MATH 372 - Salmon Draft

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Formulation, Solution, Code, Critique

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Introduction, Interpretation of results, Critique

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1 Introduction

The life of a Pacific salmon consists of a series of stages, each involved in complex ecological processes. The point of interest in our model is the dynamic exhibited between salmon and their larvae.

1.1 Pacific salmon life cycle

The Pacific salmon life cycle consists of the following stages:

1. Salmon begins in freshwater as larvae (Pacific Salmon Foundation). Unfortunately, larvae are at risk of predation by their parents.

2. The surviving larvae eventually reach maturity. After their bodies transform, they move from freshwater to saltwater, entering the sea as juvenile adults (Pacific Salmon Foundation).

3. Once fully matured, the adults will transition back to freshwater and begin the breeding process (Pacific Salmon Foundation). The majority of the previous adult population dies at this stage.

4. The cycle repeats with the new salmon population.

1.2 Questions we hope to answer

The intra-species predation present within the Pacific salmon lifecycle poses the following questions:

1. How does the larval population change as a function of adult population size and intra-species predation? How do factors such as fishing, pollution... etc affect the population?

2. What are the governing equations that describe the rate of larval decline due to cannibalism and the transition to the young adult stage?

3. Does the salmon population exhibit cyclical behavior? Can an equilibrium solution be found? How do variations in the provided proportionality constants impact the model?

The answers to these questions can be found by building a mathematical model for Pacific salmon populations.

2 Formulation

2.1 Parameters

x_n - Number of hundreds of millions of Pacific salmon at the beginning of the n^{th} cycle.

$y(t)$ - Larval population at time t in the n^{th} cycle.

α - Proportionality constant for decay of larval population.

β - Proportionality constant for reproduction of the larvae.

γ - Proportionality constant representing the fraction of larvae that survive going to sea after reaching maturity.

$t_0 < t < t_e$ - Interval of decay for larvae population during a cycle.

2.2 Assumptions

1. We do not consider the impact of external factors, such as fishing and pollution, on the salmon population in the middle of a cycle. Consequently, we assume that all salmon are taken into account when calculating the number of larvae eggs produced (y_n) in a cycle.

2. We assume the range $t_e - t_0$ remains fixed across all cycles.

3. We assume the initial population $x_0 = 1$ million (0.01 hundred million in code).

4. The entire adult salmon population dies between cycles. They are replaced by their surviving offspring that successfully reach maturity.

3 Solution

3.1 Formulation of initial model

At the end of a cycle, the entire adult salmon population dies. They are replaced by a portion of the larvae population which reach maturity. We model the number of salmon in the $n+1^{th}$ cycle using the following equation;

$$x_{n+1} = \gamma y(t_e)$$

The number of larvae produced in the n^{th} cycle is defined using the following equation. The number produced at the start of the cycle (t_0) will decay during the cycle.

$$y(t_0) = \beta x_n$$

The rate of decay in the larvae population caused by cannibalism is proportional to the number of interactions with the adult population over the course of a cycle. We model the number of larvae using the following equation;

$$\frac{dy}{dt} = -\alpha x_n \, y(t)$$

To find $y(t)$, we integrate both sides from t_0 to t:

$$\int_{y(t_0)}^{y(t)} \frac{1}{y} \, dy = \int_{t_0}^{t} -\alpha x_n \, dt.$$

The integral evaluates to the following:

$$\ln\big(y(t)\big) - \ln\big(y(t_0)\big) = -\alpha x_n (t - t_0).$$

Our previous expression is equivalent to:

$$\ln\Big(\frac{y(t)}{y(t_0)}\Big) = -\alpha x_n (t - t_0).$$

We manipulate our result to isolate y(t)

$$y(t) = y(t_0) \, \exp\big[-\alpha x_n (t - t_0)\big].$$

We are concerned with the population of larvae at time t_e. This is the population of larvae which will attempt to go out to sea. A portion of this population will eventually reach maturity and represent x_n in the next cycle.

$$y(t_e) = y(t_0) \, \exp\big[-\alpha x_n (t_e - t_0)\big].$$

We use $y(t_e)$ (salmon that reached maturity during the current cycle) in our original equation used to calculate x_{n+1}.

$$x_{n+1} = \gamma y(t_e)$$

$$x_{n+1} = \gamma[y(t_0)\ \exp\big[-\alpha x_n(t_e - t_0)\big]]$$

Using our definition of $y(t_0)$, we get the following equation. We will use this equation in subsequent sections to generate our figures and analysis.

$$x_{n+1} = \gamma(\beta x_n)\ \exp\big[-\alpha x_n(t_e - t_0)\big]$$

3.2 Finding fixed points

We find our fixed-point solution, x^*, as follows:

$$x^* = \gamma\beta x^*\ \exp\big[-\alpha x^*(t_e - t_0)\big]$$

Note, we have a trivial fixed point $x^* = 0$. We will omit the analysis of this fixed point.

Dividing both sides by x^* produces:

$$1 = \gamma\beta\ \exp\big[-\alpha x^*(t_e - t_0)\big]$$

Next, we rearrange to isolate our exponent:

$$\frac{1}{\gamma\beta} = \exp\big[-\alpha x^*(t_e - t_0)\big]$$

Finally, we isolate the remaining x^*.

$$\ln\frac{1}{\gamma\beta} = -\alpha x^*(t_e - t_0)$$

$$\ln 1 - ln\gamma\beta = -\alpha x^*(t_e - t_0)$$

$$-ln\gamma\beta = -\alpha x^*(t_e - t_0)$$

$$x^* = \frac{-ln\gamma\beta}{-\alpha(t_e - t_0)}$$

$$x^* = \frac{ln\gamma\beta}{\alpha(t_e - t_0)}$$

3.3 Explanation of solution in software

Using MATLAB, we modeled the salmon population using the equation derived in Section 3.1. The process was as follows;

1. Establish parameters α, β, γ, t_0, t_e
2. Set the number of cycles we wish to test ($N = 30$)

3. Create an array to store values of x_n for each cycle. Each entry will store the value of the n^{th} cycle. The values from previous cycles will be used in subsequent iterations.
4. Loop through N cycles. For each cycle, calculate x_{n+1} as follows:

$$x_{n+1} = \gamma\beta x_n \exp\big[-\alpha x_n(t_e - t_0)\big]$$

5. Plot all all x_n values. The x-axis corresponds to the cycle n, and the y-axis corresponds to the population at the n^{th} cycle.

3.4 Formulation of improved model

In the next iteration of our model, we attempt to model the variability of environmental factors between cycles.

As stated in Section 2.1, we have a range of permissible values for products of the following proportionality constants:

$$1 < \alpha(t_e - t_0) < 10$$

$$3 < \beta\gamma < 20$$

We define parameters A and B as follows:

$$A = \alpha(t_e - t_0)$$

$$B = \beta\gamma$$

We randomly generate a value for A and B in each cycle to model the day-to-day varaiability of environmental factors which may affect our proportionality constants. For example, a high B value may reflect an improvement in environmental conditions, which consequently increases the reproduction rate of the adult salmon population and increases the survival rate of the maturing Larvae.

We utilize the randomly selected values in our prior model as follows:

$$x_{n+1} = \gamma\beta x_n \exp\big[-\alpha x_n(t_e - t_0)\big]$$

$$x_{n+1} = Bx_n \exp\big[-Ax_n\big]$$

4 Interpretation of Results

4.1 Initial model

We ran our program using various parameter settings (for α, β and γ). Each figure is annotated with the parameters used and includes an analysis of the behaviour below.

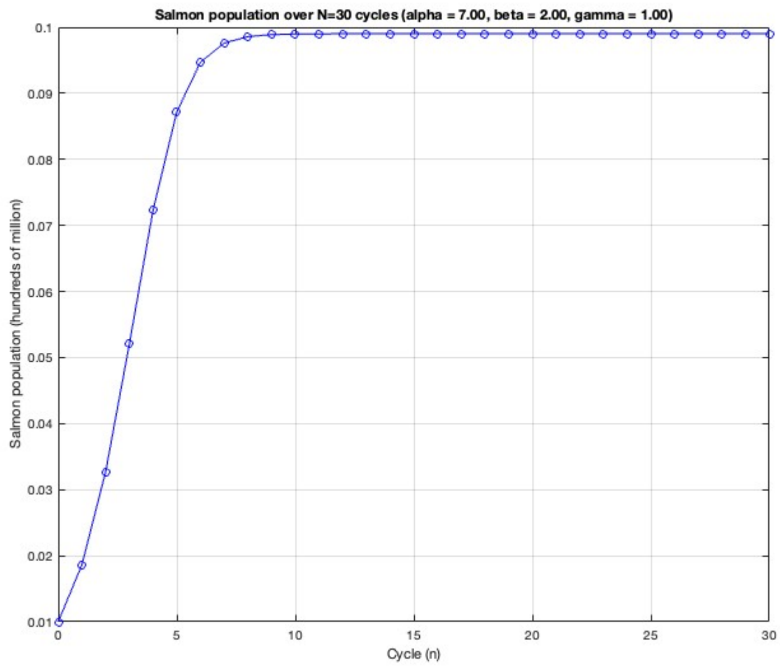

Figure 1: Fixed point = 0.0990

Using parameters $\alpha = 7$, $\beta = 2$ and $\gamma = 1$, we get a fixed point at 0.099. Figure 1 shows that the population stays relatively constant across cycles once our fixed point is reached, showing that the system is in equilibrium. This suggests that our fixed point is stable.

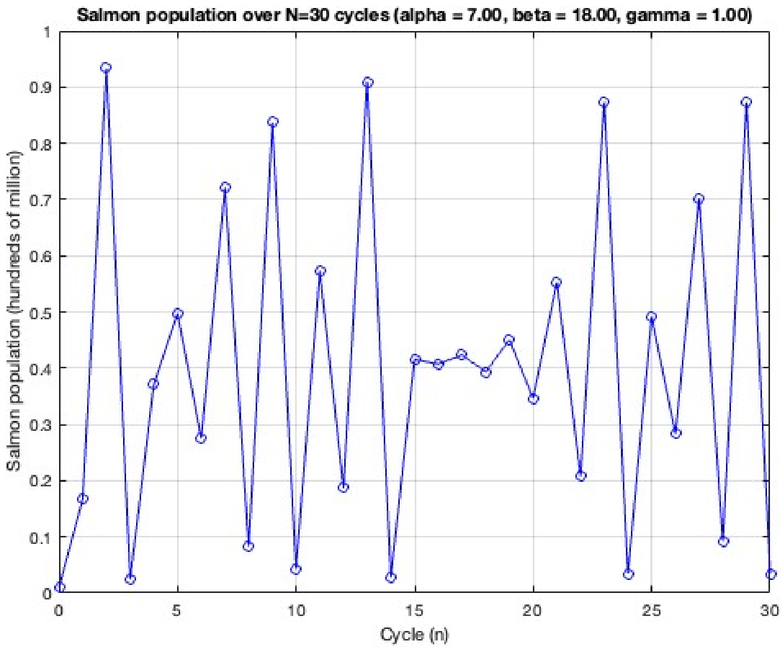

Figure 2: Fixed point = 0.4129

Figure 2 shows model behavior when parameters are $\alpha = 7$, $\beta = 18$ and $\gamma = 1$. Under these parameters, the fixed point produced is not stable. The salmon populations fluctuate indefinitely across cycles. The model illustrates how a large reproduction coefficient affects the rate of cannibalism and reproduction, leading to large spikes and dips across cycles.

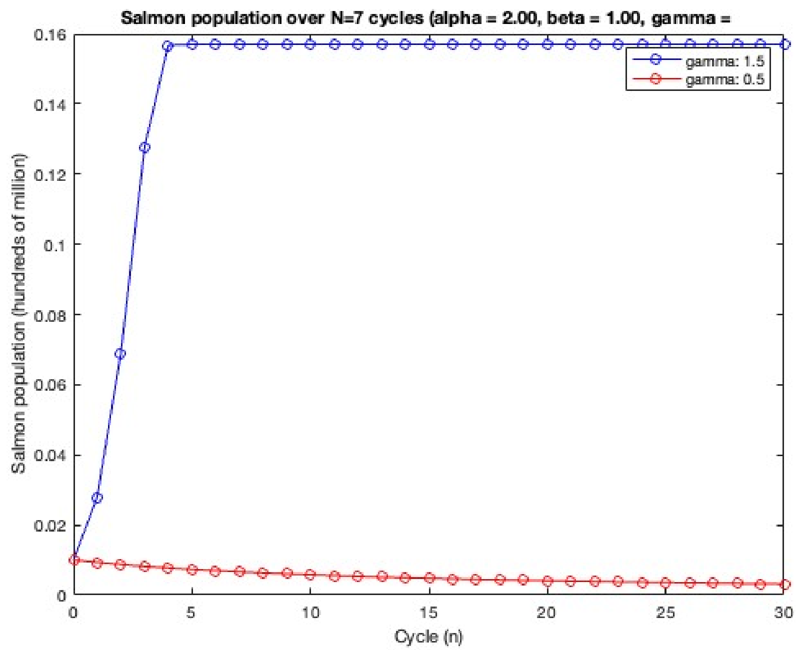

Figure 3: Gamma Comparison

Figure 3 shows model behavior when parameters are $\alpha = 7$, $\beta = 18$. We vary the values of gamma between the two displayed plots to demonstrate the effect that the survival constant has on population between cycles. When $\gamma = 1.5$, the population increases between cycles and eventually stabilizes. When $\gamma = 0.5$, the population gradually declines between cycles and eventually stabilizes at a population level lower than our initial value of x_0. This behaviour illustrates the effects of the cannibalism constant on population growth.

4.2 Improved model

4.2.1 Run 1

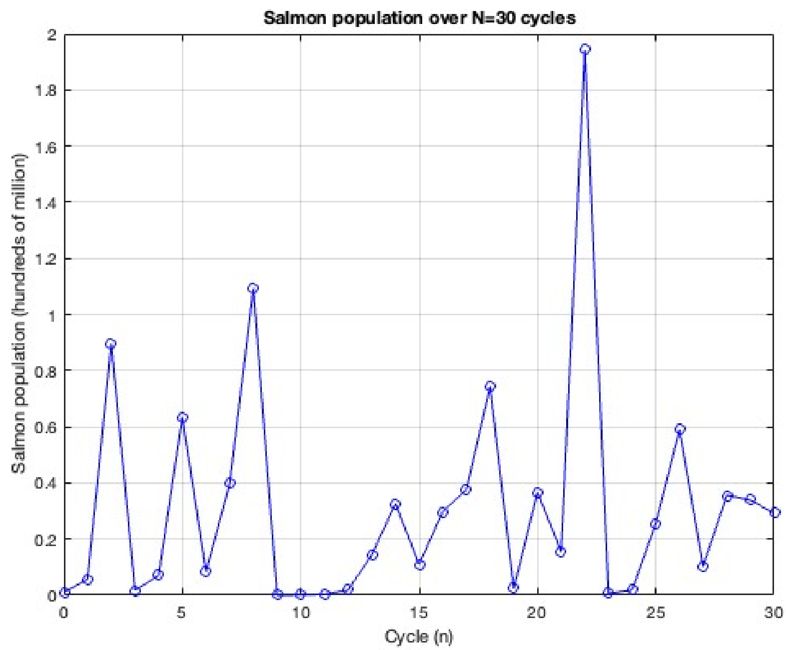

Figure 4: Population model - Randomly generated A,B values

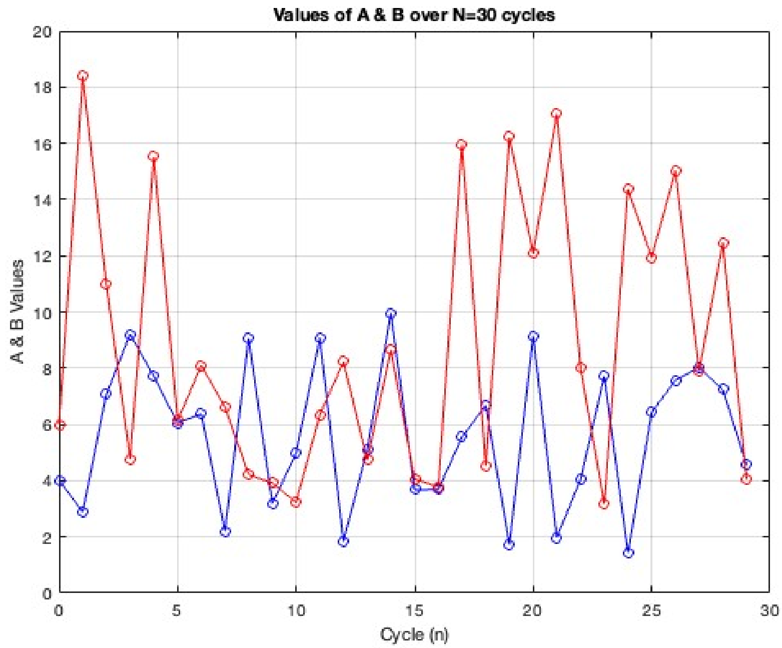

Figure 5: A, B values in each cycle for simulation shown in Figure 4

4.2.2 Run 2

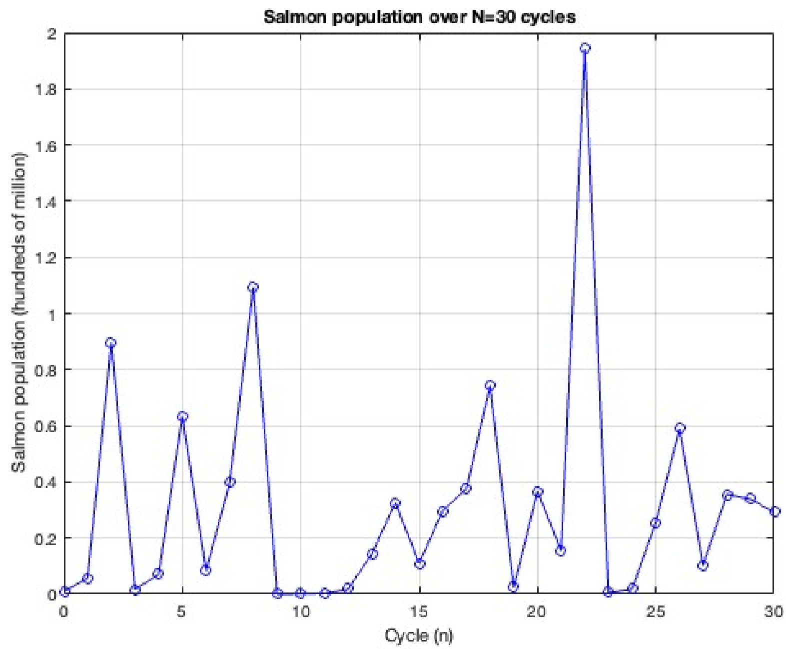

Figure 6: Population model - Randomly generated A,B values

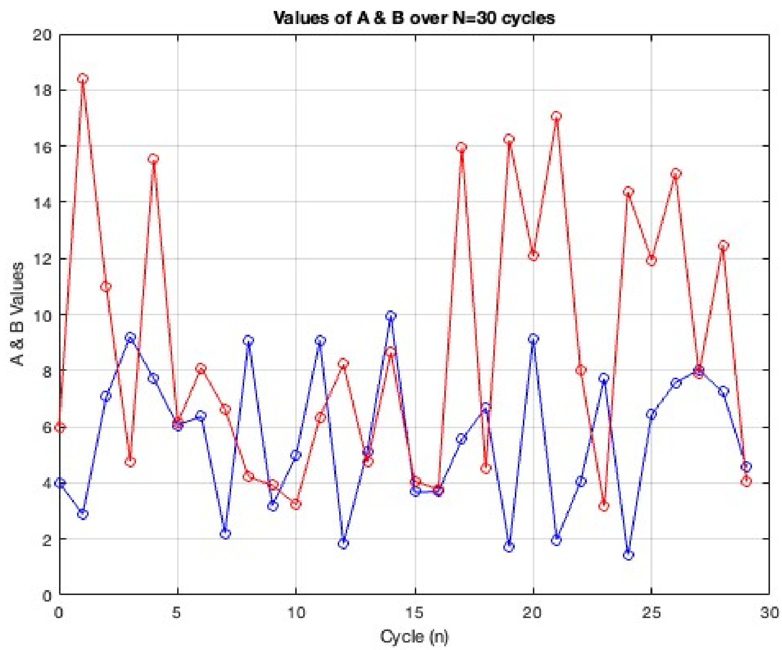

Figure 7: A, B values in each cycle for simulation shown in Figure 4

[ ] TOOD: Is there anything we should add here?

5 Critiques of Model

1. The lack of consideration for external factors such as fishing rates in the region, pollution, and other predators is a limiting factor in our model. These factors are likely to impact the rate of decay in the salmon and larvae populations.

2. We do not consider the carrying capacity and quantity of food available in the body of water in our model. We realized this too late, and didn't have time to make the appropriate revisions before the draft deadline. We will take this into consideration in future variations of the model.

3. Additional research has to be done to generate realistic figures for the rate of decay, initial population, and time per reproductive cycle. The current values were picked for convenience to demonstrate different behaviors in the model. We are unsure if this is a sensible approach.

4. The defined proportionality constants may not be constants in reality. The reproduction, survival and cannibalization rates are likely to be affected

by external factors. We believe these factors are too complex to model for the scope of this assignment; however, modeling them in a more realistic way would likely produce a more accurate representation of reality.

6 References

1. "10: Population Modeling." Biology LibreTexts, Libretexts, 23 Oct. 2023, `bio.libretexts.org/Courses/Gettysburg_College/01%3A_Ecology_for_All/10%3A_Population_modeling`.

2. "The Salmon Life Cycle." National Park Service, US Department of the Interior, `www.nps.gov/olym/learn/nature/the-salmon-life-cycle.htm`. Accessed 17 Mar. 2025.

3. "Species & Lifecycle." Pacific Salmon Foundation, 5 Sept. 2021, `psf.ca/learn/species-lifecycle/`.

4. Class notes

7 Appendix

1. MATLAB Code: `https://github.com/ethandt24/salmon_project/blob/main/salmon.m`