

Sound & Light (1)

1.1 Miscellaneous

% error = (observed - theoretical) / theoretical \* 100%

1.2 Kinematics

x = (a/2)(Δt)² + v₀Δt + x₀      v = v₀ + aΔt

v² = v₀² + 2aΔx      Δx = (v₀ + v) / 2 \* Δt

1.3 Simple Harmonic Motion

x = A cos(ωt + φ)    v = -ωA sin(ωt + φ)    a = -ω²A cos(ωt + φ)

x\_max = A      v\_max = ωA      a\_max = ω²A      F\_max = mω²A

1.3.1 Springs and Slinkies

x represents the distance from the equilibrium.

If you put a mass on top of the slinky, Δx\_eq represents the difference between the original equilibrium and the new equilibrium.

F\_s = kx = ma      F\_s\_max = kΔx\_eq = 9.8Δm

f = (1/2π)√(k/m)      T = 2π√(m/k)      ω = 2πf = √(k/m)

SPE = (1/2)kx²      KE = (1/2)mv²

TME = (1/2)kx² + (1/2)mv² = (1/2)kA² = (1/2)m v\_max²

1.3.2 Springs in parallel and series

Quantity	In Series	In Parallel
Equivalent spring constant	1/k_eq = 1/k_1 + 1/k_2	k_eq = k_1 + k_2
Deflection (elongation)	x_eq = x_1 + x_2	x_eq = x_1 = x_2
Force	F_eq = F_1 = F_2	F_eq = F_1 + F_2
Stored energy	E_eq = E_1 + E_2	E_eq = E_1 + E_2

1.3.3 Pendulums

f = (1/2π)√(g/L)      T = 2π√(L/g)

1.4 Waves

T = 1/f      v = λf      v = Δx/Δt

1.4.1 Slinkies and strings with fixed ends

F\_T = F\_s = kx      μ = m/L      v = √(F\_T/μ)

Given mass m\_T hanging below a pulley, F\_T = m\_Tg.

1.5 Standing waves

1.5.1 Open-open, closed-closed

n is the number of antinodes, or the n<sup>th</sup> harmonic.

f\_n = f\_1 n = nv/2L      f\_1 = v/2L      λ\_n = 2L/n

1.5.2 Open-closed

f\_n = f\_1 n = nv/4L      f\_1 = v/4L      λ\_n = 2L/n

1.6 Sound

1.6.1 Speed of sound

v = 331√((T\_c + 273)/273)      v ≈ 331 + 0.59T

1.6.2 Sound intensity

I = Power (W) / Area = Power (W) / (4πr²)

I\_dB = 10 log₁₀(I/I₀)      I = 10^(I\_dB/10 - 12)

1.6.3 Doppler effect

[O] → [S]    f\_o = f\_s (v + v\_o) / v      ← [O] [S]    f\_o = f\_s (v - v\_o) / v

[O] ← [S]    f\_o = f\_s v / (v - v\_s)      [O] [S] →    f\_o = f\_s v / (v + v\_s)

[O] → ← [S]    f\_o = f\_s (v + v\_o) / (v - v\_s)      [O] → [S] →    f\_o = f\_s (v + v\_o) / (v + v\_s)

← [O] ← [S]    f\_o = f\_s (v - v\_o) / (v - v\_s)      ← [O] [S] →    f\_o = f\_s (v - v\_o) / (v + v\_s)

1.6.4 Constructive and Destructive Interference (2 dimensions)

For a point on the m<sup>th</sup> antinodal/nodal line playing the same frequency with the same phase:

PD = mλ

where PD is the path length difference.

1.6.5 Beats

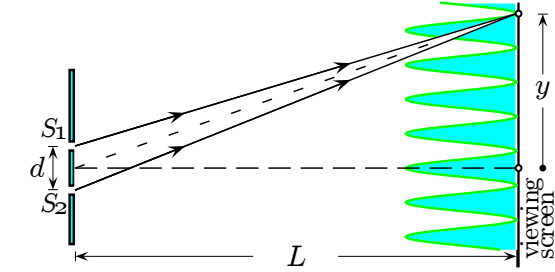
f\_B = Δf

1.7 Light

1.7.1 Speed of light

$c = 299\,792\,458 \frac{\text{m}}{\text{s}} \approx 3 * 10^8 \frac{\text{m}}{\text{s}}$

1.7.2 Two-slit experiment



$PD = \frac{dy}{L} = m\lambda$

1.7.3 Mirror

$r = 2f \quad \frac{1}{f} = \frac{1}{p} + \frac{1}{q} \quad M = \frac{h}{h_o} = \frac{-q}{p}$

In a plane mirror,  $p = -q$ .

1.7.4 Lenses

$\frac{1}{f} = (n - 1)(\frac{1}{r_1} - \frac{1}{r_2}) \quad \frac{1}{f} = \frac{1}{p} + \frac{1}{q} \quad M = \frac{h}{h_o} = \frac{-q}{p}$

Multiple lenses

$p_2 = \Delta x - q_1$

1.7.5 Mirrors and Lenses

Real  $\iff$  inverted, virtual  $\iff$  upright.

	Converging	Diverging
Mirror		
Lens		
focal length	+	-
object distance	image	image
$\infty$ to $2f$	Real smaller	Virtual smaller
$2f$ to $f$	Real larger	
$f$ to $0$	Virtual larger	
$0$ to $-f$	Virtual smaller	Virtual larger
$-f$ to $-2f$		Real larger
$-2f$ to $-\infty$		Real smaller

1.7.6 Refraction / Snell’s Law

The **normal line** is the line perpendicular to the surface which touches the intersection of the surface and the light ray.

The **incident angle** is the angle between the ray of light and the normal line.

$\theta_1$  and  $\theta_2$  are both measured from the normal line, not the surface.

Refraction occurs when the speed of light in two media are different and light hits the boundary of the two media. The frequency of the light will stay the same, but the speed, wavelength, and direction will change.

$n = \frac{c}{v} \quad n_1 \sin \theta_1 = n_2 \sin \theta_2$

1.7.7 Ray diagrams

Mechanics (2)

2.1 Miscellaneous

% error = (observed - theoretical) / theoretical \* 100%

2.2 Kinematics

x(t) = 1/2 at^2 + v0t + x0      v(t) = v0 + at

v^2 = v0^2 + 2aΔx      Δx = (v0 + v) / 2 \* Δt

2.3 Forces

F\_net = ma

FT represents tension. It always points in the direction on which the rope pulls on the object.

FN represents normal force. It takes the direction and magnitude necessary to prevent the object from passing through the surface that creates the normal force.

2.3.1 Friction

fs ≤ μsFN

The static friction takes the direction and magnitude necessary to prevent the object from moving in the component parallel to the surface, until the magnitude reaches μsFN. Upon reaching μsFN, the static friction is replaced by kinetic friction and the object starts moving:

ks = μkFN

2.3.2 Centripetal force

The centripetal force always points towards the center of the circle representing the object's path, and therefore is perpendicular to the velocity. It is just another name for the net force in the centripetal direction. The centripetal acceleration is what causes the object to rotate.

Fc = mac = m(v^2/r) = mrω^2      ac = v^2/r

2.4 Work and Energy

W = ∫a^b F(r̄) · d r̄

In one dimension:

W = ∫a^b F(x)dx

Work-energy theorem. The work is the change in kinetic energy:

W = ΔK

2.5 Simple Harmonic Motion

The object is at rest at the equilibrium position. x = 0 when the object is at equilibrium, and x represents the distance and direction from the equilibrium. When you pull it to one direction, the restoring force pulls the object back toward the equilibrium position. It oscillates back and forth, between x = -A and x = A, where A is the amplitude. The period is the amount of time to complete one oscillation, and the frequency is the amount of oscillations that happen in one second (or some other time unit).

x(t) = A cos(ωt + φ)

v(t) = -Aω sin(ωt + φ)

a(t) = -Aω^2 cos(ωt + φ)

2.5.1 Spring force

Fs = -kx

Us = -1/2 kx^2

a = -(k/m)x

v = √(k/m (A^2 - x^2))

2.6 Simple Harmonic Motion (old)

x = A cos(ωt + φ)    v = -ωA sin(ωt + φ)    a = -ω^2 A cos(ωt + φ)

x\_max = A    v\_max = ωA    a\_max = ω^2 A    F\_max = mω^2 A

2.6.1 Springs and Slinkies

x represents the distance from the equilibrium.

If you put a mass on top of the slinky, Δx\_eq represents the difference between the original equilibrium and the new equilibrium.

Fs = kx = ma      F\_s\_max = kΔx\_eq = 9.8Δm

f = 1/(2π) √(k/m)      T = 2π √(m/k)      ω = 2πf = √(k/m)

SPE = 1/2 kx^2      KE = 1/2 mv^2

TME = 1/2 kx^2 + 1/2 mv^2 = 1/2 kA^2 = 1/2 mv\_max^2

2.6.2 Springs in parallel and series

Quantity	In Series	In Parallel
Equivalent spring constant	1/k_eq = 1/k1 + 1/k2	k_eq = k1 + k2
Deflection (elongation)	x_eq = x1 + x2	x_eq = x1 = x2
Force	F_eq = F1 = F2	F_eq = F1 + F2
Stored energy	E_eq = E1 + E2	E_eq = E1 + E2

2.7 Center of mass

The center of mass of an object which consists of masses at discrete points r̄1, r̄2, r̄3, ..., r̄n is

r\_cm = (Σ mi r̄i) / Σ mi

The center of mass of a solid object with variable density is

r\_cm = (∫ r̄ dm) / ∫ dm

To find the velocity of the center of mass, replace r̄i with v̄i in the above equations.

2.7.1 Density

3D: density      ρ = M/v = dm/dV      (kg/m^3)

2D: surface density      σ = M/A = dm/dA      (kg/m^2)

1D: linear density      λ = M/L = dm/dx      (kg/m)

2.8 Momentum/Impulse

Definition of Momentum. Momentum (p̄) is defined as:

p̄ = mv̄      (Ns = kg m/s)

Definition of Impulse. Impulse is the integral of force over time. By the Impulse-Momentum Theorem, it is also the change in momentum.

J̄ = ∫ F̄ dt = Δp = F̄\_net,avg Δt

Additionally,

F̄\_net = dp/dt

Theorem (Conservation of momentum). Momentum is conserved when there are no external forces acting upon the object.

2.9 Rotational motion

2.9.1 Rotational kinematics

$\bar{\theta}$ , angle, is the rotational equivalent of position.

$\Delta\theta$  is called angular displacement.

Angular velocity =  $\vec{\omega} = \frac{d\bar{\theta}}{dt} \left( \frac{\text{rad}}{\text{s}} \right)$

Angular acceleration =  $\vec{\alpha} = \frac{d\vec{\omega}}{dt} \left( \frac{\text{rad}}{\text{s}^2} \right)$

Arc length =  $s = R\theta$

Velocity =  $v = R\omega$

Acceleration =  $a = R\alpha$

Constant acceleration

$\omega_f = \omega_0 + \alpha t$

$\Delta\theta = \omega_0 t + \frac{1}{2}\alpha t^2$

$\omega_f^2 = \omega_0^2 + 2\alpha\delta\theta$

2.9.2 Torque

$\vec{\tau} = \vec{r} \times \vec{F}$

$|\tau| = |r||F| \sin \theta$

2.9.3 Rotational inertia

Where  $r$  represents the distance between the axis of rotation and the object,  $m$  represents mass,  $I$  represents rotational inertia, and  $K$  represents rotational kinetic energy:

$I = mr^2$

$I_{\text{total}} := \sum m_i r_i^2 = \int r^2 dm$

$K = \frac{1}{2} I \omega^2$

**Parallel Axis Theorem.** If  $d$  is the distance between the axis through the center of mass and the desired axis of rotation parallel to the axis through the center of mass, then the rotational inertia through the desired axis of rotation is

$I_{\text{II}} = I_{\text{cm}} + Md^2$

2.9.4 Newton’s Second Law

$\sum \tau = I\alpha$

2.9.5 Rotational kinetic energy

*Definition of Rotational kinetic energy.*

$K_{\text{rot}} = \frac{1}{2} I \omega^2$

*Definition of Rotational work.*

$W = \int \tau d\theta = \Delta K$

2.9.6 Rolling

During rolling, the velocity of the center of mass is

$v_{\text{cm}} = R\omega$

in the direction of rolling. Add  $v_{\text{cm}} = R\omega$  to the tangential velocity,  $R\omega$  in the direction perpendicular to the radius.

When rolling at the maximum speed possible without slipping, the friction is

$f_s = \mu_s F_N$

Otherwise,

$f_s < \mu_s F_N$

2.9.7 Angular momentum

$\vec{L} = I\vec{\omega}$  for solid objects

$\vec{L} = \vec{r} \times \vec{p}$  for point masses

$L = rp \sin(\theta) = rmv \sin(\theta)$  for point masses

Note: angular velocity’s direction is specified by right-hand rule.

The derivative of angular momentum is the change in external net torque.

$\frac{d\vec{L}}{dt} = \vec{\tau}_{\text{net, ext}}$

2.9.8 Equilibrium

If  $\sum F = 0$  and  $\sum \tau = 0$ , then the object is in equilibrium and therefore is not accelerating.

If  $\sum F = 0$  and  $\sum \tau \neq 0$ , then the object is accelerating in its rotation but has constant velocity.

If  $\sum F \neq 0$  and  $\sum \tau = 0$ , then the object is linearly accelerating but is not accelerating rotationally.

If  $\sum F \neq 0$  and  $\sum \tau \neq 0$ , then the object is in complex motion.

2.10 Gravitation

Using the equations which are similar to electrostatics:

*Definition of Gravitational constant.*

$G \approx 6.67 * 10^{-11} \frac{\text{Nm}^2}{\text{kg}^2}$

*Definition of Gravitational field.* The vector field  $g$ , which applies to all objects.

The electric field points towards masses.

$g$  is also the acceleration caused by gravity.

**Newton’s law of universal gravitation.** If stationary point or spherical masses  $m_1$  and  $m_2$  are near each other but not touching, then the magnitude of the gravitational force between the two objects is

$|F_g| = \frac{Gm_1m_2}{r^2}$

**Law.** The gravitational force on a stationary point or spherical mass  $g$  is  $F_g = mg$ , where  $g$  is the strength and direction of the gravitational field.

**Law (by Newton’s law of universal gravitation).** The gravitational field at distance  $r$  due to a stationary point or spherical charge  $m$  is

$g = \frac{GM}{r^2}.$

The gravitational field at distance  $r$  due to a massive object whose total mass is  $\int dm$  is

$g = \int \frac{G dm}{r^2}$

where  $r$ ’s tail is at the mass and points towards the point for which we want to calculate  $g$ .

2.10.1 Gravitational flux

*Definition of Gravitational flux ( $\Phi$ ).* The flux through a surface with area vector  $\vec{A}$  is

$\Phi = \int \vec{g} \cdot d\vec{A}$

where  $\vec{g}$  is the gravitational field that passes through the surface, and the area vector’s magnitude is the area and direction is normal/perpendicular to the surface.

*Definition of Net flux.* The flux through any surface that encloses a charge.

**Gauss’s Law.** The net flux is proportional to the enclosed mass, and can be equated to the flux through the surface:

$\Phi_{\text{net}} = -4\pi GM = \oint \vec{g} \cdot d\vec{A}$

# Electromagnetism (3)

## 3.1 Electrostatics

*Definition of Electric charge.* A scalar quantity that can describe an object. It can be positive or negative

*Definition of Elementary charge ( $e$ ).* The smallest possible charge  $e \approx 1.6 * 10^{-19} \text{ C}$ .

### 3.1.1 Charging

**Procedure (Charging by friction).** Rub two nonconductive objects together. The one with the greater electron affinity (electronegativity) becomes negatively charged.

**Procedure (Polarization).** When two objects near each other, their charge distributions will change in order to ensure that similar charges in the different objects don't get too near each other.

### 3.1.2 Electric fields

*Definition of Permittivity of free space ( $\epsilon_0$ ).*

$$\epsilon_0 \approx 8.85 * 10^{-12} \frac{\text{C}^2}{\text{Nm}^2}$$

*Definition of Coulomb's constant ( $k$ ).*

$$k = \frac{1}{4\pi\epsilon_0} \approx 8.99 * 10^9 \frac{\text{Nm}^2}{\text{C}^2}$$

**Coulomb's Law.** If stationary point or spherical charges  $q_1$  and  $q_2$  are near each other but not touching, then the magnitude of the electrostatic force between the two objects is

$$|F_e| = \frac{k|q_1||q_2|}{r^2} = \frac{|q_1||q_2|}{4\pi\epsilon_0 r^2}$$

*Definition of Electric field.* The vector field  $E \left( \frac{\text{N}}{\text{C}} \right)$ , which applies to all objects.

The electric field points away from positive charges and towards negative charges.

**Law.** The electrostatic force on a stationary point or spherical charge  $q$  is  $F_e = Eq$ , where  $E$  is the strength and direction of the electric field.

**Law (by Coulomb's Law).** The electric field at distance  $r$  due to a stationary point or spherical charge  $Q$  is

$$E = \frac{kQ}{r^2} = \frac{Q}{4\pi\epsilon_0 r^2}$$

The electric field at distance  $r$  due to a charged object whose total charge is  $\int dq$  is

$$E = \int \frac{k dq}{r^2} = \int \frac{dq}{4\pi\epsilon_0 r^2}$$

where  $r$ 's tail is at the charge and points towards the point for which we want to calculate  $E$ .

*Definition of Charge density.*

3D: density	$\rho = \frac{Q}{v} = \frac{dq}{dV}$	$\left( \frac{\text{C}}{\text{m}^3} \right)$
2D: surface density	$\sigma = \frac{Q}{A} = \frac{dq}{dA}$	$\left( \frac{\text{C}}{\text{m}^2} \right)$
1D: linear density	$\lambda = \frac{Q}{L} = \frac{dq}{dx}$	$\left( \frac{\text{C}}{\text{m}} \right)$

### 3.1.3 Electric flux

*Definition of Electric flux ( $\Phi$ ).* The flux through a surface with area vector  $\vec{A}$  is

$$\Phi = \int \vec{E} \cdot d\vec{A} \quad \left( \frac{\text{Nm}^2}{\text{C}} \right)$$

where  $\vec{E}$  is the electric field that passes through the surface, and the area vector's magnitude is the area and direction is normal/perpendicular to the surface.

*Definition of Net flux.* The flux through any surface that encloses a charge.

**Gauss's Law.** The net flux is equivalent to the enclosed charge divided by the permittivity of free space, and can be equated to the flux through the surface:

$$\Phi_{\text{net}} = \frac{q}{\epsilon_0} = 4\pi kq = \oint \vec{E} \cdot d\vec{A}$$

Mathematics (4)

4.1 Logarithms

$\log_b(MN) = \log_b(M) + \log_b(N)$   
 $\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)$   
 $\log_b(M^p) = p \cdot \log_b(M)$   
 $\log_b(a) = \frac{\log_x(a)}{\log_x(b)}$   
 $\log_b(b) = 1$

4.2 Notation

deg  $p(x)$  means the degree of polynomial  $p$ .

LC  $p(x)$  means the leading coefficient of polynomial  $p$ .

4.3 Rational functions

For a rational function  $f(x) = \frac{p(x)}{q(x)}$ , cancel out any common factors, then:

- For all rational functions:
  - VA: roots of  $q(x)$
  - Roots: roots of  $p(x)$
- When  $\deg p(x) = \deg q(x)$ :
  - HA:  $y = \frac{\text{LC } p(x)}{\text{LC } q(x)}$
- When  $\deg p(x) < \deg q(x)$ :
  - HA:  $y = 0$
- When  $\deg p(x) > \deg q(x)$ :
  - HA: none
  - slant asymptote:  $\frac{p(x)}{q(x)}$  excluding remainder

4.4 Polynomials

4.4.1 Linear equations

Slope-intercept form:  $y = mx + b$   
Point-slope form:  $y - y_1 = m(x - x_1)$  for point  $(x, y)$   
Standard form:  $ax + by = c$

4.4.2 Quadratic equations

Standard form:  $y = ax^2 + bx + c$   
Vertex form:  $y = a(x - h)^2 + k$  for vertex  $(h, k)$   
Sum of roots:  $-\frac{b}{a}$   
Product of roots:  $\frac{c}{a}$

4.4.3 Higher-degree polynomials

In a polynomial

$a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0 = 0$

, with roots

$r_1, r_2, r_3, \dots, r_n$

then:

$r_1 + r_2 + r_3 + \dots + r_n = \sum_{k=1}^n nr_k = -\frac{a_{n-1}}{a_n}$

4.5 Sequences and Series

4.5.1 Explicit formulas

Aritmetic sequence:  $a_n = a_1 + r(n - 1)$   
Geometric sequence:  $a_n = a_1 * r^{n-1}$   
Harmonic sequence:  $a_n = \frac{1}{a_1 + r(n - 1)}$

4.5.2 Arithmetic and Geometric Series

In the following equations, substituting  $j = 1$  with  $j = 0$ ,  $j - 1$  with  $j$ , and  $a_1$  with  $a_0$  will produce the same result.

$\sum_{j=1}^n (a_1 + r(j - 1)) = \frac{n}{2}(2a_1 + (n - 1)d)$   
 $\sum_{j=1}^n (a_1 * r^{j-1}) = \frac{a_1(1 - r^n)}{1 - r}$   
 $\sum_{j=1}^\infty (a_1 * r^{j-1}) = \frac{a_1}{1 - r}$  for  $r \in [-1, 1]$

4.5.3 Special Sums

$\sum_{j=1}^n c = nc$                        $\sum_{j=1}^n ca_j = c \sum_{j=1}^n a_j$   
 $\sum_{j=1}^n (a_j + b_j) = \sum_{j=1}^n a_j + \sum_{j=1}^n b_j$                        $\sum_{j=1}^n j = \frac{n(n + 1)}{2}$   
 $\sum_{j=1}^n j^2 = \frac{n(n + \frac{1}{2})(n + 1)}{3}$                        $\sum_{j=1}^n j^3 = \frac{n^2(n + 1)^2}{4}$   
 $= \frac{n(2n + 1)(n + 1)}{6}$

4.6 Trigonometry

°	rad	sin	cos	tan
0°	0	0	1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	$\frac{\pi}{2}$	1	0	undef

4.6.1 Law of Sines and Cosines

$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$                        $c^2 = a^2 + b^2 - 2ab \cos(C)$

4.6.2 Triangle area

$K = \frac{1}{2}bh$                        $K = \frac{1}{2}bc \sin(A)$                        $K = \sqrt{s(s - a)(s - b)(s - c)}$



Calculus Theorems (5)

1 Completeness

**Theorem (Completeness of the Real Numbers).** Every nonempty subset  $S$  of  $\mathbb{R}$  which is bounded above has a least upper bound  $\sup S$ .

*Definition of Supremum* ( $\sup S$ ). A number such that

- (1)  $s \leq \sup S$  for every  $s \in S$  (which just says that  $\sup S$  is an upper bound for  $S$ )
- (2) If  $u$  is any upper bound for  $S$ , then  $\sup S \leq u$  (which says that  $\sup S$  is the least upper bound for  $S$ ).

*Definition of Infimum* ( $\inf S$ ). A number such that

- (1)  $\inf S \leq s$  for every  $s \in S$  (i.e.  $\inf S$  is an lower bound for  $S$ )
- (2) If  $l$  is any upper bound for  $S$ , then  $l \leq \inf S$  (i.e.  $\inf S$  is the greatest lower bound for  $S$ ).

**Theorem.** Every nonempty subset  $S$  of  $\mathbb{R}$  which is bounded below has a greatest lower bound.

**Theorem.** If  $\min S$  exists, then  $\min S = \inf S$ .

**Theorem.** If  $A \subset R$  and  $c \geq 0$ , and  $cA := ca : a \in A$ ,  $\sup cA = c \sup A$ .

**Theorem (Rationals between Reals).** For every two real numbers  $a$  and  $b$  with  $a < b$ , there exists a rational number  $r$  satisfying  $a < r < b$ .

**Nested Intervals Theorem.**

If  $I_n = [a_n, b_n] = \{x \in R : a_n \leq x \leq b_n\}$  s.t.  $a_n \leq a_{n+1}$  and  $b_{n+1} \leq b_n$  for  $n \in \mathbb{N}$ , so that  $I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \dots$ , then  $\bigcap_{n=1}^\infty I_n \neq \emptyset$ .

If  $\inf\{b_n - a_n\} = 0$ , then  $\bigcap_{n=1}^\infty I_n = \{x\}$ , where

$x = \sup\{a_n\} = \inf\{b_n\}.$

**Capture Theorem.** If  $A$  is a nonempty subset of  $\mathbb{R}$ , then:

- (i) If  $A$  is bounded above, then any open interval containing  $\sup A$  contains an element of  $A$ .
- (ii) Similarly, if  $A$  is bounded below, then any open interval containing  $\inf A$  contains an element of  $A$ .

**Theorem (Binary Search (Bisection Method)).** If we binary-search for  $x$  over  $I_1 = [a_1, b_1]$  for  $a_1, b_1 \in \mathbb{Q}$ , we define  $I_n$  s.t. either  $I_n := [a_{n-1}, \frac{a_{n-1}+b_{n-1}}{2}]$  or  $I_n := [\frac{a_{n-1}+b_{n-1}}{2}, a_{n-1}]$ , and we define  $a_n := \inf I_n$  and  $b_n := \sup I_n$ . We define  $A$  to be the set of all  $a_n$ , and  $B$  to be the set of all  $b_n$ .

Then, the size of  $I_n = \frac{b_1-a_1}{2^n} = b_n - a_n$ , and  $\bigcap_{n=1}^\infty I_n \{x\}$ , where

$x = \sup\{a_n\} = \inf\{b_n\}.$

2 Limits

*Definition of Limit.* If  $\lim_{x \rightarrow a} f(x) = L$ , then for any  $\varepsilon > 0$ , there exists  $\delta > 0$  s.t. for any  $x \in (a - \delta, a) \cup (a, a + \delta)$ ,  $f(x) \in (L - \varepsilon, L + \varepsilon)$ .

Alternatively,

*Definition of Limit.* If  $\lim_{x \rightarrow a} f(x) = L$ , then for any  $\varepsilon > 0$ , there exists  $\delta > 0$  s.t.  $|f(x) - L| < \varepsilon$  whenever  $0 < |x - a| < \delta$ .

*Definition of Right-sided limit.* If  $\lim_{x \rightarrow a^+} f(x) = L$ , then for any  $\varepsilon > 0$ , there exists  $\delta > 0$  s.t.  $|f(x) - L| < \varepsilon$  whenever  $0 < x - a < \delta$ .

*Definition of Left-sided limit.* If  $\lim_{x \rightarrow a^-} f(x) = L$ , then for any  $\varepsilon > 0$ , there exists  $\delta > 0$  s.t.  $|f(x) - L| < \varepsilon$  whenever  $0 < a - x < \delta$ .

**Theorem (Limit Laws).** Let  $c \in R$  be a constant and suppose the limits  $\lim_{x \rightarrow a} f(x)$  and  $\lim_{x \rightarrow a} g(x)$  exist. Then

- $\lim_{x \rightarrow a} (f(x) \pm g(x)) = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$
- $\lim_{x \rightarrow a} (cf(x)) = c \lim_{x \rightarrow a} f(x)$
- $\lim_{x \rightarrow a} (f(x)g(x)) = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x)$
- $\lim_{x \rightarrow a} f(x)g(x) = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x)$  , provided that  $\lim_{x \rightarrow a} g(x) \neq 0$
- $\lim_{x \rightarrow a} x^n = (\lim_{x \rightarrow a} x)^n$
- $\lim_{x \rightarrow a} \frac{a(x)b(x)}{c(x)b(x)} = \lim_{x \rightarrow a} \frac{a(x)}{c(x)}$

These laws also apply to one-sided limits.

**Theorem (L’Hopital’s Rule).** If  $f$  and  $g$  are differentiable and  $g'(x) \neq 0$  on an open interval  $I$  that surrounds  $a$ , and  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} \in \{ \frac{0}{0}, \pm \frac{\infty}{\infty} \}$ , then  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ .

**Theorem (Composition of Limits).** If  $f$  is continuous at  $L$  and  $\lim_{x \rightarrow a} g(x) = L$ , then  $\lim_{x \rightarrow a} f(g(x)) = f(\lim_{x \rightarrow a} g(x)) = f(L)$

**Theorem (Operations on infinity).** For  $x \in \mathbb{R}$ ,

$$\begin{aligned} \infty + x &= \infty & -\infty + x &= -\infty & \frac{x}{\pm\infty} &= 0 \end{aligned}$$
$$x * \infty = \begin{cases} \infty & \text{if } x > 0 \\ -\infty & \text{if } x < 0 \end{cases}$$

$$x * -\infty = \begin{cases} -\infty & \text{if } x > 0 \\ \infty & \text{if } x < 0. \end{cases}$$

*Definition of Indeterminate forms.* The following forms are indeterminate and you cannot evaluate them.

$$\frac{0}{0}, \frac{\pm\infty}{\pm\infty}, 0 * \pm\infty, \infty - \infty$$

**Squeeze Theorem.** Let  $f$  ,  $g$ , and  $h$  be defined for all  $x \neq a$  over an open interval containing  $a$ . If

$$f(x) \leq g(x) \leq h(x)$$

for all  $x \neq a$  in an open interval containing  $a$  and

$$\lim_{x \rightarrow a} f(x) = L = \lim_{x \rightarrow a} h(x)$$

where  $L \in \mathbb{R}$ , then  $\lim_{x \rightarrow a} g(x) = L$ .

3 Continuity

*Definition of Continuity at a point.* Function  $f$  is continuous at point  $a$  if  $\lim_{x \rightarrow a} f(x) = f(a)$ .

*Definition.*  $f$  has a **removable discontinuity** if  $\lim_{x \rightarrow a} f(x) = L \in \mathbb{R}$  (in this case either  $f(a)$  is undefined, or  $f(a)$  is defined by  $L \neq f(a)$ ).

*Definition.*  $f$  has a **jump discontinuity** if  $\lim_{x \rightarrow a^-} f(x) = L_1 \in \mathbb{R}$  and  $\lim_{x \rightarrow a^+} f(x) = L_2 \in \mathbb{R}$  but  $L_1 \neq L_2$ .

*Definition.*  $f$  has an **infinite discontinuity** at  $a$  if  $\lim_{x \rightarrow a^-} f(x) = \pm\infty$  or  $\lim_{x \rightarrow a^+} f(x) = \pm\infty$

**Intermediate Value Theorem.** If  $f$  is continuous on  $[a, b]$ , then for any real number  $L$  between  $f(a)$  and  $f(b)$  there exists at least one  $c \in [a, b]$  such that  $f(c) = L$ . In other words, if  $f$  is continuous on  $[a, b]$ , then the graph must cross the horizontal line  $y = L$  at least once between the vertical lines  $x = a$  and  $x = b$ .

**Aura Theorem.** If  $f(x)$  is continuous and  $f(a)$  is positive, then there exists an open interval containing  $a$  such that for all  $x$  in the interval,  $f(x)$  is positive.

If  $f(x)$  is continuous and  $f(a)$  is negative, then there exists an open interval containing  $a$  such that for all  $x$  in the interval,  $f(x)$  is negative.

**Bolzano’s Theorem.** Let  $f$  be a continuous function defined on  $[a, b]$ . If 0 is between  $f(a)$  and  $f(b)$ , then there exists  $x \in [a, b]$  such that  $f(x) = 0$ .



4 Derivatives

The derivative is the instantaneous rate of change, and the slope of the tangent line to the point.

Definition of Derivative  $(f'(a))$ .

$$\frac{d}{da}f(a) = f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

Theorem (Tangent line to a point). The equation of the tangent line to the point  $(a, f(a))$  is

$$y = f'(a)(x - a) + f(a)$$

Derivative Rules

Theorem (Difference Rule).

$$\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

Theorem (Sum Rule).

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

Theorem (Constant Multiple Rule).

$$\frac{d}{dx}(cf(x)) = c \frac{d}{dx}f(x)$$

Theorem (Product Rule).

$$\begin{aligned} \frac{d}{dx}(f(x)g(x)) &= f'(x)g(x) + f(x)g'(x) \\ \frac{d}{dx}(f(x)g(x)h(x)) &= f'(x)g(x)h(x) + f(x)g'(x)h(x) \\ &\quad + f(x)g(x)h'(x) \end{aligned}$$

and so on.

Theorem (Quotient Rule).

$$\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

Theorem (Power Rule).

$$\frac{d}{dx}x^n = nx^{n-1}$$

for  $n \in \mathbb{R}$

Theorem (Chain Rule).

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x) \qquad \frac{dy}{dx} = \frac{dy}{db} \frac{db}{dx}$$

Theorem (Derivative of inverse functions). Let  $x \in \mathbb{R}$  and  $f$  be a differentiable, one-to-one function at  $x$ . Then if  $f'(x) \neq 0$ , then

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}$$

Theorem (Derivatives of exponentials and logs).

$$\begin{aligned} \frac{d}{dx}e^x &= e^x & \frac{d}{dx}\ln x &= \frac{1}{x} \\ \frac{d}{dx}a^x &= a^x \ln(a) & \frac{d}{dx}\log_a x &= \frac{1}{x \ln(a)} \end{aligned}$$

Theorem (Derivatives of trig functions).

Warning:  $x$  must be an angle in radians!

$$\begin{aligned} \sin'(x) &= \cos(x) & \cos'(x) &= -\sin(x) \\ \sec'(x) &= \sec(x)\tan(x) & \csc'(x) &= -\csc(x)\cot(x) \\ \tan'(x) &= \sec(x)^2 & \cot'(x) &= -\csc(x)^2 \end{aligned}$$

$$\begin{aligned} \arcsin'(x) &= \frac{1}{\sqrt{1-x^2}} & \arccos'(x) &= -\frac{1}{\sqrt{1-x^2}} \\ \operatorname{arcsec}'(x) &= \frac{1}{|x|\sqrt{x^2-1}} & \operatorname{arccsc}'(x) &= -\frac{1}{|x|\sqrt{x^2-1}} \\ \arctan'(x) &= \frac{1}{1+x^2} & \operatorname{arccot}'(x) &= -\frac{1}{1+x^2} \end{aligned}$$

5 Derivative Applications

5.7 Mean Value Theorem

Theorem (Mean Value Theorem). If the function  $f$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ , then there exists  $c \in (a, b)$  s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{\Delta f(x)}{\Delta x} \text{ on } [a, be]$$

Theorem (Some colloraries to the MVT). If  $f(x)$  is differentiable on  $I$ , then:

- $f'(x) > 0$  for  $x \in I \iff f(x)$  is strictly increasing for  $x \in I$ .
- $f'(x) \geq 0$  for  $x \in I \iff f(x)$  is increasing or constant for  $x \in I$ .
- $f'(x) = 0$  for  $x \in I \iff f(x)$  is constant for  $x \in I$ .
- $f'(x) \leq 0$  for  $x \in I \iff f(x)$  is decreasing or constant for  $x \in I$ .
- $f'(x) < 0$  for  $x \in I \iff f(x)$  is strictly decreasing for  $x \in I$ .

5.3, 5.10, 5.11, 5.16 Extrema

Definition of Critical point of  $f$ . A number  $c$  in the domain of  $f$  where  $f'(c) = 0$  or  $f'(c)$  does not exist.

Definition of Stationary point of  $f$ . A number  $c$  in the domain of  $f$  where  $f'(c) = 0$

Fermat's Theorem. The local maxima and minima of  $f$  are critical points of  $f$ .

Exteme Value Theorem. If  $f$  is continuous on  $[a, b]$ , then it has an absolute max and an absolute min.

Theorem (Method to find absolute minima and maxima). Store the critical points of  $f$  in the array  $C$ . Then, the absolute maximum is  $\max\{f(c) : c \in C\}$  and the absolute minimum is  $\min\{f(c) : c \in C\}$ .

Theorem (First Derivative Test). If  $f$  is continuous over  $I$ , and  $c \in I$  is a critical point of  $f$ , and  $f$  is differentiable over  $I \setminus c$ , then:

- If  $f'(x)$  is decreasing at  $c$ , then  $f(c)$  is a local max.
- If  $f'(x)$  is increasing at  $c$ , then  $f(c)$  is a local min.
- If  $f'(x)$  has the same sign before and after  $c$ , then  $f(c)$  is neither a local max nor a local min.

Definition of Concavity.  $f$  is concave up on  $I$  if the tangent line to  $f$  at each point in  $I$  is lower than the graph of  $f$ .

$f$  is concave down on  $I$  if the tangent line to  $f$  at each point in  $I$  is higher than the graph of  $f$ .

Theorem (Test for Concavity). If  $f''(x) > 0$  for all  $x \in I$ , then  $f$  is concave up on  $I$ .

If  $f''(x) < 0$  for all  $x \in I$ , then  $f$  is concave down on  $I$ .

Theorem (Second Derivative Test). If  $f''$  is continuous on an interval containing  $c$ , where  $c$  is the  $x$ -value of a stationary point of  $f$ . Then,

- If  $f''(c) > 0$ , then  $f(c)$  is a local max.
- If  $f''(c) < 0$ , then  $f(c)$  is a local min.

Trimm's Single Extremum Theorem. If  $f$  is continuous on an interval  $I$ , and  $f$  has a single local extremum (max or min), then that extremum is a global max or min.

Miscellaneous

Definition of Exponential function.

$$\exp x = e^x = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n$$

6 Integrals  
Antiderivative

*Definition of Antiderivative / Indefinite Integral.* The antiderivative  $F$  of a function  $f$  is the function such that  $F'(x) = f(x)$ .

$$F(x) = \int f(x)dx$$

**Theorem (Antiderivative plus a constant).** If  $F$  is the antiderivative of a function  $f$ , then  $G(x) = F(x) + c$  where  $c \in \mathbb{R}$  is also an antiderivative.

*Definition of Integral.*

$$(f_1[a, b]) \mapsto \int_a^b f(x)dx \in \mathbb{R}$$

such that the Properties of the Integral are true.

The definite integral takes in a function and a range  $[a, b]$ , and returns a number. The indefinite integral takes in a function and returns an infinitely large set of functions (the antiderivatives).

If  $a > b$ , then  $\int_a^b f := -\int_b^a f$ .

Let  $\mathcal{R}([a, b])$  be the set of integrable functions,  $\mathcal{C}([a, b])$  be the set of continuous functions, and  $\mathcal{B}([a, b])$  be the set of bounded functions on  $[a, b]$ . Then

$$\mathcal{C}([a, b]) \subset \mathcal{R}([a, b]) \subset \mathcal{B}([a, b])$$

**Theorem (Properties of the Integral).** The integral is defined such that the following are true:

(I0) Every continuous function is integrable.

(I1) If  $f(x) = c$ , then  $\int_a^b f(x)dx = c(b - a)$

(I2) If  $f_1(x) \leq f_2(x)$ , then  $\int_a^b f_1(x)dx \leq \int_a^b f_2(x)dx$ .

(I3) For any  $a, b, c$ ,  $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ .

**Theorem (Fundamental Theorem of Calculus).** Let  $f \in \mathcal{R}([a, b])$  be some integrable function, where  $a, b \in \mathbb{R}$ . Let  $\mathcal{F}(x) = \int_a^x f$  for  $x \in [a, b]$ . Then:

- (a)  $\mathcal{F}$  is continuous for every  $c \in [a, b]$ .
- (b) If  $f$  is continuous at  $c \in [a, b]$ , then  $\mathcal{F}$  is diffentiable at  $c$ , and  $\mathcal{F}'(c) = f(c)$ .
- (c) If  $f$  is continuous on  $[a, b]$ , and  $F$  is an antiderivative of  $f$ , then  $\int_a^b f = F(b) - F(a)$ , or

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

**Theorem (Substitution Rule).** If  $g$  is a function that has a continuous derivative on an interval, another function  $f$  is continuous on the range of  $g$ , and  $F$  is an antiderivative of  $f$  on the range of  $g$ , then

$$\int f(g(x))g'(x)dx = F(g(x)) + C$$

For  $a, b \in \mathbb{R}$ ,

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du = F(g(b)) - F(g(a))$$

Let  $u := g(x)$ . Then, if  $du = E(g'(x)dx)$ , where  $E \in \mathbb{R}$ , then

$$\int Ef(u)du = E \int f(u)du = F(u) + C$$

**Theorem (Some Antiderivative Rules).**

$$\begin{aligned} \int e^x dx &= e^x + C & \int x^a dx &= \frac{x^{a+1}}{a+1} + C \text{ for } a \neq -1 \\ \int a^x dx &= \frac{a^x}{\ln(a)} + C & \int x^{-1} dx &= \ln|x| + C \end{aligned}$$

$$\int f(x) + g(x)dx = \int f(x)dx + \int g(x)dx$$

**Theorem (Integration By Parts).** If  $f$  and  $g$  are integrable functions, then

$$\int f(x)g'(x)dx = f(x)g(x) + \int f(x)g(x)dx$$

Equivalently, if  $u$  and  $v$  are integrable functions of  $x$ , then

$$\int u dv = uv - \int v du$$

Additionally, if  $f'$  and  $g'$  are continuous, then

$$\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) + \int_a^b g(x)f'(x)dx$$

**Theorem (Integrals of Trig Functions).**

$$\int \sin(x)dx = -\cos(x) + C \quad \int \cos(x)dx = \sin(x) + C$$

$$\int \tan(x)dx = -\ln|\cos(x)| + C = \ln|\sec(x)| + C$$

$$\int \cot(x)dx = \ln|\sin(x)| + C = -\ln|\csc(x)| + C$$

$$\int \sec(x)dx = \ln|\sec(x) + \tan(x)| + C$$

$$\int \csc(x)dx = -\ln|\cot(x) + \csc(x)| + C$$

**Procedure (Integrals of Powers of Trig Functions).**

To solve  $\int \sin^n(x) dx$  where  $n$  is even, substitute  $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$ .

To solve  $\int \sin^n(x) dx$  where  $n$  is odd, substitute  $\sin^2(x) = 1 - \cos^2(x)$  and perform u-sub with  $u := \cos^2(x)$  and  $du = -\sin(x) dx$ .

To solve  $\int \cos^n(x) dx$  where  $n$  is even, substitute  $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$ .

To solve  $\int \cos^n(x) dx$  where  $n$  is odd, substitute  $\cos^2(x) = 1 - \sin^2(x)$  and perform u-sub with  $u := \sin^2(x)$  and  $du = \cos(x) dx$ .

To solve  $\int \tan^n(x) dx$ , substitute  $\tan^2(x) = \sec^2(x) - 1$ . If  $n$  is odd, perform u-sub with  $u := \sec(x)$  and  $du = \sec(x) \tan(x) dx$ .

To solve  $\int \cot^n(x) dx$ , substitute  $\cot^2(x) = \csc^2(x) - 1$ . If  $n$  is odd, perform u-sub with  $u := \csc(x)$  and  $du = -\csc(x) \cot(x) dx$ .

To solve  $\int \sec^n(x) dx$  where  $n$  is even, substitute  $\sec^2(x) = \tan^2(x) + 1$ , but ensure that  $\sec^2(x) dx$  remains. Then perform u-sub with  $u := \tan(x)$  and  $du = \sec^2(x) dx$ .

To solve  $\int \sec^n(x) dx$  where  $n$  is odd, substitute  $\sec^n(x) = \frac{1}{\cos^n(x)} \frac{\cos(x)}{\cos(x)}$ . Then substitute  $\cos^2(x) = (1 + u)(1 - u)$  and u-sub with  $u := \sin(x)$  and  $du = \cos(x)dx$ . Then perform partial fraction decomposition.

To solve  $\int \csc^n(x) dx$  where  $n$  is even, substitute  $\csc^2(x) = \csc^2(x) + 1$ , but ensure that  $\sec^2(x) dx$  remains. Then perform u-sub with  $u := \cot(x)$  and  $du = -\csc^2(x) dx$ .

To solve  $\int \csc^n(x) dx$  where  $n$  is odd, substitute  $\csc^n(x) = \frac{1}{\sin^n(x)} \frac{\sin(x)}{\sin(x)}$ . Then substitute  $\sin^2(x) = (1 + u)(1 - u)$  and u-sub with  $u := \cos(x)$  and  $du = -\sin(x)dx$ . Then perform partial fraction decomposition.

Remember that  $\int \sec^2(x) dx = \tan(x)$  and  $\int \csc^2(x) dx = -\cot(x)$ .

**Procedure (Trig Substitution).**

Orig expression	Substitution	Pythagorean identity
$\sqrt{a^2 - x^2}$	$x := a \sin(\theta)$	$1 - \sin^2(\theta) = \cos^2(\theta)$
$\sqrt{a^2 + x^2}$	$x := a \tan(\theta)$	$1 + \tan^2(\theta) = \sec^2(\theta)$
$\sqrt{x^2 - a^2}$	$x := a \sec(\theta)$	$\sec^2(\theta) - 1 = \tan^2(\theta)$

Partial Fraction Decomposition

**Theorem.** Any polynomial  $Q(x)$  with real coefficients can be factored over the reals as a product of two types of factors:

- linear factors (of the form  $ax + b$ )
- irreducible quadratic factors (of the form  $ax^2 + bxx + c$ , where  $b^2 - 4ac < 0$ )

**Definition of Proper rational function.** A rational function  $\frac{P(x)}{Q(x)}$  where  $\deg P < \deg Q$ .

**Theorem.** Any rational function can be converted into a proper rational function plus a polynomial by continually long-dividing by the denominator.

**Theorem.** Let  $R(x) = \frac{P(x)}{Q(x)}$  be a proper rational function, where the denominator  $Q(x)$  has been factored into linear and irreducible quadratic factors.  $R(x)$  can be written as a sum of partial fractions, where each factor in the denominator gives rise to terms in the partial fraction decomposition:

- For each factor of the form  $(ax + b)^k$  in the denominator, add  $\sum_{i=1}^k \frac{A_i}{(ax + b)^i}$  to the partial fraction decomposition.
- For each factor of the form  $(ax^2 + bxx + c)^k$  in the denominator, add  $\sum_{i=1}^k \frac{A_ix + B_i}{(ax^2 + bxx + c)^i}$  to the partial fraction decomposition.

Riemann Sums

**Definition of Elementary function.** A function which is a polynomial, rational function, power function ( $x^a$ ), exponential function ( $a^x$ ), logarithmic functions, trigonometric and inverse trigonometric functions, or an addition, subtraction, multiplication, division, and composition of the above.

**Definition of Riemann sum.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be any bounded function and let  $P$  be a partition of  $[a, b]$ .

- A choice of a point  $x_i^* \in [x_{i-1}, x_i]$  for all  $i \in [1, n]$  is called a *tagging* of  $P$ , which we denote by  $\tau = x_1^*, \dots, x_n^*$ .
- A pair  $(P, \tau)$  is called a *tagged partition*.
- Given a bounded function  $f : [a, b] \rightarrow \mathbb{R}$  and a tagged partition  $(P, \tau)$  of  $[a, b]$ , the sum

$$R(f, P, \tau) = \sum_{i=1}^n f(x_i^*)(x_i - x_{i-1})$$

is called the *Riemann sum* of  $f$  for  $(P, \tau)$ .

**Theorem.** If  $f$  is integrable on  $[a, b]$ , then for every  $\varepsilon > 0$  there exists  $\delta > 0$  s.t. if  $P$  is a partition of  $[a, b]$  with  $|P| := \max x_i - x_{i-1} < \delta$  and  $\tau = \{x_i^*\}$ , then

$$|R(f, P, \tau) - \int_a^b f(x)dx| < \varepsilon$$

Approximate Integration

**Definition of Left-endpoint approximation.** Take  $x_i^* = x_{i-1} = a + \frac{(i-1)(b-a)}{n}$ . Then the left-endpoint approximation of the function  $f(x)$  is

$$\int_a^b f(x)dx \approx \sum_{i=1}^n f\left(a + \frac{(i-1)(b-a)}{n}\right) \frac{b-a}{n}$$

The error bound is

$$E_n^L \leq \max\{|f'(x)|\}_{x \in [a, b]} \frac{(b-a)^2}{2n}$$

**Definition of Right-endpoint approximation.** Take  $x_i^* = x_{i-1} = a + \frac{(i)(b-a)}{n}$ . Then the right-endpoint approximation of the function  $f(x)$  is

$$\int_a^b f(x)dx \approx \sum_{i=1}^n f\left(a + \frac{(i)(b-a)}{n}\right) \frac{b-a}{n}$$

The error bound is

$$E_n^R \leq \max\{|f'(x)|\}_{x \in [a, b]} \frac{(b-a)^2}{2n}$$

**Definition of Midpoint approximation.** Take  $x_i^* = x_{i-1} = a + \frac{(i-\frac{1}{2})(b-a)}{n}$ . Then the midpoint approximation of the function  $f(x)$  is

$$\int_a^b f(x)dx \approx \sum_{i=1}^n f\left(a + \frac{(i-\frac{1}{2})(b-a)}{n}\right) \frac{b-a}{n}$$

The error bound is

$$E_n^M \leq \max\{|f''(x)|\}_{x \in [a, b]} \frac{(b-a)^3}{24n^2}$$

**Definition of Trapezoidal approximation.** Then the trapezoidal approximation of the function  $f(x)$  is

$$\int_a^b f(x)dx \approx \frac{b-a}{2n} \left( f(a) + 2 \sum_{i=1}^{n-1} f\left(a + \frac{(i)(b-a)}{n}\right) + f(b) \right)$$

The error bound is

$$E_n^T \leq \max\{|f''(x)|\}_{x \in [a, b]} \frac{(b-a)^3}{12n^2}$$

**Definition of Simpson's approximation.** For even  $n$  (greater values of  $n$  give more precise more accuracy):

$$\frac{b-a}{3n} \left( f(a) + \sum_{i=1}^{n-1} (3 - (-1)^i) f\left(a + \frac{(i)(b-a)}{n}\right) + f(b) \right)$$

The error bound is

$$E_n^S \leq \max\{|f'''(x)|\}_{x \in [a, b]} \frac{(b-a)^5}{180n^4}$$

8 Integral Applications

Volumes

Generally, if  $A(x)$  is the cross-section of a solid that intersects the  $x$ -axis at  $x$ , then the volume of the solid is

$$V = \int_a^b A(x)dx$$

**Theorem (Disk Method).** Let  $f(x)$  be a continuous, nonnegative function defined on  $[a, b]$ , and  $R$  be the region bounded above by the graph of  $f(x)$  and below by the  $x$ -axis. Then, the volume of the solid of revolution formed by revolving  $R$  around the  $x$ -axis is given by

$$V = \int_a^b \pi(f(x))^2 dx$$

**Theorem (Washer Method).** Let  $f(x)$  be a continuous, nonnegative function defined on  $[a, b]$ , and  $R$  be the region bounded above by the graph of  $f(x)$  and below by the graph of  $g(x)$ . Then, the volume of the solid of revolution formed by revolving  $R$  around the  $x$ -axis is given by

$$V = \int_a^b \pi((f(x))^2 - (g(x))^2) dx$$

**Theorem (Cylindrical Shells Method).** Let  $f(x)$  be a continuous, nonnegative function defined on  $[a, b]$ , and  $R$  be the region bounded above by the graph of  $f(x)$  and below by the  $x$ -axis. Then, the volume of the solid of revolution formed by revolving  $R$  around the  $y$ -axis is given by

$$V = \int_a^b 2\pi x f(x) dx$$

Other things

**Theorem (Arc length of a curve).** The arc length of the curve  $f(x)$  on  $[a, b]$ , when  $f'(x)$  exists and is continuous on  $[a, b]$ , is

$$\int_a^b \sqrt{1 + (f'(x))^2} dx$$

**Theorem (Surface area of a solid of revolution).** Let  $f(x)$  be a continuous, nonnegative function defined on  $[a, b]$ , and  $R$  be the region bounded above by the graph of  $f(x)$  and below by the  $x$ -axis. Then, the surface area of the solid of revolution formed by revolving  $R$  around the  $x$ -axis is given by

$$S = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} dx$$

**Theorem (Mass of a thin rod).** Let there be a rod whose left end is located at  $x = a$  and whose right end is located at  $x = b$ , and which is on the  $x$ -axis. Let  $\rho(x)$  be the linear density at the point  $x = a$ . Then the mass of the rod is

$$M = \int_a^b \rho(x) dx$$

**Theorem (Mass of a thin disk).** Let there be a disk of radius  $R$  whose center is at the origin of the  $xy$ -plane. Let the mass be distributed in a rotationally-symmetric way. Let  $\rho(r)$  be the radial density at the radius  $r$ . Then the mass of the disk is

$$M = \int_0^R 2\pi r \rho(r) dr$$

**Theorem (Work).** If an object moves along the  $x$ -axis from  $a$  to  $b$ , and  $F(x)$  is the force applied to the object when the object is at the point  $x$  on the  $x$ -axis, then the work is

$$\int_a^b F(x) dx$$

**Definition of Average value of a function.** If  $f$  is continuous on  $[a, b]$ , then the average value of  $f$  on  $[a, b]$  is

$$f_{\text{avg}} := \lim_{n \rightarrow \infty} \frac{1}{b-a} \int_a^b f(x) dx$$

**Theorem (Mean Value Theorem for Integrals).** If  $f$  is continuous on  $[a, b]$ , then there exists  $c \in [a, b]$  such that

$$f(c) = f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx$$

Equivalently,

$$\int_a^b f(x) dx = f(c)(b-a)$$

If  $f$  is positive on  $[a, b]$ , then there is a number  $c$  such that the rectangle with base  $[a, b]$  and height  $f(c)$  has the same area as  $\int_a^b f(x) dx$ .

9 Improper Integrals

**Definition of Improper Integrals with Infinite Bounds.**

- (a) If  $\int_a^t f(x) dx$  exists for every  $t \geq a$ , then we define

$$\int_a^\infty f(x) dx := \lim_{t \rightarrow \infty} \int_a^t f(x) dx$$

provided that this limit exists and is finite.

- (b) If  $\int_t^b f(x) dx$  exists for every  $t \leq b$ , then we define

$$\int_{-\infty}^b f(x) dx := \lim_{t \rightarrow -\infty} \int_t^b f(x) dx$$

provided that this limit exists and is finite.

$\int_a^\infty f(x) dx$  and  $\int_{-\infty}^b f(x) dx$  are *convergent* if the corresponding limit exists and *divergent* if the limit doesn't exist.

- (c) If both  $\int_a^\infty f(x) dx$  and  $\int_{-\infty}^b f(x) dx$  are convergent, then we define

$$\begin{aligned} \int_{-\infty}^\infty f(x) dx &= \int_{-\infty}^a f(x) dx + \int_a^\infty f(x) dx \\ &= \lim_{s \rightarrow -\infty} \int_s^a f(x) dx + \lim_{t \rightarrow \infty} \int_a^t f(x) dx \end{aligned}$$

**Definition of Improper Integrals with Discontinuous Integrand.**

- (a) If  $f$  is continuous on  $[a, b)$  and is discontinuous at  $b$ , then define

$$\int_a^b f(x) dx := \lim_{t \rightarrow b^-} \int_a^t f(x) dx$$

provided that this limit exists and is finite.

- (b) If  $f$  is continuous on  $(a, b]$  and is discontinuous at  $a$ , then define

$$\int_a^b f(x) dx := \lim_{t \rightarrow a^+} \int_t^b f(x) dx$$

provided that this limit exists and is finite.

$\int_a^\infty f(x) dx$  and  $\int_{-\infty}^b f(x) dx$  are *convergent* if the corresponding limit exists and *divergent* if the limit doesn't exist.

- (c) If  $f$  has a discontinuity at  $c$ , where  $a < c < b$ , and both  $\int_a^c f(x) dx$  and  $\int_c^b f(x) dx$  are convergent, then we define

$$\begin{aligned} \int_a^b f(x) dx &= \int_a^c f(x) dx + \int_c^b f(x) dx \\ &= \lim_{s \rightarrow c^-} \int_a^s f(x) dx + \lim_{t \rightarrow c^+} \int_t^b f(x) dx \end{aligned}$$

**Theorem (Comparison Test).** Suppose  $f$  and  $g$  are continuous functions with  $f(x) \geq g(x) \geq 0$  for  $x \geq a$ .

- (a) If  $\int_a^\infty f(x) dx$  is convergent, then  $\int_a^\infty g(x) dx$  is convergent.
- (b) If  $\int_a^\infty g(x) dx$  is divergent, then  $\int_a^\infty f(x) dx$  is divergent.

Additionally,

- (a) If  $\int_{-\infty}^b f(x) dx$  is convergent, then  $\int_{-\infty}^b g(x) dx$  is convergent.
- (b) If  $\int_{-\infty}^b g(x) dx$  is divergent, then  $\int_{-\infty}^b f(x) dx$  is divergent.

**Theorem (Limit Comparison Test).** Suppose  $f(x)$  and  $g(x)$  are positive continuous functions defined on  $[a, \infty)$  such that  $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = c$  for some positive real number  $c$ . Then  $\int_a^\infty f(x) dx$  converges iff  $\int_a^\infty g(x) dx$ .

10 Differential Equations

**Definition of Differential equation.** An equation involving an unknown function  $y = f(x)$  and one or more of its derivatives.

**Definition of Solution to a differential equation.** A function  $y = f(x)$  that satisfies the differential equation when  $f$  and its derivatives are substituted into the equation.

**Procedure (Euler's Method).** To numerically approximate the solution to the differential equation  $y' = F(x, y)$  with  $y(x_0) = y_0$ ,

$$y_n = y_{n-1} + F(x_{n-1}, y_{n-1})(x_n - x_{n-1})$$

**Definition of Separable Equation.** A separable equation is a differential equation where

$$\frac{dy}{dx} = g(x)f(y)$$

for some function  $g(x)$  which depends only on  $x$  and  $f(y)$  which depends only on  $y$ .

**Theorem.** For a separable equation,

$$\int \frac{1}{f(y)} dy = \int g(x) dx$$

*Definition of Logistic Differential Equation.* For a population  $P$  which increases exponentially ( $\frac{dP}{dt} \approx kP$ ) when the population is small compared to the carrying capacity  $M$  but where the environment cannot sustain a population larger than  $M$ ,

$$\frac{dP}{dt} = kP \left(1 - \frac{P}{M}\right)$$

Then

$$P(t) = \frac{M}{1 + \left(\frac{M}{P_0} - 1\right) e^{-kt}}.$$

and

$$\frac{d^2P}{dt^2} = k^2P \left(1 - \frac{P}{M}\right) \left(1 - \frac{2P}{M}\right)$$

11 Sequences and Series

*Definition of Sequence.* A **sequence** is an ordered list of real numbers (**elements**)  $a_1, a_2, a_3, \dots$ . It can also be considered a function  $a : \mathbb{N} \rightarrow \mathbb{R}$ .

*Definition of Convergence.* A sequence converges if  $\lim_{n \rightarrow \infty} a_n$  exists.

**Theorem.** The Limit Laws and Squeeze Theorem also apply to limits of sequences.

**Theorem.** If  $\{a_n\}$  is convergent, then it is bounded.

**Theorem (Bounded Monotonic Sequence Theorem).** Every bounded monotonic sequence converges. If  $\{a_n\}$  is increasing, then  $\lim_{n \rightarrow \infty} a_n = \sup\{a_n : n \geq 1\}$ . If  $\{a_n\}$  is decreasing, then  $\lim_{n \rightarrow \infty} a_n = \inf\{a_n : n \geq 1\}$ .

*Definition of Series.* The  $n$ th **partial sum** of  $\{a_i\}_{i=1}^n$  is  $s_n := \sum_{i=1}^n a_n$ . Then the **series** is  $\sum_{i=1}^\infty a_i := \lim_{n \rightarrow \infty} s_n$

**Theorem (Convergence Test).** If the series  $\sum_{i=1}^n a_i$  is convergent, then  $\lim_{i \rightarrow \infty} a_i = 0$ .

**Theorem (Divergence Test).** If  $\lim_{i \rightarrow \infty} a_i \neq 0$ , then  $\sum_{i=1}^n a_i$  diverges.

**Theorem (Integral Test).** If  $f$  is a continuous positive decreasing function for  $x \geq 1$  and  $a_n = f(n)$  for  $n \in \mathbb{N}$ , then

$$\sum_{n=1}^\infty a_n \quad \text{and} \quad \int_1^\infty f(x) \, dx$$

either both converge or both diverge.

**Theorem (Remainder Estimate for Integral Test).** If  $f$  is a continuous positive decreasing function for  $x \geq 1$  and  $a_n = f(n)$  for  $n \in \mathbb{N}$ , then

$$\int_{n+1}^\infty f(x) \, dx \leq R_n = s - s_n = \sum_{i=n+1}^\infty a_i \leq \int_n^\infty f(x) \, dx$$

and therefore

$$s_n + \int_{n+1}^\infty f(x) \, dx \leq \sum_{i=1}^\infty a_i \leq s_n + \int_n^\infty f(x) \, dx$$

**Theorem (Comparison Test for Series).** If  $\sum a_n$  and  $\sum b_n$  are series with positive terms:

- If  $\sum b_n$  is convergent and  $a_n \leq b_n$  for all  $n$ , then  $\sum a_n$  is also convergent.
- If  $\sum b_n$  is divergent and  $a_n \geq b_n$  for all  $n$ , then  $\sum a_n$  is also convergent.

**Theorem (Limit Comparison Test for Series).** If  $\sum a_n$  and  $\sum b_n$  are series with positive terms, then

- If  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} \in \mathbb{R}^+$ , then either both series converge or both series diverge.
- If  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$  and  $\sum b_n$  converges, then  $\sum a_n$  converges.
- If  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$  and  $\sum b_n$  diverges, then  $\sum a_n$  diverges.

**Theorem (Well-known series).**  $\sum \frac{1}{n^p}$  (the  **$p$ -series**) converges if  $p > 1$  and diverges if  $p \leq 1$ .  $\sum ar^{n-1}$  (the **geometric series**) converges if  $|r| < 1$  and diverges if  $|r| \geq 1$ . If it converges,  $\sum_{n=1}^\infty ar^{n-1} = \frac{a}{1-r}$

*Definition of Alternating series.* A series  $\sum_{n=1}^\infty a_n = \sum_{n=1}^\infty (-1)^n b_n$  or  $\sum_{n=1}^\infty (-1)^{n-1} b_n$  where  $b_n > 0$ .

**Theorem (Alternating Series Test).** If the alternating series  $\sum_{n=1}^\infty (-1)^{n-1} b_n$  satisfies

- (i)  $b_{n+1} \leq b_n$  for all  $n$ , (i.e.  $\{b_n\}$  is decreasing) and
- (ii)  $\lim_{n \rightarrow \infty} b_n = 0$ ,

then the series is convergent.

**Theorem (Alternating Series Divergence Test).** If  $\lim_{n \rightarrow \infty} b_n \neq 0$ , then the series is divergent.

**Theorem (Alternating Series Estimation Theorem).** If  $s = \sum_{n=1}^\infty (-1)^{n-1} b_n$  is a series that satisfies the conditions of the Alternating Series Test, then

$$|R_n| := |s - s_n| \leq b_{n+1}$$

and therefore

$$s_n - b_{n+1} \leq s \leq s_n + b_{n+1}$$

*Definition of Absolute convergence.* A series  $\sum_{n=1}^\infty a_n$  **converges absolutely** if the series of absolute values  $\sum_{n=1}^\infty |a_n|$  converges.

*Definition of Conditional convergence.* A series **converges conditionally** if it converges but does not converge absolutely.

**Theorem (Rearrangement of terms of absolutely convergent series).** The terms of an absolutely convergent series can be rearranged without affecting the value/sum of the series.

**Theorem (Absolute convergence implies convergence).** If a series converges absolutely, then it converges.

**Theorem (Ratio Test).**

- If  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$ , then the series  $\sum_{n=1}^\infty a_n$  converges absolutely and therefore converges.
- If  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$  or  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \infty$ , then the series  $\sum_{n=1}^\infty a_n$  diverges.
- If  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$ , the Ratio Test is inconclusive.

**Theorem (Root Test).**

- If  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} < 1$ , then the series  $\sum_{n=1}^\infty a_n$  converges absolutely and therefore converges.
- If  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} > 1$  or  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \infty$ , then the series  $\sum_{n=1}^\infty a_n$  diverges.
- If  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 1$ , the Ratio Test is inconclusive.

12 Power Series

*Definition of Power series.* A power series centered at  $a$  is a series of the form

$$\sum_{k=0}^\infty c_k(x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

where  $x$  is a variable,  $a$  is a fixed real number called the *center* of the series, and the  $c_k$ s are constants called the *coefficients* of the series.

**Theorem.** For a power series  $\sum_{k=0}^\infty c_k(x-a)^k$ , there are only three possibilities:

- (i) The series converges only when  $x = a$ .
- (ii) The series converges for all  $x$ .
- (iii) There is a positive number  $R$  such that the series converges if  $|x-a| < R$  and diverges if  $|x-a| > R$ .  $R$  is called the **radius of convergence**.

**Theorem.** Suppose  $\sum_{n=0}^\infty c_n(x-a)^n$  has radius of convergence  $R > 0$ . Then  $f(x) = \sum_{n=0}^\infty c_n(x-a)^n$  is differentiable (and therefore continuous and integrable) on  $(a-R, a+R)$ , and

$$(i) \quad f'(x) = \sum_{n=0}^\infty \frac{d}{dx} c_n(x-a)^n = \sum_{n=0}^\infty n c_n(x-a)^{n-1}$$

$$(ii) \int f(x)dx = \sum_{n=0}^{\infty} \int c_n(x-a)^n = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

**Definition of Taylor series.** The **Taylor series** of a function  $f$  centered at  $a$  is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

**Definition of Maclaurin series.** The **Maclaurin series** of a function  $f$  is the Taylor series of  $f$  centered at  $a$ :

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

**Theorem.** If  $f$  has a power series representation centered at  $a$ , then that power series is equivalent to the Taylor series of  $f$  centered at  $a$ .

**Theorem (Basic Maclaurin Series).**

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for } x \in \mathbb{R}$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{for } x \in \mathbb{R}$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad \text{for } x \in \mathbb{R}$$

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \quad \text{for } |x| < 1$$

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{for } |x| < 1$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1$$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty} (n+1)x^n \quad \text{for } |x| < 1$$

**Definition of Taylor polynomial.** The  $n$ th-degree Taylor polynomial of  $f$  at  $a$  is

$$\sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k$$

**Theorem (Taylor's Theorem).** If  $f$  has  $(n+1)$  continuous derivatives on an open interval  $I$  containing  $a$ , then for all  $x$  in  $I$ ,

$$f(x) = T_n(x) + R_n(x),$$

where  $T_n$  is the  $n$ th-order Taylor polynomial for  $f$  with center  $a$  and the remainder is

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}$$

for some point  $c$  between  $x$  and  $a$ . This form of  $R_n$  is also called **Lagrange error bound** or the **Lagrange form of the remainder**.

**Theorem.** If  $R_n$  converges to 0, then

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k.$$

### 13 Polar

**Theorem.** If  $r$  is a function of  $\theta$  on the  $xy$ -plane, then

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta} \left( \frac{dy}{dx} \right)}{\frac{dx}{d\theta}}$$

**Theorem (Arc length).** If  $r$  is a function of  $\theta$  on the  $xy$ -plane, then

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta$$

and therefore

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

### 14 Vectors

$$\frac{d}{dt} \vec{r}(t) = \begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix}$$

$$\int \vec{r}(t) dt = \begin{pmatrix} \int x(t) dt \\ \int y(t) dt \\ \int z(t) dt \end{pmatrix}$$

Multivariable Calculus (6)

Colley (2012) refers to *Vector Calculus, Fourth Edition* by Susan Jane Colley.

6.1 Vectors, lines, planes

6.1.1  $\mathbb{R}^n$

*Definition of Two-dimensional real-coordinate space* ( $\mathbb{R}^2$ ).

$$\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$$

*Definition of Three-dimensional real-coordinate space* ( $\mathbb{R}^3$ ).

$$\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$$

$x, y, z$  should be presented such that the coordinate system is right-handed ( $\hat{k} = \hat{i} \times \hat{j}$  should have direction according to the right-hand rule).

*Definition of Standard basis vectors.* The standard basis vectors of a space are the unit vectors that go along the axes of the space. All vectors in that space can be expressed as sums of scalar multiples of the standard basis vectors of that space.

The standard basis vectors of  $\mathbb{R}^2$  are  $\hat{i}$  and  $\hat{j}$ , also called  $\mathbf{e}_1$  and  $\mathbf{e}_2$ .

The standard basis vectors of  $\mathbb{R}^3$  are  $\hat{i}$ ,  $\hat{j}$ , and  $\hat{k}$ , also called  $\mathbf{e}_1$ ,  $\mathbf{e}_2$ , and  $\mathbf{e}_3$ .

6.1.2 Vectors

See the properties of fields in the LinAlg notes for definitions of addition and scalar multiplication.

*Definition of Displacement vector.* The vector from the end of  $\vec{A}$  to the end of  $\vec{B}$  when their starts are in the same location.

$$\overrightarrow{AB} = \vec{B} - \vec{A}$$

6.1.3 Dot and cross products

*Definition of Dot product.* Where  $a, b \in \mathbb{R}^n$  and  $\theta$  is the angle between  $a$  and  $b$ :

$$a \cdot b = \sum a_i b_i = |a||b| \cos \theta$$

**Theorem (Properties of dot product).** For  $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^n$  and  $k \in \mathbb{R}$ :

- $\vec{a} \cdot \vec{a} = |\vec{a}|^2$
- $\vec{a} \cdot \vec{a} = 0$  iff  $\vec{a} = 0$
- Commutativity:  $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
- Distributivity:  $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
- Distributivity:  $(k\vec{a}) \cdot \vec{b} = k(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (k\vec{b})$

- $\vec{a} \cdot \vec{b} = 0$  iff  $a \perp b$ ,  $a = 0$ , or  $b = 0$ .

*Definition of Cross product.* For  $\vec{a}, \vec{b} \in \mathbb{R}^3$ , the unique vector  $a \times b$  satisfying

- $|a \times b|$  is the area of the parallelogram spanned by  $a$  and  $b$
- $a \times b = 0$  iff  $a \parallel b$ ,  $a = 0$ , or  $b = 0$ .
- $a \times b$  is orthogonal to  $a$  and  $b$ .
- $(a, b, a \times b)$  is right-handed (if the coordinate system is right-handed)

**Theorem (Properties of cross product).** For  $a, b, c \in \mathbb{R}^3$  and  $k \in \mathbb{R}$ :

- $a \times b = (-b) \times a$
- $a \times (b + c) = a \times b + a \times c$
- $(a + b) \times c = a \times c + b \times c$
- $k(a \times b) = (ka) \times b = a \times (kb)$

**Theorem (Calculation of cross product).** Where  $a, b \in \mathbb{R}^n$  and  $\theta$  is the angle between  $a$  and  $b$ :

$$\begin{aligned} a \times b &= \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= \hat{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} + \hat{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ |a \times b| &= |a||b| \sin \theta \end{aligned}$$

6.1.4 Lines

The most useful notation for a line is in parametric form:

*Definition of Parametric form of a line.* Where  $r_0 \in \mathbb{R}^3$  is a point on the line, and  $t \in \mathbb{R}^3$ :

$$r(t) = r_0 + vt$$

$t$  is called the **direction vector**.

**Theorem.** Two lines are parallel iff their direction vectors are scalar multiples of each other.

*Definition of Skew lines.* Two lines are skew iff they do not intersect but are not parallel, i.e. they lie in different parallel planes.

**Procedure (Finding the intersection of two lines).** Given lines with parametric equations

$$r_1(t) = a_1 + v_1 t \qquad r_2(t) = a_2 + v_2 t$$

solve the system of equations

$$a_1 + v_1 t_1 = a_2 + v_2 t_2$$

for  $t_1$  or  $t_2$ , then plug it in to the appropriate equation.

(Break up the equation into its  $x$ ,  $y$ , and  $z$  components, or whichever is appropriate for your coordinate space.)

6.1.5 Planes

(Section 1.5 of Colley (2012))

*Definition of Plane.* A plane  $\Pi$  is determined uniquely by a point  $P$  in the plane and a normal vector  $n$ .

A plane is the set of points  $A$  in space such that  $\vec{AP}$  is perpendicular to  $n$ .

**Theorem (Scalar equation for a plane in  $\mathbb{R}^3$ ).** If  $n \in \mathbb{R}^3$  is the **normal vector** to the plane (vector perpendicular to the plane), and  $P \in \mathbb{R}^3$  is a point on the plane:

$$n_x(x - P_x) + n_y(x - P_y) + n_z(z - P_z) = 0$$

or equivalently:

$$n_x x + n_y y + n_z z = n_x P_x + n_y P_y + n_z P_z$$

**Procedure (Equation of plane containing three points).**

If  $A, B, C \in \mathbb{R}^3$  are points on our plane, then we can find the normal vector by performing  $n = \overrightarrow{AB} \times \overrightarrow{AC} = (B - A) \times (C - A)$  (since  $\overrightarrow{AB}$  and  $\overrightarrow{AC}$  are vectors on the plane).

**Theorem (Parametric equation for a plane in  $\mathbb{R}^3$ ).** If  $a, b \in \mathbb{R}^3$  are nonparallel nonzero vectors on the plane, and  $P \in \mathbb{R}^3$  is a point on the plane, then the parametric equation for the plane is:

$$x(s, t) = P + sa + tb$$

6.1.6 Distance

(Section 1.5 of Colley (2012))

**Procedure (Distance between point and line).** Let  $P$  be the point, and  $A + Lt$  be the line. Then the distance is

$$|\overrightarrow{AP} - \text{proj}_L \overrightarrow{AP}| = \vec{P}$$

**Procedure (Distance between parallel planes).** Let  $\Pi_1$  and  $\Pi_2$  be the two planes.

If  $n$  is normal to both planes, and  $P_1 \in \Pi_1$  and  $P_2 \in \Pi_2$ , then the answer is

$$|\text{proj}_n \overrightarrow{P_1 P_2}|$$

6.1.7 Cylindrical and spherical coordinates

Trimm WS: 5.6, 5.7  
Brummet WS: 08, MVCWUP:Feb3(29-33)  
Colley (2012): 1.7

*Definition of Cylindrical coordinate.* An ordered pair  $(r, \theta, z)$  where  $r$  is the distance between the point and the  $z$ -axis,  $\theta$  is the angle counterclockwise from the positive  $x$ -axis along the  $xy$ -plane, and  $z$  is the position on the  $z$ -axis.

*Definition of Spherical coordinate.* An ordered pair  $(\rho, \phi, \theta)$ , where  $\rho$  is the distance between the point and the origin,  $\phi$  is the angle clockwise from the positive  $z$ -axis going downwards towards the  $xy$ -plane, and  $\theta$  is the angle counterclockwise from the positive  $x$ -axis along the  $xy$ -plane.

Typically we use the following restrictions:

$$\rho > 0 \qquad 0 \leq \theta \leq 2\pi \qquad 0 \leq \phi \leq \pi$$

Theorem (Useful formulas).

$$\begin{aligned} r &= \rho \sin \phi & z &= \rho \cos \phi \\ x &= \rho \sin \phi \cos \theta & y &= \rho \sin \phi \sin \theta \\ r^2 &= x^2 + y^2 & x &= r \cos \theta & y &= r \sin \theta \end{aligned}$$

6.2 Functions, limits, differentiation

6.2.1 Multivariable functions

Trimm WS: 3.1, 3.2  
Brummet WS: MVCWUP:Feb4(35-41)  
Colley (2012): 2.1

*Definition of Function.* All functions  $f : X \Rightarrow Y$  are defined by:

- A domain set  $X$
- A codomain set  $Y$
- A rule of assignment that associates a unique element  $y \in Y$  to each element  $x \in X$

*Definition of Graph.* The graph of  $f : X \subseteq \mathbb{R}^n \Rightarrow \mathbb{R}$  is the set

$$\{(x_1, \dots, x_n, f(x)) : x = (x_1, \dots, x_n)\}$$

Specifically, for  $f : \mathbb{R}^2 \Rightarrow \mathbb{R}$  the graph is the set

$$\{(x, y, z) : (x, y) \in X \text{ and } z = f(x, y)\}$$

6.2.2 Limits

Trimm WS: 3.3, DiffEq-1.0  
Brummet WS: 09, MVCWUP:Feb11(42-46)  
Colley (2012): 2.2

*Definition of Limit.*  $\lim_{\vec{x} \rightarrow \vec{a}} f(\vec{x}) = \vec{L}$  if for all  $\varepsilon > 0$ , there exists  $\delta > 0$  s.t. if  $0 < |\vec{x} - \vec{a}| < \delta$  then  $|f(\vec{a}) - \vec{L}| < \varepsilon$ .

6.2.3 Differentiation

Trimm WS: 3.4, 3.5, 3.7, DiffEq-1.0  
Brummet WS: 11, 12.5, MVCWUP:Feb12/22(48-54)  
Colley (2012): 2.3, 2.4

*Definition of Partial derivative with respect to  $x$ .* The partial derivative of  $f(x, y)$  with respect to  $x$  is

$$\lim_{h \rightarrow 0} \frac{f(a + h, b) - f(a, b)}{h}$$

Let  $z = f(x, y)$ . Then the partial derivative is denoted by

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = D_x f$$

*Definition of Partial derivative.* The partial derivative of  $f(\vec{x})$  with respect to the  $i$ th variable is

$$\frac{\partial f(\vec{x})}{\partial x_i} = \lim_{h \rightarrow 0} \frac{f\left(\begin{bmatrix} x_0 \\ \vdots \\ x_i + h \\ \vdots \\ x_n \end{bmatrix}\right) - f(\vec{x})}{h}$$

This is equivalent to letting  $F(x_i) = f(\vec{x})$  and finding  $F'(x_i)$ .

6.2.4 Miscellaneous differentiation things

Trimm WS: ???  
Brummet WS: 11, 12, MVCWUP:Feb24(57)  
Colley (2012): ???

*Definition of Gradient.*

$$\nabla f(\vec{x}) = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{bmatrix}$$

*Definition of Linear approximation.* The linear approximation or tangent plane to a function at a point is expressed by

$$L(\vec{x}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a})$$

In  $\mathbb{R}^3$ , this is equivalent to

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

6.2.5 Implicit surfaces

*Definition of Implicit surface.* A surface in  $\mathbb{R}^3$  defined by an equation which is not solved for  $x$ ,  $y$ , nor  $z$ .

We often express it as

$$F(x, y, z) = 0,$$

in which case the surface is the set of points which satisfy  $F(x, y, z) = 0$ .

**Theorem.** The gradient  $\nabla F(\vec{x})$  is the normal vector to the tangent plane to the implicit surface defined by  $F(\vec{x}) = k$ , where  $k$  is a constant.

Equivalently, if  $x_0$  is a point on the level set  $S = \{x \in X : F(x) = k\}$  where  $F : X \subseteq \mathbb{R}^n \Rightarrow \mathbb{R}$ , then the vector  $\nabla F(x_0)$  is perpendicular to  $S$ .

6.2.6 Chain rule

Trimm WS: 3.8, 6.5, DiffEq-1.0  
Brummet WS: 12, MVCWUP:Feb24(58-61)  
Colley (2012): 2.5

*Definition of Jacobian.* If  $f : X \subseteq \mathbb{R}^n \Rightarrow \mathbb{R}^m$  is a vector-valued function, then the Jacobian is

$$Df\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} \nabla f_1 \\ \nabla f_2 \\ \vdots \\ \nabla f_m \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

**Theorem (Multivariable chain rule).** Suppose  $X \subseteq \mathbb{R}^m$  and  $T \subseteq \mathbb{R}^n$  are open and  $f : X \Rightarrow \mathbb{R}^p$  and  $r : T \Rightarrow \mathbb{R}^m$  are defined so that the range  $r \subseteq X$ . If  $x$  is differentiable at  $t_0 \in T$  and  $f$  is differentiable at  $x_0 = r(t_0)$ , then the composite  $f \circ r$  is differentiable at  $t_0$ , and we have

$$\frac{dF}{dt} = \nabla f(x_0) \cdot r'(t_0)$$

Equivalently,

$$D(f \circ r)(t_0) = Df(x_0)Dr(t_0)$$

In  $\mathbb{R}^3$ ,

$$\frac{dF}{dt} = \frac{\partial F}{\partial x} \frac{dx}{dt} + \frac{\partial F}{\partial y} \frac{dy}{dt} + \frac{\partial F}{\partial z} \frac{dz}{dt}$$

6.2.7 Paths

Trimm WS: ???  
Brummet WS: MVCWUP:Feb24(55-56)  
Colley (2012): ???

*Definition of Path.* A path in  $\mathbb{R}^n$  is a function  $x : I \Rightarrow \mathbb{R}^n$ , where  $I$  is a set of scalars. If  $I = [a, b]$ , then the endpoints of the path are  $f(a)$  and  $f(b)$ .



*Definition of Tangent vector.* Given a path  $r : \mathbb{R} \Rightarrow \mathbb{R}^3$ , the tangent vector to said path at some point  $P$  is given by  $r'(t)$ , provided that  $r'(t) \neq 0$ . In  $\mathbb{R}^3$ ,

$$r'(t) = \lim_{h \rightarrow 0} \frac{r(t+h) - r(t)}{h} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{bmatrix}$$

# Linear Algebra (7)

Let  $F = \mathbb{R}$  or  $F = \mathbb{C}$ .

## 1 Linear equations

*Definition of Linear equation.* An equation that can be written in the form

$$\sum_k a_k x_k = y$$

where all  $a_k \in F$  and  $y \in F$ .

*Definition of Solution to a linear equation.* The solution to a linear equation is a set  $\{s_k\}$  such that  $\sum_k a_k s_k = y$ , i.e. substituting  $x_k = s_k$  results in the equation being true.

*Definition of Linear system.* A set of linear equations.

Let  $m$  be the number of linear equations in the system. Let  $n$  be the number of variables in the system. Then the  $j$ th equation can be written as

$$\sum_{k=1}^n A_{jk} x_k = y_j$$

Let:

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad Y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

Then the system can be written as  $AX = Y$ .

*Definition of Consistent linear system.* A system that has at least one solution.

*Definition of Linear combination.* The linear combination of the equations of a linear system is a linear equation formed by multiplying each equation by  $c_j$  where  $c_j \in F$ .

This linear combination can be written as

$$\sum_{j=1}^m \sum_{k=1}^n c_j A_{jk} x_k = \sum_{j=1}^m y_j$$

**Theorem.** All solutions of a linear system are solutions to the linear combination of the equations of the system.

*Definition of Equivalent linear systems.* Two systems are equivalent if they have the same set of solutions.

**Theorem.** Two systems are equivalent if each equation in each system is a linear combination of the equations in the other system.

### 7.0.1 Matrices and rows

*Definition of Elementary row operations.* The elementary row operations are:

*Definition of Scaling.*  $R_i \mapsto cR_i$  where  $c$  is a nonzero scalar.

*Definition of Replacement.*  $R_i \mapsto R_i + cR_j$  where  $c$  is a scalar.

*Definition of Interchange.* Swap  $R_i$  and  $R_j$ .

**Theorem (Elementary row operations are invertible).** For any elementary row operation  $e$ , there exists an elementary row operation  $e^{-1}$  such that  $e^{-1}(e(A)) = A$  for any matrix  $A$ .

*Definition of Row equivalence.* Two matrices are row-equivalent if each can be derived from the other using a finite number of elementary row operations.

*Definition of Row echelon form (REF).* A matrix is in REF if it satisfies:

1. All nonzero rows are above all rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

*Definition of Reduced row echelon form (RREF).* A matrix is in RREF if it is in REF and additionally satisfies:

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column

*Definition of Pivot position.* A location  $A_{ij}$  where  $\text{RREF}(A)_{ij}$  is a leading 1.

*Definition of Pivot column.* A column which contains a pivot position.

*Definition of Pivot.* A nonzero number at a pivot position.

**Procedure (Gauss-Jordan elimination).**

**Procedure (Gaussian elimination).** Iterate through the pivot columns of  $A$  from left to right. For each pivot column, use elementary row operations to ensure that the pivot position is nonzero and that all entries in the column below the pivot position are zero. This produces  $\text{REF}(A)$ .

**Procedure (Jordan elimination).** Iterate through the pivot columns of  $\text{REF}(A)$  from right to left. For each pivot column, use elementary row operations to ensure that all other entries in the column other than the pivot are zero and that the pivot is equal to 1. This produces  $\text{RREF}(A)$ .

*Definition of Leading variable, determined variable, basic variable.* A variable in a pivot column.

*Definition of Free variable.* A variable not in a pivot column.

### 7.0.2 Homogeneous linear systems

*Definition of Homogeneous linear system.* A system where  $y_0 = y_1 = \cdots = y_m = 0$ . It can be written as  $AX = 0$ .

**Theorem (Trivial solution).** For any homogeneous system,  $x_0 = x_1 = \cdots = x_n = 0$  is a solution to the system. Therefore, all homogeneous systems are consistent.

- Theorem.**
- (a) If there are less equations than there are variables ( $m < n$ ), then  $AX = 0$  has an infinite number of solutions.
  - (b) If there are an equal number of equations and variables, then  $A$  is row-equivalent to the  $n \times n$  identity iff  $AX = 0$  has only the trivial solution.
  - (c) If there are more equations than there are variables ( $m > n$ ), then

$$\text{RREF}(A) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

iff  $AX = 0$  has only the trivial solution.

**Procedure (Solution).** To solve a homogeneous system, perform Gauss-Jordan elimination on  $A$  so that  $R = \text{RREF}(A)$ . Then solve  $RX = 0$ . The variables which are not in pivot columns are free variables and may be set to any value, typically denoted  $u_1, u_2, \dots$ .

### 7.0.3 Inhomogeneous linear systems

**Procedure (Solution).** To solve an homogeneous system, perform Gauss-Jordan elimination on  $A' = [A|Y]$  so that  $R' = [R|Z] = \text{RREF}(A')$ . Then solve  $RX = Z$ . Note that not all inhomogeneous systems are solvable (consistent).

2 Fields

7.0.4 Real and complex numbers

*Definition of Field properties.* The following properties:

**Theorem (Properties of addition).** For all  $x, y, z \in F$ :

- (A1) Commutativity:  $x + y = y + x$
- (A2) Associativity:  $(x + y) + z = x + (y + z)$
- (A3) Identity:  $\exists 0 \in \mathbb{R}$  s.t.  $0 + x = x$
- (A4) Additive inverse: For  $x \in F$ ,  $\exists -x \in F$  s.t.  $x + (-x) = 0$

**Theorem (Properties of multiplication).** For all  $x, y, z \in F$ :

- (M1) Commutativity:  $xy = yx$
- (M2) Associativity:  $(xy)z = x(yz)$
- (M3) Identity:  $\exists 1 \in \mathbb{R}$  s.t.  $1x = x$  and  $1 \neq 0$
- (M4) Additive inverse: For  $x \in F \setminus \{0\}$ ,  $\exists x^{-1} \in F$  s.t.  $xx^{-1} = 1$

**Theorem (Distributive property).**

- (D)  $x(y + z) = xy + xz$  for all  $x, y, z \in F$ .

7.0.5 Fields

*Definition of Field.* A set  $F$  which defines the following two operations:

- Addition: an operation that maps  $x, y \in F \Rightarrow c \in F$  and satisfies the properties of addition
- Multiplication: an operation that maps  $x, y \in F \Rightarrow c \in F$  and satisfies the properties of multiplication

for which the distributive property also holds.

**Theorem.**  $\mathbb{R}$  and  $\mathbb{C}$  are fields.

*Definition of Complex number.* A number which can be defined by a pair of real numbers  $a, b$  where the value of the number is equal to  $a + bi$ .

**Theorem (Useful things about complex numbers).** Let  $z = a + bi$  and  $w = c + di$  be complex numbers. Then:

- $z + w = (a + c) + (b + d)i$
- $zw = (ac - bd) + (bc + ad)i$

*Definition of  $F^n$ .* For a field  $F$ ,  $F^n$  is the set of all ordered  $n$ -tuples of elements of  $F$ :

$$F^n := \{(x_1, \dots, x_n) : x_1, \dots, x_n \in F\}$$

*Definition of Addition in  $F^n$ .* If  $a, b \in F^n$ :

$$a + b = (a_1 + b_1, \dots, a_n + b_n)$$

Addition follows the properties of additon (A1-A4).

*Definition of Product of element of  $F$  and element of  $F^n$ .* If  $\alpha \in F$  and  $x \in F^n$ , then

$$\alpha x = (\alpha x_1, \dots, \alpha x_n)$$

2.3 Vector spaces

*Definition of Vector space.* A vector space over  $F$  is a set  $V$  with the following operations:

- **Vector addition:**  $u \in V, v \in V \mapsto (u + v) \in V$ , which satisfies the properties of addition
- **Scalar multiplication:**  $\alpha \in F, v \in V \mapsto \alpha v \in V$ , which satisfies the properties of scalar multiplication

*Definition of Properties of addition.* For all  $u, v, w \in V$ :

- (A1) Commutativity:  $u + v = v + u$
- (A2) Associativity:  $(u + v) + w = u + (v + w)$
- (A3) Identity:  $\exists 0 \in \mathbb{R}$  s.t.  $0 + u = u$
- (A4) Additive inverse: For  $u \in F$ ,  $\exists -u \in F$  s.t.  $u + (-u) = 0$

*Definition of Properties of scalar multiplication.* For all  $\alpha, \beta \in F, v, w \in V$ :

- (S1) Associativity:  $(\alpha\beta)v = \alpha(\beta v)$
- (S2) Distributivity over scalar addition:  $(\alpha + \beta)v = \alpha v + \beta v$
- (S3) Distributivity over vector additon:  $\alpha(v + w) = \alpha v + \alpha w$
- (S4) Multiplicative identity:  $1v = v$

**Theorem.**  $F^n$  is a vector space.

**Theorem.** All inverses and identities are unique in a vector space.

*Definition of  $F^\omega$ .* The set of all sequences of elements of  $F$ :

$$F^\omega := \{(x_1, x_2, \dots) : x_k \in F \text{ for } k \in \mathbb{N}\}$$

where addition and scalar multiplication are defined similarly to  $F^n$ :

$$\begin{aligned} a + b &:= (a_1 + b_1, \dots, a_n + b_n) \\ \alpha x &:= (\alpha x_1, \dots, \alpha x_n) \end{aligned}$$

*Definition of  $F^{m,n}$ .* The set of all  $m \times n$  matrices with entries in  $F$ , where addition and scalar multiplication are defined as:

$$\begin{aligned} (A + B)_{ij} &:= A_{ij} + B_{ij} \\ (\alpha A)_{ij} &:= \alpha A_{ij} \end{aligned}$$

**Theorem (Vector space of functions).** Let  $V$  be a vector space,  $S$  be a set, and

$$V^S = \{f : S \rightarrow V\}$$

(the set of all functions that map members of  $S$  to members of  $V$ ). Then  $V^S$  is a vector space, if we define

$$(f + g)(s) = f(s) + g(s) \qquad (\alpha f)(s) = \alpha(f(s))$$

7.0.6 Subspaces

*Definition of Subspace.* Let  $V$  be any vector space, and let  $W$  be a subset of  $V$ . Define vector addition and scalar multiplication on  $W$  by restricting the corresponding operations of  $V$  to  $W$ . If  $W$  is a vector space with respect to the restricted operations of  $V$ , then  $W$  is said to be a subspace of  $V$ .

*Definition of Closed.* An operation is closed under a set if applying the operation to elements of the set always results in an element of the set.

*Definition of Subspace.* A subspace of a vector space  $V$  is a subset  $W$  of  $V$  which contains the zero vector and is closed under addition and scalar multiplication.

**Theorem.** A subset  $W$  of a vector space  $V$  is a subspace iff

- (i)  $W$  is nonempty
- (ii)  $\alpha \in F$  and  $w_1, w_2 \in W$  implies  $\alpha w_1 + w_2 \in W$

Typically, we prove (i) by proving that  $0 \in W$ .

7.0.7 Subspaces of  $F^n$

**Theorem (Subspaces of  $\mathbb{R}^n$ ).**  $\mathbb{R}^n$  contains the following subspaces:

- $\{0\}$
- $\mathbb{R}^n$
- Any line through the origin
- Any plane through the origin
- etc

*Definition of Spanning.* For vectors to span a space is for it to be sufficient to be able to reach any point in the space using the vectors.

*Definition of Independence.* If a vector can be made out of other vectors, then the vector is independent

7.0.8 Intersections and unions of subspaces

**Theorem.** The intersection of any collection of subspaces of  $V$  is a subspace of  $V$ .

**Theorem.** The union of two subspaces of  $V$  is a subspace of  $V$  iff one of the subspaces is contained in the other.

*Definition of Sum of subspaces.* If  $U$  and  $W$  are subspaces of a vector space  $V$ , then

$$U + W := \{u + w : u \in U \text{ and } w \in W\}$$

**Theorem.** If  $U$  and  $W$  are subspaces of  $V$ , then  $U + W$  is the smallest subspace of  $V$  containing both  $U$  and  $W$ .

*Definition of Direct sum.* If  $V_1, \dots, V_n$  are subspaces of  $V$  such that each element of  $\bigoplus_{k=1}^n V_k = V_1 + \dots + V_n$  can be written uniquely as  $\sum_{k=1}^n v_k = v_1 + \dots + v_n$  where  $v_k \in V_k$ , then  $\bigoplus_{k=1}^n V_k$  is a **direct sum** and can be written as  $\bigoplus_{k=1}^n V_k = V_1 \oplus \dots \oplus V_n$ .

**Theorem.** Let  $V_1, \dots, V_n$  be subspaces of  $V$ . Then they are direct sums iff the only way to write  $0 = v_1 + \dots + v_n$  is to take  $v_1 = \dots = v_n = 0$ .

**Theorem.** If  $U$  and  $W$  are subspaces of  $V$ , then  $U + W$  is a direct sum iff  $U \cap W = \{0\}$ . This does not generalize to higher numbers of subspaces.

7.0.9 Subspace generated by a set

*Definition of Linear combination.* A linear combination of a collection  $v_1, \dots, v_n$  of vectors in vector space  $V$  is a vector of the form

$$\alpha_1 v_1 + \dots + \alpha_n v_n$$

where each  $\alpha_k \in F$ .

*Definition of Span.* Given  $W \subseteq V$  where  $V$  is a vector field, the set of all linear combinations of vectors in  $W$  is called the span of  $W$ .

$$\text{span}(W) := \left\{ \sum_{i=1}^n \alpha_i w_i : \alpha_i \in F, w_i \in W \right\}$$

Additionally, we define

$$\text{span}(\emptyset) = \{0\}$$

# Differential Equations (8)

A *differential equation* is an equation involving a quantity and one or more of its derivatives.

## 1 Ordinary Differential Equations

An ODE involves the derivative of the dependent variable with respect to a single independent variable.

### 8.0.1 Solving by Integration

**Definition of Solution to a differential equation.** A function  $y = f(x)$  that satisfies the differential equation when  $f$  and its derivatives are substituted into the equation.

**Procedure (Euler's Method).** To numerically approximate the solution to the differential equation  $y' = F(x, y)$  with  $y(x_0) = y_0$ ,

$$y_n = y_{n-1} + F(x_{n-1}, y_{n-1})(x_n - x_{n-1})$$

**Definition of Separable Equation.** A separable equation is a differential equation where

$$\frac{dy}{dx} = g(x)f(y)$$

for some function  $g(x)$  which depends only on  $x$  and  $f(y)$  which depends only on  $y$ .

**Theorem.** For a separable equation,

$$\int \frac{1}{f(y)} dy = \int g(x) dx$$

**Definition of Logistic Differential Equation.** For a population  $P$  which increases exponentially ( $\frac{dP}{dt} \approx kP$ ) when the population is small compared to the carrying capacity  $M$  but where the environment cannot sustain a population larger than  $M$ ,

$$\frac{dP}{dt} = kP \left(1 - \frac{P}{M}\right)$$

Then

$$P(t) = \frac{M}{1 + \left(\frac{M}{P_0} - 1\right) e^{-kt}}.$$

and

$$\frac{d^2P}{dt^2} = k^2P \left(1 - \frac{P}{M}\right) \left(1 - \frac{2P}{M}\right)$$

### 8.0.2 Initial-Value Problems

**Definition of Initial-Value problem.** Assuming the function  $f$  is continuous, then the function  $y$  is a solution of the IVP given that

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0,$$

Where  $x_0$  is called the *initial point* for the IVP and  $y_0$  is the *initial value*.

### 8.0.3 Existence and Uniqueness of Solutions

**Theorem (Existence and Uniqueness).** If  $f$  is continuous, then the function  $f$  as previously defined has at least one solution on the interval of continuity. If at least one solution exists and  $\frac{\partial f}{\partial y}$  is continuous on the same interval, the solution is unique.

### 8.0.4 Autonomous Equations

**Definition of Autonomous Equation.** An autonomous equation is an equation where the derivative of the dependent variable can be expressed as a function of the dependent variable alone, assuming continuity.

An example of an autonomous equation is

$$\frac{dy}{dx} = f(y) + g(x)$$

Whereas

$$\frac{dy}{dx} = f(y)g(x)$$

Is not autonomous.

Since  $f(y)$  is independent of  $x$ , the resulting slope field has translational symmetry across the  $x$ -axis. A snapshot of the slope field through a vertical line  $x = x_0$  is called a *phase line*. A line where the direction of the slopes is indicated with arrows either going down or up is called a *one-dimensional phase portrait* of the autonomous ODE.

At  $y$  values where  $f(y) = 0$ , when the slope is zero, those points are called *equilibrium points* or *stationary points*.

1. If surrounding solutions approach  $y = y_0$  asymptotically then the equilibrium point is *asymptotically stable* or an *attractor*.
2. If surrounding solutions move away from  $y = y_0$  then the equilibrium point is *unstable* or a *repeller*.
3. If surrounding solutions approach from one side and repel from another then the equilibrium point is *semi-stable*.

### 8.0.5 Bifurcations

A differential equation that depends on a parameter *bifurcates* if there is a qualitative change in the solutions as parameter changes, meaning that the phase line changes.

We can see this change in phase line by plotting the parameter and the solution of the differential equation. The resulting graph would look like an array of phase lines. Each point on the phase lines is called a *bifurcation point*.

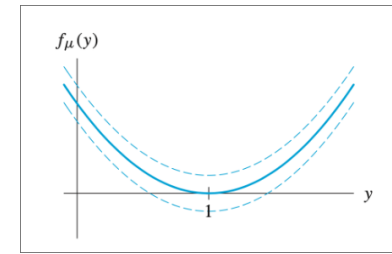


Figure: Graph of  $f_\mu(y) = y^2 - 2y + \mu$  for  $\mu < 1$ ,  $\mu = 1$ , and  $\mu > 1$ .

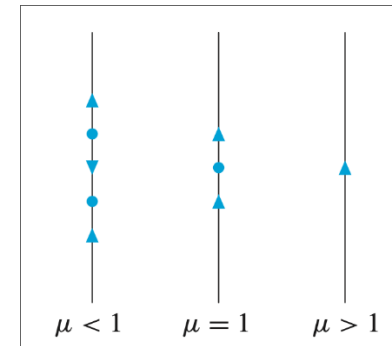


Figure: Corresponding phase portraits for  $\frac{dy}{dx} = y^2 - 2y + \mu$ .

The typical way to visualize bifurcations is through *bifurcation diagrams*. The parabola on the following figure is called a *bifurcation line*.

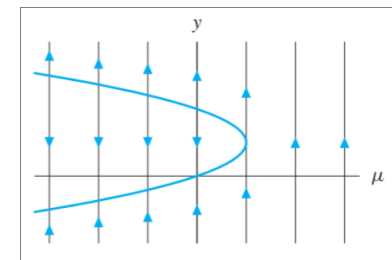


Figure: Bifurcation diagram for  $\frac{dy}{dx} = y^2 - 2y + \mu$ .

### 8.0.6 Separable Equations

**Definition of Separable first-order ODE.** An ODE is separable if it can be expressed as

$$\frac{dy}{dx} = g(x)h(y)$$

where  $g$  and  $h$  are continuous.

We can simplify the process of solving a separable equations by defining the following where  $H(y)$  and  $G(x)$  are antiderivatives of  $\frac{1}{h(y)}$  and  $g(x)$ , respectively, and  $c \in \mathbb{R}$  is an arbitrary constant.

$$H(y) = G(x) + c$$

which can then be expressed as

$$\int \frac{1}{h(y)} dy = \int g(x) dx$$

### 8.0.7 Implicitly-Defined Solutions

Sometimes a separable equation can't be solved for  $y$  explicitly. Let's define  $F(x, y)$  as

$$F(x, y) := H(y) - G(x) - c$$

so that

$$F(x, y) = 0$$

The equation above implicitly defines  $y$  as a function of  $x$  only if it follows the *Implicit Function Theorem* from Multivariable Calculus.

**Theorem (Implicit Function Theorem).** If  $F$  is defined on a disc containing  $(x_0, y_0)$ , where

1.  $F(x, y) = 0$
2.  $\frac{\partial F}{\partial x}$  and  $\frac{\partial F}{\partial y}$  are continuous on the disc
3.  $\frac{\partial F}{\partial x} \neq 0$

then the equation  $F(x, y) = 0$  defines  $y$  as a function of  $x$  on some open set containing the point  $(x_0, y_0)$ .

### 8.0.8 Singular Solutions

**Definition of Singular Solution.** A solution is singular if it cannot be obtained by any choice of  $c$  in the solution equation of the separable ODE.

When either  $h(y)$  or  $g(x)$  are equal to 0 inside a separable equation, then they would be valid solutions, but may not show up in the integration method for finding solutions as defined in the Separable Equations section. If that is the case, then those solutions are unique.

### 8.0.9 Orthogonal Trajectories

An *orthogonal trajectory* of a family of curves is a curve that intersects each curve of the family orthogonally.

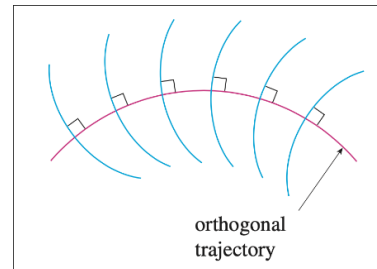


Figure: An example of an orthogonal trajectory.

For example,

$$x^2 + y^2 = r^2$$

and

$$y = mx$$

are orthogonal trajectories of each other.

To find orthogonal trajectories of an equation in terms of  $x$ ,  $y$ , and another constant variable:

1. Take derivative of both sides in respect to  $x$
2. Solve for  $\frac{dy}{dx}$  from previous equation
3. Use the original equation to eliminate the constant and write an expression that only depends on  $x$  and  $y$
4. The previous equation is a slope  $m(x, y)$ ; find the orthogonal slope at point  $(x, y)$
5. Solve the differential equation  $\frac{dy}{dx} = m(x, y)$  to find the family of orthogonal trajectories

### 8.0.10 Linear Equations

**Definition of Linear Equation.** An equation is linear if it can be expressed in the form

$$\frac{dy}{dx} = p(x)y + q(x)$$

where  $p, q : (a, b) \rightarrow \mathbb{R}, -\infty \leq a < b \leq \infty$  are continuous.

If  $q(x) = 0$ , then the linear equation is called *homogenous*. If  $p(x)$  is a constant, but not necessarily  $q(x)$ , it is called *constant-coefficient*.

### 8.0.11 Variation of Parameters

If  $y_h(x)$  is a solution of a homogenous linear equation, then  $cy_h(x)$  is also a solution of the same homogenous equation. A general solution of a homogenous linear equation is

$$y_h(x) = ce^{\int p(x) dx}$$

A general solution of nonhomogenous linear equation is

$$y(x) = ce^{\int p(x) dx} + e^{\int p(x) dx} \int e^{-\int p(x) dx} q(x) dx$$

### 8.0.12 Integrating Factors

This can be simplified by defining

$$\mu(x) := e^{-\int p(x) dx}$$

$$y(x) = \frac{1}{\mu(x)} \int \mu(x) q(x) dx$$

### 8.0.13 Singular Points

If we represent a linear equation in the form

$$a_1(x) \frac{dy}{dx} + a_0(x)y = g(x)$$

and divide both sides by  $a_1(x)$ , then  $p$  and  $q$  will be discontinuous when  $a_1(x) = 0$ . These points are called *singular points*. These discontinuous points carry over to the final solution.

### 8.0.14 Bernoulli's Equation

An ODE in the form

$$\frac{dy}{dx} = p(x)y + f(x)y^n$$

where  $n \in \mathbb{R}$ , is called *Bernoulli's equation*.

This equation is linear if  $n = 0, 1$ . By substituting  $u = y^{1-n}$ , the equation can be rewritten in a linear form as

$$\frac{du}{dx} = (1-n)(p(x)u + q(x))$$

where  $n \neq 0, 1$ .

**8.0.15   Exact Equations**

An exact equation is a differential equation in the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

where  $M, N$  are continuously differentiable. They also represent partial derivatives of the potential function  $\psi$

$$\psi_x + \psi_y \frac{dy}{dx} = 0$$

where  $\psi(x,y)$  is an existing *potential function*, as seen in Multivariable Calculus. The equation is exact only if,

- $\psi_{xy} = \psi_{yx}$
- Domain of functions above is open and topologically simply-connected (continuous)

So if the potential function exists, then the solution to the equation would be

$$\psi(x,y) = 0$$