Proposed Projects

MATH 372

Mathematical Modelling I

Winter 2025

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Mathematical and Statistical Sciences

University of Alberta

Deadline:

March 17, 2025 (First draft)

April 11, 2025 (Final draft)

Contents

1 Requirements for the projects

You may a choose project topic provided in this list, or develop your own project (these must be approved by me).

The presentations will be during the last 3-4 weeks of class.

The first draft of the project report is due on Canvas on March 17, 2025 right before the presentations start.

The final draft is due on April 11, after the presentations have ended.

1.1 The report

A project report should consist of the following components:

1. Title

2. Authors: Names and contributions of individual authors (e.g. developed the model, prepared figures, wrote the numerical code, wrote first draft, review and editing of draft)

3. Introduction. Include background, description/graphs of available data, problem description, questions to be addressed. Use non-technical terms as much as possible.

4. Formulation of the mathematical model. List variables/parameters, assumptions, restrictions, and justify the development of the equations.

5. Solution to the problem. Outline the mathematical methods used to solve the model. Details, such as complicated calculations and computer code should be provided in appendices.

6. Interpretation of results. Answer the questions posed in the introduction, summarize major findings.Use non-technical terms.

7. Critique of the model. Does the model satisfactorily answer the questions posed? Discuss the limitations of the model, possibilities to improve the model, etc. Think ahead, what would you like to do if you had more time.

8. References: Add any article, online material, books, etc, that were used during the development and writing of the project.

9. Appendices with supporting materials: Include a copy of any code used, details of calculations, etc.

   The project reports will be evaluated based on the quality of:

   - Background information and problem description
   - Formulation and sophistication of the mathematical model
   - Appropriateness and depth of the methods used to solve it
   - Interpretation of the results in non-technical terms
   - Critique of the model and suggestions of further studies
   - Quality of writing

1.2 The presentation

The purpose of the project presentation is to develop skills to communicate modelling results effectively and efficiently in a short amount of time (about 15-20 minutes).

Some thought will be required to decide which points to make. It is usually better to focus on a few solid points than to cram in many points. Presentations should be prepared for an audience that is not familiar with the project (think about how you would explain your project to your friends or parents).

The project presentation should consist of the following components:

1. Statement of the problem. What are the questions to be answered? Use plain language and nontechnical terms as much as possible.

2. Description of the model. What are the major assumptions? How is the problem translated into equations?

3. Outline of the mathematical methods used to solve the model. Give enough details to demonstrate to the audience that you know what you are doing, but not so much to put the audience to sleep!

4. Interpretation of the results. Use plain language, non-technical terms. Make sure that the questions posed at the beginning have been answered. Give some intuition as to why your solution is correct.Discuss the limitations of your solution, possible improvements to the model.

To make effective use of time and your audience's attention, make use of pictures (graphs, schematics, flow diagrams) as much as you can (a picture is worth a thousand words !).

Presentations should be prepared in electronic format (PowerPoint or pdf file).

Each project presentation will be followed by a brief questions and answer period during which members of the audience will ask questions.

Presentations will be evaluated based on the following criteria:

- Statement of the problem.
- Description of the model.
- Explanation of the mathematical methods used to solve the model.
- Interpretation of the results.
- Overall clarity of the presentation.

2 Pesticide

A farmer in BC's Okanagan Valley raises a crop over 5 months. To minimize loss of revenue, the farmer may choose to spray the crop with a pesticide once during those 5 months. Cost of spraying are a one-time cost of \$500.00 plus the price per gallon of the pesticide used. Losses due to a pest depend on time. For the population of a pest $p(t)$, the loss at that time is proportional to $p(t)$. The total amount of lost revenue includes the cost of spraying, and the total loss due to the pest (over the whole 5 months). One model for the size of the insect population is

$$p(t) = \left(p(0) + \frac{q}{r}\right) e^{rt} - \frac{q}{r} \tag{1}$$

where r is some rate constant, and q represents a boost factor, due to immigration of other insects. Data suggests that if the farmer sprays an amount of pesticide u, the effect is the following:

1. If u is less than some critical amount, then there is no effect on the pest population.

2. If u is greater than this critical amount, then the surviving population decreases as u increases.

After the farmer sprays the pesticide, the pest population grows according to the equation given above, starting from the reduced population.

Is it financially worthwhile for the farmer to spray the pesticide? If so, when should it be sprayed, and how much?

3 Scheduling

An insurance office handles two types of work: new policies and claims. There are 3 workers. Based on a study of office operations, the average work times (in minutes) for the workers are given as follows:

Worker	New policy	Claim
1	10	28
2	15	22
3	13	18

The company would like to assign a fraction of each type of task to each worker, with the goal to minimize the overall elapsed time for handling a (long) sequence of tasks. For example, if p_1 is the fraction of new policy work assigned to Worker 1, and q_1 is the fraction of claim work assigned to Worker 1, then the company would like to keep $10p_1 + 28q_1$ less than some time limit. Similarly for other workers. What would you suggest to the company?

4 Tree Harvest

A forester has to decide when to cut trees which have been planted. At a time t, the forester cuts N trees. The selling price is \$5 per cubic foot of wood. The combined planting and harvesting costs are \$50,000 plus \$20 per tree. The volume v of wood (in hundreds of cubic feet) of the tree depends on time, and satisfies the following model for tree growth:

$$\frac{dv}{dt} = k_1 v - k_2 v^2 \qquad (2)$$

$$v(0) = v_0 \qquad (3)$$

Some possible values for the model parameters are $v_0 = 0.0001$, $k_1 = 1$, and $k_2 = 0.1$. Time t is in years.

Assuming that the forester would like to plant more trees after the harvest, and harvest them at a time $2t$, etc., what would you recommend for the value of t? That is, consider several consecutive harvests.

Suppose you include inflation in your model, so that the costs and prices increase with a factor e^{rt}, with $r > 0$. Also, suppose you discount the final value of the trees (profits - costs) with a factor $e^{-\nu t}$, where $\nu > 0$ is the nominal annual interest rate. This discount factor models the fact that the trees are an investment, so their 'uncut' value is discounted by the amount of interest that could have been earned if they had been harvested and the money invested. Choose reasonable values of r and ν, and show how it affects the optimal time of harvest.

5 Recycle

The Green Supply Company manufactures plastic grocery bags and milk jugs. The company can obtain 5,000 lbs of used plastic bags, 18,000 lbs of used plastic milk jugs, and 40,000 lbs of industrial plastic scraps per week at costs of \$18, \$12, and \$10 per 100 lbs, respectively. The company has orders for 4,000 boxes of plastic bags and 80,000 milk jugs per week. One box of plastic bags requires 6 lbs of plastic, costs \$5 to manufacture, and sells for \$14. It costs \$9 and requires 14 lbs of plastic to make 100 milk jugs, and the jugs retail for \$20. The plastic bags must be at least 25% post-consumer recycled plastic (used bags or jugs) because of consumer preference, and the milk jugs must be at most 50% post-consumer recycled plastic for strength.

1. Determine the optimal mix of plastics for each product. A full analysis of the problem includes the computation of the shadow prices for each constraint.

2. A new supplier can provide industrial plastic scraps at \$8 per 100 lbs. How does this change affect the results?

3. A new customer has offered to purchase 40,000 environmentally friendly milk jugs per week for \$30 per 100 jugs. The jugs must contain at least 35% post-consumer recycled plastic. How does this change the results? Should the company accept this new customer?

6 Task Competition

Consider a machine which can perform 3 different tasks. The different tasks are brought to the machine at a rate of one per minute, but the machine processes different tasks at different rates. The machine can perform only one task at a time. While it is processing a task, the other tasks have to wait. For example, if the machine is working on a task of type 1, then the number of tasks waiting for service changes in the following way: the number of tasks of type 2 and 3 both increase at a rate of one per minute, since they are waiting for service. But the number of type 1's is changing by a rate of one per minute minus the processing rate for type 1's.

While the tasks are waiting for processing, there is a cost in lost production time which depends on the number of jobs which are waiting. For example, if there are $n = 3$ possible tasks, and at time t the machine works on task 1 while $x_i(t)$ of type i waits, then the cost at time t is

$$\sum_{i=1}^{3} c_i x_i(t) \tag{4}$$

The coefficients c_i are constants, which are different for different tasks. Consider systems with different processing rates and different costs c_i for waiting tasks. Can you find an optimal strategy for the order in which the tasks should be processed in order to minimize the total costs over a finite interval of time? Were the costs or the processing times more important in your decision?

7 Hydro-Turbine

The Great Northern Paper Company in Millinocket, Maine, operates a hydroelectric generating station on the Penebscot River. Water is piped from a dam to the power station. The rate at which the water flows through the pipe varies, depending on external conditions.

The power station has three different hydroelectric turbines, each with a known (and unique) power function that gives the amount of electric power generated as a function of the water flow arriving at the turbine. The incoming water can be apportioned in different volumes to each turbine, so the goal is to determine how to distribute water among the turbines to give the maximum total energy production for any rate of flow.

Using experimental evidence and the Bernoulli's equation, the following quadratic models were determined for the power output of each turbine:

$$\begin{aligned}
KW_1 &= \left(-18.89 + 0.1277Q_1 - 4.08 \cdot 10^{-5}Q_1^2\right)\left(170 - 1.6 \cdot 10^{-6}Q_T^2\right) \\
KW_2 &= \left(-24.51 + 0.1358Q_2 - 4.69 \cdot 10^{-5}Q_2^2\right)\left(170 - 1.6 \cdot 10^{-6}Q_T^2\right) \\
KW_3 &= \left(-27.02 + 0.1380Q_3 - 3.84 \cdot 10^{-5}Q_3^2\right)\left(170 - 1.6 \cdot 10^{-6}Q_T^2\right)
\end{aligned}$$

where

$$\begin{aligned}
Q_i &= \text{Flow through turbine } i \text{ in cubic feet per second;} \\
KW_i &= \text{Power generated by turbine } i \text{ in kilowatts;} \\
Q_T &= \text{Total flow through the station in cubic feet per second}
\end{aligned}$$

Allowable flows of operation are as follows:

$$250 \leq Q_1 \leq 1110, \tag{5}$$
$$250 \leq Q_2 \leq 1110, \tag{6}$$
$$250 \leq Q_3 \leq 1225. \tag{7}$$

Determine the flow Q_i to each turbine that will give the maximum total energy production. For which values of Q_T is the result valid?

Suppose that the incoming flow is 2500 ft^3/s, determine the distribution to the turbines and verify (by trying some nearby distributions) that your result is indeed a maximum.

Are there situations under which more power can be produced by only using one turbine? Should a flow of 1000 ft^3/s be distributed to all three turbines or routed to just one? If just one should be used, which one? What if the flow is only 600 ft^3/s ?

Perhaps for some flow levels, it would be advantageous to use two turbines. If the incoming flow is 1500 ft^3/s, which two turbines would you recommend using? How should the flow be distributed? Is using two turbines more efficient that using all three?

If the incoming flow is 3400 ft^3/s, what would you recommend to the company?

In your project report, include a memo to the operations manager of the Great Northern Paper Company that summarizes your recommended mode of operation under different flow conditions.

8 Run-Bike-Fun

The following Run-Bike-Fun sports event takes place every year in a small university town in Germany. Each participating team consists of two people. Both people have to complete a 15 km course through a combination of running and cycling. Each team has one bicycle. Only one person is allowed to ride the bicycle at any one time, but team members can switch between running and cycling as often as they wish. The first team with both partners at the finish line wins.

At the beginning of the race, one person starts riding the bicycle, and the other starts running. After some time, the cyclist gets off the bicycle, puts it down, and starts running. When the other runner reaches the bicycle, he/she picks it up, and starts cycling.

What is the optimal switching strategy? At which locations along the course should the switch(es) occur?

You may wish to begin by assuming that it takes no time to get on/off the bicycle, that both team members are x times faster at cycling compared to running, and that people run/cycle with constant velocity. Based on your own experience, estimate the value of x. When/where should you switch?

In reality, people get tired. How might you describe that? Would you use the same description for running and cycling? How does this affect the optimal strategy? Also, switching between cycling and running takes time. How does this affect the optimal strategy? What if two people with different abilities form a team?

9 Salmon

Let x_n be the number of hundreds of millions of Pacific salmon at the beginning of the nth cycle. They produce a larval population y at a time t_0, which is proportional to the number of adult salmon x_n, with proportionality constant β.

What happens to the larvae? The adults cannibalize them. Then the larval population decays at a rate which is proportional to the number of interactions with the adult population, with proportionality constant α. The larvae do not remain larvae forever, and so this decay occurs only during an interval of time $t_0 < t < t_e$, which is a portion of the cycle. After this, the young adults (the larvae that survived) go out to sea, and a fraction γ of them will survive. The other adults will die after breeding. So the number of salmon at the beginning of the next cycle will be the number of young adults that survive at sea.

Set up the equations which govern the salmon population, and determine if there is an equilibrium solution or cyclical behaviour. Some typical parameter values are $1 < \alpha(t_e - t_0) < 10$ and $3 < \beta\gamma < 20$. How could factors such as fishing, pollution, etc., affect the behaviour of the population?

10 Heart

In this project, you are asked to investigate the production of heart beats. The basic apparatus for beating of the heart is shown in Figure 1. The sinoatrial (SA) node is the pacemaker, which sends regular signals to the atrioventrical (AV) node. The AV node then tells the heart to beat if the conditions are suitable.

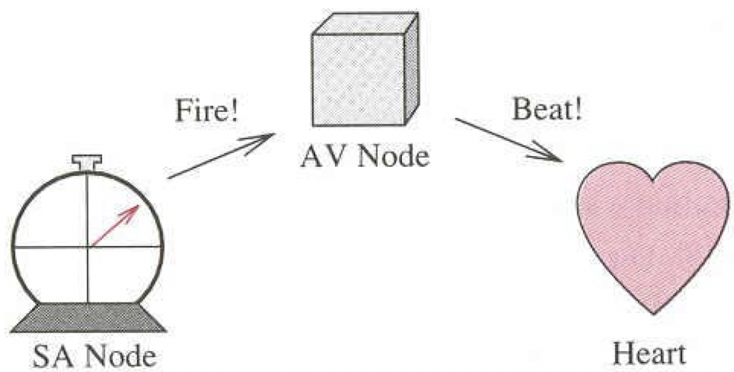

Figure 1: A mathematician's heart.

The AV node can be thought of as keeping track of the condition of the heart using an electrical potential.You may assume that the potential decreases exponentially during the time between signals from the SA node. Whether the heart beats depends on the response of the AV node when the signal arrives. If the potential is too high, it means that the heart has not had enough time to recover from the last beat, and the AV node ignores the signal. Otherwise, the AV node accepts the signal, tells the heart to beat, and increases the electrical potential (for simplicity, assume the increase is a constant).

Develop a model describing the electrical potential of the AV node. Under what conditions does your model produce regular heart beats? Investigate the production of irregular beating patterns by modifying parameters in your model. Two patterns of clinical interest are second-degree block and the Wenckebach phenomenon. Second-degree block refers to the situation in which the heart skips every other beat (that is, the AV node blocks every other signal from the SA node). The Wenckebach phenomenon refers to the situation in which the heart skips a beat every now and then, while it beats normally most of the time.

11 Arms race

In this project, we consider several models for an arms race. Consider two economically competing nations which we call Purple and Green. Both nations desire peace and hope to avoid war, but they are not pacifistic. They will not go out of their way to launch aggression, but they will not sit idly by if their country is attacked. They believe in self-defense and will fight to protect their nation and their way of life. Both nations feel that the maintenance of a large army and the stockpiling of weapons are purely "defensive" gestures when they do it, but at least somewhat "offensive" when the other side does it.

Since the two nations are in competition, there is an underlying sense of "mutual fear". The more one nation arms, the more the other nation is spurred to arm.

1. A Mutual Fear Model. Let $x(t)$ and $y(t)$ represent the yearly rates of armament expenditures of the two nations in some standardized monetary unit. To develop a model of mutual fear, we assume that each country adjusts the rate of increase or decrease of its armaments in response to the level of the other's. The simplest assumption is that each nation's rate is directly proportional to the expenditure of the other nation. That is,

$$\frac{dx}{dt} = ay \tag{8}$$

$$\frac{dy}{dt} = bx \tag{9}$$

where a and b are positive constants. Suppose the initial (at some arbitrary time $t = 0$) armament expenditures of the two nations are x_0 and y_0, respectively. Find $dy = dx$ in terms of x and y, and solve using the given initial conditions. Show that the solutions lie on a hyperbola, and interpret the results. Verify your solution numerically.

2. The Richardson Model. We now present a refinement of the mutual fear model. The mutual fear model produced a "runaway" arms race with unlimited expenditures. To prevent unlimited expenditures, we assume that excessive armament expenditures present a drag on the nation's economy so that the actual level of expenditure reduces the rate of change on the expenditure. The simplest way to model this is the assume that the rate of change for a nation is directly and negatively proportional to its own expenditure. This refines the mutual fear model to give

$$\frac{dx}{dt} = ay - mx \tag{10}$$

$$\frac{dy}{dt} = bx - ny \tag{11}$$

where a, b, m, and n are positive constants. Before proceeding with a mathematical analysis of this model, we introduce a further refinement. This refinement models any underlying grievances of each country toward the other. To model this, we introduce two additional constant terms, r and s, to the equations to obtain

$$\frac{dx}{dt} = ay - mx + r \tag{12}$$

$$\frac{dy}{dt} = bx - ny + s \tag{13}$$

A positive value of r and s indicates that there is a grievance of one country toward the other which causes an increase in the rate of arms expenditures. If r or s is negative, there there is an underlying feeling of good will, so there is a decrease in the rate of arms expenditures. This model is called Richardson's Arms Race Model in honour of Lewis F. Richardson, who considered this model in 1939 for the combatants of World War I.

Use phase plane analysis to investigate the behaviour of the Richardson model (look at the nullclines,determine steady states, and their stability). Focus on the following two scenarios:

(a) Mutual Grievances. Investigate the case where each side has a permanent underlying grievance toward the other side. In this case the parameters r and s are positive. Show that the steady state for the model either lies in the first or the third quandrant, depending upon the sign of $mn - ab$. Determine their stability. What does the model predict? Verify numerically.

(b) The Effect of Good Will. Feelings of good will are represented by one of the grievance terms r or s being negative. If both r and s are negative, show that the steady state is unstable. Further, show that if $mn - ab$ is positive, the arms race results in total disarmament of both sides. What happens if $mn - ab$ is negative? Verify your results numerically.

3. Open-ended Extensions of the Richardson Model. This part offers some suggested further extensions of the Richardson model. Some of these are much more involved than the work in parts 1 and 2 (you should view parts 1 and 2 as warm up problems for the actual project contained in part 3).

There are several directions one could choose to extend the Richardson model. Here are two. The first extension concerns a modification of the mutual fear term discussed in part 1. If we assume that there is an inherent limit to the amount a nation can spend on armaments and let K_p be the maximum expenditure of the Purple nation and K_g be the maximum expenditure of the Green nation, then the proportionality constant can be replaced by an expenditure-dependent rate. A simple way to do this is to replace exponential factors with logistic factors. Thus our arms race model becomes

$$\frac{dx}{dt} = a\left(1 - \frac{x}{K_p}\right) y - mx + r \tag{14}$$

$$\frac{dy}{dt} = b\left(1 - \frac{y}{K_g}\right) x - ny + s \tag{15}$$

Examine the stability of this "logistic" version of the Richardson model by performing an analysis similar to part 2. What does this model predict?

A second extension considers three mutually fearful nations. Each nation is spurred to arm by the expenditures of the other two. Build a Richardson model for three nations. Examine the stability of this model by performing an analysis similar to part 2. This three-nation model can be further modified if two of the nations are close allies who are not threatened by the arms buildup of each other but are threatened by the expenditures of the third.

12 Blood Alcohol

Data for this project:

1. The average human body eliminates 12 grams of alcohol per hour.

2. An average college-age male in good shape weighing K kilograms has about $0.68K$ liters of fluid in his body. A college-age female in good shape weighing K kilograms has about $0.65K$ liters of fluid in her body. People in poor shape have less.

3. One kilogram = 2.2046 pounds.

4. Threshold for legal driving: If your body fluids contain more than one gram of alcohol per liter of body fluids (or 0.1 gm/100 mL which is the usual way of reporting it), then you are too drunk to drive legally in most jurisdictions. Find out the level for Alberta, and use it in this project.

5. A blood alcohol concentration of 4.0 gm/L is likely to result in coma. A blood alcohol level of 4.5 to 5.0 gm/L is likely to result in death.

6. The alcohol content of various beverages is listed Table 1.

Table 1: Grams of alcohol for different types of drink.

Type of Drink	Grams of Alcohol
12 ounce regular beer	13.6
12 ounce light beer	11.3
4 ounce port wine	16.4
4 ounce burgundy wine	10.9
4 ounce rose wine	10.0
1.5 ounce 100-proof vodka	16.7
1.5 ounce 100-proof bourbon	16.7
1.5 ounce 80-proof vodka	13.4
1.5 ounce 80-proof bourbon	13.4

Construct a model for alcohol concentration based on the diagram shown in Figure 2 for a hypothetical person (gender and weight decided by you).

You may use a discrete-time model with a short time step (1 minute is suggested), or treat time continuously (ideally, you would do both and compare your results).

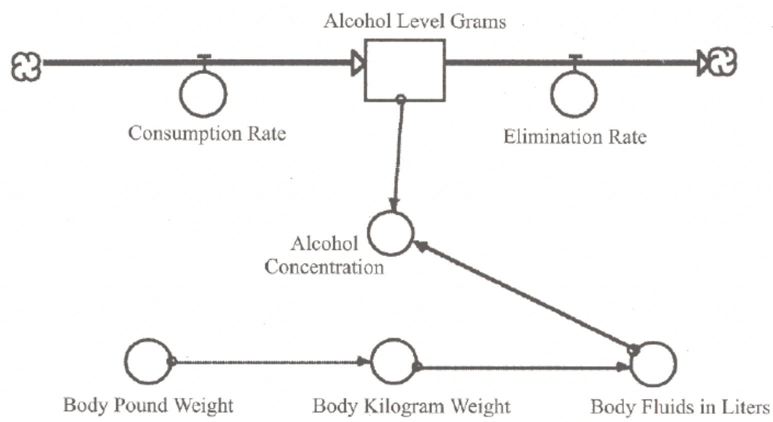

Figure 2: Compartmental diagram for the blood alcohol model.

Assume that Hypothetical arrives at a party and instantaneously downs a six-pack of beer. If Hypothetical does not enter a coma or die, how long will it be before he/she can drive home legally?

Construct a more realistic manner of consuming six beers. How does this affect Hypothetical's blood alcohol concentration? You may wish to use a piecewise defined function to model periods of drinking and non-drinking.

Construct an information sheet for distribution to your peers showing the effects of drinking style on blood alcohol concentration.

13 Carbon cycle

One class of environmental models tracks the flow of some element or compound through an ecosystem. This project examines the flow of carbon through several types of forest conditions.

Part 1. A Litter Model. We begin with a very simple model, tracking carbon levels in litter on a forest floor. In this context, litter is naturally-occurring debris such as leaves, branches, and dead falls, not beer bottles or fast-food wrappers. A boundary for the system is set up, and only litter within this region is considered. Thus, carbon is measured in density units: grams of carbon per square meter (g C/m^2). Here are the modelling assumptions:

1. Carbon continuously enters the system through litter fall at a constant rate z.

2. Carbon continuously leaves the system through two avenues - carbon dioxide produced in respiration and the conversion of litter into humus (called humification). Even though the carbon in humus has not left the physical system boundary, it is no longer in the litter, just as carbon in the trees before the leaves fell was in the physical boundary, but not yet in the litter.

3. The rate of litter fall is constant; the rate of carbon removal from both avenues is proportional to the amount of carbon present.

4. Initially, there is no carbon in the litter. This approximates the situation after a ground fire (a forest fire which burns the underbrush, but does not kill the trees).

Set up a differential equation for the carbon level, and solve it analytically. Verify numerically. For a temperate forest, it is reasonable to assume a rate of litter fall of 240 g C/m^2/yr and a proportionality constant for carbon removal of 0.4/yr. Using this information, graph a solution curve for 50 years either by numerically solving or by graphing the analytical solution using these parameters.

Part 2. Carbon in the Terrestial Biosphere. This model extends the model from part 1 to model carbon's flowing and being stored in an entire ecosystem. This ecosystem is assumed to have seven components: plants (subdivided into leaves, branches, stems, and roots), litter lying on the ground, humus, and stable humus charcoal. The amount of carbon in each of these components is given by the variables x_1, x_2, . . . , x_7, respectively. The atmosphere is, of course, another component, but due to its immense size, it is considered to have constant carbon content, unchanged either by giving carbon to plants or by absorbing carbon from the litter, humus, or stable humus charcoal. Technically the atmosphere is outside the system, so carbon to plants is carbon entering the system, and carbon out of the litter/soil components is carbon leaving the system. The parameter z denotes the carbon entering the system, and the partition parameters p_1 through p_4 indicate the percentage of z which goes into the leaves, branches, stems, and roots, respectively. The transfer coefficients k_{ij} give the rate of carbon flow from x_i to x_j (k_{i0} will denote the transfer from x_i to the atmosphere). A compartmental diagram for this system and model parameters for a variety of ecosystem types are shown in Figure 3.

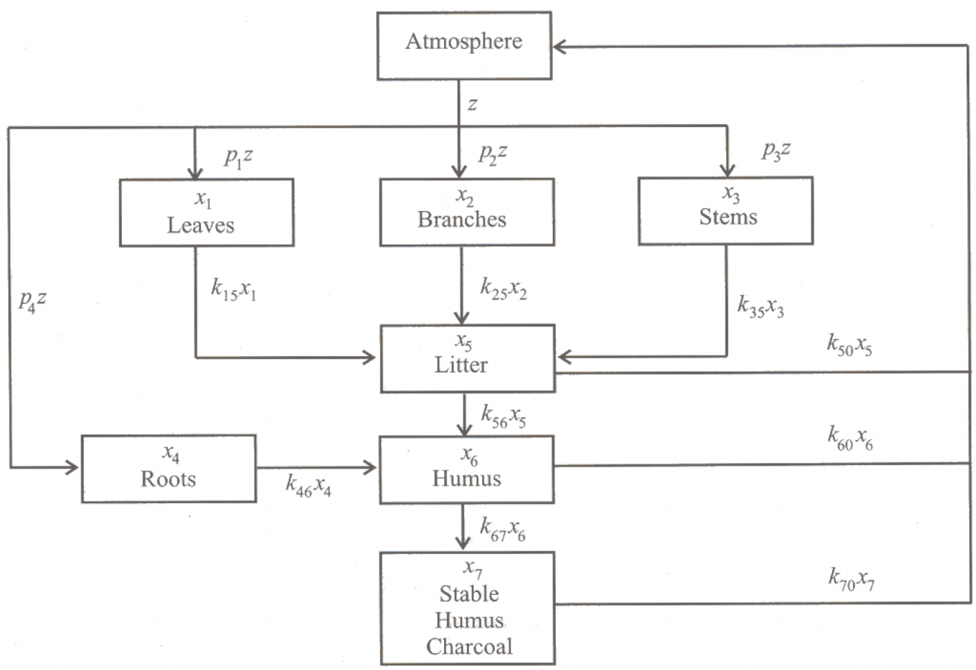

Figure 3: The terrestrial carbon system.

Tropical forest refers to tropical forest, forest plantation, shrub-dominated savannas, and chaparral. Temperate forests comprise temperate forests, boreal forests, and woodlands. Note that k_{56} and k_{k0} are not given, only an

Table 2: Parameters of carbon storage and flow in terrestrial ecosystems.

	Tropical forest	Temperate forest	Grass-land	Agri-cultural area	Human area	Tundra and semi-desert
Carbon entering System z (Gt C/yr)	27.8	8.7	10.7	7.5	0.2	2.1
Partition coefficients						
p_1 (Leaves)	0.3	0.3	0.6	0.8	0.3	0.5
p_2 (Branches)	0.2	0.2	0.0	0.0	0.2	0.1
p_3 (Stems)	0.3	0.3	0.0	0.0	0.3	0.1
p_4 (Roots)	0.2	0.2	0.4	0.2	0.2	0.3
Flows						
Leaves to litter	1.0	0.5	1.0	1.0	1.0	1.0
Branches to litter	0.1	0.1	0.1	0.1	0.1	0.1
Stems to litter	0.033	0.0166	0.02	0.02	0.02	0.02
Roots to humus	0.1	0.1	1.0	1.0	0.1	0.5
Leaving litter	1.0	0.5	0.5	1.0	0.5	0.5
Leaving humus	0.1	0.02	0.025	0.04	0.02	0.02
Charcoal to atmosphere	0.002	0.002	0.002	0.002	0.002	0.002
Humification h	0.4	0.6	0.6	0.2	0.5	0.6
Carbonization c	0.05	0.05	0.05	0.05	0.05	0.05
Areas ($10^{12}m^2$)	36.1	17.0	18.8	17.4	2.0	29.7

outflow of carbon from litter. As in part 1, carbon is lost from the litter state by either humification or respiration. The humification factor h gives the fraction of carbon leaving the litter state that goes into humus ; naturally, $1 - h$ indicates the fraction of carbon leaving the litter that does not go to humus, and hence goes to the atmosphere by respiration. Similar comments apply to k_{67} and k_{60} involving the amount of carbon leaving the humus and the coefficient of carbonization c which is the fraction going to form stable humus charcoal. The unit Gt is a Gigatonnes, or a billion tonnes, where 1 tonne (metric ton) = 1,000 kg = 1 Mg.

(a) Use the compartmental diagram to set up the system of equations for this model.

(b) For this part, use the parameters for the tropical forest ecosystem. First find the fixed points of this system. To do this, first write the answer to part (1) in the matrix form $X_0 = AX + B$. Fixed points occur when $X_0 = 0$, so solve the matrix equation $AX + B = 0$ (Matlab or Maple will come in handy here). Numerically solve the system to get graphs of each variable as a function of time. Run the simulation until all values are within 95% of their stable values. Compare the relative stabilization time for each component (how long it takes to get within 95% of the stable value). Compute the eigenvalues for the fixed point. Can you see a relationship between the size of the eigenvalues and the stabilization times?

(c) Repeat (b) for at least two other ecosystems (your choice) and compare the results.

14 Basketball Team

Suppose you are the general manager of a basketball team. The owner of the team has decided that he is willing to spend only a fixed amount in annual salaries (say K dollars). With this money, you need to sign 12 players. Your task is to decide how to allocate your money in order to win as many games as possible.

Develop a model that predicts the number of wins for a given roster. You should be able to find some data on the effect individual players have on the number of games their teams win (as well as their salaries). Keep in mind that a team typically has 5 starters, who play a lot of minutes and another 3-4 bench players who play significant minutes (but not as many as the starters). The remaining players typically play very little. The effect a player has on the total number of wins should depend on the number of minutes he plays. Is it better to pay a lot of money for a few superstar players and fill out the roster with less talented players, or should you distribute your money more evenly, in order to sign several fairly talented players(who are still not superstars)? You should also consider the effect of varying the number of minutes each player plays each game. Keeping your best players on the floor for longer might seem like a good strategy, but keep in mind that their performance may be diminished by fatigue.

15 Airline Hub

The SHORT-HOP airline is about to begin service in the north central USA, providing transportation between nine cities: Dayton (D), Fort Wayne (FW), Green Bay (GW), Grand Rapids (GR), Kenosha (K), Marquette(M), Peoria (P), South Bent(SB) and Toledo (T). As it would be very expensive to fly from each city to every other city, the company would prefer to route flights through a small number of hubs. They make the following assumptions:

1. All cities should be within 200 miles of a hub city.
2. The number of hub cities should be as small as possible. The pairwise distances between the cities (in miles) are given in Table 3

Table 3: Pairwise distances between the cities (in miles)

	D	F	GB	GR	K	M	P	SB	T
D	0	102	381	232	272	494	294	170	134
F		0	279	133	175	395	235	72	91
GB			0	159	134	143	273	215	298
GR				0	115	262	255	94	139
K					0	275	156	103	229
M						0	416	341	387
P							0	186	320
SB								0	139
T									0

1. Find an optimal solution for SHORT-HOP. Is this solution unique?

2. Can you identify a general principle or method for these types of optimization problems?

16 Don't Lek the Frogs

(Thanks to Barry Bohnet)A "lek" is a gathering of males during mating season where the males display certain traits that the females of the species are attracted to. This occurs in many species including some deer, grouse, peacocks, fish, and many others. The attractive trait could be as simple as bright colours or as elegant as a song. According to Fisher's sexy son hypothesis, this results in most of the females mating with a select group of males. The effects of this selection result in a population bottleneck which is a reduction of gene flow.DNA is the simplest unit of heredity. Most animals are diploid organisms, meaning that each individual contains two copies of chromosome. These chromosomes are long strands of DNA made up of genes. Each gene can come in a variety of forms, where each of these forms is called an allele. In the sex cells that each parent passes on to its child, there is one of the two chromosome copies. In other words each parent passes on 50% of its genes. A genotype refers to the pair of alleles that an individual has. G. H. Hardy and Wilhelm Weinberg came up with a formula known as the Hardy-Weinberg Principle:

$$p^2 + 2pq + q^2 = 1 \tag{16}$$

where p and q are the frequencies of two alleles, of one gene, present in the current generation. The frequencies p and q must add up to 1, since they are the only alleles of a certain gene. This formula gives us genotypic frequencies of the following generation.

Using the Hardy-Weinberg Principle we shall model the diversity of a bird species that uses leks in its mating patterns, and determine what type of effect this has on genotypic frequencies, what causes stable fixed points, and where they occur.

17 Hockey Coach

Assume you are a primer league hockey coach. Of course your goal is to assemble the best team possible.However, the owner can provide only a finite amount of funds to purchase players. You are faced with the challenge to hire high-level players who are all reasonably good, or to hire some few super stars but also some mediocre players.

1. The hockey league keeps statistics about each relevant or irrelevant information on players. Based on your experience and a web-search, determine relative quality measures for your players. First use very few quality measures, later use more.

2. Design a mathematical model which translates these individual measures into a quality measure for the team. Compare your quality measure with the performance history of a few teams.

3. With this model come back to the original question. Is it better to spend lots of money on super-stars,or is it better to hire a good homogeneous team?

4. Include the effect of your strategy on your fans. A super star will draw more people into the arena and your revenue from tickets, merchandise and advertisement goes up.

5. Consider extra motivation of the players for bonuses for goals or other achievements.

6. Let me know when you can start working for the Oilers.