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Modeling HW2

1a Expected obtained = 20 - Expected not obtained

$E[X] = 20 - \left(\left(\frac{19}{20}\right)^{60} \cdot 20\right) = 20 - 0.9214 = \boxed{19.0786}$

b expected number of times to get 1 unique → 1

expected number to get 2 unique 2 draws: $\frac{19}{20}$ 3 draws: $\frac{1}{20} \cdot \frac{19}{20}$ n draws: $\frac{19}{20} \cdot \left(\frac{1}{20}\right)^{n-2}$

expected number of additional to get n draws: immediate: $\frac{20-n+1}{20}$

$\hookrightarrow E[X] = \sum_{k=1}^{\infty} \frac{20-X}{20} \frac{X}{20}^{k-1} \longrightarrow \underbrace{\sum_{X=0}^{19} \sum_{k=1}^{\infty} \frac{20-X}{20} \left(\frac{X}{20}\right)^{k-1} k}_{p} = \sum_{X=0}^{19} \frac{20}{20-X}$

go from x to x+1

$p = \frac{20-X}{20} \longrightarrow$ $E[X] = \sum_{X=0}^{19} \frac{20}{20-X}$

$\mathrm{Var}(X) = \sum_{X=0}^{19} \frac{\frac{X}{20}}{\left(\frac{20-X}{20}\right)^2} = \sum_{X=0}^{19} \frac{20X}{(20-X)^2}$

3. $f_{X+Y}(t) = \int_0^t f_{X,Y}(x, t-x)\,dx = \int_0^t f_X(x) f_Y(t-x)\,dx$

$f_{Y|X}(Y|X) = \frac{f_{Y,X}(Y,X)}{f_X(X)} \longrightarrow \frac{P(X=x, Y=t-x)}{P(X+Y=t)} \longrightarrow \frac{f_X(x) f_Y(t-x)}{\int_0^t f_X(x) f_Y(t-x)\,dx}$

$\int_0^t \frac{1}{t}\,dx$

$f_{X,Y}(X,Y) = f_{X,Y}(X, t-X) = f_X(X) f_Y(t-X)$

$\frac{\cancel{\lambda^2} e^{-\lambda t}}{\cancel{\lambda^2} \int_0^t e^{-\lambda t}\,dx} \longleftarrow \frac{\lambda e^{-\lambda x} \lambda e^{-\lambda(t-x)}}{\int_0^t \lambda e^{-\lambda x} \lambda e^{-\lambda(t-x)}\,dx} \longleftarrow$ Using λ for X and Y

$\frac{e^{-\lambda t}}{e^{-\lambda t} X \Big|_0^t} = \frac{e^{-\lambda t}}{e^{-\lambda t} t - e^{-\lambda t} 0} = \frac{e^{-\lambda t}}{e^{-\lambda t} t} = \frac{1}{t} = \underbrace{\int_0^t \frac{1}{t}\,dx}_{\text{uniform distribution form 0 to t}}$

4.a $\lambda = \lambda_1 + \lambda_2$ $\quad X = x_1 + x_2$

$$p_X(X) = P(X = X) = \sum_{j=0}^{z} P(X_1 = j, X_2 = X - j) \to X_1 + X_2 = X \to \sum_{j=0}^{z} \frac{e^{-\lambda_1}\lambda_1^j}{j!} \frac{e^{-\lambda_2}\lambda_2^{z-j}}{(z-j)!}$$

$$\frac{e^{-\lambda}\lambda^x}{x!} \leftarrow \frac{e^{-\lambda}}{z!}(\lambda_1+\lambda_2)^z \leftarrow \frac{e^{-\lambda}}{z!}\sum_{j=0}^{z}\binom{z}{j}\lambda_1^j\lambda_2^{z-j} \leftarrow \sum_{j=0}^{z}\binom{z}{j}\frac{e^{-\lambda_1}\lambda_1^j e^{-\lambda_2}\lambda_2^{z-j}}{z!} \leftarrow$$

$\to$ poisson distribution, Mean and Variance are $\lambda = \lambda_1 + \lambda_2$

b. $P[X_1 = i | Y = n] \to P[X = x | Y = y] = \frac{P(X = x, Y = y)}{P(Y = y)}$ | X, Y - total customers

$\frac{\lambda_1}{\lambda_1+\lambda_2} = p \to \binom{n}{i}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^i\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-i}$

$$P(X=x) = \sum \underbrace{P(X=x, Y=y_i)} = P(X=x|Y=y_i)\cdot P(Y=y_i) \to P(X=x) = \sum P(X=x|Y=y_i)\cdot P(Y=y_i)$$

$$P(X=i) = \sum P(X=i|Y=n)\cdot P(Y=n) \to \sum_n \frac{e^{-(\lambda_1+\lambda_2)}\cancel{(\lambda_1+\lambda_2)^n}}{\cancel{n!}}\cdot\frac{\cancel{n!}}{(n-i)!\,i!}\cdot\left(\frac{\lambda_1}{\cancel{\lambda_1+\lambda_2}}\right)^i\cdot\left(\frac{\lambda_2}{\cancel{\lambda_1+\lambda_2}}\right)^{n-i}$$

$$\boxed{\begin{array}{l}\text{Store 1: poisson}(\lambda_1)\\ \text{Store 2: poisson}(\lambda_2)\end{array}} \leftarrow \frac{e^{-\lambda_1}\lambda_1^i}{i!}\underbrace{\sum_n \frac{e^{-\lambda_2}\lambda_2^{n-i}}{(n-i)!}}_{\text{Sum of Poisson}\to 1} \leftarrow \frac{e^{-(\lambda_1+\lambda_2)}\lambda_1^i}{i!}\sum_n \frac{\lambda_2^{n-i}}{(n-i)!}$$

c. $0.8 = \frac{Cov(X_1, X_2)}{\sqrt{var(X_1)\,var(X_2)}} \to Cov(X_1, X_2) = 0.8\sqrt{var(X_1)\,var(X_2)}$

$Var(X_1+X_2) = var(X_1) + var(X_2) + 2(0.8\sqrt{var(X_1)\,var(X_2)})$

$\to \boxed{\lambda_1 + \lambda_2 + 1.6\lambda_1\lambda_2} \to$ equal to mean since poisson

5.b income $\to$ poisson$(\log(1+t))\cdot p - Ct \to$ Mean trips is $\log(1+t)$

$$\frac{d}{dt}(p\log(1+t) - Ct) = 0 \leftarrow p\log(1+t) - Ct$$

$$p\cdot\frac{1}{1+t} - C = 0 \to p\cdot\frac{1}{1+t} = C$$

$$\leftarrow \frac{1}{1+t} = 0.25 \leftarrow 12\cdot\frac{1}{1+t} = 3$$

$$1 = 0.25 + 0.25t \to 0.75 = 0.25t \to \boxed{t = \frac{0.75}{0.25} = 3}$$