF}(A)$. Then solve $RX = 0$. The variables which are not in pivot columns are free variables and may be set to any value, typically denoted $u_1, u_2, \ldots$.

7.1.3 Inhomogeneous linear systems

Procedure (Solution). To solve an inhomogeneous system, perform Gauss-Jordan elimination on $A' = [A|Y]$ so that $R' = [R|Z] = \mathrm{RREF}(A')$. Then solve $RX = Z$. Note that not all inhomogeneous systems are solvable (consistent).

7.2 Fields

7.2.1 Real and complex numbers

Definition of **Field properties**. The following properties:

Theorem (Properties of addition). For all $x, y, z \in F$:

(A1) Commutativity: $x + y = y + x$

(A2) Associativity: $(x + y) + z = x + (y + z)$

(A3) Identity: $\exists 0 \in \mathbb{R}$ s.t. $0 + x = x$

(A4) Additive inverse: For $x \in F$, $\exists -x \in F$ s.t. $x + (-x) = 0$

Theorem (Properties of multiplication). For all $x, y, z \in F$:

(M1) Commutativity: $xy = yx$

(M2) Associativity: $(xy)z = x(yz)$

(M3) Identity: $\exists 1 \in \mathbb{R}$ s.t. $1x = x$ and $1 \neq 0$

(M4) Additive inverse: For $x \in F \setminus \{0\}$, $\exists x^{-1} \in F$ s.t. $xx^{-1} = 1$

Theorem (Distributive property).

(D) $x(y + z) = xy + xz$ for all $x, y, z \in F$.

7.2.2 Fields

Definition of **Field**. A set F which defines the following two operations:

- Addition: an operation that maps $x, y \in F \to c \in F$ and satisfies the properties of addition
- Multiplication: an operation that maps $x, y \in F \to c \in F$ and satisfies the properties of multiplication

for which the distributive property also holds.

Theorem. $\mathbb{R}$ and $\mathbb{C}$ are fields.

Definition of **Complex number**. A number which can be defined by a pair of real numbers a, b where the value of the number is equal to $a + bi$.

Theorem (Useful things about complex numbers). Let $z = a + bi$ and $w = c + di$ be complex numbers. Then:

- $z + w = (a + c) + (b + d)i$
- $zw = (ac - bd) + (bc + ad)i$

Definition of F^n. For a field F, F^n is the set of all ordered n-tuples of elements of F:

$$F^n := \{(x_1, \dots, x_n) : x_1, \dots, x_n \in F\}$$

Definition of **Addition in** F^n. If $a, b \in F^n$:

$$a + b = (a_1 + b_1, \dots, a_n + b_n)$$

Addition follows the properties of additon (A1-A4).

Definition of **Product of element of** F **and element of** F^n. If $\alpha \in F$ and $x \in F^n$, then

$$\alpha x = (\alpha x_1, \dots, \alpha x_n)$$

7.3 Vector spaces

2.3 Vector spaces

Definition of **Vector space**. A vector space over F is a set V with the following operations:

- **Vector addition**: $u \in V, v \in V \mapsto (u + v) \in V$, which satisfies the properties of addition
- **Scalar multiplication**: $\alpha \in F, v \in V \mapsto \alpha v \in V$, which satisfies the properties of scalar multiplication

Definition of **Properties of addition**. For all $u, v, w \in V$:

(A1) Commutativity: $u + v = v + u$

(A2) Associativity: $(u + v) + w = u + (v + w)$

(A3) Identity: $\exists 0 \in \mathbb{R}$ s.t. $0 + u = u$

(A4) Additive inverse: For $u \in F$, $\exists -u \in F$ s.t. $u + (-u) = 0$

Definition of **Properties of scalar multiplication**. For all $\alpha, \beta \in F$, $v, w \in V$:

(S1) Associativity: $(\alpha\beta)v = \alpha(\beta v)$

(S2) Distributivity over scalar addition: $(\alpha + \beta)v = \alpha v + \beta v$

(S3) Distributivity over vector additon: $\alpha(v + w) = \alpha v + \alpha w$

(S4) Multiplicative identity: $1v = v$

Theorem. F^n is a vector space.

Theorem. All inverses and identities are unique in a vector space.

Definition of F^ω. The set of all sequences of elements of F:

$$F^\omega := \{(x_1, x_2, \dots) : x_k \in F \text{ for } k \in \mathbb{N}\}$$

where addition and scalar multiplication are defined similarly to F^n:

$$a + b := (a_1 + b_1, \dots, a_n + b_n)$$
$$\alpha x := (\alpha x_1, \dots, \alpha x_n)$$

Definition of $F^{m,n}$. The set of all $m \times n$ matrices with entries in F, where addition and scalar multiplication are defined as:

$$(A + B)_{ij} := A_{ij} + B_{ij}$$
$$(\alpha A)_{ij} := \alpha A_{ij}$$

Definition of **Vector space of functions**. Let V be a vector space, S be a set, and

$$V^S = \{f : S \to V\}$$

(the set of all functions that map members of S to members of V). Then V^S is a vector space, if we define for all $p, q \in V^S$, $s \in F$,

$$(f + g)(s) = f(s) + g(s) \qquad (\alpha f)(s) = \alpha(f(s))$$

Definition of **Polynomial**. A function $p : F \to F$ is a polynomial of degree n iff there exist $c_0, \dots c_n \in F$ such that

$$p(x) = c_0 + c_1 x + c_2 x^2 + \cdots c_n x^n = \sum_{k=0}^{n} c_k x^k$$

$\mathcal{P}(F)$ is the set of all polynomials of any degree with coefficients in F. $\mathcal{P}_n(F)$ is the set of all polynomials of degree n with coefficients in F.

$\mathcal{P}(F)$ and $\mathcal{P}_n(F)$ are vector spaces if we define for all $p, q \in \mathcal{P}_n(F)$, $s \in F$,

$$(p + q)(s) = p(s) + q(s) \qquad (\alpha p)(s) = \alpha(p(s))$$

7.3.1 Subspaces

Definition of **Subspace**. Let V be any vector space, and let W be a subset of V. Define vector addition and scalar multiplication on W by restricting the corresponding operations of V to W. If W is a vector space with respect to the restricted operations of V , then W is said to be a subspace of V.

Definition of **Closed**. An operation is closed under a set if applying the operation to elements of the set always results in an element of the set.

Definition of **Subspace**. A subspace of a vector space V is a subset W of V which contains the zero vector and is closed under addition and scalar multiplication.

Theorem. A subset W of a vector space V is a subspace iff

(i) W is nonempty

(ii) $\alpha \in F$ and $w_1, w_2 \in W$ implies $\alpha w_1 + w_2 \in W$

Typically, we prove (i) by proving that $0 \in W$.

7.3.2 Subspaces of F^n

Theorem (Subspaces of $\mathbb{R}^n$). $\mathbb{R}^n$ contains the following subspaces:

- $\{0\}$
- $\mathbb{R}^n$
- Any line through the origin
- Any plane through the origin
- etc

Definition of **Spanning**. For vectors to span a space is for it to be sufficient to be able to reach any point in the space using the vectors.

Definition of **Independence**. If a vector can be made out of other vectors, then the vector is independent

7.3.3 Intersections and unions of subspaces

Theorem. The intersection of any collection of subspaces of V is a subspace of V.

Theorem. The union of two subspaces of V is a subspace of V iff one of the subspaces is contained in the other.

Definition of **Sum of subspaces**. If U and W are subspaces of a vector space V, then

$$U + W := \{u + w : u \in U \text{ and } w \in W\}$$

Theorem. If U and W are subspaces of V, then $U + W$ is the smallest subspace of V containing both U and W.

Definition of **Direct sum**. If $V_1, \dots, V_n$ are subspaces of V such that each element of $+_{k=1}^{n} V_k = V_1 + \dots + V_n$ can be written uniquely as $\sum_{k=1}^{n} v_k = v_1 + \dots + v_n$ where $v_k \in V_k$, then $+_{k=1}^{n} V_k$ is a **direct sum** and can be written as $\bigoplus_{k=1}^{n} V_k = V_1 \oplus \dots \oplus V_n$.

Theorem. Let $V_1, \dots, V_n$ be subspaces of V. Then they are direct sums iff the only way to write $0 = v_1 + \dots + v_n$ is to take $v_1 = \dots = v_n = 0$.

Theorem. If U and W are subspaces of V, then $U + W$ is a direct sum iff $U \cap W = \{0\}$. This does not generalize to higher numbers of subspaces.

7.3.4 Spanning

Definition of **Linear combination**. A linear combination of a collection $v_1, \dots, v_n$ of vectors in vector space V is a vector of the form

$$\alpha_1 v_1 + \dots + \alpha_n v_n$$

where each $\alpha_k \in F$.

Definition of **Span**. Given $W \subseteq V$ where V is a vector field, the set of all linear combinations of vectors in W is called the span of W.

$$\operatorname{span}(W) := \left\{ \sum_{i=1}^{n} \alpha_i w_i : \alpha_i \in F, w_i \in W \right\}$$

Additionally, we define

$$\operatorname{span}(\emptyset) = \{0\}$$

Definition of **Subspace generated by a set**. Given $W \subseteq V$ where V is a vector field, the subpace generated by W is the smallest subspace of V containing W, or equivalently the intersection of all subspaces of V containing W.

Theorem. The span of W is the subspace generated by W, i.e. W is the smallest subspace of V containing W.

Definition of **Spanning, spanning set**. If $\operatorname{span}(W) = V$, then W spans V and W is a spanning set for V.

Definition of **Finite-dimensional vector space**. A vector space is finite-dimensional iff it has a finite spanning set.

Otherwise, it is infinite-dimensional.

7.3.5 Linear independence

Definition of **Linear independence**. Let V be a vector space. If $W \subseteq V$ is a finite set, it is linearly independent iff the only way to write 0 as a combination

$$\alpha_1 v_1 + \dots + \alpha_m v_m = 0$$

is by taking $\alpha_1 = \dots = \alpha_m = 0$. We also define $\emptyset$ to be linearly independent.

If $W \subseteq V$ is an infinite set, it is linearly independent if every finite subset of W is linearly independent.

Theorem. If $W \subseteq V$ is linearly independent, any subset $U \subseteq W$ is linearly independent.

If one vector in W is a linear combination of the other vectors (including if $0 \in W$), then W is linearly dependent.

7.3.6 Basis

Definition. Basis A basis of V is a subset of V which is linearly independent and spans V.

Definition of **Standard basis of F^n**.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$$

Definition of $P_m(F)$.

Definition of **Standard basis of $P_m(F)$**.

7.3.7 Dimension

Plus/minus lemma. Let $S \subseteq V$ (where V is a vector space).

- If S is linearly independent, and v is not in the span of S, then $S \cup \{v\}$ is linearly independent.
- If $v \in \operatorname{span}(S \setminus \{v\})$, then $\operatorname{span}(S) = \operatorname{span}(S \setminus \{v\})$.

Theorem. Let V be a finite-dimensional vector space and $S \subseteq V$. Then,

- If $\operatorname{span}(S) = V$, then S contains a subset B which is a basis of V
- If S is linearly independent, then S can be extended to a basis of V

Theorem. Let V be a finite-dimensional vector space spanned by a set of m vectors. Then any linearly independent set of vectors in V is finite and contains no more than m elements.

7.4 Linear maps

7.4.1 Linear map

Definition of **Linear map**. Let V and W be vector spaces over F. A **linear map** (also called **linear function** or **linear transformation**) from V to W is a function $T : V \to W$ with the two properties

$$\begin{aligned} T(v_1 + v_2) &= Tv_1 + Tv_2 && \text{(additivity)} \\ T(\alpha v) &= \alpha T v && \text{(homogeneity)} \end{aligned}$$

Equivalently, it is a function with the property that

$$T(\alpha v_1 + v_2) = \alpha T v_1 + T v_2$$

Equivalently, it is a function with the property that

$$T(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 T v_1 + \alpha_2 T v_2$$

Note: Not all elementary linear functions $y = mx + b$ are linear maps! All linear maps in F^1 are of the form $y = mx$.

Definition of **Linear operator**. A function $T : V \to V$ which is a linear map.

Definition of $\mathcal{L}$. For any vector spaces V and W, $\mathcal{L}(V, W)$ is the set of all linear maps from V to W.

$\mathcal{L}(V) := \mathcal{L}(V, V)$.

$\mathcal{L}(V, W)$ is also called $\operatorname{Hom}(V, W)$.

Linear Map Lemma. Let V be a finite-dimensional vector space and W be a vector space. Suppose $\{v_1, \dots v_n\}$ is a basis of V and $w_1 \dots w_n \in W$. Then there exists a unique linear map $T : V \to W$ such that

$$Tv_k = w_k$$

for each $k \in \{1, \dots, n\}$.

Note: This means that there exists a unique linear map that maps a basis to any vectors we wish, and that a linear map is uniquely determined by its output on a basis.

Lemma. If $T : V \to W$ is a linear map, then $T(0) = 0$.

Theorem ($\mathcal{L}(V, W)$ is a vector space). If we define

$$(S + T)(v) := S + T$$
$$(\alpha T)(v) := \alpha T(v)$$

for $S, T \in \mathcal{L}(V, W)$, $\alpha \in F$, then $\mathcal{L}(V, W)$ is a vector space and is a subspace of V^W.

Definition of **Product of linear maps**. If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then we define the product $ST \in \mathcal{L}(V, W)$ by $ST := S \circ T$.

Theorem ($\mathcal{L}(V)$ is a unital associative F-algebra). The product of linear maps on V has the following properties:

- **Bilinearity**: For all $S, T_1, T_2 \in \mathcal{L}(V)$, $\alpha \in F$,
  - $S(T_1 + T_2) = ST_1 + ST_2$
  - $(S_1 + S_2)T = S_1T + S_2T$
  - $(\alpha S)T = \alpha(ST) = S(\alpha T)$
- **Associativity**: $(RS)T = R(ST)$ for all $R, S, T \in \mathcal{L}(V)$
- **Identity** (*unital*): $IT = TI = T$ for all $T \in \mathcal{L}(V)$, where I is the identity map $x \mapsto x$.

It is an *F-algebra* because it is a vector space over a field equipped with bilinear multiplication.

7.4.2 Kernel

Definition of **Kernel**. The **kernel** or **null space** of a linear map $T : V \to W$ is the set of vectors in V which T maps to the zero vector of W:

$$\operatorname{null} T = \ker T := \{v \in V : Tv = 0\} \subseteq V$$

Theorem (Kernel is subspace of domain). If $T : V \to W$, then $\ker T$ is a subspace of V.

Definition of **Injective**. A map of sets $f : A \to B$ is **injective** or **one-to-one** iff $a_1 \neq a_2$ implies $f(a_1) \neq f(a_2)$, or equivalently iff $f(a_1) = f(a_2)$ implies $a_1 = a_2$.

Theorem (Injective linear map has trivial kernel). A linear map $T : V \to W$ is injective iff $\ker T = \{0\}$.

7.4.3 Image

Definition of **Image**. Let $T : V \to W$ be a linear map. The **image** or **range** of T is the set of all outputs of T:

$$\operatorname{range} T = \operatorname{im} T := \{Tv : v \in V\} \subseteq W$$

Theorem (Image is subspace of codomain). If $T : V \to W$, then $\operatorname{im} T$ is a subspace of W.

Definition of **Surjective**. A map of sets $f : A \to B$ is surjectve iff $\operatorname{im} f = B$, or equivalently iff for every $b \in B$ there exists $a \in A$ s.t. $f(a) = b$.

Iff f is surjective, f **maps** A **onto** B.

7.4.4 Fundamental Theorem

Fundamental Theorem of Linear Maps. Let $T : V \to W$ be a linear map with V finite-dimensional. Them $\operatorname{im} T$ is finite-dimensional and

$$\dim V = \dim \ker T + \dim \operatorname{im} T$$

Theorem (Corollary to Fundamental Thm). Let V and W be finite-dimensional vector spaces, and let $T : V \to W$. Then

- If $\dim V > \dim W$, then T is not injective.
- If $\dim V < \dim W$, then T is not surjective.

7.4.5 Systems of linear equations as linear maps

Definition of **Systems of linear equations as linear maps**. For a linear equation mapping vectors in F^n to F^m

$$Ax = y$$

we can interpret this as a linear map $T_A : M^{n\times 1}(F) \to M^{m\times 1}(F)$ where $T(x) = Ax$.

Theorem. $\ker T_A$ is the solution set of the homogeneous system $Ax = 0$.

Theorem. A homogeneous system of linear equations with more variables than equations has nonzero solutions.

A system of linear equations with more equations than variables has no solution for some choice of constant terms.

7.4.6 Isomorphisms

Definition of **Bijective**. Injective and surjective.

Definition of **Isomorphism**. Let V and W be vector spaces over F. An **isomorphism from** V **to** W is a bijective linear map $T : V \to W$. Iff there exists an isomorphism from V to W, V and W are **isomorphic**, which is denoted by $V \cong W$.

Definition of **Identity map**. The identity map $\operatorname{id}_V : V \to V$ is defined such that

$$\operatorname{id}_V(v) = v \qquad \text{for } V \in V$$

Theorem (Isomorphism is an equivalence relation).

- **Reflexive**: For any vector space V, $\operatorname{id}_V : V \to V$ is an isomorphism, i.e. $V \cong V$ for every V.
- **Symmetric**: If $T : V \to W$ is an isomorphism, then $T^{-1} : W \to V$ is also an isomorphism. Thus, $V \cong W$ implies $W \cong V$.
- **Transitive**: If $T_1 : U \to V$ and $T_2 : V \to W$ are isomorphisms, then $T_2T_1 : U \to W$ is an isomorphism. Thus, $U \cong V$ and $V \cong W$ implies $U \cong W$

Therefore, isomorphism is an equivalence relation on the collection of all vector spaces over F.

Definition of **Isomorphism class**. The isomorphism class $[V]$ of a vector space V is the set of all vector spaces isomorphic to V.

Lemma. Any two isomorphism classes are disjoint or equal.

Theorem. Two finite-dimensional vector spaces over F are isomorphic iff they have the same dimension.

Equivalently, any finite-dimensional vector space V over F is isomorphic to $F^{\dim V}$.

Theorem. If V and W are finite-dimensional vector spaces of the same dimension and $T : V \to W$ is a linear map, then if T is injective or surjective, it is an isomorphism.

Lemma. $\mathcal{P}_n(F) \cong F^{n+1}$.

7.4.7 Coordinates

For the following definitions, let $B := \{v_1, \dots, v_n\}$ be a basis of V, where V is an n-dimensional vector space over F.

Definition of **Linear combination map (basis isomorphism)**. The **linear combination map** or **basis isomorphism** is the isomorphism $L_B : F^n \to V$ defined by

$$L_B(\vec{x}) = x_1v_1 + \dots + x_nv_n$$

Definition of **Coordinate isomorphism**. The **coordinate isomorphism** is the isomorphism $L_B^{-1} : V \to F^n$.

This means that for any vector space, its vectors can be expressed as vectors in F^n.

Definition of **Coordinate vector**. Let v be a vector in V. Then the coordinate vector $[v]_B \in F^n$ of v is defined as

$$[v]_B := L_B^{-1}(v)$$

Equivalently, it is the vector such that

$$L_B([v]_B) = v$$

Definition of **Ordered basis**. An ordered basis of a n-dimensional vector space V is an n-tuple which is an ordering of a basis of V.

7.4.8 Matrix of a linear map

Definition of **Matrix of a linear map**. Let $B_V = (v_1, \dots, v_n)$ be a basis of V and $B_W = (w_1, \dots, w_m)$ be a basis of W. Let $T : V \to W$ be a linear map.

Then the matrix of T with respect to B_V and B_W is denoted as

$$[T]_{B_W B_V}$$

It is the $m \times n$ matrix defined such that its jth column is the coordinate vector of Tv_j with respect to B_W:

$$\left([T]_{B_W B_V}\right)_{*j} = [Tv_j]_{B_W}$$

Equivalently, it is the $m \times n$ matrix that satisfies

$$Tv_j = \sum_{i=1}^{m} \left([T]_{B_W B_V}\right)_{ij} w_i$$

Lemma. Applying a linear map T to a vector v is equivalent to multiplying the coordinate vector of v by the matrix of T:

$$[Tv] = [T][v]$$

Definition of **Standard matrix**. The matrix of a linear map $T : F^n \to F^m$ with respect to the standard bases of F^n and F^m.

Lemma.
$$\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$$

Lemma. Composing two linear maps T_1 and T_2 is equivalent to multiplying the matrices of the two linear maps:

$$T_1 \circ T_2 = [T_1][T_2]$$

Definition of **Matrix multiplication**. The matrix multiplication of the $m \times n$ matrix A and the $n \times p$ matrix B is made by dot-producting the rows of the first by the columns of the second:

$$[AB_{ij}] = [A_{i*} \cdot B_{*j}] = \left[\sum_{k=1}^{n} A_{ik} B_{kj}\right]$$

Theorem (Properties of matrix multiplication). Let A, B, C be matrices and r and s be scalars in F. Then

$$\begin{aligned} A\,(rB + sC) &= r\,(AB) + s\,(AC) \\ (B + C)\,A &= BA + CA \\ A\,(BC) &= (AB)\,C \end{aligned}$$

Effectively, matrix multiplication is distributive and associative. It is not always commutative.

7.4.9 Fundamental matrix spaces

Given a matrix $A \in F^{m,n}$, define $T_A := x \mapsto Ax$. Then $T_A : F^n \to F^m$.

Definition of **Null space, nullity**. The **null space** of A is the kernel of T_A:

$$\operatorname{Nul} A := \ker T_A = \{x \in F^n : Ax = 0\}$$

which is the solution set of the homogeneous linear system $Ax = 0$.

$$\operatorname{nullity} A := \dim \operatorname{Nul} A$$

Definition of **Column space, rank**. The **column space** of A is the image of T_A:

$$\begin{aligned} \operatorname{Col} A := \operatorname{im} T_a &= \{T_A x : x \in F^n\} \\ &= \{Ax : x \in F^n\} \\ &= \{\textstyle\sum i = 1^n x_i A_{*i} : x_i \in F\} \\ &= \operatorname{span}\{A_{*1}, \dots, A_{*n}\} \end{aligned}$$

which is the span of the column vectors of the matrix A.

$$\operatorname{rank} A := \dim \operatorname{Col} A$$

Lemma.

$$\begin{aligned} \operatorname{rank} A &= \text{the number of pivot columns} \\ \operatorname{nullity} A &= \text{the number of non-pivot columns} \end{aligned}$$

Therefore,
$$n = \operatorname{nullity} A + \operatorname{rank} A$$

Procedure (Finding bases of $\operatorname{Nul} A$, $\operatorname{Col} A$, $\operatorname{Row} A$).

1. $R := \operatorname{RREF}(A)$
2. The pivot columns of A are a basis of $\operatorname{Col} A$.
3. The pivot rows of B are a basis of $\operatorname{Row} A$. (row space is preserved under row operations)
4. The vectors spanning $Ax = 0$ are a basis of $\operatorname{Nul} A$.

Definition of **Transpose of a matrix**. If A is an $m \times n$ matrix, then its transpose A^T is the $n \times m$ matrix obtained by interchanging the rows and columns of A:

$$[A^T_{ij}] = [A_{ji}]$$

Definition of **Row space**. The row space of A is the span of the row vectors of A:

$$\operatorname{Row} A := \operatorname{span}\{A_{1*}, \dots, A_{m*}\} = \operatorname{Col} A^T$$

Lemma.
$$\dim \operatorname{Row} A = \dim \operatorname{Col} A = \operatorname{rank} A$$

Definition of **Left null space**. The left null space of A is the null space of A^T. It is the solution set of the homogeneous linear system $A^T y = 0$ or equivalently of $y^T A = 0^T$.

7.4.10 Invertible matrices

Definition of **Invertible matrix**. Let A be an $n \times n$ matrix. If there exists an $n \times n$ matrix B s.t. $AB = BA = I$, then A is **invertible** and **non-singular** and B is the **inverse** of A:

$$A^{-1} := B$$

A matrix which is not invertible is **non-invertible** and **singular**.

Lemma. The matrix of an isomorphism of finite-dimensional vector spaces is invertible.

Lemma. If a matrix is invertible, it has a unique inverse.

Lemma. Any finite product $A_1 \cdots A_k$ of invertible $n \times n$ matrices is invertible with

$$(A_1 \cdots A_k)^{-1} = A_k^{-1} \cdots A_1^{-1}$$

Definition of **Elementary matrix**. An $n \times n$ matrix is an **elementary matrix** if it can be obtained from the $n \times n$ identity matrix by a single elementary row operation.

Theorem. If $e : F^{n,n} \to F^{n,n}$ is an elementary row operation and $A \in F^{n,n}$, then
$$e(A) = e(I)A$$

Invertible Matrix Theorem. If A is an $n \times n$ matrix, then the following conditions are equivalent:

- A is invertible.
- A is row-equivalent to the $n \times n$ identity matrix.
- A is a product of elementary matrices.

Theorem. If A is an invertible $n \times n$ matrix and $E_k \cdots E_1 A = I$ where each E_j is an elementary matrix, then $E_k \cdots E_1 I = A^{-1}$.

Lemma. If A is row-equivalent to I, then $[A|I]$ is row-equivalent to $[I|A^{-1}]$. Otherwise, A doesn't have an inverse.

Procedure (Computation of A^{-1}). Row-reduce the augmented matrix $[A|I]$. If A is row-equivalent to I, then $\mathrm{RREF}([A|I]) = [I|A^{-1}]$. Otherwise, A doesn't have an inverse.

Lemma. The linear system $Ax = y$ of n equations with n unknowns has a unique solution iff A is invertible.

7.4.11 Change of basis

Definition of **Change of basis matrix**. The **change of basis matrix** or **transition matrix** of an n-dimensional vector space V with relation to the ordered bases B and B' is the $n \times n$ matrix whose jth column is the coordinate vector of the jth vector in B with respect to B':

$$C_{*j} := [v_j]_{B'}$$

Lemma. If B and B' are bases of V, $v \in V$, and C is the transition matrix from B to B', then

$$[v]_{B'} = C[v]_B$$

C is invertible, so similarly

$$[v]_B = C^{-1}[v]_{B'}$$

Lemma. If B and B' are bases of V, $T \in \mathcal{L}(V)$, and C is the transition matrix from B to B', then

$$[T]_{B'} = C[T]_B C^{-1}$$

C is invertible, so similarly

$$[T]_B = C^{-1}[T]_{B'} C$$

Note that C^{-1} is the matrix formed by the coordinate vectors of the basis vectors of B' with respect to B.

Procedure (Finding the transition matrix). Given bases B and B', row-reduce the augmented matrix $[[w_i] \mid [v_i]]$, whose first n columns are the coordinate vectors $[w_i]$ of the vectors of B' and whose last n columns are the coordinate vectors $[v_i]$ of the vectors of B. Then the transition matrix from B to B' is the right-hand side of $\mathrm{RREF}\left([[w_i] \mid [v_i]]\right)$.

7.4.12 Similarity

Definition of **Similar matrices**. Two $n \times n$ matrices A and B are similar if there exists an invertible matrix C s.t. $B = C^{-1}AC$.

Similarity is an equivalence relation on $F^{n,n}$.

Definition of **Similarity invariant**. A property of a $n \times n$ matrix A which holds for all matrices similar to A.

Theorem (Useful similarity invariants). The following properties are similarity invariants:

1. invertibility
2. nullity
3. rank

7.4.13 Linear functionals

Definition of **Linear functional**. Let V be a vector space over F. Then a linear map $T : V \to F$ is a **linear functional** on V.

Definition of **Dual space**. Let V be a vector space over F. Then the dual space of V is the vector space of linear functionals on V, $V^* := \mathcal{L}(V, F)$.

Definition of **Kronecker delta**.

$$\delta_{ij} := \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

Definition of i**th coordinate function**.

Definition of **Dual basis**.

Lemma. Let V be an n-dimensional vector space and $B = \{v_1, \ldots v_n\}$ be a basis of V. Then there for each $i \in \{1, \ldots, n\}$ there exists a unique linear functional f_i (called the i**th coordinate function for** B) such that

$$f_i(v_j) = \delta_{ij} \qquad \text{for all } j \in \{1, \ldots, n\}$$

For any $v \in V$, we can express v as a linear combination of elements of B, where the coefficients are given by $f_i(v)$:

$$v = \sum_{j=1}^{n} f_j(v) v_j$$

Additionally,

$$B^* := f_1, \ldots, f_n$$

is a basis of V^*, and is called the **dual basis** of the basis B of V.

Any linear functional $f : V \to F$ can be written uniquely as

$$f = \sum_{i=1}^{n} f(v_i) f_i$$

and $f(v)$ can be written as

$$f(v) = \sum_{i=1}^{n} f(v_i)([v]_B)_i$$

7.4.14 Trace

Definition of **Trace of a matrix**. The trace of an $n \times n$ matrix A is the sum of the diagonal elements of A:

$$\mathrm{tr}(A) := \sum_{i=1}^{n} A_{ii}$$

$\mathrm{tr} : F^{n,n} \to F$ is a linear functional.

Theorem (Properties of the trace). Let A, B be $n \times n$ matrices and c be a scalar. Then

- $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$
- $\mathrm{tr}(cA) = ctr(A)$
- $\mathrm{tr}(A^T) = \mathrm{tr}(A)$
- $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ (this generalizes to more than 2 factors)
- The trace is similarity-invariant: if A and B are similar then $\mathrm{tr}(A) = \mathrm{tr}(B)$

7.4.15 Transpose

Definition of **Transpose of linear map**. Let V and W be vector spaces over F, and let $T \in \mathcal{L}(V, W)$. The **transpose of** T or **dual map of** T is the linear map $T^* = T^T \in \mathcal{L}(W^*, V^*)$ defined for each $f \in W^*$ by

$$T^*(f) = T^T(f) := f \circ T$$

That is, for each linear functional $f \in W^*$, $T^T(f)$ is the linear functional in V^* defined by (for each $v \in V$)

$$T^T(f)(v) = f(Tv)$$

Theorem (Properties of the transpose). For all linear maps T_1, T_2, T and $\alpha \in F$,

- $(T_1 + T_2)^T = \alpha T_1^T + T_2^T$ (the map $T \mapsto T^T$ is linear)
- $(T_1 \circ T_2)^T = T_2^T + T_1^T$

Theorem (Matrix of transpose is transpose of matrix). The matrix of T^T is the transpose of the matrix of T:

$$[T^T]_{B_W^* B_V^*} = \left([T]_{B_V B_W}\right)^T$$

Lemma. Let V and W be finite-dimensional vector spaces, $T \in \mathcal{L}(V, W)$ and $g \in W^*$. Choose ordered bases $B_V = (v_1, \dots, v_n)$ and $B_W = (w_1, \dots, w_m)$ for V and W, respectively, and let $B_V^* = (f_1, \dots, f_n)$ and $B_W^* = (g_1, \dots, g_m)$ be the corresponding dual bases of V^* and W^*. Let $S = (1)$ be the standard ordered basis of F. Then

$$[T^T g]_{B_v^*, S} = [g]_{B_W^*, S}[T]_{V,W}$$

Lemma. Let V and W be finite-dimensional vector spaces, $T \in \mathcal{L}(V, W)$, $T^T \in \mathcal{L}(W^*, V^*)$. Then:

- $$\operatorname{rank} T^T = \operatorname{rank} T \leq \min\{\dim V, \dim W\}$$
- T is injective iff T^T is surjective
- T is surjective iff T^T is injective

7.5 Multilinear algebra and determinants

7.5.1 Bilinear forms

Definition of **Bilinear form**. Let V be a vector space over F. A **bilinear form** on V is a function $B : V \times V \to F$ which is linear in each variable separately when the other variable is held constant. That is, for all $v_1, v_2, v \in V$ and $\alpha B(v, v_1) + B(v, v_2)$,

$$B(\alpha v_1 + v_2, v) = \alpha B(v_1, v) + B(v_2, v)$$

and

$$B(v, \alpha v_1 + v_2) = \alpha B(v, v_1) + B(v, v_2)$$

Lemma. If V is a vector space over F and $f, g \in \mathcal{L}(V, F)$, then $B(u, v) := f(u)g(v)$ is a bilinear form on V.

Definition. $V^{(2)}$ denotes the set of all bilinear forms on V.

Lemma. $V^{(2)}$ is a subspace of $V \times V \to F$.

Definition of **Matrix of a bilinear form**. Let B be a bilinear form on V and let $\vec{e} = (e_1, \dots e_n)$ be an ordered basis of V. Then the matrix of B with respect to $\vec{e}$ is the matrix $[B]$ defined by:

$$[B]_{ij} = B(e_i, e_j)$$

Theorem. If B is a bilinear form on V, $[B]$ is its matrix with respect to the ordered basis $\vec{e}$, $v, w \in F$, and $[v]$ and $[w]$ are the coordinate vectors of v and w with respect to the ordered basis $\vec{e}$, then

$$B(v, w) = [v]^T [B][w]$$

7.5.2 Symmetric bilinear forms

Definition of **Symmetric bilinear form**. A bilinear form $B \in V^{(2)}$ is symmetric iff $B(u, w) = B(w, u)$ for all $u, w \in V$. The set of all symmetric bilinear forms on V is denoted by $V^{(2)}_{\text{sym}}$.

Definition of **Symmetric matrix**. An $n \times n$ matrix A is symmetric iff $A^T = A$.

Theorem. If $B \in V^{(2)}$, then the following conditions are equivalent.

- B is a symmetric bilinear form on V.
- $[B]$ is a symmetric matrix for every basis of V.
- $[B]$ is a symmetric matrix for some basis of V.
- $[B]$ is a diagonal matrix for some basis of V.

7.5.3 Alternating bilinear forms

Definition of **Alternating bilinear form**. A bilinear form $B \in V^{(2)}$ is alternating if

$$B(u, w) = -B(w, u)$$

for all $u, w \in V$. The set of alternating bilinear forms on V is denoted by $V^{(2)}_{\text{alt}}$.

Definition of **Antisymmetric matrix**. A matrix A is antisymmetric iff $A^T = -A$.

Theorem. If $B \in V^{(2)}$, then the following conditions are equivalent.

- B is an alternating bilinear form on V.
- $[B]$ is an antisymmetric matrix for every basis of V.
- $B(v, v) = 0$ for all $v \in V$

Lemma. $V^{(2)} = V^{(2)}_{\text{sym}} \oplus V^{(2)}_{\text{alt}}$, and $V^{(2)}_{\text{sym}}$ and $V^{(2)}_{\text{sym}}$ are subspaces of $V^{(2)}$.

7.5.4 Multilinear forms

Definition.

$$V^m := \underbrace{V \times V \times \cdots \times V}_{m \text{ times}}$$

Definition of **Multilinear form**. An m-linear form on V is a map which is linear in each entry when all other entries are held fixed.

The set of m-linear forms on V is denoted by $V^{(m)}$.

7.5.5 Alternating multilinear forms

Definition of **Alternating multilinear form**. An m-linear form M is alternating iff

$$M(v_1, \dots, v_i, \dots, v_j, \dots v_m) = -M(v_1, \dots, v_j, \dots, v_i, \dots v_m)$$

for all i, j.

Lemma. $M \in V^{(m)}$ is alternating iff $v_i = v_j$ implies $M(v_1, \dots, v_m) = 0$ for all $i \neq j \in [1, m]$.

Lemma. If $M \in V^{(m)}_{\text{alt}}$ and $\{v_1, \dots, v_m\}$ is linearly dependent, then $M(v_1, \dots, v_m) = 0$.

Lemma. If $m > \dim V$, then $\dim V^{(m)}_{\text{alt}} = 0$.

7.5.6 Permutations

Definition of **Permutation**. A permutation of the set $A = \{1, 2, \dots, m\}$ is a bijection $\sigma : A \to A$ (equivalently, a reordering of the ordered list A). It is often denoted by a matrix

$$\begin{bmatrix} 1 & 2 & \cdots & m \\ \sigma(1) & \sigma(2) & \cdots & \sigma(m) \end{bmatrix}$$

where the inputs form the first row and the corresponding outputs form the second row.

Definition of **Group**. A group is a set G which contains a binary operation $\cdot$ such that

- **Identity**: There exists an identity element $e \in G$ such that $a \cdot e = a$
- **Inverse**: There exists an inverse a^{-1} for each $a \in G$ such that $a \cdot a^{-1} = I$
- **Associativity**: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

Theorem. The set S_m of all permutations of $A = \{1, 2, \dots, m\}$ is a group under composition. The identity element is the trivial permutation $x \mapsto x$:

$$\begin{bmatrix} 1 & 2 & \cdots & m \\ 1 & 2 & \cdots & m \end{bmatrix}$$

The inverse of a permutation is denoted σ^{-1}, and the composition of two permutations is denoted $\tau \circ \sigma = \tau\sigma$.

Definition of **Cyclic permutation, r-cycle**. A **cyclic permutation** or **r-cycle** is a permutation defined by and denoted by a list $(a_1\ a_2\ \cdots\ a_r)$ where the a_i are distinct, such that $\sigma(a_i) = a_{i+1}$ for $i \in [1, r]$, $\sigma(a_r) = a_1$, and $\sigma(a_l) = a_l$ for $l > r$.

A 2-cycle is also called a **transposition**.

Definition of **Disjoint cycles**. Cycles which share no elements in commmon.

Theorem. Any permutation σ can be written as a composition of disjoint transpositions.

Definition of **Even, odd, sign**. A permutation σ is **even** and $\operatorname{sgn}(\sigma) = +1$ iff it is a product of an even number of disjoint transpositions.

It is **odd** and $\operatorname{sgn}(\sigma) = -1$ iff it is a product of an odd number of disjoing transpositions.

Procedure (Writing a permutation as a series of disjoint cycles). Take any element a of the input set A, and apply σ repeatedly until $a = \sigma \cdots \sigma(a)$. Then we have cycle $a \mapsto \sigma(a) \mapsto \cdots \mapsto \sigma \cdots \sigma(a) = a$. Repeat for all remaing elements that are not part of this cycle.

Procedure (Writing a permutation as a series of disjoint transpositions). Apply the above procedure to the permutation. Then for each of the disjoint cycles in the series that are not transpositions, apply that procedure to each of those cycles until they are transpositions (2-cycles).

Lemma. $\operatorname{sgn}(\tau\sigma) = \operatorname{sgn}(\tau)\operatorname{sgn}(\sigma)$ for all $\tau, \sigma \in S_m$.

Lemma. An r-cycle is an even permutation iff r is odd, and an odd permutation iff r is even.

Procedure (Determining the sign of a permutation). Decompose the permutation into a product of disjoint cycles. Then the parity (oddness/evenness) of the permutation is the number of cycles of even length in its decomposition.

7.5.7 Determinant

Definition. $[v_1 \cdots v_n]$ is the matrix whose jth column is v_j.

$|v_1 \cdots v_n|$ is the determinant of said matrix.

Definition of **Determinant of a linear operator**. For $T \in \mathcal{L}(V)$, the **determinant** of T is the unique scalar such that $M_T := M \circ T = (\det T)M$ for all $M \in V_{\text{alt}}^{(\dim V)}$.

Definition of **Determinant of a square matrix**. Let n be a positive integer, A be an $n \times n$ matrix with entries in F, and $T \in \mathcal{L}(F^n)$ be the operator whose matrix with respect to the standard basis of F^n is A. Then the determinant of A is $\det A := \det T$.

Lemma. Let $(v_1, \ldots, v_n)$ be an ordered n-tuple of column vectors. Then the map $(v_1, \ldots, v_n) \mapsto |v_1 \cdots v_n|$ is an alternating n-linear form on F^n.

Lemma ($O(n!)$ method for calculating determinant). If A is an $n \times n$ matrix,

$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) A_{\sigma(1)1} \cdots A_{\sigma(n)n}$$

Theorem (Cofactor method (also $O(n!)$)). Let A be an $n \times n$ matrix. Define $A[i|j]$ to be the result of removing the ith row and jth column from A. Define the ***ij*-cofactor** of A to be

$$C_{ij} := (-1)^{i+j} \det(A[i|j])$$

Then for any row numbered i,

$$\det A = \sum_{k=1}^{n} A_{ik} C_{ik}$$

And for any column numbered j,

$$\det A = \sum_{k=1}^{n} A_{kj} C_{kj}$$

Note that the coefficient $(-1)^{i+j}$ in C_{ij} forms the following pattern:

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \ddots \end{bmatrix}$$

Lemma. If A is a triangular matrix with $\lambda_1, \ldots, \lambda_n$ on the diagonal, then $\det A = \lambda_1 \cdots \lambda_n$ ($\det A$ is the product of the elements on the diagonal).

7.5.8 Properties of the determinant

Theorem (Determinant is multiplicative).

- If $S, T \in \mathcal{L}(V)$, then $\det(ST) = (\det S)(\det T)$.
- If A and B are $n \times n$ matrices, then $\det(AB) = (\det A)(\det B)$.

Theorem. A linear operator $T \in \mathcal{L}(V)$ is an isomorphism iff $\det T \neq 0$.

If T is an isomorphism, then $\det(T^{-1}) = (\det T)^{-1}$.

Theorem. The following are equivalent:

- A is invertible.
- $\det A \neq 0$.
- The homogeneous system $AX = 0$ has only the unique solution.

Theorem (Determinant is similarity invariant). Let $T \in \mathcal{L}(V)$ and $S : W \to V$ be an invertible linear map. Then

$$\det(S^{-1}TS) = \det T$$

Theorem. For all $T \in \mathcal{L}(V)$, $\det T = \det[T]$, where $[T]$ is the matrix of T with respect to any basis of V.

Theorem. If A is a square matrix matrix, or if $A \in \mathcal{L}(V)$, then

$$\det(A^T) = \det(A)$$

Theorem (Effect of row and column operations on the determinant). Let A be an $n \times n$ matrix.

- If two rows or columns of A are equal, then $\det A = 0$.
- If B is the result of swapping two rows or columns of A, then $\det B = -\det A$.
- If B is the result of multiplying one row or column of A by the scalar λ, then $\det B = \lambda \det A$
- If B is the result of replacing a column or row by the sum of itself and a scalar multiple of another column or row, then $\det B = \det A$. (this does not apply adding a column to a scalar multiple of a row or vice-versa.)

Theorem (Effect of elementary row operations (summary)). Let A be an $n \times n$ matrix.

- **Scaling**: $R_i \mapsto cR_i \implies \det B = -\det A$.
- **Replacement**: $R_i \mapsto R_i + cR_j \implies \det B = \det A$
- **Interchange**: $R_i \leftrightarrow R_j \implies \det B = -\det A$

Procedure ($O(n^3)$ procedure for finding determinant). Row-reduce the matrix A to row-echelon form ($R := \text{REF}(A)$). Then $\det R$ is the product of the items along the diagonal of R. To find $\det A$, apply the above theorems.

7.5.9 Cramer's rule

Definition of **Adjugate matrix**. Let A be an $n \times n$ matrix. Then the adjugate matrix of A is the transpose of the matrix of cofactors, i.e.

$$\operatorname{adj} A = C^T$$

where C is defined by

$$C_{ij} = (-1)^{i+j} \det(A[i|j])$$

Lemma. Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

Theorem (Cramer's Rule). Let A be an invertible $n \times n$ matrix. For any $\vec{b} \in \mathbb{R}^n$. the unique solution $\vec{x}$ of the linear system $A\vec{x} = \vec{b}$ has entries given by

$$x_i = \frac{\det A_i(\vec{b})}{\det A}$$

where $A_i(\vec{b})$ is the result of replacing the ith column of A by $\vec{b}$.

7.6 Eigenvalues and eigenvectors

7.6.1 Eigenvalues and eigenvectors

Theorem (Diagonality condition). Let V be a finite-dimensional vector space, $B = (v_1, \dots, v_n)$ be an ordered basis for V, and $T \in \mathcal{L}(V)$. Then $[T]_B$ is diagonal iff $T(v_i) = \lambda_i v_i$ for all i, where λ_i is the ith element along the diagonal of $[T]_B$.

Definition of **Eigenvalue, eigenvector of a linear operator**. Let V be a vector space and $T \in \mathcal{L}(V)$. An **eigenvalue** of T is a scalar λ s.t. there exists a nonzero vector $v \in V$ s.t.

$$T(v) = \lambda v$$

The vector v is the **eigenvector** of T associated with the eigen value λ.

Lemma. If v is an eigenvector of T associated with eigenvalue λ and if B is a basis of V, then

$$[T(v)]_B = \lambda[v]_B$$

Definition of **Eigenvalue, eigenvector of a square matrix**. An **eigenvalue** of the $n \times n$ matrix A is a scalar λ s.t. there exists a nonzero vector $v \in \mathbb{R}^n$ s.t.

$$Av = \lambda v$$

The vector v is the **eigenvector** of A associated with the eigen value λ.

Lemma. Similar matrices have the same eigenvalues.

Any matrix representing $T \in \mathcal{L}(V)$ has the same eigenvalues as T.

Definition of **Eigenspace**. A scalar λ is an eigenvalue of A iff $Av = \lambda v$ has a nontrivial solution. This is equivalent to the homogeneous linear system (where I is the $n \times n$ identity matrix)

$$(A - \lambda I)v = 0$$

The set of all solutions to this equation is the **eigenspace** of A associated to the eigenvalue λ, which we denote by E_λ.

Lemma. If $A \in \mathbb{R}^{n \times n}$, then the eigenspace of A corresponding to λ is a subspace of $\mathbb{R}^n$.

Lemma. Let A be an $n \times n$ matrix. Then $\det(\lambda I - A)$ is a monic (leading coefficient 1) polynomial of degree n on λ.

Definition. For any $n \times n$ matrix A, the monic, degree n polynomial $\det(\lambda I - A)$ is the **characteristic polynomial** of A. The equation

$$\det(\lambda I - A) = 0$$

is the **characteristic equation** of A.

Lemma. Similar matrices have the same characteristic equation.

Lemma. The eigenvalues of a triangular matrix are the entries on the main diagonal.

Theorem (Rational root theorem). Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ be a degree-n polynomial with integer coefficients. If $x = \frac{p}{q}$ i a rational root of $p(x)$, then a_0 is divisible by p and a_n is divisible by q.

7.6.2 Diagonalizable operators

Definition of **Diagonalizable operator**. A linear operator $T \in \mathcal{L}(V)$ is said to be diagonalizable if there exists a basis consisting of eigenvectors of T. Such as basis is called an **eigenbasis**.

Lemma. If $T \in \mathcal{L}(V)$, then every set of eigenvectors corresponding to distinct eigenvalues of T is linearly independent.

Lemma. If $\dim V = n$ and $T \in \mathcal{L}(V)$ has n distinct eigenvalues, then T is diagonalizable.

Definition of **Multiplicity of a root of a polynomial**. Let $p(x)$ be a polynomial. Then an element $a \in F$ is a root of multiplicity k iff there exists a polynomial $s(x)$ such that $s(a) \neq 0$ and $p(x) = (x - a)^k s(x)$.

Definition of **Algebraic multiplicity**. The algebraic multiplicity of the eigenvalue λ_k is the multiplicity of λ_k as a root of the characteristic polynomial of A.

Definition of **Geometric multiplicity**. The geometric multiplicity of the eigenvalue λ_k dimension of the eigenspace corresponding to λ_k.

Theorem (Conditions equivalent to diagonalizability). Let V be a finite-dimensional vector space, $T \in \mathcal{L}(V)$, and $\lambda_1, \dots, \lambda_n$ be the distinct eigenvalues of T. Then the following are equivalent:

- V has a basis consisting of eigenvectors of T
- $V = E_{\lambda_1} \oplus \dots \oplus E_{\lambda_m}$
- $\dim V = \dim E_{\lambda_1} + \dots + \dim E_{\lambda_m}$
- The geometric multiplicity of each eigenvector is equal to its algebraic multiplicity.

Differential Equations (8)

A *differential equation* is an equation involving a quantity and one or more of its derivatives.

1 Ordinary Differential Equations

An ODE involves the derivative of the dependent variable with respect to a single independent variable.

8.0.1 Solving by Integration

Definition of **Solution to a differential equation**. A function $y = f(x)$ that satisfies the differential equation when f and its derivatives are substituted into the equation.

Procedure (Euler's Method). To numerically approximates the solution to the differential equation $y' = F(x, y)$ with $y(x_0) = y_0$,

$$y_n = y_{n-1} + F(x_{n-1}, y_{n-1})(x_n - x_{n-1})$$

Definition of **Separable Equation**. A separable equation is a differential equation where

$$\frac{dy}{dx} = g(x)f(y)$$

for some function $g(x)$ which depends only on x and $f(y)$ which depends only on y.

Theorem. For a separable equation,

$$\int \frac{1}{f(y)}dy = \int g(x)dx$$

Definition of **Logistic Differential Equation**. For a population P which increases exponentially ($\frac{dP}{dt} \approx kP$) when the population is small compared to the carrying capacity M but where the environment cannot sustain a population larger than M,

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

Then

$$P(t) = \frac{M}{1 + \left(\frac{M}{P_0} - 1\right)e^{-kt}}.$$

and

$$\frac{d^2P}{dt^2} = k^2P\left(1 - \frac{P}{M}\right)\left(1 - \frac{2P}{M}\right)$$

8.0.2 Initial-Value Problems

Definition of **Initial-Value problem**. Assuming the function f is continuous, then the function y is a solution of the IVP given that

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0,$$

Where x_0 is called the *initial point* for the IVP and y_0 is the *initial value.*

8.0.3 Existence and Uniqueness of Solutions

Theorem (Existence and Uniqueness). If f is continuous, then the function f as previously defined has at least one solution on the interval of continuity. If at least one solution exists and $\frac{\partial f}{\partial y}$ is continuous on the same interval, the solution is unique.

8.0.4 Autonomous Equations

Definition of **Autonomous Equation**. An autonomous equation is an equation where the derivative of the dependent variable can be expressed as a function of the dependent variable alone, assuming continuity.

An example of an autonomous equation is

$$\frac{dy}{dx} = f(y) + g(x)$$

Whereas

$$\frac{dy}{dx} = f(y)g(x)$$

Is not autonomous.

Since $f(y)$ is independent of x, the resulting slope field has translational symmetry across the x-axis. A snapshot of the slope field through a vertical line $x = x_0$ is called a *phase line.* A line where the direction of the slopes is indicated with arrows either going down or up is called a *one-dimensional phase portrait* of the autonomous ODE.

At y values where $f(y) = 0$, when the slope is zero, those points are called *equilibrium points* or *stationary points.*

1. If surrounding solutions approach $y = y_0$ asymptotically then the equilibrium point is *asymptotically stable* or an *attractor.*
2. If surrounding solutions move away from $y = y_0$ then the equilibrium point is *unstable* or a *repeller.*
3. If surrounding solutions approach from one side and repel from another then the equilibrium point is *semi-stable.*

8.0.5 Bifurcations

A differential equation that depends on a parameter *bifurcates* if there is a qualitative change in the solutions as parameter changes, meaning that the phase line changes.

We can see this change in phase line by plotting the parameter and the solution of the differential equation. The resulting graph would look like an array of phase lines. Each point on the phase lines is called a *bifurcation point.*

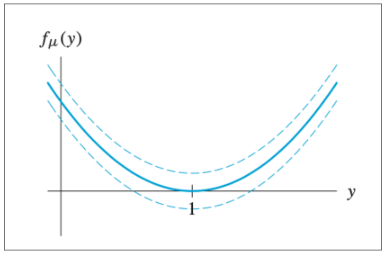


Figure: Graph of $f_\mu(y) = y^2 - 2y + \mu$ for $\mu < 1$, $\mu = 1$, and $\mu > 1$.

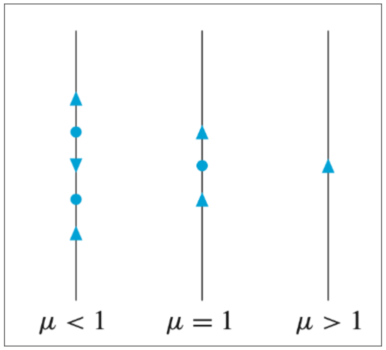


Figure: Corresponding phase portraits for $\frac{dy}{dx} = y^2 - 2y + \mu$.

The typical way to visualize bifurcations is through *bifurcation diagrams.* The parabola on the following figure is called a *bifurcation line.*

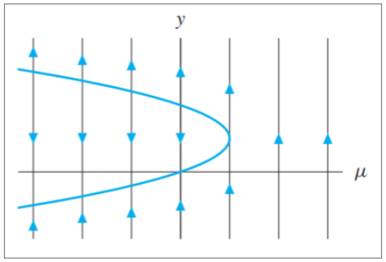


Figure: Bifurcation diagram for $\frac{dy}{dx} = y^2 - 2y + \mu$.

8.0.6 Separable Equations

Definition of **Separable first-order ODE**. An ODE is separable if it can be expressed as

$$\frac{dy}{dx} = g(x)h(y)$$

where g and h are continuous.

We can simplify the process of solving a separable equations by defining the following where $H(y)$ and $G(x)$ are antiderivatives of $\frac{1}{h(y)}$ and $g(x)$, respectively, and $c \in \mathbb{R}$ is an arbitrary constant.

$$H(y) = G(x) + c$$

which can then be expressed as

$$\int \frac{1}{h(y)} dy = \int g(x) dx$$

8.0.7 Implicitly-Defined Solutions

Sometimes a separable equation can't be solved for y explicitly. Let's define $F(x, y)$ as

$$F(x, y) := H(y) - G(x) - c$$

so that

$$F(x, y) = 0$$

The equation above implicitly defines y as a function of x only if it follows the *Implicit Function Theorem* from Multivariable Calculus.

Theorem (Implicit Function Theorem). If F is defined on a disc containing (x_0, y_0), where

1. $F(x, y) = 0$
2. $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ are continuous on the disc
3. $\frac{\partial F}{\partial x} \neq 0$

then the equation $F(x, y) = 0$ defines y as a function of x on some open set containing the point (x_0, y_0).

8.0.8 Singular Solutions

Definition of **Singular Solution**. A solution is singular if it cannot be obtained by any choice of c in the solution equation of the separable ODE.

When either $h(y)$ or $g(x)$ are equal to 0 inside a separable equation, then they would be valid solutions, but may not show up in the integration method for finding solutions as defined in the Separable Equations section. If that is the case, then those solutions are unique.

8.0.9 Orthogonal Trajectories

An *orthogonal trajectory* of a family of curves is a curve that intersects each curve of the family orthogonally.

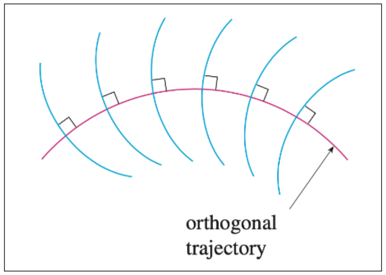


Figure: An example of an orthogonal trajectory.

For example,

$$x^2 + y^2 = r^2$$

and

$$y = mx$$

are orthogonal trajectories of each other.

To find orthogonal trajectories of an equation in terms of x, y, and another constant variable:

1. Take derivative of both sides in respect to x
2. Solve for $\frac{dy}{dx}$ from previous equation
3. Use the original equation to eliminate the constant and write an expression that only depends on x and y
4. The previous equation is a slope $m(x, y)$; find the orthogonal slope at point (x, y)
5. Solve the differential equation $\frac{dy}{dx} = m(x, y)$ to find the family of orthogonal trajectories

8.0.10 Linear Equations

Definition of **Linear Equation**. An equation is linear if it can be expressed in the from

$$\frac{dy}{dx} = p(x)y + q(x)$$

where $p, q : (a, b) \to \mathbb{R}, -\infty \leq a < b \geq \infty$ are continuous.

If $q(x) = 0$, then the linear equation is called *homogeneous*. If $p(x)$ is a constant, but not necessarily $q(x)$, it is called *constant-coefficient*.

8.0.11 Variation of Parameters

If $y_h(x)$ is a solution of a homogeneous linear equation, then $cy_h(x)$ is also a solution of the same homogeneous equation. A general solution of a homogeneous linear equation is

$$y_h(x) = ce^{\int p(x)dx}$$

A general solution of nonhomogeneous linear equation is

$$y(x) = ce^{\int p(x)dx} + e^{\int p(x)dx} \int e^{-\int p(x)dx} q(x) dx$$

8.0.12 Integrating Factors

This can be simplified by defining

$$\mu(x) := e^{-\int p(x)dx}$$

$$y(x) = \frac{1}{\mu(x)} \int \mu(x)q(x)dx$$

8.0.13 Singular Points

If we represent a linear equation in the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

and divide both sides by $a_1(x)$, then p and q will be discontinuous when $a_1(x) = 0$. These points are called *singular points*. These discontinuous points carry over to the final solution.

8.0.14 Bernoulli's Equation

An ODE in the from

$$\frac{dy}{dx} = p(x)y + f(x)y^n$$

where $n \in \mathbb{R}$, is called *Bernoulli's equation*.

This equation is linear if $n = 0, 1$. By substituting $u = y^{1-n}$, the equation can be rewritten in a linear form as

$$\frac{du}{dx} = (1 - n)(p(x)u + q(x))$$

where $n \neq 0, 1$.

8.0.15 Exact Equations

An exact equation is a differential equation in the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0$$

where M, N are continuously differentiable. They also represent partial derivatives of the potential function ψ

$$\psi_x+\psi_y\frac{dy}{dx}=0$$

where $\psi(x,y)$ is an existing *potential function*, as seen in Multivariable Calculus. The equation is exact only if,

- $\psi_{xy}=\psi_{yx}$
- Domain of functions above is open and topologically simply-connected (continuous)

So if the potential function exists, then the solution to the equation would be

$$\psi(x,y)=0$$

8.0.16 Homogeneous Equations

A real-valued function is *homogeneous of degree* α if it can be rewritten in the form

$$f(tx,ty)=t^\alpha f(x,y)$$

where $\alpha, t \in \mathbb{R}$.

A first-order differential equation of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0$$

is called *homogeneous* if both M and N are homogeneous of the same degree. To solve it, simply remember the substitution

$$y=ux$$

8.0.17 nth-Order Linear Equations

An *nth-order differential equation* is in the form

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

where the functions $a_i, g:(a,b)\to\mathbb{R}$ are continuous. An *nth-order initial value problem* (IVP) is to find the solution of the equation above where x_0 on interval I is subject to n initial conditions

$$y(x_0)=y_0, \qquad y'(x_0)=y_1, \qquad \dots, \qquad y^{n-1}(x_0)=y_{n-1}$$

Theorem. If $a_n(x)\neq 0$ on I and $x_0\in I$, then a solution y of the nth-order IVP exists on I and is unique.

8.0.18 Boundary Value Problems

A boundary value problem (BVP) is when a linear differential equation of order two or greater has the dependent variable y or its derivatives specified at different points. E.g. $y(a)=y_0, y(b)=y_1$. Those values are called *boundary conditions*. A solution to this BVP would satisfy the differential equation on I, whose graph passes through (a,y_0) and (b,y_1). Those boundary conditions can be often written as

$$\alpha_1 y(a)+\beta_1 y'(a)=\gamma_1,$$
$$\alpha_2 y(a)+\beta_2 y'(a)=\gamma_2,$$

where $\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2$ are arbitrary constants. Conclusions of the earlier defined theorem regarding IVP for nth-order equations does not apply to BVP.

8.0.19 Homogeneous (nth-Order Linear) Equations

An nth-order linear differential equation is homogeneous if $g(x)$ in

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

is equal to zero. First the associated homogeneous equation has to be solved before solving the nonhomogeneous equation.

8.0.20 Differential Operators

The symbol D is the *differential operator*, where

$$D^ny=\frac{d^ny}{dx^n}.$$

The *n-th order differential operator* or *polynomial operator* is

$$L=a_n(x)D^n+a_{n-1}(x)D^{n-1}+\cdots+a_1(x)D+a_0(x)$$

Using L, nth-order linear equations can be written as

$$Ly=0 \qquad \text{and} \qquad Ly=g(x)$$

The operator is also linear so it has the property

$$L(c_1f(x)+c_2g(x))=c_1Lf(x)+c_2Lg(x)$$

Theorem (Superposition Principle for Homogeneous Linear Equations). Let $y_1,\dots,y_k$ be solution of $Ly=0$ on interval I. Then the linear combination

$$y=c_1y_1+\cdots+c_ky_k$$

where c_i are constants, is also a solution of $Ly=0$.

By this theorem, every homogeneous nth-order linear equation has a solution of $y=0$.

8.0.21 Linear Dependence and Independence

Definition of **Linear Dependence**. A set of functions is *linearly dependent* on interval I if there exists constants c_i, not all zero, where

$$c_1f_1(c)+\cdots+c_nf_n(x)=0$$

for all x in I. If the only constants for which the equation above is satisfied are

$$c_1=c_2=\cdots=c_n=0$$

then the set is *linearly independent*.

Definition of **Wronskian**. Given that each function in $f_1,\dots,f_n$ is at least $n-1$ times differentiable, the determinant

$$W(f_1,f_2,\dots,f_n)=\begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

is called the *Wronskian* of the functions.

The set of solutions $y_1,y_2,\dots,y_n$ on interval I for a homogeneous equation is linearly independent only if $W(y_1,y_2,\dots,y_n)\neq 0$ on I.

This linearly independent set of solutions is called the *fundamental set of solutions* on the specified interval.

Theorem (Existence of a Fundamental Set). If functions a_i, g are continuous on some common interval I, and $a_n(x)\neq 0$ on I, then there exists a fundamental set of solutions of the homogeneous linear nth-order differential equation.

Given that $y_1,y_2,\dots,y_n$ is the fundamental set of solutions for the homogeneous equation on I, then its general solution is

$$y=c_1y_1(x)+c_2y_2(x)+\cdots+c_ny_n(x)$$

where c_i are arbitrary real constants.

Any function y_p, free of arbitrary parameters, that satisfies the nonhomogeneous equation, is said to be the *particular solution* of the equation.

Theorem. The general solution of the nth-order nonhomogeneous equation is of the form

$$y(x)=y_p(x)+c_1y_1(x)+c_2y_2(x)+\cdots+c_ny_n(x)$$

where y_p is the particular solution and $y_1,\dots,y_n$ are a fundamental set of solutions of the associated homogeneous equation.

8.0.22 Reduction of Order

Reduction of order is a method of substitution to reduce a homogeneous second order equation to a first order equation. This can be used to find a second nontrivial solution using the first and create a fundamental set of solutions.

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx$$

where the original equation is in the standard form

$$y'' + P(x)y' + Q(x)y = 0$$

8.0.23 Reduction of order for nonhomogeneous equations

The equation

$$y_p(x) = y_1(x) \left[\int \frac{e^{-\int P(x)dx}}{(y_1(x))^2} \left[\int e^{\int P(x)dx} y_1(x)R(x)dx \right] dx \right]$$

can be used to find the particular solution needed for general equation, as mentioned in the previous section, of the nonhomogeneous standard equation

$$y'' + P(x)y' + Q(x)y = R(x)$$

8.0.24 Solving first-order constant coefficient equations

To solve a first-order constant coefficient equation, an algebraic method can be used where in

$$ay' + by = 0$$

where a, b are constants. A general solution would be in the form

$$y = e^{-\frac{b}{a}x}$$

8.0.25 Solving second-order constant coefficient equations

To solve a second-order constant coefficient equation,

$$ay'' + by' + cy = 0$$

find the two solutions m_1 and m_2 in,

$$am^2 + bm + c = 0$$

using the quadratic formula,

$$m_1, m_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If the discriminant $b^2 - 4ac > 0$, the general solution would be,

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

otherwise, if the discriminant is zero (i.e. $m_1 = m_2$), the first solution would be,

$$y_1 = e^{m_1 x}$$

where the general solution can be found by finding y_2 using reduction of order.

If the discriminant is negative and the solution is two conjugates in the form,

$$m_1 = \alpha + i\beta$$

$$m_2 = \alpha - i\beta$$

Then the fundamental set of solutions can be written using Euler's formula,

$$y = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)$$

8.0.26 Solving higher-order constant coefficient equations

First, the auxiliary nth-degree polynomial equation must be solved,

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m^1 + a_0 = 0$$

Then, higher-order constant coefficient equations with real solutions extend the same way where,

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$$

But if any real root would have a multiplicity of $k > 1$, then for each multiplicity,

$$e^{m_1 x}, xe^{m_1 x}, \dots, x^{k-1} e^{m_1 x}$$

are additional linearly independent solutions in the fundamental set. The same pattern extends for complex roots,

$$e^{\alpha x} \cos(\beta x), xe^{\alpha x} \cos(\beta x), \dots, x^{k-1} e^{\alpha x} \cos(\beta x)$$

$$e^{\alpha x} \sin(\beta x), xe^{\alpha x} \sin(\beta x), \dots, x^{k-1} e^{\alpha x} \sin(\beta x)$$

8.0.27 Annihilation operators

Recall the form $Ly = g(x)$. Assuming coefficients are constant and if r_1 is a root of the auxiliary equation,

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0$$

then $L = (D - r_1)P(D)$, where the polynomial expression $P(D)$ is a linear differential operator of order $n - 1$.

For example, the expression,

$$y'' + 4y' + 4y = 0$$

can be represented as

$$L = D^2 + 4D + 4 = (D + 2)^2$$

$$Ly = 0$$

Definition of **Annihilation of a function**. If L is a linear differential operator with constant coefficients such that $L(f(x)) = y$, then L is the *annihilator* of the function.

- $y = k$ is annihilated by D
- $y = x^{n-1} + x^{n-2} + \cdots + c_0$ is annihilated by D^n
- $y = x^{n-1} e^{\alpha x} + \cdots + e^{\alpha x}$ is annihilated by $(D - \alpha)^n$
- $y = x^{n-1} e^{\alpha x} \cos(\beta x) + \cdots + e^{\alpha x} \cos(\beta x)$ is annihilated by $(D^2 - 2\alpha D + (\alpha^2 + \beta^2))^n$
- $y = x^{n-1} e^{\alpha x} \sin(\beta x) + \cdots + e^{\alpha x} \sin(\beta x)$ is annihilated by $(D^2 - 2\alpha D + (\alpha^2 + \beta^2))^n$

If L_1 annihilates y_1 and L_2 annihilates y_2, then the (commutative) product $L_1 L_2$ annihilates any linear combination $c_1 y_1 + c_2 y_2$. This also implies that the annihilator is not unique.

8.0.28 Method of undetermined coefficients

This method can be used to solve equations in the form

$$Ly = g(x)$$

where all coefficients are constant and $g(x)$ is a linear combination of

$$k, x^m, x^m e^{\alpha x}, x^m e^{\alpha x} \cos(\beta x), x^m e^{\alpha x} \sin(\beta x)$$

and m is nonnegative integer and $\alpha, \beta \in \mathbb{R}$.

If $g(x)$ can be annihilated by L_1, then applying it to both sides gives

$$L_1 Ly = L_1(g(x)) = 0$$

then solving for $L_1 Ly = 0$ can yield the general form of a particular solution y_p of the nonhomogeneous equation. Then substituting it into $Ly = g(x)$ will yield an explicit particular solution.

For example, to solve

$$y'' + 3y' + 2y = 4x^2$$

first find the auxiliary form of associated homogeneous equation

$$m^2 + 3m + 2 = (m + 1)(m + 2) = 0$$

and find the general solution using it

$$y = c_1 e^{-1x} + c_2 e^{-2x}$$

Then find L_1 that annihilates $g(x)$

$$L_1 = D^3$$

$$D^3(4x^2) = 0$$

So

$$L_1 Ly = D^3(D + 1)(D + 2) = 0$$

Now turn it into an auxiliary equation and solve for roots (including multiplicity above 1)

$$m^3(m+1)(m+2) = 0$$

$$m_1 = m_2 = m_3 = 0$$

$$m_4 = -1$$

$$m_5 = -2$$

Use those solutions to create a general solution

$$y = c_1 + c_2x + c_3x^2 + c_4e^{-1x} + c_5e^{-2x}$$

Since we know that the last two terms $c_4e^{-1x} + c_5e^{-2x}$are the solution of the associated homogeneous equation, the rest must include the particular solution

$$y_p = c_1 + c_2x + c_3x^2$$

You can substitute derivatives of it into the original nonhomogeneous equation and solve for the constants

$$y_p' = c_2 + 2c_3x$$

$$y_p'' = 2c_3$$

$$y_p'' + 3y_p' + 2y_p = 4x^2$$

$$(2c_3) + 3(c_2 + 2c_3x) + 2(c_1 + c_2x + c_3x^2) = 4x^2$$

Collect like terms and create a system of equations

$$(2c_3 + 3c_2 + 2c_1) + x(6c_3 + 2c_2) + x^2(2c_3 - 4) = 0$$

$$2c_3 + 3c_2 + 2c_1 = 0$$

$$6c_3 + 2c_2 = 0$$

$$2c_3 - 4 = 0$$

Therefore the general solution and final answer will be

$$y = c_1e^{-x} + c_2e^{-2x} + 7 - 6x + 2x^2$$

8.0.29 Variation of parameters for second-order equations

Theorem (Cramer's Rule). Let A be an invertible $n \times n$ matrix. For any $b \in \mathbb{R}^n$, the linear system $Ax = b$ has the unique solution

$$x_i = \frac{\det A_i(b)}{\det A}, i = 1, 2, \ldots, n$$

where $A_i(b)$ is the matrix obtained from A by replacing the ith column of A by the vector b.

Using variation of parameters, the particular solution of nonhomogeneous linear equation in the form $y'' + P(x)y' + Q(x)y = R(x)$ can be found by

$$y_p(x) = y_1(x)\int \frac{-y_2(x)R(x)}{W}dx + y_2(x)\int \frac{y_1(x)R(x)}{W}dx$$

where W is the wronskian

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

So the general solution is

$$y = c_1y_1(x) + c_2y_2(x) + y_p$$

8.0.30 Variation of parameters for higher-order equations

TODO

8.0.31 Cauchy-Euler Equation

Definition of **Cauchy-Euler Equation**. An equation in the form

$$a_nx^ny^n + a_{n-1}x^{n-1}y^{n-1} + \cdots + a_1xy' + a_0 = g(x)$$

on $(0, \infty)$ where a_n coefficients are constant, is known as a *Cauchy-Euler equation.* To find solutions on $(-\infty, 0)$, substitute $u = -x$.

You can solve Cauchy-Euler equations with the substitution $x = e^t$, which reduces it to an equation with constant coefficients, then substitute it back with $t = \ln(x)$.

$$z(t) = e^t$$

$$y'(x) = y'(e^t)e^t = z'(t)$$

$$y''(x) = y''(e^t)e^{2t} = z''(t)$$

So

$$a_2x^2y'' + a_1xy' + a_0 = g(x)$$

Turns into

$$a_2z''(t) + a_1z'(t) + a_0 = g(e^t)$$

8.0.32 Ordinary and singular points

In the standard second-order homogeneous equation

$$y'' + P(x)y' + Q(x)y = 0$$

A point x_0 is an *ordinary point* if both $P(x)$ and $Q(x)$ are analytic at x_0.

A function f is analytic at a point x_0 if it can be represented by a power series in $x - x_0$ with a positive or infinite radius of convergence. For example functions like e^x or $\sin(x)$ that can be represented with a Taylor series are analytic at any $x_0 \in \mathbb{R}$.

All points that are not ordinary are *singular points* of the equation.

Theorem (Existence of Power Series Solutions). If x_0 is an ordinary point, there exists a fundamental set of solutions in the form of power series centered at x_0, each of which converges at least on $|x - x_0| < R$, where R is the distance in complex plane from x_0 to the nearest singular point.

For example in the equation

$$(x^2 - 1)y'' + 2xy' + 6y = 0$$

At ordinary point $x = 0$, $R = 1$ since $x_0 = \pm i$ are the singular points and distance between x and x_0 is 1.

8.0.33 Power series solutions

A simple equation

$$y' - y = 0$$

can be represented with power series with

$$y = \sum_{n=0}^{\infty} c_nx^n$$

$$y' = \sum_{n=1}^{\infty} nc_nx^{n-1}$$

$$\sum_{n=1}^{\infty} nc_nx^{n-1} - \sum_{n=0}^{\infty} c_nx^n = 0$$

These series can be combined into

$$\sum_{n=0}^{\infty}((n+1)c_{n+1} - c_n)x^n = 0$$

which implies that

$$(n+1)c_{n+1} - c_n = 0, n \geq 0$$

that can be rearranged into

$$c_{n+1} = \frac{c_n}{n+1}$$

and converted to non-recursive form

$$c_n = \frac{c_0}{n!}$$

The result can be plugged back in, meaning that the final solution is

$$y = c_0\sum_{n=0}^{\infty} \frac{x^n}{n!} = c_0e^x$$

8.0.34 Laplace Transform

The Laplace transform can be defined as,

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

given that $t \geq 0$ and the integral converges. The result should be a function of s. Typically, transformed functions are denoted with their corresponding capital letter, for example,

$$\mathscr{L}\{f(t)\} = F(s)$$

Common Laplace transforms include,

$$\begin{aligned}
\mathscr{L}\{1\} &= \frac{1}{s} \\
\mathscr{L}\{t^n\} &= \frac{n!}{s^{n+1}} \\
\mathscr{L}\{e^{at}\} &= \frac{1}{s-a} \\
\mathscr{L}\{\sin(kt)\} &= \frac{k}{s^2+k^2} \\
\mathscr{L}\{\cos(kt)\} &= \frac{s}{s^2+k^2}
\end{aligned}$$

Since Laplace transform is a linear function, it satisfies the following,

$$\mathscr{L}\{\alpha f(t) + g(t)\} = \alpha\mathscr{L}\{f(t)\} + \mathscr{L}\{g(t)\}$$

To apply a Laplace transform on a piecewise function, simply split the integral into as many parts as necessary.

The Laplace transform is 1 to 1 on set of functions that are continuous on $[0, \infty)$ and does not grow faster than Me^{ct}

8.0.35 Solving ODE IVPs with Laplace transform

The Laplace transform can be used on derivatives as well,

$$\mathscr{L}\{f'(t)\} = s\mathscr{L}\{f(t)\} - f(0)$$
$$\mathscr{L}\{f''(t)\} = s^2\mathscr{L}\{f(t)\} - sf(0) - f'(0)$$

Which can be used to solve linear ODE IVPs. For instance,

$$\begin{aligned}
&a_2y'' + a_1y' + a_0y = g(t) \\
&y(0) = y_0, \qquad y'(0) = y_0'
\end{aligned}$$

By taking Laplace transform of both sides, the solution of the equation would be,

$$y(t) = \mathscr{L}^{-1}\left\{\frac{G(s) + a_2y_0s + a_2y_0' + a_1y_0}{a_2s^2 + a_1s + 1}\right\}$$

8.0.36 Laplace transform translations

Theorem (First Translation Theorem). If $\mathscr{L}\{f(t)\} = F(s)$ and $a \in \mathbb{R}$, then,

$$\mathscr{L}\{e^{at}f(t)\} = F(s-a)$$

Which can also be denoted by,

$$\mathscr{L}\{e^{at}f(t)\} = \mathscr{L}\{f(t)\}|_{s\to s-a}$$

The unit step function can be defined as,

$$\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t \leq a, \\ 1, & t \geq a \end{cases}$$

Theorem (Second Translation Theorem). Given the function,

$$f(t-a)\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t \leq a, \\ f(t-a), & t \geq a \end{cases}$$

And if $a > 0$, then,

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s)$$

The theorem can also be applied in inverse form,

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a)$$

8.0.37 Derivatives of a transform

$$\mathscr{L}\{tf(t)\} = -\frac{d}{ds}\mathscr{L}\{f(t)\}$$

Which can be extended to,

$$\mathscr{L}\{t^nf(t)\} = (-1)^n\frac{d^n}{ds^n}\mathscr{L}\{f(t)\}$$

8.0.38 The convolution theorem

If f and g are piecewise continuous functions on the interval $[0, \infty)$, then the convolution of f and g, denoted $f * g$, is the function,

$$f * g = \int_0^t f(\tau)g(t-\tau)d\tau$$

Note that convolution is a commutative operation.

Theorem (Convolution Theorem). If f, g fit convolution criteria and are of exponential order, then,

$$\mathscr{L}\{f * g\} = F(s)G(s)$$

8.0.39 Transform of an integral

$$\mathscr{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$$

8.0.40 Volterra integral equation

A Volterra integral equation for $f(t)$ is an equation in the form,

$$f(t) = g(t) + \int_0^t f(\tau)h(t-\tau)d\tau$$

Where $g(t)$ and $h(t)$ are known.

8.0.41 Transform of a periodic function

Theorem (Transform of a Periodic Function). If $f(t)$ is piecewise continuous on $[0, \infty)$ of exponential order and periodic with period T, then,

$$\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)dt$$