

Sound & Light (1)

1.1 Miscellaneous

% error = (observed - theoretical) / theoretical \* 100%

1.2 Kinematics

x = (a/2)(Δt)^2 + v\_0Δt + x\_0      v = v\_0 + aΔt

v^2 = v\_0^2 + 2aΔx      Δx = (v\_0 + v) / 2 \* Δt

1.3 Simple Harmonic Motion

x = A cos(ωt + φ)    v = -ωA sin(ωt + φ)    a = -ω^2 A cos(ωt + φ)

x\_max = A      v\_max = ωA      a\_max = ω^2 A      F\_max = mω^2 A

1.3.1 Springs and Slinkies

x represents the distance from the equilibrium.

If you put a mass on top of the slinky, Δx\_eq represents the difference between the original equilibrium and the new equilibrium.

F\_s = kx = ma      F\_s\_max = kΔx\_eq = 9.8Δm

f = (1/2π)√(k/m)      T = 2π√(m/k)      ω = 2πf = √(m/k)

SPE = (1/2)kx^2      KE = (1/2)mv^2

TME = (1/2)kx^2 + (1/2)mv^2 = (1/2)kA^2 = (1/2)mv\_max^2

1.3.2 Springs in parallel and series

Quantity	In Series	In Parallel
Equivalent spring constant	1/k_eq = 1/k_1 + 1/k_2	k_eq = k_1 + k_2
Deflection (elongation)	x_eq = x_1 + x_2	x_eq = x_1 = x_2
Force	F_eq = F_1 = F_2	F_eq = F_1 + F_2
Stored energy	E_eq = E_1 + E_2	E_eq = E_1 = E_2

1.3.3 Pendulums

f = (1/2π)√(g/L)      T = 2π√(L/g)

1.4 Waves

T = 1/f      v = λf      v = Δx/Δt

1.4.1 Slinkies and strings with fixed ends

F\_T = F\_s = kx      μ = m/L      v = √(F\_T/μ)

Given mass m\_T hanging below a pulley, F\_T = m\_T g.

1.5 Standing waves

1.5.1 Open-open, closed-closed

n is the number of antinodes, or the n^th harmonic.

f\_n = f\_1 n = (nv) / 2L      f\_1 = v / 2L      λ\_n = 2L / n

1.5.2 Open-closed

f\_n = f\_1 n = (nv) / 4L      f\_1 = v / 4L      λ\_n = 2L / n

1.6 Sound

1.6.1 Speed of sound

v = 331√((T\_c + 273) / 273)      v ≈ 331 + 0.59T

1.6.2 Sound intensity

I = (Power (W)) / Area = (Power (W)) / (4πr^2)

I\_dB = 10 log\_10(I / 10^-12)      I = 10^(I\_dB / 10 - 12)

1.6.3 Doppler effect

[O] → [S]    f\_o = f\_s (v + v\_o) / v      ← [O] [S]    f\_o = f\_s (v - v\_o) / v  
[O] ← [S]    f\_o = f\_s (v - v\_s) / v      [O] [S] →    f\_o = f\_s (v + v\_s) / v  
[O] → ← [S]    f\_o = f\_s (v + v\_o) / (v - v\_s)      [O] → [S] →    f\_o = f\_s (v + v\_o) / (v + v\_s)  
← [O] ← [S]    f\_o = f\_s (v - v\_o) / (v - v\_s)      ← [O] [S] →    f\_o = f\_s (v - v\_o) / (v + v\_s)

1.6.4 Constructive and Destructive Interference (2 dimensions)

For a point on the m^th antinodal/nodal line playing the same frequency with the same phase:

PD = mλ

where PD is the path length difference.

1.6.5 Beats

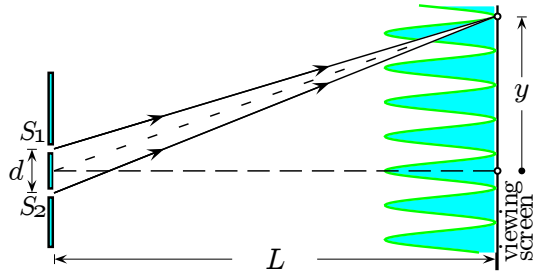
f\_B = Δf

1.7 Light

1.7.1 Speed of light

$c = 299\,792\,458 \frac{\text{m}}{\text{s}} \approx 3 * 10^8 \frac{\text{m}}{\text{s}}$

1.7.2 Two-slit experiment



$PD = \frac{dy}{L} = m\lambda$

1.7.3 Mirror

$r = 2f \quad \frac{1}{f} = \frac{1}{p} + \frac{1}{q} \quad M = \frac{h}{h_o} = \frac{-q}{p}$

In a plane mirror,  $p = -q$ .

1.7.4 Lenses

$\frac{1}{f} = (n - 1)(\frac{1}{r_1} - \frac{1}{r_2}) \quad \frac{1}{f} = \frac{1}{p} + \frac{1}{q} \quad M = \frac{h}{h_o} = \frac{-q}{p}$

Multiple lenses

$p_2 = \Delta x - q_1$

1.7.5 Mirrors and Lenses

Real  $\iff$  inverted, virtual  $\iff$  upright.

	Converging	Diverging
Mirror		
Lens		
focal length	+	-
object distance	image	image
$\infty$ to $2f$	Real smaller	Virtual smaller
$2f$ to $f$	Real larger	
$f$ to $0$	Virtual larger	
$0$ to $-f$	Virtual smaller	Virtual larger
$-f$ to $-2f$		Real larger
$-2f$ to $-\infty$		Real smaller

1.7.6 Refraction / Snell’s Law

The **normal line** is the line perpendicular to the surface which touches the intersection of the surface and the light ray.

The **incident angle** is the angle between the ray of light and the normal line.

$\theta_1$  and  $\theta_2$  are both measured from the normal line, not the surface.

Refraction occurs when the speed of light in two media are different and light hits the boundary of the two media. The frequency of the light will stay the same, but the speed, wavelength, and direction will change.

$n = \frac{c}{v} \quad n_1 \sin \theta_1 = n_2 \sin \theta_2$

1.7.7 Ray diagrams

## Mathematics (2)

### 2.1 Logarithms

$$\log_b(MN) = \log_b(M) + \log_b(N)$$

$$\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)$$

$$\log_b(M^p) = p \cdot \log_b(M)$$

$$\log_b(a) = \frac{\log_x(a)}{\log_x(b)}$$

$$\log_b(b) = 1$$

### 2.2 Notation

deg  $p(x)$  means the degree of polynomial  $p$ .

LC  $p(x)$  means the leading coefficient of polynomial  $p$ .

### 2.3 Rational functions

For a rational function  $f(x) = \frac{p(x)}{q(x)}$ , cancel out any common factors, then:

- For all rational functions:
  - VA: roots of  $q(x)$
  - Roots: roots of  $p(x)$
- When deg  $p(x) = \text{deg } q(x)$ :
  - HA:  $y = \frac{\text{LC } p(x)}{\text{LC } q(x)}$
- When deg  $p(x) < \text{deg } q(x)$ :
  - HA:  $y = 0$
- When deg  $p(x) > \text{deg } q(x)$ :
  - HA: none
  - slant asymptote:  $\frac{p(x)}{q(x)}$  excluding remainder

### 2.4 Polynomials

#### 2.4.1 Linear equations

Slope-intercept form:  $y = mx + b$

Point-slope form:  $y - y_1 = m(x - x_1)$  for point  $(x, y)$

Standard form:  $ax + by = c$

#### 2.4.2 Quadratic equations

Standard form:  $y = ax^2 + bx + c$

Vertex form:  $y = a(x - h)^2 + k$  for vertex  $(h, k)$

Sum of roots:  $-\frac{b}{a}$

Product of roots:  $\frac{c}{a}$

#### 2.4.3 Higher-degree polynomials

In a polynomial

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0$$

, with roots

$$r_1, r_2, r_3, \dots, r_n$$

then:

$$r_1 + r_2 + r_3 + \cdots + r_n = \sum_{k=1}^n nr_k = -\frac{a_{n-1}}{a_n}$$

### 2.5 Sequences and Series

#### 2.5.1 Explicit formulas

Aritmetic sequence:  $a_n = a_1 + r(n - 1)$

Geometric sequence:  $a_n = a_1 * r^{n-1}$

Harmonic sequence:  $a_n = \frac{1}{a_1 + r(n - 1)}$

#### 2.5.2 Arithmetic and Geometric Series

In the following equations, substituting  $j = 1$  with  $j = 0$ ,  $j - 1$  with  $j$ , and  $a_1$  with  $a_0$  will produce the same result.

$$\sum_{j=1}^n (a_1 + r(j - 1)) = \frac{n}{2}(2a_1 + (n - 1)d)$$

$$\sum_{j=1}^n (a_1 * r^{j-1}) = \frac{a_1(1 - r^n)}{1 - r}$$

$$\sum_{j=1}^\infty (a_1 * r^{j-1}) = \frac{a_1}{1 - r} \text{ for } r \in [-1, 1]$$

#### 2.5.3 Special Sums

$$\begin{aligned} \sum_{j=1}^n c &= nc & \sum_{j=1}^n ca_j &= c \sum_{j=1}^n a_j \\ \sum_{j=1}^n (a_j + b_j) &= \sum_{j=1}^n a_j + \sum_{j=1}^n b_j & \sum_{j=1}^n j &= \frac{n(n+1)}{2} \\ \sum_{j=1}^n j^2 &= \frac{n(n+\frac{1}{2})(n+1)}{3} & \sum_{j=1}^n j^3 &= \frac{n^2(n+1)^2}{4} \\ &= \frac{n(2n+1)(n+1)}{6} \end{aligned}$$

### 2.6 Trigonometry

°	rad	sin	cos	tan
0°	0	0	1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	$\frac{\pi}{2}$	1	0	undef

#### 2.6.1 Law of Sines and Cosines

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c} \qquad c^2 = a^2 + b^2 - 2ab \cos(C)$$

2.6.2 Triangle area

$K = \frac{1}{2}bh \qquad K = \frac{1}{2}bc\sin(A) \qquad K = \sqrt{s(s-a)(s-b)(s-c)}$

2.6.3 More identities

$(\sin A)^2 + (\cos A)^2 = 1 \qquad (\tan A)^2 + 1 = (\sec A)^2$   
 $\sin(\frac{\pi}{2} - x) = \cos(x) \qquad (\cot A)^2 + 1 = (\csc A)^2$   
 $\cos(-x) = \cos(x) \qquad \sin(-x) = \sin(x) \qquad \tan(-x) = \tan(x)$

2.6.4 Slope

Where  $\alpha$  is the angle between the line and the x-axis, and  $m$  is the slope of the line:

$m = \tan \alpha$

2.6.5 Sum and difference formulas

$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$   
 $\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B)$   
 $\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$   
 $\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$   
 $\tan(A + B) = \frac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}$   
 $\tan(A - B) = \frac{\tan(A) - \tan(B)}{1 + \tan(A)\tan(B)}$

$\sin(2A) = 2\sin(A)\cos(A)$   
 $\cos(2A) = (\cos A)^2 - (\sin A)^2 = 2(\cos A)^2 - 1 = 1 - 2(\sin A)^2$   
 $\tan(2A) = \frac{2\tan(A)}{1 - (\tan A)^2}$

2.7 Vectors

$\vec{v} + \vec{w} = \begin{bmatrix} v_x + w_x \\ v_y + w_y \\ v_z + w_z \end{bmatrix} \qquad c * \vec{v} = \begin{bmatrix} c * v_x \\ c * v_y \\ c * v_z \end{bmatrix}$   
 $\vec{v} \cdot \vec{w} = v_xw_x + v_yw_y + v_zw_z = |\vec{v}||\vec{w}|\cos(\theta)$   
 $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}|\sin(\theta) = \text{area of parallelogram}$   
 $\vec{v} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} \qquad \vec{v} \times \vec{w} \perp \vec{v} \qquad \vec{v} \times \vec{w} \perp \vec{w}$   
 $\vec{v} \perp \vec{w} \iff \vec{v} \times \vec{w} = \vec{0} \qquad \vec{v} \parallel \vec{w} \iff \vec{v} \cdot \vec{w} = 0$   
 $\hat{v} = \frac{\vec{v}}{|\vec{v}|} \qquad \text{proj}_{\vec{b}}\vec{v} = \frac{\vec{v} \cdot \vec{b}}{\vec{b} \cdot \vec{b}} * \vec{b} = (|\vec{v}|\cos(\theta))\hat{b}$

Right-hand rule

To determine the direction of  $\vec{v} \times \vec{w}$ , put the side of the right hand on  $\vec{v}$  and curl the fingers toward  $\vec{w}$ . The direction the thumb is pointing is the direction of  $\vec{v} \times \vec{w}$ .

2.8 Polar

2.8.1 Polar and Cartesian sytems

With point  $(x, y) = (r; \theta) = (r; \beta)$ , where  $\theta$  is CCW from the x-axis and  $\beta$  is a bearing, CW from the y-axis:

$x = r\cos(\theta) = r\sin(\beta) \qquad y = r\sin(\theta) = r\cos(\beta)$   
 $r = \sqrt{x^2 + y^2} \qquad \theta \equiv \arctan(\frac{y}{x}) \qquad \beta \equiv \arctan(\frac{x}{y})$

2.8.2 Converting functions

Try these substitutions in order:

$x^2 = x^2 + y^2 \qquad \tan \theta = \frac{y}{x} \qquad x = r\cos \theta \qquad y = r\sin \theta$

2.8.3 Limaçons and Petals

The function  $y = A\cos(B(\theta + C)) + D$  is equivalent to  $y = A\cos(B\theta) + D$  rotated  $C$  degrees/radians clockwise.

When  $C$  is 0 and  $B$  is 1, the x-intercepts are  $A \pm D$  and the y-intercepts are  $\pm D$ , and it forms a limaçon.

When  $C$  is 0, but  $B \neq 1$ , then this sometimes still holds. The x-intercepts may also be  $\pm A \pm D$ .

There are  $B$  petals, with the axis of the first petal on the positive x-axis.

When  $B$  is even and  $|D| < 1$ , then the number of petals is  $2B$ .

Using sin instead of cos, limaçons have their axes on the positive y-axis, while for petals, the first petal starts from the positive x-axis and curves upwards.

2.9 Complex

$\text{cis}(\theta) = e^{i\theta} = \cos(\theta) + i\sin(\theta)$

To find the  $n^{\text{th}}$  root of  $x_r \text{cis}(x_\theta)$ , solve the equation  $z_r^n \text{cis}(nz_\theta) = x_r \text{cis}(x_\theta + 360^\circ k)$  for  $k \in \mathbb{R}$ .

2.10 Function domain

Function	Domain $x$	Range $y$
$\log(x)$	$(0, \infty)$	$\mathbb{R}$
$\sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$\arcsin(x)$	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$\arccos(x)$	$[-1, 1]$	$[0, \pi]$
$\arctan(x)$	$\mathbb{R}$	$(-\frac{\pi}{2}, \frac{\pi}{2})$

Calculus Theorems (3)

1 Completeness

**Theorem (Completeness of the Real Numbers).** Every nonempty subset  $S$  of  $\mathbb{R}$  which is bounded above has a least upper bound  $\sup S$ .

*Definition of Supremum* ( $\sup S$ ). A number such that

- (1)  $s \leq \sup S$  for every  $s \in S$  (which just says that  $\sup S$  is an upper bound for  $S$ )
- (2) If  $u$  is any upper bound for  $S$ , then  $\sup S \leq u$  (which says that  $\sup S$  is the least upper bound for  $S$ ).

*Definition of Infimum* ( $\inf S$ ). A number such that

- (1)  $\inf S \leq s$  for every  $s \in S$  (i.e.  $\inf S$  is an lower bound for  $S$ )
- (2) If  $l$  is any upper bound for  $S$ , then  $l \leq \inf S$  (i.e.  $\inf S$  is the greatest lower bound for  $S$ ).

**Theorem.** Every nonempty subset  $S$  of  $\mathbb{R}$  which is bounded below has a greatest lower bound.

**Theorem.** If  $\min S$  exists, then  $\min S = \inf S$ .

**Theorem.** If  $A \subset \mathbb{R}$  and  $c \geq 0$ , and  $cA := \{ca : a \in A\}$ ,  $\sup cA = c \sup A$ .

**Theorem (Rationals between Reals).** For every two real numbers  $a$  and  $b$  with  $a < b$ , there exists a rational number  $r$  satisfying  $a < r < b$ .

**Nested Intervals Theorem.**

If  $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$  s.t.  $a_n \leq a_{n+1}$  and  $b_{n+1} \leq b_n$  for  $n \in \mathbb{N}$ , so that  $I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \dots$ , then  $\bigcap_{n=1}^\infty I_n \neq \emptyset$ .

If  $\inf\{b_n - a_n\} = 0$ , then  $\bigcap_{n=1}^\infty I_n = \{x\}$ , where

$x = \sup\{a_n\} = \inf\{b_n\}$ .

**Capture Theorem.** If  $A$  is a nonempty subset of  $\mathbb{R}$ , then:

- (i) If  $A$  is bounded above, then any open interval containing  $\sup A$  contains an element of  $A$ .
- (ii) Similarly, if  $A$  is bounded below, then any open interval containing  $\inf A$  contains an element of  $A$ .

**Theorem (Binary Search (Bisection Method)).** If we binary-search for  $x$  over  $I_1 = [a_1, b_1]$  for  $a_1, b_1 \in \mathbb{Q}$ , we define  $I_n$  s.t. either  $I_n := [a_{n-1}, \frac{a_{n-1}+b_{n-1}}{2}]$  or  $I_n := [\frac{a_{n-1}+b_{n-1}}{2}, a_{n-1}]$ , and we define  $a_n := \inf I_n$  and  $b_n := \sup I_n$ . We define  $A$  to be the set of all  $a_n$ , and  $B$  to be the set of all  $b_n$ .

Then, the size of  $I_n = \frac{b_1-a_1}{2^n} = b_n - a_n$ , and  $\bigcap_{n=1}^\infty I_n \{x\}$ , where

$x = \sup\{a_n\} = \inf\{b_n\}$ .

2 Limits

*Definition of Limit.* If  $\lim_{x \rightarrow a} f(x) = L$ , then for any  $\varepsilon > 0$ , there exists  $\delta > 0$  s.t. for any  $x \in (a - \delta, a) \cup (a, a + \delta)$ ,  $f(x) \in (L - \varepsilon, L + \varepsilon)$ .

Alternatively,

*Definition of Limit.* If  $\lim_{x \rightarrow a} f(x) = L$ , then for any  $\varepsilon > 0$ , there exists  $\delta > 0$  s.t.  $|f(x) - L| < \varepsilon$  whenever  $0 < |x - a| < \delta$ .

*Definition of Right-sided limit.* If  $\lim_{x \rightarrow a^+} f(x) = L$ , then for any  $\varepsilon > 0$ , there exists  $\delta > 0$  s.t.  $|f(x) - L| < \varepsilon$  whenever  $0 < x - a < \delta$ .

*Definition of Left-sided limit.* If  $\lim_{x \rightarrow a^-} f(x) = L$ , then for any  $\varepsilon > 0$ , there exists  $\delta > 0$  s.t.  $|f(x) - L| < \varepsilon$  whenever  $0 < a - x < \delta$ .

**Theorem (Limit Laws).** Let  $c \in \mathbb{R}$  be a constant and suppose the limits  $\lim_{x \rightarrow a} f(x)$  and  $\lim_{x \rightarrow a} g(x)$  exist. Then

- $\lim_{x \rightarrow a} (f(x) \pm g(x)) = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$
- $\lim_{x \rightarrow a} (cf(x)) = c \lim_{x \rightarrow a} f(x)$
- $\lim_{x \rightarrow a} (f(x)g(x)) = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x)$
- $\lim_{x \rightarrow a} f(x)g(x) = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x)$  , provided that  $\lim_{x \rightarrow a} g(x) \neq 0$
- $\lim_{x \rightarrow a} x^n = (\lim_{x \rightarrow a} x)^n$
- $\lim_{x \rightarrow a} \frac{a(x)b(x)}{c(x)b(x)} = \lim_{x \rightarrow a} \frac{a(x)}{c(x)}$

These laws also apply to one-sided limits.

**Theorem (L'Hopital's Rule).** If  $f$  and  $g$  are differentiable and  $g'(x) \neq 0$  on an open interval  $I$  that surrounds  $a$ , and  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} \in \{\frac{0}{0}, \pm \frac{\infty}{\infty}\}$ , then  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ .

**Theorem (Composition of Limits).** If  $f$  is continuous at  $L$  and  $\lim_{x \rightarrow a} g(x) = L$ , then  $\lim_{x \rightarrow a} f(g(x)) = f(\lim_{x \rightarrow a} g(x)) = f(L)$

**Theorem (Operations on infinity).** For  $x \in \mathbb{R}$ ,

$$\begin{aligned} \infty + x &= \infty & -\infty + x &= -\infty & \frac{x}{\pm \infty} &= 0 \\ x * \infty &= \begin{cases} \infty & \text{if } x > 0 \\ -\infty & \text{if } x < 0 \end{cases} \end{aligned}$$

$$x * -\infty = \begin{cases} -\infty & \text{if } x > 0 \\ \infty & \text{if } x < 0. \end{cases}$$

*Definition of Indeterminate forms.* The following forms are indeterminate and you cannot evaluate them.

$$\frac{0}{0}, \frac{\pm \infty}{\pm \infty}, 0 * \pm \infty, \infty - \infty$$

**Squeeze Theorem.** Let  $f$  ,  $g$ , and  $h$  be defined for all  $x \neq a$  over an open interval containing  $a$ . If

$$f(x) \leq g(x) \leq h(x)$$

for all  $x \neq a$  in an open interval containing  $a$  and

$$\lim_{x \rightarrow a} f(x) = L = \lim_{x \rightarrow a} h(x)$$

where  $L \in \mathbb{R}$ , then  $\lim_{x \rightarrow a} g(x) = L$ .

3 Continuity

*Definition of Continuity at a point.* Function  $f$  is continuous at point  $a$  if  $\lim_{x \rightarrow a} f(x) = f(a)$ .

*Definition.*  $f$  has a **removable discontinuity** if  $\lim_{x \rightarrow a} f(x) = L \in \mathbb{R}$  (in this case either  $f(a)$  is undefined, or  $f(a)$  is defined by  $L \neq f(a)$ ).

*Definition.*  $f$  has a **jump discontinuity** if  $\lim_{x \rightarrow a^-} f(x) = L_1 \in \mathbb{R}$  and  $\lim_{x \rightarrow a^+} f(x) = L_2 \in \mathbb{R}$  but  $L_1 \neq L_2$ .

*Definition.*  $f$  has an **infinite discontinuity** at  $a$  if  $\lim_{x \rightarrow a^-} f(x) = \pm \infty$  or  $\lim_{x \rightarrow a^+} f(x) = \pm \infty$

**Intermediate Value Theorem.** If  $f$  is continuous on  $[a, b]$ , then for any real number  $L$  between  $f(a)$  and  $f(b)$  there exists at least one  $c \in [a, b]$  such that  $f(c) = L$ . In other words, if  $f$  is continuous on  $[a, b]$ , then the graph must cross the horizontal line  $y = L$  at least once between the vertical lines  $x = a$  and  $x = b$ .

**Aura Theorem.** If  $f(x)$  is continuous and  $f(a)$  is positive, then there exists an open interval containing  $a$  such that for all  $x$  in the interval,  $f(x)$  is positive.

If  $f(x)$  is continuous and  $f(a)$  is negative, then there exists an open interval containing  $a$  such that for all  $x$  in the interval,  $f(x)$  is negative.

**Bolzano's Theorem.** Let  $f$  be a continuous function defined on  $[a, b]$ . If 0 is between  $f(a)$  and  $f(b)$ , then there exists  $x \in [a, b]$  such that  $f(x) = 0$ .

4 Derivatives

The derivative is the instantaneous rate of change, and the slope of the tangent line to the point.

Definition of Derivative ( $f'(a)$ ).

$$\frac{d}{da}f(a) = f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

Theorem (Tangent line to a point). The equation of the tangent line to the point  $(a, f(a))$  is

$$y = f'(a)(x - a) + f(a)$$

Derivative Rules

Theorem (Difference Rule).

$$\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

Theorem (Sum Rule).

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

Theorem (Constant Multiple Rule).

$$\frac{d}{dx}(cf(x)) = c\frac{d}{dx}f(x)$$

Theorem (Product Rule).

$$\begin{aligned} \frac{d}{dx}(f(x)g(x)) &= f'(x)g(x) + f(x)g'(x) \\ \frac{d}{dx}(f(x)g(x)h(x)) &= f'(x)g(x)h(x) + f(x)g'(x)h(x) \\ &\quad + f(x)g(x)h'(x) \end{aligned}$$

and so on.

Theorem (Quotient Rule).

$$\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

Theorem (Power Rule).

$$\frac{d}{dx}x^n = nx^{n-1}$$

for  $n \in \mathbb{R}$

Theorem (Chain Rule).

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x) \qquad \frac{dy}{dx} = \frac{dy}{db} \frac{db}{dx}$$

Theorem (Derivative of inverse functions). Let  $x \in \mathbb{R}$  and  $f$  be a differentiable, one-to-one function at  $x$ . Then if  $f'(x) \neq 0$ , then

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}$$

Theorem (Derivatives of exponentials and logs).

$$\begin{aligned} \frac{d}{dx}e^x &= e^x & \frac{d}{dx}\ln x &= \frac{1}{x} \\ \frac{d}{dx}a^x &= a^x \ln(a) & \frac{d}{dx}\log_a x &= \frac{1}{x \ln(a)} \end{aligned}$$

Theorem (Derivatives of trig functions).

Warning:  $x$  must be an angle in radians!

$$\begin{aligned} \sin'(x) &= \cos(x) & \cos'(x) &= -\sin(x) \\ \sec'(x) &= \sec(x)\tan(x) & \csc'(x) &= -\csc(x)\cot(x) \\ \tan'(x) &= \sec(x)^2 & \cot'(x) &= -\csc(x)^2 \\ \arcsin'(x) &= \frac{1}{\sqrt{1-x^2}} & \arccos'(x) &= -\frac{1}{\sqrt{1-x^2}} \\ \operatorname{arcsec}'(x) &= \frac{1}{|x|\sqrt{x^2-1}} & \operatorname{arccsc}'(x) &= -\frac{1}{|x|\sqrt{x^2-1}} \\ \arctan'(x) &= \frac{1}{1+x^2} & \operatorname{arccot}'(x) &= -\frac{1}{1+x^2} \end{aligned}$$

5 Derivative Applications

5.7 Mean Value Theorem

Theorem (Mean Value Theorem). If the function  $f$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ , then there exists  $c \in (a, b)$  s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{\Delta f(x)}{\Delta x} \text{ on } [a, be]$$

Theorem (Some colloraries to the MVT). If  $f(x)$  is differentiable on  $I$ , then:

- $f'(x) > 0$  for  $x \in I \iff f(x)$  is strictly increasing for  $x \in I$ .
- $f'(x) \geq 0$  for  $x \in I \iff f(x)$  is increasing or constant for  $x \in I$ .
- $f'(x) = 0$  for  $x \in I \iff f(x)$  is constant for  $x \in I$ .
- $f'(x) \leq 0$  for  $x \in I \iff f(x)$  is decreasing or constant for  $x \in I$ .
- $f'(x) < 0$  for  $x \in I \iff f(x)$  is strictly decreasing for  $x \in I$ .

5.3, 5.10, 5.11, 5.16 Extrema

Definition of Critical point of  $f$ . A number  $c$  in the domain of  $f$  where  $f'(c) = 0$  or  $f'(c)$  does not exist.

Definition of Stationary point of  $f$ . A number  $c$  in the domain of  $f$  where  $f'(c) = 0$

Fermat’s Theorem. The local maxima and minima of  $f$  are critical points of  $f$ .

Exteme Value Theorem. If  $f$  is continuous on  $[a, b]$ , then it has an absolute max and an absolute min.

Theorem (Method to find absolute minima and maxima). Store the critical points of  $f$  in the array  $C$ . Then, the absolute maximum is  $\max\{f(c) : c \in C\}$  and the absolute minimum is  $\min\{f(c) : c \in C\}$ .

Theorem (First Derivative Test). If  $f$  is continuous over  $I$ , and  $c \in I$  is a critical point of  $f$ , and  $f$  is differentiable over  $I \setminus c$ , then:

- If  $f'(x)$  is decreasing at  $c$ , then  $f(c)$  is a local max.
- If  $f'(x)$  is increasing at  $c$ , then  $f(c)$  is a local min.
- If  $f'(x)$  has the same sign before and after  $c$ , then  $f(c)$  is neither a local max nor a local min.

Definition of Concavity.  $f$  is concave up on  $I$  if the tangent line to  $f$  at each point in  $I$  is lower than the graph of  $f$ .

$f$  is concave down on  $I$  if the tangent line to  $f$  at each point in  $I$  is higher than the graph of  $f$ .

Theorem (Test for Concavity). If  $f''(x) > 0$  for all  $x \in I$ , then  $f$  is concave up on  $I$ .

If  $f''(x) < 0$  for all  $x \in I$ , then  $f$  is concave down on  $I$ .

Theorem (Second Derivative Test). If  $f''$  is continuous on an interval containing  $c$ , where  $c$  is the  $x$ -value of a stationary point of  $f$ . Then,

- If  $f''(c) > 0$ , then  $f(c)$  is a local max.
- If  $f''(c) < 0$ , then  $f(c)$  is a local min.

Trimm’s Single Extremum Theorem. If  $f$  is continuous on an interval  $I$ , and  $f$  has a single local extremum (max or min), then that extremum is a global max or min.

5 Integrals  
Antiderivative

*Definition of **Antiderivative / Indefinite Integral**.* The antiderivative  $F$  of a function  $f$  is the function such that  $F'(x) = f(x)$ .

$$F(x) = \int f(x)dx$$

**Theorem (Antiderivative plus a constant).** If  $F$  is the antiderivative of a function  $f$ , then  $G(x) = F(x) + c$  where  $c \in \mathbb{R}$  is also an antiderivative.

*Definition of **Integral**.*

$$(f_1[a, b]) \mapsto \int_a^b f(x)dx \in \mathbb{R}$$

such that the Properties of the Integral are true.

The definite integral takes in a function and a range  $[a, b]$ , and returns a number. The indefinite integral takes in a function and returns an infinitely large set of functions (the antiderivatives).

If  $a > b$ , then  $\int_a^b f := -\int_b^a f$ .

Let  $\mathcal{R}([a, b])$  be the set of integrable functions,  $\mathcal{C}([a, b])$  be the set of continuous functions, and  $\mathcal{B}([a, b])$  be the set of bounded functions on  $[a, b]$ . Then

$$\mathcal{C}([a, b]) \subset \mathcal{R}([a, b]) \subset \mathcal{B}([a, b])$$

**Theorem (Properties of the Integral).** The integral is defined such that the following are true:

- (I0) Every continuous function is integrable.
- (I1) If  $f(x) = c$ , then  $\int_a^b f(x)dx = c(b - a)$
- (I2) If  $f_1(x) \leq f_2(x)$ , then  $\int_a^b f_1(x)dx \leq \int_a^b f_2(x)dx$ .
- (I3) For any  $a, b, c$ ,  $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ .

**Theorem (Fundamental Theorem of Calculus).** Let  $f \in \mathcal{R}([a, b])$  be some integrable function, where  $a, b \in \mathbb{R}$ . Let  $\mathcal{F}(x) = \int_a^x f$  for  $x \in [a, b]$ . Then:

- (a)  $\mathcal{F}$  is continuous for every  $c \in [a, b]$ .
- (b) If  $f$  is continuous at  $c \in [a, b]$ , then  $\mathcal{F}$  is diffentiable at  $c$ , and  $\mathcal{F}'(c) = f(c)$ .
- (c) If  $f$  is continuous on  $[a, b]$ , and  $F$  is an antiderivative of  $f$ , then  $\int_a^b f = F(b) - F(a)$ , or

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

**Theorem (Substitution Rule).** If  $g$  is a function that has a continuous derivative on an interval, another function  $f$  is continuous on the range of  $g$ , and  $F$  is an antiderivative of  $f$  on the range of  $g$ , then

$$\int f(g(x))g'(x)dx = F(g(x)) + C$$

For  $a, b \in \mathbb{R}$ ,

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du = F(g(b)) - F(g(a))$$

Let  $u := g(x)$ . Then, if  $du = E(g'(x)dx)$ , where  $E \in \mathbb{R}$ , then

$$\int Ef(u)du = E \int f(u)du = F(u) + C$$

**Theorem (Some Antiderivative Rules).**

$$\begin{aligned} \int e^x dx &= e^x + C & \int x^a dx &= \frac{x^{a+1}}{a+1} + C \text{ for } a \neq -1 \\ \int a^x dx &= \frac{a^x}{\ln(a)} + C & \int x^{-1} dx &= \ln|x| + C \end{aligned}$$

$$\int f(x) + g(x)dx = \int f(x)dx + \int g(x)dx$$

**Theorem (Integration By Parts).** If  $f$  and  $g$  are integrable functions, then

$$\int f(x)g'(x)dx = f(x)g(x) + \int f(x)g(x)dx$$

Equivalently, if  $u$  and  $v$  are integrable functions of  $x$ , then

$$\int u dv = uv - \int v du$$

Additionally, if  $f'$  and  $g'$  are continuous, then

$$\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) + \int_a^b g(x)f'(x)dx$$

**Theorem (Integrals of Trig Functions).**

$$\int \sin(x)dx = -\cos(x) + C \quad \int \cos(x)dx = \sin(x) + C$$

$$\int \tan(x)dx = -\ln|\cos(x)| + C = \ln|\sec(x)| + C$$

$$\int \cot(x)dx = \ln|\sin(x)| + C = -\ln|\csc(x)| + C$$

$$\int \sec(x)dx = \ln|\sec(x) + \tan(x)| + C$$

$$\int \csc(x)dx = -\ln|\cot(x) + \csc(x)| + C$$

**Procedure (Integral of Powers of Sine and Cosine).** Consider  $\int \sin^m(x) \cos^n(x)dx$ , where m,n are nonnegative integers.

If m is odd, then  $m = 2k + 1$  for some integer k. Then

$$\int \sin^m(x) \cos^m(x)dx = -\int (1 - u^2)^k u^n du$$

where  $u = \cos x$  and  $du = -\sin x dx$ .

If n is odd, then  $m = 2k + 1$  for some integer k. Then

$$\int \sin^m(x) \cos^m(x)dx = \int u^m (1 - u^2)^k du$$

where  $u = \sin x$  and  $du = \cos x dx$ .

If both m and n are even, use the power-reducing identities

$$\cos^2 x = \frac{1 + \cos(2x)}{2} \quad \text{and} \quad \sin^2 x = \frac{1 - \cos(2x)}{2}$$

to reduce the degree of the integrand. Expand the result and apply this procedure again.

**Theorem (Multiplications of Sinusoidal Functions).**

$$\begin{aligned} \int \sin(mx) \cos(nx)dx &= -\frac{\cos((n+m)x)}{2(n+m)} - \frac{\cos((m-n)x)}{2(m-n)} \\ \int \sin(mx) \sin(nx)dx &= \frac{\sin((m-n)x)}{2(m-n)} - \frac{\sin((n+m)x)}{2(n+m)} \\ \int \cos(mx) \cos(nx)dx &= \frac{\sin((n+m)x)}{2(n+m)} + \frac{\sin((m-n)x)}{2(m-n)} \end{aligned}$$

**Theorem (Tangent/Secant Reduction Formulas).**

$$\begin{aligned} \int \sec^n x \, dx &= \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx \\ \int \csc^n x \, dx &= -\frac{1}{n-1} \csc^{n-2} x \cot x + \frac{n-2}{n-1} \int \csc^{n-2} x \, dx \\ \int \tan^n x \, dx &= \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x \, dx \\ \int \cot^n x \, dx &= -\frac{1}{n-1} \cot^{n-1} x - \int \cot^{n-2} x \, dx \end{aligned}$$

**Procedure (Trig Substitution).**

Orig expression	Substitution	Pythagorean identity
$\sqrt{a^2 - x^2}$	$x := a \sin(\theta)$	$1 - \sin^2(\theta) = \cos^2(\theta)$
$\sqrt{a^2 + x^2}$	$x := a \tan(\theta)$	$1 + \tan^2(\theta) = \sec^2(\theta)$
$\sqrt{x^2 - a^2}$	$x := a \sec(\theta)$	$\sec^2(\theta) - 1 = \tan^2(\theta)$

Partial Fraction Decomposition

**Theorem.** Any polynomial  $Q(x)$  with real coefficients can be factored over the reals as a product of two types of factors:

- linear factors (of the form  $ax + b$ )
- irreducible quadratic factors (of the form  $ax^2 + bxx + c$ , where  $b^2 - 4ac < 0$ )

*Definition of Proper rational function.* A rational function  $\frac{P(x)}{Q(x)}$  where  $\deg P < \deg Q$ .

**Theorem.** Any rational function can be converted into a proper rational function plus a polynomial by continually long-dividing by the denominator.

**Theorem.** Let  $R(x) = \frac{P(x)}{Q(x)}$  be a proper rational function, where the denominator  $Q(x)$  has been factored into linear and irreducible quadratic factors.  $R(x)$  can be written as a sum of partial fractions, where each factor in the denominator gives rise to terms in the partial fraction decomposition:

- For each factor of the form  $(ax + b)^k$  in the denominator, add  $\sum_{i=1}^k \frac{A_i}{(ax + b)^i}$  to the partial fraction decomposition.
- For each factor of the form  $(ax^2 + bx + c)^k$  in the denominator, add  $\sum_{i=1}^k \frac{A_ix + B_i}{(ax^2 + bx + c)^i}$  to the partial fraction decomposition.

Riemann Sums

*Definition of Elementary function.* A function which is a polynomial, rational function, power function ( $x^a$ ), exponential function ( $a^x$ ), logarithmic functions, trigonometric and inverse trigonometric functions, or an addition, subtraction, multiplication, division, and composition of the above.

*Definition of Riemann sum.* Let  $f : [a, b] \rightarrow \mathbb{R}$  be any bounded function and let  $P$  be a partition of  $[a, b]$ .

- A choice of a point  $x_i^* \in [x_{i-1}, x_i]$  for all  $i \in [1, n]$  is called a *tagging* of  $P$ , which we denote by  $\tau = x_1^*, \dots, x_n^*$ .
- A pair  $(P, \tau)$  is called a *tagged partition*.
- Given a bounded function  $f : [a, b] \rightarrow \mathbb{R}$  and a tagged partition  $(P, \tau)$  of  $[a, b]$ , the sum

$$R(f, P, \tau) = \sum_{i=1}^n f(x_i^*)(x_i - x_{i-1})$$

is called the *Riemann sum* of  $f$  for  $(P, \tau)$ .

**Theorem.** If  $f$  is integrable on  $[a, b]$ , then for every  $\varepsilon > 0$  there exists  $\delta > 0$  s.t. if  $P$  is a partition of  $[a, b]$  with  $|P| := \max x_i - x_{i-1} < \delta$  and  $\tau = \{x_i^*\}$ , then

$$|R(f, P, \tau) - \int_a^b f(x)dx| < \varepsilon$$

Approximate Integration

*Definition of Left-endpoint approximation.* Take  $x_i^* = x_{i-1} = a + \frac{(i-1)(b-a)}{n}$ . Then the left-endpoint approximation of the function  $f(x)$  is

$$\int_a^b f(x)dx \approx \sum_{i=1}^n f(a + \frac{(i-1)(b-a)}{n}) \frac{b-a}{n}$$

The error bound is

$$E_n^L \leq \max\{|f'(x)|\}_{x \in [a, b]} \frac{(b-a)^2}{2n}$$

*Definition of Right-endpoint approximation.* Take  $x_i^* = x_{i-1} = a + \frac{(i)(b-a)}{n}$ . Then the right-endpoint approximation of the function  $f(x)$  is

$$\int_a^b f(x)dx \approx \sum_{i=1}^n f(a + \frac{(i)(b-a)}{n}) \frac{b-a}{n}$$

The error bound is

$$E_n^R \leq \max\{|f'(x)|\}_{x \in [a, b]} \frac{(b-a)^2}{2n}$$

*Definition of Midpoint approximation.* Take  $x_i^* = x_{i-1} = a + \frac{(i-\frac{1}{2})(b-a)}{n}$ . Then the midpoint approximation of the function  $f(x)$  is

$$\int_a^b f(x)dx \approx \sum_{i=1}^n f(a + \frac{(i-\frac{1}{2})(b-a)}{n}) \frac{b-a}{n}$$

The error bound is

$$E_n^M \leq \max\{|f''(x)|\}_{x \in [a, b]} \frac{(b-a)^3}{24n^2}$$

*Definition of Trapezoidal approximation.* Then the trapezoidal approximation of the function  $f(x)$  is

$$\int_a^b f(x)dx \approx \frac{b-a}{2n} (f(a) + 2 \sum_{i=1}^{n-1} f(a + \frac{(i)(b-a)}{n}) + f(b))$$

The error bound is

$$E_n^T \leq \max\{|f''(x)|\}_{x \in [a, b]} \frac{(b-a)^3}{12n^2}$$

*Definition of Simpson's approximation.* For even  $n$  (greater values of  $n$  give more precise more accuracy):

$$\frac{b-a}{3n} \left( f(a) + \sum_{i=1}^{n-1} (3 - (-1)^i) f\left(a + \frac{(i)(b-a)}{n}\right) + f(b) \right)$$

The error bound is

$$E_n^S \leq \max\{|f''''(x)|\}_{x \in [a, b]} \frac{(b-a)^5}{180n^4}$$

8 Integral Applications

Volumes

Generally, if  $A(x)$  is the cross-section of a solid that intersects the  $x$ -axis at  $x$ , then the volume of the solid is

$$V = \int_a^b A(x)dx$$

**Theorem (Disk Method).** Let  $f(x)$  be a continuous, nonnegative function defined on  $[a, b]$ , and  $R$  be the region bounded above by the graph of  $f(x)$  and below by the  $x$ -axis. Then, the volume of the solid of revolution formed by revolving  $R$  around the  $x$ -axis is given by

$$V = \int_a^b \pi(f(x))^2 dx$$

**Theorem (Washer Method).** Let  $f(x)$  be a continuous, nonnegative function defined on  $[a, b]$ , and  $R$  be the region bounded above by the graph of  $f(x)$  and below by the graph of  $g(x)$ . Then, the volume of the solid of revolution formed by revolving  $R$  around the  $x$ -axis is given by

$$V = \int_a^b \pi((f(x))^2 - (g(x))^2) dx$$

**Theorem (Cylindrical Shells Method).** Let  $f(x)$  be a continuous, nonnegative function defined on  $[a, b]$ , and  $R$  be the region bounded above by the graph of  $f(x)$  and below by the  $x$ -axis. Then, the volume of the solid of revolution formed by revolving  $R$  around the  $x$ -axis is given by

$$V = \int_a^b 2\pi x(f(x))dx$$



## Other things

**Theorem (Arc length of a curve).** The arc length of the curve  $f(x)$  on  $[a, b]$ , when  $f'(x)$  exists and is continuous on  $[a, b]$ , is

$$\int_a^b \sqrt{1 + (f'(x))^2} dx$$

**Theorem (Surface area of a solid of revolution).** Let  $f(x)$  be a continuous, nonnegative function defined on  $[a, b]$ , and  $R$  be the region bounded above by the graph of  $f(x)$  and below by the  $x$ -axis. Then, the surface area of the solid of revolution formed by revolving  $R$  around the  $x$ -axis is given by

$$S = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} dx$$

**Theorem (Mass of a thin rod).** Let there be a rod whose left end is located at  $x = a$  and whose right end is located at  $x = b$ , and which is on the  $x$ -axis. Let  $\rho(x)$  be the linear density at the point  $x = a$ . Then the mass of the rod is

$$M = \int_a^b \rho(x) dx$$

**Theorem (Mass of a thin disk).** Let there be a disk of radius  $R$  whose center is at the origin of the  $xy$ -plane. Let the mass be distributed in a rotationally-symmetric way. Let  $\rho(r)$  be the radial density at the radius  $r$ . Then the mass of the disk is

$$M = \int_0^R 2\pi r \rho(r) dr$$

**Theorem (Work).** If an object moves along the  $x$ -axis from  $a$  to  $b$ , and  $F(x)$  is the force applied to the object when the object is at the point  $x$  on the  $x$ -axis, then the work is

$$\int_a^b F(x) dx$$