

Mathematics (1)

1.1 Logarithms

$$\log_b(MN) = \log_b(M) + \log_b(N)$$
$$\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)$$
$$\log_b(M^p) = p \cdot \log_b(M)$$
$$\log_b(a) = \frac{\log_x(a)}{\log_x(b)}$$
$$\log_b(b) = 1$$

1.2 Notation

deg  $p(x)$  means the degree of polynomial  $p$ .

LC  $p(x)$  means the leading coefficient of polynomial  $p$ .

1.3 Rational functions

For a rational function  $f(x) = \frac{p(x)}{q(x)}$ , cancel out any common factors, then:

- For all rational functions:
  - VA: roots of  $q(x)$
  - Roots: roots of  $p(x)$
- When  $\deg p(x) = \deg q(x)$ :
  - HA:  $y = \frac{\text{LC } p(x)}{\text{LC } q(x)}$
- When  $\deg p(x) < \deg q(x)$ :
  - HA:  $y = 0$
- When  $\deg p(x) > \deg q(x)$ :
  - HA: none
  - slant asymptote:  $\frac{p(x)}{q(x)}$  excluding remainder

1.4 Polynomials

1.4.1 Linear equations

Slope-intercept form:  $y = mx + b$

Point-slope form:  $y - y_1 = m(x - x_1)$  for point  $(x, y)$

Standard form:  $ax + by = c$

1.4.2 Quadratic equations

Standard form:  $y = ax^2 + bx + c$

Vertex form:  $y = a(x - h)^2 + k$  for vertex  $(h, k)$

Sum of roots:  $-\frac{b}{a}$

Product of roots:  $\frac{c}{a}$

1.4.3 Higher-degree polynomials

In a polynomial

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0$$

, with roots

$$r_1, r_2, r_3, \dots, r_n$$

then:

$$r_1 + r_2 + r_3 + \cdots + r_n = \sum_{k=1} nr_k = -\frac{a_{n-1}}{a_n}$$

1.5 Sequences and Series

1.5.1 Explicit formulas

Aritmetic sequence:  $a_n = a_1 + r(n - 1)$

Geometric sequence:  $a_n = a_1 * r^{n-1}$

Harmonic sequence:  $a_n = \frac{1}{a_1 + r(n - 1)}$

1.5.2 Arithmetic and Geometric Series

In the following equations, substituting  $j = 1$  with  $j = 0$ ,  $j - 1$  with  $j$ , and  $a_1$  with  $a_0$  will produce the same result.

$$\sum_{j=1}^n(a_1 + r(j - 1)) = \frac{n}{2}(2a_1 + (n - 1)d)$$
$$\sum_{j=1}^n(a_1 * r^{j-1}) = \frac{a_1(1 - r^n)}{1 - r}$$
$$\sum_{j=1}^\infty(a_1 * r^{j-1}) = \frac{a_1}{1 - r} \text{ for } r \in [-1, 1]$$

1.5.3 Special Sums

$$\sum_{j=1}^nc = nc$$
$$\sum_{j=1}^nca_j = c\sum_{j=1}^na_j$$
$$\sum_{j=1}^n(a_j + b_j) = \sum_{j=1}^na_j + \sum_{j=1}^nb_j$$
$$\sum_{j=1}^nj = \frac{n(n + 1)}{2}$$
$$\sum_{j=1}^nj^2 = \frac{n(n + \frac{1}{2})(n + 1)}{3}$$
$$\sum_{j=1}^nj^3 = \frac{n^2(n + 1)^2}{4}$$

1.6 Trigonometry

°	rad	sin	cos	tan
0°	0	0	1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	$\frac{\pi}{2}$	1	0	undef

1.6.1 Law of Sines and Cosines

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$$
$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

1.6.2 Triangle area

$$K = \frac{1}{2}bh$$
$$K = \frac{1}{2}bc\sin(A)$$
$$K = \sqrt{s(s - a)(s - b)(s - c)}$$

1.6.3 More identities

$$(\sin A)^2 + (\cos A)^2 = 1$$

$$(\tan A)^2 + 1 = (\sec A)^2$$

$$\sin(\frac{\pi}{2} - x) = \cos(x)$$

$$(\cot A)^2 + 1 = (\csc A)^2$$

$$\cos(-x) = \cos(x)$$

$$\sin(-x) = -\sin(x)$$

$$\tan(-x) = -\tan(x)$$

1.6.4 Slope

Where  $\alpha$  is the angle between the line and the x-axis, and  $m$  is the slope of the line:

$$m = \tan \alpha$$

1.6.5 Sum and difference formulas

$$\sin(A + B) = \sin(A) \cos(B) + \cos(A) \sin(B)$$

$$\sin(A - B) = \sin(A) \cos(B) - \cos(A) \sin(B)$$

$$\cos(A + B) = \cos(A) \cos(B) - \sin(A) \sin(B)$$

$$\cos(A - B) = \cos(A) \cos(B) + \sin(A) \sin(B)$$

$$\tan(A + B) = \frac{\tan(A) + \tan(B)}{1 - \tan(A) \tan(B)}$$

$$\tan(A - B) = \frac{\tan(A) - \tan(B)}{1 + \tan(A) \tan(B)}$$

$$\sin(2A) = 2 \sin(A) \cos(A)$$

$$\cos(2A) = (\cos A)^2 - (\sin A)^2 = 2(\cos A)^2 - 1 = 1 - 2(\sin A)^2$$

$$\tan(2A) = \frac{2 \tan(A)}{1 - (\tan A)^2}$$

1.7 Vectors

$$\vec{v} + \vec{w} = \begin{bmatrix} v_x + w_x \\ v_y + w_y \\ v_z + w_z \end{bmatrix}$$

$$c * \vec{v} = \begin{bmatrix} c * v_x \\ c * v_y \\ c * v_z \end{bmatrix}$$

$$\vec{v} \cdot \vec{w} = v_x w_x + v_y w_y + v_z w_z = |\vec{v}| |\vec{w}| \cos(\theta)$$

$$|\vec{v} \times \vec{w}| = |\vec{v}| |\vec{w}| \sin(\theta) = \text{area of parallelogram}$$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$$\vec{v} \times \vec{w} \perp \vec{v} \quad \vec{v} \times \vec{w} \perp \vec{w}$$

$$\vec{v} \perp \vec{w} \iff \vec{v} \times \vec{w} = \vec{0} \quad \vec{v} \parallel \vec{w} \iff \vec{v} \cdot \vec{w} = 0$$

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|} \quad \text{proj}_{\vec{b}} \vec{v} = \frac{\vec{v} \cdot \vec{b}}{\vec{b} \cdot \vec{b}} * \vec{b} = (|\vec{v}| \cos(\theta)) \hat{b}$$

Right-hand rule

To determine the direction of  $\vec{v} \times \vec{w}$ , put the side of the right hand on  $\vec{v}$  and curl the fingers toward  $\vec{w}$ . The direction the thumb is pointing is the direction of  $\vec{v} \times \vec{w}$ .

1.8 Polar

1.8.1 Polar and Cartesian sytems

With point  $(x, y) = (r; \theta) = (r; \beta)$ , where  $\theta$  is CCW from the x-axis and  $\beta$  is a bearing, CW from the y-axis:

$$x = r \cos(\theta) = r \sin(\beta)$$

$$y = r \sin(\theta) = r \cos(\beta)$$

$$r = \sqrt{x^2 + y^2}$$

$$\theta \equiv \arctan(\frac{y}{x}) \quad \beta \equiv \arctan(\frac{x}{y})$$

1.8.2 Converting functions

Try these substitutions in order:

$$x^2 = x^2 + y^2$$

$$\tan \theta = \frac{y}{x}$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

1.8.3 Limaçons and Petals

The function  $y = A \cos(B(\theta + C)) + D$  is equivalent to  $y = A \cos(B\theta) + D$  rotated  $C$  degrees/radians clockwise.

When  $C$  is 0 and  $B$  is 1, the x-intercepts are  $A \pm D$  and the y-intercepts are  $\pm D$ , and it forms a limaçon.

When  $C$  is 0, but  $B \neq 1$ , then this sometimes still holds. The x-intercepts may also be  $\pm A \pm D$ .

There are  $B$  petals, with the axis of the first petal on the positive x-axis.

When  $B$  is even and  $|D| < 1$ , then the number of petals is  $2B$ .

Using sin instead of cos, limaçons have their axes on the positive y-axis, while for petals, the first petal starts from the positive x-axis and curves upwards.

1.9 Complex

$$\text{cis}(\theta) = e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

To find the  $n^{\text{th}}$  root of  $x_r \text{cis}(x_\theta)$ , solve the equation  $z_r^n \text{cis}(nz_\theta) = x_r \text{cis}(x_\theta + 360^\circ k)$  for  $k \in \mathbb{R}$ .

1.10 Function domain

Function	Domain $x$	Range $y$
$\log(x)$	$(0, \infty)$	$\mathbb{R}$
$\sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$\arcsin(x)$	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$\arccos(x)$	$[-1, 1]$	$[0, \pi]$
$\arctan(x)$	$\mathbb{R}$	$(-\frac{\pi}{2}, \frac{\pi}{2})$

Calculus Theorems (2)

1 Completeness

1.3 Completeness

**Theorem (Completeness of the Real Numbers).** Every nonempty subset  $S$  of  $\mathbb{R}$  which is bounded above has a least upper bound  $\sup S$ .

*Definition of Supremum* ( $\sup S$ ). A number such that

- (1)  $s \leq \sup S$  for every  $s \in S$  (which just says that  $\sup S$  is an upper bound for  $S$ )
- (2) If  $u$  is any upper bound for  $S$ , then  $\sup S \leq u$  (which says that  $\sup S$  is the least upper bound for  $S$ ).

*Definition of Infimum* ( $\inf S$ ). A number such that

- (1)  $\inf S \leq s$  for every  $s \in S$  (i.e.  $\inf S$  is an lower bound for  $S$ )
- (2) If  $l$  is any upper bound for  $S$ , then  $l \leq \inf S$  (i.e.  $\inf S$  is the greatest lower bound for  $S$ ).

**Theorem.** Every nonempty subset  $S$  of  $\mathbb{R}$  which is bounded below has a greatest lower bound.

**Theorem.** If  $\min S$  exists, then  $\min S = \inf S$ .

**Theorem.** If  $A \subset \mathbb{R}$  and  $c \geq 0$ , and  $cA := \{ca : a \in A\}$ ,  $\sup cA = c \sup A$ .

1.4 Consequences of Completeness

**Theorem (Rationals between Reals).** For every two real numbers  $a$  and  $b$  with  $a < b$ , there exists a rational number  $r$  satisfying  $a < r < b$ .

1.5 Nested Intervals Theorem

**Nested Intervals Theorem.**

If  $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$  s.t.  $a_n \leq a_{n+1}$  and  $b_{n+1} \leq b_n$  for  $n \in \mathbb{N}$ , so that  $I_1 \subseteq I_2 \subseteq I_3 \subseteq I_4 \subseteq \dots$ , then  $\bigcap_{n=1}^\infty I_n \neq \emptyset$ .

If  $\inf\{b_n - a_n\} = 0$ , then  $\bigcap_{n=1}^\infty I_n \{x\}$ , where

$x = \sup\{a_n\} = \inf\{b_n\}$ .

1.6 Capture Theorem

**Capture Theorem.** If  $A$  is a nonempty subset of  $\mathbb{R}$ , then:

- (i) If  $A$  is bounded above, then any open interval containing  $\sup A$  contains an element of  $A$ .
- (ii) Similarly, if  $A$  is bounded below, then any open interval containing  $\inf A$  contains an element of  $A$ .

1.7 Binary Search

If we binary-search for  $x$  over  $I_1 = [a_1, b_1]$  for  $a_1, b_1 \in \mathbb{Q}$ , we define  $I_n$  s.t. either  $I_n := [a_{n-1}, \frac{a_{n-1}+b_{n-1}}{2}]$  or  $I_n := [\frac{a_{n-1}+b_{n-1}}{2}, a_{n+1}]$ , and we define  $a_n := \inf I_n$  and  $b_n := \sup I_n$ . We define  $A$  to be the set of all  $a_n$ , and  $B$  to be the set of all  $b_n$ .

Then, the size of  $I_n = \frac{b_1-a_1}{2^n} = b_n - a_n$ , and  $\bigcap_{n=1}^\infty I_n \{x\}$ , where  $x = \sup\{a_n\} = \inf\{b_n\}$ .

2 Limits

2.4  $\varepsilon$ - $\delta$  definition of a Limit

*Definition of Limit.* If  $\lim_{x \rightarrow a} f(x) = L$ , then for any  $\varepsilon > 0$ , there exists  $\delta > 0$  s.t. for any  $x \in (a - \delta, a) \cup (a, a + \delta)$ ,  $f(x) \in (L - \varepsilon, L + \varepsilon)$ .

Alternatively,

*Definition of Limit.* If  $\lim_{x \rightarrow a} f(x) = L$ , then for any  $\varepsilon > 0$ , there exists  $\delta > 0$  s.t. for any  $|f(x) - L| < \varepsilon$  whenever  $0 < |x - a| < \delta$ .

2.6 Limit Laws

**Theorem (Limit Laws).** Let  $c \in \mathbb{R}$  be a constant and suppose the limits  $\lim_{x \rightarrow a} f(x)$  and  $\lim_{x \rightarrow a} g(x)$  exist. Then

- (i)  $\lim_{x \rightarrow a} (f(x) \pm g(x)) = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$
- (ii)  $\lim_{x \rightarrow a} (cf(x)) = c \lim_{x \rightarrow a} f(x)$
- (iii)  $\lim_{x \rightarrow a} (f(x)g(x)) = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x)$
- (iv)  $\lim_{x \rightarrow a} f(x)g(x) = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x)$  , provided that  $\lim_{x \rightarrow a} g(x) \neq 0$
- (v) See (i).
- (vi)  $\lim_{x \rightarrow a} x^n = (\lim_{x \rightarrow a} x)^n$
- (vii)  $\lim_{x \rightarrow a} \sqrt{f(x)} = \sqrt{\lim_{x \rightarrow a} f(x)}$
- (viii)  $\lim_{x \rightarrow a} \frac{a(x)b(x)}{c(x)b(x)} = \lim_{x \rightarrow a} \frac{a(x)}{c(x)}$

**Theorem (Operations on infinity).** For  $x \in \mathbb{R}$ ,

$$\begin{aligned} \infty + x &= \infty \\ -\infty + x &= -\infty \\ x * \infty &= \begin{cases} \infty & \text{if } x > 0 \\ -\infty & \text{if } x < 0 \end{cases} \\ x * -\infty &= \begin{cases} -\infty & \text{if } x > 0 \\ \infty & \text{if } x < 0. \end{cases} \end{aligned}$$

$$\frac{x}{\pm \infty} = 0$$

*Definition of Indeterminate forms.* The following forms are indeterminate and you cannot evaluate them.

$$\frac{0}{0}, \frac{\pm \infty}{\pm \infty}, 0 * \pm \infty, \infty - \infty$$

Other theorems

**Composite Function Theorem.** If  $f$  is continuous at  $L$  and  $\lim_{x \rightarrow a} g(x) = L$ , then  $\lim_{x \rightarrow a} f(g(x)) = f(\lim_{x \rightarrow a} g(x)) = f(L)$

2.12 Squeeze Theorem

**Squeeze Theorem.** Let  $f$  ,  $g$ , and  $h$  be defined for all  $x \neq a$  over an open interval containing  $a$ . If

$$f(x) \leq g(x) \leq h(x)$$

for all  $x \neq a$  in an open interval containing  $a$  and

$$\lim_{x \rightarrow a} f(x) = L = \lim_{x \rightarrow a} h(x)$$

where  $L \in \mathbb{R}$ , then  $\lim_{x \rightarrow a} g(x) = L$ .

3 Continuity

*Definition of Continuity at a point.* Function  $f$  is continuous at point  $a$  if  $\lim_{x \rightarrow a} f(x) = f(a)$ .

*Definition.*  $f$  has a **removable discontinuity** if  $\lim_{x \rightarrow a} f(x) = L \in \mathbb{R}$  (in this case either  $f(a)$  is undefined, or  $f(a)$  is defined by  $L \neq f(a)$ ).

*Definition.*  $f$  has a **jump discontinuity** if  $\lim_{x \rightarrow a^-} f(x) = L_1 \in \mathbb{R}$  and  $\lim_{x \rightarrow a^+} f(x) = L_2 \in \mathbb{R}$  but  $L_1 \neq L_2$ .

*Definition.*  $f$  has an **infinite discontinuity** at  $a$  if  $\lim_{x \rightarrow a^-} f(x) = \pm \infty$  or  $\lim_{x \rightarrow a^+} f(x) = \pm \infty$

**Intermediate Value Theorem.** If  $f$  is continuous on  $[a, b]$ , then for any real number  $L$  between  $f(a)$  and  $f(b)$  there exists at least one  $c \in [a, b]$  such that  $f(c) = L$ . In other words, if  $f$  is continuous on  $[a, b]$ , then the graph must cross the horizontal line  $y = L$  at least once between the vertical lines  $x = a$  and  $x = b$ .

**Aura Theorem.** If  $f(x)$  is continuous and  $f(a)$  is positive, then there exists an open interval containing  $a$  such that for all  $x$  in the interval,  $f(x)$  is positive.

If  $f(x)$  is continuous and  $f(a)$  is negative, then there exists an open interval containing  $a$  such that for all  $x$  in the interval,  $f(x)$  is negative.

**Bolzano's Theorem.** Let  $f$  be a continuous function defined on  $[a, b]$ . If 0 is between  $f(a)$  and  $f(b)$ , then there exists  $x \in [a, b]$  such that  $f(x) = 0$ .

4 Derivatives

The derivative is the instantaneous rate of change, and the slope of the tangent line to the point.

*Definition of Derivative* ( $f'(a)$ ).

$$\frac{d}{da} f(a) = f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

**Theorem (Tangent line to a point).** The equation of the tangent line to the point  $(a, f(a))$  is

$$y = f'(a)(x - a) + f(a)$$

Derivative Rules

**Theorem (Difference Rule).**

$$\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx} f(x) - \frac{d}{dx} g(x)$$

**Theorem (Sum Rule).**

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$

**Theorem (Constant Multiple Rule).**

$$\frac{d}{dx}(cf(x)) = c \frac{d}{dx} f(x)$$

**Theorem (Product Rule).**

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

**Theorem (Quotient Rule).**

$$\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

**Theorem (Power Rule).**

$$\frac{d}{dx} x^n = nx^{n-1}$$

for  $n \in \mathbb{R}$

**Theorem (Chain Rule).**

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x) \qquad \frac{dy}{dx} = \frac{dy}{db} \frac{db}{dx}$$

**Theorem (Derivative of inverse functions).** Let  $x \in \mathbb{R}$  and  $f$  be a differentiable, one-to-one function at  $x$ . Then if  $f'(x) \neq 0$ , then

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}$$

**Theorem (Derivatives of exponentials and logs).**

$$\begin{aligned} \frac{d}{dx} e^x &= e^x & \frac{d}{dx} \ln x &= \frac{1}{x} \\ \frac{d}{dx} a^x &= a^x \ln(a) & \frac{d}{dx} \log_a x &= \frac{1}{x \ln(a)} \end{aligned}$$

**Theorem (Derivatives of trig functions).**

$$\begin{aligned} \sin'(x) &= \cos(x) & \cos'(x) &= -\sin(x) \\ \sec'(x) &= \sec(x) \tan(x) & \csc'(x) &= -\csc(x) \cot(x) \\ \tan'(x) &= \sec(x)^2 & \cot'(x) &= -\csc(x)^2 \end{aligned}$$

$$\begin{aligned} \arcsin'(x) &= \frac{1}{\sqrt{1-x^2}} & \arccos'(x) &= -\frac{1}{\sqrt{1-x^2}} \\ \operatorname{arcsec}'(x) &= \frac{1}{|x|\sqrt{x^2-1}} & \operatorname{arccsc}'(x) &= -\frac{1}{|x|\sqrt{x^2-1}} \\ \arctan'(x) &= \frac{1}{1+x^2} & \operatorname{arccot}'(x) &= -\frac{1}{1+x^2} \end{aligned}$$