Calculus Theorems (1)

1 Completeness

1.3 Completeness

Theorem (Completeness of the Real Numbers). Every nonempty subset S of $\mathbb{R}$ which is bounded above has a least upper bound $\sup S$.

Definition of **Supremum** ($\sup S$). A number such that

(1) $s \leq \sup S$ for every $s \in S$ (which just says that $\sup S$ is an upper bound for S)

(2) If u is any upper bound for S, then $\sup S \leq u$ (which says that $\sup S$ is the least upper bound for S).

Definition of **Infimum** ($\inf S$). A number such that

(1) $\inf S \leq s$ for every $s \in S$ (i.e. $\inf S$ is an lower bound for S)

(2) If l is any upper bound for S, then $l \leq \inf S$ (i.e. $\inf S$ is the greatest lower bound for S).

Theorem. Every nonempty subset S of $\mathbb{R}$ which is bounded below has a greatest lower bound.

Theorem. If $\min S$ exists, then $\min S = \inf S$.

Theorem. If $A \subset R$ and $c \geq 0$, and $cA := ca : a \in A$, $\sup cA = c \sup A$.

1.4 Consequences of Completeness

Theorem (Rationals between Reals). For every two real numbers a and b with $a < b$, there exists a rational number r satisfying $a < r < b$.

2 Limits

2.4 ε-δ definition of a Limit

Definition of **Limit**. If $\lim\limits_{x \to a} f(x) = L$, then for any $\varepsilon > 0$, there exists $\delta > 0$ s.t. for any $x \in (a - \delta, a) \cup (a, a + \delta)$, $f(x) \in (L - \varepsilon, L + \varepsilon)$.

Alternatively,

Definition of **Limit**. If $\lim\limits_{x \to a} f(x) = L$, then for any $\varepsilon > 0$, there exists $\delta > 0$ s.t. for any $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.

2.6 Limit Laws

Theorem (Limit Laws). Let $c \in R$ be a constant and suppose the limits $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. Then

(i) $\lim\limits_{x \to a} (f(x) \pm g(x)) = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x)$

(ii) $\lim\limits_{x \to a} (cf(x)) = c \lim\limits_{x \to a} f(x)$

(iii) $\lim\limits_{x \to a} (f(x)g(x)) = \lim\limits_{x \to a} f(x) \lim\limits_{x \to a} g(x)$

(iv) $\lim\limits_{x \to a} f(x)g(x) = \lim\limits_{x \to a} f(x) \lim\limits_{x \to a} g(x)$, provided that $\lim\limits_{x \to a} g(x) \neq 0$

(v) See (i).

(vi) $\lim\limits_{x \to a} x^n = (\lim\limits_{x \to a} x)^n$

(vii) $\lim\limits_{x \to a} \sqrt{f(x)} = \sqrt{\lim\limits_{x \to a} f(x)}$

(viii) $\lim\limits_{x \to a} \dfrac{a(x)b(x)}{c(x)b(x)} = \lim\limits_{x \to a} \dfrac{a(x)}{c(x)}$

Theorem (Operations on infinity). For $x \in \mathbb{R}$,

$$\infty + x = \infty$$

$$-\infty + x = -\infty$$

$$x * \infty = \begin{cases} \infty & \text{if } x > 0 \\ -\infty & \text{if } x < 0 \end{cases}$$

$$x * -\infty = \begin{cases} -\infty & \text{if } x > 0 \\ \infty & \text{if } x < 0. \end{cases}$$

$$\frac{x}{\pm\infty} = 0$$

Definition of **Indeterminate forms**. The following forms are indeterminate and you cannot evaluate them.

$$\frac{0}{0}, \frac{\pm\infty}{\pm\infty}, 0 * \pm\infty, \infty - \infty$$

2.12 Squeeze Theorem

Squeeze Theorem. Let f , g, and h be defined for all $x \neq a$ over an open interval containing a. If

$$f(x) \leq g(x) \leq h(x)$$

for all $x \neq a$ in an open interval containing a and

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$$

where $L \in \mathbb{R}$, then $\lim_{x \to a} g(x) = L$.

3 Continuity

Definition of **Continuity at a point**. Function f is continuous at point a if $\lim\limits_{x \to a} f(x) = f(a)$.

Definition. f has a **removable discontinuity** if $\lim\limits_{x \to a} f(x) = L \in \mathbb{R}$ (in this case either $f(a)$ is undefined, or $f(a)$ is defined by $L \neq f(a)$).

Definition. f has a **jump discontinuity** if $\lim\limits_{x \to a^-} f(x) = L_1 \in \mathbb{R}$ and $\lim\limits_{x \to a^+} f(x) = L_2 \in \mathbb{R}$ but $L1 \neq L2$.

Definition. f has an **infinite discontinuity** at a if $\lim\limits_{x \to a^-} f(x) = \pm\infty$ or $\lim\limits_{x \to a^+} f(x) = \pm\infty$

Intermediate Value Theorem. If f is continuous on $[a, b]$, then for any real number L between $f(a)$ and $f(b)$ there exists at least one $c \in [a, b]$ such that $f(c) = L$. In other words, if f is continuous on $[a, b]$, then the graph must cross the horizontal line $y = L$ at least once between the vertical lines $x = a$ and $x = b$.

Aura Theorem. If $f(x)$ is continuous and $f(a)$ is positive, then there exists an open interval containing a such that for all x in the interval, $f(x)$ is positive.

If $f(x)$ is continuous and $f(a)$ is negative, then there exists an open interval containing a such that for all x in the interval, $f(x)$ is negative.

Bolzano's Theorem. Let f be a continuous function defined on $[a, b]$. If 0 is between $f(a)$ and $f(b)$, then there exists $x \in [a, b]$ such that $f(x) = 0$.

4 Derivatives

The derivative is the instantaneous rate of change, and the slope of the tangent line to the point.

Definition of **Derivative** ($f'(a)$).

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

Theorem (Tangent line to a point). The equation of the tangent line to the point $(a, f(a))$ is

$$y = f'(a)(x - a) + f(a)$$