# Mathematics (1)

## 1.1 Logarithms

$$\log_b(MN) = \log_b(M) + \log_b(N)$$
$$\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)$$
$$\log_b(M^p) = p \cdot \log_b(M)$$
$$\log_b(a) = \frac{\log_x(a)}{\log_x(b)}$$
$$\log_b(b) = 1$$

## 1.2 Notation

deg $p(x)$ means the degree of polynomial $p$.

LC $p(x)$ means the leading coefficient of polynomial $p$.

## 1.3 Rational functions

For a rational function $f(x) = \frac{p(x)}{q(x)}$, cancel out any common factors, then:

- When deg $p(x) =$ deg $q(x)$:
  - HA: $y = \frac{\text{LC } p(x)}{\text{LC } q(x)}$
  - VA: roots of $q(x)$
- When deg $p(x) <$ deg $q(x)$:
  - HA: $y = 0$
  - x-intercept: roots of $p(x)$
  - VA: roots of $q(x)$
- When deg $p(x) >$ deg $q(x)$:
  - HA: none
  - slant asymptote: $\frac{p(x)}{q(x)}$ excluding remainder
  - VA: roots of $q(x)$

## 1.4 Polynomials

### 1.4.1 Linear equations

Slope-intercept form: $y = mx + b$

Point-slope form: $y - y_1 = m(x - x_1)$ for point $(x, y)$

Standard form: $ax + by = c$

### 1.4.2 Quadratic equations

Standard form: $y = ax^2 + bx + c$

Vertex form: $y = a(x - h)^2 + k$ for vertex $(h, k)$

Sum of roots: $\frac{-b}{a}$

Product of roots: $\frac{c}{a}$

### 1.4.3 Higher-degree polynomials

In a polynomial

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

, with roots

$$r_1, r_2, r_3, \ldots, r_n$$

then:

$$r_1 + r_2 + r_3 + \cdots + r_n = \sum_{k=1} nr_k = -\frac{a_{n-1}}{a_n}$$

## 1.5 Sequences and Series

### 1.5.1 Explicit formulas

Aritmetic sequence: $a_n = a_1 + r(n-1)$

Geometric sequence: $a_n = a_1 * r^{n-1}$

Harmonic sequence: $a_n = \frac{1}{a_1 + r(n-1)}$

### 1.5.2 Arithmetic and Geometric Series

In the following equations, substituting $j = 1$ with $j = 0$, $j - 1$ with $j$, and $a_1$ with $a_0$ will produce the same result.

$$\sum_{j=1}^{n}(a_1 + r(j-1)) = \frac{n}{2}(2a_1 + (n-1)d)$$
$$\sum_{j=1}^{n}(a_1 * r^{j-1}) = \frac{a_1(1 - r^n)}{1 - r}$$
$$\sum_{j=1}^{\infty}(a_1 * r^{j-1}) = \frac{a_1}{1 - r} \text{ for } r \in [-1, 1]$$

### 1.5.3 Special Sums

$$\sum_{j=1}^{n} c = nc \qquad \sum_{j=1}^{n} ca_j = c\sum_{j=1}^{n} a_j$$
$$\sum_{j=1}^{n}(a_j + b_j) = \sum_{j=1}^{n} a_j + \sum_{j=1}^{n} b_j \qquad \sum_{j=1}^{n} j = \frac{n(n+1)}{2}$$
$$\sum_{j=1}^{n} j^2 = \frac{n(n+\frac{1}{2})(n+1)}{3} \qquad \sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4}$$

## 1.6 Trigonometry

| ° | rad | sin | cos | tan |
|---|---|---|---|---|
| 0° | $0$ | $0$ | $1$ | $0$ |
| 30° | $\frac{\pi}{6}$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{3}}$ |
| 45° | $\frac{\pi}{4}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | $1$ |
| 60° | $\frac{\pi}{3}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\sqrt{3}$ |
| 90° | $\frac{\pi}{2}$ | $1$ | $0$ | undef |

### 1.6.1 Law of Sines and Cosines

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c} \qquad c^2 = a^2 + b^2 - 2ab\cos(C)$$

### 1.6.2 Triangle area

$$K = \frac{1}{2}bh \qquad K = \frac{1}{2}bc\sin(A) \qquad K = \sqrt{s(s-a)(s-b)(s-c)}$$

### 1.6.3 More identities

$$(\sin A)^2 + (\cos A)^2 = 1 \qquad (\tan A)^2 + 1 = (\sec A)^2$$
$$\sin(\frac{\pi}{2} - x) = \cos(x) \qquad (\cot A)^2 + 1 = (\csc A)^2$$
$$\cos(-x) = \cos(x) \qquad \sin(-x) = \sin(x) \qquad \tan(-x) = \tan(x)$$

### 1.6.4 Slope

Where $\alpha$ is the angle between the line and the x-axis, and $m$ is the slope of the line:

$$m = \tan\alpha$$

### 1.6.5 Sum and difference formulas

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$
$$\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B)$$
$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$
$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$
$$\tan(A+B) = \frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$$
$$\tan(A-B) = \frac{\tan(A)-\tan(B)}{1+\tan(A)\tan(B)}$$

$$\sin(2A) = 2\sin(A)\cos(A)$$
$$\cos(2A) = (\cos A)^2 - (\sin A)^2 = 2(\cos A)^2 - 1 = 1 - 2(\sin A)^2$$
$$\tan(2A) = \frac{2\tan(A)}{1-(\tan A)^2}$$

## 1.7 Vectors

$$\vec{v}+\vec{w} = \begin{bmatrix} v_x + w_x \\ v_y + w_y \\ v_z + w_z \end{bmatrix} \qquad c * \vec{v} = \begin{bmatrix} c * v_x \\ c * v_y \\ c * v_z \end{bmatrix}$$

$$\vec{v}\cdot\vec{w} = v_x w_x + v_y w_y + v_z w_z = |\vec{v}||\vec{w}|\cos(\theta)$$
$$|\vec{v}\times\vec{w}| = |\vec{v}||\vec{w}|\sin(\theta) = \text{area of parallelogram}$$

$$\vec{v}\times\vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} \qquad \vec{v}\times\vec{w} \perp \vec{v} \qquad \vec{v}\times\vec{w} \perp \vec{w}$$

$$\vec{v}\perp\vec{w} \iff \vec{v}\times\vec{w} = \vec{0} \qquad \vec{v} \parallel \vec{w} \iff \vec{v}\cdot\vec{w} = 0$$

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|} \qquad \text{proj}_{\vec{b}}\vec{v} = \frac{\vec{v}\cdot\vec{b}}{\vec{b}\cdot\vec{b}} * \vec{b} = (|\vec{v}|\cos(\theta))$$

### Right-hand rule

To determine the direction of $\vec{v}\times\vec{w}$, put the side of the right hand on $\vec{v}$ and curl the fingers toward $\vec{w}$. The direction the thumb is pointing is the direction of $\vec{v}\times\vec{w}$.

## 1.8 Polar

### 1.8.1 Polar and Cartesian sytems

With point $(x, y) = (r; \theta) = (r; \beta)$, where $\theta$ is CCW from the $x$-axis and $\beta$ is a bearing, CW from the $y$-axis:

$$x = r\cos(\theta) = r\sin(\beta) \qquad y = r\sin(\theta) = r\cos(\beta)$$
$$r = \sqrt{x^2+y^2} \qquad \theta \equiv \arctan(\frac{y}{x}) \quad \beta \equiv \arctan(\frac{x}{y})$$

### 1.8.2 Converting functions

Try these substitutions in order:

$$x^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad x = r\cos\theta \qquad y = r\sin\theta$$

### 1.8.3 Limaçons and Petals

The function $y = A\cos(B(\theta + C)) + D$ is equivalent to $y = A\cos(B\theta) + D$ rotated $C$ degrees/radians clockwise.

When $C$ is 0 and $B$ is 1, the x-intercepts are $A \pm D$ and the y-intercepts are $\pm D$, and it forms a limaçon.

When $C$ is 0, but $B \neq 1$, then this sometimes still holds. The x-intercepts may also be $\pm A \pm D$.

There are $B$ petals, with the axis of the first petal on the positive x-axis.

When $B$ is even and $|D| < 1$, then the number of petals is $2B$.

Using sin instead of cos, limaçons have their axes on the positive y-axis, while for petals, the first petal starts from the positive x-axis and curves upwards.

## 1.9 Complex

$$\text{cis}(\theta) = e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

To find the $n^{\text{th}}$ root of $x_r \text{cis}(x_\theta)$, solve the equation $z_r^n \text{cis}(nz_\theta) = x_r \text{cis}(x_\theta + 360°k)$ for $k \in \mathbb{R}$.

## 1.10 Function domain

| Function | Domain $x$ | Range $y$ |
|---|---|---|
| $\log(x)$ | $(0, \infty)$ | $\mathbb{R}$ |
| $\sqrt{x}$ | $[0, \infty)$ | $[0, \infty)$ |
| $\arcsin(x)$ | $[-1, 1]$ | $[-\frac{\pi}{2}, \frac{\pi}{2}]$ |
| $\arccos(x)$ | $[-1, 1]$ | $[0, \pi]$ |
| $\arctan(x)$ | $\mathbb{R}$ | $(-\frac{\pi}{2}, \frac{\pi}{2})$ |

# Calculus Theorems (2)

## 1 Completeness

### 1.3 Completeness

**Theorem (Completeness of the Real Numbers).** Every nonempty subset $S$ of $\mathbb{R}$ which is bounded above has a least upper bound $\sup S$.

*Definition of* **Supremum** ($\sup S$). A number such that

(1) $s \leq \sup S$ for every $s \in S$ (which just says that $\sup S$ is an upper bound for $S$)

(2) If $u$ is any upper bound for S, then $\sup S \leq u$ (which says that $\sup S$ is the least upper bound for $S$).

*Definition of* **Infimum** ($\inf S$). A number such that

(1) $\inf S \leq s$ for every $s \in S$ (i.e. $\inf S$ is an lower bound for $S$)

(2) If $l$ is any upper bound for $S$, then $l \leq \inf S$ (i.e. $\inf S$ is the greatest lower bound for $S$).

**Theorem.** Every nonempty subset $S$ of $\mathbb{R}$ which is bounded below has a greatest lower bound.

**Theorem.** If $\min S$ exists, then $\min S = \inf S$.

**Theorem.** If $A \subset R$ and $c \geq 0$, and $cA := ca : a \in A$, $\sup cA = c \sup A$.

### 1.4 Consequences of Completeness

**Theorem (Rationals between Reals).** For every two real numbers $a$ and $b$ with $a < b$, there exists a rational number $r$ satisfying $a < r < b$.

### 1.5 Nested Intervals Theorem

**Nested Intervals Theorem.**
If $I_n = [a_n, b_n] = \{x \in R : a_n \leq x \leq b_n\}$ s.t. $a_n \leq a_{n+1}$ and $b_{n+1} \leq b_n$ for $n \in \mathbb{N}$, so that $I_1 \subseteq I_2 \subseteq I_3 \subseteq I_4 \subseteq \ldots$, then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

If $\inf\{b_n - a_n\} = 0$, then $\bigcap_{n=1}^{\infty} I_n \{x\}$, where $x = \sup\{a_n\} = inf\{b_n\}$.

### 1.6 Capture Theorem

**Capture Theorem.** If $A$ is a nonempty subset of $\mathbb{R}$, then:

(i) If $A$ is bounded above, then any open interval containing $\sup A$ contains an element of $A$.

(ii) Similarly, if $A$ is bounded below, then any open interval containing $\inf A$ contains an element of $A$.

### 1.7 Binary Search

If we binary-search for $x$ over $I_1 = [a_1, b_1]$ for $a_1, b_1 \in \mathbb{Q}$, we define $I_n$ s.t. either $I_n := [a_{n-1}, \frac{a_{n-1}+b_{n-1}}{2}]$ or $I_n := [\frac{a_{n-1}+b_{n-1}}{2}, a_{n+1}]$, and we define $a_n := \inf I_n$ and $b_n := \sup I_n$. We define $A$ to be the set of all $a_n$, and $B$ to be the set of all $b_n$.

Then, the size of $I_n = \frac{b_1 - a_1}{2^n} = b_n - a_n$, and $\bigcap_{n=1}^{\infty} I_n\{x\}$, where $x = \sup\{a_n\} = inf\{b_n\}$.

## 2 Limits

### 2.4 $\varepsilon$-$\delta$ definition of a Limit

*Definition of* **Limit**. If $\lim_{x \to a} f(x) = L$, then for any $\varepsilon > 0$, there exists $\delta > 0$ s.t. for any $x \in (a - \delta, a) \cup (a, a + \delta)$, $f(x) \in (L - \varepsilon, L + \varepsilon)$.

Alternatively,

*Definition of* **Limit**. If $\lim_{x \to a} f(x) = L$, then for any $\varepsilon > 0$, there exists $\delta > 0$ s.t. for any $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.

### 2.6 Limit Laws

**Theorem (Limit Laws).** Let $c \in R$ be a constant and suppose the limits $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. Then

(i) $\lim_{x \to a} (f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

(ii) $\lim_{x \to a} (cf(x)) = c \lim_{x \to a} f(x)$

(iii) $\lim_{x \to a} (f(x)g(x)) = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$

(iv) $\lim_{x \to a} f(x)g(x) = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$ , provided that $\lim_{x \to a} g(x) \neq 0$

(v) See (i).

(vi) $\lim_{x \to a} x^n = (\lim_{x \to a} x)^n$

(vii) $\lim_{x \to a} \sqrt{f(x)} = \sqrt{\lim_{x \to a} f(x)}$

(viii) $\lim_{x \to a} \frac{a(x)b(x)}{c(x)b(x)} = \lim_{x \to a} \frac{a(x)}{c(x)}$

**Theorem (Operations on infinity).** For $x \in \mathbb{R}$,

$$\infty + x = \infty$$

$$-\infty + x = -\infty$$

$$x * \infty = \begin{cases} \infty & \text{if } x > 0 \\ -\infty & \text{if } x < 0 \end{cases}$$

$$x * -\infty = \begin{cases} -\infty & \text{if } x > 0 \\ \infty & \text{if } x < 0. \end{cases}$$

$$\frac{x}{\pm\infty} = 0$$

*Definition of* **Indeterminate forms**. The following forms are indeterminate and you cannot evaluate them.

$$\frac{0}{0}, \frac{\pm\infty}{\pm\infty}, 0 * \pm\infty, \infty - \infty$$

### Other theorems

**Composite Function Theorem.** If $f$ is continuous at $L$ and $\lim_{x \to a} g(x) = L$, then $\lim_{x \to a} f(g(x) = f(lim_{x \to a} g(x))) = f(L)$

### 2.12 Squeeze Theorem

**Squeeze Theorem.** Let $f$ , $g$, and $h$ be defined for all $x \neq a$ over an open interval containing $a$. If

$$f(x) \leq g(x) \leq h(x)$$

for all $x \neq a$ in an open interval containing $a$ and

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$$

where $L \in \mathbb{R}$, then $\lim_{x \to a} g(x) = L$.

## 3 Continuity

*Definition of* **Continuity at a point**. Function $f$ is continuous at point $a$ if $\lim_{x \to a} f(x) = f(a)$.

*Definition.* $f$ has a **removable discontinuity** if $\lim_{x \to a} f(x) = L \in \mathbb{R}$ (in this case either $f(a)$ is undefined, or $f(a)$ is defined by $L \neq f(a)$).

*Definition.* $f$ has a **jump discontinuity** if $\lim_{x \to a^-} f(x) = L_1 \in \mathbb{R}$ and $\lim_{x \to a^+} f(x) = L_2 \in \mathbb{R}$ but $L1 \neq L2$.

*Definition.* $f$ has an **infinite discontinuity** at $a$ if $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$

**Intermediate Value Theorem.** If $f$ is continuous on $[a, b]$, then for any real number $L$ between $f(a)$ and $f(b)$ there exists at least one $c \in [a, b]$ such that $f(c) = L$. In other words, if $f$ is continuous on $[a, b]$, then the graph must cross the horizontal line $y = L$ at least once between the vertical lines $x = a$ and $x = b$.

**Aura Theorem.** If $f(x)$ is continuous and $f(a)$ is positive, then there exists an open interval containing $a$ such that for all $x$ in the interval, $f(x)$ is positive.

If $f(x)$ is continuous and $f(a)$ is negative, then there exists an open interval containing $a$ such that for all $x$ in the interval, $f(x)$ is negative.

**Bolzano's Theorem.** Let $f$ be a continuous function defined on $[a, b]$. If 0 is between $f(a)$ and $f(b)$, then there exists $x \in [a, b]$ such that $f(x) = 0$.

## 4 Derivatives

The derivative is the instantaneous rate of change, and the slope of the tangent line to the point.

*Definition of* **Derivative** ($f'(a)$).

$$\frac{d}{da}f(a) = f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

**Theorem (Tangent line to a point).** The equation of the tangent line to the point $(a, f(a))$ is

$$y = f'(a)(x-a) + f(a)$$

### Derivative Rules

**Theorem (Difference Rule).**

$$\frac{d}{dx}(f(x)-g(x)) = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

**Theorem (Sum Rule).**

$$\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

**Theorem (Constant Multiple Rule).**

$$\frac{d}{dx}(cf(x)) = c\frac{d}{dx}f(x)$$

**Theorem (Product Rule).**

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

**Theorem (Quotient Rule).**

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

**Theorem (Power Rule).**

$$\frac{d}{dx}x^n = nx^{n-1}$$

for $n \in \mathbb{R}$

**Theorem (Chain Rule).**

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

**Theorem (Derivatives of exponentials and logs).**

$$\begin{aligned} \frac{d}{dx}e^x &= e^x & \frac{d}{dx}\ln x &= \frac{1}{x} \\ \frac{d}{dx}a^x &= a^x\ln(a) & \frac{d}{dx}\log_a x &= \frac{1}{x\ln(a)} \end{aligned}$$

**Theorem (Derivatives of trig functions).**

$$\begin{aligned} \sin'(x) &= \cos(x) & \cos'(x) &= -\sin(x) \\ \sec'(x) &= \sec(x)\tan(x) & \csc'(x) &= -\csc(x)\cot(x) \\ \tan'(x) &= \sec(x)^2 & \cot'(x) &= -\csc(x)^2 \end{aligned}$$